# OBSERVABILITY INEQUALITY OF BACKWARD STOCHASTIC HEAT EQUATIONS FOR MEASURABLE SETS AND ITS APPLICATIONS 

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#### Abstract

This paper aims to provide directly the observability inequality of backward stochastic heat equations for measurable sets. As an immediate application, the null controllability of the forward heat equations is obtained. Moreover, an interesting relaxed optimal actuator location problem is formulated, and the existence of its solution is proved. Finally, the solution is characterized by a Nash equilibrium of the associated game problem.


## 1. Introduction

Observability inequality is an important and powerful tool for the study of stabilization and controllability problems of partial differential equations. However, most of related works for heat equations concern with the internal control living on an open subset. Recently, the authors in [3, 14] establish the observability inequality of the heat equation for the measurable subsets, and show the null controllability with controls restricted over these sets. This generalization facilitates the study of the optimal actuator location problem for a wider class of equations. For example, compared to the one dimensional case studied in [1] and a special class of controlled domains considered in [8], the authors in [7] investigate the optimal actuator location of the minimum norm controls for heat equations in arbitrary dimensions, and the actuator domain is only required to have a prescribed Lebesgue measure.

One of the main contributions of this paper is the direct derivation of the observability inequality for stochastic backward heat equations for measurable subsets, which is considered very challenging and difficult in [18, page 99 and page 108-110]. By duality we obtain the null controllability for the corresponding forward equation. Our results extend the deterministic case to the stochastic counterpart. It is worth noting that we cannot simply mimic the calculations in the deterministic case by applying the time change technique, and treat the backward and forward equations in the same way, since adaptedness is always required in the stochastic system. On the other hand, our observability estimate also recovers the result in [11, Proposition 4.1], where only open controlled domain is considered, and the result is obtained by null controllability. For more general stochastic parabolic equations, but with two controls, we refer the reader to the work in [16].

[^0]As an important application, we consider the optimal actuator location of the minimum norm control problem for internal null controllable stochastic heat equations. In fact, the actuator location problem for deterministic equations has been widely studied; see for example, [1, [5, 7, 15, and also numerical research in 12, 13, 17. To the best of our knowledge, this paper is the first attempt to consider the shape optimization for the stochastic system. We show the existence of the minimum norm control, which is done by solving a variational problem with suitable norms guaranteed by the observability inequality. Then we prove the existence of the relaxed optimal actuator location and characterize the solution of the relaxed problem via a Nash equilibrium.

Before we state our main theorems, let us introduce necessary notations.
Let $T>0$ be a fixed positive time constant, and $D$ be a bounded domain in $\mathbb{R}^{d}$ with a $C^{2}$ boundary $\partial D$. Let $E$ and $G$ be measurable subsets with positive measures of $[0, T]$ and $D$, respectively.

Throughout this paper, we denote by $(\cdot, \cdot)$ the inner product in $L^{2}(D)$, and denote by $\|\cdot\|$ the norm induced by $(\cdot, \cdot)$. We also use the notations $(\cdot, \cdot)_{G}$ and $\|\cdot\|_{G}$ for the inner product and the norm defined on $L^{2}(G)$, respectively. We denote by $|\cdot|$ the Lebesgue measure on $\mathbb{R}^{d}$.

Let $\mathbb{F}=\left(\Omega, \mathcal{F},\left\{\mathcal{F}_{t}\right\}_{t \geq 0}, \mathbb{P}\right)$ be a stochastic basis with usual conditions. On $\mathbb{F}$, we define a standard scalar Wiener process $W=\{w(t)\}_{t \geq 0}$. For simplicity, we assume that the filtration $\left\{\mathcal{F}_{t}\right\}_{t \geq 0}$ is generated by $W$.

Given a Hilbert space $H$, we denote by $L_{\mathcal{F}}^{2}(0, T ; H)$ the Banach space consisting of all $H$-valued $\left\{\mathcal{F}_{t}\right\}_{t \geq 0}$-adapted processes $X$ such that the square of the canonical norm $\mathbb{E}\|X(\cdot)\|_{L^{2}(0, T ; H)}^{2}<\infty$; denote by $L_{\mathcal{F}}^{\infty}(0, T ; H)$ the Banach space consisting of all $H$-valued $\left\{\mathcal{F}_{t}\right\}_{t \geq 0}$-adapted bounded processes, with the essential supremum norm; and denote by $L_{\mathcal{F}}^{2}(\Omega ; C([0, T] ; H))$ the Banach space consisting of all $H$-valued $\left\{\mathcal{F}_{t}\right\}_{t \geq 0^{-}}$-adapted continuous processes $X$ such that the square of the canonical norm $\mathbb{E}\|X(\cdot)\|_{C(0, T ; H)}^{2}<\infty$. For any $t \in[0, T]$, the space $L^{2}\left(\Omega, \mathcal{F}_{t}, \mathbb{P} ; H\right)$ consists of all $H$-valued $\mathcal{F}_{t}$-measurable random variables with finite second moments.

Let $A$ be an unbounded linear operator on $L^{2}(D)$ :

$$
\mathcal{D}(A)=H^{2}(D) \cap H_{0}^{1}(D), \quad A v=\Delta v, \quad \forall v \in \mathcal{D}(A)
$$

The goal of this paper is to derive directly the observability inequality for the following backward stochastic heat equation

$$
\left\{\begin{array}{l}
d z=-A z d t-a(t) Z d t+Z d w(t), \quad t \in(0, T)  \tag{1.1}\\
z(T)=\eta
\end{array}\right.
$$

where $a \in L_{\mathcal{F}}^{\infty}(0, T ; \mathbb{R})$. For each $\eta \in L^{2}\left(\Omega, \mathcal{F}_{T}, \mathbb{P} ; L^{2}(D)\right)$, it is known (see for example [9, 6) that the equation (1.1) admits a unique solution $(z, Z)$ in the space of $\left(L_{\mathcal{F}}^{2}\left(\Omega ; C\left(0, T ; L^{2}(D)\right)\right) \cap L_{\mathcal{F}}^{2}\left(0, T ; H_{0}^{1}(D)\right)\right) \times L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right)$.

The following is our main theorem.
Theorem 1.1. Let $D$ be a bounded domain in $\mathbb{R}^{d}$ with a $C^{2}$ boundary. Let $x_{0} \in D$ and $R \in(0,1]$ such that $B_{4 R}\left(x_{0}\right) \subseteq D$. Suppose $G$ is a subset of $D$ with positive measure, contained in $B_{R}\left(x_{0}\right)$, and $E$ is measurable subset of $[0, T]$ with positive
measure. Then there exists a constant $C=C(T, D, R,|G|,|E|)$ such that the following observability inequality holds: for any $\eta \in L^{2}\left(\Omega, \mathcal{F}_{T}, \mathbb{P} ; L^{2}(D)\right)$,

$$
\begin{equation*}
\mathbb{E}\|z(0 ; T, \eta)\|^{2} \leq C\left(\int_{E}\left(\mathbb{E}\|z(t ; T, \eta)\|_{G}^{2}\right)^{1 / 2} d t\right)^{2} \tag{1.2}
\end{equation*}
$$

As a result, we obtain the null controllability for a class of forward stochastic heat equations:

$$
\left\{\begin{array}{l}
d y=A y d t+\chi_{E} \chi_{G} u(t) d t+a(t) y d w(t), \quad t \in(0, T)  \tag{1.3}\\
y(0)=y_{0}
\end{array}\right.
$$

Theorem 1.2. The equation (1.3) is $L^{\infty}$-null controllable. That is, for each initial data $y_{0} \in L^{2}\left(\Omega, \mathcal{F}_{0}, \mathbb{P} ; L^{2}(D)\right)$, there is a control $u$ in the space $L_{\mathcal{F}}^{\infty}\left(0, T ; L^{2}(D)\right)$ such that the solution $y$ of the equation (1.3) satisfies $y_{T}=0$ in $D, \mathbb{P}$-a.s. Moreover, the control $u$ satisfies the following estimate

$$
\begin{equation*}
\mathbb{E}\|u\|_{L^{\infty}\left(0, T ; L^{2}(D)\right)}^{2} \leq C \mathbb{E}\left\|y_{0}\right\|^{2} \tag{1.4}
\end{equation*}
$$

The rest of the paper is organized as follows. In Section 2, we prove our main theorems. In Section 3, we discuss the relaxed optimal actuator location problem. More specifically, we state and formulate the problem in Section 3.1. In Section 3.2 , we show the existence of the optimal minimal norm control. In Section 3.3 the existence of relaxed optimal actuator location is proved. Finally, Section 3.4 provides the characterization of the solution of the relaxed optimal actuator location problem by a Nash equilibrium. For completeness, we include some basics of two person zero sum game in Appendix.

## 2. Observability Inequality and Null Controllability

In this section, we will prove our main therorem and provide the observability inequality (1.2). By duality, the equivalence between the null controllability of the equation (1.3) and the observability estimate for the adjoint equation (1.1) is obtained. As a result, we obtain Theorem 1.2

Let us start with some notations. we write

$$
0<\lambda_{1} \leq \lambda_{2} \leq \cdots
$$

for the eigenvalues of $-\Delta$ with the zero Dirichlet boundary condition over $\partial D$, and $\left\{e_{j}\right\}_{j \geq 1}$ for the orthonormal basis for $L^{2}(D)$. For each $\lambda>0$, we define

$$
\mathcal{E}_{\lambda} f=\sum_{\lambda_{j} \leq \lambda}\left(f, e_{j}\right) e_{j}, \text { and } \mathcal{E}_{\lambda}^{\perp} f=\sum_{\lambda_{j}>\lambda}\left(f, e_{j}\right) e_{j}
$$

Now recall an important spectral inequality used later in this paper; see Theorem 5 in [3].

Lemma 2.1. Let $D$ be a bounded domain in $\mathbb{R}^{d}$ with a $C^{2}$ boundary. Let $x_{0} \in D$ and $R \in(0,1]$ such that $B_{4 R}\left(x_{0}\right) \subseteq D$. Suppose $G$ is a subset of $D$ with positive measure, contained in $B_{R}\left(x_{0}\right)$. Then there exists a positive constant $N=$ $N(D, R,|G|)$ such that

$$
\begin{equation*}
\left\|\mathcal{E}_{\lambda} \eta\right\|^{2} \leq N \exp (N \sqrt{\lambda})\left\|\mathcal{E}_{\lambda} \eta\right\|_{G}^{2}, \quad \forall \eta \in L^{2}(D), \quad \lambda>0 \tag{2.1}
\end{equation*}
$$

Set $\tau=\|a\|_{L_{\mathcal{F}}^{\infty}(0, T ; \mathbb{R})}^{2}$.
Let us denote by $z(\cdot ; T, \eta)$ the solution of equation (1.1) given the terminal condition $\eta=z(T)$. By linearity, it is easy to check that

$$
\begin{align*}
& z\left(t ; T, \mathcal{E}_{\lambda} \eta\right)=\sum_{\lambda_{j} \leq \lambda} z_{j}\left(t ; T, \eta_{j}\right) e_{j}=\mathcal{E}_{\lambda} z(t ; T, \eta)  \tag{2.2}\\
& z\left(t ; T, \mathcal{E}_{\lambda}^{\perp} \eta\right)=\sum_{\lambda_{j}>\lambda} z_{j}\left(t ; T, \eta_{j}\right) e_{j}=\mathcal{E}_{\lambda}^{\perp} z(t ; T, \eta), \tag{2.3}
\end{align*}
$$

where $\eta_{j}=\left(\eta, e_{j}\right)$ and $\left(z_{j}\left(\cdot ; T, \eta_{j}\right), Z_{j}\left(\cdot ; T, \eta_{j}\right)\right)$ is the solution of the following backward stochastic differential equation

$$
\left\{\begin{array}{l}
d z_{j}=\lambda_{j} z_{j} d t-a(t) Z_{j} d t+Z_{j} d W(t), \quad t \in(0, T)  \tag{2.4}\\
z_{j}(T)=\eta_{j}
\end{array}\right.
$$

Lemma 2.2. Given any $\eta$ in the space of $L^{2}\left(\Omega, \mathcal{F}_{T}, \mathbb{P} ; L^{2}(D)\right)$, we have for each $t \in[0, T]$,

$$
\begin{equation*}
\mathbb{E}\left\|z\left(t ; T, \mathcal{E}_{\lambda}^{\perp} \eta\right)\right\|^{2} \leq e^{(-2 \lambda+\tau)(T-t)} \mathbb{E}\|\eta\|^{2} \tag{2.5}
\end{equation*}
$$

Proof. Applying Itô formula to $\exp [(2 \lambda-\tau)(T-t)]\left\|z\left(t ; T, \mathcal{E}_{\lambda}^{\perp}\right)\right\|^{2}$, we obtain

$$
\begin{aligned}
& \left\|\mathcal{E}_{\lambda}^{\perp} \eta\right\|^{2}-e^{(2 \lambda-\tau)(T-t)}\left\|z\left(t ; T, \mathcal{E}_{\lambda}^{\perp} \eta\right)\right\|^{2} \\
= & \int_{t}^{T} e^{(2 \lambda-\tau)(T-s)}[2(z, A z)-2 a(s)(z, Z)] d s \\
& +\int_{t}^{T} e^{(2 \lambda-\tau)(T-s)}\|Z\|^{2} d s+\int_{t}^{T} e^{(2 \lambda-\tau)(T-s)} 2(z, Z) d W(s) \\
& \quad-\int_{t}^{T} e^{(2 \lambda-\tau)(T-s)}(2 \lambda-\tau)\left\|z\left(s ; T, \mathcal{E}_{\lambda}^{\perp} \eta\right)\right\|^{2} d s
\end{aligned}
$$

Taking the expectation, it follows from equality (2.3) that

$$
\begin{aligned}
& \quad \mathbb{E}\left\|\mathcal{E}_{\lambda}^{\perp} \eta\right\|^{2}-e^{(2 \lambda-\tau)(T-t)} \mathbb{E}\left\|z\left(t ; T, \mathcal{E}_{\lambda}^{\perp} \eta\right)\right\|^{2} \\
& =\mathbb{E} \int_{t}^{T} e^{(2 \lambda-\tau)(T-s)}\left(2 \sum_{\lambda_{j}>\lambda} \lambda_{j}\left(z^{j}\right)^{2}-2 a(s)(z, Z)\right. \\
& \left.\quad \quad+\|Z\|^{2}-(2 \lambda-\tau)\left\|z\left(s ; T, \mathcal{E}_{\lambda}^{\perp} \eta\right)\right\|^{2}\right) d t \\
& \geq \mathbb{E} \int_{t}^{T} e^{(2 \lambda-\tau)(T-s)}(-2 a(s)(z, Z) \\
& \left.\quad \quad+\|Z\|^{2}+\tau\left\|z\left(s ; T, \mathcal{E}_{\lambda}^{\perp} \eta\right)\right\|^{2}\right) d t \\
& \geq \mathbb{E} \int_{t}^{T} e^{(2 \lambda-\tau)(T-s)}\left(\left(\|Z\|-\mid a(s)\| \| z\left(s ; T, \mathcal{E}_{\lambda}^{\perp} \eta\right) \|\right)^{2}\right. \\
& \left.\quad \quad+\left(\tau-|a(s)|^{2}\right)\left\|z\left(s ; T, \mathcal{E}_{\lambda}^{\perp} \eta\right)\right\|^{2}\right) d t \\
& \geq 0
\end{aligned}
$$

which implies the inequality (2.5).
Next, we provide an interpolation inequality.

Proposition 2.3. For $\eta \in L^{2}\left(\Omega, \mathcal{F}_{T}, \mathbb{P} ; L^{2}(D)\right)$, and $t \in[0, T)$, there exists $a$ constant $K=K(T, D, R,|G|)$ such that

$$
\begin{equation*}
\mathbb{E}\|z(t ; T, \eta)\|^{2} \leq K \exp \left(K(T-t)^{-1}\right)\left(\mathbb{E}\|z(t ; T, \eta)\|_{G}^{2}\right)^{1 / 2}\left(\mathbb{E}\|\eta\|^{2}\right)^{1 / 2} \tag{2.6}
\end{equation*}
$$

Proof. Set $z=z(\cdot ; T, \eta)$, then it follows from the spectral estimate (2.1) that

$$
\begin{aligned}
\mathbb{E}\left\|\mathcal{E}_{\lambda} z(t)\right\|^{2} & \leq N \exp (N \sqrt{\lambda}) \mathbb{E}\left\|\mathcal{E}_{\lambda} z(t)\right\|_{G}^{2} \\
& \leq N \exp (N \sqrt{\lambda})\left(\mathbb{E}\|z(t)\|_{G}^{2}+\mathbb{E}\left\|\mathcal{E}_{\lambda}^{\perp} z(t)\right\|_{G}^{2}\right)
\end{aligned}
$$

for some constant $N=N(D, R,|G|)$. Therefore, by the decay estimate (2.5) we obtain that

$$
\begin{aligned}
& \mathbb{E}\|z(t)\|^{2}=\mathbb{E}\left\|\mathcal{E}_{\lambda} z(t)\right\|^{2}+\mathbb{E}\left\|\mathcal{E}_{\lambda}^{\perp} z(t)\right\|^{2} \\
& \leq N \exp (N \sqrt{\lambda})\left(\mathbb{E}\|z(t)\|_{G}^{2}+\mathbb{E}\left\|\mathcal{E}_{\lambda}^{\perp} z(t)\right\|_{G}^{2}\right)+\mathbb{E}\left\|\mathcal{E}_{\lambda}^{\perp} z(t)\right\|^{2} \\
& \leq 2 N \exp (N \sqrt{\lambda})\left(\mathbb{E}\|z(t)\|_{G}^{2}+\mathbb{E}\left\|\mathcal{E}_{\lambda}^{\perp} z(t)\right\|^{2}\right) \\
& \leq 2 N \exp (N \sqrt{\lambda})\left(\mathbb{E}\|z(t)\|_{G}^{2}+e^{(-2 \lambda+\tau)(T-t)} \mathbb{E}\|\eta\|^{2}\right) \\
& \leq 2 N e^{\tau T} \exp (N \sqrt{\lambda})\left(\mathbb{E}\|z(t)\|_{G}^{2}+e^{(-2 \lambda(T-t))} \mathbb{E}\|\eta\|^{2}\right) \\
& =2 N e^{\tau T} \exp (N \sqrt{\lambda}-\lambda(T-t))\left(e^{\lambda(T-t)} \mathbb{E}\|z(t)\|_{G}^{2}+e^{-\lambda(T-t)} \mathbb{E}\|\eta\|^{2}\right)
\end{aligned}
$$

It is easy to verify that for all $\lambda>0$,

$$
N \sqrt{\lambda}-\lambda(T-t) \leq \frac{N^{2}}{4(T-t)}
$$

Hence, there exists a constant $K=K(T, D, R,|G|)$ such that

$$
\mathbb{E}\|z(t)\|^{2} \leq K \exp \left(K(T-t)^{-1}\right)\left[e^{\lambda(T-t)} \mathbb{E}\|z(t)\|_{G}^{2}+e^{-\lambda(T-t)} \mathbb{E}\|\eta\|^{2}\right]
$$

which is equivalent to

$$
\begin{equation*}
\mathbb{E}\|z(t)\|^{2} \leq K \exp \left(K(T-t)^{-1}\right)\left[\varepsilon^{-1} \mathbb{E}\|z(t)\|_{G}^{2}+\varepsilon \mathbb{E}\|\eta\|^{2}\right], \quad \forall \varepsilon \in(0,1) \tag{2.7}
\end{equation*}
$$

Noting that $\mathbb{E}\|z(t)\|^{2} \leq C \mathbb{E}\|\eta\|^{2}$, where $C$ is a constant depending on $T$, we see that the inequality (2.7) holds for all $\varepsilon>0$. Finally, minimizing (2.7) with respect to $\varepsilon$ leads to the desired estimate (2.6).

We are now ready to prove Theorem 1.1
Proof of Theorem 1.1. Let $\ell \in(0, T)$ be any Lebesgue point of $E$. Then for each constant $q \in(0,1)$ which is to be fixed later, there exists a monotone increasing sequence $\left\{\ell_{m}\right\}_{m \geq 1}$ in $(\ell, T)$ such that

$$
\begin{gather*}
\lim _{m \rightarrow+\infty} \ell_{m}=\ell \\
\ell_{m+2}-\ell_{m+1}=q\left(\ell_{m+1}-\ell_{m}\right), \forall m \geq 1 \tag{2.8}
\end{gather*}
$$

and

$$
\left|E \cap\left(\ell_{m}, \ell_{m+1}\right)\right| \geq \frac{\ell_{m+1}-\ell_{m}}{3}, \forall m \geq 1
$$

Set

$$
\tau_{m}=\ell_{m+1}-\frac{\ell_{m+1}-\ell_{m}}{6}, \forall m \geq 1
$$

For each $t \in\left(\ell_{m}, \tau_{m}\right)$, by the interpolation inequality (2.6), we have

$$
\mathbb{E}\|z(t)\|^{2} \leq K \exp \left(K\left(\ell_{m+1}-t\right)^{-1}\right)\left(\mathbb{E}\|z(t)\|_{G}^{2}\right)^{1 / 2}\left(\mathbb{E}\left\|z\left(\ell_{m+1}\right)\right\|^{2}\right)^{1 / 2}
$$

Since

$$
\ell_{m+1}-t \geq \ell_{m+1}-\tau_{m}=\frac{\ell_{m+1}-\ell_{m}}{6}
$$

and for some constant $C=C(T), \mathbb{E}\left\|z\left(\ell_{m}\right)\right\|^{2} \leq C \mathbb{E}\|z(t)\|^{2}$, there exists a constant $C=C(T, D, R,|G|)$ such that for all $m \geq 1$, and $t \in\left(\ell_{m}, \tau_{m}\right)$,

$$
\mathbb{E}\left\|z\left(\ell_{m}\right)\right\|^{2} \leq C e^{\frac{C}{\ell_{m+1}-\ell_{m}}}\left(\mathbb{E}\|z(t)\|_{G}^{2}\right)^{1 / 2}\left(\mathbb{E}\left\|z\left(\ell_{m+1}\right)\right\|^{2}\right)^{1 / 2}
$$

which implies for each $\varepsilon>0$,

$$
\mathbb{E}\left\|z\left(\ell_{m}\right)\right\|^{2} \leq \varepsilon^{-1} C e^{\frac{C}{\ell_{m+1}-\ell_{m}}} \mathbb{E}\|z(t)\|_{G}^{2}+\varepsilon \mathbb{E}\left\|z\left(\ell_{m+1}\right)\right\|^{2},
$$

by the Cauchy inequality with $\varepsilon$. Equivalently, we have

$$
\begin{equation*}
A_{m} \leq \varepsilon^{-1} C e^{\frac{C}{\ell_{m+1}-\ell_{m}}} B(t)+\varepsilon A_{m+1} \tag{2.9}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{m}=\left(\mathbb{E}\left\|z\left(\ell_{m}\right)\right\|^{2}\right)^{1 / 2}, B(t)=\left(\mathbb{E}\|z(t)\|^{2}\right)^{1 / 2} \tag{2.10}
\end{equation*}
$$

Integrating the previous inequality (2.9) over $E \cap\left(\ell_{m}, \tau_{m}\right)$, and noting that

$$
\begin{aligned}
\left|E \cap\left(\ell_{m}, \tau_{m}\right)\right| & =\left|E \cap\left(\ell_{m}, \ell_{m+1}\right)\right|-\left|E \cap\left(\tau_{m}, \ell_{m+1}\right)\right| \\
& \geq \frac{\ell_{m+1}-\ell_{m}}{3}-\frac{\ell_{m+1}-\ell_{m}}{6} \\
& =\frac{\ell_{m+1}-\ell_{m}}{6},
\end{aligned}
$$

we have that for each $\varepsilon>0$

$$
A_{m} \leq \varepsilon A_{m+1}+\varepsilon^{-1} C e^{\frac{C}{\ell_{m+1}-\ell_{m}}} \int_{\ell_{m}}^{\ell_{m+1}} \chi_{E} B(t) d t
$$

Multiplying the above inequality by $\varepsilon \exp \left(-C /\left(\ell_{m+1}-\ell_{m}\right)\right)$, and replacing $\varepsilon$ by $\sqrt{\varepsilon}$ lead to

$$
\sqrt{\varepsilon} e^{-\frac{C}{\ell_{m+1}-\ell_{m}}} A_{m} \leq \varepsilon e^{-\frac{C}{\ell_{m+1}-\ell_{m}}} A_{m+1}+C \int_{\ell_{m}}^{\ell_{m+1}} \chi_{E} B(t) d t
$$

Finally choosing $\varepsilon=\exp \left(-1 /\left(\ell_{m}-\ell_{m+1}\right)\right)$ in the above inequality, we get

$$
e^{-\frac{C+1 / 2}{\ell_{m+1}-\ell_{m}}} A_{m}-e^{-\frac{C+1}{\ell_{m+1}-\ell_{m}}} A_{m+1} \leq C \int_{\ell_{m}}^{\ell_{m+1}} \chi_{E} B(t) d t
$$

Now, choosing $q=\frac{C+1 / 2}{C+1}$ in (2.8), we have

$$
e^{-\frac{C+1 / 2}{\ell_{m+1}-\ell_{m}}} A_{m}-e^{-\frac{C+1 / 2}{\ell_{m+2}-\ell_{m+1}}} A_{m+1} \leq C \int_{\ell_{m}}^{\ell_{m+1}} \chi_{E} B(t) d t
$$

Summing the above inequality from $m=1$ to $+\infty$, we have

$$
A_{1} \leq C e^{\frac{C+1 / 2}{\ell_{2}-\ell_{1}}} \int_{\ell}^{\ell_{1}} \chi_{E} B(t) d t
$$

By the substitution (2.10), we obtain

$$
\mathbb{E}\left\|z\left(\ell_{1}\right)\right\|^{2} \leq C e^{\frac{C+1}{\ell_{2}-\ell_{1}}}\left(\int_{\ell}^{\ell_{1}} \chi_{E}\left(\mathbb{E}\|z(t)\|_{G}^{2}\right)^{1 / 2} d t\right)^{2}
$$

which implies the observability inequality (1.2), completing the proof.

Next, by the standard duality augment, we have the following equivalence between the null controllability of the equation (1.3) and the observability inequality for the adjoint equation (1.1).
Proposition 2.4. For any $T>0$, the equation (1.3) is null controllable at time $T$ with the control $u$ in the space of $L_{\mathcal{F}}^{\infty}\left(0, T ; L^{2}(D)\right)$ such that the estimate (1.4) holds if and only if there exists $C>0$ such that the solution of the adjoint equation (1.1) satisfies the observability inequality (1.2).

We omit the proof here, and refer the reader to, for example [10, Proposition 1.1]. Then Theorem 1.2 is a direct consequence of Theorem 1.1 and Proposition 2.4

## 3. A Relaxed Optimal Actuator Location Problem

3.1. Problem formulation. In the sequel, we assume $E=[0, T]$.

Now we consider the following norm optimal control problem

$$
\begin{equation*}
N(G)=\inf \left\{\mathbb{E}\|u\|_{L^{2}((0, T) \times D)}^{2} \mid y(T ; G, u)=0 \text { in } D, \mathbb{P} \text {-a.s. }\right\} \tag{3.1}
\end{equation*}
$$

where $y(\cdot ; G, u)$ is the solution of equation (1.3). In the problem (3.1), we say $u$ is an admissible control if $u \in L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right)$ and $y(T ; G, u)=0$ in $D$, $\mathbb{P}$-a.s.; we say $u^{*}$ is an optimal minimal norm control if $u^{*}$ is an admissible control such that $N(G)$ is achieved.

Remark 3.1. It is obvious that minimizing $\mathbb{E}\|u\|_{L^{2}((0, T) \times D)}^{2}$ is equivalent to minimizing $\mathbb{E}\|u\|_{L^{2}((0, T) \times D)}$. Thus, the problem we consider is a natural generalization of the usual norm optimal control problem in the deterministic case.

Given $\alpha \in(0,1)$, let

$$
\begin{equation*}
\mathcal{W}=\{G \subseteq D \mid G \text { is Lebesgue measurable with }|G|=\alpha|D|\} \tag{3.2}
\end{equation*}
$$

where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}^{d}$.
A classical optimal actuator location of the minimal norm control problem is to seek a set $G^{*} \in \mathcal{W}$ such that

$$
\begin{equation*}
N\left(G^{*}\right)=\inf _{G \in \mathcal{W}} N(G) \tag{3.3}
\end{equation*}
$$

If such a $G^{*}$ exists, we say that $G^{*}$ is an optimal actuator location of the optimal minimal norm controls. Any optimal minimal norm control $u^{*}$ satisfying

$$
\mathbb{E}\left\|u^{*} \chi_{G^{*}}\right\|_{L^{2}((0, T) \times D)}^{2}=N\left(G^{*}\right),
$$

is called an optimal control with respect to the optimal actuator location $G^{*}$.
The existence of the optimal actuator location $G^{*}$ is generally not guaranteed because of the absence of the compactness of $\mathcal{W}$. For this reason, we consider instead a relaxed problem. To this end, define

$$
\begin{equation*}
\mathcal{B}=\left\{\beta \in L^{\infty}(D ;[0,1])\left|\|\beta\|^{2}=\alpha\right| D \mid\right\} . \tag{3.4}
\end{equation*}
$$

Note that the set $\mathcal{B}$ is a relaxation of the set $\left\{\chi_{G} \mid G \in \mathcal{W}\right\}$.
For any $\beta \in \mathcal{B}$, consider the following equation

$$
\left\{\begin{array}{l}
d y=A y d t+\beta u(t) d t+a(t) y d w(t), \quad t \in(0, T)  \tag{3.5}\\
y(0)=y_{0}
\end{array}\right.
$$

We denote by $y(\cdot ; \beta, u)$ the solution of equation (3.5), and say the system (3.5) null controllable if there exists $u \in L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right)$ such that $y(T ; \beta, u)=0$ in $D, \mathbb{P}$-a.s. Accordingly, the problem (3.1) is replaced by

$$
\begin{equation*}
N(\beta)=\inf \left\{\mathbb{E}\|u\|_{L^{2}((0, T) \times D)}^{2} \mid y(T ; \beta, u)=0 \text { in } D, \mathbb{P} \text {-a.s. }\right\}, \tag{3.6}
\end{equation*}
$$

and the classical optimal actuator location problem (3.3) is changed into the following relaxed problem

$$
\begin{equation*}
N\left(\beta^{*}\right)=\inf _{\beta \in \mathcal{B}} N(\beta) \tag{3.7}
\end{equation*}
$$

Any solution $\beta^{*}$ to the problem (3.7) is called a relaxed optimal actuator location of the optimal minimal norm controls.

Now we study the controllability of the relaxed system (3.5) with the same adjoint equation (1.1), and make sure that the set on the right hand side of (3.6) is not empty. In fact, the null controllability is equivalent to the following observability inequality, as we have done in the proof of Theorem 1.2 .

Lemma 3.2. The system (1.1) is exactly observable, i.e., there exists a constant $C>0$, independent of $\beta$, but possibly depending on $\alpha$ such that for all $\eta \in L^{2}\left(\Omega, \mathcal{F}_{T}, \mathbb{P} ; L^{2}(D)\right)$ and $\beta \in \mathcal{B}$,

$$
\begin{equation*}
\mathbb{E}\|z(0 ; T, \eta)\|^{2} \leq C \int_{0}^{T} \mathbb{E}\|\beta z(t ; T, \eta)\|^{2} d t \tag{3.8}
\end{equation*}
$$

Proof. By Theorem 1.1, for each $G \in \mathcal{W}$, there exists a constant $C>0$ such that the solution of equation (1.1) satisfies

$$
\begin{equation*}
\mathbb{E}\|z(0 ; T, \eta)\|^{2} \leq C \int_{0}^{T}\left\|\chi_{G} z(t ; T, \eta)\right\|^{2} d t \tag{3.9}
\end{equation*}
$$

for all $\eta \in L^{2}\left(\Omega, \mathcal{F}_{T}, \mathbb{P}\right)$. Moreover, the constant $C$ only depends on the measure of the set $G$.

For any $\beta \in \mathcal{B}$, let

$$
\gamma=\frac{|\{\beta \geq \sqrt{\alpha / 2}\}|}{|D|}
$$

Since

$$
\begin{aligned}
\alpha \cdot|D| & =\int_{D} \beta^{2} d x=\int_{\{\beta \geq \sqrt{\alpha / 2}\}} \beta^{2} d x+\int_{\{\beta<\sqrt{\alpha / 2}\}} \beta^{2} d x \\
& \leq|\{\beta \geq \sqrt{\alpha / 2}\}|+\frac{\alpha}{2} \cdot|\{\beta<\sqrt{\alpha / 2}\}|
\end{aligned}
$$

we have

$$
\gamma \cdot|D|+\frac{\alpha}{2}(1-\gamma) \cdot|D|
$$

and consequently,

$$
\gamma \geq \frac{\alpha}{2-\alpha}
$$

Therefore, we obtain

$$
\begin{equation*}
|\{\beta \geq \sqrt{\alpha / 2}\}| \geq \frac{\alpha}{2-\alpha} \cdot|D| \tag{3.10}
\end{equation*}
$$

for all $\beta \in \mathcal{B}$. It then follows from inequality (3.9) with $G=\{\beta \geq \sqrt{\alpha / 2}\}$ that

$$
\begin{aligned}
\mathbb{E}\|z(0 ; T, \eta)\|^{2} & \leq C \mathbb{E} \int_{0}^{T} \int_{D} \chi_{\{\beta \geq \sqrt{\alpha / 2}\}} z^{2}(t ; T, \eta) d x d t \\
& \leq C \mathbb{E} \int_{0}^{T} \int_{D} \chi_{\{\beta \geq \sqrt{\alpha / 2}\}}\left(\frac{\beta}{\sqrt{\alpha / 2}}\right)^{2} z^{2}(t ; T, \eta) d x d t \\
& =C \cdot \frac{2}{\alpha} \cdot \mathbb{E} \int_{0}^{T} \int_{D} \chi_{\{\beta \geq \sqrt{\alpha / 2}\}} \beta^{2} z^{2}(t ; T, \eta) d x d t \\
& \leq C \int_{0}^{T} \mathbb{E}\|\beta z(t ; T, \eta)\|^{2} d t
\end{aligned}
$$

which completes the proof.
Remark 3.3. The observability inequality (3.8) is in fact an $L^{2}$ estimate, and it is sufficient for our purpose in this section, though we have an $L^{1}$ estimate in (1.2).
3.2. The optimal minimal norm control. In general, it is not easy (or impossible) to solve the problem (3.6) directly; see [8] for a special class of subdomains. Instead, let us introduce a functional

$$
\begin{equation*}
\mathcal{J}(\eta ; \beta)=\frac{1}{2} \int_{0}^{T} \mathbb{E}\|\beta z(t ; \eta)\|^{2} d t+\mathbb{E}\left(y_{0}, z(0 ; \eta)\right) \tag{3.11}
\end{equation*}
$$

and propose the following variational problem

$$
\begin{equation*}
\mathcal{J}(\beta)=\inf _{\eta \in L^{2}\left(\Omega, \mathcal{F}_{T}, \mathbb{P} ; L^{2}(D)\right)} \mathcal{J}(\eta ; \beta) . \tag{3.12}
\end{equation*}
$$

Here and what follows, we simply set $z(\cdot ; \eta)=z(\cdot ; T, \eta)$ for the solution of the adjoint equation (1.1) with the terminal condition $z(T)=\eta$. We will show later the equivalence between the problem (3.12) and the problem (3.6).

To this end, denote by

$$
\begin{equation*}
X=\left\{z(\cdot ; \eta) \mid \eta \in L^{2}\left(\Omega, \mathcal{F}_{T}, \mathbb{P} ; L^{2}(D)\right)\right\} \tag{3.13}
\end{equation*}
$$

and for each $\beta \in \mathcal{B}$, define $F_{\beta}: X \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
F_{\beta}(z)=\left(\mathbb{E} \int_{0}^{T}\|\beta z\|^{2} d t\right)^{1 / 2} \tag{3.14}
\end{equation*}
$$

It follows from the observability inequality (3.8) that $F_{\beta}$ is indeed a norm on space $X$. We denote by $\overline{X_{\beta}}$ the completion of the space $X$ under the norm $F_{\beta}$. The following proposition provides us a description of $\overline{X_{\beta}}$.
Lemma 3.4. Under an isomorphism, any element of $\overline{X_{\beta}}$ can be expressed as a process $\varphi \in L_{\mathcal{F}}^{2}\left(\Omega ; C\left([0, T) ; L^{2}(D)\right)\right)$, which satisfies

$$
\begin{equation*}
d \varphi=-A \varphi d t-a(t) Z d t+Z d w(t) \tag{3.15}
\end{equation*}
$$

for some $Z \in L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right)$ in $L^{2}\left(0, T ; L^{2}(D)\right), \mathbb{P}$-a.s. Moreover, $\beta \varphi=\lim _{n \rightarrow \infty} \beta z\left(\cdot ; \eta_{n}\right)$ for some sequence $\left\{\eta_{n}\right\} \subseteq L^{2}\left(\Omega, \mathcal{F}_{T}, \mathbb{P}\right)$ in $L^{2}\left(\Omega ; L^{2}((0, T) \times D)\right)$.
Proof. Let $\bar{\varphi} \in\left(\overline{X_{\beta}}, \overline{F_{\beta}}\right)$, where $\left(\overline{X_{\beta}}, \overline{F_{\beta}}\right)$ is the completion of $\left(X, F_{\beta}\right)$. Then there exists a sequence $\left\{\eta_{n}\right\} \subseteq L^{2}\left(\Omega, \mathcal{F}_{T}, \mathbb{P}\right)$ such that

$$
\overline{F_{\beta}}\left(z\left(\cdot ; \eta_{n}\right)-\bar{\varphi}\right) \rightarrow 0, \text { as } n \rightarrow \infty
$$

from which, one has

$$
F_{\beta}\left(z\left(\cdot ; \eta_{n}\right)-z\left(\cdot ; \eta_{m}\right)\right)=\overline{F_{\beta}}\left(z\left(\cdot ; \eta_{n}\right)-z\left(\cdot ; \eta_{m}\right)\right) \rightarrow 0 \text { as } n, m \rightarrow \infty
$$

In other words,

$$
\begin{equation*}
\mathbb{E} \int_{0}^{T}\left\|\beta z\left(t ; \eta_{n}\right)-\beta z\left(t ; \eta_{m}\right)\right\|^{2} d t \rightarrow 0 \text { as } n, m \rightarrow \infty \tag{3.16}
\end{equation*}
$$

Hence, there exists $\hat{\varphi} \in L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right)$ such that

$$
\begin{equation*}
\beta z\left(\cdot, \eta_{n}\right) \rightarrow \hat{\varphi} \text { strongly in } L^{2}(\Omega ;(0, T) \times D) \tag{3.17}
\end{equation*}
$$

Now choose a strictly increasing sequence $\left\{T_{k}\right\} \subseteq(0, T)$ such that $T_{k} \rightarrow T$ as $k \rightarrow \infty$. Set $\left(z_{n}, Z_{n}\right)=\left(z\left(\cdot ; \eta_{n}\right), Z_{n}\left(\cdot ; \eta_{n}\right)\right)$, i.e., the solution of equation (1.1) with the terminal condition $z_{n}(T)=\eta_{n}$.
(a) For $T_{1}$. By the observability inequality (3.8) and (3.16),

$$
\mathbb{E}\left\|z\left(T_{2}, \eta_{n}\right)\right\|^{2} \leq C_{1} \mathbb{E} \int_{T_{2}}^{T}\left\|\beta z\left(t ; \eta_{n}\right)\right\|^{2} d t \leq C_{1} \mathbb{E} \int_{0}^{T}\left\|\beta z\left(t ; \eta_{n}\right)\right\|^{2} d t \leq C_{1}
$$

for all $n \geq 1$. Then there exist a subsequence $\left\{z\left(T_{2}, \eta_{1 n}\right)\right\}$ of $\left\{z\left(T_{2}, \eta_{n}\right)\right\}$ and $z_{T_{2}, 1} \in L^{2}\left(\Omega, \mathcal{F}_{T_{2}}, \mathbb{P}\right)$ such that

$$
z\left(T_{2}, \eta_{1 n}\right) \rightarrow z_{T_{2}, 1} \text { weakly in } L^{2}(\Omega \times D)
$$

Consequently, there exist a subsequence $\left\{\left(z_{1 n}, Z_{1 n}\right)\right\}$ of $\left\{\left(z_{n}, Z_{n}\right)\right\}$ and $\left(\varphi_{1}, \psi_{1}\right)$ in the space of $L_{\mathcal{F}}^{2}\left(\Omega ; C\left(\left[0, T_{2}\right] ; L^{2}(D)\right)\right) \times L_{\mathcal{F}}^{2}\left(0, T_{2} ; L^{2}(D)\right)$ solving the adjoint equation (1.1) with the terminal conditions $z_{1 n}\left(T_{2}\right)=z\left(T_{2}, \eta_{1 n}\right)$ and $\varphi_{1}\left(T_{2}\right)=z_{T_{2}, 1}$, respectively, and

$$
\left(z_{1 n}, Z_{1 n}\right) \rightarrow\left(\varphi_{1}, \psi_{1}\right) \text { weakly in } L^{2}\left(\Omega ; C\left(\left[0, T_{2}\right] ; L^{2}(D)\right)\right) \times L^{2}\left(\Omega ;\left(0, T_{2}\right) \times D\right)
$$

In particular,

$$
\begin{equation*}
\left(z_{1 n}, Z_{1 n}\right) \rightarrow\left(\varphi_{1}, \psi_{1}\right) \text { weakly in } L^{2}\left(\Omega ; C\left(\left[0, T_{1}\right] ; L^{2}(D)\right)\right) \times L^{2}\left(\Omega ;\left(0, T_{1}\right) \times D\right) \tag{3.18}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta z_{1 n} \rightarrow \beta \varphi_{1} \text { weakly in } L^{2}\left(\Omega ;\left(0, T_{1}\right) \times D\right) \tag{3.19}
\end{equation*}
$$

Thus, it follows from (3.17) and (3.19) that

$$
\beta \varphi_{1}=\hat{\varphi} \text { in } L^{2}\left(\Omega ;\left(0, T_{1}\right) \times D\right)
$$

(b) For $T_{2}$. In the same spirit of (a), we can find a subsequence $\left\{\left(z_{2 n}, Z_{2 n}\right)\right\}$ of $\left\{\left(z_{1 n}, Z_{1 n}\right)\right\}$, and $\left(\varphi_{2}, \psi_{2}\right)$ in the space of $L_{\mathcal{F}}^{2}\left(\Omega ; C\left(\left[0, T_{3}\right] ; L^{2}(D)\right)\right) \times L_{\mathcal{F}}^{2}\left(0, T_{3} ; L^{2}(D)\right)$ solving the adjoint equation (1.1) with the terminal conditions $z_{2 n}\left(T_{3}\right)=z\left(T_{3}, \eta_{2 n}\right)$ and $\varphi_{1}\left(T_{3}\right)=z_{T_{3}, 2}$, respectively, and

$$
\left(z_{2 n}, Z_{2 n}\right) \rightarrow\left(\varphi_{2}, \psi_{2}\right) \text { weakly in } L^{2}\left(\Omega ; C\left(\left[0, T_{3}\right] ; L^{2}(D)\right)\right) \times L^{2}\left(\Omega ;\left(0, T_{3}\right) \times D\right)
$$

where $\left\{\eta_{2 n}\right\}$ is a subsequence of $\left\{\eta_{1 n}\right\}$ such that $z\left(T_{3} ; \eta_{2 n}\right)$ converges weakly to $z_{T_{3}, 2}$ in $L^{2}(\Omega \times D)$. Then it follows from (3.17), (3.18) and (3.19) that

$$
\left(\varphi_{2}, \psi_{2}\right) \upharpoonright_{\left[0, T_{1}\right]}=\left(\varphi_{1}, \psi_{1}\right)
$$

and

$$
\beta \varphi_{2}=\hat{\varphi} \text { in } L^{2}\left(\Omega ;\left(0, T_{2}\right) \times D\right)
$$

(c) In general, we obtain a sequence $\left\{\left(\varphi_{k}, \psi_{k}\right)\right\}$ satisfies for each $k \geq 1$ that

- $\left\{\left(\varphi_{k}, \psi_{k}\right)\right\} \in L_{\mathcal{F}}^{2}\left(\Omega ; C\left(\left[0, T_{k+1}\right] ; L^{2}(D)\right)\right) \times L_{\mathcal{F}}^{2}\left(0, T_{k+1} ; L^{2}(D)\right)$;
- $\left(\varphi_{k+1}, \psi_{k+1}\right) \upharpoonright_{\left[0, T_{k}\right]}=\left(\varphi_{k}, \psi_{k}\right)$;
- $\left\{\left(\varphi_{k}, \psi_{k}\right)\right\}$ satisfies (3.15) on $\left(0, T_{k+1}\right)$;
- $\beta \varphi_{k}=\hat{\varphi}$ in $L^{2}\left(\Omega ;\left(0, T_{k}\right) \times D\right)$.

Now define

$$
(\varphi(t), Z(t))=\left(\varphi_{k}(t), \psi_{k}(t)\right), t \in\left[0, T_{k}\right]
$$

Then $(\varphi(t), Z(t)) \in L_{\mathcal{F}}^{2}\left(\Omega ; C\left([0, T) ; L^{2}(D)\right)\right) \times L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right)$ satisfies equation (3.15), and

$$
\beta \varphi=\hat{\varphi}=\lim _{n \rightarrow \infty} \beta z\left(\cdot ; \eta_{n}\right)
$$

Under an isometric isomorphism, we can identify $\bar{\varphi}$ by $\varphi$. The proof is completed.

Remark 3.5. The element $\varphi$ in $\overline{X_{\beta}}$ is not necessarily in the space of $L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right)$, but $\beta \varphi \in L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right)$ for $\beta \in \mathcal{B}$. Also, because of the isomorphism, we can write $\overline{F_{\beta}}(\varphi)=\left(\mathbb{E} \int_{0}^{T}\|\beta \varphi\|^{2} d t\right)^{1 / 2}$.

Next, let us introduce an auxiliary operator

$$
\begin{equation*}
\mathcal{T}_{\beta}: \beta \overline{X_{\beta}} \subseteq L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right) \rightarrow L^{2}\left(\Omega ; L^{2}(D)\right), \quad \beta \varphi \rightarrow \varphi(0) \tag{3.20}
\end{equation*}
$$

By Lemma 3.4 the operator $\mathcal{T}_{\beta}$ is well defined, and it is bounded as well. In fact, if we consider the equation (1.1) on the interval $(0, T / 2)$, then by the observability inequality (3.8), we have

$$
\mathbb{E}\|\varphi(0)\|^{2} \leq C \mathbb{E} \int_{0}^{T / 2}\|\beta \varphi\|^{2} d t \leq C \mathbb{E} \int_{0}^{T}\|\beta \varphi\|^{2} d t
$$

Then the functional $\mathcal{J}(\beta)$ defined in (3.12) can be written as

$$
\begin{aligned}
\mathcal{J}(\beta) & =\inf _{z \in X}\left[\frac{1}{2} \int_{0}^{T} \mathbb{E}\|\beta z\|^{2} d t+\mathbb{E}\left(y_{0}, z(0)\right)\right] \\
& =\inf _{\varphi \in \overline{X_{\beta}}}\left[\frac{1}{2} \int_{0}^{T} \mathbb{E}\|\beta \varphi\|^{2} d t+\mathbb{E}\left(y_{0}, \mathcal{T}_{\beta}(\beta \varphi)\right)\right] \\
& =\inf _{\varphi \in \overline{X_{\beta}}}\left[\frac{1}{2} \int_{0}^{T} \mathbb{E}\|\beta \varphi\|^{2} d t+\int_{0}^{T} \mathbb{E}\left(\mathcal{T}_{\beta}^{*} y_{0}, \beta \varphi\right) d t\right]
\end{aligned}
$$

Set $y_{0, \beta}=\mathcal{T}_{\beta}^{*} y_{0}$, and thus the problem (3.12) is equivalent to the following problem

$$
\begin{equation*}
\mathcal{V}(\beta)=\inf _{\varphi \in \bar{X}_{\beta}}\left[\frac{1}{2} \int_{0}^{T} \mathbb{E}\|\beta \varphi\|^{2} d t+\int_{0}^{T} \mathbb{E}\left(y_{0, \beta}, \beta \varphi\right) d t\right] \tag{3.21}
\end{equation*}
$$

The key motivation of this transformation is that the functional on the right hand side of the problem (3.21) is coercive in $\varphi$ with respect to the norm $\overline{F_{\beta}}$, but in general, $\mathcal{J}(\eta ; \beta)$ in (3.11) does not satisfy such a condition. The next theorem characterizes the minmal norm control of problem (3.6) in terms of the solution of the problem (3.21).
Theorem 3.6. Fix $\beta \in \mathcal{B}$. Suppose $y_{0} \in L^{2}\left(\Omega, \mathcal{F}_{0}, \mathbb{P} ; L^{2}(D)\right)$. Then problem (3.21) admits a unique solution $\varphi^{*}$. Moreover, the control defined by

$$
\begin{equation*}
u^{*}=\beta \varphi^{*} \tag{3.22}
\end{equation*}
$$

is the minimal norm optimal control to the problem (3.6), and

$$
\begin{equation*}
N(\beta)=\int_{0}^{T} \mathbb{E}\left\|\beta \varphi^{*}\right\|^{2} d t \tag{3.23}
\end{equation*}
$$

Proof. It is obvious that the functional on the right hand side of (3.21) is continuous, strictly convex and coercive in $\varphi$ with respect to the norm $\overline{F_{\beta}}$. Therefore, the problem (3.21) admits a unique solution, denoted by $\varphi^{*}$.

It follows from Lemma 3.4 that the control $u^{*}=\beta \varphi^{*}$ is well defined and $u^{*} \in$ $L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right)$. We claim first that $u^{*}$ is a control driving the solution $y$ of equation (3.5) to rest at time $T$. In fact, by the optimality of $\varphi^{*}$, we obtain the following Euler-Lagrange equation to the variational problem (3.21):

$$
\begin{equation*}
\int_{0}^{T} \mathbb{E}\left(u^{*}, \beta \psi\right) d t+\int_{0}^{T} \mathbb{E}\left(y_{0, \beta}, \beta \psi\right) d t=0, \text { for all } \psi \in \overline{X_{\beta}} \tag{3.24}
\end{equation*}
$$

Taking $\psi=z(\cdot ; \eta) \in X$ for any $\eta \in L^{2}\left(\Omega ; L^{2}(D)\right)$, a straightforward computation and Itô formula imply that

$$
y\left(T ; \beta, u^{*}\right)=0 \text { in } D, \mathbb{P} \text {-a.s. }
$$

Next, we will show that $u^{*}$ is optimal in the sense that

$$
\begin{equation*}
\int_{0}^{T} \mathbb{E}\left\|u^{*}\right\|^{2} d t \leq \int_{0}^{T} \mathbb{E}\|\hat{u}\|^{2} d t \tag{3.25}
\end{equation*}
$$

for any $\hat{u} \in L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right)$ such that $y(T ; \beta, \hat{u})=0$ in $D, \mathbb{P}$-a.s.
Without loss of generality, we assume $u^{*} \neq 0$. By Itô formula, we have

$$
\int_{0}^{T} \mathbb{E}(\hat{u}, \beta z) d t+\mathbb{E}\left(y_{0}, z(0)\right)=0, \text { for all } z \in X
$$

or equivalently

$$
\int_{0}^{T} \mathbb{E}(\hat{u}, \beta z) d t+\int_{0}^{T} \mathbb{E}\left(y_{0, \beta}, \beta z\right) d t=0, \text { for all } z \in X
$$

which, together with equality (3.24), implies

$$
\begin{equation*}
\int_{0}^{T} \mathbb{E}\left(u^{*}, \beta z\right) d t=\int_{0}^{T} \mathbb{E}(\hat{u}, \beta z) d t, \text { for all } z \in X \tag{3.26}
\end{equation*}
$$

By the density argument, the equality (3.26) still holds for all $\psi \in \overline{X_{\beta}}$. Thus, replacing $z$ in (3.26) by $\varphi^{*}$ gives

$$
\int_{0}^{T} \mathbb{E}\left\|u^{*}\right\|^{2} d t=\int_{0}^{T} \mathbb{E}\left(u^{*}, \hat{u}\right) d t \leq\left(\int_{0}^{T} \mathbb{E}\left\|u^{*}\right\|^{2} d t\right)^{1 / 2}\left(\int_{0}^{T} \mathbb{E}\|\hat{u}\|^{2} d t\right)^{1 / 2}
$$

Therefore, the inequality (3.25) is true and this concludes the proof.
From above, we can describe the relation between $\mathcal{V}(\beta)$ (or equivalently $\mathcal{J}(\beta)$ ) and $N(\beta)$.
Corollary 3.7. Let $\beta \in \mathcal{B}$ and $y_{0} \in L^{2}\left(\Omega, \mathcal{F}_{0}, \mathbb{P} ; L^{2}(D)\right)$. Then

$$
\begin{equation*}
\mathcal{V}(\beta)=-\frac{1}{2} N(\beta) \tag{3.27}
\end{equation*}
$$

where $\mathcal{V}(\beta)$ and $N(\beta)$ are defined as in (3.21), and (3.6), respectively.

Proof. Let $\varphi^{*}$ be a solution of the problem (3.21) such that

$$
\mathcal{V}(\beta)=\frac{1}{2} \int_{0}^{T} \mathbb{E}\left\|\beta \varphi^{*}\right\|^{2} d t+\int_{0}^{T} \mathbb{E}\left(y_{0, \beta}, \beta \varphi^{*}\right) d t
$$

On the other hand, it follows from the Euler-Lagrange equation (3.24) that

$$
\int_{0}^{T} \mathbb{E}\left(y_{0, \beta}, \beta \varphi^{*}\right) d t=-\int_{0}^{T} \mathbb{E}\left\|\beta \varphi^{*}\right\|^{2} d t
$$

Thus, by (3.23) we have

$$
\mathcal{V}(\beta)=-\frac{1}{2} \int_{0}^{T} \mathbb{E}\left\|\beta \varphi^{*}\right\|^{2} d t=-\frac{1}{2} N(\beta)
$$

3.3. Existence of relaxed optimal actuator location. Now we are ready to show the existence of relaxed optimal actuator location of the optimal minimal norm controls, i.e., we can find $\beta^{*} \in \mathcal{B}$ such that $N\left(\beta^{*}\right)=\inf _{\beta \in \mathcal{B}} N(\beta)$. To this end, define

$$
\begin{equation*}
\Theta=\left\{\theta \in L^{\infty}(D ;[0,1])\left|\int_{D} \theta(x) d x=\alpha\right| D \mid\right\} \tag{3.28}
\end{equation*}
$$

It is clear that

$$
\begin{equation*}
\beta^{2} \in \Theta \text { for any } \beta \in \mathcal{B}, \text { and } \theta^{1 / 2} \in \mathcal{B} \text { for all } \theta \in \Theta \tag{3.29}
\end{equation*}
$$

Then it follows from the relation (3.27) that

$$
\begin{aligned}
& \inf _{\beta \in \mathcal{B}} \frac{1}{2} N(\beta)=\inf _{\beta \in \mathcal{B}}-\mathcal{V}(\beta)=\inf _{\beta \in \mathcal{B}}-\mathcal{J}(\beta) \\
= & \inf _{\beta \in \mathcal{B}} \sup _{z \in X}\left[\frac{1}{2} \int_{0}^{T} \mathbb{E}\|\beta z\|^{2} d t+\mathbb{E}\left(y_{0}, z(0)\right)\right] \\
= & \inf _{\theta \in \Theta} \sup _{z \in X}\left[-\frac{1}{2} \mathbb{E} \int_{0}^{T} \int_{D} \theta z^{2} d x d t-\mathbb{E}\left(y_{0}, z(0)\right)\right] \\
= & \inf _{\theta \in \Theta} \sup _{z \in X} F(\theta, z),
\end{aligned}
$$

where the functional $F$ is defined by

$$
\begin{equation*}
F(\theta, z)=-\frac{1}{2} \mathbb{E} \int_{0}^{T} \int_{D} \theta z^{2} d x d t-\mathbb{E}\left(y_{0}, z(0)\right) \tag{3.30}
\end{equation*}
$$

Therefore, seeking a minimizer $\beta^{*} \in \mathcal{B}$ for $N(\beta)$ amounts to finding a minimizer $\theta^{*} \in \Theta$ for $\sup _{z \in X} F(\theta, z)$.

Let us equip $L^{\infty}(D)$ with the weak* topology. Then $\Theta$ is compact in $L^{\infty}(D)$.
Lemma 3.8. Given $y_{0} \in L^{2}\left(\Omega ; \mathcal{F}_{0}, \mathbb{P} ; L^{2}(D)\right)$ and $z \in X$. Then the functional $F(\cdot, z): \Theta \rightarrow \mathbb{R} \cup\{+\infty\}$ defined in (3.30) is sequentially weakly* lower semicontinuous.

Proof. Suppose there is a sequence $\left\{\theta_{n}\right\} \subseteq \Theta$ such that

$$
\theta_{n} \rightarrow \theta \text { weakly }^{*} \text { in } L^{\infty}(D)
$$

Then for any $t \in[0, T]$, we have

$$
\lim _{n \rightarrow \infty} \mathbb{E} \int_{D} \theta_{n} z^{2}(t) d x=\mathbb{E} \int_{D} \theta z^{2}(t) d x \leq \mathbb{E}\|z(t)\|^{2}
$$

Since $\int_{0}^{T} \mathbb{E}\|z(t)\|^{2} d t<\infty$, it follows from the Dominated Convergence Theorem, and (3.30) that

$$
\lim _{n \rightarrow \infty} F\left(\theta_{n}, z\right)=F(\theta, z)
$$

So $F(\cdot, z)$ is sequentially weakly* continuous, and in particular, lower semi-continuous.

It is obvious that the functional $F(\cdot, z)$ is linear in $\theta$ for any $z \in X$, so it is convex. Then it follows from Proposition 2.31 in [2, page 62] that $F(\cdot, z)$ is weakly* lower semi-continuous. Under the weak* topology in $L^{\infty}(D), F(\cdot, z)$ is lower semicontinuous, so is $\sup _{z \in X} F(\cdot, z)$. Together with the fact that $\Theta$ is compact in $L^{\infty}(D)$, we claim that there exists $\theta^{*} \in \Theta$ minimizing $\sup _{z \in X} F(\cdot, z)$ by Theorem 38.B in [19, page 152]. Equivalently, we obtain the following theorem of existence to conclude this subsection.

Theorem 3.9. Suppose $y_{0} \in L^{2}\left(\Omega, \mathcal{F}_{0}, \mathbb{P} ; L^{2}(D)\right)$. Then the problem (3.7) admits a solution $\beta^{*} \in \mathcal{B}$, i.e.,

$$
N\left(\beta^{*}\right)=\inf _{\beta \in \mathcal{B}} N(\beta) .
$$

3.4. Characterization via Nash equilibrium. Now we define a non-negative nonlinear functional $F_{\Theta}$ on $X$ by

$$
\begin{equation*}
F_{\Theta}(z):=\sup _{\theta \in \Theta} F_{\theta^{1 / 2}}(z), z \in X \tag{3.31}
\end{equation*}
$$

where $F_{\theta^{1 / 2}}$ is defined as in (3.14). Since $F_{\theta^{1 / 2}}$ is a norm on $X$ for each $\theta \in \Theta, F_{\Theta}$ is also a norm on $X$. Thus, $\left(X, F_{\Theta}\right)$ is a normed space, and we denote by $\left(\overline{X_{\Theta}}, \overline{F_{\Theta}}\right)$ its completion.

Along the same line in the proof of Lemma 3.4, we have the following similar result.

Lemma 3.10. Under an isomorphism, any element of $\overline{X_{\Theta}}$ can be expressed as a process $\varphi \in L_{\mathcal{F}}^{2}\left(\Omega ; C\left([0, T) ; L^{2}(D)\right)\right)$, which satisfies

$$
\begin{equation*}
d \varphi=-A \varphi d t-a(t) Z d t+Z d w(t) \tag{3.32}
\end{equation*}
$$

for some $Z \in L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right)$ in $L^{2}\left(0, T ; L^{2}(D)\right), \mathbb{P}$-a.s. Moreover, $F_{\Theta}(\varphi)=$ $\lim _{n \rightarrow \infty} F_{\Theta}\left(z\left(\cdot ; \eta_{n}\right)\right)$ for some sequence $\left\{\eta_{n}\right\} \subseteq L^{2}\left(\Omega, \mathcal{F}_{T}, \mathbb{P}\right)$.

By Lemma 3.10, we have the following inclusion relation:

$$
\begin{equation*}
\overline{X_{\Theta}} \subseteq L_{\mathcal{F}}^{2}\left(0, T ; L^{2}(D)\right) \tag{3.33}
\end{equation*}
$$

In fact, suppose that $n_{0} \in \mathbb{N}$ so that $n_{0} \geq 1 / \alpha$. Then there are $n_{0}$ measurable subsets $G_{1}, \cdots, G_{n_{0}}$ of $D$ such that

$$
G_{j} \in \mathcal{W}, 1 \leq j \leq n_{0}, \text { and } \bigcup_{j=1}^{n_{0}} G_{j}=D
$$

Then the inclusion relation follows from

$$
\begin{aligned}
\int_{0}^{T} \mathbb{E}\|\varphi\|^{2} d t & =\int_{0}^{T} \mathbb{E}\left\|\varphi \sum_{j=1}^{n_{0}} \chi_{G_{j}}\right\|^{2} d t \\
& \leq n_{0} \sum_{j=1}^{n_{0}} \int_{0}^{T} \mathbb{E}\left\|\varphi \chi_{G_{j}}\right\|^{2} d t \\
& \leq n_{0} \sum_{j=1}^{n_{0}}{\bar{F}_{\Theta}}^{2}(\varphi)=n_{0}^{2}{\bar{F}_{\Theta}}^{2}(\varphi)
\end{aligned}
$$

On the other hand, it is obvious that $\overline{F_{\Theta}}(\varphi) \leq \int_{0}^{T} \mathbb{E}\|\varphi\|^{2} d t$. Thus, $F_{\Theta}$ and $\left(\mathbb{E}\|\cdot\|_{L^{2}((0, T) \times D)}^{2}\right)^{1 / 2}$ are equivalent norms on $X$.

In this subsection, we solve the following Nash equilibrium problem of two-person zero-sum game (see Appendix): to find $\bar{\theta} \in \Theta, \bar{\varphi} \in \overline{X_{\Theta}}$ such that

$$
\begin{equation*}
F(\bar{\theta}, \bar{\varphi})=\sup _{\varphi \in \overline{X_{\Theta}}} F(\bar{\theta}, \varphi)=\inf _{\theta \in \Theta} F(\theta, \bar{\varphi}) \tag{3.34}
\end{equation*}
$$

where $F(\theta, \varphi)$ is defined as in (3.30). This requires by Theorem A. 2 in Appendix that we solve the following two problems

$$
\begin{equation*}
\inf _{\theta \in \Theta} \sup _{\varphi \in \overline{X_{\Theta}}} F(\theta, \varphi) \tag{3.35}
\end{equation*}
$$

and

$$
\begin{equation*}
\sup _{\varphi \in \overline{X_{\Theta}}} \inf _{\theta \in \Theta} F(\theta, \varphi) \tag{3.36}
\end{equation*}
$$

and verify the equality (3.34).
In fact, the problem (3.35) is solved by choosing $\bar{\theta}=\left(\beta^{*}\right)^{2}$, where $\beta^{*} \in \mathcal{B}$ is a solution of the problem (3.7), guaranteed by Theorem 3.9. To see this clearly, recall that $X$ is dense in $\overline{X_{\beta}}$, and for each $\theta \in \Theta$ with $\theta=\beta^{2}$

$$
\sup _{\varphi \in X} F(\theta, \varphi)=\sup _{\varphi \in \overline{X_{\beta}}} F(\theta, \varphi)
$$

where we use the fact that $F(\theta, \cdot)$ is continuous with respect to the norm $\overline{F_{\beta}}$, the completion of $F_{\beta}$ in (3.14). On the other hand, since for each $\beta \in \mathcal{B}$, we have

$$
\int_{0}^{T} \mathbb{E}\|\beta z\|^{2} d t \leq F_{\Theta}^{2}(z), \forall z \in X
$$

which implies

$$
\overline{X_{\Theta}} \subseteq \overline{X_{\beta}}, \quad \forall \beta \in \mathcal{B}
$$

Therefore,

$$
\begin{equation*}
\sup _{z \in X} F(\theta, z) \leq \sup _{\varphi \in \overline{X_{\ominus}}} F(\theta, \varphi) \leq \sup _{\varphi \in \overline{X_{\beta}}} F\left(\beta^{2}, \varphi\right)=\sup _{z \in X} F(\theta, z) \tag{3.37}
\end{equation*}
$$

and thus

$$
\inf _{\beta \in \mathcal{B}} \frac{1}{2} N(\beta)=\inf _{\beta \in \mathcal{B}}-\mathcal{J}(\beta)=\inf _{\theta \in \Theta} \sup _{z \in X} F(\theta, z)=\inf _{\theta \in \Theta} \sup _{\varphi \in \overline{X_{\Theta}}} F(\theta, \varphi)
$$

So the problem (3.35) is solved by Theorem 3.9.

To solve the problem (3.36) is to find $\bar{\varphi} \in \overline{X_{\Theta}}$ such that

$$
\inf _{\theta \in \Theta} F(\theta, \bar{\varphi})=\sup _{\varphi \in \overline{X_{\Theta}}} \inf _{\theta \in \Theta} F(\theta, \varphi),
$$

or equivalently,

$$
\begin{equation*}
\sup _{\theta \in \Theta}-F(\theta, \bar{\varphi})=\inf _{\varphi \in \overline{X_{\Theta}}} \sup _{\theta \in \Theta}-F(\theta, \varphi) \tag{3.38}
\end{equation*}
$$

Lemma 3.11. For any $y_{0} \in L^{2}\left(\Omega, \mathcal{F}_{0}, \mathbb{P} ; L^{2}(D)\right)$, the problem (3.38) admits a unique solution.

Proof. Define the functional $F: \overline{X_{\Theta}} \rightarrow \mathbb{R}$ by

$$
F(\varphi):=\sup _{\theta \in \Theta}-F(\theta, \varphi) .
$$

Then

$$
F(\varphi)=\sup _{\theta \in \Theta} \frac{1}{2} \int_{0}^{T} \mathbb{E}\left\|\theta^{1 / 2} \varphi\right\|^{2} d t+\mathbb{E}\left(y_{0}, \varphi(0)\right)={\overline{F_{\Theta}}}^{2}(\varphi)+\mathbb{E}\left(y_{0}, \varphi(0)\right) .
$$

It is clear that $F$ is strictly convex in $\varphi$. To show continuity and coercivity, we consider the equation (3.32) on the time interval $(0, T / 2)$. Then by the observability inequality (3.8), we have for all $\beta \in \mathcal{B}$ and $\varphi \in \overline{X_{\Theta}}$

$$
\begin{equation*}
\mathbb{E}\|\varphi(0)\|^{2} \leq C \mathbb{E} \int_{0}^{T / 2}\|\beta \varphi\|^{2} d t \leq C \mathbb{E} \int_{0}^{T}\|\beta \varphi\|^{2} d t \leq C{\overline{F_{\Theta}}}^{2}(\varphi) \tag{3.39}
\end{equation*}
$$

Thus, by Cauchy-Schwartz inequality, we have $\left|\mathbb{E}\left(y_{0}, \varphi(0)\right)\right| \leq C \overline{F_{\Theta}}(\varphi)$. Now suppose there exists a sequence $\left\{\varphi_{n}\right\} \subseteq \overline{X_{\Theta}}$ such that $\varphi_{n} \rightarrow \varphi$ in $\overline{X_{\Theta}}$, i.e., $\overline{F_{\Theta}}\left(\varphi_{n}-\varphi\right) \rightarrow 0$, then

$$
\begin{aligned}
\left|F\left(\varphi_{n}\right)-F(\varphi)\right| & \leq\left|{\overline{F_{\Theta}}}^{2}\left(\varphi_{n}\right)-{\overline{F_{\Theta}}}^{2}(\varphi)\right|+\left|\mathbb{E}\left(y_{0},\left(\varphi_{n}-\varphi\right)(0)\right)\right| \\
& \leq C\left|\overline{F_{\Theta}}\left(\varphi_{n}\right)-\overline{F_{\Theta}}(\varphi)\right|+C \overline{F_{\Theta}}\left(\varphi_{n}-\varphi\right) \\
& \leq C \overline{F_{\Theta}}\left(\varphi_{n}-\varphi\right) \rightarrow 0
\end{aligned}
$$

which implies that $F$ is continuous. Finally, it follows from (3.39) that

$$
F(\varphi) \geq{\overline{F_{\Theta}}}^{2}(\varphi)-C \overline{F_{\Theta}}(\varphi)
$$

and so $F$ is coercive. Hence, the problem (3.38) has a unique solution.
Now it remains to show the equality (3.34) holds. To this end, denote by

$$
\begin{equation*}
U^{+}=\inf _{\theta \in \Theta} \sup _{z \in X} F(\theta, z), U^{-}=\sup _{z \in X} \inf _{\theta \in \Theta} F(\theta, z) \tag{3.40}
\end{equation*}
$$

where $F$ is defined in (3.30). Let $\mathcal{K}$ be the collection of all the finite subsets of $X$. For any $K \in \mathcal{K}$, set

$$
\begin{equation*}
U_{K}=\inf _{\theta \in \Theta} \sup _{z \in K} F(\theta, z), \hat{U}:=\sup _{K \in \mathcal{K}} U_{K} . \tag{3.41}
\end{equation*}
$$

Then it is easy to verify that

$$
\begin{equation*}
U^{-} \leq \hat{U} \leq U^{+} \tag{3.42}
\end{equation*}
$$

Furthermore, we can obtain the equalities in (3.42).
Proposition 3.12. Define $U^{-}$and $U^{+}$as in (3.40), then

$$
U^{-}=U^{+}
$$

Proof. We first show that $U^{+} \leq \hat{U}$.
Given any $K \in \mathcal{K}$, using a similar argument to the one above Theorem 3.9, we can find $\theta_{K} \in \Theta$ such that

$$
\sup _{z \in K} F\left(\theta_{K}, z\right)=\inf _{\theta \in \Theta} \sup _{z \in K} F(\theta, z)=U_{K}
$$

This, together with the definition of $\hat{U}$ in (3.41), enables us to derive

$$
\begin{equation*}
F\left(\theta_{K}, z\right) \leq U_{K} \leq \hat{U}, \text { for all } z \in K \tag{3.43}
\end{equation*}
$$

Let $z \in X$, define

$$
S_{z}:=\{\theta \in \Theta \mid F(\theta, z) \leq \hat{U}\}
$$

It follows from (3.43) that the set $S_{z}$ is not empty, and

$$
\begin{equation*}
\left\{\theta_{K}\right\} \subseteq \bigcap_{z \in K} S_{z} \neq \emptyset \tag{3.44}
\end{equation*}
$$

In addition, since $F(\cdot, z)$ is weakly* lower semi-continuous, $S_{z}$ is weakly* closed in $L^{\infty}(D)$. By the compactness of $\Theta$ under the weak* topology of $L^{\infty}(D)$, we have

$$
\bigcap_{z \in X} S_{z} \neq \emptyset
$$

Thus, there exists $\hat{\theta} \in \Theta$ such that $\sup _{z \in Z} F(\hat{\theta}, z) \leq \hat{U}$, and so

$$
U^{+}=\inf _{\theta \in \Theta} \sup _{z \in X} F(\theta, z) \leq \hat{U}
$$

Next, we show $U^{-}=\hat{U}$.
It is clear that both $\Theta$ and $X$ are convex sets. Note that $F(\theta, \cdot)$ is convex for each $\theta \in \Theta$ and $F(\cdot, z)$ is convex (in fact, it is linear) for each $z \in X$. By Proposition 8.3 in 4, page 132], $U^{-}=\hat{U}$.

Therefore, we have

$$
U^{+} \leq U^{-}
$$

which, together with (3.42) implies $U^{-}=U^{+}$.
Again, it follows from (3.37) that

$$
U^{+}=\inf _{\theta \in \Theta} \sup _{\varphi \in \overline{X_{\Theta}}} F(\theta, \varphi)
$$

On the other hand,

$$
U^{-}=\sup _{z \in X} \inf _{\theta \in \Theta} F(\theta, z)=\sup _{\varphi \in \overline{X_{\Theta}}} \inf _{\theta \in \Theta} F(\theta, \varphi) \leq \inf _{\theta \in \Theta} \sup _{\varphi \in \overline{X_{\Theta}}} F(\theta, \varphi)
$$

and thus by $U^{-}=U^{+}$we obtain

$$
\sup _{\varphi \in \overline{X_{\Theta}}} \inf _{\theta \in \Theta} F(\theta, \varphi)=\inf _{\theta \in \Theta} \sup _{\varphi \in \bar{X}_{\Theta}} F(\theta, \varphi)
$$

Summarizing the previous analysis, we arrive at the following theorem.
Theorem 3.13. The problem (3.34) admits at least one Nash equilibrium. Specifically, $\bar{\varphi}$ is a solution of the problem (3.38) and $\beta^{*}$ is a solution of the relaxed optimal actuator location problem (3.7), if and only if the pair $\left(\bar{\theta}=\left(\beta^{*}\right)^{2}, \bar{\varphi}\right)$ is a Nash equilibrium.

Consequently, we can characterize the solution of the relaxed optimal location problem (3.7) via a Nash equilibrium.

Theorem 3.14. There exists at least one solution of the problem (3.7). In addition, $\beta^{*}$ is a relaxed optimal actuator location of the optimal minimal norm controls if and only if there exists $\bar{\varphi} \in \overline{X_{\Theta}}$ such that the pair $\left(\beta^{*}, \bar{\varphi}\right)$ is a Nash equilibrium of the following two-person zero-sum game problem: to find $\left(\beta^{*}, \bar{\varphi}\right) \in \mathcal{B} \times \overline{X_{\Theta}}$ such that

$$
\begin{align*}
& \frac{1}{2} \int_{0}^{T} \mathbb{E}\left\|\beta^{*} \bar{\varphi}\right\|^{2} d t+\mathbb{E}\left(y_{0}, \bar{\varphi}(0)\right)=\sup _{\beta \in \mathcal{B}}\left[\frac{1}{2} \int_{0}^{T} \mathbb{E}\|\beta \bar{\varphi}\|^{2} d t+\mathbb{E}\left(y_{0}, \bar{\varphi}(0)\right)\right] \\
& \frac{1}{2} \int_{0}^{T} \mathbb{E}\left\|\beta^{*} \bar{\varphi}\right\|^{2} d t+\mathbb{E}\left(y_{0}, \bar{\varphi}(0)\right)=\inf _{\varphi \in \overline{X_{\Theta}}}\left[\frac{1}{2} \int_{0}^{T} \mathbb{E}\left\|\beta^{*} \varphi\right\|^{2} d t+\mathbb{E}\left(y_{0}, \varphi(0)\right)\right] \tag{3.45}
\end{align*}
$$

## Appendix A. Appendix

Let us recall some basics for the two-person zero-sum game problem; for more details, see for example [4, Chapter 8].

There are two players: Emil and Francis. Consider a real-valued function $f$ : $E \times F \rightarrow \mathbb{R}$, where $f(x, y)$ is both the loss of Emil by taking the strategy $x$ from her strategy set $E$ and the gain of Francis by taking the strategy $y$ from his strategy set $F$ (the sum of the gains is zero). Emil wants to minimize the function $f$, while Francis wants to maximize $f$. The most important concept in the two-person zero-sum game is the Nash equilibrium.

Definition A.1. Suppose that $E$ and $F$ are strategy sets of Emil and Francis, respectively. Let $f: E \times F \rightarrow \mathbb{R}$ be an index cost functional. We say $(\bar{x}, \bar{y}) \in E \times F$ is a Nash equilibrium, if

$$
f(\bar{x}, y) \leq f(\bar{x}, \bar{y}) \leq f(x, \bar{y}), \forall x \in E, y \in F
$$

The following result is well known, see, for instance, Proposition 8.1 in 4, page 121], which says seeking a Nash equilibrium is equivalent to solving a minimax and a maxmini problems, respectively, so that the extremes achieved are the same.

Theorem A.2. The pair $(\bar{x}, \bar{y})$ is a Nash equilibrium if and only if

$$
\sup _{y \in F} f(\bar{x}, y)=V^{+}, \quad \inf _{x \in E} f(x, \bar{y})=V^{-}
$$

and $V^{+}=V^{-}$, where

$$
V^{+}:=\inf _{x \in E} \sup _{y \in F} f(x, y), \quad V^{-}:=\sup _{y \in F} \inf _{x \in E} f(x, y)
$$

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