# Parameterized Complexity of Directed Steiner Tree on Sparse Graphs 

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#### Abstract

We study the parameterized complexity of the directed variant of the classical Steiner Tree problem on various classes of directed sparse graphs. While the parameterized complexity of Steiner Tree parameterized by the number of terminals is well understood, not much is known about the parameterization by the number of non-terminals in the solution tree. All that is known for this parameterization is that both the directed and the undirected versions are $\mathrm{W}[2]$-hard on general graphs, and hence unlikely to be fixed parameter tractable (FPT). The undirected Steiner Tree problem becomes FPT when restricted to sparse classes of graphs such as planar graphs, but the techniques used to show this result break down on directed planar graphs.

In this article we precisely chart the tractability border for Directed Steiner Tree (DST) on sparse graphs parameterized by the number of non-terminals in the solution tree. Specifically, we show that the problem is fixed parameter tractable on graphs excluding a topological minor, but becomes W[2]-hard on graphs of degeneracy 2. On the other hand we show that if the subgraph induced by the terminals is required to be acyclic then the problem becomes FPT on graphs of bounded degeneracy.

We further show that our algorithm achieves the best possible running time dependence on the solution size and degeneracy of the input graph, under standard complexity theoretic assumptions. Using the ideas developed for DST, we also obtain improved algorithms for Dominating SET on sparse undirected graphs. These algorithms are asymptotically optimal.


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## 1 Introduction

In the Steiner Tree problem we are given as input a $n$-vertex graph $G=(V, E)$ and a set $T \subseteq V$ of terminals. The objective is to find a subtree $S T$ of $G$ spanning $T$ that minimizes the number of vertices in $S T$. Steiner Tree is one of the most intensively studied graph problems in Computer Science. Steiner trees are important in various applications such as VLSI routings [28], phylogenetic tree reconstruction [26] and network routing [31]. We refer to the book of Prömel and Steger [38] for an overview of the results on, and applications of the Steiner Tree problem. The Steiner Tree problem is known to be NP-hard [20], and remains hard even on planar graphs [19]. The minimum number of non-terminals can be approximated to within $O(\log n)$, but cannot be approximated to $o(\log t)$, where $t$ is the number of terminals, unless $\mathrm{P} \subseteq \mathrm{DTIME}\left[n^{\text {polylog } n}\right]$ (see [29]). Furthermore the weighted variant of Steiner Tree remains APX-complete, even when the graph is complete and all edge costs are either 1 or 2 (see [3).

In this paper we study a natural generalization of Steiner Tree to directed graphs, from the perspective of parameterized complexity. The goal of parameterized complexity is to find ways of solving NP-hard problems more efficiently than by brute force. The aim is to restrict the combinatorial explosion in the running time to a parameter that is much smaller than the input size for many input instances occurring in practice. Formally, a parameterization of a problem is the assignment of an integer $k$ to each input instance and we say that a parameterized problem is fixed-parameter tractable (FPT) if there is an algorithm that solves the problem in time $f(k) \cdot|I|^{O(1)}$, where $|I|$ is the size of the input instance and $f$ is an arbitrary computable function depending only on the parameter $k$. Above FPT, there exists a hierarchy of complexity classes, known as the W-hierarchy. Just as NP-hardness is used as an evidence that a problem is probably not polynomial time solvable, showing that a parameterized problem is hard for one of these classes gives evidence that the problem is unlikely to be fixed-parameter tractable. The main classes in this hierarchy are:

$$
\mathrm{FPT} \subseteq \mathrm{~W}[1] \subseteq \mathrm{W}[2] \subseteq \cdots \subseteq \mathrm{W}[\mathrm{P}] \subseteq \mathrm{XP}
$$

The principal analogue of the classical intractability class NP is $\mathrm{W}[1]$. In particular, this means that an FPT algorithm for any W[1]-hard problem would yield a $O\left(f(k) n^{c}\right)$ time algorithm for every problem in the class $\mathrm{W}[1] . X \mathrm{P}$ is the class of all problems that are solvable in time $O\left(n^{g(k)}\right)$. Here, $g$ is some (usually computable) function. For more background on parameterized complexity the reader is referred to the monographs [13, 16, 36]. We consider the following directed variant of Steiner Tree.

```
Directed Steiner Tree (DST)
Parameter: k
Input: A directed graph D=(V,A), a root vertex r\inV, a set T\subseteqV\{r} of terminals and an
integer }k\in\mathbb{N}\mathrm{ .
Question: Is there a set S\subseteqV\(T\cup{r}) of at most k vertices such that the digraph D[S\cupT\cup{r}]
contains a directed path from r to every terminal }t\inT\mathrm{ ?
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The DST problem is well studied in approximation algorithms, as the problem generalizes several important connectivity and domination problems on undirected as well as directed graphs [6, 12, 23, [25, 39, 40. These include Group Steiner Tree, Node Weighted Steiner Tree, TSP and Connected Dominating Set. However, this problem has so far largely been ignored in the realm of parameterized complexity. The aim of this paper is to fill this gap.

It follows from the reduction presented in [34] that DST is W[2]-hard on general digraphs. Hence we do not expect FPT algorithms to exist for these problems, and so we turn our attention to classes of sparse digraphs. Our results give a nearly complete picture of the parameterized complexity of DST on sparse digraphs. Specifically, we prove the following results. We use the $O^{*}$ notation to suppress factors polynomial in the input size.



Figure 1 A summary of the main results in the paper

1. There is a $O^{*}\left(2^{O(h k)}\right)$-time algorithm for DST on digraphs excluding $K_{h}$ as a minor ${ }^{1}$ Here $K_{h}$ is a clique on $h$ vertices.
2. There is a $O^{*}\left(f(h)^{k}\right)$-time algorithm for DST on digraphs excluding $K_{h}$ as a topological minor.
3. There is a $O^{*}\left(2^{O(h k)}\right)$-time algorithm for DST on digraphs excluding $K_{h}$ as a topological minor if the graph induced on terminals is acyclic.
4. DST is W[2]-hard on 2-degenerated digraphs if the graph induced on terminals is allowed to contain directed cycles.
5. There is a $O^{*}\left(2^{O(d k)}\right)$-time algorithm for DST on $d$-degenerated graphs if the graph induced on terminals is acyclic, implying that DST is FPT parameterized by $k$ on $o(\log n)$-degenerated graph classes. This yields the first FPT algorithm for Steiner Tree on undirected d-degenerate graphs.
6. For any constant $c>0$, there is no $f(k) n^{o\left(\frac{k}{\log k}\right)}$-time algorithm on graphs of degeneracy $c \log n$ even if the graph induced on terminals is acyclic, unless the Exponential Time Hypothesis [27] (ETH) fails.

Our algorithms for DST hinge on a novel branching which exploits the domination-like nature of the DST problem. The branching is based on a new measure which seems useful for various connectivity and domination problems on both directed and undirected graphs of bounded degeneracy. We demonstrate the versatility of the new branching by applying it to the Dominating Set problem on graphs excluding a topological minor and more generally, graphs of bounded degeneracy. The well-known Dominating SET problem is defined as follows.

## Parameter: $k$

Input: An undirected graph $G=(V, E)$, and an integer $k \in \mathbb{N}$.
Question: Is there a set $S \subseteq V$ of at most $k$ vertices such that every vertex in $G$ is either in $S$ or adjacent to a vertex in $S$ ?
Our $O^{*}\left(2^{O(d k)}\right)$-time algorithm for Dominating Set on $d$-degenerated graphs improves over the $O^{*}\left(k^{O(d k)}\right)$ time algorithm by Alon and Gutner [2]. It turns out that our algorithm is essentially optimal - we show that assuming the ETH, the running time dependence of our algorithm on the degeneracy of the input graph and solution size $k$ can not be significantly improved. Using these ideas we also obtain a polynomial time $O\left(d^{2}\right)$ factor approximation algorithm for Dominating Set on $d$-degenerate graphs. We give survey of existing literature on Dominating Set and the results for it in Section 4 We believe that our new branching and corresponding measure will turn out to be useful for several other problems on sparse (di)graphs.

[^1]Related Results. Though the parameterized complexity of DST has so far been largely ignored, it has not been left completely unexplored. In particular the classical dynamic programming algorithm by Dreyfus and Wagner [14] from 1972 solves Steiner Tree in time $O^{*}\left(3^{t}\right)$ where $t$ is the number of terminals in the input graph. The algorithm can also be used to solve DST within the same running time, and may be viewed as a FPT algorithm for Steiner Tree and DST if the number of terminals in the instance is the parameter. Fuchs et al. [18] improved the algorithm of Dreyfus and Wagner and obtained an algorithm with running time $O^{*}\left((2+\epsilon)^{t}\right)$, for any constant $\epsilon>0$. More recently, Björklund, Husfeldt, Kaski, and Koivisto [4] obtained an $O^{*}\left(2^{t}\right)$ time algorithm for the cardinality version of Steiner Tree. Finally, Nederlof [35] obtained an algorithm running in $O^{*}\left(2^{t}\right)$ and polynomial space. All of these algorithms can also be modified to work for DST.

For most hard problems, the most frequently studied parameter in parameterized complexity is the size or quality of the solution. For Steiner Tree and DST, however, this is not the case. The non-standard parameterization of the problem by the number of terminals is well-studied, while the standard parameterization by the number of non-terminals in the solution tree has been left unexplored, aside from the simple W[2]-hardness proofs [34]. Steiner-type problems in directed graphs from parameterized perspective were studied in [24] in arc-weighted setting, but the paper focuses more on problems in which the required connectivity among the terminals is more complicated than just a tree.

For Steiner Tree parameterized by the solution size $k$, there is a simple (folklore) FPT algorithm on planar graphs. The algorithm is based on the fact that planar graphs have the diameter-treewidth property [15], the fact that Steiner Tree can be solved in polynomial time on graphs of bounded treewidth [9] along with a simple preprocessing step. In this step, one contracts adjacent terminals to single vertices and removes all vertices at distance at least $k+1$ from any terminal. For DST, however, this preprocessing step breaks down. Thus, previous to this work, nothing is known about the standard parameterization of DST aside from the W[2]-hardness result on general graphs.

## 2 Preliminaries

Given a digraph $D=(V, A)$, for each vertex $v \in V$, we define $N^{+}(v)=\{w \in V \mid(v, w) \in A\}$ and $N^{-}(v)=\{w \in H \mid(w, v) \in A\}$. In other words, the sets $N^{+}(v)$ and $N^{-}(v)$ are the set of out-neighbors and in-neighbors of $v$, respectively.

Degeneracy of an undirected graph $G=(V, E)$ is defined as the least number $d$ such that every subgraph of $G$ contains a vertex of degree at most $d$. Degeneracy of a digraph is defined to be the degeneracy of the underlying undirected graph. We say that a class of $($ di $)$ graphs $\mathcal{C}$ is $o(\log n)$ degenerated if there is a function $f(n)=o(\log n)$ such that every (di)graph $G \in \mathcal{C}$ is $f(|V(G)|)$ degenerated.

In a directed graph, we say that a vertex $u$ dominates a vertex $v$ if there is an $\operatorname{arc}(u, v)$ and in an undirected graph, we say that a vertex $u$ dominates a vertex $v$ if there is an edge $(u, v)$ in the graph.

Given a vertex $v$ in a directed graph $D$, we define the operation of short-circuiting across $v$ as follows. We add an arc from every vertex in $N^{-}(v)$ to every vertex in $N^{+}(v)$ and delete $v$.

For a set of vertices $X \subseteq V(G)$ such that $G[X]$ is connected we denote by $G / X$ the graph obtained by contracting edges of a spanning tree of $G[X]$ in $G$.

Given an instance ( $D, r, T, k$ ) of DST, we say that a set $S \subseteq V \backslash(T \cup\{r\})$ of at most $k$ vertices is $a$ solution to this instance if in the digraph $D[S \cup T \cup\{r\}]$ there is a directed path from $r$ to every terminal $t \in T$.
Minors and Topological Minors. For a graph $G=(V, E)$, a graph $H$ is a minor of $G$ if $H$ can be obtained from $G$ by deleting vertices, deleting edges, and contracting edges. We denote that $H$ is a minor of $G$ by $H \preceq G$. A mapping $\varphi: V(H) \rightarrow 2^{V(G)}$ is a model of $H$ in $G$ if for every $u, v \in V(H)$ with $u \neq v$ we have $\varphi(u) \cap \varphi(v)=\emptyset, G[\varphi(u)]$ is connected, and, if $\{u, v\}$ is an edge of $H$, then there are $u^{\prime} \in \varphi(u)$ and $v^{\prime} \in \varphi(v)$ such that $\left\{u^{\prime}, v^{\prime}\right\} \in E(G)$. It is known, that $H \preceq G$ iff $H$ has a model in $G$.


A subdivision of a graph $H$ is obtained by replacing each edge of $H$ by a non-trivial path. We say that $H$ is a topological minor of $G$ if some subgraph of $G$ is isomorphic to a subdivision of $H$ and denote it by $H \preceq_{T} G$. In this paper, whenever we make a statement about a directed graph having (or being) a minor of another graph, we mean the underlying undirected graph. A graph $G$ excludes graph $H$ as a (topological) minor if $H$ is not a (topological) minor of $G$. We say that a class of graphs $\mathcal{C}$ excludes $o(\log n)$-sized (topological) minors if there is a function $f(n)=o(\log n)$ such that for every graph $G \in \mathcal{C}$ we have that $K_{f(|V(G)|)}$ is not a (topological) minor of $G$.
Tree Decompositions. A tree decomposition of a graph $G=(V, E)$ is a pair $(M, \beta)$ where $M$ is a rooted tree and $\beta: V(M) \rightarrow 2^{V}$, such that:

1. $\bigcup_{t \in V(M)} \beta(t)=V$.
2. For each edge $(u, v) \in E$, there is a $t \in V(M)$ such that both $u$ and $v$ belong to $\beta(t)$.
3. For each $v \in V$, the nodes in the set $\{t \in V(M) \mid v \in \beta(t)\}$ form a connected subtree of $M$.

The following notations are the same as that in [22]. Given a tree decomposition of graph $G=(V, E)$, we define mappings $\sigma, \gamma, \alpha: V(M) \rightarrow 2^{V}$ by letting for all $t \in V(M)$,

$$
\begin{aligned}
& \sigma(t)= \begin{cases}\emptyset & \text { if } t \text { is the root of } M \\
\beta(t) \cap \beta(s) & \text { if } s \text { is the parent of } t \text { in } M\end{cases} \\
& \gamma(t)=\bigcup_{u \text { is a descendant of } t} \beta(u) \\
& \alpha(t)=\gamma(t) \backslash \sigma(t) .
\end{aligned}
$$

Let $(M, \beta)$ be a tree decomposition of a graph $G$. The width of $(M, \beta)$ is $\min \{|\beta(t)|-1 \mid t \in V(M)\}$, and the adhesion of the tree decomposition is $\max \{|\sigma(t)| \mid t \in V(M)\}$. For every node $t \in V(M)$, the torso at $t$ is the graph

$$
\tau(t):=G[\beta(t)] \cup E(K[\sigma(t)]) \cup \bigcup_{u \text { child of } t} E(K[\sigma(u)]) .
$$

Again, by a tree decomposition of a directed graph, we mean a tree decomposition for the underlying undirected graph.

## 3 DST on sparse graphs

In this section, we introduce our main idea and use it to design algorithms for the Directed Steiner Tree problem on classes of sparse graphs. We begin by giving a $O^{*}\left(2^{O(h k)}\right)$ algorithm for DST on $K_{h}$-minor free graphs. Following that, we give a $O^{*}\left(f(h)^{k}\right)$ algorithm for DST on $K_{h}$-topological minor free graphs for some $f$. Then, we show that in general, even in 2 degenerated graphs, we cannot expect to have an FPT algorithm for DST parameterized by the solution size. Finally, we show that when the graph induced on the terminals is acyclic, then our ideas are applicable and we can give a $O^{*}\left(2^{O(h k)}\right)$ algorithm on $K_{h}$-topological minor free graphs and a $O^{*}\left(2^{O(d k)}\right)$ algorithm on $d$-degenerated graphs.

### 3.1 DST on minor free graphs

We begin with a polynomial time preprocessing which will allow us to identify a special subset of the terminals with the property that it is enough for us to find an arborescence from the root to these terminals.

- Rule 1. Given an instance $(D, r, T, k)$ of DST, let $C$ be a strongly connected component with at least 2 vertices in the graph $D[T]$. Then, contract $C$ to a single vertex $c$, to obtain the graph $D^{\prime}$ and return the instance $\left(D^{\prime}, r, T^{\prime}=(T \backslash C) \cup\{c\}, k\right)$.

Correctness. Suppose $S$ is a solution to $(D, r, T, k)$. Then there is a directed path from $r$ to every terminal $t \in T$ in the digraph $D[S \cup T \cup\{r\}]$. Contracting the vertices of $C$ will preserve this path. Hence, $S$ is also a solution for $\left(D^{\prime}, r, T^{\prime}, k\right)$.

Conversely, suppose $S$ is a solution for $\left(D^{\prime}, r, T^{\prime}, k\right)$. If the path $P$ from $r$ to some $t \in T^{\prime} \backslash C$ in $D^{\prime}\left[S \cup T^{\prime} \cup\{r\}\right]$ contains $c$, then there must be a path from $r$ to some vertex $x$ of $C$ and a path (possibly trivial) from some vertex $y \in C$ to $t$ in $D[S \cup T \cup\{r\}]$. As there is a path between any $x$ and $y$ in $D[C]$, concatenating these three paths results in a path from $r$ to $t$ in $D[S \cup T \cup\{r\}]$. Hence, $S$ is also a solution to $(D, r, T, k)$.

- Proposition 1. Given an undirected graph $G=(V, E)$ which excludes $K_{h}$ as a minor for some $h$, and a vertex subset $X \subseteq V$ inducing a connected subgraph of $G$, the graph $G / X$ also excludes $K_{h}$ as a minor.

We call an instance reduced if Rule 1 cannot be applied to it. Given an instance ( $D, r, T, k$ ), we first apply Rule 1 exhaustively to obtain a reduced instance. Since the resulting graph still excludes $K_{h}$ as a minor (by Proposition 11, we have not changed the problem and hence, for ease of presentation, we denote the reduced instance also by $(D, r, T, k)$. We call a terminal vertex $t \in T$ a source-terminal if it has no in-neighbors in $D[T]$. We use $T_{0}$ to denote the set of all source-terminals. Since for every terminal, the graph $D[T]$ contains a path from some source terminal to this terminal, we have the following observation.

- Observation 1. Let ( $D, r, T, k$ ) be a reduced instance and let $S \subseteq V$. Then the digraph $D[S \cup T \cup\{r\}]$ contains a directed path from $r$ to every terminal $t \in T$ if and only if it contains a directed path from $r$ to every source-terminal $t \in T_{0}$.

The following is an important subroutine of our algorithm.

- Lemma 2. Let $D$ be a digraph, $r \in V(D), T \subseteq V(D) \backslash\{r\}$ and $T_{0} \subseteq T$. There is an algorithm which can find a minimum size set $S \subseteq V(D)$ such that there is path from $r$ to every $t \in T_{0}$ in $D[T \cup\{r\} \cup S]$ in time $O^{*}\left(2^{\left|T_{0}\right|}\right)$.

Proof. Nederlof [35] gave an algorithm to solve the Steiner Tree problem on undirected graphs in time $O^{*}\left(2^{t}\right)$ where $t$ is the number of terminals. Misra et al. [33] observed that the same algorithm can be easily modified to solve the DST problem in time $O^{*}\left(2^{t}\right)$ with $t$ being the number of terminals. In our case, we create an instance of the DST problem by taking the same graph, defining the set of terminals as $T_{0}$ and for every vertex $t \in T \backslash T_{0}$, short-circuiting across this vertex. Clearly, a $k$-sized solution to this instance gives a $k$-sized solution to the original problem. To actually find the set of minimum size, we can first find its size by a binary search and then delete one by one the non-terminals, if their deletion does not increase the size of the minimum solution.

We call the algorithm from Lemma $2 \operatorname{Nederlof}\left(D, r, T, T_{0}\right)$.
We also need the following structural claim regarding the existence of low degree vertices in graphs excluding $K_{h}$ as a topological minor.

- Lemma 3. Let $G=(V, E)$ be an undirected graph excluding $K_{h}$ as a topological minor and let $X, Y \subseteq V$ be two disjoint vertex sets. If every vertex in $X$ has at least $h-1$ neighbors in $Y$, then there is a vertex in $Y$ with at most ch neighbors in $X \cup Y$ for some constant $c$.

Proof. It was proved in [5, 30, that there is a constant $a$ such that any graph that does not contain $K_{h}$ as a topological minor is $d=a h^{2}$-degenerated. Consider the graph $H_{0}=G[X \cup Y] \backslash E(X)$. We construct a sequence of graphs $H_{0}, \ldots, H_{l}$, starting from $H_{0}$ and repeating an operation which ensures that any graph in the sequence excludes $K_{h}$ as a topological minor. The operation is defined as follows. In graph $H_{i}$, pick a vertex $x \in X$. As it has degree at least $h-1$ in $Y$ and there is no $K_{h}$ topological minor in $H_{i}$, it has two neighbors $y_{1}$ and $y_{2}$ in $Y$, which are non-adjacent. Remove $x$ from $H$ and add the edge $\left(y_{1}, y_{2}\right)$ to obtain the graph $H_{i+1}$. By repeating this operation, we finally obtain a graph $H_{l}$ where the set $X$ is empty. As the graph $H_{l}$ still excludes $K_{h}$ as a topological minor, it
is $d$-degenerated, and hence it has at most $d|Y|$ edges. In the sequence of operations, every time we remove a vertex from $X$, we added an edge between two vertices of $Y$. Hence, the number of vertices in $X$ in $H_{0}$ is bounded by the number of edges within $Y$ in $H_{l}$, which is at most $d|Y|$. As $H_{0}$ is also $d$-degenerated, it has at most $d(|X|+|Y|)=d(d+1)|Y|$ edges. Therefore, there is a vertex in $Y$ incident on at most $2 d(d+1)=2 a h^{2}\left(a h^{2}+1\right) \leq c h^{4}$ edges where $c=4 a^{2}$. This concludes the proof of the lemma.

The following proposition allows us to apply Lemma 3 in the case of graphs excluding $K_{h}$ as a minor.

- Proposition 2. If a graph $G$ exludes $K_{h}$ as a minor, it also excludes $K_{h}$ as a topological minor.

Let $(D, r, T, k)$ be a reduced instance of DST, $Y \subseteq V \backslash T$ be a set of non-terminals representing a partial solution and $d_{b}$ be some fixed positive integer. We define the following sets of vertices (see Fig. 22.

- $T_{1}=T_{1}(Y)$ is the set of source terminals dominated by $Y$.
- $B_{h}=B_{h}\left(Y, d_{b}\right)$ is the set of non-terminals which dominate at least $d_{b}+1$ terminals in $T_{0} \backslash T_{1}$.
- $B_{l}=B_{l}\left(Y, d_{b}\right)$ is the set of non-terminals which dominate at most $d_{b}$ terminals in $T_{0} \backslash T_{1}$.
- $W_{h}=W_{h}\left(Y, d_{b}\right)$ is the set of terminals in $T_{0} \backslash T_{1}$ which are dominated by $B_{h}$.
- $W_{l}=W_{l}\left(Y, d_{b}\right)=T_{0} \backslash\left(T_{1} \cup W_{h}\right)$ is the set of source terminals which are not dominated by $Y$ or $B_{h}$.
Note that the sets are pairwise disjoint. The constant $d_{b}$ is introduced to describe the algorithm in a more general way so that we can use it in further sections of the paper. Throughout this section, we will have $d_{b}=h-2$.
- Lemma 4. Let $(D, r, T, k)$ be a reduced instance of DST, $Y \subseteq V \backslash T, d_{b} \in \mathbb{N}$, and $T_{1}, B_{h}, B_{l}, W_{h}$, and $W_{l}$ as defined above. If $\left|W_{l}\right|>d_{b}(k-|Y|)$, then the given instance does not admit a solution containing $Y$.

Proof. This follows from the fact that any non-terminal from $V \backslash\left(B_{h} \cup Y\right)$ in the solution, which dominates a vertex in $W_{l}$ can dominate at most $d_{b}$ of these vertices. Since the solution contains at most $k-|Y|$ such non-terminals, at most $d_{b}(k-|Y|)$ of these vertices can be dominated. This completes the proof.

- Lemma 5. Let $(D, r, T, k)$ be a reduced instance of DST, $Y \subseteq V \backslash T, d_{b} \in \mathbb{N}$, and $T_{1}, B_{h}, B_{l}, W_{h}$, and $W_{l}$ as defined above. If $B_{h}$ is empty, then there is an algorithm which can test if this instance has a solution containing $Y$ in time $O^{*}\left(2^{d_{b}(k-|Y|)+|Y|}\right)$.

Proof. We use Lemma 2 and test whether $\left|\operatorname{Nederlof}\left(D, r, T \cup Y, Y \cup\left(T_{0} \backslash T_{1}\right)\right)\right| \leq k$. We know that $|Y| \leq k$ and, by Lemma 4 , we can assume that $\left|T_{0} \backslash T_{1}\right| \leq d_{b}(k-|Y|)$. Therefore, the size of $Y \cup\left(T_{0} \backslash T_{1}\right)$ is bounded by $|Y|+d_{b}(k-|Y|)$, implying that we can solve the DST problem on this instance in time $O^{*}\left(2^{d_{b}(k-|Y|)+|Y|}\right)$. This completes the proof of the lemma.

We now proceed to the main algorithm of this subsection.

- Theorem 6. DST can be solved in time $O^{*}\left(3^{h k+o(h k)}\right)$ on graphs excluding $K_{h}$ as a minor.

Proof. Let $T_{0}$ be the set of source terminals of this instance. The algorithm we describe takes as input a reduced instance ( $D, r, T, k$ ), a vertex set $Y$ and a positive integer $d_{b}$ and returns a smallest solution for the instance which contains $Y$ if such a solution exists. If there is no solution, then the algorithm returns a dummy symbol $S_{\infty}$. To simplify the description, we assume that $\left|S_{\infty}\right|=\infty$. The algorithm is a recursive algorithm and at any stage of the recursion, the corresponding recursive step returns the smallest set found in the recursions initiated in this step. We start with $Y$ being the empty set.

By Lemma 4 , if $\left|W_{l}\right|>d_{b}(k-|Y|)$, then there is no solution containing $Y$ and hence we return $S_{\infty}$ (see Algorithm 3.1. If $B_{h}$ is empty, then we apply Lemma 5 to solve the problem in time $O^{*}\left(2^{d_{b} k}\right)$.

```
Input : An instance \((D, r, T, k)\) of DST, degree bound \(d_{b}\), set \(Y\)
Output: A smallest solution of size at most \(k\) and containing \(Y\) for the instance ( \(D, r, T, k\) ) if
    it exists and \(S_{\infty}\) otherwise
Compute the sets \(B_{h}, B_{l}, Y, W_{h}, W_{l}\)
if \(\left|W_{l}\right|>d(k-|Y|)\) then return \(S_{\infty}\)
else if \(B_{h}=\emptyset\) then
    \(S \leftarrow \operatorname{Nederlof}\left(D, r, T \cup Y, W_{l} \cup Y\right)\).
    if \(|S|>k\) then \(S \leftarrow S_{\infty}\)
    return \(S\)
end
else
    \(S \leftarrow S_{\infty}\)
    Find vertex \(v \in W_{h}\) with the least in-neighbors in \(B_{h}\).
    for \(u \in B_{h} \cap N^{-}(v)\) do
        \(Y^{\prime} \leftarrow Y \cup\{u\}\),
        \(S^{\prime} \leftarrow\) DST-SOLVE \(\left((D, r, T, k), d_{b}, Y\right)\).
        if \(\left|S^{\prime}\right|<|S|\) then \(S \leftarrow S^{\prime}\)
    end
    \(D^{\prime} \leftarrow D \backslash\left(B_{h} \cap N^{-}(v)\right)\)
    \(S^{\prime} \leftarrow\) DST-SOLVE \(\left(\left(D^{\prime}, r, T, k\right), d_{b}, Y\right)\).
    if \(\left|S^{\prime}\right|<|S|\) then \(S \leftarrow S^{\prime}\)
    return \(S\)
end
```

Algorithm 3.1: Algorithm DST-solve for DST on graphs excluding $K_{h}$ as a minor

If $B_{h}$ is non-empty, then we find a vertex $v \in W_{h}$ with the least in-neighbors in $B_{h}$. Suppose it has $d_{w}$ of them.

We then branch into $d_{w}+1$ branches described as follows. In the first $d_{w}$ branches, we move a vertex $u$ of $B_{h}$ which is an in-neighbor of $v$, to the set $Y$. Each of these branches is equivalent to picking one of the in-neighbors of $v$ from $B_{h}$ in the solution. We then recurse on the resulting instance. In the last of the $d_{w}+1$ branches, we delete from the instance non-terminals in $B_{h}$ which dominate $v$ and recurse on the resulting instance. Note that in the resulting instance of this branch, we have $v$ in $W_{l}(Y)$.

Correctness. At each node of the recursion tree, we define a measure $\mu(I)=d_{b}(k-|Y|)-\left|W_{l}\right|$. We prove the correctness of the algorithm by induction on this measure. In the base case, when $d_{b}(k-|Y|)-\left|W_{l}\right|<0$, then the algorithm is correct (by Lemma 4). Now, we assume as induction hypothesis that the algorithm is correct on instances with measure less than some $\mu \geq 0$. Consider an instance $I$ such that $\mu(I)=\mu$. Since the branching is exhaustive, it is sufficient to show that the algorithm is correct on each of the child instances. To show this, it is sufficient to show that for each child instance $I^{\prime}, \mu\left(I^{\prime}\right)<\mu(I)$. In the first $d_{w}$ branches, the size of the set $Y$ increases by 1 , and the size of the set $W_{l}$ does not decrease. Hence, in each of these branches, $\mu\left(I^{\prime}\right) \leq \mu(I)-d_{b}$. In the final branch, though the size of the set $Y$ remains the same, the size of the set $W_{l}$ increases by at least 1. Hence, in this branch, $\mu\left(I^{\prime}\right) \leq \mu(I)-1$. Thus, we have shown that in each branch, the measure drops, hence completing the proof of correctness of the algorithm.

Analysis. Since $D$ exludes $K_{h}$ as a minor, Lemma 3 combined with the fact that we set $d_{b}=h-2$, implies that $d_{w}^{\max }=c h^{4}$, for some $c$, is an upper bound on the maximum $d_{w}$ which can appear during the execution of the algorithm. We first bound the number of leaves of the recursion tree as follows. The number of leaves is bounded by $\sum_{i=0}^{d_{b} k}\binom{d_{b} k}{i}\left(d_{w}^{\max }\right)^{k-\frac{i}{d_{b}}}$. To see this, observe that each branch of


Figure 2 An illustration of the sets defined in Theorem 6
the recursion tree can be described by a length $-d_{b} k$ vector as shown in the correctness paragraph. We then select $i$ positions of this vector on which the last branch was taken. Finally for $k-\frac{i}{d_{b}}$ of the remaining positions, we describe which of the first at most $d_{w}^{\max }$ branches was taken. Any of the first $d_{w}^{\max }$ branches can be taken at most $k-\frac{i}{d_{b}}$ times if the last branch is taken $i$ times.

The time taken along each root to leaf path in the recursion tree is polynomial, while the time taken at a leaf for which the last branch was taken $i$ times is $O^{*}\left(2^{d_{b}\left(k-\left(k-\frac{i}{d_{b}}\right)\right)+k-\frac{i}{d_{b}}}\right)=O^{*}\left(2^{i+k}\right)$ (see Lemmata 4 and 55. Hence, the running time of the algorithm is

$$
O^{*}\left(\sum_{i=0}^{d_{b} k}\binom{d_{b} k}{i}\left(d_{w}^{\max }\right)^{k-\frac{i}{d_{b}}} \cdot 2^{i+k}\right)=O^{*}\left(\left(2 d_{w}^{\max }\right)^{k} \cdot \sum_{i=0}^{d_{b} k}\binom{d_{b} k}{i} \cdot 2^{i}\right)=O^{*}\left(\left(2 d_{w}^{\max }\right)^{k} \cdot 3^{d_{b} k}\right) .
$$

For $d_{b}=h-2$ and $d_{w}^{\max }=c h^{4}$ this is $O^{*}\left(3^{h k+o(h k)}\right)$. This completes the proof of the theorem.

- Lemma 7. For every function $g(n)=o(\log n)$, there is a function $f(k)$ such that for every $k$ and $n$ we have $2^{g(n) k} \leq f(k) \cdot n$.

Proof. We know that there is a function $f^{\prime}(k)$ such that for every $n>f^{\prime}(k)$ we have $g(n)<(\log n) / k$. Now let $f(k)$ be the function defined as $f(k)=\max _{1 \leq n \leq f^{\prime}(k)}\left\{2^{g(n) k}\right\}$. Then, for every $k$ if $n \leq f^{\prime}(k)$ then $2^{g(n) k} \leq \max _{1 \leq n \leq f^{\prime}(k)}\left\{2^{g(n) k}\right\}=f(k)$ while for $n>f^{\prime}(k)$ we have $2^{g(n) k} \leq 2^{(\log n / k) \cdot k}=2^{\log n}=$ $n$. Hence, indeed, $2^{g(n) k} \leq f(k) \cdot n$ for every $n$ and $k$.

Theorem 6 along with Lemma 7 has the following corollary.

- Corollary 8. If $\mathcal{C}$ is a class of digraphs excluding o( $\log n)$-sized minors, then DST parameterized by $k$ is FPT on $\mathcal{C}$.


### 3.2 DST on graphs excluding topological minors

We begin by observing that on graphs excluding $K_{h}$ as a topological minor, we cannot apply Rule 1 since contractions may create new topological minors. Hence, we do not have the notion of a source terminal, which was crucial in designing the algorithm for this problem on graphs excluding minors. However, we will use a decomposition theorem of Grohe and Marx ([22], Theorem 4.1) to obtain a number of subproblems where we will be able to apply all the ideas developed in the previous subsection, and finally use a dynamic programming approach over this decomposition to combine the solutions to the subproblems.

- Theorem 9. (Global Structure Theorem, [22]) For every $h \in \mathbb{N}$, there exists constants $a(h), b(h)$, $c(h), d(h), e(h)$, such that the following holds. Let $H$ be a graph on $h$ vertices. Then, for every
graph $G$ with $H \npreceq_{T} G$, there is a tree decomposition $(M, \beta)$ of adhesion at most a(h) such that for all $t \in V(M)$, one of the following three conditions is satisfied:

1. $|\beta(t)| \leq b(h)$.
2. $\tau(t)$ has at most $c(h)$ vertices of degree larger than $d(h)$.
3. $K_{e(h)} \npreceq \tau(t)$.

Furthermore, there is an algorithm that, given graphs $G, H$ of sizes $n$, $h$, respectively, in time $f(h) n^{O(1)}$ for some computable function $f$, computes either such a decomposition $(M, \beta)$ or a subdivision of $H$ in $G$.

Let $(M, \beta)$ a tree decomposition given by the above theorem. Without loss of generality we assume, that for every $t \in V(M)$ we have $r \in \beta(t)$. This might increase $a(h), b(h), c(h)$, and $e(h)$ by at most one. For the rest of this subsection we work with this tree decomposition.

- Theorem 10. DST can be solved in time $O^{*}\left(f(h)^{k}\right)$ on graphs excluding $K_{h}$ as a topological minor.

Proof. Our algorithm is based on dynamic programming over the tree decomposition $(M, \beta)$. For $t \in V(M)$ let $T_{\sigma(t)}=(T \cup\{r\}) \cap \sigma(t)$ and $T_{\gamma(t)}=(T \cup\{r\}) \cap \gamma(t)$. For every $t \in V(M)$ we have one table $\mathrm{Tab}_{t}$ indexed by $(R, F)$, where $T_{\sigma(t)} \subseteq R \subseteq \sigma(t)$ and $F$ is a set of arcs on $R$. The index of a table represents the way a possible solution tree can cross the cut-set $\sigma(t)$. More precisely, we look for a set $S \subseteq \alpha(t)$ such that in the digraph $D\left[T_{\gamma(t)} \cup R \cup S\right] \cup F$ there is a directed path from $r$ to every $t^{\prime} \in T_{\gamma(t)} \cup R$. In such a case we say that $S$ is good for $t, R, F$.

For each index $R, F$ we store in $\operatorname{Tab}_{t}(R, F)$ one good set $S$ of minimum size. If no such set exists, or $|S|>k$ for any such set, we set $\left|\operatorname{Tab}_{t}(R, F)\right|=\infty$ and use the dummy symbol $S_{\infty}$ in place of the set. Naturally, $S_{\infty} \cup S=S_{\infty}$ for any set $S$. Furthermore, if $\operatorname{Tab}_{t}(R, F)=S \neq S_{\infty}$ we let $\kappa_{t}(R, F)$ be the set of arcs on $R$ such that $(u, v) \in \kappa_{t}(R, F)$ iff there is a directed path from $u$ to $v$ in $D\left[T_{\gamma(t)} \cup R \cup \operatorname{Tab}_{t}(R, F)\right]$. Note also, that if $\left|\operatorname{Tab}_{t}(R, F)\right|=0$ then $\kappa_{t}(R, F)$ only depends on $R$, not on $F$. As $\sigma$ of the root node of $M$ is $\emptyset$, the only entry of Tab for root is an optimal Steiner tree in $D$. Let us denote by $g(h)$ the maximum number of entries of the table $T a b_{t}$ over $t \in V(M)$. It is easy to see that $g(h) \leq 2^{a(h)+a(h)^{2}}$.

The algorithm to fill the tables proceeds bottom-up along the tree decomposition and we assume that by the time we start filling the table for $t$, the tables for all its proper descendants have already been already filled. We now describe the algorithm to fill the table for $t$, distinguishing three cases, based on the type of node $t$ (see Theorem 9 ).

### 3.2.1 Case 1: $\tau(t)$ has at most $c(h)$ vertices of degree larger than $d(h)$.

In this case we use Algorithm 3.2. For each $R$ and $F$ it first removes the irrelevant parts of the graph and then branches on the non-terminal vertices of high degree. Following that, it invokes Algorithm 3.3. Note that, since $t$ can have an unbounded number of children in $M$ we cannot afford to guess the solution for each of them. Hence SatisfyChildrenSD only branches on the solution which is taken from the children which need at least one private vertex of the solution. After a solution is selected for all such children, it uses Rule 1 and unless the number of obtained source terminals is too big, in which case there is no solution for the branch, it uses the modified algorithm of Nederlof as described in Lemma 5

For the proof of the correctness of the algorithm, we need several observations and lemmas.

- Observation 11. Let $(D, r, T, k)$ be an instance of DST. For every $x$ in $V \backslash(T \cup\{r\})$ we have that $S$ is a solution for $(D, r, T \cup\{x\}, k-1)$ if and only if $S \cup\{x\}$ is a solution for $(D, r, T, k)$ and $x$ is reachable from $r$ in $D[T \cup S \cup\{r, x\}]$. In particular, if $S \cup\{x\}$ is a minimal solution for $(D, r, T, k)$, then $S$ is a minimal solution for $(D, r, T \cup\{x\}, k-1)$.
- Lemma 12. Let $\left(D^{\prime}, r, T^{\prime}, k\right)$ be an instance of $D S T$ and $M, \beta$ be a tree decomposition for $D$ rooted at $t$. Let $s$ be a child of $t$ for which the condition on line 2 of Algorithm 3.3 is satisfied. Let $\left(D^{\prime \prime}, r, T^{\prime \prime}, k^{\prime}\right)$ be the instance as formed by lines 710 of the algortihm on $s$ for some $R^{\prime}$ and $F^{\prime}$. If $S$ is a solution for $\left(D^{\prime \prime}, r, T^{\prime \prime}, k^{\prime}\right)$ then $S^{\prime}=S \cup \operatorname{Tab}_{s}\left(R^{\prime}, F^{\prime}\right) \cup\left(R^{\prime} \backslash T_{\sigma(s)}^{\prime}\right)$ is a solution for $\left(D^{\prime}, r, T^{\prime}, k\right)$.


```
foreach \(R\) with \(T_{\sigma(t)} \subseteq R \subseteq \sigma(t)\) do
foreach \(F \subseteq R^{2}\) do
    \(D^{\prime} \leftarrow D[\alpha(t) \cup R] \cup F\).
    \(S \leftarrow S_{\infty}\).
    \(B \leftarrow\left\{v \mid v \in\left(\beta(t) \cap V\left(D^{\prime}\right)\right) \backslash(T \cup R) \& \operatorname{deg}_{\tau(t)}(v)>d(h)\right\}\).
    foreach \(Y \subseteq B\) with \(|Y| \leq k\) do
        \(D^{\prime \prime} \leftarrow D^{\prime} \backslash(B \backslash Y)\).
        \(M^{\prime} \leftarrow\) subtree of \(M\) rooted at \(t\).
        Let \(\beta^{\prime}: V\left(M^{\prime}\right) \rightarrow 2^{V\left(D^{\prime \prime}\right)}\) be such that \(\beta^{\prime}(s)=\beta(s) \cap V\left(D^{\prime \prime}\right)\) for every \(s\) in \(V\left(M^{\prime}\right)\).
        \(S^{\prime} \leftarrow \operatorname{SatifyChildrenSD}\left(D^{\prime \prime}, r, T \cup Y \cup R \backslash\{r\}, k-|Y|, M^{\prime}, \beta^{\prime}\right) \cup Y\).
        if \(\left|S^{\prime}\right|<|S|\) then \(S \leftarrow S^{\prime}\)
    end
    \(\operatorname{Tab}_{t}(R, F) \leftarrow S\).
14 end
```

Algorithm 3.2: Algorithm SmallDeg to fill $\mathrm{Tab}_{t}$ if all but few vertices of the bag have small degrees.

```
Input : An instance \(\left(D^{\prime}, r, T^{\prime}, k\right)\) of DST , a tree decomposition \((M, \beta)\) rooted at \(t\)
Output: A smallest solution to the instance or \(S_{\infty}\) if all solutions are larger than \(k\)
if \(k<0\) then return \(S_{\infty}\)
if \(\exists s\) child of \(t\) such that \(\mid \operatorname{Tab}_{s}\left(T_{\sigma(s)}^{\prime},\{r\} \times\left(T^{\prime} \cap \sigma(s)\right) \mid>0\right.\) then
    \(S \leftarrow S_{\infty}\).
    foreach \(R^{\prime}\) s.t. \(T_{\sigma(s)}^{\prime} \subseteq R^{\prime} \subseteq \sigma(s)\) do
    foreach \(F^{\prime} \subseteq\left(R^{\prime}\right)^{2}\) do
    if \(\left|T a b_{s}\left(R^{\prime}, F^{\prime}\right) \cup\left(R^{\prime} \backslash T_{\sigma(s)}^{\prime}\right)\right| \leq k\) then
        \(\hat{D} \leftarrow D^{\prime} \backslash\left(\gamma(s) \backslash R^{\prime}\right)\).
        \(D^{\prime \prime} \leftarrow \hat{D} \cup \kappa_{s}\left(R^{\prime}, F^{\prime}\right)\).
        \(T^{\prime \prime} \leftarrow\left(T^{\prime} \cap V\left(D^{\prime \prime}\right)\right) \cup\left(R^{\prime} \backslash T_{\sigma(s)}^{\prime}\right)\).
        \(k^{\prime} \leftarrow k-\left|\operatorname{Tab}_{s}\left(R^{\prime}, F^{\prime}\right)\right|-\left|R^{\prime} \backslash T_{\sigma(s)}^{\prime}\right|\).
        \(M^{\prime} \leftarrow M\) with the subtree rooted at \(s\) removed.
        Let \(\beta^{\prime}: V\left(M^{\prime}\right) \rightarrow 2^{V\left(D^{\prime \prime}\right)}\) be such that \(\beta^{\prime}(s)=\beta(s) \cap V\left(D^{\prime \prime}\right)\) for every \(s\) in \(V\left(M^{\prime}\right)\).
        \(S^{\prime} \leftarrow \operatorname{SATISFyChildrenSD}\left(D^{\prime \prime}, r, T^{\prime \prime}, k^{\prime}, M^{\prime}, \beta^{\prime}, Y^{\prime}\right) \cup \operatorname{Tab}_{s}\left(R^{\prime}, F^{\prime}\right) \cup\left(R^{\prime} \backslash T_{\sigma(s)}^{\prime}\right)\).
        if \(\left|S^{\prime}\right|<|S|\) then \(S \leftarrow S^{\prime}\)
    end
else
    Apply Rule 1 exhaustively to ( \(D^{\prime}, r, T^{\prime}, k\) ) to obtain ( \(D^{\prime \prime}, r, T^{\prime \prime}, k\) ).
    Denote by \(T_{0}\) the source terminals in ( \(D^{\prime \prime}, r, T^{\prime \prime}, k\) ).
    if \(\left|T_{0}\right|>k \cdot \max \{d(h), a(h)\}\) then \(S \leftarrow S_{\infty}\)
    else \(S \leftarrow\) Nederlof \(\left(D^{\prime \prime}, r, T^{\prime \prime}, T_{0}\right)\).
end
if \(|S|>k\) then \(S \leftarrow S_{\infty}\)
return \(S\)
```

Algorithm 3.3: Function SatisfyChildrenSD ( $D^{\prime}, r, T^{\prime}, k, M, \beta$ ) doing the main part of the work of Algortihm 3.2

Proof. Obviously $\left|S^{\prime}\right| \leq k$. We have to show that every vertex $t^{\prime}$ in $T^{\prime}$ is reachable from $r$ in $D^{\prime}\left[T^{\prime} \cup\{r\} \cup S^{\prime}\right]$. For a vertex $t^{\prime}$ in $T^{\prime \prime}$ there is a path from $r$ to $t^{\prime}$ in $D^{\prime \prime}$. If this path contains an arc $(u, v)$ in $\kappa\left(R^{\prime}, F^{\prime}\right)$, then there is a path from $u$ to $v$ in $D\left[T_{\gamma(s)}^{\prime} \cup R^{\prime} \cup \operatorname{Tab}_{s}\left(R^{\prime}, F^{\prime}\right)\right]$, and we can replace the $\operatorname{arc}(u, v)$ with this path, obtaining (possibly after shortcutting) a path in $D^{\prime}\left[T^{\prime} \cup\{r\} \cup S^{\prime}\right]$. For a vertex $t^{\prime}$ in $T^{\prime} \cap \gamma(s)$ there is a path $P$ from $r$ to $t^{\prime}$ in $D\left[T_{\gamma(s)}^{\prime} \cup R^{\prime} \cup \operatorname{Tab}_{s}\left(R^{\prime}, F^{\prime}\right)\right] \cup F^{\prime}$ as the $\mathrm{Tab}_{s}$ was filled correctly. Let $r^{\prime}$ be the last vertex of $R^{\prime}$ on $P$. Replacing the part of $P$ from $r$ to $r^{\prime}$ by a path in $D^{\prime}\left[T^{\prime} \cup\{r\} \cup S^{\prime}\right]$ obtained in the previous step, we get (possibly after shortcutting) a path from $r$ to $t^{\prime}$ in $D^{\prime}\left[T^{\prime} \cup\{r\} \cup S^{\prime}\right]$ as required.

- Observation 13. If $s$ is a child of $t$ and there are some $R^{\prime}$ and $F^{\prime}$ such that $\left|\mathrm{Tab}_{s}\left(R^{\prime}, F^{\prime}\right)\right|=0$ then every vertex in $T^{\prime} \cap \gamma(s)$ is reachable from some vertex in $R^{\prime}$ in $D^{\prime}\left[R^{\prime} \cup T_{\gamma(s)}^{\prime}\right]$. Furthermore, if there is an $F^{\prime}$ such that $\left|\operatorname{Tab}_{s}\left(R^{\prime}, F^{\prime}\right)\right|=0$, then $\mid \operatorname{Tab}_{s}\left(R^{\prime},\{r\} \times\left(R^{\prime} \backslash\{r\}\right) \mid=0\right.$, and also $\mid \operatorname{Tab}_{s}\left(R^{\prime \prime},\{r\} \times\right.$ $\left(R^{\prime \prime} \backslash\{r\}\right) \mid=0$ for every $R^{\prime} \subseteq R^{\prime \prime} \subseteq \sigma(s)$.

Proof. The first statement follows from the definition of $\mathrm{Tab}_{s}$, as $F^{\prime}$ is a set of arcs on $R^{\prime}$. The second part is a direct consequence of the first.

- Observation 14. Suppose the condition on line 2 of Algorithm 3.3 is not satisfied, $\left(D^{\prime \prime}, r, T^{\prime \prime}, k\right)$ is obtained from ( $D^{\prime}, r, T^{\prime}, k$ ) by exhaustive application of Rule 1 and $T_{0}$ is the set of source terminals. Let $t_{0} \in T_{0}$ be obtained by contracting a strongly connected component $C$ of $D^{\prime}\left[T^{\prime}\right]$. If there is a vertex $u \in C \cap \gamma(s)$ for some child $s$ of $t$, then there is a vertex $v \in C \cap \sigma(s)$.

Proof. Since the condition is not satisfied, it follows from Observation 13 that there is a path from some $v \in \sigma(s)$ to $u$ in $D^{\prime}\left[T^{\prime} \cup\{r\}\right]$. Since $t_{0}$ is a source terminal, this path has to be fully contained in $C$.

- Lemma 15. Suppose the condition on line 2 of Algorithm 3.3 is not satisfied, $\left(D^{\prime \prime}, r, T^{\prime \prime}, k\right)$ is obtained from ( $\left.D^{\prime}, r, T^{\prime}, k\right)$ by exhaustive application of Rule 1 and $T_{0}$ is the set of source terminals. A vertex $x$ in $\alpha(s) \backslash\left(T^{\prime} \cup\{r\}\right)$ for some child $s$ of $t$ can dominate at most $a(h)$ vertices of $T_{0}$. $A$ vertex $x$ in $\beta(t) \backslash\left(T^{\prime} \cup\{r\}\right)$ can dominate at most $\operatorname{deg}_{\tau(t)}(x)$ vertices of $T_{0}$.

Proof. If $x \in V\left(D^{\prime}\right) \backslash T^{\prime}$ dominates a vertex $t_{0} \in T_{0}$, then $t_{0}$ was obtained by contracting some strongly connected component $C$ of $D^{\prime}\left[T^{\prime}\right]$ and there is an $y \in C \cap N_{D^{\prime}}^{+}(x)$. If $x$ is in $\alpha(s)$, then $N_{D^{\prime}}^{+}(x) \subseteq \gamma(s), C$ contains a vertex of $\sigma(s)$ due to Observation 14 and, hence, there can be at most $a(h)$ such $t_{0}$ 's. If $x$ is in $\beta(t)$, then either $y$ is also in $\beta(t)$, in which case the edge $x y$ is in $\tau(t)$, or $y$ is in $\alpha(s)$ for some child $s$ of $t$. In this case $x$ is in $\sigma(s), C$ contains a vertex $y^{\prime}$ of $\sigma(s)$ due to Observation 14, and we can account $t_{0}$ to the edge $x y^{\prime}$ of $\tau(t)$.

- Lemma 16. Let $\left(D^{\prime}, r, T^{\prime}, k\right)$ be an instance of $D S T$ and $(M, \beta)$ be a tree decomposition for $D^{\prime}$ rooted at $t$. Let s be a child of $t$ for which the condition on line 2 of Algorithm 3.3 is satisfied. Let $S$ be a solution for $\left(D^{\prime}, r, T^{\prime}, k\right), R^{\prime}=\left(T^{\prime} \cup S \cup\{r\}\right) \cap \sigma(s)$ and $F^{\prime}$ be the set of arcs (uv) on $R$ such that $v$ is reachable from $u$ in $D^{\prime}\left[\left(T^{\prime} \cup S \cup\{r\}\right) \backslash \alpha(s)\right]$. Let $\left(D^{\prime \prime}, r, T^{\prime \prime}, k^{\prime}\right)$ be the instance as formed by lines $\sqrt[7]{10}$ on $s$ for $R^{\prime}$ and $F^{\prime}$. Then $S \backslash \alpha(s) \backslash\left(R^{\prime} \backslash T_{\sigma(s)}^{\prime}\right)$ is a solution for $\left(D^{\prime \prime}, r, T^{\prime \prime}, k^{\prime}\right)$, while $(S \cap \alpha(s))$ is good for $s, R^{\prime}, F^{\prime}$.

Proof. We first show that every vertex of $\left(T^{\prime} \cup S\right) \backslash \alpha(s)$ is reachable from $r$ in $D^{\prime \prime}=D^{\prime}\left[\left(T^{\prime} \cup S \cup\{r\}\right) \backslash\right.$ $\alpha(s)] \cup \kappa\left(R^{\prime}, F^{\prime}\right)$. For every vertex $t^{\prime} \in R^{\prime}$ there is a path from $r$ to $t^{\prime}$ in $D^{\prime}\left[T_{\gamma(s)}^{\prime} \cup R^{\prime} \cup \operatorname{Tab}_{s}\left(R^{\prime}, F^{\prime}\right)\right] \cup F^{\prime}$ since $\mathrm{Tab}_{s}$ is correctly filled. Replacing the parts of this path in $\alpha(s)$ by arcs of $\kappa\left(R^{\prime}, F^{\prime}\right)$ and arcs of $F^{\prime}$ by paths in $D^{\prime}\left[\left(T^{\prime} \cup S \cup\{r\}\right) \backslash \alpha(s)\right]$ one obtains a path in $D^{\prime \prime}$ to every vertex of $R^{\prime}$. Now for every vertex $t^{\prime}$ in $\left(T^{\prime} \cup S\right) \backslash \alpha(s)$ there is a path $P$ from $r$ to $t$ in $D^{\prime}\left[T^{\prime} \cup S \cup\{r\}\right]$. Let $r^{\prime}$ be the last vertex of $P$ in $R^{\prime}$. Then concatenating the path from $r$ to $r^{\prime}$ obtained in the previous step with the part of $P$ between $r^{\prime}$ and $t^{\prime}$ we get a path from $r$ to $t^{\prime}$ in $D^{\prime \prime}$.

We have shown, that $S \backslash \alpha(s)$ is a solution for $\left(D^{\prime \prime}, r, T^{\prime}, k-\left|\operatorname{Tab}_{s}\left(R^{\prime}, F^{\prime}\right)\right|\right)$. It remains to use Observation 11 to show that $S \backslash \alpha(s) \backslash\left(R^{\prime} \backslash T_{\sigma(s)}^{\prime}\right)$ is a solution for $\left(D^{\prime \prime}, r, T^{\prime \prime}, k^{\prime}\right)$. The second claim
follows from that there is a path from $r$ to every $t^{\prime} \in\left(T^{\prime} \cup S\right) \cap \gamma(s) \supseteq T_{\gamma(s)}^{\prime} \cup R^{\prime} \backslash\{r\}$ in $D^{\prime}\left[T^{\prime} \cup S \cup\{r\}\right]$ and the parts of it outside $\gamma(s)$ can be replaced by arcs of $F^{\prime}$.

- Lemma 17. Let $\left(D^{\prime}, r, T^{\prime}, k\right)$ be an instance of DST and let $(M, \beta)$ be a tree decomposition for $D^{\prime}$. If there is a solution $S$ of size at most $k$ for $\left(D^{\prime}, r, T^{\prime}, k\right)$, then the invocation SatisfyChildrenSD $\left(D^{\prime}, r, T^{\prime}, k, M, \beta\right)$ returns a set $S^{\prime}$ not larger than $S$.

Proof. We prove the claim by induction on the depth of the recursion. Note that the depth is bounded by the number of children of $t$ in $M$. Suppose first that the condition on line 2 is not satisfied and $\left|T_{0}\right|>k \cdot \max \{d(h), a(h)\}$. As no vertex can dominate more than $\max \{d(h), a(h)\}$ vertices of $T_{0}$ by Lemma 15, there is a vertex of $T_{0}$ not dominated by $S$, which is a contradiction. If $\left|T_{0}\right| \leq k \cdot \max \{d(h), a(h)\}$, it follows from the optimality of the modified Nederlof's algorithm (see Lemma 22 and Observation 1 that $\left|S^{\prime}\right| \leq|S|$.

Now suppose that the condition on line 2 is satisfied for some $s$. Let $R^{\prime}, F^{\prime}$ and $\left(D^{\prime \prime}, r, T^{\prime \prime}, k^{\prime}\right)$ be as in Lemma 16. Then $S \backslash \alpha(s) \backslash\left(R^{\prime} \backslash T_{\sigma(s)}^{\prime}\right)$ is a solution for ( $D^{\prime \prime}, r, T^{\prime \prime}, k^{\prime}$ ), while $S \cap \alpha(s)$ is good for $s, R^{\prime}, F^{\prime}$. Therefore $|S \cap \alpha(s)| \leq\left|\operatorname{Tab}_{s}\left(R^{\prime}, F^{\prime}\right)\right|$ as $\mathrm{Tab}_{s}$ is filled correctly by assumption. Moreover $\operatorname{SatiSFyChildrenSD}\left(D^{\prime \prime}, r, T^{\prime \prime}, k^{\prime}, M, \beta\right)$ will return a set $S^{\prime}$ with $\left|S^{\prime}\right| \leq\left|S \backslash \alpha(s) \backslash\left(R^{\prime} \backslash T_{\sigma(s)}^{\prime}\right)\right|$ due to the induction hypothesis. Together we get that $\left|S^{\prime} \cup \operatorname{Tab}_{s}\left(R^{\prime}, F^{\prime}\right) \cup\left(R^{\prime} \backslash T_{\sigma(s)}^{\prime}\right)\right| \leq|S|$ and therefore also the set returned by SatisfyChildrenSD is not larger than $S$.

Now we are ready to prove the correctness of the algorithm. We first show that if the algorithm stores a set $S \neq S_{\infty}$ in $\operatorname{Tab}_{t}(R, F)$, then $S \subseteq \alpha(t)$ and in the digraph $D\left[T_{\gamma(t)} \cup R \cup S\right] \cup F$ there is a directed path from $r$ to every $t^{\prime} \in T_{\gamma(t)} \cup R$. This will follow from Observation 11 if we prove that SatisfyChildrenSD $\left(D^{\prime}, r, T^{\prime}, k, M, \beta\right)$ returning a set $S \neq S_{\infty}$ implies that $S$ is a solution for $\left(D^{\prime}, r, T^{\prime}, k\right)$. We prove this claim by induction on the depth of the recursion. If the condition on line 2 is not satisfied (and hence there is no recursion) the claim follows from the correctness of the modified version of Nederlof's algorithm (see Lemma 2) and Observation 1 If the condition is satisfied, then the claim follows from Lemma 12 and the induction hypothesis.

In order to prove that the set stored is minimal, assume that there is a set $S \subseteq \alpha(t)$ of size at most $k$ which is good for $t, R, F$. Let $B=\left\{v \mid v \in\left(\beta(t) \cap V\left(D^{\prime}\right)\right) \backslash(T \cup R) \& \operatorname{deg}_{\tau(t)}(v)>d(h)\right\}$, $Y=S \cap B$ and $D^{\prime \prime}=D^{\prime} \backslash(B \backslash Y)$. Without loss of generality we can assume that $S$ is minimal and, therefore, $S \backslash Y$ is a solution for $\left(D^{\prime \prime}, r, T \cup Y \cup R \backslash\{r\}, k-|Y|\right)$ by Observation 11 Hence SatifyChildrenSD $\left(D^{\prime \prime}, r, T \cup Y \cup R \backslash\{r\}, k-|Y|, M^{\prime}, \beta^{\prime}\right) \cup Y$ returns a set $S^{\prime}$ not larger than $S \backslash Y$ due to Lemma 17 and the set stored in $\operatorname{Tab}_{t}(R, F)$ is not larger than $\left|S^{\prime} \cup Y\right|=|S|$ finishing the proof of correctness.

As for the time complexity, observe first, that the bottleneck of the running time of Algorithm 3.2 is the at most $2^{c(h)}$ calls of Algorithm 3.3. Therefore, we focus our attention on the running time of Algorithm 3.3. Note that in each recursive call of SatisfyChildrenSD, by Observation 13, as the condition on line 2 is satisfied, either $\left|\operatorname{Tab}_{s}\left(R^{\prime}, F^{\prime}\right)\right|>0$ or $\left|R^{\prime} \backslash T_{\sigma(s)}^{\prime}\right|>0$ and thus, $k^{\prime}<k$. There are at most $g(h)$ recursive calls for one call of the function. The time spent by SatisfyChildrenSD on instance with parameter $k$ is at most the maximum of $g(h)$ times the time spent on instances with parameter $k-1$ and the time spend by the modified algorithm of Nederlof on an instance with at $\operatorname{most} k \cdot \max \{d(h), a(h)\}$ source terminals. As the time spent for $k<0$ is constant, we conclude that the running time in Case 1 can be bounded by $O^{*}\left(\left(\max \left\{g(h), 2^{\max \{d(h), a(h)\}}\right\}\right)^{k}\right)$.

### 3.2.2 Case 2: $K_{e(h)} \npreceq \tau(t)$.

The overall strategy in this case is similar to that in the previous case. Basically all the work is done by Algorithm 3.4 SatisfyChildrenMF()), which is a slight modification of the function SatisfyChildrenSD. The modification is limited to the else branch of the condition on line 2 that is, to lines 17,20 where Algorithm 3.1 (developed in Section 3.1) is used instead of the modified version of Nederlof's algorithm. For every $R$ and $F$ we now simply store in $\left|\operatorname{Tab}_{t}(R, F)\right|$ the result of

SAtisfyChildrenMF $\left(D^{\prime}, r, T \cup R \backslash\{r\}, k, M^{\prime}, \beta^{\prime}\right)$, where $D^{\prime}=D[\alpha(t) \cup R] \cup F, M^{\prime}$ is the subtree of $M$ rooted at $t$ and $\beta^{\prime}: V\left(M^{\prime}\right) \rightarrow 2^{V\left(D^{\prime}\right)}$ is such that $\beta^{\prime}(s)=\beta(s) \cap V\left(D^{\prime}\right)$ for every $s$ in $V\left(M^{\prime}\right)$.

```
16 else
    Apply Rule 1 exhaustively to ( \(D^{\prime}, r, T^{\prime}, k\) ) to obtain ( \(D^{\prime \prime}, r, T^{\prime \prime}, k\) ).
    Denote by \(T_{0}\) the source terminals in ( \(\left.D^{\prime \prime}, r, T^{\prime \prime}, k\right)\).
    \(B_{h} \leftarrow\) non-terminals with degree at least \(\max \{e(h)-1, a(h)+1\}\) in \(T_{0}\).
    \(B_{l} \leftarrow\) non-terminals with degree at most \(\max \{e(h)-2, a(h)\}\) in \(T_{0}\).
    \(W_{h} \leftarrow\) source terminals with an in-neighbor in \(B_{h}\).
    \(S \leftarrow\) DST-solve \(\left(\left(D^{\prime \prime}, r, T^{\prime \prime}, k\right), \max \{e(h)-2, a(h)\}, B_{h}, B_{l}, \emptyset, W_{h}, T_{0} \backslash W_{h}\right)\).
end
return \(S\)
```

Algorithm 3.4: Part of the function SatisfyChildrenMF $\left(D^{\prime}, r, T^{\prime}, k, M, \beta\right)$ which differs from the appropriate part of the function SATISFYChildrenSD.

For the analysis, we need most of the lemmata proved for Case 1. To prove a running time upper bound we also need the following lemma.

- Lemma 18. Suppose the condition on line 2 of Algorithm 3.4 is not satisfied, $\left(D^{\prime \prime}, r, T^{\prime \prime}, k\right)$ is obtained from $\left(D^{\prime}, r, T^{\prime}, k\right)$ by exhaustive application of Rule 1, $T_{0}$ is the set of source terminals and $B_{h}$ and $W_{h}$ are defined as on lines 19 21. Then $D^{\prime \prime}\left[B_{h} \cup W_{h}\right] \preceq \tau(t)$.

Proof. As proven in Lemma 15 the non-terminals in $\gamma(t) \backslash \beta(t)$ can only dominate at most $a(h)$ vertices in $T_{0}$. Therefore $B_{h} \subseteq \beta(t)$. By Lemma 14 each vertex $t^{\prime}$ in $T_{0}$ was obtained by contracting a strongly connected component $C\left(t^{\prime}\right)$ which contains at least one vertex of $\beta(t)$. Finally, if $b \in B_{h}$ dominates $w \in W_{h}$, but there is no edge between $b$ and $\beta(t) \cap C(w)$ in $D^{\prime}$, then there is a child $s$ of $t$ such that $b \in \sigma(s), C(w) \cap \alpha(s) \neq \emptyset$, thus there is $y \in C(w) \cap \sigma(s)$ and by is an edge of $\tau(t)$ as $\sigma(s)$ is a clique in $\tau(t)$. By the same argument $C(w) \cap \tau(t)$ is connected for every $w \in W_{h}$ and $D^{\prime \prime}\left[B_{h} \cup W_{h}\right]$ has a model in $\tau(t)$.

The proof of correctness of the algorithm is similar to that in Case 1. By Observation 11, it is enough to prove that if SatisfyChildrenMF $\left(D^{\prime}, r, T^{\prime}, k, M, \beta\right)$ returns a set $S \neq S_{\infty}$ then $S$ is a smallest solution for $\left(D^{\prime}, r, T^{\prime}, k\right)$. We prove this claim again by induction on the depth of the recursion. If the condition on line 2 is not satisfied (and hence there is no recursion) the claim follows from the correctness of the algorithm DST-sOLVE proved in Section 3.1. If the condition is satisfied, it follows from Lemma 12 and the induction hypothesis that the set returned is indeed a solution. The minimality in this case is proved exactly the same way as in Lemma 17

As for the time complexity, let us first find a bound $d_{w}^{\max }$ for DST-solve in the case the condition on line 2 is not satisfied. Lemma 18 implies that $K_{e(h)} \npreceq D^{\prime \prime}\left[B_{h} \cup W_{h}\right]$ in this case. Using Lemma 3 . we can derive an upper bound $d_{w}^{\max } \leq c \cdot e(h)^{4}$ for some constant $c$. It follows then from the proof in Section 3.1 that DST-SOLVE runs in $O^{*}\left(\left(2^{O\left(d_{b}\right)} \cdot d_{w}^{\max }\right)^{k}\right)$ time and, as $d_{b}=\max \{e(h)-2, a(h)\}$, there is a constant $g^{\prime}(h)$, such that the running time of DST-sOLVE can be bounded by $O^{*}\left(\left(g^{\prime}(h)\right)^{k}\right)$. From this, similarly as in Case 1, it is easy to conclude, that the running time of the overall algorithm for Case 2 can be bounded by $O^{*}\left(\left(\max \left\{g(h), g^{\prime}(h)\right\}\right)^{k}\right)$.

### 3.2.3 Case 3: $|\beta(t)| \leq b(h)$.

If $|\beta(t)| \leq b(h)$, then no vertex in $\tau(t)$ has degree larger than $b(h)-1$, and $K_{b(h)+1} \npreceq \tau(t)$. Therefore, in this case, either of the two approaches described above can be used. This completes the proof of Theorem 10


Figure 3 A instance of SET COVER reduced to an instance of DST. The red vertices are the terminals and the green vertices are the non-terminals.

### 3.3 DST on $d$-degenerated graphs

Since DST has a $O^{*}(f(k, h))$ algorithm on graphs excluding minors and topological minors, a natural question is if DST has a $O^{*}(f(k, d))$ algorithm on $d$-degenerated graphs. However, we show that in general, we cannot expect an algorithm of this form even for an arbitrary 2-degenerated graph.

- Theorem 19. DST parameterized by $k$ is W[2]-hard on 2-degenerated graphs.

Proof. The proof is by a parameterized reduction from Set Cover. Given an instance $(\mathcal{U}, \mathcal{F}=$ $\left.\left\{F_{1}, \ldots, F_{m}\right\}, k\right)$ of Set Cover, we construct an instance of DST as follows. Corresponding to each set $F_{i}$, we have a vertex $f_{i}$ and corresponding to each element $u \in \mathcal{U}$, we add a directed cycle $C_{u}$ of length $l_{u}$ where $l_{u}$ is the number of sets in $\mathcal{F}$ which contain $u$ (see Fig. 3). For each cycle $C_{u}$, we add an arc from each of the sets containing $u$, to a unique vertex of $C_{u}$. Since $C_{u}$ has $l_{u}$ vertices, this is possible. Finally, we add another directed cycle $C$ of length $m+1$ and for each vertex $f_{i}$, we add an arc from a unique vertex of $C$ to $f_{i}$. Again, since $C$ has length $m+1$, this is possible. Finally, we set as the root $r$, the only remaining vertex of $C$ which does not have an arc to some $f_{i}$ and we set as terminals all the vertices involved in a directed cycle $C_{u}$ for some $u$ and all the vertices in the cycle $C$ except the root $r$. It is easy to see that the resulting digraph has degeneracy 3 . Finally, we subdivide every edge which lies in a cycle $C_{u}$ for some $u$, or on the cycle $C$ and add the new vertices to the terminal set. This results in a digraph $D$ of degeneracy 2 . Let $T$ be the set of terminals as defined above. This completes the construction. We claim that $(\mathcal{U}, \mathcal{F}, k)$ is a Yes instance of Set Cover iff ( $D, r, T, k$ ) is a Yes instance of DST.

Suppose that $(\mathcal{U}, \mathcal{F}, k)$ is a Yes instance and let $F \subseteq \mathcal{F}$ be a solution. Consider the set $F_{v}=$ $\left\{f_{i} \mid F_{i} \in F\right\}$. Clearly, $|F| \leq k$ and $F$ is a solution for the instance ( $D, r, T, k$ ) as all the terminals are reachable from $r$ in $D[F \cup T \cup\{r\}]$.

Conversely, suppose that $F_{v}$ is a solution for $(D, r, T, k)$. Since the only non-terminals are the vertices corresponding to the sets in $\mathcal{F}$, we define a set $F \subseteq \mathcal{F}$ as $F=\left\{F_{i} \mid f_{i} \in F_{v}\right\}$. Clearly $|F| \leq k$. We claim that $F$ is a solution for the Set Cover instance $(\mathcal{U}, \mathcal{F}, k)$. Since there are no edges between the cycles $C_{u}$ or $C$ in the instance of DST, for every $u$, it must be the case that $F_{v}$ contains some vertex $f_{i}$ which has an arc to a vertex in the cycle $C_{u}$. But the corresponding set $F_{i}$ will cover the element $u$ and we have defined $F$ such that $F_{i} \in F$. Hence, $F$ is indeed a solution for the instance $(\mathcal{U}, \mathcal{F}, k)$. This completes the proof.

In the instance of DST obtained in the above reduction, it seems that the presence of directed cycles in the subgraph induced by the terminals plays a major role in the hardness of this instance. We formally show that this is indeed the case by presenting an FPT algorithm for DST for the case the digraph induced by the terminals is acyclic.

- Theorem 20. DST can be solved in time $O^{*}\left(2^{(d k)}\right)$ on d-degenerated graphs if the digraph induced by the terminals is acyclic.

Proof. As the digraph induced by terminals is acyclic, Rule 1 does not apply and the instance is reduced. Therefore we can directly execute the algorithm DST-solve on it. We set the degree bound to $d_{b}=d$. Note that if the set $B_{h}$ and $W_{h}$ created by the algorithm fulfill the invariants, then, as the digraph induced by $W_{h} \cup B_{h}$ is $d$-degenerated and the degree of every vertex in $B_{h}$ is at least $d_{b}+1=d+1$, there must be a vertex $v \in W_{h}$ with at most $d$ (in-)neighbors in $B_{h}$. Therefore we have $d_{w}^{\max }=d$ and according to the analysis from Section 3.1 the algorithm runs in $O^{*}\left(\left(2^{O\left(d_{b}\right)} \cdot d_{w}^{\max }\right)^{k}\right)=O^{*}\left(2^{O(d) k}\right)$ time.

Theorem 20 combined with Lemma 7 results in the following corollary.

- Corollary 21. If $\mathcal{C}$ is an $o(\log n)$-degenerated class of digraphs, then DST parameterized by $k$ is FPT on $\mathcal{C}$ if the digraph induced by terminals is acyclic.

Before concluding this section, we also observe that analogous to the algorithms in Theorems 6 and 20. we can show that in the case when the digraph induced by terminals is acyclic, the DST problem admits an algorithm running in time $O^{*}\left(2^{O(h k)}\right)$ on graphs excluding $K_{h}$ as a topological minor.

- Theorem 22. DST can be solved in time $O^{*}\left(2^{O(h k)}\right)$ on graphs excluding $K_{h}$ as a topological minor if the digraph induced by terminals is acyclic.

Combined with Lemma 7 Theorem 22 has the following corollary.

- Corollary 23. If $\mathcal{C}$ is a class of digraphs excluding o $(\log n)$-sized topological minors, then DST parameterized by $k$ is FPT on $\mathcal{C}$ if the digraph induced by terminals is acyclic.


### 3.4 Hardness of DST

In this section, we show that the algorithm given in Theorem 20 is essentially the best possible with respect to the dependency on the degeneracy of the graph and the solution size. We begin by proving a lower bound on the time required by any algorithm for DST on graphs of degeneracy $O(\log n)$. Our starting point is the known result for the following problem.

```
Partitioned Subgraph Isomorphism (PSI)
Input: Undirected graphs \(H=\left(V_{H}, E_{H}\right)\) and \(G=\left(V_{G}=\left\{g_{1}, \ldots, g_{l}\right\}, E_{G}\right)\) and a
    coloring function col : \(V_{H} \rightarrow[l]\).
Question: \(\quad\) Is there an injection \(\phi: V_{G} \rightarrow V_{H}\) such that for every \(i \in[l], \operatorname{col}\left(\phi\left(g_{i}\right)\right)=i\) and
    for every \(\left(g_{i}, g_{j}\right) \in E_{G},\left(\phi\left(g_{i}\right), \phi\left(g_{j}\right)\right) \in E_{H}\) ?
```

We need the following lemma by Marx [32].

- Lemma 24. (Corollary 6.3, [32]) Partitioned Subgraph Isomorphism cannot be solved in time $f(k) n^{o\left(\frac{k}{\log k}\right)}$ where $f$ is an arbitrary function and $k=\left|E_{G}\right|$ is the number of edges in the smaller graph $G$ unless ETH fails.

Using the above lemma, we will first prove a similar kind of hardness for a restricted version of SET Cover (Lemma 25). Following that, we will reduce this problem to an instance of DST to prove the hardness of the problem on graphs of degeneracy $O(\log n)$.


Figure 4 An illustration of the sets in the reduced instance, corresponding to the graph $G$ (on the left). The blue sets at position $(i, i)$ correspond to vertices of $H$ with color $i$ and the green sets at position $(i, j)$ correspond to edges of $H$ between color classes $i$ and $j$.

- Lemma 25. There is a constant $\gamma$ such that SET Cover with size of each set bounded by $\gamma \log m$ cannot be solved in time $f(k) m^{o\left(\frac{k}{\log k}\right)}$, unless ETH fails, where $k$ is the size of the solution and $m$ is the size of the family of sets.

Proof. Let $\left(H=\left(V_{H}, E_{H}\right), G=\left(V_{G}, E_{G}\right), c o l\right)$ be an instance of Partitioned Subgraph Isomorphism where $\left|V_{G}\right|=l$ and the function col : $V_{H} \rightarrow[l]$ is a coloring (not necessarily proper) of the vertices of $H$ with colors from $[l]$. We call the set of vertices of $H$ which have the same color, a color class. We assume without loss of generality that there are no isolated vertices in $G$. Let $n$ be the number of vertices of $H$. For each vertex of color $i$ in $H$, we assign a $\log n$-sized subset of $2 \log n$. Since $\binom{2 \log n}{\log n} \geq n$, this is possible. Let this assignment be represented by the function $i d: V_{H} \rightarrow 2^{[2 \log n]}$.

Recall that the vertices of $G$ are numbered $g_{1}, \ldots, g_{l}$ and we are looking for a colorful subgraph of $H$ isomorphic to $G$ such that the vertex from color class $i$ is mapped to the vertex $g_{i}$.

We will list the sets of the Set Cover instance and then we will define the set of elements contained in each set. For each pair $(i, j)$ such that there is an edge between $g_{i}$ and $g_{j}$, and for every edge between vertices $u$ and $v$ in $V_{H}$ such that $\operatorname{col}(u)=i, \operatorname{col}(v)=j$, we have a set $F_{u v}^{i j}$. For each $i \in[l]$, for each $v \in V_{H}$ such that $\operatorname{col}(v)=i$, we have a set $F_{v v}^{i i}$. The notation is chosen is such way that we can think of the sets as placed on a $l \times l$ grid, where the sets $F_{u v}^{i j}$ for a fixed $i$ and $j$ are placed at the position $(i, j)$ (see Fig. 4). Observe that many sets can be placed at a position and it may also be the case that some positions of the grid do not have a set placed on them. Let $\mathcal{F}$ be the family of sets defined as above.

A position $(i, j)$ which has a set placed on it is called non-empty and empty otherwise. Without loss of generality, we assume that if there are $i \neq j$ such that there is an edge between $g_{i}$ and $g_{j}$ in $G$, then the position $(i, j)$ is non-empty. Two non-empty positions $(i, j)$ and $\left(i^{\prime}, j\right)$ are said to be consecutive if there is no non-empty position $\left(i^{\prime \prime}, j\right)$ where $i<i^{\prime \prime}<i^{\prime}$. Similarly, two non-empty positions $(i, j)$ and $\left(i, j^{\prime}\right)$ are said to be consecutive if there is no non-empty position $\left(i, j^{\prime \prime}\right)$ where $j<j^{\prime \prime}<j^{\prime}$. Note that consecutive positions are only defined along the same row or column.

We now define the universe $\mathcal{U}$ as follows. For every non-empty position $(i, j)$, we have an element $s^{(i, j)}$. For every $\left(i_{1}, j_{1}\right)$ and $\left(i_{2}, j_{2}\right)$ such that they are consecutive, we have a set $\mathcal{U}^{\left(i_{1}, j_{1}\right)\left(i_{2}, j_{2}\right)}$ of $2 \log n$ elements $\left\{u_{1}^{\left(i_{1}, j_{1}\right)\left(i_{2}, j_{2}\right)} \ldots, u_{2}^{\left(i_{1}, j_{1}\right)\left(i_{2}, j_{2}\right)}\right\}$. An element $u_{a}^{\left(i_{1}, j_{1}\right)\left(i_{2}, j_{2}\right)}$ is said to correspond to $i d(u)$ for some vertex $u$ if $a \in i d(u)$.

We will now define the elements contained within each set. For each non-empty position $(i, j)$, add the element $s^{(i, j)}$ to every $F_{u v}^{i j}$ for all (possible) $u, v$. Now, fix $1 \leq i \leq l$. Let $\left(i, j_{1}\right)$ and $\left(i, j_{2}\right)$ be consecutive positions where $j_{1}<j_{2}$. For each set $F_{u v}^{i j_{1}}$, we add the elements $\left\{u_{a}^{\left(i, j_{1}\right)\left(i, j_{2}\right)} \mid a \notin i d(u)\right\}$
and for each set $F_{u v}^{i j_{2}}$, we add the the elements $\left\{u_{a}^{\left(i, j_{1}\right)\left(i, j_{2}\right)} \mid a \in i d(u)\right\}$.
Similarly, fix $1 \leq j \leq l$. Let $\left(i_{1}, j\right)$ and $\left(i_{2}, j\right)$ be consecutive positions where that $i_{1}<i_{2}$. For each set $F_{u v}^{i_{1} j}$, we add the elements $\left\{u_{a}^{\left(i_{1}, j\right)\left(i_{2}, j\right)} \mid a \notin i d(u)\right\}$ and for each set $F_{u v}^{i_{2} j}$, we add the the elements $\left\{u_{a}^{\left(i_{1}, j\right)\left(i_{2}, j\right)} \mid a \in i d(u)\right\}$.

This completes the construction of the SET Cover instance. We first prove the following lemma regarding the constructed instance, which we will then use to show the correctness of the reduction.

- Lemma 26. Suppose ( $i, j_{1}$ ) and ( $i, j_{2}$ ) are two consecutive positions where $j_{1}<j_{2}$ and $\left(i_{1}, j\right)$ and $\left(i_{2}, j\right)$ are two consecutive positions where $i_{1}<i_{2}$.

1. The elements in $\mathcal{U}^{\left(i, j_{1}\right)\left(i, j_{2}\right)}$ can be covered by precisely one set from $\left(i, j_{1}\right)$ and one set from $\left(i, j_{2}\right)$ iff the two sets are of the form $F_{u v}^{i j_{1}}$ and $F_{u v^{\prime}}^{i j_{2}}$.
2. The elements in $\mathcal{U}^{\left(i_{1}, j\right)\left(i_{2}, j\right)}$ can be covered by precisely one set from $\left(i_{1}, j\right)$ and one set from $\left(i_{2}, j\right)$ iff the two sets are of the form $F_{u v}^{i_{1} j}$ and $F_{u^{\prime} v}^{i_{2} j}$.

Proof. We prove the first statement. The proof of the second is analogous. Observe that, by the construction, the only sets which can cover elements in $\mathcal{U}^{\left(i, j_{1}\right)\left(i, j_{2}\right)}$ are sets from $\left(i, j_{1}\right)$ and $\left(i, j_{2}\right)$.

Suppose that the elements in $\mathcal{U}^{\left(i, j_{1}\right)\left(i, j_{2}\right)}$ are covered by precisely one set from $\left(i, j_{1}\right)$ and one from $\left(i, j_{2}\right)$ and the two sets are of the form $F_{u v}^{i j_{1}}$ and $F_{u^{\prime} v^{\prime}}^{i j_{2}}$ where $u \neq u^{\prime}$. By the construction, the set $F_{u v}^{i j_{1}}$ covers the elements of $\mathcal{U}^{\left(i, j_{1}\right)\left(i, j_{2}\right)}$ which do not correspond to $i d(u)$ and the set $F_{u^{\prime} v^{\prime}}^{i j_{2}}$ covers the elements of $\mathcal{U}^{\left(i, j_{1}\right)\left(i, j_{2}\right)}$ which correspond to $i d\left(u^{\prime}\right)$. Since $\operatorname{col}(u)=\operatorname{col}\left(u^{\prime}\right)$ (implied by the construction), $i d(u) \neq i d\left(u^{\prime}\right)$. Since $|i d(u)|=\left|i d\left(u^{\prime}\right)\right|$, it must be the case that there is an element of [ $2 \log n$ ], say $x$, which is in $i d(u)$ but not in $i d\left(u^{\prime}\right)$. But then, it must be the case that the element $u_{x}^{\left(i, j_{1}\right)\left(i, j_{2}\right)}$ is left uncovered by both $F_{u v}^{i j_{1}}$ and $F_{u^{\prime} v^{\prime}}^{i j_{2}}$, a contradiction.

Conversely, consider two sets of the form $F_{u v}^{i j_{1}}$ and $F_{u v^{\prime}}^{i j_{2}}$. We claim that these two sets together cover the elements in $\mathcal{U}^{\left(i, j_{1}\right)\left(i, j_{2}\right)}$. But this is true since $F_{u v}^{i j_{1}}$ covers the elements of $\mathcal{U}^{\left(i, j_{1}\right)\left(i, j_{2}\right)}$ which do not correspond to $i d(u)$ and $F_{u v^{\prime}}^{i j_{2}}$ covers the elements of $\mathcal{U}^{\left(i, j_{1}\right)\left(i, j_{2}\right)}$ which do correspond to $i d(u)$. This completes the proof of the lemma.

We claim the instance $(H, G, c o l)$ is a Yes instance of PSI iff the instance $\left(\mathcal{U}, \mathcal{F}, k^{\prime}\right)$ is a Yes instance of Set Cover, where $k^{\prime}=2\left|E_{G}\right|+\left|V_{G}\right|$. Suppose that $(H, G, c o l)$ is a Yes instance, $\phi$ is its solution, and let $v_{i}=\phi\left(g_{i}\right)$. We claim that the sets $F_{v_{i} v_{j}}^{i j}$, where $(i, j)$ is a non-empty position, form a solution for the Set Cover instance. Since we have picked a set from every non-empty position $(i, j)$, the elements $s^{(i, j)}$ are all covered. But since the sets we picked from any two consecutive positions match premise of Lemma 26 the elements corresponding to the consecutive positions are also covered.

Conversely, suppose that the Set Cover instance is a Yes instance and let $\mathcal{F}^{\prime}$ be a solution. Since we must pick at least one set from each non-empty position (we have to cover the vertices $s^{(i, j)}$ ), and the number of non-empty positions equals $k^{\prime}$, we must have picked exactly one set from each non-empty position. Let $v_{i}$ be the vertex corresponding to the set picked at position $(i, i)$. We define the function $\phi$ as $\phi\left(g_{i}\right)=v_{i}$. Clearly, $\phi$ is an injection with $\operatorname{col}\left(\phi\left(g_{i}\right)\right)=i$. It remains to show that for every $g_{i}, g_{j}$, if $\left(g_{i}, g_{j}\right) \in E_{G}$, then there is an edge between $v_{i}$ and $v_{j}$. To show this, we need to show that the set picked from position $(i, j)$ has to be exactly $F_{v_{i} v_{j}}^{i j}$. By Lemma 26 the sets picked from row $i$ are of the form $F_{v_{i}, v}^{i j}$, for any $j$ and $v$ and the sets picked from column $j$ are of the form $F_{v, v_{j}}^{i j}$, for any $i$ and $v$. Hence, the set picked from position $(i, j)$ can only be $F_{v_{i} v_{j}}^{i j}$. Thus, there is an edge between $v_{i}$ and $v_{j}$ in $H$ and $\phi$ is indeed a homomorphism. This completes the proof of equivalence of the two instances.
Since $G$ contains no isolated vertex, we have $l=O(k)$ and, thus, $k^{\prime}=\Theta(k)$. Observe that the number of sets $m$ in the SET Cover instance is $\left|V_{H}\right|+2\left|E_{H}\right|$, that is, $n \leq m$ and $m=O\left(n^{2}\right)$. Observe that each set contains at most $4 \log n+1$ elements, one of the form $s^{(i, j)}$ and $\log n$ for each of the at most four consecutive positions the set can be a part of. Since the number of sets $m$ is at least $n$, there is a constant $\gamma$ such that the number of elements in each set is bounded by $\gamma \log m$. Finally, since
$m=O\left(n^{2}\right)$, an algorithm for SET COVER of the form $f(k) m^{o\left(\frac{k}{\log k}\right)}$ implies an algorithm of the form $f(k) n^{o\left(\frac{k}{\log k}\right)}$ for PSI. This completes the proof of the lemma.

Now we are ready to prove the main theorem of this section.

- Theorem 27. DST cannot be solved in time $f(k) n^{o\left(\frac{k}{\log k}\right)}$ on $c \log n$-degenerated graphs for any constant $c>0$ even if the digraph induced by terminals is acyclic, where $k$ is the solution size and $f$ is an arbitrary function, unless ETH fails.

Proof. The proof is by a reduction from the restricted version of Set Cover shown to be hard in Lemma 25 Fix a constant $c>0$ and let $\left(\mathcal{U}=\left\{u_{1}, \ldots, u_{n}\right\}, \mathcal{F}=\left\{F_{1}, \ldots, F_{m}\right\}, k\right)$ be an instance of Set Cover, where the size of any set is at most $\gamma \log m$, for some constant $\gamma$. For each set $F_{i}$, we have a vertex $f_{i}$. For each element $u_{i}$, we have a vertex $x_{i}$. If an element $u_{i}$ is contained in set $F_{j}$, then we add an arc $\left(f_{j}, x_{i}\right)$. Further, we add another vertex $r$ and add arcs $\left(r, f_{i}\right)$ for every $i$. Finally, we add $m^{2 \gamma / c}$ isolated vertices. This completes the construction of the digraph $D$. We set $T=\left\{x_{1}, \ldots, x_{n}\right\} \cup\{r\}$ as the set of terminals and $r$ as the root.

We claim that $(\mathcal{U}, \mathcal{F}, k)$ is a Yes instance of Set Cover iff $(D, r, T, k)$ is a Yes instance of DST. Suppose that $\left\{F_{1}, \ldots, F_{k}\right\}$ is a set cover for the given instance. It is easy to see that the vertices $\left\{f_{1}, \ldots, f_{k}\right\}$ form a solution for the DST instance.

Conversely, suppose that $\left\{f_{1}, \ldots, f_{k}\right\}$ is a solution for the DST instance. Since the only way that $r$ can reach a vertex $x_{i}$ is through some $f_{j}$, and the construction implies that $u_{i} \in F_{j}$, the sets $\left\{F_{1}, \ldots, F_{k}\right\}$ form a set cover for $(\mathcal{U}, \mathcal{F}, k)$. This concludes the proof of equivalence of the two instances.

We claim that the degeneracy of the graph $D$ is $c \log n_{1}+1$. First, we show that the degeneracy of the graph $D$ is bounded by $\gamma \log m+1$. This follows from that each vertex $f_{i}$ has total degree at most $\gamma \log m+1$ and if a subgraph contains none of these vertices, then it contains no edges. Now, $n_{1}$ is at least $m^{2 \gamma / c}$. Hence, $\log n_{1} \geq(2 \gamma / c) \log m$ and the degeneracy of the graph is at most $\gamma \log m+1 \leq c \cdot(2 \gamma / c) \log m \leq c \log n_{1}$. Finally, since each vertex $f_{i}$ is adjacent to at most $\gamma \log m+1$ vertices, $n_{1}=O\left(m \log m+m^{2 \gamma / c}\right)$ and, thus, it is polynomial in $m$. Hence, an algorithm for DST of the form $f(k) n_{1}^{o\left(\frac{k}{\log k}\right)}$ implies an algorithm of the form $f(k) m^{o\left(\frac{k}{\log k}\right)}$ for the SET COVER instance. This concludes the proof of Theorem 27

Combining the Theorem 27 with Lemma 7 we get the following corollary.

- Corollary 28. There are no two functions $f$ and $g$ such that $g(d)=o(d)$ and there is an algorithm for DST running in time $O^{*}\left(2^{g(d) f(k)}\right)$ unless ETH fails.

To examine the dependency on the solution size we utilize the following theorem.

- Theorem 29. ([27]) There is a constant c such that Dominating SEt does not have an algorithm running in time $O^{*}\left(2^{o(n)}\right)$ on graphs of maximum degree $\leq c$ unless ETH fails.

From Theorem 29, we can infer the following corollary.

- Corollary 30. There are no two functions $f$ and $g$ such that $f(k)=o(k)$ and there is an algorithm for DST running in time $O^{*}\left(2^{g(d) f(k)}\right)$, unless ETH fails.

Proof. We use the following standard reduction from Dominating Set to DST. Let $(G=(V, E), k)$ be an instance of Dominating Set with the maximum degree of $G$ bounded by some constant $c$. We can assume that the number of vertices $n$ of the graph $G$ is at most $c k+k$, since otherwise, it is a trivial No instance. Let $D=\left(V^{\prime}, A\right)$ be the digraph defined as follows. We set $V^{\prime}=V \times\{1,2\} \cup\{r\}$. There is an arc in $A$ from $(u, 1)$ to $(v, 2)$ if either $u=v$ or there is an edge between $u$ and $v$ in $E$. Finally, there is an arc from $r$ to $(v, 1)$ for every $v \in V$. We let $T=\{(v, 2) \mid v \in V\}$.

It is easy to check that $S \subseteq V$ is a solution to the instance $(G, k)$ of Dominating Set if and only if $S \times\{1\}$ is a solution to the instance $(D, r, T, k)$ of DST. As the vertices $(v, 2)$ have degree at most

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$c+1, D$ is $c+1$-degenerated. Since the reduction is polynomial time, preserves $k$ and $k=\Theta(n)$, an algorithm for DST running in time $O^{*}\left(2^{g(d) f(k)}\right)$ for some $f(k)=o(k)$ would solve Dominating SET on graphs of maximum degree $\leq c$ in time $O^{*}\left(2^{g(c+1) f(k)}\right)=O^{*}\left(2^{o(n)}\right)$, and, hence, ETH fails by Theorem 29

## 4 Applications to Dominating Set

In this section, we adapt the ideas used in the algorithms for DST to design improved algorithms for the Dominating Set problem and some variants of it in subclasses of degenerated graphs.

### 4.1 Introduction for Dominating Set

On general graphs Dominating Set is W[2]-complete [13]. However, there are many interesting graph classes where FPT-algorithms exist for Dominating Set. The project of expanding the horizon where FPT algorithms exist for Dominating Set has produced a number of cutting-edge techniques of parameterized algorithm design. This has made Dominating Set a cornerstone problem in parameterized complexity. For an example the initial study of parameterized subexponential algorithms for Dominating Set, on planar graphs [1, 17] resulted in the development of bidimensionality theory characterizing a broad range of graph problems that admit efficient approximation schemes, subexponential time FPT algorithms and efficient polynomial time pre-processing (called kernelization) on minor closed graph classes [10, 11]. Alon and Gutner [2] and Philip, Raman, and Sikdar [37] showed that Dominating Set problem is FPT on graphs of bounded degeneracy and on $K_{i, j}$-free graphs, respectively.

Numerous papers also concerned the approximability of Dominating Set. It follows from [7] that Dominating Set on general graphs can approximated to within roughly $\ln (\Delta(G)+1)$, where $\Delta(G)$ is the maximum degree in the graph $G$. On the other hand, it is NP-hard to approximate Dominating Set in bipartite graphs of degree at most $B$ within a factor of $(\ln B-c \ln \ln B)$, for some absolute constant $c$ [8]. Note that a graph of degree at most $B$ excludes $K_{B+2}$ as a topological minor, and, hence, the hardness also applies to graphs excluding $K_{h}$ as a topological minor. While a polynomial time approximation scheme (PTAS) is known for $K_{h}$-minor-free graphs [21], we are not aware of any constant factor approximation for Dominating SET on graphs excluding $K_{h}$ as a topological minor or $d$-degenerated graphs.

Based on the ideas from previous sections, we develop an algorithm for Dominating Set. Our algorithm for Dominating SET on $d$-degenerated graphs improves over the $O^{*}\left(k^{O(d k)}\right)$ time algorithm by Alon and Gutner [2]. In fact, it turns out that our algorithm is essentially optimal - we show that, assuming the ETH, the running time dependence of our algorithm on the degeneracy of the input graph and solution size $k$ cannot be significantly improved. Furthermore, we also give a factor $O\left(d^{2}\right)$ approximation algorithm for Dominating Set on $d$-degenerated graphs. A list of our results for Dominating Set is given below.

1. There is a $O^{*}\left(3^{h k+o(h k)}\right)$-time algorithm for Dominating Set on graphs excluding $K_{h}$ as a topological minor.
2. There is a $O^{*}\left(3^{d k+o(d k)}\right)$-time algorithm for Dominating Set on $d$-degenerated graphs. This implies that Dominating SET is FPT on $o(\log n)$-degenerated classes of graphs.
3. For any constant $c>0$, there is no $f(k) n^{o\left(\frac{k}{\log k}\right)}$-time algorithm on graphs of degeneracy $c \log n$ unless ETH fails.
4. There are no two functions $f$ and $g$ such that $g(d)=o(d)$ and there is an algorithm for DominATING SET running in time $O^{*}\left(2^{g(d) f(k)}\right)$, unless ETH fails.
5. There are no two functions $f$ and $g$ such that $f(k)=o(k)$ and there is an algorithm for Dominating SET running in time $O^{*}\left(2^{g(d) f(k)}\right)$, unless ETH fails.
6. There is a $O(d n \log n)$ time factor $O\left(d^{2}\right)$ approximation algorithm for Dominating SEt on $d$ degenerated graphs.

### 4.2 Dominating Set on graphs of bounded degeneracy

We begin by giving an algorithm for Dominating Set running in time $O^{*}\left(3^{h k+o(h k)}\right)$ in graphs excluding $K_{h}$ as a topological minor. This improves over the $O^{*}\left(2^{O(k h \log h)}\right)$ algorithm of [2]. Though the algorithm we give here is mainly built on the ideas developed for the algorithm for DST, the algorithm has to be modified slightly in certain places in order to fit this problem. We also stress this an example of how these ideas, with some modifications, can be made to fit problems other than DST. We begin by proving lemmata required for the correctness of the base cases of our algorithm.

- Lemma 31. Let $(G, k)$ be an instance of Dominating Set. Let $Y \subseteq V$ be a set of vertices and let $B$ be the set of vertices (other than $Y$ ) dominated by $Y$. Let $W$ be the set of vertices of $G$ not dominated by $Y, B_{h}$ be the set of vertices of $B$ which dominate at least $d+1$ terminals in $W$ for some constant $d, B_{l}$ be the rest of the vertices of $B, W_{h}$ be the vertices of $W$ which have neighbors in $B_{h} \cup W . I f\left|W \backslash W_{h}\right|>d(k-|Y|)$, then the given instance does not admit a solution which contains $Y$.

Proof. Let $W \backslash W_{h}=W_{l}$. Since any vertex in $W_{l}$ does not have neighbors in $B_{h} \cup W \cup Y, G\left[W_{l}\right]$ is an independent set, and the only vertices which can dominate a vertex in $W_{l}$ are either itself, or a vertex of $B_{l}$. Any vertex of $W_{l}$ can only dominate itself (vertices of $B_{l}$ are already dominated by $Y$ ) and any vertex of $B_{l}$ can dominate at most $d$ vertices of $W$. Hence, if $\left|W_{l}\right|>d(k-|Y|), W_{l}$ cannot be dominated by adding $k-|Y|$ vertices to $Y$. This completes the proof.

- Lemma 32. Let $(D, k)$ be an instance of Dominating Set. Let $Y \subseteq V$ be a set of vertices and let $B$ be the set of vertices (other than $Y$ ) dominated by $Y$. Let $W$ be the set of vertices of $G$ not dominated by $Y, B_{h}$ be the set of vertices of $B$ which dominate at least $d+1$ terminals in $W$ for some constant $d, B_{l}$ be the rest of the vertices of $B, W_{h}$ be the vertices of $W$ which have neighbors in $B_{h} \cup W$.

If $B_{h} \cup W_{h}$ is empty, then there is an algorithm which can test if this instance has a solution containing $Y$ in time $O^{*}\left(2^{d(k-|Y|)}\right)$.

Proof. The premises of the lemma imply that the only potentially non-empty sets are $Y, B_{l}$ and $W=W_{l}$. If $W$ is empty, we are done. Suppose $W$ is non-empty. We know that $|Y| \leq k$ and, by Lemma 31 we can assume that $|W| \leq d(k-|Y|)$. Since $W$ is an independent set, the only vertices the vertices of $W$ can dominate (except for vertices dominated by $Y$ ), are themselves. Thus it is never worse to take a neighbor of them in the solution if they have one. The isolated vertices from $W$ can be simply moved to $Y$, as we have to take them into the solution. Now, it remains to find a set of vertices of $B_{l}$ of the appropriate size such that they dominate the remaining vertices of $W$. But in this case, we can reduce it to an instance of DST and apply the algorithm from Lemma 2 The reduction is as follows. Consider the graph $G\left[B_{l} \cup W\right]$. Remove all edges between vertices in $B_{l}$. Let this graph be $G_{d s t}$. Simply add a new vertex $r$ and add directed edges from $r$ to every vertex in $B_{l}$. Also orient all edges between $B_{l}$ and $W$, from $B_{l}$ to $W$. Now, $W$ is set as the terminal set. Now, since $|W| \leq d k$, it is easy to see that the algorithm $\operatorname{Nederlof}\left(G_{d s t}, r, W, W\right)$ runs in time $O^{*}\left(2^{d(k-|Y|)}\right)$. This completes the proof of the lemma.

- Theorem 33. Dominating SET can be solved in time $O^{*}\left(3^{h k+o(h k)}\right)$ on graphs excluding $K_{h}$ as a topological minor.

Proof. The algorithm we describe takes as input an instance $(G, k)$, and vertex sets $B, W, B_{h}, B_{l}$, $W_{h}, W_{l}$ and $Y$ and returns a smallest solution for the instance, which contains $Y$ if such a solution exists. If there is no such solution, then the algorithm returns a dummy symbol $S_{\infty}$. To simplify the description, we assume that $\left|S_{\infty}\right|=\infty$. The algorithm is a recursive algorithm and at any stage of recursion, the corresponding recursive step returns the smallest set found in the recursions initiated in this step. For the purpose of this proof, we will always have $d_{b}=h-2$. Initially, all the sets mentioned above are empty.
At any point, while updating these sets, we will maintain the following invariants (see Fig. 5.


Figure 5 An illustration of the sets defined in Theorem 33

- The sets $B_{h}, B_{l}, W_{h}, W_{l}$ and $Y$ are pairwise disjoint.
- The set $Y$ has size at most $k$.
- $B$ is the set of vertices dominated by $Y$.
- $W$ is the set of vertices not dominated by $Y$.
- $B_{h}$ is the set of vertices of $B$ which dominate at least $d_{b}+1$ vertices of $W$.
- $B_{l}$ is the set of vertices of $B$ which dominate at most $d_{b}$ vertices of $W$.
- $W_{h}$ is the set of vertices of $W$ which have a neighbor in $B_{h} \cup W$.
- $W_{l}$ are the remaining vertices of $W$.

Observe that the sets $B_{l}, W_{h}$ and $W_{l}$ correspond to the sets $B_{l}, W_{h}$ and $W \backslash W_{h}$ defined in the statement of Lemma 31. Hence, by Lemma 31, if $\left|W_{l}\right|>d_{b}(k-|Y|)$, then there is no solution containing $Y$ and hence we return $S_{\infty}$ (see Algorithm 4.1). If $B_{h} \cup W_{h}$ is empty, then we apply Lemma 32 to solve the problem in time $O^{*}\left(2^{d_{b}(k-|Y|)}\right)$. If $B_{h} \cup W_{h}$ is non-empty, then we find a vertex $v \in W_{h}$ with the least neighbors in $B_{h} \cup W_{h}$. Let $N$ be the set of these neighbors and let $|N|=d_{w}$.

We then branch into $d_{w}+2$ branches described as follows. In the first $d_{w}+1$ branches, we move a vertex $u$ of $N \cup\{v\}$, to the set $Y$, and perform the following updates. We move the vertices of $W$ which are adjacent to $u$, to $B$, and update the remaining sets in a way that maintains the invariants mentioned above. More precisely, set $B_{h}$ as the set of vertices of $B$ which dominate at least $d_{b}+1$ vertices of $W, B_{l}$ as the set of vertices of $B$ which dominate at most $d_{b}$ vertices of $W, W_{h}$ as the set of vertices of $W$ which have a neighbor in $B_{h} \cup W$, and $W_{l}$ as the rest of the vertices of $W$. Finally, we recurse on the resulting instance.

In the last of the $d_{w}+2$ branches, that is, the branch where we have guessed that none of the vertices of $N$ are in the dominating set, we move the vertex $v$ to $W_{l}$, delete the vertices of $N \cap B_{h}$ and the edges from $v$ to $N \cap W_{h}$, to obtain the graph $G^{\prime}$. Starting from here, we then update all the sets in a way that the invariants are maintained. More precisely, set $B$ as the set of vertices dominated by $Y, W$ as the set of vertices not dominated by $Y$, set $B_{h}$ as the set of vertices of $B$ which dominate at least $d_{b}+1$ vertices of $W, B_{l}$ as the set of vertices of $B$ which dominate at most $d_{b}$ vertices of $W$, $W_{h}$ as the set of vertices of $W$ which have a neighbor in $B_{h} \cup W$, and $W_{l}$ as the rest of the vertices of $W$. Finally, we recurse on the resulting instance.

Correctness. At each node of the recursion tree, we define a measure $\mu(I)=d_{b}(k-|Y|)-\left|W_{l}\right|$. We prove the correctness of the algorithm by induction on this measure. In the base case, when $d_{b}(k-|Y|)-\left|W_{l}\right|<0$, then the algorithm is correct (by Lemma 31). Now, we assume as induction hypothesis that the algorithm is correct on instances with measure less than some $\mu \geq 0$. Consider an instance $I$ such that $\mu(I)=\mu$. Since the branching is exhaustive, it is sufficient to show that

```
Input : An instance \((G, k)\) of Dominating Set, degree bound \(d_{b}\), sets \(B_{h}, B_{l}, Y, W_{h}, W_{l}\)
Output: A smallest solution of size at most \(k\) and containing \(Y\) for the instance \((D, k)\) if it
    exists and \(S_{\infty}\) otherwise
if \(\left|W_{l}\right|>d_{b}(k-|Y|)\) then return \(S_{\infty}\)
else if \(B_{h}=\emptyset\) then
    \(S \leftarrow \operatorname{NedERLoF}\left(G_{d s t}, r, W, W\right)\).
    if \(|S|>k\) then \(S \leftarrow S_{\infty}\)
    return \(S\)
end
else
    \(S \leftarrow S_{\infty}\)
    Find vertex \(v \in W_{h}\) with the least neighbors in \(B_{h} \cup W_{h}\).
    for \(u \in\left(\left(B_{h} \cup W_{h}\right) \cap N[v]\right)\) do
        \(Y=Y \cup\{u\}\), perform updates to get \(B_{h}^{\prime}, B_{l}^{\prime}, W_{h}^{\prime}, W_{l}^{\prime}\).
        \(S^{\prime} \leftarrow \mathrm{DS}-\operatorname{solve}\left((G, k), d_{b}, B_{h}^{\prime}, B_{l}^{\prime}, Y, W_{h}^{\prime}, W_{l}^{\prime}\right)\).
        if \(\left|S^{\prime}\right|<|S|\) then \(S \leftarrow S^{\prime}\)
    end
    \(W_{l} \leftarrow W_{l} \cup\{v\}\), perform updates to get new graph \(G^{\prime}\) and sets \(B_{h}^{\prime}, B_{l}^{\prime}, W_{h}^{\prime}, W_{l}^{\prime}\).
    \(S^{\prime} \leftarrow \mathrm{DS}-\operatorname{solvE}\left(\left(G^{\prime}, k\right), d_{b}, B_{h}^{\prime}, B_{l}^{\prime}, Y, W_{h}^{\prime}, W_{l}\right)\).
    if \(\left|S^{\prime}\right|<|S|\) then \(S \leftarrow S^{\prime}\)
    return \(S\)
end
```

Algorithm 4.1: Algorithm DS-solve for Dominating Set
the algorithm is correct on each of the child instances. To show this, it is sufficient to show that for each child instance $I^{\prime}, \mu\left(I^{\prime}\right)<\mu(I)$. In the first $d_{w}+1$ branches, the size of the set $Y$ increases by 1 , and the size of the set $W_{l}$ does not decrease ( $W_{l}$ has no neighbors in $B_{h} \cup W_{l}$ ). Hence, in each of these branches, $\mu\left(I^{\prime}\right) \leq \mu(I)-d_{b}$. In the final branch, though the size of the set $Y$ remains the same, the size of the set $W_{l}$ increases by at least 1 . Hence, in this branch, $\mu\left(I^{\prime}\right) \leq \mu(I)-1$. Thus, we have shown that in each branch, the measure drops, hence completing the proof of correctness of the algorithm.

Analysis. Since $D$ exludes $K_{h}$ as a topological minor, Lemma 3 combined with the fact that we set $d_{b}=h-2$, implies that $d_{w}^{\max }=c h^{4}$, for some $c$, is an upper bound on the maximum $d_{w}$ which can appear during the execution of the algorithm. We first bound the number of leaves of the recursion tree as follows. The number of leaves is bounded by $\sum_{i=0}^{d_{b} k}\binom{d_{b} k}{i}\left(d_{w}^{\max }+1\right)^{k-\frac{i}{d_{b}}}$. To see this, observe that each branch of the recursion tree can be described by a length- $d_{b} k$ vector as shown in the correctness paragraph. We then select $i$ positions of this vector on which the last branch was taken. Finally for $k-\frac{i}{d_{b}}$ of the remaining positions, we describe which of the first at most $\left(d_{w}^{\max }+1\right)$ branches was taken. Any of the first $d_{w}^{\max }+1$ branches can be taken at most $k-\frac{i}{d_{b}}$ times if the last branch is taken $i$ times.

The time taken along each root to leaf path in the recursion tree is polynomial, while the time taken at a leaf for which the last branch was taken $i$ times is $O^{*}\left(2^{d_{b}\left(k-\left(k-\frac{i}{d_{b}}\right)\right)}\right)=O^{*}\left(2^{i+k}\right)$ ( see Lemmata 31 and 32). Hence, the running time of the algorithm is
$O^{*}\left(\sum_{i=0}^{d_{b} k}\binom{d_{b} k}{i}\left(d_{w}^{\max }+1\right)^{k-\frac{i}{d_{b}}} \cdot 2^{i+k}\right)=O^{*}\left(\left(2 d_{w}^{\max }+2\right)^{k} \cdot \sum_{i=0}^{d_{b} k}\binom{d_{b} k}{i} \cdot 2^{i}\right)=O^{*}\left(\left(2 d_{w}^{\max }+2\right)^{k} \cdot 3^{d_{b} k}\right)$.
For $d_{b}=h-2$ and $d_{w}^{\max }=c h^{4}$ this is $O^{*}\left(3^{h k+o(h k)}\right)$. This completes the proof of the theorem.

Observe that, when Algorithm 4.1 is run on a graph of degeneracy $d$, setting $d_{b}=d$, combined with the simple fact that $d_{w}^{\max } \leq d$ gives us the following theorem.

- Theorem 34. Dominating Set can be solved in time $O^{*}\left(3^{d k+o(d k)}\right)$ on graphs of degeneracy $d$.

From the above two theorems and Lemma 7, we have the following corollary.

- Corollary 35. If $\mathcal{C}$ is a class of graphs excluding $o(\log n)$-sized topological minors or an o $(\log n)$ degenerated class of graphs, then Dominating SEt parameterized by $k$ is FPT on $\mathcal{C}$.


### 4.3 Hardness

- Theorem 36. Dominating Set cannot be solved in time $f(k) n^{o\left(\frac{k}{\log k}\right)}$ on $c \log n$-degenerated graphs for any constant $c>0$, where $k$ is the solution size and $f$ is an arbitrary function, unless ETH fails.

Proof. The proof is by a reduction from the restricted version of Set Cover shown to be hard in Lemma 25. Fix a constant $c>0$ and let $\left(\mathcal{U}=\left\{u_{1}, \ldots, u_{n}\right\}, \mathcal{F}=\left\{F_{1}, \ldots, F_{m}\right\}, k\right)$ be an instance of Set Cover, where the size of any set is at most $\gamma \log m$ for some constant $\gamma$. For each set $F_{i}$, we have a vertex $f_{i}$. For each element $u_{i}$, we have a vertex $x_{i}$. If an element $u_{i}$ is contained in the set $F_{j}$, then we add an edge $\left(f_{j}, x_{i}\right)$. Further, we add two vertices $r$ and $p$ and add edges $\left(r, f_{i}\right)$ for every $i$ and an edge $(r, p)$. Finally, we add star of size $m^{2 \gamma / c}$ centered in $q$ disjoint from the rest of the graph. This completes the construction of the graph $G$.

We claim that $(\mathcal{U}, \mathcal{F}, k)$ is a Yes instance of Set Cover iff $(G, k+2)$ is a Yes instance of Dominating Set. Suppose that $\left\{F_{1}, \ldots, F_{k}\right\}$ is a set cover for the given instance. It is easy to see that the vertices $q, r, f_{1}, \ldots, f_{k}$ form a solution for the Dominating Set instance.

In the converse direction, since one of $p$ and $r$ and at least one vertex of the star must be in any dominating set, we assume without loss of generality that $r$ and $q$ are contained in the minimum dominating set. Also, since $r$ dominates any vertex $f_{i}$, we may also assume that the solution is disjoint from the $x_{i}$ 's. This is because, if $x_{i}$ was in the solution, we can replace it with an adjacent $f_{j}$ to get another solution of the same size. Hence, we suppose that $\left\{q, r, f_{1}, \ldots, f_{k}\right\}$ is a solution for the Dominating Set instance. Since the only way that some vertex $x_{i}$ can be dominated is by some $f_{j}$, and the construction implies that $u_{i} \in F_{j}$, the sets $\left\{F_{1}, \ldots, F_{k}\right\}$ form a set cover for $(\mathcal{U}, \mathcal{F}, k)$. This concludes the proof of equivalence of the two instances.

We claim that the degeneracy of the graph $G$ is bounded by $c \log n_{1}$, where $n_{1}$ is the number of vertices in the graph $G$. First, we claim that the degeneracy of the graph $G$ is bounded by $\gamma \log m+1$. This follows from that each vertex $f_{i}$ has total degree at most $\gamma \log m+1$, each leaf of the star has degree 1 and if a subgraph contains none of these vertices, then it contains no edges. Now, $n_{1}$ is at least $m^{2 \gamma / c}$. Hence, $\log n_{1} \geq(2 \gamma / c) \log m$ and the degeneracy of the graph is at most $\gamma \log m+1 \leq c \cdot(2 \gamma / c) \log m \leq c \log n_{1}$. Finally, since each vertex $f_{i}$ is incident to at most $\gamma \log m+1$ vertices, $n_{1}=O\left(m \log m+m^{2 \gamma / c}\right)$ and, thus, it is polynomial in $m$. Hence, an algorithm for Dominating SET of the form $f(k) n_{1}^{o\left(\frac{k}{\log k}\right)}$ implies an algorithm of the form $f(k) m^{o\left(\frac{k}{\log k}\right)}$ for the Set Cover instance. This concludes the proof of the theorem.

As a corollary of Theorem 36, and Lemma 7 we have the following corollary.

- Corollary 37. There are no two functions $f$ and $g$ such that $g(d)=o(d)$ and there is an algorithm for Dominating Set running in time $O^{*}\left(2^{g(d) f(k)}\right)$ unless ETH fails.

From Theorem 29, we can infer the following corollary.

- Corollary 38. There are no two functions $f$ and $g$ such that $f(k)=o(k)$ and there is an algorithm for Dominating Set running in time $O^{*}\left(2^{g(d) f(k)}\right)$, unless ETH fails.

Proof. Suppose that there were an algorithm for Dominating Set running in time $O^{*}\left(2^{g(d) f(k)}\right)$, where $f(k)=o(k)$. Consider an instance $(G, k)$ of Dominating Set where $G$ is a graph with

```
Input : An Undirected graph \(G=(V, E)\) without isolated vertices
Output: A dominating set for \(G\) of size at most \(O\left(d^{2}\right)\) times the size of a minimum
        dominating set
\(Y \leftarrow \emptyset\)
\(W_{h} \leftarrow V\)
while \(W_{h} \neq \emptyset\) do
    Find vertex \(v \in W_{h}\) with the least neighbors in \(B_{h} \cup W_{h}\).
    \(Y \leftarrow Y \cup\left(\left(B_{h} \cup W_{h}\right) \cap N(v)\right)\)
    \(B \leftarrow\) vertices in \(V \backslash Y\) with a neighbor in \(Y\)
    \(W \leftarrow V \backslash(Y \cup B)\)
    \(B_{h} \leftarrow\) vertices in \(B\) with at least \(d+1\) neighbors in \(W\)
    \(B_{l} \leftarrow B \backslash B_{h}\)
    \(W_{h} \leftarrow\) vertices in \(W\) with a neighbor in \(B_{h}\) or \(W\)
    \(W_{l} \leftarrow W \backslash W_{h}\)
end
\(Y \leftarrow Y \cup W_{l}\)
return \(Y\)
```

Algorithm 4.2: Algorithm DS-APPRox for Dominating Set
maximum degree $c$. We can assume that the number of vertices of the graph is at most $c k+k$, since otherwise, it is a trivial No instance. Hence, $k=\Theta(n)$ and thus, an algorithm running in time $O^{*}\left(2^{g(d) f(k)}\right)$ will run in time $O^{*}\left(2^{g(c) f(k)}\right)=O^{*}\left(2^{o(n)}\right)$, and, by Theorem 29. ETH fails.

Thus, Corollaries 37 and 38 together show that, unless ETH fails, our algorithm for Dominating Set has the best possible dependence on both the degeneracy and the solution size.

### 4.4 Approximating Dominating SET on graphs of bounded degeneracy

In this section, we adapt the ideas developed in the previous subsections, to design a polynomial-time $O\left(d^{2}\right)$-approximation algorithm for the Dominating Set problem on $d$-degenerated graphs.

- Theorem 39. There is a $O(d n \log n)$-time $d^{2}$ - approximation algorithm for the Dominating Set problem on d-degenerated graphs.

Proof. The approximation algorithm is based on our FPT algorithm, but whenever the branching algorithm would branch, we take all candidates into the solution and cycle instead of recursing. During the execution of the algorithm the partial solution is kept in the set $Y$ and vertex sets $B, W, B_{h}, B_{l}$, $W_{h}$, and $W_{l}$ are updated with the same meaning as in the branching algorithm. In the base case, all vertices of $W_{l}$ are taken into the solution. This last step could be replaced by searching a dominating set for the vertices in $W_{l}$ by some approximation algorithm for Set Cover. While this would probably improve the performance of the algorithm in practice, it does not improve the theoretical worst case bound.
Correctness. It is easy to see that the set $Y$ output by the algorithm is a dominating set for $G$. Now let $Q$ be an optimal dominating set for $G$. We want to show that $|Y|$ is at most $d^{2}$ times $|Q|$. In particular, we want to account every vertex of $Y$ to some vertex of $Q$, which "should have been chosen instead to get the optimal set." Before we do that let us first observe, how the vertices can move around the sets during the execution of the algorithm. Once a vertex is added to $Y$ it is never removed. Hence, once a vertex is moved from $W$ to $B$, it is never moved back. But then the vertices in $B$ are only losing neighbors in $W$, and once they get to $B_{l}$ they are never moved anywhere else. Thus, vertices in $W_{l}$ can never get a new neighbor in $W$ or $B_{h}$ and they also stay in $W_{l}$ until Step 13

The vertices of $B_{h}$ can get to $Y$ or $B_{l}$ and the vertices of $W_{h}$ can get to any other set during the execution of the algorithm.

Now let $v$ be the vertex found in Step 4 of Algorithm 4.2 and $w$ be the vertex dominating it in $Q$. As in the branching algorithm, we know that $\left|\left(B_{h} \cup W_{h}\right) \cap N(v)\right| \leq d$ since the subgraph of $G$ induced by $\left(B_{h} \cup W_{h}\right)$ is $d$-degenerated. Hence at most $d$ vertices are added to $Y$ in Step 5 of the algorithm. We charge the vertex $w$ for these at most $d$ vertices and make the vertex $v$ responsible for that. Finally, in Step 13 let $v \in W_{l}$ and $w$ be a vertex which dominates it in $Q$. We charge $w$ for adding $v$ to $Y$ and make $v$ responsible for it. Obviously, for each vertex added to $Y$, some vertex is responsible and some vertex of $Q$ is charged. It remains to count for how many vertices can be a vertex of $Q$ charged.

Observe that whenever a vertex is responsible for adding some vertices to $Y$, it is $W$ and after adding these vertices it becomes dominated and, hence, moved to $B$. Therefore, each vertex becomes responsible for adding vertices only once and, thus, each vertex is responsible for adding at most $d$ vertices to $Y$. Now let us distinguish in which set a vertex $w$ of $Q$ is, when it is first charged. If $w$ is first charged in Step 5 of the algorithm, then a vertex $v \in W_{h}$ is responsible for that and $w$ has to be either in $W_{h}, B_{h}$, or $B_{l}$, as vertices in $W_{h}$ do not have neighbors elsewhere. In the first two cases, if $w \neq v$, then it is moved to $Y$ and hence has no neighbors in $W$ anymore. If $v=w$, then it is moved to $B_{l}$, and it has no neighbors in $W$, as all of them are moved to $Y$. Thus, in these cases, after the step is done, $w$ has no neighbors in $W$ and, hence, is never charged again. If $w$ is in $B_{l}$, then it has at most $d$ neighbors in $W$ and, as each of them is responsible for adding at most $d$ vertices to $Y, w$ is charged for at most $d^{2}$ vertices. If a vertex $w$ is first charged in Step 13 , then it is $B_{l}$ or $W_{l}$, has at most $d$ neighbors in $W_{l}$, each of them being responsible for addition of exactly one vertex, so it is charged for addition of at most $d$ vertices. It follows, that every vertex of $Q$ is charged for addition of at most $d^{2}$ to $Y$ and, therefore, $|Y|$ is at most $d^{2}$ times $|Q|$.
Running time analysis. To see the running time, observe first that by the above argument, every vertex is added to each of the sets at most once. Also a vertex is moved from one set to another only if some of its neighbors is moved to some other set or it is selected in Step 4 Hence, we might think of a vertex sending a signal to all its neighbor, once it is moved to another set. There are only constantly many signals and each of them is sent at most once over each edge in each direction. Hence, all the updations of the sets can be done in $O(m)=O(d n)$-time, as the graph is $d$-degenerate. Also each vertex in $W_{h}$ can keep its number of neighbors in $W_{h} \cup B_{h}$ and update it whenever it receives a signal from some of its neighbors about being moved out of $W_{h} \cup B_{h}$. We can keep a heap of the vertices in $W_{h}$ sorted by a degree in $W_{h} \cup B_{h}$ and update it in $O(\log n)$ time whenever the degree of some of the vertices change. This means $O(d n \log n)$ time to keep the heap through the algorithm. Using the heap, the vertex $v$ in Step 4 can be found in $O(\log n)$-time in each iteration. As in each iteration at least one vertex is added to $Y$, there are at most $n$ iterations and the total running time is $O(d n \log n)$.

This algorithm can be also used for $K_{h}$-minor-free and $K_{h}$-topological-minor free graphs, yielding $O\left(h^{2} \cdot \log h\right)$-approximation and $O\left(h^{4}\right)$-approximation, as these graphs are $O(h \cdot \sqrt{\log h})$ and $O\left(h^{2}\right)$ degenerated, respectively. As far as we know, this is also the first constant factor approximation for dominating set in $K_{h}$-topological-minor free graphs. Although PTAS is known for $K_{h}$-minor-free graphs [21, our algorithm can be still of interest due to its simplicity and competitive running time.

## 5 Conclusions

We gave the first FPT algorithms for the Steiner Tree problem on directed graphs excluding a fixed graph as a (topological) minor, and then extended the results to directed graphs of bounded degeneracy. We mention that the same approach also gives us FPT algorithms for Dominating Set and some of its variants, for instance Connected Dominating Set and Total Dominating Set. Finally, in the process of showing the optimality of our algorithm, we showed that for any constant $c$, DST is not expected to have an algorithm of the form $f(k) n^{o\left(\frac{k}{\log k}\right)}$ on $o(\log n)$-degenerated graphs.


It would be interesting to either improve this lower bound, or prove the tightness of this bound by giving an algorithm with a matching running time.

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[^1]:    ${ }^{1}$ When we say that a digraph excludes a fixed (undirected) graph as a minor or a topological minor, or that the digraph has degeneracy $d$ we mean that the statement is true for the underlying undirected graph.

