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HYBRIDIZATION NUMBER ON THREE ROOTED BINARY TREES IS EPT*

LEO VAN IERSEL[†], STEVEN KELK[‡], NELA LEKIĆ[‡], CHRIS WHIDDEN[§], AND
NORBERT ZEH[¶]

Abstract. Phylogenetic networks are leaf-labeled directed acyclic graphs that are used to describe nontreelike evolutionary histories and are thus a generalization of phylogenetic trees. The hybridization number of a phylogenetic network is the sum of all in-degrees minus the number of nodes plus one. The hybridization number problem takes as input a collection of rooted binary phylogenetic trees and asks to construct a phylogenetic network that contains an embedding of each of the input trees and has the smallest possible hybridization number. We present an algorithm for the hybridization number problem on three binary phylogenetic trees on n leaves that runs in time $O(c^k \text{poly}(n))$ with k the hybridization number of an optimal network and c some (astronomical) constant. For the case of two trees, an algorithm with running time $O(3.18^k n)$ was proposed before, whereas an algorithm with running time $O(c^k \text{poly}(n))$, also called an EPT algorithm, had prior to this article remained elusive for more than two trees. The algorithm for two trees uses the close connection to acyclic agreement forests to achieve a linear exponent in the running time, while previous algorithms for more than two trees (explicitly or implicitly) relied on a brute force search through all possible underlying network topologies, leading to running times that are not $O(c^k \text{poly}(n))$ for any c . The connection to acyclic agreement forests is much weaker for more than two trees, so even given the right agreement forest, the reconstruction of the network poses major challenges. We prove novel structural results that allow us to reconstruct a network without having to guess the underlying topology. Our techniques generalize to more than three input trees with the exception of one key lemma that maps nodes in the network to tree nodes in order to minimize the amount of guessing involved in constructing the network. The main open problem therefore is to prove results that establish such a mapping for more than three trees.

Key words. hybridization number, rooted phylogenetic tree, rooted phylogenetic network, reticulate evolution, agreement forest, fixed parameter tractability

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1. Introduction. In computational biology the evolutionary history of a set of contemporary species (or *taxa*) is often modeled as a *rooted phylogenetic tree*. Informally this is a rooted tree in which the leaves are bijectively labeled by the taxa

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[†]Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5, 2600 AA Delft, the Netherlands (l.j.v.iersel@gmail.com).

[‡]Department of Knowledge Engineering (DKE), Maastricht University, P.O. Box 616, 6200 MD Maastricht, the Netherlands (steven.kelk@maastrichtuniversity.nl, nela.lekic@maastrichtuniversity.nl).

[§]Fred Hutchinson Cancer Research Center, Seattle, WA 98109 (whidden@cs.dal.ca). This author is a Simons Foundation Fellow of the Life Sciences Research Foundation.

[¶]Faculty of Computer Science, Dalhousie University, 6050 University Ave, Halifax, NS B3H 1W5, Canada (nzeh@cs.dal.ca).

and edges are directed away from the root, reflecting the direction of evolution [22]. Nodes of out-degree two or higher model the points in history at which a common ancestor of a subset of the taxa differentiated into two or more sublineages. The central problem in phylogenetics is to recover the topology of the “true” phylogenetic tree, given only information about the taxa, often DNA data. This is a challenging computational problem and has been the topic of intensive research during the last 40 years [9]. Recently our understanding of evolutionary mechanisms has deepened, and there is growing awareness that evolution is not always treelike [1]. In particular, due to *reticulate phenomena* such as hybridization and horizontal gene transfer [19], the evolution of a set of species is sometimes better modeled as a *rooted phylogenetic network* [10], essentially a generalization of phylogenetic trees to directed acyclic graphs (DAGs). In such graphs, nodes with indegree two or higher, known as *reticulations*, represent the points at which two or more lineages merge rather than diversify.

The study of rooted phylogenetic networks is comparatively new and has given rise to many novel and hard combinatorial optimization problems [10]. In this article we focus on the hybridization number problem, originally introduced in [2, 3], which is one of the most well-studied phylogenetic network problems to date. Here we are given a set of rooted phylogenetic trees $\mathcal{T}$, on the same set of taxa X , and the goal is to construct a phylogenetic network—henceforth called a *hybridization network*—that contains an image of each of the input trees, while minimizing the hybridization number k of the network. If we restrict (without loss of generality) to networks with maximum in-degree two, the hybridization number is simply equal to the number of reticulation nodes. We defer exact definitions to the preliminaries. See Figure 1 for an example of a hybridization network (with hybridization number three) for three input trees.

The holy grail for this problem is to develop algorithms that can cope with many input trees and nonbinary input trees [19] (and to take different causes of incongruence into account, see, e.g., [25]). However, thus far most algorithmic research has focused on the simplest possible case: $|\mathcal{T}| = 2$, and both input trees are binary. Unfortunately even this version of the problem is NP-hard and APX-hard [5], with similar (in)approximability properties to the classical problem directed feedback vertex set [17]. Fortunately the binary two-tree problem is fixed parameter tractable (FPT) in k . (See [7, 20] for an introduction to fixed parameter tractability.) This result was initially established via kernelization—the problem has a quadratic kernel [4]—but the theoretical state of the art is an algorithm based on bounded-search with running time $O(3.18^k \cdot \text{poly}(n))$ [23], where $n = |X|$. The comparative tractability of the problem, both in theory and practice (see, e.g., [6] for a fast implementation), stems from the essentially one-to-one relationship between solutions to the two-tree problem and the *maximum acyclic agreement forest* (MAAF) problem. In the latter problem (originally introduced in [3]) one is required to cut the two input trees into common components so that the number of components is minimized and there are no cyclical dependencies between components. The MAAF abstraction gives a useful static characterization of the two-tree hybridization number problem [5]. In particular, in the two-tree case the MAAF abstraction essentially allows us to bypass the problem of actually constructing the hybridization network: it can easily be constructed in polynomial time from the components of the MAAF. The MAAF abstraction, and related FPT results, also hold in the case of two *nonbinary* trees, albeit with significant technical complications [18, 21].

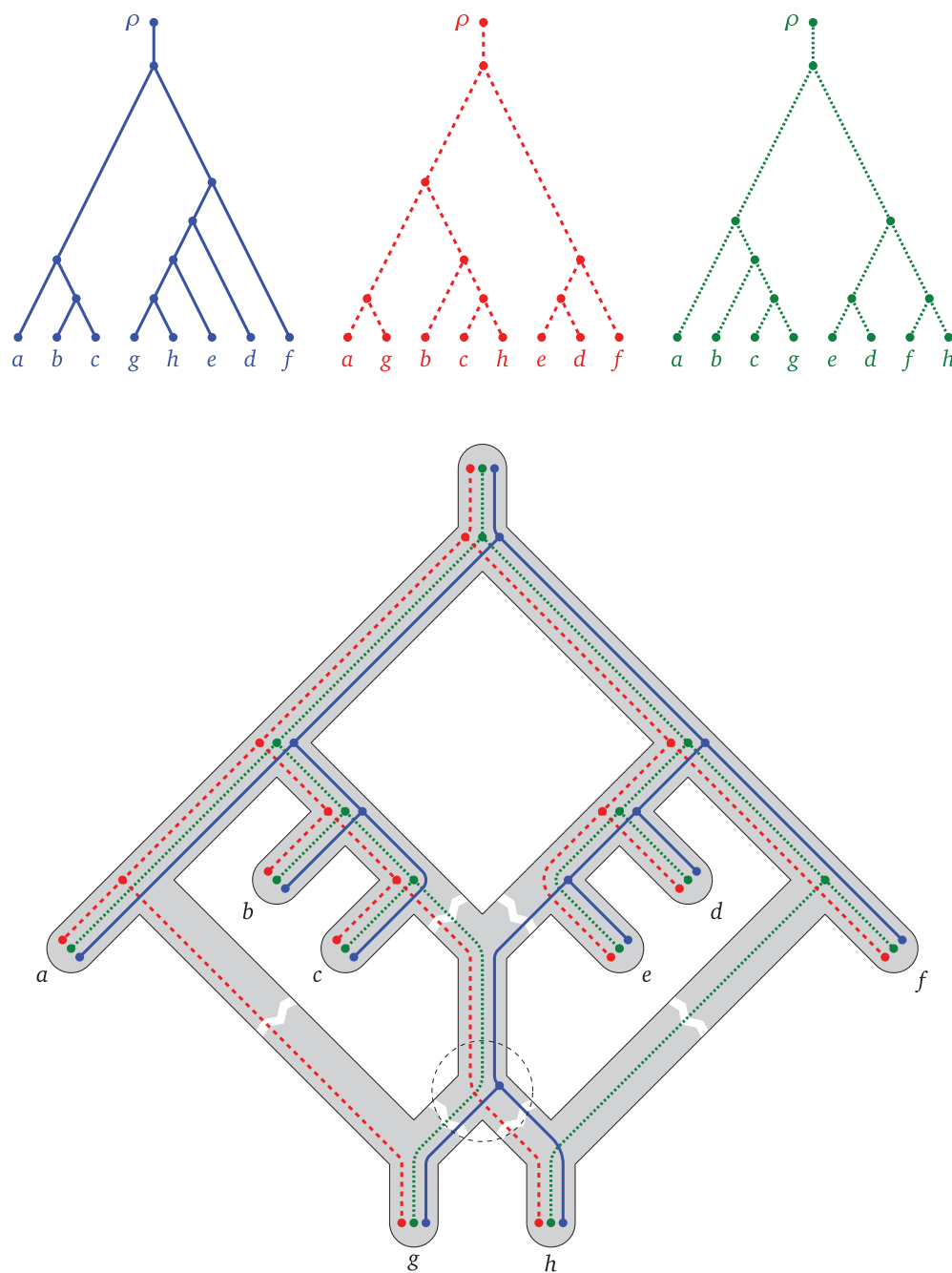


FIG. 1. A hybridization network for three trees which contains an invisible component (inside the dashed circle). It can be shown that any hybridization network for these trees contains an invisible component. However, the single node inside this component can be identified because it corresponds to a node of the blue solid input tree.

For $|\mathcal{T}| > 2$ the situation becomes more complex, however, even when restricted to binary trees¹ and $|\mathcal{T}| = 3$. The MAAF abstraction weakens significantly and cannot (obviously) be used to generate optimal solutions to the hybridization number problem. Without the MAAF abstraction it seems that we have to explicitly confront the challenge of actually constructing the hybridization network itself. This is a theoretically daunting challenge, since the space of DAGs is huge. The good news is that for $|\mathcal{T}| > 2$ the problem nevertheless remains FPT in k [13, 14]. The bad news is that none of these results satisfactorily address the problem of actually constructing the network. The FPT result in [13] gives a quadratic kernel but does not describe a (good) algorithm for solving the kernel. The bounded-search FPT result in [14], based on [15], does actually construct the network but has an astronomical running time. The running time is so large because it brute forces over the space of all possible *generators*, i.e., possible “backbone topologies” of the network [15, 16], a space which is not known to be $O(c^k)$, and continues with a tower of guesses, which is not $O(c^k)$, for each such generator. At present, therefore, the only FPT algorithms for the case of three binary trees are either kernelizations or bounded-search algorithms with an exponential dependency on k with a nonlinear exponent. Several exponential-time algorithms do exist, such as [24] and the algorithm discussed in [13], but using them to solve a kernelized hybridization number instance unfortunately does not help for two reasons. First, the size of the best-known kernel (i.e., the number n of leaves of a kernelized instance) is quadratic and not linear in k . Second, no previously known exponential-time algorithm has an $O(c^n)$ running time. Therefore, the challenge is to determine whether an algorithm with running time $O(c^k \cdot \text{poly}(n))$ exists for the case of three binary trees. In other words, is the problem EPT [8]?

In this article we answer this challenge positively. Although the constant c that we find is astronomical—1,609,891,840—it represents a significant development in our understanding of the underlying combinatorial structure of the hybridization number problem. We show that, although it is not clear how a MAAF can be pieced together into an optimal solution to the hybridization number problem, it is still possible to identify in $O(c^k \text{poly}(n))$ time a (not necessarily maximum) acyclic agreement forest that does have this property. Having found the appropriate acyclic agreement forest, we use deep insights into the structure of optimal hybridization networks to piece the components of the forest together into a network. The difficulty of this step comes from the fact that, unlike in the two-tree case, it is no longer possible to avoid having nodes in the network that are separated from all leaves by hybridization edges and that are hence not represented in the agreement forest. The main insight helping to overcome this problem is that, in the case $|\mathcal{T}| = 3$, there always exists an optimal hybridization network such that each of its out-degree-2 nodes corresponds to nodes of one or more of the input trees; see Figure 1. This enables us to keep the combinatorial explosion in the number of possible network topologies under control.

Note that our algorithm can be viewed as a structural generalization of existing algorithms for two trees, which also separate the identification of the underlying acyclic agreement forest and the construction of the network into two phases. In the case of two trees the second phase is polynomial and it is comparatively easy to obtain $O(c^k \text{poly}(n))$ running times for the first phase. In fact, although our overall result at present only holds for the case $|\mathcal{T}| = 3$, the results for the first phase hold without

¹For the rest of the introduction we focus only on the case of binary trees—see [11] and [12] for an overview of recent nonbinary results. The nonbinary case is a generalization of the binary case and therefore inherits all the negative results, but not the positive results, of the binary problem.

modification for the case $|\mathcal{T}| > 3$. As we demonstrate, the only barrier to extending our result is the fact that, for $|\mathcal{T}| > 3$, the combinatorial insight mentioned in the previous paragraph no longer holds. Indeed, there are two new challenges stemming from this article: first, to adapt and generalize the combinatorial insight so that the wider result can be extended to four or more trees, and second, to significantly optimize the constant c in our running time. How close can we get to the competitive $O(3.18^k \cdot \text{poly}(n))$ running time achieved in the case of two trees?

The structure of the remainder of this article is as follows. In section 2, we present the necessary definitions. Section 3 shows how we can guess the underlying acyclic agreement forest of an optimal hybridization network in $O(c^k \text{poly}(n))$ time. Then we define a notion of “tight” networks in section 4 (basically, networks where each out-degree-2 node corresponds to a node of at least one of the input trees) and show that we may restrict our attention to tight networks as long as there are at most three trees in the input. Subsequently, section 5 shows how such a tight network can be reconstructed from an acyclic agreement forest and the input trees in $O(c^k \text{poly}(n))$ time. Finally, we present our conclusions in section 6 and give an example of the algorithm in the appendix.

2. Preliminaries. A *rooted phylogenetic X -tree* is a rooted tree with no nodes with in-degree 1 and out-degree 1, a node with in-degree 0 and out-degree 1 (the root), and leaves bijectively labeled with the elements of a finite set X . Such a tree is called *binary* if all inner nodes except the root have in-degree 1 and out-degree 2. From now on we will refer to a rooted binary phylogenetic X -tree as a *tree* for short, since we only consider rooted binary trees that are all on the same set X . The convention that roots have out-degree 1 is not essential but for technical convenience.

A rooted phylogenetic network (on X) is a DAG with no nodes with in-degree 1 and out-degree 1, a single in-degree-0 node (the root) with out-degree 1 and leaves bijectively labeled with the elements of X . Rooted phylogenetic networks will be called *networks* for short. We identify each leaf of a tree or network with its label and call directed edges *edges* for short. We see the root of a tree or network as a leaf and assume without loss of generality that it is labeled ρ . We call a network *binary* if every nonleaf node has total degree 3 and all leaves have degree 1.

We call network nodes with in-degree 1 and out-degree at least 2 *split nodes*, while nodes with in-degree at least 2 are called *reticulation nodes*, or *reticulations* for short. The *hybridization number* (often also called reticulation number) of a binary network N is defined as the number of reticulation nodes of N . For a general network the hybridization number is given by the sum $\sum (d^-(v) - 1)$ over all nodes v of N with in-degree $d^-(v)$ at least 2. For a tree T and a set $X' \subseteq X$, we define $T(X')$ as the minimal subtree of T that contains all elements of X' , and $T|X'$ as the result of suppressing all nodes of $T(X')$ with in- and out-degree 1. The set of leaves of a tree T is denoted $L(T)$.

We say that a tree T is *displayed* by a network N if T can be obtained from a subgraph of N by contracting edges. Given a set $\mathcal{T}$ of rooted phylogenetic trees, the minimum hybridization problem asks to find a phylogenetic network N that displays each tree in $\mathcal{T}$ such that the hybridization number of N is minimized. We say that N is a *hybridization network* for a set $\mathcal{T}$ of input trees if N displays all $T \in \mathcal{T}$. In addition, we say that the *hybridization number* of a set of input trees $\mathcal{T}$ is the hybridization number of a hybridization network for $\mathcal{T}$ that has the lowest hybridization number over all hybridization networks for $\mathcal{T}$. It is well known, and easy to see, that if there exists a hybridization network for $\mathcal{T}$, there also exists a binary hybridization network

for $\mathcal{T}$ with the same hybridization number (see, e.g., [14]). Therefore, all hybridization networks are from now on assumed to be binary.

Let T be a rooted, binary phylogenetic X -tree and S a rooted, binary phylogenetic X' -tree for some $X' \subseteq X$. We say that S is a *pendant subtree* of T if it is a subtree that can be detached from T by deleting a single edge. For a set $\mathcal{T}$ of phylogenetic X -trees and $X' \subseteq X$, a *common pendant subtree* of $\mathcal{T}$ is a rooted phylogenetic X' -tree that is a pendant subtree of each tree in $\mathcal{T}$. A common pendant subtree is called *trivial* if it consists of a single leaf. Let T be a tree, let $(x_1, x_2, \dots, x_q)$ be a tuple of elements of X with $q \geq 1$, and let p_i denote the parent of x_i in T . We say that the tuple $(x_1, x_2, \dots, x_q)$ is a *chain* of T if either $(p_q, p_{q-1}, \dots, p_1)$ is a directed path in T , or $(p_q, p_{q-1}, \dots, p_2)$ is a directed path in T and $p_1 = p_2$. A *common chain* of a set $\mathcal{T}$ of trees is a maximal tuple $(x_1, x_2, \dots, x_q)$ that is a chain of each tree in $\mathcal{T}$.

Related to the hybridization number problem is a concept of agreement forests. A forest is a collection of trees, which we will call components rather than trees to avoid confusion with the input trees. We say that a forest $\mathcal{F}$ is a *forest for* a tree T if $T|L(F)$ is isomorphic to F for all $F \in \mathcal{F}$ and the trees $\{T(L(F)) \mid F \in \mathcal{F}\}$ are node-disjoint subtrees of T whose leaf-set union equals $L(T)$. By this definition, if $\mathcal{F}$ is a forest for some tree T , then $\{L(F) \mid F \in \mathcal{F}\}$ is a partition of the leaf set of T . It will indeed sometimes be useful to see a forest as a partition of the leaves and sometimes to see it as a collection of trees. If $\mathcal{T}$ is a set of trees, then a forest $\mathcal{F}$ is an *agreement forest* of $\mathcal{T}$ if it is a forest for each $T \in \mathcal{T}$. Note that these definitions only apply to binary trees.

We define the *inheritance graph* $IG(\mathcal{T}, \mathcal{F})$ of an agreement forest $\mathcal{F}$ of a set $\mathcal{T}$ of trees as the directed graph whose node set is the set of components of $\mathcal{F}$ and whose edge set contains an edge (F, F') precisely if there is a directed path from the root of $T(L(F))$ to the root of $T(L(F'))$ in at least one tree $T \in \mathcal{T}$. An agreement forest $\mathcal{F}$ of $\mathcal{T}$ is called an *acyclic agreement forest* (AAF) if the graph $IG(\mathcal{T}, \mathcal{F})$ does not contain any directed cycles.

The last definition we need is the notion of generators (see, e.g., [15]), which we use to describe the underlying structure of networks without nontrivial pendant subtrees. A (binary) *r-reticulation generator* is defined as an acyclic directed multigraph with a single root with in-degree 0 and out-degree 1 and exactly r nodes with in-degree 2 and out-degree at most 1; all other nodes have in-degree 1 and out-degree 2. If N is a network, then the *underlying generator* of N is the generator obtained from N by deleting all leaves and suppressing all in-degree-1 out-degree-1 nodes. The *sides* of a generator are its edges (the *edge sides*) and its nodes with in-degree 2 and outdegree 0 (the *node sides*). Thus, each leaf of a network N is on a certain side of its underlying generator. See Figure 2 for an example.

3. Guessing the AAF. Let $\mathcal{T}$ be a collection of input trees. Without loss of generality we will assume that $\mathcal{T}$ contains no nontrivial common pendant subtrees (because each such subtree can be replaced by a single leaf). In this section, we show how we can guess an AAF from which we can build an optimal hybridization network for $\mathcal{T}$. To make this precise, we define the *deletion forest* of a network N as the forest obtained from N by deleting all the edges entering reticulation nodes, deleting all resulting connected components that do not contain any taxa, and then taking the partition of the taxa induced by the remaining connected components. Note that for a given network the deletion forest is uniquely defined. We start by proving the following lemma.

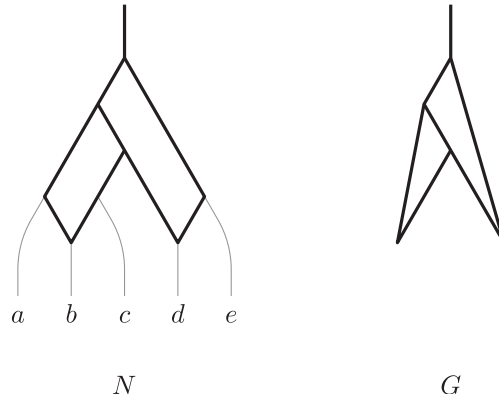


FIG. 2. The graph G is the generator of the network N . It has 9 sides: 7 edge sides and 2 node sides. The leaves b and d are on node sides while a , c , and e are on edge sides.

LEMMA 1. Given a hybridization network N with hybridization number k for a set $\mathcal{T}$ of input trees, the deletion forest of N is an AAF of $\mathcal{T}$ with at most $k + 1$ components.

Proof. We first show that the deletion forest contains at most $k + 1$ components. To see this, note that N contains exactly k reticulation nodes. For a reticulation node r , let $X(r)$ be the set of taxa that can be reached from r by directed paths that start at r and which do not intersect with any reticulation apart from r . (Possibly, $X(r) = \emptyset$.) By construction, none of the edges on these directed paths are deleted when the deletion forest is created. Hence, all the taxa in $X(r)$ will be in the same connected component. Similarly, if $X(\rho)$ denotes the set of taxa reachable by directed paths that start at the root and which do not intersect with any reticulations, then the taxa in $X(\rho)$ will also be together in the same connected component. Note that the deletion forest $\mathcal{F}$ of N (seeing it as a partition of the taxa) is the collection containing $X(\rho)$ if $X(\rho) \neq \emptyset$ and $X(r)$ for each reticulation r for which $X(r) \neq \emptyset$. Hence, the deletion forest contains at most $k + 1$ components. Moreover, for each $F \in \mathcal{F}$, each input tree must yield the same subtree when restricted to the subset of taxa of F because N displays all the input trees. In addition, for each input tree $T \in \mathcal{T}$, the subtrees $\{T(L(F)) \mid F \in \mathcal{F}\}$ are node-disjoint, again because N displays T . It follows that the deletion forest $\mathcal{F}$ of N is indeed an agreement forest of the input trees, with at most $k + 1$ components. Moreover, it is clearly acyclic since the network is acyclic. $\square$

As a consequence of Lemma 1, we will from now on refer to the deletion forest of a network as its *deletion AAF*. Next we show how to guess the deletion AAF of some optimal hybridization network for the input trees. More precisely, we show how to construct a set of AAFs containing at least one AAF with this property. In sections 4 and 5 we will show how to determine from which AAF(s) in the set we can build an optimal hybridization network.

LEMMA 2. Let k be the hybridization number of the set $\mathcal{T}$ of input trees. Then, in time $O(c^k \cdot \text{poly}(n))$, we can find a set of AAFs containing at least one deletion AAF of some hybridization network for $\mathcal{T}$ with hybridization number k .

Proof. Consider an arbitrary input tree $T \in \mathcal{T}$. Observe that an AAF of $\mathcal{T}$ with $k' + 1$ components can be obtained from T by deleting exactly k' edges and

taking the partition of the taxa induced by the resulting connected components. By Lemma 1, the deletion AAF of an optimal hybridization network for $\mathcal{T}$ has at most $k+1$ components. The goal therefore is to locate the at most k edges that need to be deleted from T in order to obtain the deletion AAF of some optimal hybridization network for $\mathcal{T}$.

Let N be a hybridization network for $\mathcal{T}$ with hybridization number k and consider its underlying generator, which is a k -reticulation generator and hence has at most k node sides and at most $4k-1$ edge sides [13]. It follows that there are at most $5k-1$ common chains of $\mathcal{T}$, because any two taxa on the same edge side of N are in the same common chain [13]. The set of common chains is unambiguously defined by the set of input trees and can be computed in polynomial time, and no two chains can share a taxon. Moreover, in [13] it is proved that if two or more taxa of a common chain are on a single edge side of the underlying generator, the entire chain can safely be moved onto that edge side. That is, the new network still displays the input trees and has a hybridization number no higher than the old network.

This means that we can assume the existence of an optimal network N' such that for each common chain there are exactly two possibilities: (1) the chain is on a single side of the underlying generator, or (2) each taxon of the chain is on a different side of the underlying generator. For each chain we can guess which of the two cases holds, using at most 2^{5k-1} guesses for the entire set of chains. Since, as mentioned before, any two taxa that are on the same side belong to a common chain, it follows that each side of (the underlying generator of) N' contains a complete case-1 chain, a single taxon (which is either in a case-2 chain or a singleton-chain), or no taxa at all.

Now, assume that we have identified the correct set of guesses describing the behavior of the common chains in N' . It remains to show that we can identify the correct set of edges to delete in T to obtain the deletion AAF corresponding to N' . Observe that for each case-1 chain it is not necessary to delete any of the internal edges of the chain in T . This is because we have correctly identified that the entire chain is attached to a single edge side of the generator and thus that it belongs to a single component of the deletion AAF. For each of the other edges in T we simply guess whether to delete it or not. Fortunately, there are not too many of these edges. Specifically, recall that each side contains either a case-1 chain or a single taxon and that the number of sides is at most $5k-1$. Hence, if we collapse each case-1 chain C into a new taxon x_C , which is permitted because we will never cut its internal edges, there are in total at most $5k-1$ taxa left. A binary tree with $5k-1$ taxa has $10k-4$ edges. By guessing for each of these edges whether or not to delete it, we observe that, in total, the deletion AAF of N' can be located in time at most $O(2^{5k} \cdot 2^{10k} \cdot \text{poly}(n))$. $\square$

4. Tight networks. In this section we give the only lemma that is specific to three trees. We describe a transformation from a hybridization network to a structure, called a tight network with embedded trees, that has some desirable properties. We prove that the transformation preserves the hybridization number, so we are allowed to concentrate on tight networks in the case of three trees. For ease of notation, we will sometimes identify a directed graph with its edge set.

A *tight network with embedded trees (TNET)* for a set $\mathcal{T}$ of phylogenetic trees over a label set X is a pair $\mathcal{H} = (H, \mathcal{E})$ with the following properties:

- (i) H is a DAG. We call its sources roots and its sinks leaves.
- (ii) Every root of H has one child.
- (iii) The leaves of H are labeled bijectively with the leaf labels in X .

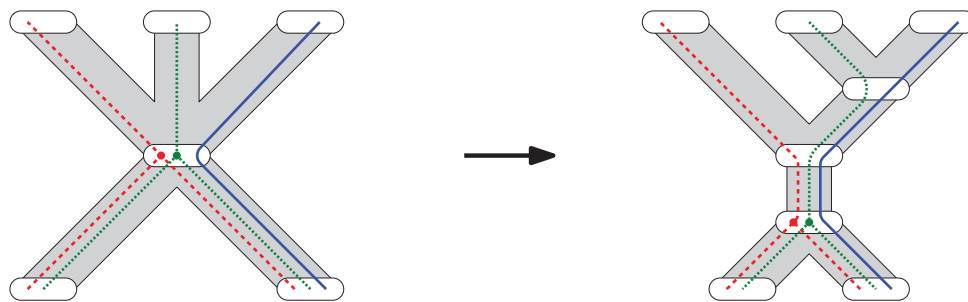


FIG. 3. The first two steps of transforming a TNET into a hybridization network: expanding nodes that are both reticulation and split nodes and refining reticulations.

- (iv) $\mathcal{E} = \{H(T) \subseteq H \mid T \in \mathcal{T}\}$ and, for all $T \in \mathcal{T}$, $H(T)$ is an *image* of T , that is, T can be obtained from $H(T)$ by suppressing degree-2 nodes.
- (v) Every tree image $H(T) \in \mathcal{E}$ contains an edge incident to a root of H .
- (vi) $H = \bigcup_{T \in \mathcal{T}} H(T)$, that is, every edge of H belongs to at least one tree image in $\mathcal{E}$.
- (vii) Every nonleaf nonroot node of H has exactly two children.
- (viii) For every nonleaf nonroot node, there exists a tree image $H(T) \in \mathcal{E}$ that contains both its child edges.

We represent the tree images in $\mathcal{E}$ by associating a unique color with each tree $T \in \mathcal{T}$ and coloring every edge in $H(T)$ with this color. We call the color associated with tree T *color* T . We use $C(e)$ to denote the color set of an edge e of H , that is, the set of trees $T \in \mathcal{T}$ whose images $H(T) \in \mathcal{E}$ include e . A TNET for the input trees in Figure 11 is shown in Figure 14. A corresponding hybridization network is shown in Figure 15.

The hybridization network *induced* by a TNET $(H', \mathcal{E})$ is the hybridization network obtained by applying the following transformations to H' :

- We replace every node x that is both a reticulation node and a split node with two nodes x_t and x_b , change the bottom endpoints of x 's parent edges to x_t , change the top endpoints of x 's child edges to x_b , and add an edge from x_t to x_b .
- We replace every reticulation node with more than two parents with a chain of binary reticulation nodes. See Figure 3 for an illustration of these first two steps.
- As long as there are at least two roots, we choose two such roots r_1 and r_2 , change the top endpoint of r_1 's child edge to r_2 , and add an edge from r_1 to r_2 . This reduces the number of roots by one, so we eventually obtain a network with a single root. See Figure 4.
- For every leaf x with more than one parent, we create a new node x' , change the bottom endpoint of every parent edge of x to x' , and add an edge from x' to x .

The *deletion AAF* of a TNET $(H, \mathcal{E})$ is defined to be the deletion AAF of the hybridization network induced by $(H, \mathcal{E})$. The *hybridization number* of a TNET $(H, \mathcal{E})$ is (as for networks) defined to be the sum $\sum (d^-(v) - 1)$ over all nodes v of H with indegree $d^-(v)$ at least 2.

LEMMA 3. If $|\mathcal{T}| = 3$, then there exists a hybridization network H with a hybridization number k for $\mathcal{T}$ if and only if there exists a TNET $\mathcal{H} = (H', \mathcal{E})$ with a

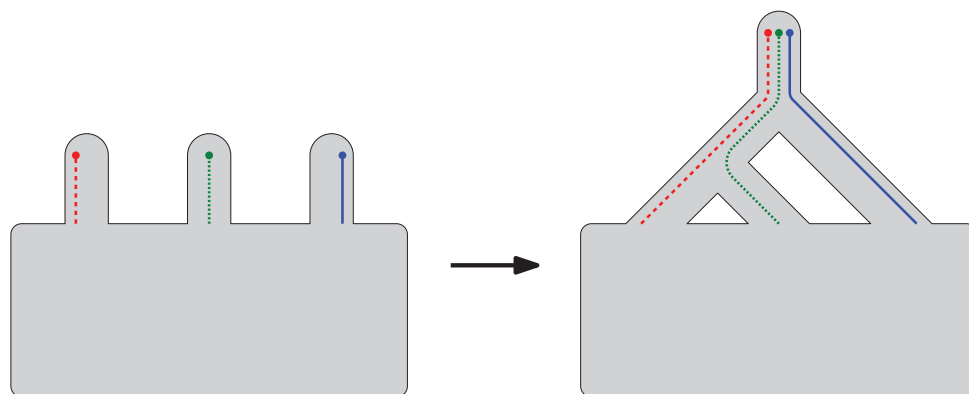


FIG. 4. The third step of transforming a TNET into a hybridization network: combining multiple roots.

hybridization number k for $\mathcal{T}$. Moreover, if such a network H exists, then there exists such a TNET $\mathcal{H}$ with the same deletion AAF.

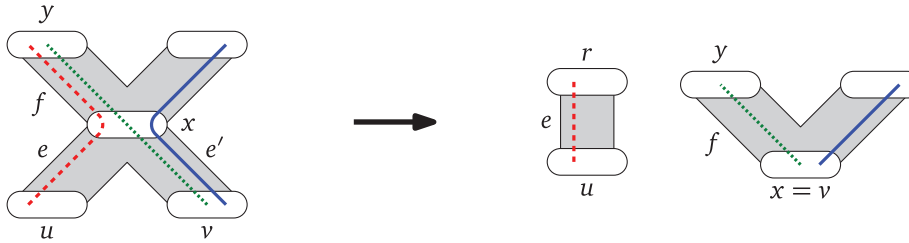
Proof. First suppose that there exists a TNET $\mathcal{H} = (H', \mathcal{E})$ with a hybridization number k for $\mathcal{T}$. Then the hybridization network H induced by $\mathcal{H}$ has the same hybridization number as H' , and it is easy to see that H displays the trees in $\mathcal{T}$, given that H' displays these trees. It follows directly from the definition of the deletion AAF of a TNET that H and $\mathcal{H}$ have the same deletion AAF.

Now assume we are given a hybridization network H with hybridization number k for $\mathcal{T}$. Since H displays all trees in $\mathcal{T}$, we can choose a tree image $H(T)$ for every tree $T \in \mathcal{T}$ that includes the root of H . Then we set $\mathcal{H} = (H, \mathcal{E})$. $\mathcal{H}$ satisfies conditions (i)–(v) of a TNET but may violate the remaining three conditions. Next we describe transformations that we apply to $\mathcal{H}$ to ensure it satisfies these remaining conditions without introducing any violations of the conditions that $\mathcal{H}$ already satisfies and without increasing the hybridization number of $\mathcal{H}$. Thus, after applying these transformations, we obtain a TNET with hybridization number at most k for $\mathcal{T}$.

Condition (vi). Deleting all edges of H that are not contained in $\bigcup_{T \in \mathcal{T}} H(T)$ does not violate conditions (i)–(v) and establishes condition (vi). Since it also does not increase the hybridization number of $\mathcal{H}$, we obtain a network with hybridization number at most k that satisfies conditions (i)–(vi).

Condition (vii). As long as there is a *nonsplitting* nonroot node x , that is, a nonroot node with only one child (which may arise after the modifications from the previous paragraph), we contract the edge e between x and its child in H and in every tree image $H(T) \in \mathcal{E}$ that includes e , and merge any parallel edges this may create. Each such contraction reduces the number of nonsplitting nonroot nodes by one, does not introduce any violations of conditions (i)–(vi), and does not increase the hybridization number of $\mathcal{H}$. Thus, we eventually obtain a network with a hybridization number at most k that satisfies conditions (i)–(vii).

Condition (viii). By condition (vii), every nonroot node of H is a split node. We call it a *true split node* if it also satisfies condition (viii) and a *fake split node* otherwise. We also call a true split node a *T-split node* if the tree image $H(T)$ contains both its child edges. The *weight* of a node x is the number of trees $T \in \mathcal{T}$ such that x is a *T-split node*. The weight of a path is the sum of the weights of the nodes on the path. Now we define the *potential* ϕ_x of a fake split node to be one plus the maximum weight

FIG. 5. Eliminating a fake split node x below a root y .

of a path from a root to x . All other nodes have potential 0. The potential of the network is $\Phi := \sum_x \phi_x$, where the sum is taken over all nodes of the network. Since every fake split node has a positive potential, a network has potential $\Phi = 0$ if and only if it contains no fake split node, that is, if and only if it satisfies condition (viii). Next we describe transformations that decrease the potential of the network without increasing its hybridization number or introducing any violations of conditions (i)–(vii). Thus, after repeating this transformation as often as possible, we obtain a network with hybridization number at most k and which satisfies conditions (i)–(viii), so it is a TNET with hybridization number at most k for $\mathcal{T}$.

While the network contains fake split nodes, there exists such a node x all of whose parents are true split nodes or roots. (Simply remove all roots, true split nodes, and leaves from H and choose x to be an in-degree-0 node of the resulting subgraph of H .) Since the color sets of x 's child edges are disjoint and x has two child edges, one of these edges, e , must have exactly one color: $C(e) = \{T\}$ for some $T \in \mathcal{T}$.² Let f be x 's parent edge that has color T . By the choice of x , the top endpoint y of f is a root or a true split node.

If y is a root (see Figure 5), we remove T from the color set of f , create a new root node r , change e 's top endpoint to r , and remove f and its top endpoint y if the color set of f is now empty and restore condition (vii). It is easily verified that this does not introduce any violations of conditions (i)–(vi) and does not increase the hybridization number of the network. It also does not increase the potential of any node and reduces the number of fake split nodes by one. The potential of the network therefore decreases.

If y is a true split node, we distinguish two cases. The first case is that y is a T' -split node for some $T' \neq T$. This case is illustrated in Figure 6. Figure 6(a) depicts the subcase when y is not also a T -split node, while Figure 6(b) depicts the subcase when y is also a T -split node. Both cases can be handled in a similar way.

Let g be the parent edge of y whose color set includes T , and let f' be f 's sibling edge. We divide g into two edges g_t and g_b with g_t above g_b and denote their common endpoint by z . We remove T from the color set of f , set $C(g_t) := C(g)$ and $C(g_b) := C(g) \setminus \{T\}$ if $T \notin C(f')$ and $C(g_b) := C(g)$ if $T \in C(f')$, change the top endpoint of e to z , remove all edges whose color sets are now empty (this can only be f and g_b), and finally restore condition (vii). Again, it is easily verified that this does not introduce any violations of conditions (i)–(vi) and does not increase the hybridization number of the network. It also does not increase the potential of any

²This is the only place in the entire paper where we use that $|\mathcal{T}| = 3$. All other arguments are easily seen to generalize to more than 3 trees. Alas, this argument is crucial because our algorithm does not work without condition (viii).

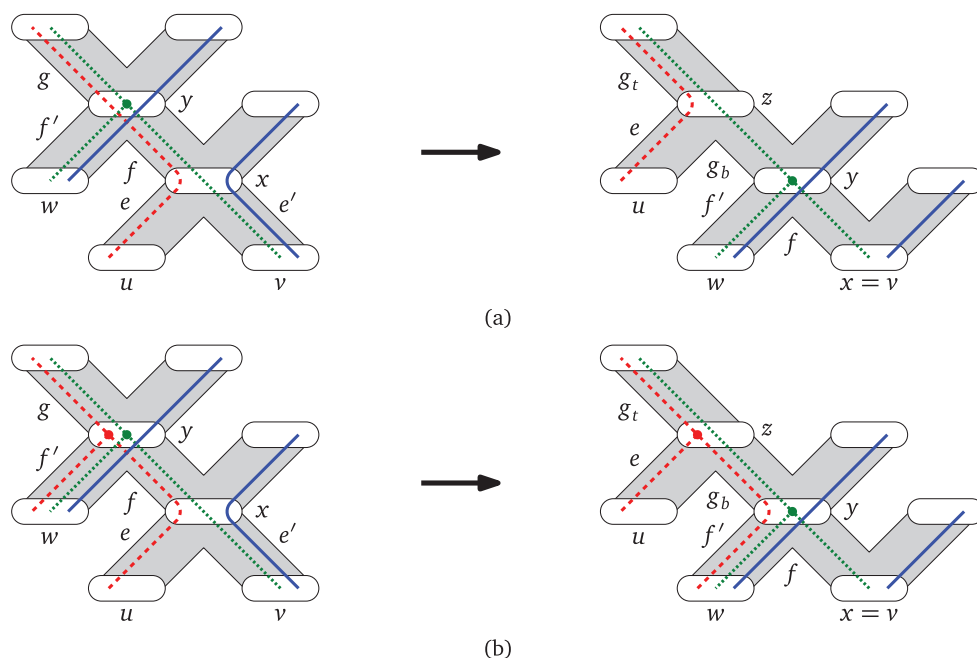


FIG. 6. Eliminating a fake split node x below a split node y that is a T' -split node with T' not equal to the color T (red, dashed) of the monochromatic edge below x . (a) y is not a T -split node. (b) y is a T -split node.

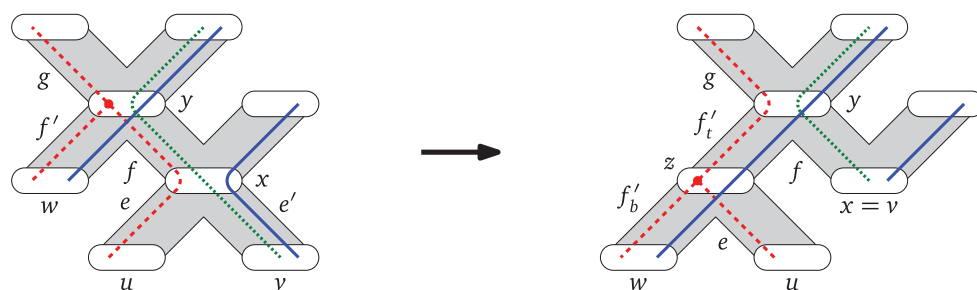


FIG. 7. Eliminating a fake split node x below a split node y that is not a T' -split node for any T' that is not equal to the color T (red, dashed) of the monochromatic edge below x .

node and eliminates x from the network (because x has only one child edge e' after changing e 's top endpoint and hence e' is being contracted). The only new node is z . If $T \in C(f')$, then z is a true split node and its contribution to the network's potential is 0. Thus, since x is eliminated from the network, the network's potential decreases. If $T \notin C(f')$, then z is a fake split node. However, its potential ϕ_z is less than ϕ_x because for every path from a root to z in the modified network, there exists a corresponding path from this root to x in the original network that has a greater weight because it contains the true split node y . Thus, once again, the potential of the network decreases.

Finally, if y is not a T' -split node for any $T' \neq T$ (see Figure 7), it must be a T -split node. As before, let g be the parent edge of y whose color set includes T , and let f' be f' 's sibling edge. We subdivide f' into f'_t and f'_b , where f'_t is above f'_b , and let z be the

newly created node. We change the top endpoint of e to z , remove T from the color set of f , remove all edges whose color sets are now empty, and restore condition (vii). This transformation maintains conditions (i)–(vi) and does not increase the hybridization number of the network. It eliminates x from the network (because it has only one child edge e' after changing the top endpoint of e) and makes y a fake split node. However, the potential of y in the modified network is less than the potential of x in the original network. To see this, let P be a path of maximum weight from a root to y . Then the potential of y in the modified network is one plus the weight of P . In the original network, the path P extended by the edge (y, x) is then a path from a root to x , and its weight is one higher than the weight of P because it contains the true split node y . Hence, the potential of x in the original network is at least one higher than the potential of y in the modified network. Since the potential of all other nodes remains the same or decreases, the potential of the network decreases.

Let $\mathcal{H} = (H', \mathcal{E}')$ be the TNET eventually obtained by the above transformations and let H be the original hybridization network. It remains to show that $\mathcal{H}$ and H have the same deletion AAF. To this end, observe that, if F is a component of the deletion AAF of H and x, y are two taxa in F , then a network edge e belongs to the path from x to y in the image $H(T)$ of some tree $T \in \mathcal{T}$ if and only if it belongs to the path from x to y in every image $H(T')$, $T' \in \mathcal{T}$. Now it suffices to verify that, in each of the transformations in Figures 5–7, none of the edges that are destroyed or created belongs to all three tree images, so each transformation leaves the deletion AAF unchanged. $\square$

By Lemma 3, it is sufficient if our algorithm can construct a TNET of the three input trees. In the next section, we show how to do this.

5. Reconstructing a tight network. Let $\mathcal{H} = (H, \mathcal{E})$ be a TNET for $\mathcal{T}$, let F be its deletion AAF, and let k be its hybridization number. For each tree $T \in \mathcal{T}$, let $I(T)$ be the set of nodes in T that do not belong to any path between two leaves x and y in the same AAF component (considering the root as a leaf). In a sense, these nodes are “invisible” in F . The *extended AAF* F^* of $\mathcal{T}$ is defined as $F \cup I$, where $I := \bigcup_{T \in \mathcal{T}} I(T)$. We will refer to the elements of F^* as components. Let C be a component of F^* . Hence, C is either an AAF component or a node in I . If C is an AAF component, then r_C denotes the root of C . If C is a node of I , then r_C is equal to the node C . In either case, we will refer to r_C as the root of C .

By the following lemma, the size of I is at most $3(k-1)$ if $\mathcal{T}$ contains at most three trees.

LEMMA 4. *For any $T \in \mathcal{T}$, $|I(T)| \leq k-1$.*

Proof. Let $T \in \mathcal{T}$. Since the root of T is a leaf after omitting directions, we can see T as an unrooted tree. Since F is an AAF of $\mathcal{T}$ and $T \in \mathcal{T}$, we know that F can be obtained by deleting a set E^* of k edges from T and then taking the partition of the leaves induced by the resulting connected components. Let $\mathcal{C}$ be the partition of the nodes of $I(T)$ induced by the connected components of the subgraph of T containing the vertices of $I(T)$ and all edges between them that are not in E^* . See Figure 8 for an example. For each $C \in \mathcal{C}$, let $\delta(C)$ denote the set of edges with exactly one endpoint in C . Then, at most one of the edges in $\delta(C)$ is not contained in E^* because otherwise at least one vertex of C would be on a path between two vertices in the same AAF component, which is not possible by the definition of $I(T)$. Hence, at least $|C|+1$ of the $|C|+2$ edges in $\delta(C)$ are in E^* . Moreover, since T is a tree, at

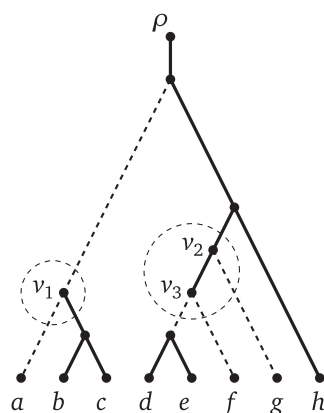


FIG. 8. Example for the proof of Lemma 4. The set E^* consists of the $k = 5$ dashed edges. The two elements of $\mathcal{C}$ are indicated by dashed circles.

most $|\mathcal{C}| - 1$ edges can be in $\delta(C) \cap \delta(C')$ for two different $C, C' \in \mathcal{C}$. Hence,

$$|E^*| \geq \sum_{C \in \mathcal{C}} (|C| + 1) - (|\mathcal{C}| - 1) = |I(T)| + 1.$$

Since $|E^*| = k$, the lemma follows. $\square$

We will construct $\mathcal{H}$ from F^* and $\mathcal{T}$ with the help of a *guess* of the structure of $\mathcal{H}$. In particular we construct $\mathcal{H}$ by “gluing together” the components of F^* . Our guess concerns how this gluing is to be done. Under the embedding of $\mathcal{T}$ in H described by $\mathcal{E}$, the root r_C of every component $C \in F^*$ has a unique image $H(r_C)$ in H : If $C \in F$, this is true because F is the deletion AAF of $\mathcal{H}$. If $r_C = C \in I(T)$, for some $T \in \mathcal{T}$, r_C is a split node of T and thus has a unique image $H(r_C)$ in $H(T)$ which is also a node of H . Our guess for r_C defines the “wiring” of the in-edges of $H(r_C)$; we call it the *wiring guess* for r_C . Since the color sets of the parent edges of every node in H are disjoint, $H(r_C)$ has between one and three parent edges. The first part of the wiring guess for r_C is the number of parent edges of $H(r_C)$. The second part of the wiring guess for r_C is which of these in-edges is included in which tree image. Finally, observe that the top endpoint x of each parent edge of $H(r_C)$ must once again be a T' -split node for at least one $T' \in \mathcal{T}$. The third part of the wiring guess for r_C determines such a tree T' for each parent edge of $H(r_C)$. We will assume without loss of generality that any two nodes $r_C, r_{C'}$ for which $H(r_C)$ and $H(r_{C'})$ have a common parent x both guess the same tree T' as the tree for which x is a T' -split node.

First consider a component root r_C that is a node in $I(T)$ for some $T \in \mathcal{T}$. Note that (i) at least one parent edge of $H(r_C)$ must have color T because $H(r_C)$ is a nonroot node of $H(T)$, and (ii) the top endpoint of a parent edge e of $H(r_C)$ can be a T' -split node only for trees T' such that $T' \in C(e)$. This gives the 17 possible wiring guesses for r_C shown in Figure 9.

If r_C is an AAF root, the set of possible wiring guesses is more restricted. Since $H(r_C)$ is contained in every tree image, the only valid wiring guesses are the ones where the union of the color sets of the parent edges of $H(r_C)$ contains all three colors. This reduces the number of possible wiring guesses for AAF roots to 10 (see again Figure 9). Finally note that, when r_C is the root ρ of the trees, there is only a single wiring guess.

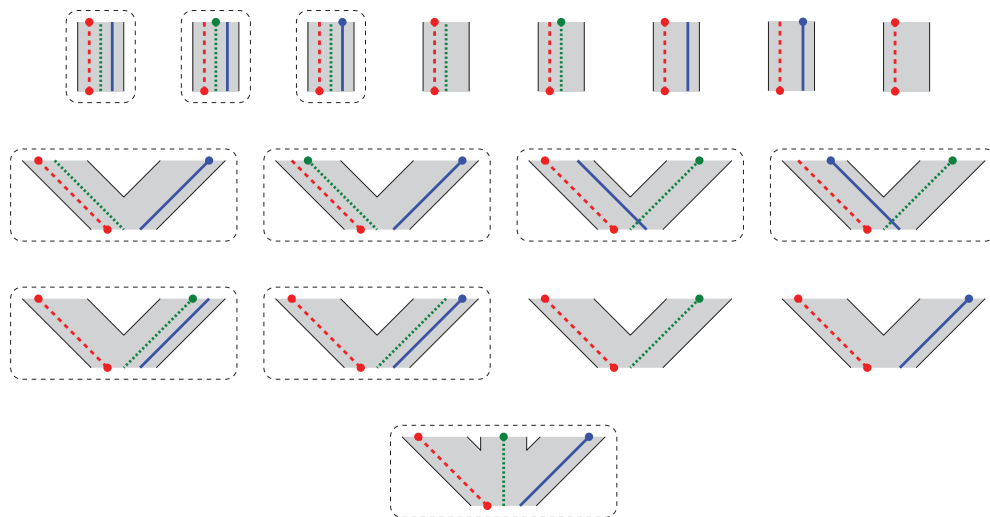


FIG. 9. The 17 possible wiring guesses for the root r_C of a component C of the extended AAF F^* that is a node in $I(T)$, with T the red dashed tree. The guess of the split node at the top of each edge is indicated by a dot. Valid guesses for a root r_C of an AAF component are indicated by dashed boxes around them.

Our guess $\mathcal{G}$ of the structure of $\mathcal{H}$ consists of the wiring guesses for all roots r_C , $C \in F^*$.

Since we have 17 wiring guesses to choose from for each component in I and 10 wiring guesses to choose from for each component in F that does not contain the tree root ρ , there are $10^{|F|-1} \cdot 17^{|I|}$ possible guesses $\mathcal{G}$. Since $|F| \leq k+1$ and $|I| \leq 3(k-1)$ by Lemma 4, the number of possible guesses $\mathcal{G}$ is bounded by $10^k \cdot 17^{3(k-1)} = 49130^k/4913$.

Our algorithm considers each guess in $\mathcal{G}$ in turn and attempts to construct a TNET $\mathcal{H}$ from $(\mathcal{G}, F^*, \mathcal{T})$ in polynomial time. We call $(\mathcal{G}, F^*, \mathcal{T})$ the TNET's *description*. To establish our algorithm's correctness, we prove that, given the description $(\mathcal{G}, F^*, \mathcal{T})$ of a TNET $\mathcal{H}$ for $\mathcal{T}$, our algorithm succeeds in constructing a TNET $\mathcal{H}'$ for $\mathcal{T}$ with description $(\mathcal{G}, F^*, \mathcal{T})$ that differs from $\mathcal{H}$ only in insignificant details and in particular has the same hybridization number as $\mathcal{H}$. Our proof is divided into two lemmas. The first one shows that two TNETs with the same description are essentially the same. We make this precise below. The second one shows that, given the description of a TNET, we can construct a TNET with this description in polynomial time. Note that not every description $(\mathcal{G}, F^*, \mathcal{T})$ is necessarily a valid description of any TNET. If there is no TNET with the given description, then our algorithm will determine this in polynomial time. The description of this algorithm is given in the proofs of Lemmas 5 and 6 below. Appendix A provides an example of the operation of the algorithm.

Given a TNET $\mathcal{H} = (H, \mathcal{E})$ for $\mathcal{T}$, we obtain a DAG $\tilde{H}$ by contracting the image of every component of the deletion AAF F of H into a single node, keeping parallel edges this creates. We call $\tilde{H}$ the *signature* of H . For an example, see Figures 13 and 14 in the appendix. It is easy to see that the node set of $\tilde{H}$ is $\{H(r_C) \mid C \in F^*\}$ and that $\tilde{H}$ has the same hybridization number as H . Note that two nodes $C \in I(T)$, $C' \in I(T')$ might map to the same node $H(r_C) = H(r'_C)$ of $\tilde{H}$. We label every node

x of $\tilde{H}$ with the set of roots r_C , $C \in F^*$, such that $x = H(r_C)$. We call roots that label the same node of $\tilde{H}$ *buddies* (of each other). Note that roots of components of F have no buddies.

We will use the following notion of the “attached subtrees” of an AAF component C in a tree T . For each edge (u, v) of T for which u is contained in $T(C)$ but v is not contained in $T(C)$, we say that the subtree of T induced by u, v and all nodes reachable from v is a subtree of T attached to C .

In addition, we will use the following notion of the “pendant subtrees” represented by an edge $e = (H(r_C), H(r_{C'}))$ of the signature $\tilde{H}$. For each $T \in C(e)$, the subtree of T induced by $r_{C'}$, the parent of r_C , and all nodes reachable from $r_{C'}$ is said to be a *pendant subtree* of T represented by the edge e of $\tilde{H}$.

LEMMA 5. *The signature $\tilde{H}$ of a TNET $\mathcal{H} = (H, \mathcal{E})$ can be uniquely reconstructed in polynomial time from the description $(\mathcal{G}, F^*, \mathcal{T})$ of $\mathcal{H}$.*

Proof. We define a DAG D whose nodes are the roots of components of F^* . There is an edge from a root r_C to a root $r_{C'}$ if and only if there exists a tree $T \in \mathcal{T}$ such that $r_{C'}$ is an ancestor of r_C in T and there exists no component root $r_{C''}$ that is an ancestor of r_C and a descendant of $r_{C'}$ in T . If this is the case, we call r_C a *direct descendant* of $r_{C'}$ in T . (Note that the edge in D is directed from the descendant to the ancestor.) We incrementally construct a topological ordering of D , define U_i to be the first i nodes in this topological ordering, define D_i to be the subgraph of D induced by U_i , and set $V_i := \{H(r_C) \mid r_C \in U_i\}$. $\bar{D}_i$ is the subgraph of D induced by all nodes of D not in U_i . Let the partial signature $\tilde{H}_i$ be the graph obtained as follows from the subgraph of $\tilde{H}$ induced by V_i : For each parent edge e in $\tilde{H}$ of a node $H(r_C) \in V_i$ whose top endpoint does not belong to V_i , we adding a new root and an edge $\tilde{e}$ from this new root to $H(r_C)$ and give $\tilde{e}$ the same color set as e (which is determined by the wiring guess for r_C). For an example, see Figure 13 in the appendix. We call an edge whose top endpoint is a root a *root edge*.

We assign these new roots colors, where the color of the top endpoint of edge $\tilde{e}$ is T if the bottom endpoint of $\tilde{e}$ guesses that the top endpoint of e is a T -split node.

We will show that, from $D_i, \tilde{H}_i$, and $\mathcal{G}$, we can determine a unique nonempty set of nodes U' of $\bar{D}_i$, the set U^+ of all buddies of nodes in U' and an extended partial signature $\tilde{H}_j$ corresponding to the graph D_j induced by $U_j := U_i \cup U' \cup U^+$.

Once $D_i = D$, we thus obtain $\tilde{H}_i = \tilde{H}$, that is, we are able to reconstruct the signature $\tilde{H}$ only given the description of $\mathcal{H}$.

We initialize the reconstruction by defining D_0 and $\tilde{H}_0$ to be empty digraphs. Now consider an iteration with input digraph D_i and partial signature $\tilde{H}_i$. For an in-degree-0 node r_C of $\bar{D}_i$ that belongs to $I(T)$ for some $T \in \mathcal{T}$, observe that both child edges of $H(r_C)$ in $\tilde{H}$ have corresponding root edges in $\tilde{H}_i$ (by the construction of $\tilde{H}_i$). For such a node r_C , we call r_C *free* if the top endpoints of both of these root edges are colored T . For an in-degree-0 node r_C of $\bar{D}_i$ that is a root of a component of F , we say that r_C is free if, for every in-edge e of r_C in D , the corresponding root edge $\tilde{e}$ of $\tilde{H}_i$ has the property that, for every $T \in C(\tilde{e})$, the pendant subtree of T represented by $\tilde{e}$ is attached to the AAF component C in T . All other nodes of $\bar{D}_i$ are nonfree. We claim that at least one of the nodes in $\bar{D}_i$ is free and that every free node can determine its buddies in $\tilde{H}$ and can augment the partial signature $\tilde{H}_i$ to $\tilde{H}_j$.

Consider a node x of the signature $\tilde{H}$ that does not belong to $\tilde{H}_i$ and all of whose children do belong to $\tilde{H}_i$. Since $\tilde{H}$ is a DAG, such a node exists. Node x is a T -split node for some $T \in \mathcal{T}$ and is thus the image $H(r_C)$ of the root r_C of a component in $F \cup I(T)$.

First consider the case that $C \in F$. Observe that, because $\tilde{H}$ is the signature of a TNET $\mathcal{H}$ for $\mathcal{T}$, the subtrees attached to C in all input trees are exactly the pendant subtrees represented by the child edges of $H(r_C)$ in $\tilde{H}$. Since these child edges are root edges of $\tilde{H}_i$, r_C is free. Node r_C can trivially identify its set of buddies because it has no buddies besides itself. We obtain D_j by adding r_C to D_i and obtain $\tilde{H}_j$ from $\tilde{H}_i$ by locating the root edges of $\tilde{H}_i$ that correspond to child edges of $H(r_C)$ in $\tilde{H}$, merging the top endpoints of these root edges to a single node $H(r_C)$ and adding new root edges entering $H(r_C)$ according to r_C 's wiring guess. This gives a new graph $D_j \supset D_i$ and the corresponding graph $\tilde{H}_j$.

Now consider the case that $r_C \in I(T)$. In this case, both child edges of r_C in $\tilde{H}$ correspond to root edges of $\tilde{H}_i$. Since these two edges have a common top endpoint in $\tilde{H}$, their top endpoints in $\tilde{H}_i$ must have the same color T' . This implies that $H(r_C)$ is a T' -split node and is thus the image $H(r_{C'})$ of a node $r_{C'} \in I(T')$. Since the child edges of $H(r_{C'})$ in $\tilde{H}$ belong to $\tilde{H}_i$, both in-neighbors of $r_{C'}$ in D belong to D_i . Thus, $r_{C'}$ is free. Since $H(r_C) = H(r_{C'})$, we may assume that $r_{C'} = r_C$ (because in at least one set of guesses this is the case). We obtain $\tilde{H}_j$ from $\tilde{H}_i$ by locating the two root edges of $\tilde{H}_i$ that correspond to child edges of $H(r_C)$ in $\tilde{H}$, merging the top endpoints of these edges to a single node $H(r_C)$, and adding new root edges entering $H(r_C)$ according to r_C 's wiring guess. Now consider the child edges of $H(r_C)$ in $\tilde{H}_j$. For each tree $T' \in \mathcal{T}$ such that these edges are both colored T' , $H(r_C)$ is the image $H(r_{C'})$ of a node in $I(T')$. The nodes of $\bar{D}_i$ that satisfy this condition are the buddies of r_C , and we add them to D_i along with r_C to obtain D_j .

We have shown that every node $H(r_C)$ of $\tilde{H}$ whose children belong to $\tilde{H}_i$ corresponds to a free node r_C in $\bar{D}_i$ and that r_C can determine its set of buddies and can construct the extended partial signature $\tilde{H}_j$ corresponding to the digraph $D_j \supset D_i$ obtained by adding these buddies to D_i . If r_C is an AAF root, it has no buddies besides itself. If $r_C \in I(T)$ for some $T \in \mathcal{T}$, observe that none of its buddies is free because it belongs to a tree $T' \neq T$ but the top endpoints of the child edges of $H(r_C)$ have color T in $\tilde{H}_i$. Thus, to prove that *every* free node can determine its set of buddies and can construct $\tilde{H}_j$ from $\tilde{H}_i$, it suffices to show that there is no free node r_C such that $H(r_C)$ has a child not in $\tilde{H}_i$.

Assume there exists such a node r_C . Then r_C has in-degree 0 in $\bar{D}_i$ because otherwise it is not free. First assume r_C is the root of a component in F . Let $H(r_{C'})$ be an in-neighbor of $H(r_C)$ in $\tilde{H}$ that does not belong to $\tilde{H}_i$, and let T be an arbitrary color T in the color set of the edge e between $H(r_C)$ and $H(r_{C'})$ in $\tilde{H}$. Since r_C has in-degree 0 in $\bar{D}_i$, the pendant subtree of T represented by e is also represented by some root edge f of $\tilde{H}_i$. Thus, $\tilde{H}$ contains a unique path from e to f and every node on this path is a T' -split node for some $T' \neq T$. This implies in particular that the top endpoint of edge f is a T' -split node, $T, T' \in C(f)$, and the pendant subtree of T' represented by f is not a subtree attached to C in T' . Thus, since f represents the same in-edge of r_C in D as e , r_C is not free, a contradiction.

If $r_C \in I(T)$, for some tree $T \in \mathcal{T}$, we choose an in-neighbor $H(r_{C'})$ and a root edge f of $\tilde{H}_i$ as in the case when $r_C \in F$. Since the top endpoint of edge f is not a T -split node, its color in $\tilde{H}_i$ must be $T' \neq T$. Thus, since f is the root edge of $\tilde{H}_i$ representing one of the in-edges of r_C in D , r_C is not free, again a contradiction. $\square$

By Lemma 5, it suffices to provide a polynomial-time algorithm that decides whether there exists a TNET with description $(\mathcal{G}, F^*, \mathcal{T})$ and, if so, construct any such TNET. Our next lemma states that such an algorithm exists.

LEMMA 6. *Given a description $(\mathcal{G}, F^*, \mathcal{T})$, it takes polynomial time to decide whether there exists a TNET with this description and, if so, to construct such a TNET.*

Proof. By Lemma 5, there exists a polynomial-time algorithm for constructing the signature $\tilde{H}$ of a TNET $\mathcal{H} = (H, \mathcal{E})$ with description $(\mathcal{G}, F^*, \mathcal{T})$, if such a TNET exists. This algorithm provides a first test that can be used to reject invalid descriptions: If the algorithm reaches an iteration where $\bar{D}_i$ is nonempty but none of its nodes is free, then the description is invalid because the proof of Lemma 5 shows that, if $(\mathcal{G}, F^*, \mathcal{T})$ is the description of a TNET, then there exists a free node in each iteration. Thus, the algorithm aborts and rejects the description. If the algorithm does not reject the description, its output is a signature $\tilde{H}$ that respects all wiring guesses in $\mathcal{G}$. However, this signature may not correspond to a network H that displays all input trees. Next we present a polynomial-time algorithm for constructing $\mathcal{H} = (H, \mathcal{E})$ from $(\tilde{H}, F^*, \mathcal{T})$ or determining that no TNET $\mathcal{H} = (H, \mathcal{E})$ with description $(\mathcal{G}, F^*, \mathcal{T})$ exists.

The nodes of $\tilde{H}$ are of two types: images of AAF roots and images of nodes in I . To obtain a TNET $\mathcal{H} = (H, \mathcal{E})$ with description $(\mathcal{G}, F^*, \mathcal{T})$ from $\tilde{H}$, we let H initially be equal to $\tilde{H}$ and replace each AAF root image with the AAF component it represents. We process these AAF root images bottom-up, that is, in reverse topological order.³ Consider such an image $H(r_C)$ of the root r_C of an AAF component C , and let E be the set of child edges of $H(r_C)$ in $\tilde{H}$. We remove the edges in E from $\tilde{H}$ and attach C below r_C , setting $C(e) = \mathcal{T}$ for every edge e of C . Our goal now is to reattach the edges in E to edges of C so that, for all $T \in \mathcal{T}$, the descendant edges of $H(r_C)$ with color T form an image of the pendant subtree of T with root r_C . For each tree T , observe that the edges in E colored T represent the subtrees attached to C in T . We need to attach these edges to C in H so that each edge branches off the same edge of C as in T and edges that branch off the same edge of C in T do so in the same order as in T .

First we test every edge $e \in E$ whether e branches off the same edge of C in every tree $T \in C(e)$. If this is the case for all edges $e \in E$, then we can partition E into subsets E_f , one per edge f of C , such that all edges in E_f branch off edge f . If there is an edge e that branches off some edge f in a tree $T \in C(e)$ and off a different edge f' in a tree $T' \in C(e)$, it is impossible to attach this edge to C in a way that satisfies both constraints. Since the edges of T and T' represented by e are determined by $\tilde{H}$, which in turn is uniquely defined by $(\mathcal{G}, F^*, \mathcal{T})$, there is thus no network $\mathcal{H}$ with description $(\mathcal{G}, F^*, \mathcal{T})$, so the algorithm aborts and reports that there is no such network.

Given the partition of E into subsets E_f such that the edges in E_f branch off edge f , we need to attach the edges in each such set E_f in an ordering consistent with the input trees. Let $E_{f,T}$ be the subset of edges in E_f that have color T . Tree T determines the ordering in which these edges need to be attached to f . We define a DAG D_f whose nodes represent the edges in E_f and which has an edge (g, g') precisely if there is a tree T such that $g, g' \in E_{f,T}$ and g branches off f in T above g' . There exists an ordering in which to attach the edges of E_f to f so that the ordering constraints imposed by all trees in $\mathcal{T}$ are satisfied if and only if D_f is acyclic. Moreover, if D_f is indeed acyclic, then a topological ordering of D_f provides a valid ordering in which the edges can be attached. Thus, we can test whether such an ordering exists and, if so, compute such an ordering in time $O(|E_f|)$ per edge f . If no such ordering exists,

³This is not really essential, but it simplifies the description of the algorithm.

the algorithm once again aborts and reports that there is no TNET with description $(\mathcal{G}, F^*, \mathcal{T})$. If we can find a correct ordering of the edges attached to each edge f of C , then the replacement of r_C with C in this manner results in a network where all descendant edges of $H(r_C)$ with color T form an image of the pendant subtree of T with root r_C for all $T \in \mathcal{T}$. Thus, after replacing each AAF root r_C with its corresponding component C in this fashion, we obtain a TNET $\mathcal{H} = (H, \mathcal{E})$ with description $(\mathcal{G}, F^*, \mathcal{T})$. $\square$

To summarize, the overall strategy for trying to reconstruct a hybridization network H with hybridization number k for an input set $\mathcal{T}$ of three trees is as follows. First, we guess the deletion AAF F of H in time $O(2^{5k} \cdot 2^{10k} \cdot \text{poly}(n))$. Then we identify the set I of invisible nodes and add them to F , obtaining the extended AAF F^* , and guess $\mathcal{G}$, the wiring of each component of F^* . The total number of possible guesses for $\mathcal{G}$ is $10^k \cdot 17^{3(k-1)} = 49130^k/4913$. For each possible description $(\mathcal{G}, F^*, \mathcal{T})$, we try to construct the signature $\tilde{H}$ of a TNET with this description using Lemma 5 in polynomial time. For each signature $\tilde{H}$, we decide whether there exists a TNET with this description (see Lemma 6), again in polynomial time. Once a correct TNET has been found, it can be expanded to the hybridization network H (see section 4). The overall running time is $O(2^{5k} \cdot 2^{10k} \cdot 10^k \cdot 17^{3(k-1)} \cdot \text{poly}(n))$, which is $O(1,609,891,840^k \cdot \text{poly}(n))$.

6. Conclusions. For two trees, a hybridization network can easily be constructed in polynomial time once the AAF is known. No guessing is required since the AAF carries all the necessary information. For more than two trees, it seemed natural enough to try to guess the wiring structure that determines how the AAF components need to be glued together into a network. For any constant number of trees, there are only a constant number of choices for the wiring of the root of each component, so with $O(k)$ components, one would obtain an $O(c^k \cdot \text{poly}(n))$ -time algorithm. Unfortunately, guessing the wiring structure of AAF components turned out not to be enough, even for three trees, because there are examples of three input trees such that every optimal network displaying these trees contains an *invisible component*: a group of nodes that are isolated from all taxa once all hybridization edges are deleted; see Figure 1 in the introduction. We call these components invisible because they are not represented in any form in the AAF.

Guessing the number and structure of these invisible components seems extremely challenging. In this paper, we showed that one can get away without having to guess these components in the case of three trees because, for three trees, these components are not invisible after all: They may not be represented in the AAF, but they are present as nodes in the three input trees, at least as long as we consider only tight networks (and we have shown that we may do this without loss of generality). This is the key to our $O(c^k \cdot \text{poly}(n))$ -time algorithm for three trees. Unfortunately, it appears that we simply scraped by. While the framework of our algorithm extends to more than three trees, it seems that already for four trees, there are input instances where the optimal network includes truly invisible nodes: nodes whose only purpose is to change the way in which edges of the tree images are braided together along network edges; see Figure 10. Thus, the main open problem is to discover structural properties that, while unlikely to eliminate the need to guess the existence of these braiding structures in the network altogether, at least limit the number of possible guesses to be explored.

Another interesting question that arises from our work is whether guessing the wiring of the extended AAF components as part of the reconstruction is necessary at

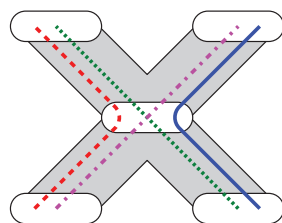


FIG. 10. A truly invisible node of a network with four trees embedded in it. None of the four trees branches in this network node.

all, at least in the case of three trees. Since all these components are visible in the input trees, it could be possible that one can construct the entire network directly from the AAF, that is, as in the case of two trees, the hard core of the problem is finding the right AAF. However, we conjecture that this is not the case: i.e., that even given the deletion AAF of an optimal hybridization network for the three input trees, it remains NP-hard to find the network.

Appendix A. An example of constructing a network from its description. This appendix provides an example of the construction described in the proofs of Lemmas 5 and 6. Consider the description $(\mathcal{G}, F^*, \mathcal{T})$ in Figure 11. The TNET in Figure 14 has this description. We first show how to construct its signature using the construction in the proof of Lemma 5. This signature is shown in Figure 13. Then we discuss how to construct the TNET in Figure 14 from this signature. The hybridization network induced by this TNET is shown in Figure 15.

Lemma 5: Constructing the signature. The DAG D used in the construction of the signature $\tilde{H}$ of any TNET with the description in Figure 11 is shown in Figure 12. This DAG represents the adjacency of components of F^* in the three input trees in $\mathcal{T}$.

Before the first iteration, we have $\bar{D}_0 = D$. The only nodes with in-degree 0 in $\bar{D}_0$ are b , c , and d . They are all free because they have no in-edges at all. Assume we pick b as the first node to add to $\tilde{H}$. We create the node $H(b)$ and add parent edges according to b 's wiring guess to obtain the partial network $\tilde{H}_1$ in Figure 13. Since b is the root of an AAF component, it has no buddies, so D_1 has the node set $\{b\}$.

$\bar{D}_1$ has two nodes of in-degree 0, namely, c and d . Both are again free. Assume we choose c as the next node to add to D_1 to obtain D_2 . Since c is again the root of an AAF component, it has no buddies, so the node set of D_2 is $\{b, c\}$. We construct the graph $\tilde{H}_2$ in Figure 14 from $\tilde{H}_1$ by creating a node $H(c)$ and adding parent edges of this node according to c 's wiring guess.

$\bar{D}_2$ has two nodes of in-degree 0, namely, d and v_3 . Both nodes are free: d is free because it is the root of an AAF component; v_3 is free because $\tilde{H}_2$ has two root edges above $H(b)$ and $H(c)$, which are children of v_3 in the green dotted tree, and the top endpoints of both edges are colored green by the wiring guesses for b and c . Let us assume we choose v_3 as the next node to add to D_2 to obtain D_3 . We create the node $H(v_3)$ by identifying the top endpoints of the two green dotted parent edges of $H(b)$ and $H(c)$ and add a parent edge above $H(v_3)$ according to v_3 's wiring guess. This gives the graph $\tilde{H}_3$ shown in Figure 13. Since the only tree common to the color sets of the parent edges of $H(b)$ and $H(c)$ is the green dotted one, v_3 has no buddies. Thus, D_3 has the node set $\{b, c, v_3\}$.

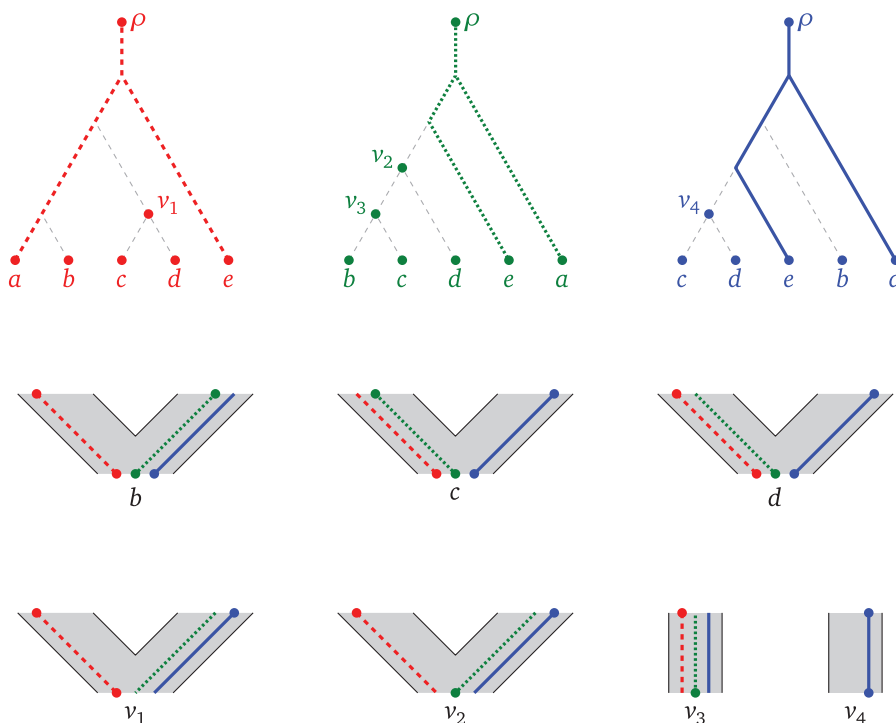


FIG. 11. The input to the network reconstruction, including the AAF F (shown in bold in the three input trees), the set of invisible nodes $I = \{v_1, \dots, v_4\}$, and the wiring guesses for the resulting set of components of F^* . There is no guess for the component $\{a, e, \rho\}$ because it includes the root ρ of the three trees.

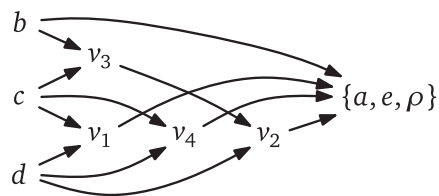


FIG. 12. The DAG D used in the reconstruction of the signature $\tilde{H}$ in Figure 13 from the description in Figure 11.

$\bar{D}_3$ has d as its only in-degree-0 node, and d is free. We obtain D_4 by adding d to D_3 . Node d has no buddies because it is the root of an AAF component. To obtain $\tilde{H}_4$ from $\tilde{H}_3$, we create the node $H(d)$ and add parent edges according to d 's wiring guess.

$\bar{D}_4$ has three nodes of in-degree 0, namely, v_1, v_2 , and v_4 . The two child edges of v_1 are represented by the red dashed parent edges of $H(v_3)$ and $H(d)$ in $\tilde{H}_4$. According to the wiring guesses for v_3 and d , their top endpoints are red. Since v_1 belongs to the red dashed tree, v_1 is free. The same two edges also represent the child edges of v_2 . However, v_2 is green and thus is not free. Finally, the two child edges of v_4 are represented by the blue solid parent edges of $H(c)$ and $H(d)$, both of which have blue top endpoints according to the wiring guesses for c and d . Thus, v_4 is also free.

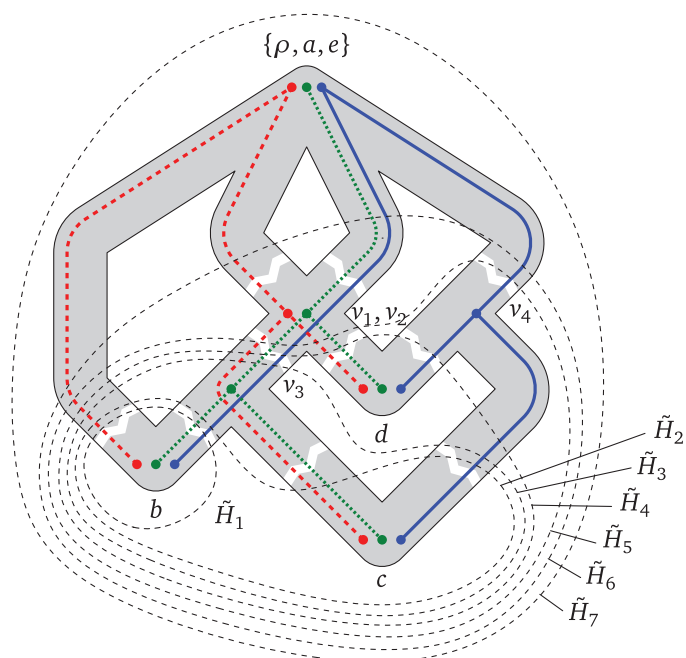


FIG. 13. The signature $\tilde{H}$ of any TNET with the description $(\mathcal{G}, F^*, \mathcal{T})$ shown in Figure 11. The partial signatures $\tilde{H}_1, \tilde{H}_2, \dots, \tilde{H}_7 = \tilde{H}$ constructed incrementally are indicated by dashed lines (edges that are partly in the indicated region are also contained in the partial signature).

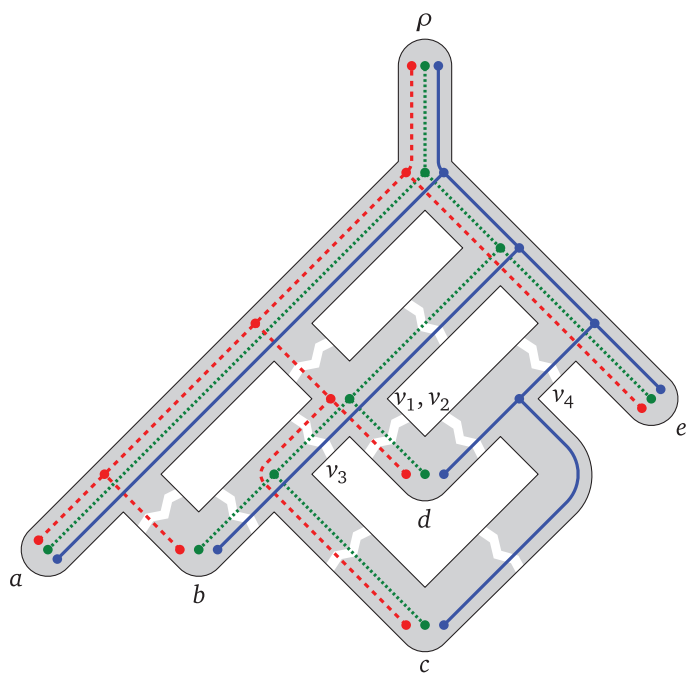


FIG. 14. The TNET obtained from the signature in Figure 13 by expanding the components of F .

Assume we choose v_4 as the next node to add to D_4 to obtain D_5 . We create the node $H(v_4)$ in $\tilde{H}_5$ by identifying the top endpoints of the blue solid parent edges of $H(c)$ and $H(d)$ and then create a parent edge of $H(v_4)$ according to v_4 's wiring guess. This produces the graph $\tilde{H}_5$ in Figure 13. Since the color sets of the two child edges of $H(v_4)$ have only the blue solid tree in common, v_4 has no buddies and D_5 has node set $\{b, c, d, v_3, v_4\}$.

Nodes v_1 and v_2 are the only nodes of in-degree 0 in $\bar{D}_5$. Just as in $\bar{D}_4$, v_1 is free and v_2 is not. Thus, we choose v_1 as the node to add to D_5 to obtain D_6 . The two child edges of v_1 are represented by the red dashed parent edges of $H(v_3)$ and $H(d)$ in $\tilde{H}_5$. We identify their top endpoints to create the node $H(v_1)$ and add parent edges according to the wiring guess for v_1 . This produces the graph $\tilde{H}_6$ in Figure 13. Now observe that the two child edges of $H(v_1)$ in $\tilde{H}_6$ are also colored green (dotted). Thus, we identify the node that is the parent of the two edges of the green dotted tree represented by these child edges, which is node v_2 . Node v_2 becomes a buddy of v_1 and is added to D_5 along with v_1 to obtain D_6 . Thus, D_6 has the node set $\{b, c, d, v_1, v_2, v_3, v_4\}$. Note that making v_1 and v_2 buddies does not create any conflicts because they both have the same wiring guess in $\mathcal{G}$.

Finally, the only node remaining in $\bar{D}_6$ is $\{a, e, \rho\}$. The root edges of $\tilde{H}_6$ represent exactly the set of pendant edges of the AAF component $\{a, e, \rho\}$ in the three input trees, so $\{a, e, \rho\}$ is free in $\bar{D}_6$. We create a node $H(\{a, e, \rho\})$ in $\tilde{H}_7$ by identifying the top endpoints of all root edges of $\tilde{H}_6$. Since there is no wiring guess for $\{a, e, \rho\}$ in $\mathcal{G}$, we do not add any parent edges to $H(\{a, e, \rho\})$, and $\tilde{H}_7 = \tilde{H}$ is the final signature.

It is easily verified that we would have obtained the exact same signature had we chosen different nodes to add to D_i in iterations where $\bar{D}_i$ contained more than one free node.

Lemma 6: Expanding AAF components. In our example, the only nontrivial AAF component to be expanded is the component $\{a, e, \rho\}$. This component has two non-root edges. Let f_a be the parent edge of a , and let f_e be the parent edge of e in this component. In $\tilde{H}$, the node $H(\{a, e, \rho\})$ has four child edges: a red dashed parent edge e_1 of $H(b)$, a red dashed parent edge e_2 of $H(v_1) = H(v_2)$, a green-blue (dotted-solid) parent edge e_3 of $H(v_1) = H(v_2)$, and a blue solid parent edge e_4 of $H(v_4)$. Edges e_1 and e_2 attach to f_a in the red dashed tree and do not represent any edges in any other trees, so we add them to E_{f_a} . Edge e_3 attaches to f_e in the green dotted and the blue solid trees, so there is no conflict and we add it to E_{f_e} . Edge e_4 attaches to f_e in the blue solid tree and does not represent any edge in any other tree, so we add it to E_{f_e} .

The DAG D_{f_a} has two nodes representing edges e_1 and e_2 with an edge from e_2 to e_1 because e_2 attaches to f_a above e_1 in the red dashed tree. A topological ordering of D_{f_a} places these edges in the order $\langle e_2, e_1 \rangle$, and this is the order in which we attach e_2 and e_1 to f_a .

The DAG D_{f_e} has two nodes representing edges e_3 and e_4 with an edge from e_3 to e_4 because e_3 attaches to f_e above e_4 in the blue solid tree. The green dotted tree does not impose any conflicting ordering constraints because only edge e_3 belongs to this tree. A topological ordering of D_{f_e} places e_3 and e_4 in the order $\langle e_3, e_4 \rangle$, and this is the order in which we attach e_3 and e_4 to f_e . The result is the TNET shown in Figure 14.

Finally, the hybridization network induced by this TNET is shown in Figure 15.

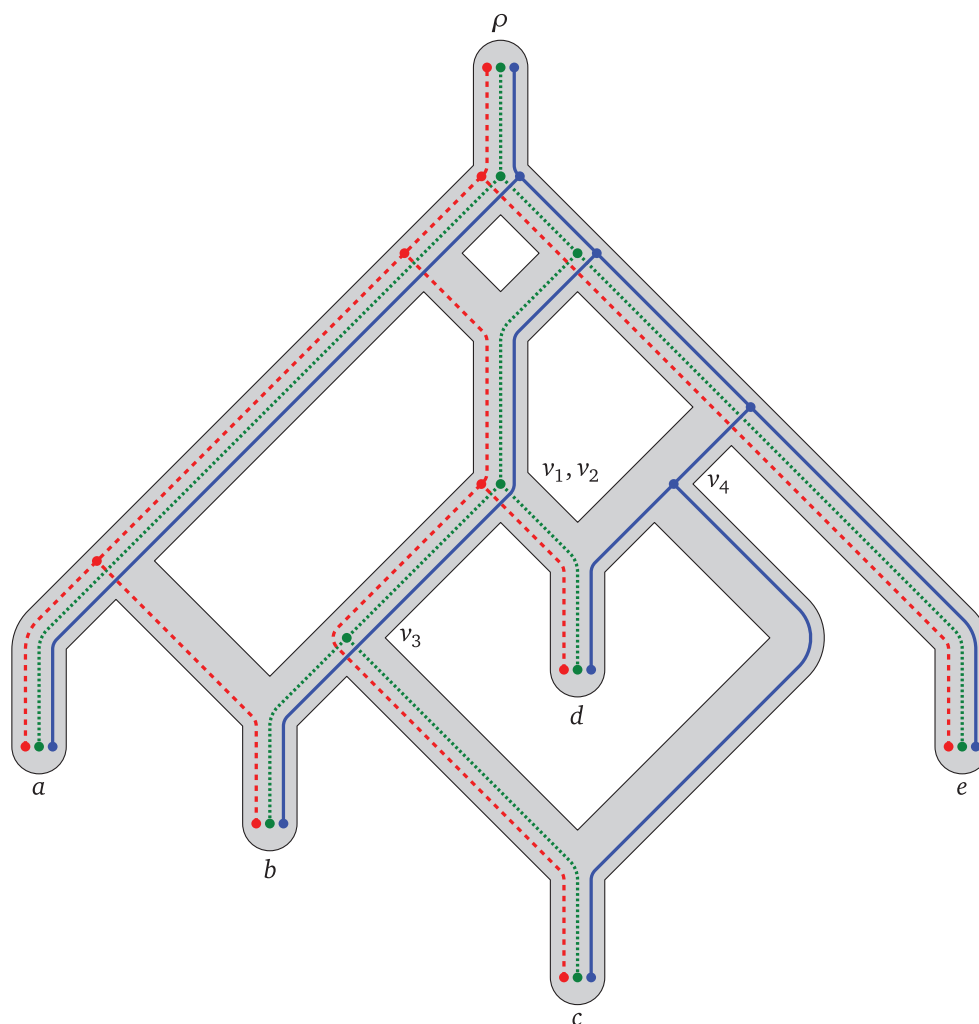


FIG. 15. The hybridization network induced by the TNET in Figure 14, obtained by separating reticulations, split nodes, and leaves.

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