# Bounds on equiangular lines and on related spherical codes 

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#### Abstract

An $L$-spherical code is a set of Euclidean unit vectors whose pairwise inner products belong to the set $L$. We show, for a fixed $0<\alpha, \beta<1$, that the size of any $[-1,-\beta] \cup\{\alpha\}$-spherical code is at most linear in the dimension.

In particular, this bound applies to sets of lines such that every two are at a fixed angle to each another.


## 1 Introduction

Background A set of lines in $\mathbb{R}^{d}$ is called equiangular, if the angle between any two of them is the same. Equivalently, if $P$ is the set of unit direction vectors, the corresponding lines are equiangular with the angle $\arccos \alpha$ if $\left\langle v, v^{\prime}\right\rangle \in\{-\alpha, \alpha\}$ for any two distinct vectors $v, v^{\prime} \in P$. The second equivalent way of defining equiangular lines is via the Gram matrix. Let $M$ be the matrix whose columns are the direction vectors. Then $M^{T} M$ is a positive semidefinite matrix whose diagonal entries are 1's, and each of whose off-diagonal entries is $-\alpha$ or $\alpha$. Conversely, any such matrix of size $m$ and rank $d$ gives rise to $m$ equiangular lines in $\mathbb{R}^{d}$.

Equiangular lines have been extensively studied following the works of van Lint and Seidel [10], and of Lemmens and Seidel [8]. Let $N(d)$ be the maximum number of equiangular lines in $\mathbb{R}^{d}$. Let $N_{\alpha}(d)$ be the maximum number of equiangular lines with the angle arccos $\alpha$. The values of $N(d)$ are known exactly for $d \leq 13$, for $d=15$, for $21 \leq d \leq 41$ and for $d=43[1,5]$. When $d$ is large, the only known upper bound on $N(d)$ is due to Gerzon (see [8, Theorem 3.5]) and asserts that
$N(d) \leq d(d+1) / 2$ with equality only if $d=2,3$ or $d+2$ is a square of an odd integer.
A remarkable construction of de Caen[3] shows that $N(d) \geq \frac{2}{9}(d+1)^{2}$ for $d$ of the form $d=6 \cdot 4^{i}-1$. A version of de Caen's construction suitable for other values of $d$ has been given by Greaves, Koolen, Munemasa and Szöllösi [5]. See also the work of Jedwab and Wiebe [6] for an alternative construction of $\Theta\left(d^{2}\right)$ equiangular lines. In these constructions the inner product $\alpha$ tends to 0 as dimension grows.

Previously known bounds on $N_{\alpha}(d)$ The first bound is the so-called relative bound (see [10, Lemma 6.1] following [8, Theorem 3.6])

$$
N_{\alpha}(d) \leq d \frac{1-\alpha^{2}}{1-d \alpha^{2}} \quad \text { if } d<1 / \alpha^{2}
$$

[^0]While useful in small dimensions, it gives no information for a fixed $\alpha$ and large $d$. The second bound is

$$
N_{\alpha}(d) \leq 2 d \quad \text { unless } 1 / \alpha \text { is an odd integer [8, Theorem 3.4]. }
$$

This bound can be further improved to $\frac{3}{2}(d+1)$ unless $\frac{1}{2 \alpha}+\frac{1}{2}$ is an algebraic integer of degree 2 , see [2, Subsection 2.3].

Finally, the values of $N_{1 / 3}(d)$ and $N_{1 / 5}(d)$ for a large $d$ have been completely determined:

$$
\begin{array}{lll}
N_{1 / 3}(d)=2 d-2 & \text { for } d \geq 15 & {[8, \text { Theorem 4.5], }} \\
N_{1 / 5}(d)=\lfloor 3(d-1) / 2\rfloor & \text { for all sufficiently large } d & {[9] \text { and [5, Corollary 6.6]. }}
\end{array}
$$

New bound We will show that $N_{\alpha}(d)$ is linear for every $\alpha$. In fact, we will prove a result in greater generality. Following [4], we call a set of unit vectors $P$ an $L$-spherical code if $\left\langle v, v^{\prime}\right\rangle \in L$ for every pair of distinct points $v, v^{\prime} \in P$. In this language, a set of equiangular lines is a $\{-\alpha, \alpha\}$-spherical code. Let $N_{L}(d)$ be the maximum cardinality of an $L$-spherical code in $\mathbb{R}^{d}$.

Theorem 1. For every fixed $0<\beta \leq 1$ there exists a constant $c_{\beta}$ such that for any $L$ of the form $L=[-1,-\beta] \cup\{\alpha\}$ we have $N_{L}(d) \leq c_{\beta} d$.

We make no effort to optimize the constant $c_{\beta}$ that arises from our proof, as it is huge. We speculate about the optimal bounds on $N_{L}(d)$ in section 3 . We do not know if the constant $c_{\beta}$ can be replaced by an absolute constant that is independent of $\beta$, i.e., whether $N_{L}(d) \leq c d+o_{\beta}(d)$ holds.

The rest of the paper is organized as follows. In the next section we prove Theorem 1 and in the concluding section we discuss possible generalizations and strengthenings of Theorem 1.

## 2 Proof of Theorem 1

Proof sketch The idea behind the proof of Theorem 1 builds upon the argument of Lemmens and Seidel for $N_{1 / 3}(d)$. Before going into the details, we outline the argument.

Let $L=[-1,-\beta] \cup\{\alpha\}$, and let $P$ be an $L$-spherical code whose size we wish to bound. Define a graph $G$ on the vertex set $P$ by connecting $v$ and $v^{\prime}$ by an edge if $\left\langle v, v^{\prime}\right\rangle \in[-1,-\beta]$. In their treatment of $N_{1 / 3}(d)$ Lemmens and Seidel consider the largest clique in $G$, and carefully analyze how the rest of the graph attaches to that clique. In contrast, in our argument we consider the largest independent set $I$ of $G$, and show that almost every other vertex is incident to nearly all vertices of $I$. Iterating this argument inside the common neighborhood of $I$ we can build a large clique in $G$. As the clique size is bounded by a function of $\beta$, that establishes the theorem.

Proof details For the remainder of the section, $L, P$ and $G$ will be as defined as in the preceding proof sketch. The following two well-known lemmas bound the sizes of cliques and independent sets in $G$ :

Lemma 2. Suppose $u_{1}, \ldots, u_{n}$ are $n$ vectors of norm at most 1 satisfying $\left\langle u_{i}, u_{j}\right\rangle \leq-\gamma$. Then $n \leq 1 / \gamma+1$.

Proof. This follows from $0 \leq\left\|\sum u_{i}\right\|^{2}=\sum_{i, j}\left\langle u_{i}, u_{j}\right\rangle \leq n-\gamma n(n-1)$.

## Lemma 3.

i. Every independent set in $G$ is linearly independent. In particular, the graph $G$ contains no independent set on more than d vertices.
ii. The graph $G$ contains no clique on more than $1 / \beta+1$ vertices.

Proof. i) Let $p_{1}, \ldots, p_{n}$ be the points of the independent set. Suppose that $\sum c_{i} p_{i}=0$. Taking an inner product with $p_{j}$ we obtain $0=(1-\alpha) c_{j}+\alpha \sum c_{i}$ implying that all $c_{i}$ 's are equal. The result follows since $(1-\alpha)+n \alpha \neq 0$.
ii) This is a special case of the preceding lemma.

In the next two lemmas we analyze how the vertices of $G$ attach to an independent set.
Lemma 4. Suppose that $M$ is a matrix with linearly independent column vectors $p_{1}, \ldots, p_{n}$. Suppose that $v, v^{\prime} \in \operatorname{span}\left\{p_{1}, \ldots, p_{n}\right\}$ are points satisfying $\left\langle p_{i}, v\right\rangle=s_{i}$ and $\left\langle p_{i}, v^{\prime}\right\rangle=s_{i}^{\prime}$ for some column vectors $s=\left(s_{1}, \ldots, s_{n}\right)^{T}$ and $s^{\prime}=\left(s_{1}^{\prime}, \ldots, s_{n}^{\prime}\right)^{T}$. Then $\left\langle v, v^{\prime}\right\rangle=s^{T}\left(M^{T} M\right)^{-1} s^{\prime}$.

Proof. By passing to a subspace we may assume that $p_{1}, \ldots, p_{n}$ span $\mathbb{R}^{n}$, and so $M$ is invertible. As $s=M^{T} v$ and $s^{\prime}=M^{T} v^{\prime}$, we infer that $\left\langle v, v^{\prime}\right\rangle=v^{T} v^{\prime}=\left(\left(M^{T}\right)^{-1} s\right)^{T}\left(M^{T}\right)^{-1} s^{\prime}=s^{T}\left(M^{T} M\right)^{-1} s^{\prime}$.

The following lemma is the geometric heart of the proof. In its special case $v=v^{\prime}$, the lemma bounds degrees from certain vertices into an independent set. More precisely, let $I$ be a sufficiently large independent set. We will show later (in Lemma 6) that the vertices, the norm of whose projection on span $I$ exceeds $\alpha^{1 / 2}$, are few. The straightforward, but slightly messy calculations in the following lemma characterize the vertices with such projections in terms of their degree into $I$. The case $v \neq v^{\prime}$ is not needed when $P$ comes from a set of equiangular lines, but is required to establish Theorem 1 in its full generality.

Lemma 5. Let $t=1 / \beta+1$. There exists $n_{0}=n_{0}(\beta)$ and $\varepsilon=\varepsilon(\beta)>0$ such that the following holds. Suppose that $p_{1}, \ldots, p_{n}$ is an independent set in $G$ of size $n$, and suppose that points $p, p^{\prime} \in P$ are adjacent to the same $m$ vertices among $p_{1}, \ldots, p_{n}$. Assume $0<m<n-t$ and $n \geq n_{0}$. Write $p=v+u$ and $p^{\prime}=v^{\prime}+u^{\prime}$ where $v, v^{\prime} \in \operatorname{span}\left\{p_{1}, \ldots, p_{n}\right\}$ and $u, u^{\prime}$ are both orthogonal to $\operatorname{span}\left\{p_{1}, \ldots, p_{n}\right\}$. Then $\left\langle v, v^{\prime}\right\rangle \geq \alpha+\varepsilon$.

Proof. For the duration of this proof, $I$ denotes the identity matrix, and $J$ denotes the all-1 matrix. Let $M$ be the matrix comprised of column vectors $p_{1}, p_{2}, \ldots, p_{n}$. Since points $p_{1}, \ldots, p_{n}$ are linearly independent (by Lemma 3), the condition of the preceding lemma is fulfilled. We have $M^{T} M=$ $\alpha J+(1-\alpha) I$. One can verify that its inverse is given by

$$
\begin{equation*}
(1-\alpha)\left(M^{T} M\right)^{-1}=I-\phi J \quad \text { with } \quad \phi \stackrel{\text { def }}{=} \frac{\alpha}{1+(n-1) \alpha} . \tag{1}
\end{equation*}
$$

Note that $\phi \leq 1 / n$ since $\alpha \leq 1$.
Without loss of generality, $p_{1}, \ldots, p_{m}$ are the $m$ vertices that $p$ and $p^{\prime}$ are adjacent to. This means that $s \stackrel{\text { def }}{=} M^{T} v$ and $s^{\prime} \stackrel{\text { def }}{=} M^{T} v^{\prime}$ are of the form $s=\left(-\beta_{1}, \ldots,-\beta_{m}, \alpha, \ldots, \alpha\right)$ and $s^{\prime}=$ $\left(-\beta_{1}^{\prime}, \ldots,-\beta_{m}^{\prime}, \alpha, \ldots, \alpha\right)$ for some $\beta_{1}, \beta_{1}^{\prime}, \ldots, \beta_{m}, \beta_{m}^{\prime} \in[\beta, 1]$. From Lemma 4 and (1) it follows that

$$
\begin{equation*}
(1-\alpha)\left\langle v, v^{\prime}\right\rangle=\alpha^{2}(n-m)+\sum_{i=1}^{m} \beta_{i} \beta_{i}^{\prime}-\phi\left(\sum_{i=1}^{n} s_{i}\right)\left(\sum_{i=1}^{n} s_{i}^{\prime}\right) . \tag{2}
\end{equation*}
$$

We claim that, subject to the constraint $\beta_{1}, \beta_{1}^{\prime}, \ldots, \beta_{m}, \beta_{m}^{\prime} \in[\beta, 1]$, the right side of (2) is minimized when all the $\beta_{i}$ 's and all the $\beta_{i}^{\prime}$ 's are equal to $\beta$. Indeed, since $[\beta, 1]^{2 m}$ is compact, the minimum is actually attained. Assume that $\left(\beta_{1}, \beta_{1}^{\prime}, \ldots, \beta_{m}, \beta_{m}^{\prime}\right)$ is the vector achieving the minimum, and let $j$ be the index for which $\beta_{j}^{\prime}$ is the largest. Then the derivative of the right side of (2) with respect to $\beta_{j}^{\prime}$ is

$$
\beta_{j}-\phi \sum \beta_{i}+(n-m) \alpha \geq \beta_{j}-\frac{1}{n} \sum \beta_{i}>\beta_{j}-\frac{1}{m} \sum \beta_{i} \geq 0 .
$$

By the optimality assumption on the vector $\left(\beta_{1}, \beta_{1}^{\prime}, \ldots, \beta_{m}, \beta_{m}^{\prime}\right)$ this implies that $\beta_{j}^{\prime}=\beta$. From the choice of $j$ it then follows that $\beta_{i}^{\prime}=\beta$ for all $i$. Similarly, $\beta_{i}=\beta$ for all $i$.

We thus deduce that

$$
(1-\alpha)\left\langle v, v^{\prime}\right\rangle \geq \alpha^{2}(n-m)+\beta^{2} m-\phi((n-m) \alpha-m \beta)^{2} .
$$

Let $R(m, n)$ denote the right side of preceding inequality. Let $t^{*} \stackrel{\text { def }}{=} \frac{(1-\alpha)(\alpha-\beta)}{\alpha(\alpha+\beta)}$. We have

$$
t^{*}=\frac{(1-\alpha)(\alpha-\beta)}{\alpha(\alpha+\beta)}<\frac{1-\alpha}{\alpha+\beta}<\frac{1}{\beta}=t-1 .
$$

Thus to prove the lemma, it is enough to show that $R(m, n) \geq(1-\alpha) \alpha+\varepsilon$ whenever $1 \leq m \leq n-t^{*}-1$ and $n \geq n_{0}$ for suitable $n_{0}$ and $\varepsilon$.

The expression $R(m, n)$ is a quadratic polynomial in $m$. A simple calculation shows that it satisfies $R(m, n)=R\left(n-t^{*}-m, n\right)$, and in particular that the maximum of $R(m, n)$ for a fixed $n$ is at the point $m_{\max } \stackrel{\text { def }}{=}\left(n-t^{*}\right) / 2$, which is inside the interval $\left[1, n-t^{*}-1\right]$. Furthermore, at the boundary points of the interval we have

$$
R(1, n)=R\left(n-t^{*}-1, n\right)=\alpha(1-\alpha)+(\alpha+\beta)^{2}-\frac{\alpha(1+\beta)^{2}}{1+\alpha(n-1)}
$$

Let $n_{0}=1+8 / \beta^{2}$. Since $\frac{\alpha(1+\beta)^{2}}{1+\alpha(n-1)}<\frac{(1+\beta)^{2}}{n-1} \leq \frac{4}{n-1}$, for $n \geq n_{0}$ and $1 \leq m \leq n-t^{*}-1$ we have the inequality $R(m, n) \geq R(1, n)>\alpha(1-\alpha)+\frac{1}{2}(\alpha+\beta)^{2}$. In particular $\left\langle v, v^{\prime}\right\rangle>\alpha+\varepsilon$ holds under the same conditions on $n$ and $m$, where $\varepsilon=\frac{1}{2} \beta^{2}$.

Lemma 6. Suppose $p_{1}, \ldots, p_{n}$ is an independent set in $G$. Suppose $p^{(1)}, \ldots, p^{(m)} \in P$ are points of the form $p^{(i)}=v^{(i)}+u^{(i)}$ with $v^{(i)} \in \operatorname{span}\left\{p_{1}, \ldots, p_{n}\right\}$ and $u^{(i)} \perp \operatorname{span}\left\{p_{1}, \ldots, p_{n}\right\}$ and $\left\langle v^{(i)}, v^{(j)}\right\rangle>\alpha+\varepsilon$ for all $i, j$. Then $m \leq 1 / \varepsilon+1$.

Proof. From $\left\langle p^{(i)}, p^{(j)}\right\rangle=\left\langle v^{(i)}, v^{(j)}\right\rangle+\left\langle u^{(i)}, u^{(j)}\right\rangle$ and $\left\langle p^{(i)}, p^{(j)}\right\rangle \in[-1,-\beta] \cup\{\alpha\}$, we deduce that $\left\langle u^{(i)}, u^{(j)}\right\rangle<-\varepsilon$. The result then follows from Lemma 2.

The combinatorial part of the argument is contained in the next result.
Lemma 7. Suppose $\delta>0$ is given. Then there exists a constant $M(\beta, \delta)$ such that the following holds. Let $U \subset P$ be arbitrary. Suppose $I$ is a maximum-size independent subset of $U$. Then there is a subset $U^{\prime} \subset U \backslash I$ of size $\left|U^{\prime}\right| \geq|U|-M|I|$ such that every vertex of $U^{\prime}$ is adjacent to at least $(1-\delta)|I|$ vertices of $I$.

Proof. Let $t, \varepsilon$ and $n_{0}$ be as in Lemma 5, and put $n=\max \left(n_{0},\lceil t / \delta\rceil\right)$. Denote by $R$ the least integer such that every graph on $R$ vertices contains either an independent set of size $n+1$ or a clique of size at least $1 / \beta+2$ (such an $R$ exists by Ramsey's theorem; furthermore, it satisfies $R \leq\left({ }_{n}^{n+1 / \beta+1}\right)$ ). Let

$$
\begin{aligned}
M & =\max \left(R,(1 / \varepsilon+1) 2^{n}\right), \\
N & =|I| .
\end{aligned}
$$

If $|U|<M$, then $|U|-M|I|$ is negative, and the lemma is vacuous. So, assume $|U| \geq M$. In particular, $|U| \geq R$, and since by Lemma 3 the set $U$ contains no clique of size greater $1 / \beta+1$, we conclude that $N \geq n+1$.

Arrange the elements of $I$ on a circle, and consider all $N$ circular intervals containing $n$ vertices of $I$. Let $S_{1}, S_{2}, \ldots, S_{N}$ be these intervals, in order.

We declare a vertex $p \in U \backslash I$ to be $i$-bad if it is adjacent to between 1 and $n-t$ vertices of $S_{i}$. For a set $T \subset S_{i}$, we call an $i$-bad vertex $p$ to be of type $T$ if $T$ is precisely the set of neighbors of $p$ in the set $S_{i}$. Let $B_{i, T}$ be the set of all $i$-bad vertices of type $T$, and let $B_{i}=\bigcup_{T} B_{i, T}$ be the set of all $i$-bad vertices. By Lemmas 5 and 6 we have $\left|B_{i, T}\right| \leq 1 / \varepsilon+1$ for every $T$, and so

$$
\left|B_{i}\right| \leq(1 / \varepsilon+1)\left(2^{n}-1\right)
$$

Let $B=\bigcup B_{i}$ be the set of bad vertices. Hence, $|B| \leq N(1 / \varepsilon+1)\left(2^{n}-1\right)$, and $|B \cup I| \leq M N$.
Consider a vertex $p \in U \backslash I$ that is good, i.e., $p \notin B$. Since $I$ is a maximal independent set, $p$ is adjacent to at least one vertex of $I$. Say $p$ is adjacent to a vertex of $S_{i}$ for some $i$. Since $p$ is good, $p$ must in fact be adjacent to at least $n-t$ vertices of $S_{i}$. As $S_{i}$ shares $n-1$ vertices with both $S_{i-1}$ and $S_{i+1}$, we are impelled to conclude that $p$ must be adjacent to some of the vertices of $S_{i-1}$ and of $S_{i+1}$. Repeating this argument we conclude that $p$ is non-adjacent to at most $t$ elements from among any interval of length $n$. In particular, $p$ is adjacent to at least $N(1-t / n)$ vertices of $I$. As $p$ is an arbitrary good vertex and $t / n \leq \delta$, the lemma follows.

We are now ready to complete the proof of Theorem 1. Indeed, with foresight we set

$$
\begin{aligned}
B & =\lceil 1 / \beta+1\rceil \\
\delta & =1 /(B+1)^{2} .
\end{aligned}
$$

and let $M$ be as in the proceeding lemma. Put $U_{0}=P$ and let $I_{0}$ be a maximal independent set in $U_{0}$. By the preceding lemma, there exists $U_{1} \subset U_{0} \backslash I_{0}$ such that every vertex of $U_{1}$ is adjacent to $(1-\delta)\left|I_{0}\right|$ vertices of $I_{0}$ and $\left|U_{1}\right| \geq\left|U_{0}\right|-M\left|I_{0}\right|$. In view of Lemma $3,\left|U_{1}\right| \geq\left|U_{0}\right|-M d$. Let $I_{1}$ be a maximal independent set in $U_{1}$. Repeating this argument, we obtain a nested sequence of sets $U_{0} \supset U_{1} \supset$ and a corresponding sequence of independent sets $I_{0}, I_{1}, \ldots$ such that
i. $\left|U_{i}\right| \geq\left|U_{i-1}\right|-M d$ for each $i=1,2, \ldots$,
ii. For $r<s$, each vertex in $I_{s}$ is adjacent to at least $(1-\delta)\left|I_{r}\right|$ vertices of $I_{r}$.

We claim that $|P| \leq B M d$, which would be enough to complete the proof of Theorem 1. Indeed, suppose for the sake of contradiction that $|P|>B M d$. Then $I_{0}, \ldots, I_{B}$ are non-empty. Pick vertices $v_{0}, \ldots, v_{B}$ uniformly at random from $I_{0}, \ldots, I_{B}$ respectively. Since, for every $i \neq j$, the pair $v_{i} v_{j}$ is
an edge with probability at least $1-\delta$, it follows that $v_{0}, \ldots, v_{B}$ is a clique with probability at least $1-\binom{B+1}{2} \delta>0$. In particular, $G$ then contains a clique of size $B+1>1 / \beta+1$, contrary to Lemma 3 . The contradiction shows that $|P| \leq B M d$, completing the proof of Theorem 1.

## 3 Open problems

- I know of only one asymptotic lower bound on $N_{L}$. It is a version of [5, Proposition 5.12] that is also implicit in the bound for $N_{1 / 3}(d)$ in [8]. Denote by $I_{n}$ the identity matrix of size $n$, and by $J_{n}$ the all-one matrix of size $n$. Then the matrix $M=(r-1) I_{r t}-\left(J_{r}-I_{r}\right) \otimes I_{t}$ is a positive semidefinite matrix of nullity $t$, it has $(r-1)$ 's on the diagonal, and its off-diagonal entries are 0 and -1 . Hence, $\frac{1}{r-1+\tau}\left(M+\tau J_{r t}\right)$ is a Gram matrix of a $\left\{-\frac{1-\tau}{r-1+\tau}, \frac{\tau}{r-1+\tau}\right\}$-code in $\mathbb{R}^{(r-1) t+1}$ of size $r$. So, $N_{L}(d) \geq \frac{r}{r-1} d+O(1)$ for $L=\left\{-\frac{1-\tau}{r-1+\tau}, \frac{\tau}{r-1+\tau}\right\}$. For $\tau=1 / 2$, this yields a family of equiangular lines. The results in $[8,9,5]$ suggest that this bound is sharp.

Conjecture 8. For an integer $r \geq 2$, the maximum number of equiangular lines with angle $\arccos \frac{1}{2 r-1}$ is $N_{1 /(2 r-1)}(d)=\frac{r}{r-1} d+O(1)$ as $d$ tends to infinity.

In contrast, one can show that the bound implicit in the proof of Theorem 1 is $2^{O\left(1 / \beta^{2}\right)} d$.

- Informally, it is natural to think of Theorem 1 as a juxtaposition of two trivial results from Lemma 3: $N_{[-1,-\beta]}(d)=O(1)$ and $N_{\{\alpha\}}(d)=O(d)$. Since $N_{\left\{\alpha_{1}, \ldots, \alpha_{k}\right\}}(d)=O\left(d^{k}\right)$ for any real numbers $\alpha_{1}, \ldots, \alpha_{k}$ (see [2, Proposition 1]) this motivates the following conjecture.

Conjecture 9. Suppose $\alpha_{1}, \ldots, \alpha_{k}$ are any $k$ real numbers, and $L=[-1,-\beta] \cup\left\{\alpha_{1}, \ldots, \alpha_{k}\right\}$. Then $N_{L}(d) \leq c_{\beta, k} d^{k}$.

It is conceivable that in this case even $N_{L}(d) \leq c_{\beta} N_{\left\{\alpha_{1}, \ldots, \alpha_{k}\right\}}(d)$ might be true.
Added in revision: Conjecture 9 has been resolved by Keevash and Sudakov [7]

- I cannot rule out the possibility that for a fixed $\alpha$ the size of any $[-1,0) \cup\{\alpha\}$-code is at most linear in the dimension.

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## References

[1] Alexander Barg and Wei-Hsuan Yu. New bounds for equiangular lines. In Discrete geometry and algebraic combinatorics, volume 625 of Contemp. Math., pages 111-121. Amer. Math. Soc., Providence, RI, 2014. arXiv:1311.3219.
[2] Boris Bukh. Ranks of matrices with few distinct entries. arXiv:1508.00145, 2015.
[3] D. de Caen. Large equiangular sets of lines in Euclidean space. Electron. J. Combin., 7:Research Paper 55, 3 pp. (electronic), 2000. http://www.combinatorics.org/Volume_7/Abstracts/v7i1r55.html.
[4] P. Delsarte, J. M. Goethals, and J. J. Seidel. Spherical codes and designs. Geometriae Dedicata, 6(3):363-388, 1977.
[5] Gary Greaves, Jacobus H. Koolen, Akihiro Munemasa, and Ferenc Szöllösi. Equiangular lines in Euclidean spaces. arXiv:1403.2155, January 2015.
[6] Jonathan Jedwab and Amy Wiebe. Large sets of complex and real equiangular lines. J. Combin. Theory Ser. A, 134:98-102, 2015. arXiv:1501.05395.
[7] Peter Keevash and Benny Sudakov. Bounds for spherical codes. arXiv:1602.07645, February 2016.
[8] P. W. H. Lemmens and J. J. Seidel. Equiangular lines. J. Algebra, 24:494-512, 1973.
[9] A. Neumaier. Graph representations, two-distance sets, and equiangular lines. Linear Algebra Appl., 114/115:141-156, 1989.
[10] J. H. van Lint and J. J. Seidel. Equilateral point sets in elliptic geometry. Nederl. Akad. Wetensch. Proc. Ser. A 69=Indag. Math., 28:335-348, 1966.


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