THE BLOW-UP OF ELECTROMAGNETIC FIELDS IN 3-DIMENSIONAL INVISIBILITY CLOAKING FOR MAXWELL'S EQUATIONS

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Abstract: Transformation optics constructions have allowed the design of cloaking devices that steer electromagnetic, acoustic and quantum waves around a region without penetrating it, so that this region is hidden from external observations. The proposed material parameters are anisotropic, and singular at the interface between the cloaked region and the cloaking device. The presence of these singularities causes various mathematical problems and physical effects on the interface surface. In this paper, we analyze the 3-dimensional cloaking for Maxwell's equations when there are sources or sinks present inside the cloaked region. In particular, we consider nonsingular approximate invisibility cloaks based on the truncation of the singular transformations. We analyze the limit of solutions when the approximate cloaking approaches the ideal cloaking in the sense of distributions. We show that the solutions in the approximate cloaks converge to a distribution that contains Dirac's delta distribution supported on the interface surface. In particular, this implies that the limit of solutions are not measurable functions, making them outside of those classes of functions that have earlier been used in the models of the ideal invisibility cloaks. Also, we give a rigorous meaning for the "extraordinary surface voltage effect" considered in physical literature of invisibility cloaks.

Key words: Invisibility cloaking, Maxwell's equations, transformation optics.

1. INTRODUCTION

The transformation optics-based cloaking design, since proposed in 2006 in [19, 31], has attracted most attention among many theoretical proposals for realizing invisibility, with widely reported experiments and data accumulated due to the development of metamaterials. The basic idea is that, the coordinate transformation invariance for certain systems, such as those describing electromagnetic/optic or acoustic wave propagations, makes modifying the background medium of a local region Ω in a certain way undetectable when observe far away. In particular, when singular spatial transformations are used, like the one that blows up a point into a small bounded domain D, this undetectable customization of medium (known as the push-forward by the singular transformation) can have the structure of a fixed layer $\Omega \setminus \overline{D}$ (the cloaking device) surrounding "arbitrary" media in D (the cloaked region). Namely, whatever in D along with the cloaking device is invisible! Such singular coordinate transformations to create a "hidden pocket" surrounded by a layer of degenerate material, were proposed in 2003 in [14] for electrostatics as a counter-example of uniqueness of an inverse problem, known as the Calderón's

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problem. In [19, 31] a customized cloaking layer was proposed to be created of metamaterials. Such layer should be designed so that it bends the light rays or detour the electromagnetic waves away from the cloaked region and return them back, as if they penetrate the region straightly, making observation outside indistinguishable from that of the empty background space.

The main difficulty in analyzing this scheme rigorously lies on the prescribed singular (non-regular) medium in the cloaking layer, see discussion in [3, 9, 10]. To be more precise, we consider in this paper the electromagnetic cloaking, where the prescribed medium in the cloaking layer $\Omega \setminus \overline{D}$ has electric permittivity $\tilde{\varepsilon}(x)$, magnetic permeability $\tilde{\mu}(x)$. Since they are the push-forward of the ambient medium parameters by a singular transformation (detailed in Section 2), one of the eigenvalues of $\tilde{\varepsilon}(x)$ and $\tilde{\mu}(x)$ degenerates to zero at the exterior of the cloaking interface ∂D^+ . Suppose D is filled with an arbitrary medium $(\mu_1(x), \varepsilon_1(x))$ to be cloaked. Let us consider the waves propagating in domain $\Omega \subset \mathbb{R}^3$ consisting of medium whose permittivity and permeability are given by $(\tilde{\mu}, \tilde{\varepsilon})$ in $\Omega \setminus \overline{D}$ (the cloaking layer/device) and (μ_1, ε_1) in \overline{D} (the cloaked region). Such cloaking gives counterexamples for uniqueness of the inverse problems for Maxwell's equations. As we want to consider also cloaking of active bodies, we add a current source \tilde{J} in the domain D. To study this, we will consider the case when $\mu_1(x) = \mu_0$ and $\varepsilon_1(x) = \varepsilon_0$ are constants. We note that the results of this paper can with small modifications be extended to the case where $\mu_1(x) - \mu_0$ and $\varepsilon_1(x) - \varepsilon_0$ are compactly supported functions in D.

Since $(\tilde{\mu}, \tilde{\varepsilon})$ are singular near ∂D , the key mathematical question is, in what sense the solutions to Maxwell's equations

$$\operatorname{curl} \widetilde{E} - i\omega \widetilde{B} = 0, \quad \operatorname{curl} \widetilde{H} + i\omega \widetilde{D} = \widetilde{J}, \quad \operatorname{on} \ \Omega \setminus \overline{D} \cup D,$$

exist. Here the constitutive relations are given by

$$\widetilde{D} = \begin{cases} \widetilde{\varepsilon}\widetilde{E} & \text{on } \Omega \backslash \overline{D}, \\ \varepsilon_0 \widetilde{E} & \text{on } D. \end{cases} \qquad \widetilde{B} = \begin{cases} \widetilde{\mu}\widetilde{H} & \text{on } \Omega \backslash \overline{D}, \\ \mu_0 \widetilde{H} & \text{on } D. \end{cases}$$

Moreover, one would like to specify what kind of boundary conditions should appear at the interface ∂D , where the singularity happens. There have been many different proposals to model the above Maxwell's equations with singular coefficients and different boundary conditions have been proposed, depending on what spaces the solutions of the equations are assumed to belong. In the present paper our aim is to consider approximative cloaks and consider what is the limit of the solutions as it approaches an ideal cloaking. Before formulating our results, let us review the earlier proposed models.

In [7], the concept of finite energy solution (FES) was formulated. These are distributional solutions to Maxwell's equations for which all physical fields, that is, the electric and magnetic fields \tilde{E} and \tilde{H} as well as the electromagnetic fluxes \tilde{D} and \tilde{B} are \mathbb{C}^3 -valued measurable functions on Ω . The "tilde" here refers to the fact that these solutions satisfy Maxwell's equation with the cloaking material parameters $\tilde{\varepsilon}$ and $\tilde{\mu}$. Moreover, the energy norm with degenerate weight is required to be finite, that is

$$\int_{\Omega\setminus\overline{D}} \left(\widetilde{\varepsilon}^{jk} \widetilde{E}_j \overline{\widetilde{E}_k} + \widetilde{\mu}^{jk} \widetilde{H}_j \overline{\widetilde{H}_k} \right) \, dV + \int_D \left(\varepsilon_0^{jk} \widetilde{E}_j \overline{\widetilde{E}_k} + \mu_0^{jk} \widetilde{H}_j \overline{\widetilde{H}_k} \right) \, dV < \infty,$$

where, as everywhere below, we have used the Einstein summating convention by summing over indecies j and k that appear both as sub- and superindecies. It is shown in [7] that a hidden boundary condition must be satisfied by the FES at the interior of the interface, namely

$$\nu \times \widetilde{E}|_{\partial D^-} = 0, \qquad \nu \times \widetilde{H}|_{\partial D^-} = 0.$$

These, physically known as PEC and PMC conditions, form an over-determined set of boundary conditions for Maxwell's equations in D, and could be only possibly satisfied for cloaking passive media (i.e., $\tilde{J} = 0$ or \tilde{J} is a non-radiating source that would produce in the free space \mathbb{R}^3 a compactly supported electromagnetic field). But for generic \tilde{J} , including ones arbitrarily close to 0, there is no FES-solution.

In [33, 34], solutions with various boundary conditions at ∂D^{\pm} are considered in the context of self-adjoint extensions of the Maxwell's operator, that are compatible with energy conservation. There, the fields \tilde{E} and \tilde{H} are assumed to be measurable functions that are in the domain of the closed quadratic form, as well as in the domain of the corresponding self-adjoint operator. We call these the operator theoretic (OT) solutions. It is shown there that the Maxwell operator Awith degenerate coefficients $\tilde{\epsilon}(x)$ and $\tilde{\mu}(x)$ is essentially self-adjoint and its unique self-adjoint extension has a domain $\mathcal{D}(A) \subset L^2(\Omega)$ such that $(E, H) \in \mathcal{D}(A)$ have one-sided traces (from the outside of D) satisfying

$$\nu \times \widetilde{E}|_{\partial D^+} = 0, \qquad \nu \times \widetilde{H}|_{\partial D^+} = 0,$$

due to the degeneracy of parameters at ∂D^+ . With regular medium to be cloaked in D, functions $(E, H) \in \mathcal{D}(A)$ have also one-side traces, from the inside of D, satisfying

(1)
$$\nu \cdot (\operatorname{curl} \widetilde{H})|_{\partial D^{-}} = \nu \cdot (\operatorname{curl} \widetilde{E})|_{\partial D^{-}} = 0.$$

Notice that this is from the point of view that self-adjoint extensions in the domain Ω are direct sums of those in $\Omega \setminus \overline{D}$ and D, hence the solutions are considered completely decoupled from each other.

In [8, 12, 13, 15, 18, 21, 25, 26, 27, 28], for the purpose of both physical realization and analysis of the waves behavior, the regularization of the singular scheme is studied as the fundamentals of approximate cloaking design. In this paper, we will consider the same approximation scheme as in [21] for electromagnetic waves. The detailed non-singular approximation will be introduced in Section 2. Roughly speaking, instead of using the transformation that blows up a point into the cloaked region $D = B_1$ where $B_R \subset \mathbb{R}^3$ is the ball of radius R centred at the origin, in this idealized spherical geometric setting, we use a non-singular transformation that blows up a small ball B_{ρ} with radius ρ (the regularization parameter) into B_1 . The transformation medium is then regular for fixed $\rho > 0$ and appears as a small inhomogeneity in the empty space. The well-understood solutions for regular media provide a tool to scrutinize the extreme case, that is taking the limit $\rho \to 0$.

As seen above, there are different, partly contradicting, alternatives for rigorous models of ideal invisibility cloaks. As physical attempts to build up an invisibility cloak from metamaterial are always based on approximate constructions, we propose here a change of the point of view for defining a model of an ideal invisibility cloak: We consider a limit of the solutions in an approximate cloak as $\rho \to 0$,

in the sense of distributions (i.e., generalized functions), and describe the plausible solutions in an ideal cloak as these limits of the solutions. We show that the solutions in the approximative cloaking structures converge to a distribution that contains Dirac's delta distribution supported on the interface ∂D . This implies that the set of measurable functions might be too small as the domain of the singular Maxwell's operator, when considered as the limit of electromagnetic approximate cloaking. We note that the ideal and the approximate cloaks we study here have no energy absorption, that is, the conductivity is zero.

The results of the paper can be summarised as follows: Let E_{ρ} and H_{ρ} denote the electric and the magnetic fields in an approximative cloak in Ω as described in (15), satisfying the boundary condition $\nu \times \tilde{E}_{\rho}|_{\partial B_2} = f$. We consider the limit, in the sense of distributions, $\tilde{E}_0 = \lim_{\rho \to 0} \tilde{E}_{\rho}$ and $\tilde{H}_0 = \lim_{\rho \to 0} \tilde{H}_{\rho}$ of the solutions of the Maxwell equations, that is, the limit when the approximative cloak approaches a perfect cloak. Then

- (1) We show that the limit $(\tilde{E}_0, \tilde{H}_0)$ is not in the class of finite energy solutions. This is compatible with the non-existence result of finite energy solutions given in [7].
- (2) We show that the limit (E₀, H₀) is not a measureble function and thus it does not belong in the class of operator theoretic solutions studied in [33]. However, if (E₀, H₀) is decomposed to a sum of a measurable function (E_m, H_m) and a delta-distribution supported on the surface ∂D, the function part (E_m, H_m) satisfies the non-standard boundary conditions (1) on ∂D described in [33], that is, certain traces of (E_m, H_m) are zero. Surprisingly, we observe that the corresponding traces of the fields E_ρ and H_ρ on ∂D do not converge to zero, see Remark 4.6.
- (3) The convergence of the solutions in the so-called virtual space and physical space are different: In the physical space the limit of $(\tilde{E}_{\rho}, \tilde{H}_{\rho})$ contains a delta-distribution part but in the virtual space, images of the fields given in (17), converge in the L^2 -sense as described in [21].
- (4) The blow up of the wave, that is, the appearance of the delta-distribution in the limit $\rho \to 0$ happens with all frequencies. Thus the observed blow up is different to the destruction of the cloaking effect appearing at the eigenfrequencies of the inside of the cloak studied in [8, 12, 15].
- (5) The high concentration of waves on ∂D has been observed in the physical literature by Zhang et. al. [36], and is called the "extraordinary surface voltage effect". We give a rigorous formulation, as a convergence result in the weak topology of $H^{-1}(\Omega)$, of this effect.

We note that in this paper we consider only the most commonly used approximative cloaks that are equivalent to multiplying the singular electric permittivity $\tilde{\varepsilon}(x)$ and magnetic permeability $\tilde{\mu}(x)$ by a characteristic function. Such approximation could be implemented also in different ways, see [2, 4, 5, 8, 15, 18, 17, 16, 21, 25, 26, 27, 28], and an interesting question is how the delta-distribution part of the limit solution (E_0, H_0) depends on how the approximation is used. However, this question is outside the scope of this paper.

The rest of this paper is organized as following. In Section 2, we describe in details the transformation optics-based ideal cloaking and the regularized approximate cloaking in the spherical geometric setting. The main result is formulated

in Theorem 2.1 and the proof is presented in Section 4, where the Dirac's delta distribution is proved rigorously to appear at the exterior of the interface ∂D^+ , as a limit of the high concentrated electromagnetic waves in the approximate cloak. In Section 3, an example is presented to demonstrate the physical motivation and insight of the existence of such condition. Our estimates rely heavily on the spherical harmonics and asymptotic of Bessel and Hankel functions, of which we provide some basics in the Appendix.

2. Singular ideal electromagnetic cloaking and the regularization

How the transformation invariance of the system amounts to the undetectability of certain non-ambient media with respect to the observation is best seen when formulated as inverse boundary value problems for underlying PDEs (for nearfield measurements) or inverse scattering problems (for far-field measurements). This is also why the layer-structured medium for cloaking was first studied in [14] for electrostatics as a counter-example of uniqueness of an inverse problem, known as the Calderón's problem. Here we consider a similar construction for electromagnetism, that is for Maxwell's equations. In particular, we prescribe timeharmonic incident waves with time frequency ω , which results in the time harmonic equations as following.

Let Ω denote a bounded domain in \mathbb{R}^3 with smooth boundary. Consider Maxwell's equations for time-harmonic electric and magnetic fields (E, H), viewed as 1-forms,

(2)
$$\nabla \times E = i\omega B, \quad \nabla \times H = -i\omega D + J \quad \text{in } \Omega$$

where the 2-forms B, D and J denote the magnetic induction, the electric displacement and the current density respectively, Furthermore, we specify the constitutive relations as

$$(3) D = \varepsilon E, B = \mu H,$$

where ε and μ represent the permittivity and permeability of the material in Ω . In the following we assume that ε, μ are in $L^{\infty}(\Omega)^{3\times 3}$, and satisfy

(4)
$$c_m |\xi|^2 \le \xi^T \varepsilon(x) \xi \le c_M |\xi|^2, \quad c_m |\xi|^2 \le \xi^T \mu(x) \xi \le c_M |\xi|^2$$

for some constants $c_m, c_M > 0$ and all $x \in \Omega$ and $\xi \in \mathbb{R}^3 \setminus \{0\}$, We remark that (4) are physical conditions for *regular* EM media. Given $J \in L^2(\Omega)^3$, we denote by $\mathcal{C}^{\omega}_{\varepsilon,\mu,J}$ a subset of $H^{-\frac{1}{2}}(\text{Div};\partial\Omega) \times H^{-\frac{1}{2}}(\text{Div};\partial\Omega)$ given by

$$\mathcal{C}^{\omega}_{\varepsilon,\mu,J} := \left\{ (\nu \times E|_{\partial\Omega}, \nu \times H|_{\partial\Omega}) \mid E, H \in H(\operatorname{curl}; \Omega) \text{ satisfy (2) and (3) with } J \right\}$$

where

$$H(\operatorname{curl};\Omega) := \{ \mathbf{u} \in L^2(\Omega)^3 ; \operatorname{curl} \mathbf{u} \in L^2(\Omega)^3 \},\$$

and

 $H^{-\frac{1}{2}}(\text{Div};\partial\Omega) := \{\mathbf{f} \in H^{-\frac{1}{2}}(\partial\Omega)^3; \mathbf{f} \cdot \nu = 0 \text{ a.e. on } \partial\Omega \text{ and } \text{Div } \mathbf{f} \in H^{-\frac{1}{2}}(\partial\Omega)\}$ with Div denoting the surface divergence on $\partial\Omega$. Also, we denote $\mathcal{C}^{\omega}_{\varepsilon,\mu} := \mathcal{C}^{\omega}_{\varepsilon,\mu,J}$ with J = 0.

The set $C^{\omega}_{\varepsilon,\mu,J}$ is known as the *Cauchy data set* which encodes the full exterior (boundary) measurements of electromagnetic fields. The inverse problem is then to understand the dependence of $C^{\omega}_{\varepsilon,\mu,J}$ on the parameters (ε,μ) , in order to recover the latter from the former. Knowing this, the design of invisibility is to find the

parameters to be prescribed in a cloaking device such that its Cauchy data is indistinguishable from that of the vacuum background, independent of the object to be cloaked in the device.

First, let us define $M^{-T} = (M^{-1})^T$ for an matrix M. Consider a transformation $x = F(y) : \Omega \to \widetilde{\Omega}$ between two bounded domains $\Omega, \widetilde{\Omega} \subset \mathbb{R}^3$ with smooth boundaries. Assume F is bi-Lipschitz and orientation-preserving, and denote by $M := DF(y) = (\frac{\partial x_i}{\partial y_j})_{i,j=1}^3$ the Jacobian matrix of F. The pull-back fields of the solution $(E, H) \in H(\operatorname{curl}; \Omega) \times H(\operatorname{curl}; \Omega)$ to (2) by F^{-1} , are given in $\widetilde{\Omega}$ by

(5)
$$\widetilde{E}(x) = (F^{-1})^* E(x) := (M^{-T}E) \circ F^{-1}(x), \widetilde{H}(x) = (F^{-1})^* H(x) := (M^{-T}H) \circ F^{-1}(x),$$

(6)
$$\widetilde{J}(x) = (F^{-1})^* J(x) = ([\det(M)]^{-1} M J) \circ F^{-1}(x).$$

Then we have that $(E, H) \in H(\operatorname{curl}; \Omega) \times H(\operatorname{curl}; \Omega)$ satisfies Maxwell's equations

(7)
$$\widetilde{\nabla} \times \widetilde{E} = i\omega \widetilde{\mu} \widetilde{H}, \quad \widetilde{\nabla} \times \widetilde{H} = -i\omega \widetilde{\varepsilon} \widetilde{E} + \widetilde{J} \quad \text{in } \widetilde{\Omega},$$

where $\nabla \times$ denotes the curl in the *x*-coordinates, and $\tilde{\varepsilon}, \tilde{\mu}$ are the push-forwards of ε, μ via *F*, defined by

(8)
$$\widetilde{\varepsilon}(x) = F_* \varepsilon(x) := ([\det(M)]^{-1} M \cdot \varepsilon \cdot M^T) \circ F^{-1}(x),$$

and similarly for $\tilde{\mu} = F_*\mu$. Moreover, if we assume $F|_{\partial\Omega} = \text{Id}$, using Green's identity, it is directly verified that

$$\mathcal{C}^{\omega}_{\varepsilon,\mu,J} = \mathcal{C}^{\omega}_{F_*\varepsilon,F_*\mu,(F^{-1})^*J}.$$

This summarizes the basics of transformation optics in a rather general setting, which we shall make essential use of in the construction of singular ideal cloaking and regularized approximate cloaking. Throughout the paper, we will focus on a spherical geometry of design.

Next, we consider singular coordinate transformations. Consider the map

(9)
$$F_1: B_2 \setminus \{0\} \to B_2 \setminus \overline{B_1}, \quad F_1(y) = \left(1 + \frac{1}{2}|y|\right) \frac{y}{|y|}, \quad 0 < |y| < 2$$

which blows up the origin to B_1 while keeping the boundary ∂B_2 fixed. In the cloaking layer $x \in B_2 \setminus \overline{B_1}$, we prescribe the EM material parameters given by

(10)
$$\widetilde{\mu}(x) = \widetilde{\varepsilon}(x) = (F_{1*}I)(x) := \left. \frac{(DF_1) \cdot I \cdot (DF_1)^T}{|\det(DF_1)|}(y) \right|_{y = F_1^{-1}(x)}$$

where I is the identity matrix, representing the homogeneous vacuum background material. In the region B_1 to be cloaked, we consider an arbitrary but *regular* EM medium (ε_0, μ_0) satisfying (4), i.e., $\tilde{\mu}(x) = \mu_0(x)$ and $\tilde{\varepsilon}(x) = \varepsilon_0(x)$ for $x \in B_1$, which can be viewed as the push-forwards of (μ_0, ε_0) in B_1 by $F_2 = \text{Id}$. We denote the "glued" transformation by

(11)
$$F = (F_1, F_2): (B_2 \setminus \{0\}, B_1) \to (B_2 \setminus \overline{B_1}, B_1).$$

Noticing that F_1 is a radial dilation and by some simple calculations, we have that

(12)
$$\widetilde{\mu}(x) = \widetilde{\varepsilon}(x) = 2 \frac{(|x|-1)^2}{|x|^2} \mathbf{e}_r + 2\mathbf{e}_\theta, \quad 1 < |x| < 2,$$

where \mathbf{e}_r and \mathbf{e}_{θ} are respectively, the unit projections along radial and angular directions, i.e., $\mathbf{e}_r = I - \hat{x}\hat{x}^T$, $\mathbf{e}_{\theta} = \hat{x}\hat{x}^T$, $\hat{x} = x/|x|$. It is readily seen that as one approaches the cloaking interface ∂B_1^+ the medium in the cloaking device becomes singular, in the sense that $\tilde{\varepsilon}$ and $\tilde{\mu}$ no longer satisfy the condition (4) (the eigenvalue along the radial direction degenerates).

2.1. Construction of regularized approximate cloaking. For approximate acoustic cloaking by regularization, Kohn et al., in [15], proposed blowing up a small ball B_{ρ} to B_1 using a nonsingular transformation F_{ρ} which degenerates to the singular transformation F_1 in (9) as $\rho \to 0$, while Greenleaf et al., in [8], proposed truncating the singular medium in (12) to $B_2 \setminus \overline{B_R}$ for R > 1. For the present study, we shall focus on the 'blow-up- B_{ρ} -to- B_1 ' regularization.

Let $0 < \rho < 1$ denote a regularizing parameter and set

(13)
$$a = \frac{2(1-\rho)}{2-\rho}, \quad b = \frac{1}{2-\rho}$$

Consider the nonsingular transformation from B_2 to B_2 defined by

(14)
$$x := F_{\rho}(y) = \begin{cases} F_{\rho}^{(1)}(y) = (a+b|y|)\frac{y}{|y|}, & \text{for } \rho < |y| < 2, \\ F_{\rho}^{(2)}(y) = \frac{y}{\rho}, & \text{for } ||y| \le \rho. \end{cases}$$

Our approximate cloaking device is obtained by the push-forward of a homogeneous medium in $B_2 \setminus \overline{B_{\rho}}$ by $F_{\rho}^{(1)}$. Suppose we hide a regular but arbitrary uniform EM medium (ε_0, μ_0) in the cloaked region B_1 . Then the corresponding EM material parameter in B_2 in the *physical space* is

(15)
$$(\widetilde{\varepsilon}_{\rho}(x), \widetilde{\mu}_{\rho}(x)) = \begin{cases} ((F_{\rho}^{(1)})_* I, (F_{\rho}^{(1)})_* I), & \text{for } 1 < |x| < 2, \\ (\varepsilon_0, \mu_0), & \text{for } |x| < 1. \end{cases}$$

The EM fields $(\tilde{E}_{\rho}, \tilde{H}_{\rho}) \in H(\operatorname{curl}; B_2) \times H(\operatorname{curl}; B_2)$ corresponding to $\{B_2; \tilde{\varepsilon}_{\rho}, \tilde{\mu}_{\rho}\}$ then satisfy Maxwell's equations

(16)
$$\begin{cases} \nabla \times \widetilde{E}_{\rho} = i\omega \widetilde{\mu}_{\rho} \widetilde{H}_{\rho}, \quad \nabla \times \widetilde{H}_{\rho} = -i\omega \widetilde{\varepsilon}_{\rho} \widetilde{E}_{\rho} + \widetilde{J} \quad \text{in } B_2, \\ \nu \times \widetilde{E}_{\rho}|_{\partial B_2} = f \in H^{-1/2}(\text{Div}; \partial B_2) \end{cases}$$

where $\widetilde{J} \in (L^2(B_1))^3$ is the current source. At this point, we assume that it is supported in a smaller ball B_{r_1} with radius $0 < r_1 < 1$. Then the pull-back EM fields

(17)
$$(E_{\rho}, H_{\rho}) = ((F_{\rho})^* \widetilde{E}_{\rho}, (F_{\rho})^* \widetilde{H}_{\rho}) \in H(\operatorname{curl}; B_2) \times H(\operatorname{curl}; B_2)$$

satisfy Maxwell's equations in the *virtual space* of y with parameters

$$(\varepsilon_{\rho}(y), \mu_{\rho}(y)) = \begin{cases} (I, I) & \rho < |y| < 2\\ ((F_{\rho}^{(2)})^* \varepsilon_0, (F_{\rho}^{(2)})^* \mu_0) & |y| < \rho \end{cases}$$

and

$$J = (F_{\rho}^{(2)})^* \widetilde{J}.$$

Moreover, the observation $C^{\omega}_{\tilde{\epsilon}_{\rho},\tilde{\mu}_{\rho}}$ of the whole cloaking object $\{B_2;\tilde{\epsilon}_{\rho},\tilde{\mu}_{\rho}\}$ is shown to be identical to $C^{\omega}_{\epsilon_{\rho},\mu_{\rho}}$, that of a small inhomogeneity of radius ρ in the background vacuum space. In particular, the regularized cloaking is expected to converge to the ideal cloaking as ρ shrinks. The order of such convergence was discussed in [15, 21] for both Helmholtz equations and time-harmonic Maxwell's equations. Here we reproduce some calculations from [21] for Maxwell's equations in order to pursue our discussion.

Due to the two-layered structure, the following notations for EM fields will be adopted

$$\widetilde{E}_{\rho} := \begin{cases} \widetilde{E}_{\rho}^{+}(x), & \text{for } x \in B_2 \setminus \overline{B_1}, \\ \widetilde{E}_{\rho}^{-}(x), & \text{for } x \in B_1, \end{cases} \qquad \widetilde{H}_{\rho} := \begin{cases} \widetilde{H}_{\rho}^{+}(x), & \text{for } x \in B_2 \setminus \overline{B_1}, \\ \widetilde{H}_{\rho}^{-}(x), & \text{for } x \in B_1, \end{cases}$$

in the *physical space* and

$$E_{\rho} := \begin{cases} E_{\rho}^{+}(y), & \text{for } y \in B_2 \setminus \overline{B_{\rho}}, \\ E_{\rho}^{-}(y), & \text{for } y \in B_{\rho}, \end{cases} \qquad H_{\rho} := \begin{cases} H_{\rho}^{+}(y), & \text{for } y \in B_2 \setminus \overline{B_{\rho}}, \\ H_{\rho}^{-}(y), & \text{for } y \in B_{\rho}, \end{cases}$$

in the virtual space, and the fields satisfy the following transmission problems

(18)
$$\begin{cases} \nabla \times \widetilde{E}_{\rho}^{+} = i\omega\widetilde{\mu}_{\rho}(x)\widetilde{H}_{\rho}^{+}, \quad \nabla \times \widetilde{H}_{\rho}^{+} = -i\omega\widetilde{\varepsilon}_{\rho}(x)\widetilde{E}_{\rho}^{+} \quad \text{in } B_{2}\backslash\overline{B_{1}}, \\ \nabla \times \widetilde{E}_{\rho}^{-} = i\omega\mu_{0}\widetilde{H}_{\rho}^{-}, \quad \nabla \times \widetilde{H}_{\rho}^{-} = -i\omega\varepsilon_{0}\widetilde{E}_{\rho}^{-} + \widetilde{J} \quad \text{in } B_{1}, \\ \nu \times \widetilde{E}_{\rho}^{+}|_{\partial B_{1}^{+}} = \nu \times \widetilde{E}_{\rho}^{-}|_{\partial B_{1}^{-}}, \quad \nu \times \widetilde{H}_{\rho}^{+}|_{\partial B_{1}^{+}} = \nu \times \widetilde{H}_{\rho}^{-}|_{\partial B_{1}^{-}}, \\ \nu \times \widetilde{E}_{\rho}^{+}|_{\partial B_{2}} = f. \end{cases}$$

and

(19)
$$\begin{cases} \nabla \times E_{\rho}^{+} = i\omega H_{\rho}^{+}, \quad \nabla \times H_{\rho}^{+} = -i\omega E_{\rho}^{+} \quad \text{in } B_{2} \setminus \overline{B_{\rho}}, \\ \nabla \times E_{\rho}^{-} = i\omega \mu_{\rho}(y) H_{\rho}^{-}, \quad \nabla \times H_{\rho}^{-} = -i\omega \varepsilon_{\rho}(y) E_{\rho}^{-} + J \quad \text{in } B_{\rho}, \\ \nu \times E_{\rho}^{+}|_{\partial B_{\rho}^{+}} = \nu \times E_{\rho}^{-}|_{\partial B_{\rho}^{-}}, \quad \nu \times H_{\rho}^{+}|_{\partial B_{\rho}^{+}} = \nu \times H_{\rho}^{-}|_{\partial B_{\rho}^{-}}, \\ \nu \times E_{\rho}^{+}|_{\partial B_{2}} = f. \end{cases}$$

Now we are ready to present our main theorem.

Theorem 2.1. Let E and H be 1-forms satisfying the Maxwell's equations on $B_2 \setminus \{0\}$, at frequency $\omega > 0$ which is not an eigenvalue for the background Maxwell's operator,

(20)
$$\begin{cases} \nabla \times E = i\omega H, \quad \nabla \times H = -i\omega E \quad on \ B_2 \\ \nu \times E|_{\partial B_2} = f, \end{cases}$$

and let E_0 and H_0 be solutions to

(21)
$$\begin{cases} \nabla \times E_0 = i\omega\mu_0 H_0, \quad \nabla \times H_0 = -i\omega\varepsilon_0 E_0 + \widetilde{J} \quad on \ B_1 \\ \nu \cdot E_0|_{\partial B_1} = \nu \cdot H_0|_{\partial B_1} = 0 \end{cases}$$

where \widetilde{J} is supported on B_{r_1} for some $r_1 < 1$. Moreover, suppose that \widetilde{E} and \widetilde{H} belong to $L^1(B_2; \mathbb{R}^3)$ such that

(22)
$$(\widetilde{E}, \widetilde{H}) = \begin{cases} (F_{1*}E, F_{1*}H), & \text{in } B_2 \setminus \overline{B_1} \\ (E_0, H_0), & \text{in } B_1 \end{cases}$$

where F_1 is the singular transformation given by (9). Then we have

(23)
$$\widetilde{E}_{\rho} \rightharpoonup \widetilde{E} + \alpha[J] \delta_{\partial B_1} \quad \widetilde{H}_{\rho} \rightharpoonup \widetilde{H} + \beta[J] \delta_{\partial B_1} \quad as \ \rho \to 0$$

in the weak topology of $H^{-1}(B_2)$, where $\delta_{\partial B_1}$ is the Dirac's delta distribution supported on ∂B_1 and $\alpha[J]$, $\beta[J]$ are smooth and depending on the source term J (see Remark 4.5). Moreover, we have (see Remark 4.2)

(24)
$$\nu \cdot (\operatorname{curl} \widetilde{H})|_{\partial B_1^-} = \nu \cdot (\operatorname{curl} \widetilde{E})|_{\partial B_1^-} = 0$$

and

(25)
$$\nu \times \tilde{H}|_{\partial B_1^+} = \nu \times \tilde{E}|_{\partial B_1^+} = 0.$$

We note that despite (25),

(26)
$$\mathbf{h}_t := \lim_{\rho \to 0} \nu \times \widetilde{H}_{\rho}|_{\partial B_1} \quad \text{and} \quad \mathbf{e}_t := \lim_{\rho \to 0} \nu \times \widetilde{E}_{\rho}|_{\partial B_1}.$$

may be non-zero. (See Remark 4.6)

3. MOTIVATION: A PHYSICAL EXAMPLE OF SCATTERING FROM THE HALF-SPACE.

Before the analysis of the waves in an approximative cloak, let us us consider as a motivation on a simple example of scattering of a plane wave from homogeneous half-space that resembles the approximate cloak near the cloaking surface. To begin with, we decompose the space \mathbb{R}^3 to half spaces $U_+ = \{(x, y, z); x > 0\}$ and $U_- = \{(x, y, z); x < 0\}$, and their interface $\Sigma = \{(x, y, z); x = 0\}$. We assume that the electromagnetic parameters in U_- correspond to vacuum, that is, $\varepsilon_- = I$ and $\mu_- = I$. In U_+ the electromagnetic parameters are given by constant matrices,

$$\varepsilon_+ = \operatorname{diag} (\varepsilon_x^+, \varepsilon_y^+, \varepsilon_z^+) \quad \mu_+ = \operatorname{diag} (\mu_x^+, \mu_y^+, \mu_z^+),$$

Below, $\hat{e}_x = (1, 0, 0)$, $\hat{e}_y = (0, 1, 0)$, and $\hat{e}_z = (0, 0, 1)$ are the unit coordinate vectors.

To study the case that is close to the 3-dimensional approximate cloak studied in this paper, we let

$$\varepsilon_x^+ = \mu_x^+ = 2\rho^2, \quad \varepsilon_y^+ = \mu_y^+ = 2, \quad \varepsilon_z^+ = \mu_z^+ = 2.$$

To illustrate, we consider an incident plane wave whose magnetic field is parallel to y-axis. It comes from the domain U_{-} to the interface Σ with non-zero incident angle and scatters and refracts at the interface. More precisely, we let magnetic field be $H_{\rho}(x, z) = h(x, z)\hat{e}_y$, where

$$h(x,z) = h^+(x,z) := h^+ e^{i(k_x^+ x + k_z^+ z)} \hat{e}_y$$
 for $x > 0$

and

$$h(x,z) = h^{-}(x,z) := (h^{in}e^{i(k_{x}^{-}x + k_{z}^{-}z)} + h^{sc}e^{i(-k_{x}^{-}x + k_{z}^{-}z)})\hat{e}_{y} \quad \text{ for } x < 0.$$

Recall that this corresponds to the decomposition of the wave in U_{-} into a sum of incident and reflected plane waves. At frequency $\omega > 0$, given the incident amplitude $h^{in} \in \mathbb{C}$ and the z-component of the incident wave vector $k_z^+ \in (0, \omega)$, we consider the behavior of the transmitted wave in U_{-} with different values of $\rho > 0$.

The field $H_{\rho}(x,z)$ satisfies Maxwell's equations in U_{\pm} if

(27)
$$\frac{1}{\mu_y} \left[\frac{\partial}{\partial x} \left(\frac{1}{\varepsilon_z} \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{1}{\varepsilon_x} \frac{\partial h}{\partial z} \right) \right] + \omega^2 h = 0.$$

This implies

(28)
$$\frac{1}{\mu_y^{\pm}\varepsilon_x^{\pm}}(k_z^{\pm})^2 + \frac{1}{\mu_y^{\pm}\varepsilon_z^{\pm}}(k_x^{\pm})^2 = \omega^2.$$

Correspondingly, the electric field is of the form $E_{\rho} = E_x \hat{e}_x + E_z \hat{e}_z$ where

$$E_x = -i\omega^{-1}\frac{1}{\varepsilon_x}\frac{\partial h}{\partial z}, \quad E_z = i\omega^{-1}\frac{1}{\varepsilon_z}\frac{\partial h}{\partial x}.$$

Now the transmission conditions on Σ ,

$$\begin{aligned} \hat{e}_x \times H_\rho|_{\Sigma^-} &= \hat{e}_x \times H_\rho|_{\Sigma^+}, \\ \hat{e}_x \times E_\rho|_{\Sigma^-} &= \hat{e}_x \times E_\rho|_{\Sigma^+} \end{aligned}$$

give $k_z^- = k_z^+$ and

(29)
$$h^{in} + h^{sc} = h^+,$$

(30)
$$k_x^-(h^{in} - h^{sc}) = \frac{1}{2}k_x^+ h^+$$

Since $k_z^- = k_z^+$, equation (28) implies

$$k_x^+ = \sqrt{\mu_y^+ \varepsilon_z^+ \left(\omega^2 - \frac{1}{\mu_y^+ \varepsilon_x^+} (k_z^+)^2\right)} = \sqrt{4\omega^2 - \frac{1}{\rho^2} (k_z^+)^2} = ik_z^+ \rho^{-1} (1 + O(\rho^{-1})).$$

Note that when $\rho \to 0$, the expression under the square root becomes negative. We have choosen above the positive sign for the imaginary part of the square. Denote below $t(\rho) = \text{Im } k_x^+$. Then $t(\rho) \to \infty$ as $\rho \to 0$.

By solving (29) and (30) we obtain

$$h^{+} = \frac{4k_{x}^{-}}{2k_{x}^{-} + k_{x}^{+}}h^{in}, \quad h^{sc} = -\left(1 - \frac{4k_{x}^{-}}{2k_{x}^{-} + k_{x}^{+}}\right)h^{in}.$$

Thus

$$H_{\rho}(x,z) = \frac{4k_x^- h^{in}}{2k_x^- + t(\rho)i} e^{-t(\rho)x + ik_z^- z} \hat{e}_y, \quad \text{for } x > 0,$$

$$H_{\rho}(x,z) = \left(e^{ik_x^- x + ik_z^- z} - \left(1 - \frac{4k_x^-}{2k_x^- + t(\rho)i}\right) e^{-ik_x^- x + ik_z^- z}\right) h^{in} \hat{e}_y, \quad \text{for } x < 0.$$

Next, let $\chi_{\mathbb{R}_+}(s)$ be the characteristic function of the set \mathbb{R}_+ . Since

$$\lim_{t \to \infty} \chi_{\mathbb{R}_+}(s) \frac{1}{t} e^{-ts} = \delta_0(s)$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R})$, we see that

$$\lim_{\rho \to 0} H_{\rho}(x,z) = \chi_{\mathbb{R}_{-}}(x) (e^{ik_{x}^{-}x + ik_{z}^{-}z} - e^{-ik_{x}^{-}x + ik_{z}^{-}z}) h^{in} \hat{e}_{y} - 4ik_{x}^{-} h^{in} \delta_{0}(x) \hat{e}_{y},$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R}^3)$. This means that the magnetic field H_{ρ} tends to a generalized function that is a sum of a measurable function and a delta distribution as $\rho \to 0$. Note that here the delta distribution component is caused by the concentration of the waves in a thin layer near the interface in the region U_- . Note that the blow up of the fields at the interface, that causes the delta distribution to appear, causes the boundary also to reflect the incoming wave perfectly with the reflection coefficient -1. Similar considerations to the above one, with slightly different setting where μ is constant in the whole space, are done in [36]. Also, the physical interpretation of the blow-up of the fields at the interface as infinite polarization of the material is analyzed in [36] in detail. Our aim is to show that similar phenomenon appears in the 3-dimensional approximate cloak as the cloak tends to the ideal one and analyze the convergence of the electromagnetic fields in the sense of distributions.

4. Limiting behavior at the interface

4.1. Spherical harmonic expansions. Through out the rest of the paper, we assume that the cloaked medium is uniform, i.e., ε_0 and μ_0 are constants. Then the EM fields $(\tilde{E}_{\rho}^-, \tilde{H}_{\rho}^-)$ and (E_{ρ}^+, H_{ρ}^+) both admit the spherical harmonic expansions, given by

(31)

$$\begin{split} \widetilde{E}_{\rho}^{-} &= \varepsilon_{0}^{-1/2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \alpha_{n}^{m} M_{n,k\omega}^{m} + \beta_{n}^{m} \nabla \times M_{n,k\omega}^{m} + p_{n}^{m} N_{n,k\omega}^{m} + q_{n}^{m} \nabla \times N_{n,k\omega}^{m} \\ \widetilde{H}_{\rho}^{-} &= \frac{1}{ik\omega} \mu_{0}^{-1/2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} k^{2} \omega^{2} \beta_{n}^{m} M_{n,k\omega}^{m} + \alpha_{n}^{m} \nabla \times M_{n,k\omega}^{m} \\ &+ k^{2} \omega^{2} q_{n}^{m} N_{n,k\omega}^{m} + p_{n}^{m} \nabla \times N_{n,k\omega}^{m}, \end{split}$$

for $x \in B_1 \setminus \overline{B_{r_1}}$, where $k := (\mu_0 \varepsilon_0)^{1/2}$, and

$$E_{\rho}^{+} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \gamma_{n}^{m} M_{n,\omega}^{m} + \eta_{n}^{m} \nabla \times M_{n,\omega}^{m} + c_{n}^{m} N_{n,\omega}^{m} + d_{n}^{m} \nabla \times N_{n,\omega}^{m},$$

$$H_{\rho}^{+} = \frac{1}{i\omega} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \omega^{2} \eta_{n}^{m} M_{n,\omega}^{m} + \gamma_{n}^{m} \nabla \times M_{n,\omega}^{m} + \omega^{2} d_{n}^{m} N_{n,\omega}^{m} + c_{n}^{m} \nabla \times N_{n,\omega}^{m}$$
(32)

for $y \in B_2 \setminus \overline{B_{\rho}}$, where the basis vectors are defined in Appendix A by (67). Notice that the last two terms in the expansion of \widetilde{E}_{ρ}^- with respect to $N_{n,k\omega}^m$ and $\nabla \times N_{n,k\omega}^m$ represent the radiating EM field generated by \widetilde{J} . Hence, the coefficients p_n^m and q_n^m are determined.

In terms of the vector spherical harmonics (69), we have the expansion of the boundary condition

(33)
$$f = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} S_n (f_{nm}^{(1)} U_n^m + f_{nm}^{(2)} V_n^m).$$

where $S_n := \sqrt{n(n+1)}$. By (74), (75) and the boundary condition $\nu \times E_{\rho}^+|_{\partial B_2} = f$, we obtain

(34)
$$c_n^m h_n^{(1)}(2\omega) + \gamma_n^m j_n(2\omega) = f_{nm}^{(1)}, \quad d_n^m \mathcal{H}_n(2\omega) + \eta_n^m \mathcal{J}_n(2\omega) = 2f_{nm}^{(2)}.$$

The transmission conditions in (18) reads

(35)
$$\hat{x} \times \widetilde{E}_{\rho}^{+}|_{\partial B_{1}^{+}} = \rho(\hat{y} \times E_{\rho}^{+}|_{\partial B_{\rho}^{+}}) = \hat{x} \times \widetilde{E}_{\rho}^{-}|_{\partial B_{1}^{-}},$$

which gives

(36)
$$\rho c_n^m h_n^{(1)}(\omega \rho) + \rho \gamma_n^m j_n(\omega \rho) = \varepsilon_0^{-1/2} (\alpha_n^m j_n(k\omega) + p_n^m h_n^{(1)}(k\omega)),$$
$$d_n^m \mathcal{H}_n(\omega \rho) + \eta_n^m \mathcal{J}_n(\omega \rho) = \varepsilon_0^{-1/2} (\beta_n^m \mathcal{J}_n(k\omega) + q_n^m \mathcal{H}_n(k\omega)).$$

Similarly, the transmission condition on the magnetic field implies

(37)
$$kc_n^m \mathcal{H}_n(\omega\rho) + k\gamma_n^m \mathcal{J}_n(\omega\rho) = \mu_0^{-1/2} (\alpha_n^m \mathcal{J}_n(k\omega) + p_n^m \mathcal{H}_n(k\omega)),$$
$$\rho d_n^m h_n^{(1)}(\omega\rho) + \rho \eta_n^m j_n(\omega\rho) = \mu_0^{-1/2} (k\beta_n^m j_n(k\omega) + kq_n^m h_n^{(1)}(k\omega)).$$

Solving (36) and (37), we have

(38)
$$c_n^m = t_1 \gamma_n^m + t_1' p_n^m, \quad \alpha_n^m = t_2 \gamma_n^m + t_2' p_n^m, \\ d_n^m = t_3 \eta_n^m + t_3' q_n^m, \quad \beta_n^m = t_4 \eta_n^m + t_4' q_n^m,$$

where

(39)

$$t_{1} := \frac{1}{D_{n}} \left[\varepsilon_{0}^{-1/2} k \mathcal{J}_{n}(\omega\rho) j_{n}(k\omega) - \mu_{0}^{-1/2} \rho j_{n}(\omega\rho) \mathcal{J}_{n}(k\omega) \right],$$

$$t_{2} := \frac{1}{D_{n}} \left[k \rho \mathcal{J}_{n}(\omega\rho) h_{n}^{(1)}(\omega\rho) - k \rho j_{n}(\omega\rho) \mathcal{H}_{n}(\omega\rho) \right],$$

$$t_{3} := \frac{1}{D_{n}'} \left[\mu_{0}^{-1/2} k \mathcal{J}_{n}(\omega\rho) j_{n}(k\omega) - \varepsilon_{0}^{-1/2} \rho j_{n}(\omega\rho) \mathcal{J}_{n}(k\omega) \right],$$

$$t_{4} := \frac{1}{D_{n}'} \left[\rho \mathcal{J}_{n}(\omega\rho) h_{n}^{(1)}(\omega\rho) - \rho j_{n}(\omega\rho) \mathcal{H}_{n}(\omega\rho) \right];$$

 $\quad \text{and} \quad$

(40)
$$t_{1}' := \frac{1}{D_{n}} \left[h_{n}^{(1)}(k\omega) \mathcal{J}_{n}(k\omega) - \mathcal{H}_{n}(k\omega) j_{n}(k\omega) \right],$$
$$t_{2}' := \frac{1}{D_{n}} \left[\varepsilon_{0}^{-1/2} k h_{n}^{(1)}(k\omega) \mathcal{H}_{n}(\omega\rho) - \mu_{0}^{-1/2} \rho \mathcal{H}_{n}(k\omega) h_{n}^{(1)}(\omega\rho) \right],$$
$$t_{3}' := \frac{1}{D_{n}'} \left[\mathcal{J}_{n}(k\omega) h_{n}^{(1)}(k\omega) - \mathcal{H}_{n}(k\omega) j_{n}(k\omega) \right],$$
$$t_{4}' := \frac{1}{D_{n}'} \left[\mu_{0}^{-1/2} k h_{n}^{(1)}(k\omega) \mathcal{H}_{n}(\omega\rho) - \varepsilon_{0}^{-1/2} \rho \mathcal{H}_{n}(k\omega) h_{n}^{(1)}(\omega\rho) \right],$$

with

(41)
$$D_n = \mu_0^{-1/2} \rho h_n^{(1)}(\omega \rho) \mathcal{J}_n(k\omega) - \varepsilon_0^{-1/2} k \mathcal{H}_n(\omega \rho) j_n(k\omega),$$
$$D'_n = \varepsilon_0^{-1/2} \rho h_n^{(1)}(\omega \rho) \mathcal{J}_n(k\omega) - \mu_0^{-1/2} k \mathcal{H}_n(\omega \rho) j_n(k\omega).$$

Plugging into (34), we obtain

(42)
$$\gamma_n^m = \frac{f_{nm}^{(1)} - p_n^m t_1' h_n^{(1)}(2\omega)}{t_1 h_n^{(1)}(2\omega) + j_n(2\omega)}, \quad \eta_n^m = \frac{2f_{nm}^{(2)} - t_3' q_n^m \mathcal{H}_n(2\omega)}{t_3 \mathcal{H}_n(2\omega) + \mathcal{J}_n(2\omega)}.$$

In this notes, we are interested in the effect of an active source cloaked in B_1 , hence assuming no incident wave for simplicity, i.e., the case of zero boundary condition f = 0. Therefore, we have

(43)
$$\gamma_n^m = \frac{-p_n^m t_1' h_n^{(1)}(2\omega)}{t_1 h_n^{(1)}(2\omega) + j_n(2\omega)}, \quad \eta_n^m = \frac{-t_3' q_n^m \mathcal{H}_n(2\omega)}{t_3 \mathcal{H}_n(2\omega) + \mathcal{J}_n(2\omega)}.$$

We will also need the following asymptotic estimates derived from (77) for $n \ge 0$

$$t_{3} = \frac{i\pi(n+1)}{\Gamma(n+1/2)\Gamma(n+3/2)n} \left(\frac{\omega}{2}\right)^{2n+1} \rho^{2n+1} (1+O_{\rho\to 0}(\rho)),$$

$$t_{4} = \frac{(2n+1)\sqrt{\pi}}{\Gamma(n+3/2)\mu_{0}^{-1/2}k\omega n j_{n}(k\omega)} \left(\frac{\omega}{2}\right)^{n+1} \rho^{n+1} (1+O_{\rho\to 0}(\rho)),$$

$$(44)$$

$$t_{3}' = \frac{2i\sqrt{\pi} \left[\mathcal{J}_{n}(k\omega)h_{n}^{(1)}(k\omega) - \mathcal{H}_{n}(k\omega)j_{n}(k\omega)\right]}{\Gamma(n+1/2)\mu_{0}^{-1/2}kn j_{n}(k\omega)} \left(\frac{\omega}{2}\right)^{n+1} \rho^{n+1} (1+O_{\rho\to 0}(\rho)),$$

$$t_{4}' = -\frac{h_{n}^{(1)}(k\omega)}{j_{n}(k\omega)} (1+O_{\rho\to 0}(\rho)),$$

where $|O_{\rho\to 0}(\rho)| \leq C(n)\rho$ for some constant C(n) depending on n.

It suffices to show the proof for the electric field only, and that of magnetic field is obtained by symmetry.

4.2. Normal components. Consider

(45)
$$\int_{B_2 \setminus \overline{B_{r_1}}} (\hat{x} \cdot \widetilde{E}_{\rho}) \varphi \, dx = \int_{B_1 \setminus \overline{B_{r_1}}} (\hat{x} \cdot \widetilde{E}_{\rho}^-) \varphi \, dx + \int_{B_2 \setminus \overline{B_1}} (\hat{x} \cdot \widetilde{E}_{\rho}^+) \varphi \, dx.$$

for $\varphi \in H_0^1(B_2)$, which admits the spherical expansion

(46)
$$\varphi(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \varphi_n^m(|x|) Y_n^m(\hat{x}), \quad x \in B_2.$$

and necessarily

(47)
$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \int_{0}^{2} |\varphi_{n}^{m}(r)|^{2} r dr < \infty, \qquad \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \int_{0}^{2} \left| \frac{d}{dr} \varphi_{n}^{m}(r) \right|^{2} r dr < \infty.$$

In the physical region $r_1 < |x| < 1$, by (74) and (76), we have

(48)
$$\hat{x} \cdot \tilde{E}_{\rho}^{-} = \varepsilon_{0}^{-1/2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} S_{n}^{2} \frac{1}{|x|} \left[\beta_{n}^{m} j_{n}(k\omega|x|) + q_{n}^{m} h_{n}^{(1)}(k\omega|x|) \right] Y_{n}^{m}(\hat{x}).$$

Then the first integral of (45) is given by

$$\int_{B_1 \setminus \overline{B_{r_1}}} (\hat{x} \cdot \widetilde{E}_{\rho}^{-}) \varphi \, dx = \int_{r_1}^1 \sum_{n=1}^\infty \sum_{m=-n}^n \widetilde{\psi}_{n,\rho}^m(\widetilde{r}) \, d\widetilde{r}$$

where

$$\widetilde{\psi}_{n,\rho}^{m}(\widetilde{r}) := S_{n}^{2} \varepsilon_{0}^{-1/2} \left[\beta_{n}^{m} j_{n}(k\omega \widetilde{r}) + q_{n}^{m} h_{n}^{(1)}(k\omega \widetilde{r}) \right] \varphi_{n}^{m}(\widetilde{r}) \widetilde{r}.$$

Moreover, we can show

Lemma 4.1. Suppose that \widetilde{J} is supported in B_{r_1} for $r_1 < 1$. Then we have

(49)
$$\lim_{\rho \to 0} \int_{B_1 \setminus \overline{B_{r_1}}} (\hat{x} \cdot \widetilde{E}_{\rho}^-) \varphi \, dx = \int_{r_1}^1 \sum_{n=1}^\infty \sum_{m=-n}^n \lim_{\rho \to 0} \widetilde{\psi}_{n,\rho}^m(\widetilde{r}) \, d\widetilde{r}.$$

Proof. Given that the current source \widetilde{J} is supported on B_{r_1} , the radiation part (only depending on J) of \widetilde{E}_{ρ}^- , that is,

(50)
$$\sum_{n=1}^{\infty} \sum_{m=-n}^{n} S_n^2 \varepsilon_0^{-1/2} q_n^m h_n^{(1)}(k\omega \widetilde{r}) \varphi_n^m(\widetilde{r}) \widetilde{r}$$

is C^{∞} for $|x| > r_1$, hence is uniformly convergent for $\tilde{r} \in [r_1, 1]$ and independent of ρ . Furthermore, we have

$$\left|S_{n}^{2}q_{n}^{m}h_{n}^{(1)}(k\omega\widetilde{r})\widetilde{r}\right| \lesssim_{M} (1+n)^{-M}, \quad \text{for } \widetilde{r} > r_{1}$$

for arbitrary M > 0, where $A \leq_M B$ represents $A \leq C_M B$ for a constant C_M only depending on M. Using (77) (or equivalently (78)), let $N_1 > 0$ be such that $N_1 >> k\omega$, then for $n > N_1$, $|h_n^{(1)}(k\omega \tilde{r})|$ is monotonically decreasing for $\tilde{r} \in [r_1, 1]$, implying

(52)
$$|q_n^m| \lesssim_M S_n^{-2} \frac{1}{|h_n^{(1)}(k\omega r_1)|} (1+n)^{-M}.$$

By (38) and (43) we have

$$\beta_{n,\rho}^m = \frac{(t_3t_4' - t_4t_3')\mathcal{H}_n(2\omega) + t_4'\mathcal{J}_n(2\omega)}{t_3\mathcal{H}_n(2\omega) + \mathcal{J}_n(2\omega)}q_n^m.$$

From (44), for $n \ge 0$

$$(t_3t'_4 - t_4t'_3) = -i\pi \frac{n+1}{n\Gamma(n+1/2)\Gamma(n+3/2)} \frac{h_n^{(1)}(k\omega)}{j_n(k\omega)} \left(\frac{\omega}{2}\right)^{2n+1} \rho^{2n+1} \left(1 + O_{\rho \to 0}(\rho)\right),$$

which implies for n > 0

$$\beta_{n,\rho}^m = -\frac{h_n^{(1)}(k\omega)}{j_n(k\omega)} q_n^m \left(1 + O_{\rho \to 0}(\rho)\right).$$

Therefore,

(53)
$$\beta_{n,0}^m = \lim_{\rho \to 0} \beta_{n,\rho}^m = -\frac{h_n^{(1)}(k\omega)}{j_n(k\omega)} q_n^m$$

and the convergence is uniform. As a consequence, for $\rho << 1 << N_1 < n$ and $\widetilde{r} \in [r_1, 1]$, the other term of $\widetilde{\psi}^m_{n,\rho}$ can be bounded by

$$\begin{split} \left| S_n^2 \varepsilon_0^{-1/2} \beta_{n,\rho}^m j_n(k\omega \widetilde{r}) \varphi_n^m(\widetilde{r}) \widetilde{r} \right| &= \left| \frac{\beta_{n,\rho}^m j_n(k\omega \widetilde{r})}{q_n^m h_n^{(1)}(k\omega \widetilde{r})} \right| \left| S_n^2 \varepsilon_0^{-1/2} q_n^m h_n^{(1)}(k\omega \widetilde{r}) \varphi_n^m(\widetilde{r}) \widetilde{r} \right| \\ &\lesssim \left| \frac{j_n(k\omega \widetilde{r}) h_n^{(1)}(k\omega)}{j_n(k\omega) h_n^{(1)}(k\omega \widetilde{r})} \right| \left| S_n^2 \varepsilon_0^{-1/2} q_n^m h_n^{(1)}(k\omega \widetilde{r}) \varphi_n^m(\widetilde{r}) \widetilde{r} \right| \\ &\lesssim \left| S_n^2 \varepsilon_0^{-1/2} q_n^m h_n^{(1)}(k\omega \widetilde{r}) \varphi_n^m(\widetilde{r}) \widetilde{r} \right|. \end{split}$$

Therefore, the series converges uniformly (independent of ρ) and by dominated convergence theorem the lemma is proved.

Remark 4.2. The proof actually shows that

$$\lim_{\rho \to 0} \int_{B_1 \setminus \overline{B_{r_1}}} (\hat{x} \cdot \widetilde{E}_{\rho}^-) \varphi \, dx = \int_{r_1}^1 (\hat{x} \cdot \widetilde{E}_0^-) \varphi \, dx$$

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(51)

The word "should" is removed.- Matti

where

$$\hat{x} \cdot \widetilde{E}_0^{-}(x) := \varepsilon_0^{-1/2} \sum_{n=1}^{\infty} \sum_{m=-n}^n S_n^2 \frac{1}{|x|} \left[\beta_{n,0}^m j_n(k\omega|x|) + q_n^m h_n^{(1)}(k\omega|x|) \right] Y_n^m(\hat{x}).$$

By the uniform convergence of (53) and estimate (52), we have

$$\hat{x} \cdot \widetilde{E}_0^-|_{\partial B_1^-} = 0.$$

Notice that the medium inside B_1 is regular and \tilde{J} is supported away from ∂B_1 , this proves (24) in Theorem 2.1, which is consistent to Weder's definition [33].

Now we focus on the second integral of (45). First of all,

(54)
$$\int_{B_2 \setminus \overline{B_1}} (\hat{x} \cdot \widetilde{E}_{\rho}^+) \varphi \, dx = \int_{B_2 \setminus \overline{B_\rho}} \frac{1}{b} (\hat{y} \cdot E_{\rho}^+(y)) \varphi \left(F_{\rho}(y)\right) \left| DF_{\rho}(y) \right| \, dy$$
$$= \int_{\rho}^{2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \psi_{n,\rho}^m(r) \, dr$$

where

$$\psi_{n,\rho}^{m}(r) := S_{n}^{2} \left[d_{n,\rho}^{m} h_{n}^{(1)}(\omega r) + \eta_{n,\rho}^{m} j_{n}(\omega r) \right] \varphi_{n}^{m}(a+br) \ \frac{(a+br)^{2}}{r}$$

and $d_{n,\rho}^m := d_n^m$ and $\eta_{n,\rho}^m := \eta_n^m$ to indicate the ρ dependence. We separate the series into two parts: let $N_2 > 0$ be such that $\max\{2\omega, k\omega\} << N_1 < N_2$ (as defined in (77)) and consider

$$I_1(\rho) := \int_{\rho}^2 \sum_{n=1}^{N_2} \sum_{m=-n}^n \psi_{n,\rho}^m(r) \, dr, \qquad I_2(\rho) := \int_{\rho}^2 \sum_{n=N_2+1}^\infty \sum_{m=-n}^n \psi_{n,\rho}^m(r) \, dr.$$

First we have

Lemma 4.3.

(55)
$$\lim_{\rho \to 0} I_1(\rho) = \sum_{n=1}^{N_2} \sum_{m=-n}^n S_n^2 \frac{\mu_0^{1/2} \left[\mathcal{J}_n(k\omega) h_n^{(1)}(k\omega) - \mathcal{H}_n(k\omega) j_n(k\omega) \right]}{kn(n+1)j_n(k\omega)} q_n^m \varphi_n^m(1).$$

Proof. We first consider the limit

(56)
$$\lim_{\rho \to 0} \int_{\rho}^{2} \sum_{n=1}^{N_{2}} \sum_{m=-n}^{n} S_{n}^{2} \eta_{n,\rho}^{m} j_{n}(\omega r) \varphi_{n}^{m}(a+br) \frac{(a+br)^{2}}{r} dr$$
$$= \sum_{n=1}^{N_{2}} \sum_{m=-n}^{n} S_{n}^{2} \lim_{\rho \to 0} \eta_{n,\rho}^{m} \int_{\rho}^{2} j_{n}(\omega r) \varphi_{n}^{m}(a+br) \frac{(a+br)^{2}}{r} dr.$$

From (43), (38) and (44), for $\rho << 1$ and $0 \le n \le N_2$,

(57)
$$\begin{aligned} |\eta_n^m| &= \left| \frac{-t_3' \mathcal{H}_n(2\omega) q_n^m}{t_3 \mathcal{H}_n(2\omega) + \mathcal{J}_n(2\omega)} \right| \lesssim_{N_2} |q_n^m| \rho^{n+1}, \\ |d_n^m| &= \left| \frac{t_3' \mathcal{J}_n(2\omega) q_n^m}{t_3 \mathcal{H}_n(2\omega) + \mathcal{J}_n(2\omega)} \right| \lesssim_{N_2} |q_n^m| \rho^{n+1}. \end{aligned}$$

Therefore, by Cauchy-Schwartz and (47)

$$\left|\eta_{n,\rho}^{m}\int_{\rho}^{2}j_{n}(\omega r)\varphi_{n}^{m}(a+br)\frac{(a+br)^{2}}{r}\ dr\right|\lesssim\rho^{n+1/2}\to0,\qquad\text{as}\quad\rho\to0.$$

This implies that the limit (56) is 0.

To take care of the other terms in I_1 , we choose $1 >> \rho_1 > \rho$ and consider

$$\sum_{n=1}^{N_2} \sum_{m=-n}^n S_n^2 \lim_{\rho \to 0} d_{n,\rho}^m \left(\int_{\rho}^{\rho_1} + \int_{\rho_1}^2 \right) h_n^{(1)}(\omega r) \varphi_n^m (a+br) \frac{(a+br)^2}{r} dr.$$

By (57), it is easy to see

$$\sum_{n=1}^{N_2} \sum_{m=-n}^n S_n^2 \lim_{\rho \to 0} d_{n,\rho}^m \int_{\rho_1}^2 h_n^{(1)}(\omega r) \varphi_n^m (a+br) \frac{(a+br)^2}{r} dr = 0.$$

To see that the terms integrated on (ρ, ρ_1) converge to the right hand side of (55), we first apply integration by parts

$$d_{n,\rho}^{m} \int_{\rho}^{\rho_{1}} h_{n}^{(1)}(\omega r) \varphi_{n}^{m}(a+br) \frac{(a+br)^{2}}{r} dr = d_{n,\rho}^{m} \varphi_{n}^{m}(1) A_{n}(\rho) - d_{n,\rho}^{m} \int_{\rho}^{\rho_{1}} \varphi_{n}^{m'}(a+br) b A_{n}(r) dr$$

where

$$A_n(r) := \int_r^{\rho_1} h_n^{(1)}(\omega s)(a+bs)^2 s^{-1} \, ds.$$

For $0 \le n \le N_2$, and $r < \rho_1 << 1$

$$A_n(r) = -\frac{i}{2\sqrt{\pi}}\Gamma(n+1/2)\left(\frac{2}{\omega}\right)^{n+1} \int_r^{\rho_1} (a+bs)^2 s^{-(n+2)} (1+O(s)) ds$$
$$= -\frac{i}{2\sqrt{\pi}}\Gamma(n+1/2)\left(\frac{2}{\omega}\right)^{n+1} \frac{a^2}{n+1} r^{-(n+1)} (1+O_{r\to 0}(r^{-n})).$$

Therefore, as $\rho \to 0$,

$$A_n(\rho) = -\frac{i\Gamma(n+1/2)}{2\sqrt{\pi}(n+1)} \left(\frac{2}{\omega}\right)^{n+1} \rho^{-(n+1)} \left(1 + O_{\rho \to 0}(\rho)\right)$$

and by (57)

$$\left| d_{n,\rho}^m \int_{\rho}^{\rho_1} \varphi_n^{m\prime}(a+br) b A_n(r) \ dr \right| \lesssim \left(\int_{1}^{2} |\varphi_n^{m\prime}(\tilde{r})|^2 \tilde{r} \ d\tilde{r} \right)^{1/2} \left(\int_{\rho}^{\rho_1} |A_n(r)|^2 \frac{b}{a+br} \ dr \right)^{1/2}$$
$$\lesssim_{\varphi} \rho^{1/2} \to 0 \quad \text{as } \rho \to 0.$$

Together we have

$$\lim_{\rho \to 0} I_1(\rho) = \sum_{n=1}^{N_2} \sum_{m=-n}^n S_n^2 \{\lim_{\rho \to 0} d_{n,\rho}^m A_n(\rho)\} \varphi_n^m(1).$$

From (57) and (44), we further have as $\rho \to 0$, (58)

$$d_{n,\rho}^{m} = \frac{2i\sqrt{\pi} \left[\mathcal{J}_{n}(k\omega)h_{n}^{(1)}(k\omega) - \mathcal{H}_{n}(k\omega)j_{n}(k\omega) \right]}{\Gamma(n+1/2)\mu_{0}^{-1/2}knj_{n}(k\omega)} \left(\frac{\omega}{2}\right)^{n+1} q_{n}^{m}\rho^{n+1} \left(1 + O_{\rho \to 0}(\rho)\right),$$

implying

$$\lim_{\rho \to 0} d_{n,\rho}^m A_n(\rho) = \frac{\mu_0^{1/2} \left[\mathcal{J}_n(k\omega) h_n^{(1)}(k\omega) - \mathcal{H}_n(k\omega) j_n(k\omega) \right]}{kn(n+1)j_n(k\omega)} q_n^m.$$

Therefore, (55) is proved.

Before considering the tail term $I_2(\rho)$, let us define

$$B_n(r) := \int_r^2 h_n^{(1)}(\omega s)(a+bs)^2 s^{-1} \, ds, \quad r > \rho.$$

for $n > N_2$ (so we can use (78)), similar to $A_n(r)$, we have

(59)
$$B_n(\rho) = -\frac{i\Gamma(n+1/2)}{2\sqrt{\pi}(n+1)} \left(\frac{2}{\omega}\right)^{n+1} \rho^{-(n+1)} \left(1 + O_{\rho \to 0}(\rho)\right),$$

and

(60)
$$B_n(r) = -\frac{i}{2\sqrt{\pi}}\Gamma(n+1/2)\left(\frac{2}{\omega}\right)^{n+1}\frac{a^2}{n+1}r^{-(n+1)}\left(1+O_{r\to 0}(r^{-n})\right)$$

for $n > N_2$.

Lemma 4.4.

(61)

$$\lim_{\rho \to 0} I_2(\rho) = \sum_{n=N_2+1}^{\infty} \sum_{m=-n}^n S_n^2 \frac{\mu_0^{1/2} \left[\mathcal{J}_n(k\omega) h_n^{(1)}(k\omega) - \mathcal{H}_n(k\omega) j_n(k\omega) \right]}{kn(n+1)j_n(k\omega)} q_n^m \varphi_n^m(1).$$

Remark 4.5. Along with Lemma 4.3, this shows that the limit of the normal component of the exterior field is some function (or distribution) times $\delta(\tilde{r}-1)$. The smoothness of this function (the strength of the delta singularity) is estimated by the growth of the coefficient with respect to $n \gg 1$. For $n \ge N_2$, by (77), (80) and (52), we have

$$\begin{vmatrix} S_n^2 \frac{\mu_0^{1/2} \left[\mathcal{J}_n(k\omega) h_n^{(1)}(k\omega) - \mathcal{H}_n(k\omega) j_n(k\omega) \right]}{kn(n+1)j_n(k\omega)} \\ = \frac{\mu_0^{1/2} \frac{(2n+1)\Gamma(n+1/2)}{2\Gamma(n+3/2)} (k\omega)^{-1}}{kn(n+1) \frac{\sqrt{\pi}}{2\Gamma(n+1/2)} \left(\frac{k\omega}{2}\right)^n} S_n^2 |q_n^m| \left(1 + O_{n \to \infty}(\frac{1}{n}) \right) \lesssim_M \frac{2n+1}{kn(n+1)} r_1^{n+1} (1+n)^{-M} dx + \frac{1}{2} \frac{2n+1}{kn(n+1)} \frac{2n+1}{kn(n+1)} \frac{2n+1}{kn(n+1)} r_1^{n+1} (1+n)^{-M} dx + \frac{1}{2} \frac{2n+1}{kn(n+1)} \frac{2n+1}$$

for any M > 0.

Proof of Lemma 4.4. Taking into account the *n*-dependence, we have from (43), (38) and (44),

(62)
$$|\eta_{n,\rho}^m| \lesssim \Gamma(n+1/2)\Gamma(n+3/2)\omega^{-n}|q_n^m| \left(\frac{\rho}{k\omega}\right)^{n+1}, \quad |d_{n,\rho}^m| \lesssim |q_n^m| \left(\frac{\rho}{k}\right)^{n+1}$$

where the general constants associated to \leq are independent of n.

First to show

(63)
$$I_2(\rho) = \sum_{n=N_2+1}^{\infty} \sum_{m=-n}^{n} \Psi_{n,\rho}^m, \qquad \Psi_{n,\rho}^m := \int_{\rho}^{2} \psi_{n,\rho}^m(r) \ dr,$$

by (62) and (77), we have for $r \in [\rho, 2]$ and $n > N_2$

$$\left| d_{n,\rho}^{m} h_{n}^{(1)}(\omega r) + \eta_{n,\rho}^{m} j_{n}(\omega r) \right| \lesssim \Gamma(n+1/2) \left(\frac{2}{k\omega} \right)^{n+1} |q_{n}^{m}| \rho^{n+1} r^{-(n+1)}.$$

By the estimates (52) for q_n^m and $N_1 < N_2$,

$$|h_n^{(1)}(k\omega r_1)| \gtrsim \Gamma(n+1/2) \left(\frac{2}{k\omega r_1}\right)^{n+1},$$

we have

$$\int_{\rho}^{2} |\psi_{n,\rho}^{m}(r)| dr \lesssim S_{n}^{2} \Gamma(n+1/2) \left(\frac{2}{k\omega}\right)^{n+1} |q_{n}^{m}| \rho^{n+1} \int_{\rho}^{2} |\varphi_{n}^{m}(a+br)| r^{-n-2} dr$$
$$\lesssim_{M} (1+n)^{-M} r_{1}^{n+1} \rho^{-1/2},$$

implying that the series is uniformly convergent. So (63) is valid.

To show

(64)
$$\lim_{\rho \to 0} \sum_{n=N_2+1}^{\infty} \sum_{m=-n}^{n} \Psi_{n,\rho}^m = \sum_{n=N_2+1}^{\infty} \sum_{m=-n}^{n} \lim_{\rho \to 0} \Psi_{n,\rho}^m$$

we write for $n \ge N_2$

$$\begin{split} \Psi^m_{n,\rho} &= S_n^2 \eta^m_{n,\rho} \int_{\rho}^2 j_n(\omega r) \varphi^m_n(a+br)(a+br)^2 r^{-1} \ dr \\ &+ S_n^2 d^m_{n,\rho} \int_{\rho}^2 h_n^{(1)}(\omega r) \varphi^m_n(a+br)(a+br)^2 r^{-1} \ dr \\ &:= \Psi^m_{1,n,\rho} + \Psi^m_{2,n,\rho}. \end{split}$$

By (62) and (77), we have

$$\begin{split} |\Psi_{1,n,\rho}^{m}| &\lesssim S_{n}^{2} |q_{n}^{m}| \Gamma(n+1/2) \left(\frac{\rho}{2k\omega}\right)^{n+1} \int_{\rho}^{2} r^{n-1} |\varphi_{n}^{m}(a+br)| (a+br)^{2} dr \\ &\lesssim S_{n}^{2} |q_{n}^{m}| \frac{\Gamma(n+1/2)}{2n-1} \left(\frac{\rho}{2k\omega}\right)^{n+1} \left(2^{n} + O(\rho^{n-1/2})\right) \\ &\lesssim S_{n}^{2} |q_{n}^{m}| \frac{\Gamma(n+1/2)}{n} \left(\frac{\rho}{k\omega}\right)^{n+1} \end{split}$$

where the general constants are independent of $n > N_2$ and ρ . The right hand side is summable with respect to n uniformly in ρ , using (52). Moreover, it converges to 0 as $\rho \to 0$. Therefore,

(65)
$$\lim_{\rho \to 0} \sum_{n=N_2+1}^{\infty} \sum_{m=-n}^{n} \Psi_{1,n,\rho}^m = \sum_{n=N_2+1}^{\infty} \sum_{m=-n}^{n} \lim_{\rho \to 0} \Psi_{1,n,\rho}^m = 0.$$

For the second term, by the integration by parts, we have

$$\Psi_{2,n,\rho}^{m} = S_{n}^{2} d_{n,\rho}^{m} \left\{ \varphi_{n}^{m}(1) B_{n}(\rho) - \int_{\rho}^{2} \varphi_{n}^{m'}(a+br) b B_{n}(r) \ dr \right\}.$$

Combining (62) and (77), we obtain

$$\begin{split} \left| S_n^2 d_{n,\rho}^m B_n(\rho) \varphi_n^m(1) \right| &\lesssim S_n^2 \left(\frac{2}{k\omega}\right)^{n+1} \frac{\Gamma(n+1/2)}{n+1} |q_n^m \varphi_n^m(1)| \\ &\lesssim \left(\frac{2}{k\omega}\right)^{n+1} \frac{\Gamma(n+1/2)}{n+1} \frac{1}{|h_n^{(1)}(k\omega)|} |S_n^2 q_n^m h_n^{(1)}(k\omega) \varphi_n^m(1)| \\ &\lesssim |S_n^2 q_n^m h_n^{(1)}(k\omega) \varphi_n^m(1)| \end{split}$$

which is summable by that of (50). Also, by (62), (60) and (51), one has

$$\begin{aligned} \left| S_n^2 d_{n,\rho}^m \int_{\rho}^{2} \varphi_n^{m'}(a+br) b B_n(r) \, dr \right| &\lesssim S_n^2 |q_n^m| \left(\frac{\rho}{k}\right)^{n+1} \frac{\Gamma(n+1/2)}{n+1} \left(\frac{2}{\omega}\right)^{n+1} \rho^{-(n+1/2)} \\ &\lesssim S_n^2 |q_n^m| \frac{\Gamma(n+1/2)}{n+1} \left(\frac{2}{k\omega}\right)^{n+1} \rho^{1/2} \lesssim_M (1+n)^{-(M+1)} \rho^{1/2} \end{aligned}$$

By Lebesgue dominated convergence theorem, we have

$$\lim_{\rho \to 0} \sum_{n=N_2+1}^{\infty} \sum_{m=-n}^{n} \Psi_{2,n,\rho}^m = \sum_{n=N_2+1}^{\infty} \sum_{m=-n}^{n} \lim_{\rho \to 0} \Psi_{2,n,\rho}^m = \sum_{n=N_2+1}^{\infty} \sum_{m=-n}^{n} S_n^2 B_n^m \varphi_n^m(1)$$

where

$$B_n^m := \lim_{\rho \to 0} d_{n,\rho}^m B_n(\rho)$$
$$= \frac{2i\sqrt{\pi} \left[\mathcal{J}_n(k\omega) h_n^{(1)}(k\omega) - \mathcal{H}_n(k\omega) j_n(k\omega) \right]}{\Gamma(n+1/2)\mu_0^{-1/2} kn j_n(k\omega)} \left(\frac{\omega}{2}\right)^{n+1} q_n^m \lim_{\rho \to 0} [\rho^{n+1} B_n(\rho)]$$

by (58), which proves the lemma by (59).

$$T_{n,m}^{(1)}(k,\omega,J) := \lim_{\rho \to 0} \beta_{n,\rho}^m \mathcal{J}_n(k\omega) + q_n^m \mathcal{H}_n(k\omega) = \beta_{n,0}^m \mathcal{J}_n(k\omega) + q_n^m \mathcal{H}_n(k\omega) \neq 0$$

where $\beta_{n,0}^m = -\frac{h_n^{(1)}(k\omega)}{j_n(k\omega)}q_n^m$ as in (53). Notice the convergence is uniform in n. Therefore, we have

(66)
$$\lim_{\rho \to 0} \hat{x} \times \tilde{E}_{\rho}^{\pm}|_{\partial B_1} = \varepsilon_0^{-1/2} \sum_{n=1}^{\infty} \sum_{m=-n}^n S_n(T_{n,m}^{(1)} V_n^m + T_{n,m}^{(2)} U_n^m)$$

where

$$T^{(2)}_{n,m}(k,\omega,J):=\lim_{\rho\to 0}\alpha^m_{n,\rho}j_n(k\omega)+p^m_nh^{(1)}_n(k\omega)$$

Similarly, we can obtain the tangential magnetic boundary condition at the interface as well.

APPENDIX A. SPHERICAL HARMONICS AND BESSEL FUNCTIONS

Our arguments rely heavily on expanding the EM fields into series of spherical wave functions. To that end, we introduce for $n \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$,

(67)
$$M_{n,\omega}^{m}(x) := \nabla \times \{xj_{n}(\omega|x|)Y_{n}^{m}(\hat{x})\},\$$
$$N_{n,\omega}^{m}(x) := \nabla \times \{xh_{n}^{(1)}(\omega|x|)Y_{n}^{m}(\hat{x})\},\$$

where $\omega \in \mathbb{R}$ and $\hat{x} = x/|x|$ for $x \in \mathbb{R}^3$. Here, $Y_n^m(\hat{x})$ are spherical harmonics and $h_n^{(1)}(t) := j_n(t) + iy_n(t)$ with $j_n(t)$ and $y_n(t)$, for $t \in \mathbb{R}$, being the spherical Bessel functions of the first and second kind, respectively. The following facts about these functions are useful in our estimates.

Set $S_n = \sqrt{n(n+1)}$. Define

(68)
$$\mathcal{J}_n(t) := j_n(t) + t j'_n(t), \qquad \mathcal{H}_n(t) := h_n^{(1)}(t) + t h_n^{(1)'}(t)$$

where j'_n and ${h_n^{(1)}}'$ are the derivatives of j_n and $h_n^{(1)}$. We introduce the vector spherical harmonics

(69)
$$U_n^m(\hat{x}) := \frac{1}{S_n} \text{Grad } Y_n^m(\hat{x}), \qquad V_n^m(\hat{x}) := \nu \times U_n^m$$

where Grad denotes the surface gradient. They satisfy

(70)
$$\hat{x} \times V_n^m(\hat{x}) = -U_n^m(\hat{x}), \qquad \hat{x} \times U_n^m(\hat{x}) = V_n^m(\hat{x}).$$

Moreover, we can rewrite $M^m_{n,\zeta}$ and $N^m_{n,\zeta}$ as

(71)
$$M_{n,\omega}^{m}(x) = -S_{n}j_{n}(\omega|x|)V_{n}^{m}(\hat{x}),$$
$$N_{n,\omega}^{m}(x) = -S_{n}h_{n}^{(1)}(\omega|x|)V_{n}^{m}(\hat{x}).$$

Moreover, we have

(72)
$$\nabla \times M_{n,\omega}^{m}(x) = S_{n}|x|^{-1}\mathcal{J}_{n}(\omega|x|)U_{n}^{m}(\hat{x}) + S_{n}^{2}|x|^{-1}j_{n}(\omega|x|)Y_{n}^{m}(\hat{x})\hat{x},$$
$$\nabla \times N_{n,\omega}^{m}(x) = S_{n}|x|^{-1}\mathcal{H}_{n}(\omega|x|)U_{n}^{m}(\hat{x}) + S_{n}^{2}|x|^{-1}h_{n}^{(1)}(\omega|x|)Y_{n}^{m}(\hat{x})\hat{x}.$$

Alternatively, we also have by (69)

(73)
$$\nabla \times M_{n,\omega}^m(x) = |x|^{-1} \mathcal{J}_n(\omega|x|) \nabla Y_n^m(\hat{x}) + S_n^2 |x|^{-1} j_n(\omega|x|) Y_n^m(\hat{x}) \hat{x},$$
$$\nabla \times N_{n,\omega}^m(x) = |x|^{-1} \mathcal{H}_n(\omega|x|) \nabla Y_n^m(\hat{x}) + S_n^2 |x|^{-1} h_n^{(1)}(\omega|x|) Y_n^m(\hat{x}) \hat{x}.$$

It is easy to see that

(74)
$$\hat{x} \times M_{n,\omega}^m(x) = S_n j_n(\omega|x|) U_n^m(\hat{x}), \quad \hat{x} \times N_{n,\omega}^m(x) = S_n h_n^{(1)}(\omega|x|) U_n^m(\hat{x}) \hat{x} \cdot M_{n,\omega}^m(x) = \hat{x} \cdot N_{n,\omega}^m(x) = 0,$$

and

(75)
$$\begin{cases} \hat{x} \times (\nabla \times M_{n,\omega}^m(x)) = S_n |x|^{-1} \mathcal{J}_n(\omega |x|) V_n^m(\hat{x}), \\ \hat{x} \times (\nabla \times N_{n,\omega}^m(x)) = S_n |x|^{-1} \mathcal{H}_n(\omega |x|) V_n^m(\hat{x}), \end{cases}$$

(76)
$$\begin{cases} \hat{x} \cdot (\nabla \times M_{n,\omega}^m(x)) = S_n^2 |x|^{-1} j_n(\omega |x|) Y_n^m(\hat{x}), \\ \hat{x} \cdot (\nabla \times N_{n,\omega}^m(x)) = S_n^2 |x|^{-1} h_n^{(1)}(\omega |x|) Y_n^m(\hat{x}). \end{cases}$$

The spherical Bessel functions are given by

$$j_n(t) = \sqrt{\pi/(2t)} J_{n+1/2}(t), \qquad y_n(t) = \sqrt{\pi/(2t)} Y_{n+1/2}(t)$$

where $J_{n+1/2}(t)$ and $Y_{n+1/2}(t)$ are the standard Bessel functions. More specifically,

$$j_0(t) = \frac{\sin t}{t}, \qquad h_0^{(1)}(t) = \frac{\sin t}{t} - i\frac{\cos t}{t}$$

Let $\Gamma(n+1/2) := \frac{(2n-1)!!}{2^n} \sqrt{\pi}$. From their series representations, we obtain that (77)

$$j_n(t) \approx \frac{\sqrt{\pi}}{2\Gamma(n+3/2)} \left(\frac{t}{2}\right)^n, \qquad h_n^{(1)}(t) \approx -i\frac{\Gamma(n+1/2)}{2\sqrt{\pi}} \left(\frac{2}{t}\right)^{n+1} \qquad \text{for } n >> t$$

in the following sense

(78)
$$\begin{cases} j_n(t) = \frac{\sqrt{\pi}}{2\Gamma(n+3/2)} \left(\frac{t}{2}\right)^n \left(1+O(1/n)\right) \\ \text{as } n \to \infty, \text{ uniformly for } t \text{ on } a \text{ compact subset of } \mathbb{R}, \\ h_n^{(1)}(t) = -i \frac{\Gamma(n+1/2)}{2\sqrt{\pi}} \left(\frac{2}{t}\right)^{n+1} \left(1+O(1/n)\right) \\ \text{as } n \to \infty, \text{ uniformly for } t \text{ on } a \text{ compact subset of } (0,\infty). \end{cases}$$

and for each n > 0,

(79)
$$\begin{cases} j_n(t) = \frac{\sqrt{\pi}}{2\Gamma(n+3/2)} \left(\frac{t}{2}\right)^n \left(1 + O_{t\to0}(t)\right), \\ h_n^{(1)}(t) = -i\frac{\Gamma(n+1/2)}{2\sqrt{\pi}} \left(\frac{2}{t}\right)^{n+1} \left(1 + O_{t\to0}(t)\right) \end{cases}$$

where $|O_{t\to0}(t)| \leq C(n)t$ as $t \to 0$ for some constant C(n) > 0 depending on n. Due to (78), one can easily obtain a uniform C independent of n, hence replace $O_n(t)$ by O(t). (Notice that such uniformity for t small is corresponding graphically to the spreading out shape of $j_n(t)$ and $y_n(t)$ with respect to n, i.e., less oscillatory for larger n.) In the same sense, we have when n >> t,

(80)
$$\mathcal{J}_n(t) \approx \frac{\sqrt{\pi}(n+1)}{2\Gamma(n+3/2)} \left(\frac{t}{2}\right)^n, \qquad \mathcal{H}_n(t) \approx i \frac{\Gamma(n+1/2)n}{2\sqrt{\pi}} \left(\frac{2}{t}\right)^{n+1}$$

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