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A COMPLEX VERSION OF THE CAHN–HILLIARD EQUATION FOR GRAYSCALE IMAGE INPAINTING

LAURENCE CHERFILS¹, HUSSEIN FAKIH², AND ALAIN MIRANVILLE²

ABSTRACT. Our aim in this article is to propose a generalization of the Bertozzi–Esedoglu–Gillette–Cahn–Hilliard equation, introduced for binary image inpainting, for grayscale image inpainting. In particular, we consider the solution to the corresponding Cahn–Hilliard inpainting model as a complex valued function. We are interested in the study of the well-posedness and of the asymptotic behavior, in terms of finite-dimensional attractors, of the associated dynamical system. We have to face two major difficulties here. The first one comes from the fact that we no longer have the conservation of mass, i.e., of the spatial average of the order parameter u , contrary to the classical Cahn–Hilliard equation. The second one is due to the estimates on the nonlinear terms, combined with the fact that the order parameter u is complex valued. We finally give numerical simulations which confirm and extend previous ones on the efficiency of the binary model.

1. INTRODUCTION

Image inpainting involves filling in parts of an image or a video from the surrounding area. It is essentially some type of interpolation. Its applications include restoration of old paintings by museum artists [31], removing scratches from old photographs [9], altering scenes in photographs [46], and restoration of motion pictures [49].

The work of Bertalmio et al. in [4] is very important, as it proposed a new direction in image inpainting by considering PDE’s models. In particular, there, the authors proposed boundary conditions for PDE’s image inpainting models. These boundary conditions consist of the constant grayscale image intensity and the direction of the isophote vectors at the boundary of the inpainting region. The isophote vector $\nabla^\perp u$ is the orthogonal gradient vector (e.g., in $\mathbb{R}^2$, $\nabla^\perp u = (-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x})$).

Furthermore, Bertalmio et al. proposed in [3] a new PDE’s model for image inpainting based on the Navier–Stokes equations. More precisely, the idea is to use a reformulation of the Navier–Stokes equations with an anisotropic term (to keep the no slip boundary conditions) and, of course, a fidelity term (such a term is essential in image inpainting, as it forces the solution to the PDE’s to stay close to the original image outside the inpainting region).

Esedoglu and Shen [33] proposed two PDE’s models for image inpainting based on the Mumford–Shah model. The first one is obtained by a simple modification of the fidelity term, while the second one is obtained by considering the Euler’s elastic approximation

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(other PDE's inpainting models can be found in, e.g., [1], [14], [15], [16], [32], [39], [40], [41], [54], and [65]).

A simplified version of the Euler's elastic model is the Ginzburg–Landau energy. In particular, the Cahn–Hilliard equation can be derived as an H^{-1} –gradient flow of the Ginzburg–Landau energy, as proved by Fife [37].

The Cahn–Hilliard equation plays an important role in materials science and describes phase separation processes. This can be observed, e.g., when a binary alloy is cooled down sufficiently. One then observes a partial nucleation (i.e., the apparition of nucleides in the material) or a total nucleation, the so-called spinodal decomposition: the material quickly becomes inhomogeneous, forming a fine-grained structure in which each of the two components appears more or less alternatively. In a second stage, which is called coarsening and occurs at a slower time scale, these microstructures coarsen. Such phenomena play an essential role in the mechanical properties of the material, e.g., strength. We refer the reader to, e.g., [11], [12], [20], [29], [48], [50], [52], [53], [59], and [60] for more details.

It is also interesting to note that the Cahn–Hilliard equation, or some of its variants, is relevant in other contexts, in which phase separation and coarsening/clustering processes can be observed or come into play. We can mention, for instance, population dynamics (see [23]), bacterial films (see [47]), wound healing and tumor growth (see [21], [36], [45], [55], and [56]), thin films (see [61] and [67]), image processing and inpainting (see [5], [6], [10], [13], [17], and [25]), and even the rings of Saturn (see [68]) and the clustering of mussels (see [51]).

Binary image inpainting with fourth-order PDE's. Bertozzi, Esedoglu, and Gillette proposed in [5] the following two-scale variant of the Cahn–Hilliard equation:

$$(1.1) \quad \frac{\partial u}{\partial t} + \varepsilon \Delta^2 u - \frac{1}{\varepsilon} \Delta f(u) + \lambda_0 \chi_{\Omega \setminus D}(x)(u - h) = 0, \quad \varepsilon > 0, \quad \lambda_0 > 0,$$

in view of applications to binary image inpainting. Here, $h = h(x)$ is a given (damaged) image and $D \subset \Omega$ is the inpainting region (Ω is the total region). Furthermore, the term $\lambda_0 \chi_{\Omega \setminus D}(x)(u - h)$ is the fidelity term (χ denotes the indicator function). There are several motivations to use such a term (instead of, e.g., a condition of the form $u = h$ outside the inpainting domain), in particular, in view of the analysis of the model. Indeed, no regularity assumption on D is necessary and no perfect h outside D is required (it could, for instance, be noisy). Finally, the nonlinear term f is regular and cubic, typically, $f(s) = 4s^3 - 6s^2 + 2s$. The idea in this model is to solve (1.1) up to steady state in order to obtain an inpainted version $u(x)$ of $h(x)$.

This equation was studied, endowed with Neumann boundary conditions, in [6], [7], [10], [17], and [18].

Well-posedness results for (1.1) have been obtained in [6] (see also [24] for the study of the nonlocal version of the problem and [10] for the study of the stationary problem).

Furthermore, the asymptotic behavior, in terms of finite-dimensional attractors, of (1.1) has been studied in [17]. More precisely, such sets provide information on all the possible dynamics of the system and are expected to have a rich geometric structure; they contain in particular all steady states and heteroclinic orbits. Moreover, the finite-dimensionality means, roughly speaking, that, even though the initial phase space is infinite-dimensional,

the reduced dynamics can be characterized by a finite number of parameters. We refer the reader to, e.g., [26], [57], and [66] for discussions on this subject.

Finally, the existence of local (in time) solutions to (1.1) with logarithmic nonlinear terms has been studied in [18] (note indeed that the original Cahn–Hilliard equation was actually proposed with thermodynamically relevant logarithmic nonlinear terms which follow from a mean-field model; regular (and, in particular, cubic) nonlinear terms are approximations of such logarithmic nonlinear terms). In that case, we can obtain better results than those obtained with polynomial nonlinear terms in [17], as far as the convergence time is concerned, in particular. We also note that double obstacle nonlinear terms give better results than those obtained with polynomial nonlinear terms, see [7].

Color image inpainting with fourth-order PDE's. We proposed in [19] the following color inpainting model:

$$(1.2) \quad \frac{\partial c_i}{\partial t} = \Delta \mu_i + \lambda_0 \chi_{\Omega \setminus D}(x)(h_i - c_i), \quad i = 1, \dots, n,$$

$$(1.3) \quad \mu_i = \frac{\partial F(c)}{\partial c_i} - \varepsilon^2 \Delta c_i - \frac{1}{n} \sum_{i=1}^n \frac{\partial F(c)}{\partial c_i}, \quad i = 1, \dots, n,$$

where $F(c) = \frac{1}{n} \sum_{i=1}^n c_i^2(1 - c_i)^2$, $h = (h_1, \dots, h_n)$ being the original image and n the number of colors in the original image h . System (1.2)–(1.3) is endowed with Neumann boundary conditions. We studied in [19] the well-posedness of the problem, as well as the existence of finite-dimensional attractors. Furthermore, we obtained consistency results with the diphasic model. Finally, we gave numerical simulations which confirm that the one step algorithm with threshold proposed in [17] is efficient, also in the context of multi-color inpainting.

Grayscale image inpainting with fourth-order PDE's. The authors in [10] proposed the following model (total variation in H^{-1}):

$$(1.4) \quad \frac{\partial u}{\partial t} - \Delta p + \lambda_0 \chi_{\Omega \setminus D}(x)(u - h) = 0, \quad p \in \partial TV(u),$$

where

$$TV(u) = \begin{cases} |Du| & \text{if } |u(x)| \leq 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise,} \end{cases}$$

in view of applications to grayscale image inpainting. In particular, they proved that the corresponding stationary problem has a bounded variation solution.

Furthermore, the authors in [64] proposed two grayscale inpainting models. The first one consists in generalizing the Bertozzi–Esedoglu–Gillette–Cahn–Hilliard model by splitting the grayscale image bit-wise into channels,

$$u(x) \text{ approximated by } \sum_{k=1}^n u_k(x) 2^{-(k-1)},$$

where $n > 0$ is the number of channels, and then in applying the Bertozzi–Esedoglu–Gillette–Cahn–Hilliard equation to each binary channel u_k separately. The second one is

based on the LCIS (Low Curvature Image Simplifier) model. This latter model reads

$$\frac{\partial u}{\partial t} + \operatorname{div}(g(\Delta u) \nabla \Delta u) + \lambda_0 \chi_{\Omega \setminus D}(x)(u - h) = 0,$$

where g is the thresholding function, $g(s) = \frac{1}{1+s^2}$. Since $g(\Delta u) \nabla \Delta u = \nabla(\arctan(\Delta u))$, we can rewrite the equation as follows:

$$\frac{\partial u}{\partial t} + \Delta(\arctan(\Delta u)) + \lambda_0 \chi_{\Omega \setminus D}(x)(u - h) = 0.$$

Note that this model can be seen as a generalization of the Rudin–Osher–Fatemi ([63]) and Perona–Malik ([62]) models.

Finally, the authors in [8] proposed system (1.2)–(1.3) in view of applications to grayscale image inpainting.

Proposed model. Let $u = u_1 + iu_2$ be the phase variable and Ω be a bounded domain in $\mathbb{R}^N$ ($N \leq 3$) with a regular boundary Γ . We postulate that the free energy can be written as follows (see also [42]):

$$(1.5) \quad \mathcal{E}(u) = \int_{\Omega} \left[\frac{\varepsilon}{2} | -i \nabla u |^2 + \frac{1}{4\varepsilon} |u|^4 - \frac{1}{2\varepsilon} |u|^2 \right] dx.$$

Here and below, we denote by $|\cdot|$ the modulus of a complex number.

Assuming proper boundary conditions, the variation of the energy (1.5) with respect to the phase field u is given by

$$(1.6) \quad \partial \mathcal{E}(u) = \int_{\Omega} \left[-\varepsilon \Delta u + \frac{1}{\varepsilon} |u|^2 u - \frac{1}{\varepsilon} u \right] \partial u dx,$$

which yields

$$(1.7) \quad \frac{\partial \mathcal{E}(u)}{\partial u} = -\varepsilon \Delta u + \frac{1}{\varepsilon} |u|^2 u - \frac{1}{\varepsilon} u,$$

and the descent equation can be written as follows:

$$(1.8) \quad \frac{\partial u}{\partial t} = -\frac{\partial \mathcal{E}(u)}{\partial u} = \varepsilon \Delta u - \frac{1}{\varepsilon} |u|^2 u + \frac{1}{\varepsilon} u.$$

Equation (1.8) can be seen as a complex version of the Ginzburg–Landau (Allen–Cahn) equation. In particular, Grossauer and Scherzer [42] proposed this equation in view of applications to image processing and, more precisely, to image inpainting. In that case, they proposed to treat the image data as complex: the real part of $u(x, t)$ corresponds to the actual grayscale value of the image at (x, t) , while the imaginary part is such that $u(x, t)$ is on the boundary of a circle of radius 1 in the complex plane. This leads to a coupled system of equations for $u_1(x, t)$ and $u_2(x, t)$ which Grossauer and Scherzer solved by an explicit numerical scheme, assuming Dirichlet boundary conditions, $u(x, t)|_{\partial D} = u(x, 0)|_{\partial D}$. Finally, the authors in [2] proposed a mesh adaptation strategy to solve (1.8) by a finite element scheme.

The complex version of the Cahn–Hilliard equation can be derived exactly as the standard Cahn–Hilliard equation, namely,

$$\frac{\partial u}{\partial t} = \Delta \frac{\partial \mathcal{E}}{\partial u},$$

hence

$$(1.9) \quad \frac{\partial u}{\partial t} + \varepsilon \Delta^2 u - \frac{1}{\varepsilon} \Delta (|u|^2 u - u) = 0.$$

We can note that, as in the case of the complex Ginzburg–Landau equation (1.8), the minima of the nonlinear term satisfy $|u| = 1$.

Now, in order for such a model to have applications to image inpainting, we need to add a fidelity term to keep the solution constructed close to the given image in the complement of the inpainting domain, where image information is available. Let h_1 be the given (damaged) image, $h_1 : \Omega \rightarrow [-1, 1]$. We then introduce the function h ,

$$h : \Omega \rightarrow \mathbb{C}, \quad h(x) = h_1(x) + ih_2(x),$$

where h satisfies the constraint $|h| = 1$, which yields

$$h_2(x) = \sqrt{1 - (h_1(x))^2}.$$

We assume that $h_1 \in L^2(\Omega)$, so that $h_2 \in L^2(\Omega)$ and $h \in L^2(\Omega; \mathbb{C})$; for simplicity, we will also denote by $L^2(\Omega)$ the L^2 –functions with values in $\mathbb{C}$ (i.e., $L^2(\Omega; \mathbb{C})$; a similar notation will hold for the corresponding Sobolev spaces). We finally consider the following complex Bertozzi–Esedoglu–Gillette–Cahn–Hilliard model:

$$(1.10) \quad \frac{\partial u}{\partial t} + \varepsilon \Delta^2 u - \frac{1}{\varepsilon} \Delta f(u) + \lambda_0 \chi_{\Omega \setminus D}(x)(u - h) = 0,$$

where $D \Subset \Omega$ is the inpainting domain, $\varepsilon, \lambda_0 > 0$, and $f(z) = |z|^2 z - z$, $z \in \mathbb{C}$. This equation is endowed with Neumann boundary conditions as in the original Bertozzi–Esedoglu–Gillette–Cahn–Hilliard equation,

$$(1.11) \quad \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0, \quad \text{on } \Gamma.$$

Furthermore, we will assume, in the numerical simulations (but not in the mathematical analysis), that the imaginary part of the initial datum is given by

$$(1.12) \quad u_2(0, x) = \mathbf{Im}[u(0, x)] = \sqrt{1 - (\mathbf{Re}[u(0, x)])^2} = \sqrt{1 - (u_1(0, x))^2}.$$

Here and below, $\mathbf{Re}[\varphi]$ and $\mathbf{Im}[\varphi]$ denote the real and the imaginary parts of φ , respectively.

One main idea when considering Cahn–Hilliard models in image inpainting is to keep the continuity of the isophote vectors at the boundary of the inpainting domain of the constructed image (see [5] and [6]), which is much desirable. Furthermore, compared with other models, the Cahn–Hilliard equation is more efficient, as far as the convergence time is concerned, and allows to reconstruct images with large inpainting regions.

The total variation (in H^{-1}) model proposed in [10] and the LCIS model proposed in [64] lose some inpainting advantages of the Cahn–Hilliard inpainting. Furthermore, the bit-wise model proposed in [64] and the vector-valued model proposed in [8], while preserving the inpainting advantages of the Bertozzi–Esedoglu–Gillette–Cahn–Hilliard model, may be too heavy numerically and may not always be applicable when the number of channels

(i.e., of shades of gray) in the original damaged image is large, since, in that case, we need to compute the n channels u_k , $k = 1, \dots, n$.

Our main aim in this article is to propose a simple model for grayscale image inpainting which preserves the inpainting advantages obtained with the Bertozzi–Esedoglu–Gillette–Cahn–Hilliard model, i.e., it is not too heavy numerically and is fast, as far as the convergence time is concerned. In particular, we can recover the binary inpainting results obtained in [5] and [6] (see Figure 1) and we only need to compute two solutions (the real and imaginary parts of the order parameter) whatever the number of channels in the damaged image is.

As already encountered in the binary model in [17], one essential difficulty to study the well-posedness and the existence of finite-dimensional attractors is that we no longer have the conservation of mass, i.e., of the spatial average of the order parameter, contrary to the original Cahn–Hilliard equation. A second essential difficulty, when compared to (1.1), is that (1.10) is complex valued (note indeed that (1.1) and (1.10) have exactly the same structure). This makes the derivation of estimates on the nonlinear term (and, consequently, the proof of dissipativity; note that the dissipativity is also used in an essential way in the proof of existence of a solution) much more delicate. This is addressed in Sections 2 to 4. We can also note that the functional framework considered in [19] for the color inpainting model is different from what we have here, since, there, the sum of the components is equal to one.

As far as the numerical simulations are concerned, the authors in [6] give numerical evidence that the steady states of the modified Cahn–Hilliard equation (1.1) are not unique. More precisely, they depend on the initial condition inside the inpainting domain. Furthermore, computing inpainted images with a small ε only may not allow to extend the level lines into the missing domain as desired. This problem can a priori also appear when solving equation (1.10). Therefore, we follow the strategy devised in [5] and [6], i.e., we consider a dynamic two steps scheme involving the diffuse interface thickness ε . More precisely, we consider a large value of ε to connect the edges and then a smaller one in order to obtain the inpainting results. Furthermore, when solving (1.10), the solutions may also regularize outside the inpainting domain. Since, in image inpainting, the image is known outside the inpainting region, we use, in the final inpainting results, the information on the image known outside the inpainting domain to obtain better results (see Section 5). In particular, we give numerical simulations with the two steps scheme which preserve the efficiency of the binary inpainting and also confirm that (1.10) gives indeed good results in the context of grayscale inpainting.

Notation. We denote by $\langle u \rangle$ the spatial average of a function $u \in L^1(\Omega)$,

$$\langle u \rangle = \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} u(x) dx,$$

and set

$$v = u - \langle u \rangle,$$

v being the mean-free part of u . Furthermore, $\bar{u}$ denotes the conjugate of u .

We then set

$$\dot{H}^{-1}(\Omega) = \{\varphi \in H^{-1}(\Omega), \langle \varphi, 1 \rangle_{H^{-1}(\Omega), H^1(\Omega)} = 0\}$$

and

$$\dot{L}^2(\Omega) = \{\varphi \in L^2(\Omega), \langle \varphi \rangle = 0\}.$$

We denote by $((\cdot, \cdot))$ the usual L^2 -Hermitian product, with associated norm $\|\cdot\|$; in particular, for two complex valued functions φ and ψ in $L^2(\Omega)$,

$$((\varphi, \psi)) = \int_{\Omega} \varphi \bar{\psi} dx.$$

We further set $\|\cdot\|_{-1} = \|(-\Delta)^{-1} \cdot\|$, where $(-\Delta)^{-1}$ denotes the inverse minus Laplace operator associated with Neumann boundary conditions and acting on functions with null spatial average. More generally, $\|\cdot\|_X$ denotes the norm on the Banach space X .

Throughout the article, the same letters c and c' denote (generally positive) constants which may vary from line to line, or even in a same line. Similarly, the same letter Q denotes monotone increasing (with respect to each argument) functions which may vary from line to line, or even in a same line. In general, these quantities depend on ε and λ_0 .

2. A PRIORI ESTIMATES

We assume in Sections 2 to 4 that $h \in L^2(\Omega)$; in particular, h does not necessarily satisfy the constraint $|h| = 1$.

Estimates on the nonlinear term. We first have the

Proposition 2.1. *Let $u \in L^4(\Omega)$. Then, there exists $c_0 > 0$ such that*

$$\mathbf{Re} \left[((f(v + \langle u \rangle) - f(\langle u \rangle), v)) \right] \geq c_0 \int_{\Omega} \left(|v|^4 + |v|^2 |\langle u \rangle|^2 + \left(\mathbf{Re}[\bar{v} \langle u \rangle] \right)^2 \right) dx - \|v\|^2.$$

Proof. We first have

$$\begin{aligned} f(v + \langle u \rangle) - f(\langle u \rangle) &= (|v|^2 + |\langle u \rangle|^2 + 2\mathbf{Re}[v]\mathbf{Re}[\langle u \rangle] + 2\mathbf{Im}[v]\mathbf{Im}[\langle u \rangle] - 1)v \\ &\quad + (|v|^2 + 2\mathbf{Re}[v]\mathbf{Re}[\langle u \rangle] + 2\mathbf{Im}[v]\mathbf{Im}[\langle u \rangle])\langle u \rangle. \end{aligned}$$

Noting that

$$\mathbf{Re}[\bar{v} \langle u \rangle] = \mathbf{Re}[v]\mathbf{Re}[\langle u \rangle] + \mathbf{Im}[v]\mathbf{Im}[\langle u \rangle],$$

it follows that

$$\begin{aligned} \left[f(v + \langle u \rangle) - f(\langle u \rangle) \right] \bar{v} &= (|v|^2 + |\langle u \rangle|^2 + 2\mathbf{Re}[\bar{v} \langle u \rangle] - 1)|v|^2 \\ &\quad + (|v|^2 + 2\mathbf{Re}[\bar{v} \langle u \rangle])\langle u \rangle \bar{v} \end{aligned}$$

and

$$\begin{aligned} \mathbf{Re} \left[(f(v + \langle u \rangle) - f(\langle u \rangle)) \bar{v} \right] &= (|v|^2 + |\langle u \rangle|^2 + 2\mathbf{Re}[\bar{v} \langle u \rangle] - 1)|v|^2 \\ &\quad + (|v|^2 + 2\mathbf{Re}[\bar{v} \langle u \rangle])\mathbf{Re}[\bar{v} \langle u \rangle]. \end{aligned}$$

Integrating over Ω , we obtain

$$\begin{aligned} & \mathbf{Re} \left[((f(v + \langle u \rangle) - f(\langle u \rangle), v)) \right] \\ &= \int_{\Omega} \left[|v|^4 + |v|^2 |\langle u \rangle|^2 + 2(\mathbf{Re}[\bar{v} \langle u \rangle])^2 \right] dx + 3 \int_{\Omega} \mathbf{Re}[\bar{v} \langle u \rangle] |v|^2 dx - \|v\|^2 \\ &\geq \int_{\Omega} \left[|v|^4 + |v|^2 |\langle u \rangle|^2 + 2(\mathbf{Re}[\bar{v} \langle u \rangle])^2 \right] dx - 3 \int_{\Omega} |\mathbf{Re}[\bar{v} \langle u \rangle]| |v|^2 dx - \|v\|^2. \end{aligned}$$

Using properties of complex numbers, namely,

$$|\mathbf{Re}[z]| \leq |z| \quad \text{and} \quad |zz'| = |z||z'|, \quad \forall z, z' \in \mathbb{C},$$

we find

$$\begin{aligned} 3 \int_{\Omega} |\mathbf{Re}[\bar{v} \langle u \rangle]| |v|^2 dx &= 2 \int_{\Omega} |\mathbf{Re}[\bar{v} \langle u \rangle]| |v|^2 dx + \int_{\Omega} |\mathbf{Re}[\bar{v} \langle u \rangle]| |v|^2 dx \\ &\leq \alpha \int_{\Omega} |v|^4 dx + \beta \int_{\Omega} |\mathbf{Re}[\bar{v} \langle u \rangle]|^2 dx + \gamma \int_{\Omega} |v|^4 dx + \zeta \int_{\Omega} |v|^2 |\langle u \rangle|^2 dx, \end{aligned}$$

where $\alpha, \beta, \gamma, \zeta$ are nonnegative constants which satisfy $\alpha + \gamma < 1$, $\beta < 2$, and $\zeta < 1$. The last inequality has been obtained by employing Young's inequality (e.g., $\alpha = \frac{21}{41}$, $\beta = \frac{41}{21}$, $\gamma = \frac{3}{8}$, and $\zeta = \frac{2}{3}$). We finally deduce that

$$\mathbf{Re} \left[((f(v + \langle u \rangle) - f(\langle u \rangle), v)) \right] \geq c_0 \int_{\Omega} \left(|v|^4 + |v|^2 |\langle u \rangle|^2 + (\mathbf{Re}[\bar{v} \langle u \rangle])^2 \right) dx - \|v\|^2,$$

where $c_0 > 0$. □

Proposition 2.2. *Let $w \in H^1(\Omega)$. Then,*

$$\mathbf{Re} \left[((\nabla f(w), \nabla w)) \right] \geq -\|\nabla w\|^2.$$

Proof. We first note that

$$((\nabla f(w), \nabla w)) = \int_{\Omega} |w|^2 |\nabla w|^2 dx + 2 \sum_{j=1}^N \int_{\Omega} \mathbf{Re} \left[w \frac{\partial \bar{w}}{\partial x_j} \right] \left(w \frac{\partial \bar{w}}{\partial x_j} \right) dx - \|\nabla w\|^2,$$

hence

$$\begin{aligned} \mathbf{Re} \left[((\nabla f(w), \nabla w)) \right] &= \int_{\Omega} |w|^2 |\nabla w|^2 dx + 2 \sum_{j=1}^N \int_{\Omega} \left(\mathbf{Re} \left[w \frac{\partial \bar{w}}{\partial x_j} \right] \right)^2 dx - \|\nabla w\|^2 \\ &\geq -\|\nabla w\|^2. \end{aligned}$$
□

Absorbing set in $\dot{H}^{-1}(\Omega)$. We first recall that $\mathcal{B}_0$ is an absorbing set if it is bounded and if, for every bounded subset $\mathcal{B}$, $\exists t_0 = t_0(\mathcal{B})$ such that $u_0 \in \mathcal{B}$ and $t \geq t_0 \implies u(t) \in \mathcal{B}_0$, u_0 being the initial datum. When such a set exists, the associated dynamical system is called dissipative.

Integrating (1.10) over Ω , we have

$$(2.1) \quad \frac{d}{dt} \langle u \rangle + \lambda_0 \langle \chi_{\Omega \setminus D}(x)(u - h) \rangle = 0.$$

We thus rewrite (1.10) as follows:

$$(2.2) \quad \frac{\partial v}{\partial t} + \varepsilon \Delta^2 v - \frac{1}{\varepsilon} \Delta f(u) + \lambda_0 \chi_{\Omega \setminus D}(x)(u - h) - \lambda_0 \langle \chi_{\Omega \setminus D}(x)(u - h) \rangle = 0$$

and (2.2) is equivalent to

$$(2.3) \quad \begin{aligned} & \frac{\partial(-\Delta)^{-1}v}{\partial t} - \varepsilon \Delta v + \frac{1}{\varepsilon} (f(v + \langle u \rangle) - \langle f(u) \rangle) \\ & + \lambda_0 (-\Delta)^{-1} \left(\chi_{\Omega \setminus D}(x)(u - h) - \langle \chi_{\Omega \setminus D}(x)(u - h) \rangle \right) = 0. \end{aligned}$$

We further have the

Lemma 2.3. *For every $u \in L^4(\Omega)$, there holds*

$$\begin{aligned} & \lambda_0 \left| \mathbf{Re} \left[(((-\Delta)^{-1}(\chi_{\Omega \setminus D}(x)(u - h) - \langle \chi_{\Omega \setminus D}(x)(u - h) \rangle), v)) \right] \right| \\ & \leq \frac{c_0}{2\varepsilon} \int_{\Omega} \left(|v|^4 + |v|^2 |\langle u \rangle|^2 + \left(\mathbf{Re}[\bar{v} \langle u \rangle] \right)^2 \right) dx + c \|h\|^2 + c'. \end{aligned}$$

Proof. We first note that

$$\begin{aligned} & \lambda_0 \left| \mathbf{Re} \left[(((-\Delta)^{-1}(\chi_{\Omega \setminus D}(x)(u - h) - \langle \chi_{\Omega \setminus D}(x)(u - h) \rangle), v)) \right] \right| \\ & \leq \lambda_0 \left| (((-\Delta)^{-1}(\chi_{\Omega \setminus D}(x)(u - h) - \langle \chi_{\Omega \setminus D}(x)(u - h) \rangle), v)) \right|. \end{aligned}$$

Since $\langle v \rangle = 0$, we have

$$\begin{aligned} & \lambda_0 \left| (((-\Delta)^{-1}(\chi_{\Omega \setminus D}(x)(u - h) - \langle \chi_{\Omega \setminus D}(x)(u - h) \rangle), v)) \right| \\ & = \lambda_0 \left| ((\chi_{\Omega \setminus D}(x)(u - h), (-\Delta)^{-1}v)) \right| \end{aligned}$$

and Hölder's inequality yields

$$\begin{aligned}
& \lambda_0 \left| ((\chi_{\Omega \setminus D}(x)(u - h), (-\Delta)^{-1}v)) \right| \\
& \leq c \|u - h\| \|v\| \\
& \leq c(\|v\|^2 + |\langle u \rangle| \|v\|) + c' \|h\|^2 \\
& \leq \frac{c_0}{2\varepsilon} \int_{\Omega} \left(|v|^4 + |v|^2 |\langle u \rangle|^2 + \left(\mathbf{Re}[\bar{v} \langle u \rangle] \right)^2 \right) dx + c \|h\|^2 + c'.
\end{aligned}$$

□

Proposition 2.4. *Problem (1.10) has an absorbing set in $\dot{H}^{-1}(\Omega)$.*

Proof. We multiply (2.3) by $\bar{v}$ and have, integrating over Ω and by parts and taking the real part,

$$\begin{aligned}
(2.4) \quad & \frac{1}{2} \frac{d}{dt} \|v\|_{-1}^2 + \varepsilon \|\nabla v\|^2 + \frac{1}{\varepsilon} \mathbf{Re} \left[((f(v + \langle u \rangle) - f(\langle u \rangle), v)) \right] \\
& + \lambda_0 \mathbf{Re} \left[(((-\Delta)^{-1}(\chi_{\Omega \setminus D}(x)(u - h) - \langle \chi_{\Omega \setminus D}(x)(u - h) \rangle), v)) \right] = 0.
\end{aligned}$$

It follows from Proposition 2.1 and Lemma 2.3 that

$$\begin{aligned}
(2.5) \quad & \frac{1}{2} \frac{d}{dt} \|v\|_{-1}^2 + \varepsilon \|\nabla v\|^2 + \frac{c_0}{2\varepsilon} \int_{\Omega} \left(|v|^4 + |v|^2 |\langle u \rangle|^2 + \left(\mathbf{Re}[\bar{v} \langle u \rangle] \right)^2 \right) dx \\
& \leq c(\|v\|^2 + \|h\|^2 + 1),
\end{aligned}$$

hence

$$(2.6) \quad \frac{d}{dt} \|v\|_{-1}^2 + \varepsilon \|\nabla v\|^2 + \frac{c_0}{\varepsilon} \int_{\Omega} \left(|v|^4 + |v|^2 |\langle u \rangle|^2 + \left(\mathbf{Re}[\bar{v} \langle u \rangle] \right)^2 \right) dx \leq c.$$

In particular, it follows from (2.6) that

$$(2.7) \quad \frac{d}{dt} \|v\|_{-1}^2 + c \|v\|_{-1}^2 \leq c', \quad c > 0.$$

We finally deduce from Gronwall's lemma that

$$(2.8) \quad \|v\|_{-1}^2 \leq e^{-ct} \|v_0\|_{-1}^2 + c', \quad c > 0, \quad \forall t \geq 0,$$

where the constants c and c' are independent of v_0 and t , hence the result.

□

Absorbing set in $\dot{L}^2(\Omega)$. As a consequence of (2.6) and (2.8), we have the

Corollary 2.5. *We assume that v_0 belongs to a bounded subset $\mathcal{B}$ of $\dot{H}^{-1}(\Omega)$. Then, there exists $t_0 = t_0(\mathcal{B}) \geq 0$ such that, for every $t \geq t_0$, there holds*

$$\int_t^{t+r} \|\nabla v\|^2 ds \leq c(r)$$

and

$$\int_t^{t+r} \int_{\Omega} \left(|v|^4 + |v|^2 |\langle u \rangle|^2 + \left(\mathbf{Re}[\bar{v} \langle u \rangle] \right)^2 \right) dx ds \leq c(r),$$

$r > 0$ given.

We further have the

Lemma 2.6. *For every $u \in L^4(\Omega)$, there holds*

$$\begin{aligned} & \lambda_0 \left| \mathbf{Re} \left[((\chi_{\Omega \setminus D}(x)(u - h) - \langle \chi_{\Omega \setminus D}(x)(u - h) \rangle, v)) \right] \right| \\ & \leq \frac{c_0}{2\varepsilon} \int_{\Omega} \left(|v|^4 + |v|^2 |\langle u \rangle|^2 + \left(\mathbf{Re}[\bar{v} \langle u \rangle] \right)^2 \right) dx + c \|h\|^2 + c'. \end{aligned}$$

Remark 2.7. The proof of Lemma 2.6 is similar to the one of Lemma 2.3.

Proposition 2.8. *Problem (1.10) has an absorbing set in $\dot{L}^2(\Omega)$.*

Proof. We multiply (2.2) by $\bar{v}$ and have

$$\begin{aligned} (2.9) \quad & \frac{1}{2} \frac{d}{dt} \|v\|^2 + \varepsilon \|\Delta v\|^2 + \frac{1}{\varepsilon} \mathbf{Re} \left[((\nabla f(u), \nabla u)) \right] \\ & + \lambda_0 \mathbf{Re} \left[((\chi_{\Omega \setminus D}(x)(u - h) - \langle \chi_{\Omega \setminus D}(x)(u - h) \rangle, v)) \right] = 0. \end{aligned}$$

It follows from Proposition 2.2 and Lemma 2.6 that

$$\begin{aligned} (2.10) \quad & \frac{1}{2} \frac{d}{dt} \|v\|^2 + \varepsilon \|\Delta v\|^2 \leq \frac{1}{\varepsilon} \|\nabla v\|^2 + c \|h\|^2 + c' \\ & + \frac{c_0}{2\varepsilon} \int_{\Omega} \left(|v|^4 + |v|^2 |\langle u \rangle|^2 + \left(\mathbf{Re}[\bar{v} \langle u \rangle] \right)^2 \right) dx. \end{aligned}$$

We then deduce from Corollary 2.5 and the uniform Gronwall's lemma that

$$(2.11) \quad \|v\|^2 \leq c, \quad \forall t \geq t_0 + r,$$

where the constant c is independent of v_0 and t . Finally, integrating (2.6) and (2.10) between 0 and $t_0 + r$ and using (2.11), we obtain

$$(2.12) \quad \|v\|^2 \leq Q(\|v_0\|), \quad \forall t \geq 0.$$

□

Dissipativity of the spatial average. We have the

Proposition 2.9. *Let u be a solution to (1.10). Then, for every $u_0 \in L^2(\Omega)$, the spatial average of the order parameter satisfies*

$$|\langle u \rangle| \leq \left(Q(\|v_0\|) + |\langle u_0 \rangle| \right) e^{-ct} + c', \quad c > 0, \quad \forall t \geq 0,$$

where $c = c(\varepsilon, \lambda_0)$ and $c' = c'(\varepsilon, \lambda_0)$ are two constants which are independent of u_0 and t .

Proof. Setting $u = \langle u \rangle + v$ in (2.1), we have

$$\frac{d}{dt} \langle u \rangle + \frac{\lambda_0}{\mathbf{Vol}(\Omega)} \int_{\Omega \setminus D} (\langle u \rangle + v - h) dx = 0.$$

Therefore,

$$\frac{d}{dt} \langle u \rangle + \xi \langle u \rangle = -\frac{\lambda_0}{\mathbf{Vol}(\Omega)} \int_{\Omega \setminus D} (v - h) dx,$$

where $\xi = \frac{\lambda_0 \mathbf{Vol}(\Omega \setminus D)}{\mathbf{Vol}(\Omega)}$, hence

$$\frac{d}{dt}(e^{\xi t} \langle u \rangle) = -\frac{\lambda_0}{\mathbf{Vol}(\Omega)} e^{\xi t} \int_{\Omega \setminus D} (v - h) dx$$

and

$$\langle u \rangle = e^{-\xi t} \langle u_0 \rangle - \frac{\lambda_0}{\mathbf{Vol}(\Omega)} e^{-\xi t} \int_0^t e^{\xi s} \int_{\Omega \setminus D} (v - h) dx ds.$$

Noting that

$$|z + z'| \leq |z| + |z'|, \quad \forall z, z' \in \mathbb{C},$$

we find

$$|\langle u \rangle| \leq e^{-\xi t} |\langle u_0 \rangle| + c e^{-\xi t} \int_0^t e^{\xi s} (\|v\| + \|h\|) ds, \quad \forall t \geq 0,$$

where $c = \frac{\lambda_0}{\mathbf{Vol}(\Omega)^{\frac{1}{2}}}$. We then deduce from (2.11) and (2.12) that

$$|\langle u \rangle| \leq \left(Q(\|v_0\|) + |\langle u_0 \rangle| \right) e^{-ct} + c', \quad c > 0, \quad \forall t \geq 0,$$

where the constants c and c' are independent of u_0 and t , hence the result. □

Absorbing set in $H^2(\Omega)$. We have the

Lemma 2.10. *Let u be a regular solution to (1.10). Then,*

$$\left| \mathbf{Re}[(\Delta f(u), \Delta^2 u)] \right| \leq \frac{\varepsilon^2}{4} \|\Delta^2 u\|^2 + c$$

and

$$\left| \mathbf{Re}[(\chi_{\Omega \setminus D}(x)(u - h), \Delta^2 u)] \right| \leq \frac{\varepsilon}{4\lambda_0} \|\Delta^2 u\|^2 + c(\|u\|^2 + \|h\|^2).$$

Proof. First, observe that

$$\begin{aligned} \left| \mathbf{Re}[(\Delta f(u), \Delta^2 u)] \right| &\leq \left| (\Delta f(u), \Delta^2 u) \right| \\ &\leq \|\Delta f(u)\| \|\Delta^2 u\| \\ &\leq \frac{2}{\varepsilon^2} \|\Delta f(u)\|^2 + \frac{\varepsilon^2}{8} \|\Delta^2 u\|^2. \end{aligned}$$

We further have (see, e.g., [66])

$$\|\Delta f(u)\|^2 \leq \frac{\varepsilon^4}{16} \|\Delta^2 u\|^2 + c$$

and, owing to Young's inequality,

$$\left| \mathbf{Re}[(\Delta f(u), \Delta^2 u)] \right| \leq \frac{\varepsilon^2}{4} \|\Delta^2 u\|^2 + c.$$

On the other hand, we have

$$\begin{aligned} \left| \mathbf{Re}[(\chi_{\Omega \setminus D}(x)(u-h), \Delta^2 u)] \right| &\leq \left| ((\chi_{\Omega \setminus D}(x)(u-h), \Delta^2 u)) \right| \\ &\leq \|u-h\| \|\Delta^2 u\| \\ &\leq \frac{\varepsilon}{4\lambda_0} \|\Delta^2 u\|^2 + c(\|u\|^2 + \|h\|^2). \end{aligned}$$

□

Lemma 2.11. *Let u be a solution to (1.10). Then,*

$$\frac{d}{dt} |\langle u \rangle|^2 \leq c(\|v\|^2 + |\langle u \rangle|^2 + \|h\|^2).$$

Proof. We first have

$$\frac{d}{dt} |\langle u \rangle|^2 = \frac{d}{dt} \left(\mathbf{Re}[\langle u \rangle] \right)^2 + \frac{d}{dt} \left(\mathbf{Im}[\langle u \rangle] \right)^2.$$

Noting that, owing to (2.1),

$$\frac{d}{dt} \mathbf{Re}[\langle u \rangle] = -\lambda_0 \langle \chi_{\Omega \setminus D}(x) \mathbf{Re}[u-h] \rangle$$

and

$$\frac{d}{dt} \mathbf{Im}[\langle u \rangle] = -\lambda_0 \langle \chi_{\Omega \setminus D}(x) \mathbf{Im}[u-h] \rangle,$$

it follows that

$$\begin{aligned} \frac{d}{dt} \left(\mathbf{Re}[\langle u \rangle] \right)^2 &= 2 \mathbf{Re}[\langle u \rangle] \frac{d}{dt} \mathbf{Re}[\langle u \rangle] \\ &\leq c |\langle u \rangle| \left| \int_{\Omega \setminus D} \mathbf{Re}[u-h] dx \right| \\ &\leq c(\|v\|^2 + |\langle u \rangle|^2 + \|h\|^2) \end{aligned}$$

and, similarly,

$$\frac{d}{dt} \left(\mathbf{Im}[\langle u \rangle] \right)^2 \leq c(\|v\|^2 + |\langle u \rangle|^2 + \|h\|^2).$$

We finally deduce that

$$\frac{d}{dt} |\langle u \rangle|^2 \leq c(\|v\|^2 + |\langle u \rangle|^2 + \|h\|^2).$$

□

Proposition 2.12. *Problem (1.10) has an absorbing set in $H^2(\Omega)$ which is compact in $L^2(\Omega)$.*

Proof. We multiply (1.10) by $\Delta^2 \bar{u}$ and have

$$(2.13) \quad \frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 + \varepsilon \|\Delta^2 u\|^2 + \frac{1}{\varepsilon} \mathbf{Re}[(\Delta f(u), \Delta^2 u)] + \lambda_0 \mathbf{Re}[(\chi_{\Omega \setminus D}(x)(u-h), \Delta^2 u)] = 0,$$

hence, owing to Lemma 2.10,

$$(2.14) \quad \frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 + \frac{\varepsilon}{2} \|\Delta^2 u\|^2 \leq c(\|v\|^2 + |\langle u \rangle|^2 + \|h\|^2).$$

It then follows from Lemma 2.11 that

$$(2.15) \quad \frac{d}{dt}(\|\Delta u\|^2 + |\langle u \rangle|^2) + \varepsilon \|\Delta^2 u\|^2 \leq c(\|v\|^2 + |\langle u \rangle|^2 + \|h\|^2), \quad \forall t \geq 0.$$

Assuming that u_0 belongs to a bounded subset $\mathcal{B}$ of $L^2(\Omega)$, we further note that Proposition 2.8 and Proposition 2.9 yield that there exists $t_1 = t_1(\mathcal{B}) \geq 0$ such that

$$(2.16) \quad \|v\|^2 \leq c \quad \text{and} \quad |\langle u \rangle|^2 \leq c, \quad \forall t \geq t_1,$$

where the constant c is independent of u_0 and t . We also have, integrating (2.10) over $(t, t+r)$, for $r \in (0, 1)$ fixed, and owing to (2.11),

$$(2.17) \quad \int_t^{t+r} \|\Delta u\|^2 ds \leq c(r).$$

We finally deduce from (2.15), (2.16), (2.17), and the uniform Gronwall's lemma that

$$(2.18) \quad \|u\|_{H^2(\Omega)}^2 \leq c, \quad \forall t \geq t_1 + r, \quad r > 0 \text{ fixed},$$

where the constant c is independent of u_0 and t . □

3. WELL-POSEDNESS AND EXISTENCE OF THE GLOBAL ATTRACTOR

a) Well-posedness.

Proposition 3.1. *Let z and z' be regular solutions to (1.10). Then, there exists a non-negative constant $c_0 > 0$ such that*

$$\mathbf{Re}[(f(z) - f(z'), z - z')] \geq c_0 \int_{\Omega} \left(|z - z'|^4 + |z|^2 |z - z'|^2 + \left(\mathbf{Re}[z(\overline{z - z'})] \right)^2 \right) dx - \|z - z'\|^2$$

and

$$\begin{aligned} & \left| \mathbf{Re}[(f(z) - f(z'), \langle z - z' \rangle)] \right| \\ & \leq c(\|z\|_{L^4(\Omega)}^2 + \|z'\|_{L^4(\Omega)}^2 + 1)(\|z - z'\|^2 + |\langle z - z' \rangle|^2). \end{aligned}$$

Proof. Setting $w = z - z'$, we have

$$((f(z) - f(z'), z - z')) = (|z|^2 z - |z - w|^2(z - w) - w, w)$$

and, using the fact that

$$|z - w|^2 = |z|^2 + |w|^2 - 2\mathbf{Re}[z\bar{w}],$$

we find

$$((f(z) - f(z'), z - z')) = (|w|^2 w - |w|^2 z + |z|^2 w + 2\mathbf{Re}[z\bar{w}]z - 2\mathbf{Re}[z\bar{w}]w - w, w).$$

It thus follows that

$$\begin{aligned} & \mathbf{Re}[(f(z) - f(z'), z - z')] = \\ & \int_{\Omega} \left(|w|^4 + |z|^2 |w|^2 + 2\left(\mathbf{Re}[z\bar{w}]\right)^2 \right) dx - 3 \int_{\Omega} |w|^2 \mathbf{Re}[z\bar{w}] dx - \int_{\Omega} |w|^2 dx, \end{aligned}$$

which yields, owing to Young's inequality (see the proof of Proposition 2.1),

$$\mathbf{Re}[(f(z) - f(z'), z - z')] \geq c_0 \int_{\Omega} \left(|w|^4 + |z|^2 |w|^2 + \left(\mathbf{Re}[z\bar{w}] \right)^2 \right) dx - \int_{\Omega} |w|^2 dx,$$

where $c_0 > 0$.

On the other hand, we have

$$\begin{aligned} \mathbf{Re}[(f(z) - f(z'), \langle z - z' \rangle)] &= \mathbf{Re}[\langle z - z' \rangle] \int_{\Omega} \mathbf{Re}[f(z) - f(z')] dx \\ &\quad + \mathbf{Im}[\langle z - z' \rangle] \int_{\Omega} \mathbf{Im}[f(z) - f(z')] dx. \end{aligned}$$

Observe now that

$$|f'(z)| \leq 3|z|^2 + 1.$$

Therefore,

$$\begin{aligned} &\left| \mathbf{Re}[\langle z - z' \rangle] \int_{\Omega} \mathbf{Re}[f(z) - f(z')] dx \right| \\ &= \left| \mathbf{Re}[\langle z - z' \rangle] \int_{\Omega} \mathbf{Re}[w] \int_0^1 \mathbf{Re}[f'(z + s(z - z'))] ds dx \right| \\ &\leq c |\langle w \rangle| \int_{\Omega} (|z|^2 + |z'|^2 + 1) |w| dx \\ &\leq c (\|z\|_{L^4(\Omega)}^2 + \|z'\|_{L^4(\Omega)}^2 + 1) (\|w\|^2 + |\langle w \rangle|^2). \end{aligned}$$

Similarly,

$$\begin{aligned} &\left| \mathbf{Im}[\langle z - z' \rangle] \int_{\Omega} \mathbf{Im}[f(z) - f(z')] dx \right| \\ &\leq c (\|z\|_{L^4(\Omega)}^2 + \|z'\|_{L^4(\Omega)}^2 + 1) (\|w\|^2 + |\langle w \rangle|^2), \end{aligned}$$

so that

$$\begin{aligned} &\left| \mathbf{Re}[(f(z) - f(z'), \langle z - z' \rangle)] \right| \\ &\leq c (\|z\|_{L^4(\Omega)}^2 + \|z'\|_{L^4(\Omega)}^2 + 1) (\|w\|^2 + |\langle w \rangle|^2). \end{aligned}$$

□

Theorem 3.2. *For every $u_0 \in L^2(\Omega)$ and every $T > 0$, the initial-boundary value problem associated with (1.10) has a unique solution u such that*

$$u \in L^\infty([0, T], L^2(\Omega)) \cap L^2([0, T], H^2(\Omega)) \cap L^4([0, T], L^4(\Omega)).$$

Proof. We first note that it is easy to show that the operator $(-\Delta)^{-1}$ is self-adjoint and compact in $\dot{L}^2(\Omega)$. There thus exists an orthonormal basis of $\dot{L}^2(\Omega)$ associated with the eigenvalues λ_j , $j \geq 1$, of $-\Delta$,

$$\begin{cases} \lambda_1 = \inf_{\varphi \in (H^1(\Omega) \cap \dot{L}^2(\Omega)) \setminus \{0\}} \frac{\|\varphi\|_{H^1(\Omega)}^2}{\|\varphi\|^2}, \\ -\Delta N_j = \lambda_j N_j, \quad N_j \in H^2(\Omega) \cap \dot{L}^2(\Omega), \quad \frac{\partial N_j}{\partial \nu} = 0, \text{ on } \Gamma, \quad j \geq 1, \\ 0 < \lambda_1 \leq \lambda_2 \leq \dots, \end{cases}$$

where the family $\{N_j\}_j$ is assumed to be normalized in the norm of $\dot{L}^2(\Omega)$, i.e.,

$$((N_i, N_j)) = \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by E_m the space

$$E_m = \text{span}\{N_1, N_2, \dots, N_m\}$$

and by P_m the orthogonal projection from $H^1(\Omega) \cap \dot{L}^2(\Omega)$ onto E_m ,

$$P_m h = \sum_{j=1}^m ((h, N_j)) N_j.$$

The variational formulation of the problem reads: Find $u_m : [0, T] \rightarrow \langle u_m \rangle + E_m$, $u_m(t) = \langle u_m(t) \rangle + v_m(t) = \langle u_m(t) \rangle + \sum_{j=1}^m u_{m_j}(t) N_j$, such that

$$\begin{aligned} (3.1) \quad & \frac{d}{dt} \left(\left((-\Delta)^{-1} \sum_{i=1}^m u_{m_i}(t) N_i, N_j \right) \right) \\ & + \varepsilon \left(\left(\sum_{i=1}^m u_{m_i}(t) \nabla N_i, \nabla N_j \right) \right) + \frac{1}{\varepsilon} ((f(\langle u_m \rangle + v_m), N_j)) \\ & - \frac{1}{\varepsilon} ((f(\langle u_m \rangle + v_m), N_j)) + \lambda_0 (((-\Delta)^{-1} (\chi_{\Omega \setminus D}(x) (\langle u_m \rangle + v_m - h)), N_j)) \\ & - \lambda_0 (((-\Delta)^{-1} (\chi_{\Omega \setminus D}(x) (\langle u_m \rangle + v_m - h)), N_j)) = 0, \quad j = 1, \dots, m, \end{aligned}$$

$$(3.2) \quad \frac{d}{dt} \langle u_m \rangle + \xi \langle u_m \rangle = -\frac{\lambda_0}{\text{Vol}(\Omega)} \int_{\Omega \setminus D} (v_m - h) dx,$$

$$(3.3) \quad v_m(0) = P_m u_0,$$

$$(3.4) \quad \langle u_m(0) \rangle = \langle u_0 \rangle.$$

We can rewrite (3.1) and (3.3) as

$$(3.5) \quad \begin{aligned} & \frac{\partial(\Delta)^{-1}v_m}{\partial t} - \varepsilon\Delta v_m + \frac{1}{\varepsilon}(f(\langle u_m \rangle + v_m) - \langle f(\langle u_m \rangle + v_m) \rangle) \\ & + \lambda_0(\Delta)^{-1}(\chi_{\Omega \setminus D}(x)(\langle u_m \rangle + v_m - h) - \langle \chi_{\Omega \setminus D}(x)(\langle u_m \rangle + v_m - h) \rangle) = 0, \quad \text{in } E'_m, \end{aligned}$$

$$(3.6) \quad v_m(0) = P_m u_0,$$

and we finally rewrite equation (3.1) as follows:

$$M^{-1} \frac{dY}{dt} + \varepsilon MY + H(Y) = 0,$$

where $M = -((\Delta N_i, N_j))_{i,j=1,\dots,m}$, $Y = \begin{pmatrix} u_{m_1} \\ \vdots \\ u_{m_m} \end{pmatrix}$, and

$$H(Y) = \begin{pmatrix} ((\frac{1}{\varepsilon}(f(\langle u_m \rangle + v_m) - \langle f(\langle u_m \rangle + v_m) \rangle) + \lambda_0(-\Delta)^{-1}\Xi_m, N_1)) \\ \vdots \\ ((\frac{1}{\varepsilon}(f(\langle u_m \rangle + v_m) - \langle f(\langle u_m \rangle + v_m) \rangle) + \lambda_0(-\Delta)^{-1}\Xi_m, N_m)) \end{pmatrix},$$

where $\Xi_m = \chi_{\Omega \setminus D}((\langle u_m \rangle + v_m - h)) - \langle \chi_{\Omega \setminus D}((\langle u_m \rangle + v_m - h)) \rangle$. Note that $\langle u_m \rangle$ can be expressed as a function of v_m by solving (3.2). The matrix M is invertible and positive definite and $H(Y)$ depends continuously on Y . Applying Cauchy's theorem, we find that there exists a time $t_m \in (0, T)$ and a solution Y to

$$\frac{dY}{dt} + \varepsilon M^2 Y + M H(Y) = 0$$

on the time interval $[0, t_m[$. Having v_m , we then deduce $\langle u_m \rangle$ (from (3.2)).

It follows from the a priori estimates derived in the previous section for the solution $u_m(t)$ ($u(t)$ being replaced by $u_m(t)$; note that, in that case, these formal a priori estimates are now fully justified within the Galerkin approximation) that any local solution to (1.10) is actually a global solution defined on the whole interval $[0, T]$.

It then follows from the a priori estimates that, up to a subsequence which we do not relabel,

$$(3.7) \quad u_m \rightharpoonup u \text{ weakly in } L^2(0, T, H^2(\Omega)),$$

$$(3.8) \quad \frac{\partial u_m}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ weakly in } L^{\frac{4}{3}}(0, T, W^{-2, \frac{4}{3}}(\Omega)),$$

as $m \rightarrow \infty$. It follows from (3.7), (3.8), and the Aubin-Lions compactness theorem that

$$(3.9) \quad u_m \rightarrow u \text{ strongly in } L^{\frac{4}{3}}(0, T, L^{\frac{4}{3}}(\Omega))$$

and $u_m(x, t) \rightarrow u(x, t)$ a.e. $(x, t) \in \Omega \times [0, T]$. Moreover, since f is a continuous,

$$f(u_m(x, t)) \rightarrow f(u(x, t)) \text{ a.e.}$$

and, since $f(u_m)$ is bounded in $L^{\frac{4}{3}}(\Omega_T)$, $\Omega_T = \Omega \times (0, T)$,

$$f(u_m) \rightharpoonup f(u) \text{ weakly in } L^{\frac{4}{3}}(\Omega_T),$$

owing to the weak dominated convergence theorem. Finally, we deduce that $(-\Delta)^{-1} \frac{\partial v_m}{\partial t} \rightharpoonup (-\Delta)^{-1} \frac{\partial v}{\partial t}$ weakly in $L^{\frac{4}{3}}(\Omega_T)$. Thus, passing to the limit in (3.5), we obtain

$$(3.10) \quad \begin{aligned} & \frac{\partial(-\Delta)^{-1}v}{\partial t} - \varepsilon \Delta u + \frac{1}{\varepsilon}(f(u) - \langle f(u) \rangle) \\ & + \lambda_0(-\Delta)^{-1}(\chi_{\Omega \setminus D}(x)(u - h) - \langle \chi_{\Omega \setminus D}(x)(u - h) \rangle) = 0, \quad \text{in } L^{\frac{4}{3}}(\Omega_T). \end{aligned}$$

Finally, we easily pass to the limit in (3.2), at least in a weak sense, i.e., solving explicitly this ODE.

Let now z and w be two solutions to (1.10) with initial data z_0 and w_0 , respectively. We set $u = z - w$ and $u_0 = z_0 - w_0$ and have

$$(3.11) \quad \frac{\partial u}{\partial t} + \varepsilon \Delta^2 u - \frac{1}{\varepsilon} \Delta(f(z) - f(w)) + \lambda_0 \chi_{\Omega \setminus D}(x)u = 0, \quad \text{in } \Omega,$$

$$(3.12) \quad \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0, \quad \text{on } \Gamma,$$

$$(3.13) \quad u|_{t=0} = u_0 = z_0 - w_0, \quad \text{in } \Omega.$$

Integrating (3.11) over Ω , we obtain

$$(3.14) \quad \frac{d}{dt} \langle u \rangle + \lambda_0 \langle \chi_{\Omega \setminus D}(x)u \rangle = 0,$$

hence, setting again $v = u - \langle u \rangle$,

$$(3.15) \quad \frac{\partial v}{\partial t} + \varepsilon \Delta^2 v - \frac{1}{\varepsilon} \Delta(f(z) - f(w)) + \lambda_0(\chi_{\Omega \setminus D}(x)u - \langle \chi_{\Omega \setminus D}(x)u \rangle) = 0.$$

We multiply (3.15) by $(-\Delta)^{-1} \bar{v}$ and find, noting that

$$((\langle f(z) - f(w) \rangle, v)) = 0,$$

the differential equality

$$(3.16) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|v\|_{-1}^2 + \varepsilon \|\nabla v\|^2 + \frac{1}{\varepsilon} \mathbf{Re}[(f(z) - f(w), v)] \\ & + \lambda_0 \mathbf{Re}[((-\Delta)^{-1}(\chi_{\Omega \setminus D}(x)u - \langle \chi_{\Omega \setminus D}(x)u \rangle), v)] = 0. \end{aligned}$$

Note that

$$\begin{aligned}
\left| \lambda_0(((-\Delta)^{-1}(\chi_{\Omega \setminus D}(x)u - \langle \chi_{\Omega \setminus D}(x)u \rangle), v)) \right| &= \left| \lambda_0((\chi_{\Omega \setminus D}(x)u, (-\Delta)^{-1}v)) \right| \\
&\leq c\|u\|\|v\| \\
&\leq c(\|v\|^2 + |\langle u \rangle|^2),
\end{aligned}$$

which yields, owing to Proposition 3.1,

$$\begin{aligned}
(3.17) \quad \frac{1}{2} \frac{d}{dt} \|v\|_{-1}^2 + \varepsilon \|\nabla v\|^2 + \frac{c_0}{\varepsilon} \int_{\Omega} \left(|u|^4 + |z|^2 |u|^2 + \left(\mathbf{Re}[z(\bar{u})] \right)^2 \right) dx \\
\leq c(\|z\|_{L^4(\Omega)}^2 + \|w\|_{L^4(\Omega)}^2 + 1)(\|v\|^2 + |\langle u \rangle|^2).
\end{aligned}$$

We further multiply (3.14) by $\langle \bar{u} \rangle$ and have

$$\begin{aligned}
(3.18) \quad \frac{1}{2} \frac{d}{dt} |\langle u \rangle|^2 &\leq \lambda_0 |\langle \chi_{\Omega \setminus D}(x)u \rangle| |\langle \bar{u} \rangle| \\
&\leq c\|u\| |\langle u \rangle| \\
&\leq c(\|v\|^2 + |\langle u \rangle|^2)
\end{aligned}$$

and, recalling the interpolation inequality

$$\|v\|^2 \leq c\|v\|_{-1} \|\nabla v\|,$$

it follows from (3.17) and (3.18) that

$$\begin{aligned}
(3.19) \quad \frac{d}{dt} (\|v\|_{-1}^2 + |\langle u \rangle|^2) + 2\varepsilon \|\nabla v\|^2 + \frac{2c_0}{\varepsilon} \int_{\Omega} \left(|u|^4 + |z|^2 |u|^2 + \left(\mathbf{Re}[z(\bar{u})] \right)^2 \right) dx \\
\leq c(\|z\|_{L^4(\Omega)}^4 + \|w\|_{L^4(\Omega)}^4 + 1)(\|v\|_{-1}^2 + |\langle u \rangle|^2).
\end{aligned}$$

We finally deduce from Gronwall's lemma that

$$(3.20) \quad \|z(t) - w(t)\|_{H^{-1}}^2 \leq Q(T, \|z_0\|, \|w_0\|) \|z_0 - w_0\|_{H^{-1}}^2, \quad 0 \leq t \leq T,$$

hence the uniqueness, as well as the continuous dependence with respect to the initial data in the H^{-1} -norm. $\square$

b) Existence of the global attractor. It follows from Theorem 3.2 and (3.20) that we have the continuous (with respect to the H^{-1} -norm) semigroup

$$S(t) : L^2(\Omega) \rightarrow L^2(\Omega), \quad u_0 \mapsto u(t), \quad t \geq 0$$

(i.e. $S(0) = I$, $S(t+s) = S(t) \circ S(s)$, $t, s \geq 0$). We then deduce from Proposition 2.12 that $S(t)$ possesses a bounded absorbing set $\mathcal{B}_0$ which is compact in $L^2(\Omega)$ and bounded in $H^2(\Omega)$. We thus deduce from standard results (see, e.g., [57] and [66]) the following theorem.

Theorem 3.3. *The semigroup $S(t)$ possesses the connected global attractor $\mathcal{A}$ such that $\mathcal{A}$ is compact in $L^2(\Omega)$ and bounded in $H^2(\Omega)$.*

Remark 3.4. It is easy to see that we can assume, without loss of generality, that $\mathcal{B}_0$ is positively invariant by $S(t)$, i.e., $S(t)\mathcal{B}_0 \subset \mathcal{B}_0$, $\forall t \geq 0$.

4. EXISTENCE OF EXPONENTIAL ATTRACTORS

Let z and w be two solutions to (1.10) with initial data z_0 and w_0 , respectively. We set $u = z - w$ and $u_0 = z_0 - w_0$ and have

$$(4.1) \quad \frac{\partial u}{\partial t} + \varepsilon \Delta^2 u - \frac{1}{\varepsilon} \Delta(f(z) - f(w)) + \lambda_0 \chi_{\Omega \setminus D}(x)u = 0, \quad \text{in } \Omega,$$

$$(4.2) \quad \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0, \quad \text{on } \Gamma,$$

$$(4.3) \quad u|_{t=0} = u_0 = z_0 - w_0, \quad \text{in } \Omega.$$

Furthermore, it is sufficient here to take initial data belonging to the bounded absorbing set $\mathcal{B}_0$ defined in the previous section.

Lemma 4.1. *Let $u = z - w$ be a solution to (4.1)–(4.3). Then, we have*

$$-\mathbf{Re}[(\nabla(f(z) - f(w)), \nabla u)] \leq c\|u\|_{H^1(\Omega)}^2 + \varepsilon^2 \|\Delta u\|^2.$$

Proof. We first note that

$$f(z) - f(w) = |u|^2 u - |u|^2 z + |z|^2 u + 2\mathbf{Re}[z\bar{u}]z - 2\mathbf{Re}[z\bar{u}]u - u.$$

It follows from Proposition 2.2 that

$$\mathbf{Re}[(\nabla(|u|^2 u - u), \nabla u)] \geq -\|\nabla u\|^2.$$

Besides from Proposition 2.12 and the continuous embedding $H^2(\Omega) \subset \mathcal{C}(\bar{\Omega})$ we have that

$$\begin{aligned} & \left| \mathbf{Re}[(-|u|^2 z + |z|^2 u + 2\mathbf{Re}[z\bar{u}]z - 2\mathbf{Re}[z\bar{u}]u, \Delta u)] \right| \\ & \leq \left| ((-|u|^2 z + |z|^2 u + 2\mathbf{Re}[z\bar{u}]z - 2\mathbf{Re}[z\bar{u}]u, \Delta u)) \right| \\ & \leq 3\||u|^2 z + |z|^2 u\| \|\Delta u\| \\ & \leq c(\|u\|_{L^4(\Omega)}^2 \|z\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)}^2 \|u\|) \|\Delta u\| \\ & \leq c\|u\|_{H^1(\Omega)}^4 + \varepsilon^2 \|\Delta u\|^2. \end{aligned}$$

Observe now that

$$\begin{aligned} \|u\|_{H^1(\Omega)}^4 & \leq c(\|z\|_{H^1}^2 + \|w\|_{H^1}^2) \|u\|_{H^1}^2 \\ & \leq c\|u\|_{H^1}^2, \end{aligned}$$

we infer

$$\left| \mathbf{Re}[(-|u|^2 z + |z|^2 u + 2\mathbf{Re}[z\bar{u}]z - 2\mathbf{Re}[z\bar{u}]u, \Delta u)] \right| \leq c\|u\|_{H^1(\Omega)}^2 + \varepsilon^2 \|\Delta u\|^2.$$

We finally deduce that

$$\begin{aligned}
& -\mathbf{Re}[(\nabla(f(z) - f(w)), \nabla u)] \\
& = -\mathbf{Re}[(\nabla(|u|^2 u - u), \nabla u)] - \mathbf{Re}[((-|u|^2 z + |z|^2 u + 2\mathbf{Re}[z\bar{u}]z - 2\mathbf{Re}[z\bar{u}]u, \Delta u))] \\
& \leq \|\nabla u\|^2 + \left| \mathbf{Re}[((-|u|^2 z + |z|^2 u + 2\mathbf{Re}[z\bar{u}]z - 2\mathbf{Re}[z\bar{u}]u, \Delta u))] \right| \\
& \leq c\|u\|_{H^1(\Omega)}^2 + \varepsilon^2 \|\Delta u\|^2.
\end{aligned}$$

□

Proposition 4.2. *Let $u = z - w$ be a solution to (4.1)–(4.3). Then, there exist two constants c and c' which only depend on $\mathcal{B}_0$ such that*

$$\|z - w\|^2 \leq \frac{c}{t} e^{c't} \|z_0 - w_0\|_{H^{-1}(\Omega)}^2, \quad \forall t > 0.$$

Proof. We multiply (4.1) by $t\bar{u}$ to find

$$(4.4) \quad \frac{1}{2} \frac{d}{dt} (t\|u\|^2) + \varepsilon t \|\Delta u\|^2 + \frac{t}{\varepsilon} \mathbf{Re}[(\nabla(f(z) - f(w)), \nabla u)] + \lambda_0 t \|\chi_{\Omega \setminus D}(x)u\|^2 = \frac{1}{2} \|u\|^2,$$

which yields, owing to Lemma 4.1,

$$(4.5) \quad \frac{d}{dt} (t\|u\|^2) + \lambda_0 t \|\chi_{\Omega \setminus D}(x)u\|^2 \leq ct\|u\|_{H^1(\Omega)}^2 + \|u\|^2.$$

Note that (3.19) yields

$$\begin{aligned}
(4.6) \quad & \frac{d}{dt} (\|v\|_{-1}^2 + |\langle u \rangle|^2) + 2\varepsilon (\|\nabla v\|^2 + |\langle u \rangle|^2) \\
& + \frac{2c_0}{\varepsilon} \int_{\Omega} \left(|u|^4 + |z|^2 |u|^2 + \left(\mathbf{Re}[z(\bar{u})] \right)^2 \right) dx \leq c (\|v\|_{-1}^2 + |\langle u \rangle|^2).
\end{aligned}$$

By using Gronwall's lemma, on account of (4.6), we find

$$(4.7) \quad \|z(t) - w(t)\|_{H^{-1}(\Omega)}^2 \leq ce^{c't} \|z_0 - w_0\|_{H^{-1}(\Omega)}^2.$$

Therefore, integrating (4.6) over $(0, t)$, we obtain, owing to (4.7),

$$(4.8) \quad \int_0^t \|u\|_{H^1(\Omega)}^2 ds \leq ce^{c't} \|u_0\|_{H^{-1}(\Omega)}^2.$$

Finally, using Gronwall's lemma, on account of (4.5), and employing (4.8) we infer

$$(4.9) \quad \|z(t) - w(t)\|^2 \leq \frac{c}{t} e^{c't} \|z_0 - w_0\|_{H^{-1}(\Omega)}^2, \quad \forall t > 0.$$

□

Proposition 4.3. *The semigroup $S(t)$ is uniformly Hölder continuous on $[0, T] \times \mathcal{B}_0$ in the topology of $H^{-1}(\Omega)$, i.e.,*

$$\|S(t_1)z_0 - S(t_2)w_0\|_{H^{-1}(\Omega)} \leq c(\|z_0 - w_0\|_{H^{-1}(\Omega)} + |t_1 - t_2|^{\frac{1}{2}}),$$

where the constant c only depends on T and $\mathcal{B}_0$.

Proof. The Lipschitz continuity with respect to the initial data is an immediate corollary of (3.20). Actually, it suffices to prove the Hölder continuity with respect to time. We have

$$\begin{aligned} \|u(t_1) - u(t_2)\|_{H^{-1}(\Omega)} &= \left\| \int_{t_1}^{t_2} \frac{\partial u}{\partial t} d\tau \right\|_{H^{-1}(\Omega)} \\ (4.10) \quad &\leq \left| \int_{t_1}^{t_2} \left\| \frac{\partial u}{\partial t} \right\|_{H^{-1}(\Omega)} d\tau \right| \\ &\leq |t_1 - t_2|^{\frac{1}{2}} \left| \int_{t_1}^{t_2} \left\| \frac{\partial u}{\partial t} \right\|_{H^{-1}(\Omega)}^2 d\tau \right|^{\frac{1}{2}}. \end{aligned}$$

Multiplying (2.3) by $\frac{\partial \bar{v}}{\partial t}$, we find, noting that $\langle \frac{\partial \bar{v}}{\partial t} \rangle = 0$,

$$\begin{aligned} (4.11) \quad &\frac{\varepsilon}{2} \frac{d}{dt} \|\nabla u\|^2 + \left\| \frac{\partial v}{\partial t} \right\|_{-1}^2 + \frac{1}{\varepsilon} \mathbf{Re}[(f(u), \frac{\partial v}{\partial t})] \\ &+ \lambda_0 \mathbf{Re}[(\chi_{\Omega \setminus D}(x)(u - h), (-\Delta)^{-1} \frac{\partial v}{\partial t})] = 0. \end{aligned}$$

Here,

$$\begin{aligned} &\left| \lambda_0 \mathbf{Re}[(\chi_{\Omega \setminus D}(x)(u - h), (-\Delta)^{-1} \frac{\partial v}{\partial t})] \right| \\ &= \left| \lambda_0 ((\chi_{\Omega \setminus D}(x)(u - h), (-\Delta)^{-1} \frac{\partial v}{\partial t})) \right| \\ &\leq c \|u - h\| \left\| \frac{\partial v}{\partial t} \right\|_{-1} \\ &\leq c(\|v\|^2 + |\langle u \rangle|^2 + \|h\|^2) + \frac{1}{4} \left\| \frac{\partial v}{\partial t} \right\|_{-1}^2. \end{aligned}$$

Furthermore,

$$\left| \frac{1}{\varepsilon} \mathbf{Re}[(f(u), \frac{\partial v}{\partial t})] \right| \leq c \|\nabla f(u)\| \left\| \frac{\partial v}{\partial t} \right\|_{-1},$$

which yields

$$(4.12) \quad \varepsilon \frac{d}{dt} \|\nabla u\|^2 + \left\| \frac{\partial v}{\partial t} \right\|_{-1}^2 \leq c(\|v\|^2 + |\langle u \rangle|^2 + \|h\|^2) + c' \|\nabla f(u)\|^2.$$

From (3.14), we have that

$$\left\langle \frac{\partial u}{\partial t} \right\rangle \leq c \|u\|$$

and, (4.12) can be written as

$$(4.13) \quad \varepsilon \frac{d}{dt} \|\nabla u\|^2 + \left\| \frac{\partial v}{\partial t} \right\|_{-1}^2 + \left\langle \frac{\partial u}{\partial t} \right\rangle^2 \leq c(\|v\|^2 + |\langle u \rangle|^2 + \|h\|^2) + c' \|\nabla f(u)\|^2.$$

Note that

$$\begin{aligned} \|\nabla f(u)\| &= \|f'(u)\nabla u\| \\ &\leq Q(\|u\|_{L^\infty(\Omega)}) \|\nabla u\| \\ &\leq c \|\nabla u\|, \end{aligned}$$

owing to the definition of $\mathcal{B}_0$ and the continuous embedding $H^2(\Omega) \subset \mathcal{C}(\overline{\Omega})$. We then deduce from (2.18) and (4.13) that

$$(4.14) \quad \int_{t_1}^{t_2} \left\| \frac{\partial u}{\partial t} \right\|_{H^{-1}(\Omega)}^2 \leq c,$$

where the constant c only depends on $\mathcal{B}_0$ and T such that $t_1, t_2 \in [0, T]$, so that (4.10) yields

$$(4.15) \quad \|u(t_1) - u(t_2)\|_{H^{-1}(\Omega)} \leq c |t_1 - t_2|^{\frac{1}{2}},$$

where the constant c only depends on $\mathcal{B}_0$ and T such that $t_1, t_2 \in [0, T]$. □

We finally deduce from Theorem 3.2, Proposition 4.2, and Proposition 4.3 the following result (see, e.g., [26] and [27]).

Theorem 4.4. *The semigroup $S(t)$ possesses an exponential attractor $\mathcal{M} \subset \mathcal{B}_0$, i.e.,*

- (i) $\mathcal{M}$ is compact in $H^{-1}(\Omega)$;
- (ii) $\mathcal{M}$ is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall t \geq 0$;
- (iii) $\mathcal{M}$ has finite fractal dimension in $H^{-1}(\Omega)$;
- (iii) $\mathcal{M}$ attracts exponentially fast the bounded subsets of $L^2(\Omega)$,

$$\begin{aligned} \forall B \subset L^2(\Omega) \text{ bounded, } \quad \text{dist}_{H^{-1}(\Omega)}(S(t)B, \mathcal{M}) &\leq Q(\|B\|_{L^2(\Omega)}) e^{-\xi t}, \\ &\xi > 0, \quad t \geq 0, \end{aligned}$$

where the constant ξ is independent of B and $\text{dist}_{H^{-1}(\Omega)}$ denotes the Hausdorff semidistance between sets defined by

$$\text{dist}_{H^{-1}(\Omega)}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_{H^{-1}(\Omega)}.$$

Remark 4.5. Setting $\tilde{\mathcal{M}} = S(1)\mathcal{M}$, we can prove that $\tilde{\mathcal{M}}$ is an exponential attractor for $S(t)$, but now in the topology of $L^2(\Omega)$ (see, e.g., [28]).

Since $\mathcal{M}$ (or $\tilde{\mathcal{M}}$) is a compact attracting set, we deduce from Theorem 4.4 the following corollary.

Corollary 4.6. *The semigroup $S(t)$ possesses the finite-dimensional global attractor $\mathcal{A} \subset \mathcal{B}_0$.*

Remark 4.7. i) We can also follow [6] to prove the continuity of the isophote vectors at the boundary of the inpainting domain for problem (1.10) when λ_0 is large.

ii) An important particular case of problem (1.10) arises when the initial imaginary part vanishes (i.e., $u_2(0, x) = 0$), meaning that the damaged image (binary image in that case) has two values, -1 and 1 . In that case, numerical simulations (we have not been able to prove it) show that the imaginary part of the solution remains zero for all times.

iii) We studied the (global) dynamics of the system when λ_0 and ε are constants. More generally, all constants here and in the previous sections depend on λ_0 and ε and grow as these quantities go to $+\infty$ and 0 , respectively. In particular, the dependence on λ_0 is an important issue in view of inpainting applications (see [6]) and a natural question is whether one can have an upper bound on the dimension of the global attractor which is independent of this quantity (here, the upper bound that we obtain explodes as λ_0 goes to $+\infty$). A natural (but involved, see, e.g., [66]) problem would be to find a lower bound on this dimension (possibly, in terms of λ_0). This will be investigated elsewhere.

5. NUMERICAL SIMULATIONS

The author in [34] and [35] proposed a convexity splitting scheme for gradient flow-derived equations which results in an unconditionally gradient stable time discretization scheme. In particular, the scheme is stable for any arbitrarily large time step. Here, the idea consists in dividing the energy functional into two parts, a convex one and a concave one. Then, the convex part is treated implicitly, while the concave one is treated explicitly. This scheme has been used as a specific fast solver for the Bertozzi–Esedoglu–Gillette–Cahn–Hilliard equation, see [5], [6], [7], and [10].

As in the case of the Bertozzi–Esedoglu–Gillette–Cahn–Hilliard equation, (1.10) is not a gradient flow. However, it is the sum of a gradient descent with respect to the H^{-1} -inner product for the original Cahn–Hilliard energy and a gradient descent with respect to the L^2 -inner product for the additional fidelity energy. We recall that the complex Cahn–Hilliard energy is given by

$$\mathcal{E}(u) = \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} \left(\frac{1}{4} |u|^4 - \frac{1}{2} |u|^2 \right) \right) dx,$$

while the additional fidelity energy reads

$$\mathcal{E}'(u) = \frac{\lambda_0}{2} \int_{\Omega \setminus D} |u - h|^2 dx.$$

In view of this, we write $\mathcal{E}(u) = \mathcal{E}_1(u) - \mathcal{E}_2(u)$ and $\mathcal{E}'(u) = \mathcal{E}'_1(u) - \mathcal{E}'_2(u)$, where

$$\mathcal{E}_1(u) = \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{c_1}{2} |u|^2 \right) dx,$$

$$\begin{aligned}\mathcal{E}_2(u) &= \int_{\Omega} \left(-\frac{1}{\varepsilon} \left(\frac{1}{4}|u|^4 - \frac{1}{2}|u|^2 \right) + \frac{c_1}{2}|u|^2 \right) dx, \\ \mathcal{E}'_1(u) &= \int_{\Omega} \frac{c_2}{2}|u|^2 dx,\end{aligned}$$

and

$$\mathcal{E}'_2(u) = \int_{\Omega} \left(-\frac{\lambda}{2}|u - h|^2 + \frac{c_2}{2}|u|^2 \right) dx,$$

c_1 and c_2 being nonnegative constants which need to be chosen large enough so that the energies $\mathcal{E}_1(u)$, $\mathcal{E}_2(u)$, $\mathcal{E}'_1(u)$, and $\mathcal{E}'_2(u)$ are convex.

As far as the Euler time discretization for this problem is concerned, the time step $\delta t > 0$ is fixed. The resulting time-stepping scheme reads

$$\frac{u_h^{n+1} - u_h^n}{\delta t} = -\nabla_{H^{-1}}(\mathcal{E}_1(u_h^{n+1}) - \mathcal{E}_2(u_h^n)) - \nabla_{L^2}(\mathcal{E}'_1(u_h^{n+1}) - \mathcal{E}'_2(u_h^n)),$$

where $\nabla_{H^{-1}}$ and ∇_{L^2} denote the gradient descents with respect to the H^{-1} - and L^2 -inner products, respectively. This translates into a numerical scheme of the form

$$\frac{u_h^{n+1} - u_h^n}{\delta t} + \varepsilon \Delta^2 u_h^{n+1} - c_1 \Delta u_h^{n+1} + c_2 u_h^{n+1} = \frac{1}{\varepsilon} \Delta f(u_h^n) + \lambda_0(\chi_{\Omega \setminus D}(x)(h - u_h^n)) - c_1 \Delta u_h^n + c_2 u_h^n.$$

Therefore, using the operator splitting (see [22], [30], [38], and [44]), the final scheme reads

$$\begin{aligned}\frac{1}{\delta t}((u_h^{n+1}, \phi)) &+ ((\nabla \mu_h^{n+1}, \nabla \phi)) + c_1((\nabla u_h^{n+1}, \nabla \phi)) + c_2((u_h^{n+1}, \phi)) \\ &= \frac{1}{\delta t}((u_h^n, \phi)) + \lambda_0((\chi_{\Omega \setminus D}(x)(h(x) - u_h^n), \phi)) \\ &\quad + c_1((\nabla u_h^n, \nabla \phi)) + c_2((u_h^n, \phi)), \quad \forall \phi \in H^1(\Omega), \\ ((\mu_h^{n+1}, \psi)) - \varepsilon((\nabla u_h^{n+1}, \nabla \psi)) &= \frac{1}{\varepsilon}((f(u_h^n), \psi)), \quad \forall \psi \in H^1(\Omega).\end{aligned}$$

In this scheme, we use a $P1$ -finite element for the space discretization. The numerical simulations are performed with the software Freefem++ (see [43]).

Having this convexity splitting in mind, we follow the dynamic two-steps numerical algorithm proposed in [5] and [6] to obtain good inpainting results. The idea here is that the edges in the image should be connected with a large value of the diffuse interface thickness ε and then, switching the interface diffuse thickness ε to a smaller value (which depends on the mesh discretization), we obtain the final inpainting result. In the grayscale inpainting tests below, we further use the information outside the inpainting domain in the given image to obtain better inpainting results.

Binary image test. The gray region in Figure 1(a) corresponds to the inpainting region. We run the complex version of the modified Cahn–Hilliard equation with $\varepsilon = 0.1$. We are close to a steady state at $t = 592$, as shown in Figure 1(b) (real part of the solution which is equal to the solution, see Remark 4.7 (ii)). Then, we take $\varepsilon = 0.001$ and run again the complex version of the modified Cahn–Hilliard equation, taking Figure 1(b) as an initial datum. We are close to a steady state at $t = 677$, as shown in Figure 1(c) (real part of the solution which is equal to the solution, see Remark 4.7 (ii)). In this test, $\lambda_0 = 90000$, $\delta t = 1$, $c_1 = 3000$, and $c_2 = 270000$. Furthermore, Ω is a $(0, 0.5) \times (0, 0.5)$ -square. The

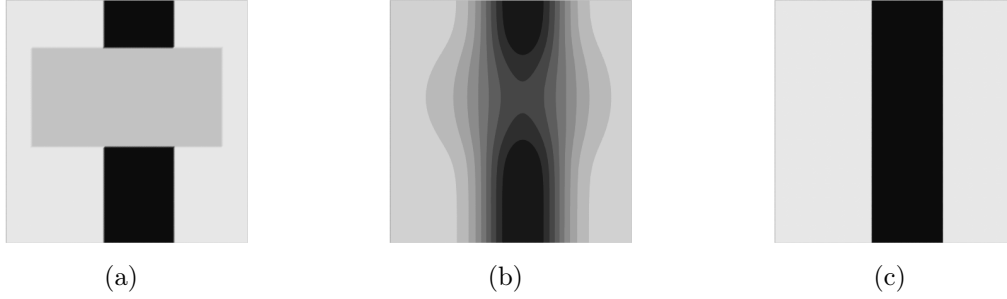


FIGURE 1. (a) Inpainting region in gray. (b) Solution at $t = 592$, $\varepsilon = 0.1$. (c) Solution at $t = 677$, $\varepsilon = 0.001$.

triangulation is obtained by dividing it into 100×100 rectangles and by dividing each rectangle along the same diagonal.

Grayscale image tests. Figure 2 shows the efficiency of our proposed model in disocclusion applications (disocclusion means the recovery of hidden parts of objects in an image by interpolation from the neighborhood of the occluded area; for more details, see, e.g., [54]). Figure 2(a) and Figure 2(b) correspond to the damaged image and the masked part, respectively. We then run the complex version of the modified Cahn–Hilliard equation with a large value of the diffuse interface thickness ($\varepsilon = 0.1$). We are close to a steady state at $t = 99$, as shown in Figure 2(c) (real part), and switch ε to a smaller value ($\varepsilon = 0.001$). We are close to a steady state at $t = 875$, as shown in Figure 2(d) (real part) and in Figure 2(e) (imaginary part). Finally, we use the information on the image in $\Omega \setminus D$ given in the damaged image 2(a) in the inpainting result 2(d) to obtain the final inpainting result in Figure 2(f). Here, $\lambda_0 = 100000$, $\delta t = 1$, $c_1 = 3000$, and $c_2 = 300000$.

Figure 3 shows the efficiency of our proposed model to remove texts. Figure 3(a) and Figure 3(b) correspond to the damaged image and the masked part, respectively. As above, we first run the complex version of the modified Cahn–Hilliard equation with a large value of the diffuse interface thickness ($\varepsilon = 0.1$). We are close to a steady state at $t = 273$ and then switch ε to a smaller value ($\varepsilon = 0.001$). We are close to a steady state at $t = 2265$, as shown in Figure 3(c). Finally, we use the information on the image outside D given in the damaged image 3(a) in the inpainting result 3(c) to obtain the final inpainting result in Figure 3(d). Here, $\lambda_0 = 10000$, $\delta t = 1$, $c_1 = 3000$, and $c_2 = 30000$.

In our final test, Figure 4(a) and Figure 4(b) correspond to the damaged image and the masked part, respectively. As above, we first run the complex version of the modified Cahn–Hilliard equation with a large value of the diffuse interface thickness ($\varepsilon = 0.1$). We are close to a steady state at $t = 115$ and then switch ε to a smaller value ($\varepsilon = 0.001$). We are close to a steady state at $t = 1162$, as shown in Figure 4(c). Finally, we use the information on the image in $\Omega \setminus D$ given in the damaged image 4(a) in the inpainting result 4(c) to obtain the final inpainting result in Figure 4(d). Here, $\lambda_0 = 100000$, $\delta t = 1$, $c_1 = 3000$, and $c_2 = 300000$.

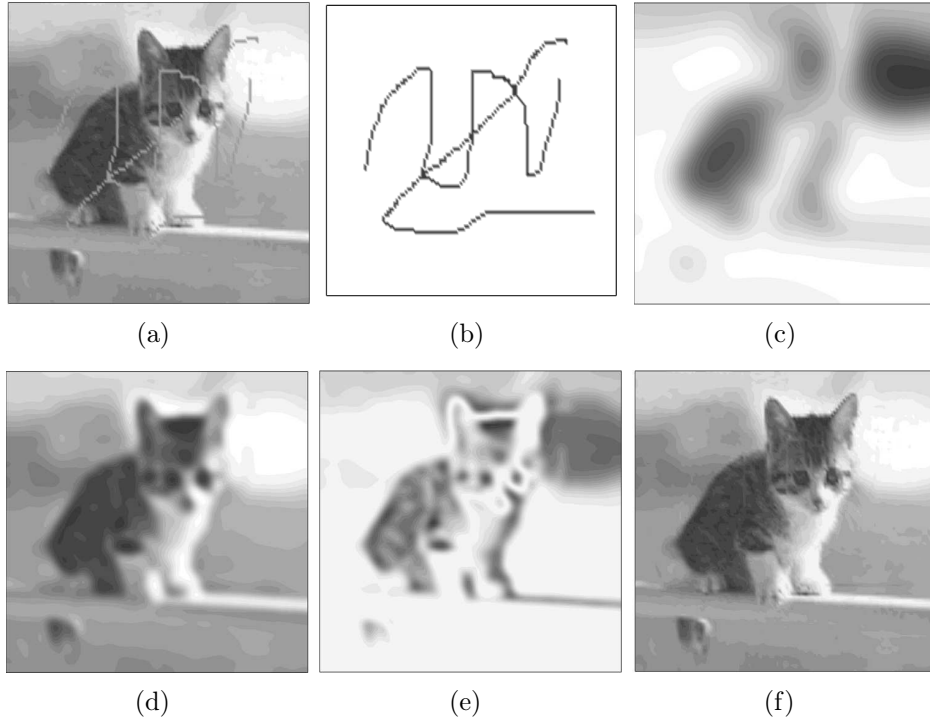


FIGURE 2. (a) Damaged image (128×102 pixels). (b) Masked part. (c) Real part of the solution at $t = 99$ with $\varepsilon = 0.1$. (d) Real part of the solution at $t = 875$ with $\varepsilon = 0.001$. (e) Imaginary part of the solution at $t = 875$ with $\varepsilon = 0.001$. (f) Using the information of the image outside D given in (a) in the inpainting result (d).

Conclusion. We proposed in this article a complex version of the Bertozzi–Esedoglu–Gillette–Cahn–Hilliard equation, introduced for binary image inpainting, in view of applications to grayscale image inpainting. Though the equation has the same structure as the binary model, we had, owing to the fact that the order parameter is now complex valued, to derive proper and careful estimates on the nonlinear term. In particular, we proved the existence and uniqueness of solutions, as well as the existence of the finite-dimensional global attractor. Furthermore, we obtained numerical simulations which show the efficiency of the model.

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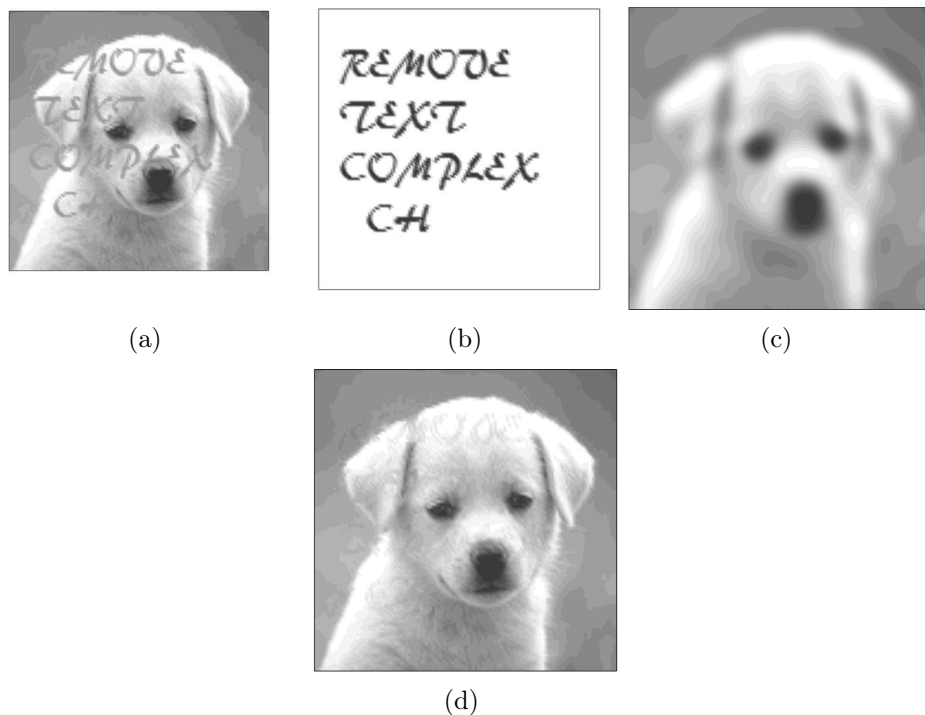


FIGURE 3. (a) Damaged image (128×102 pixels). (b) Masked part. (c) Real part of the solution. (d) Using the information of the image outside D given in (a) in the inpainting result (c).

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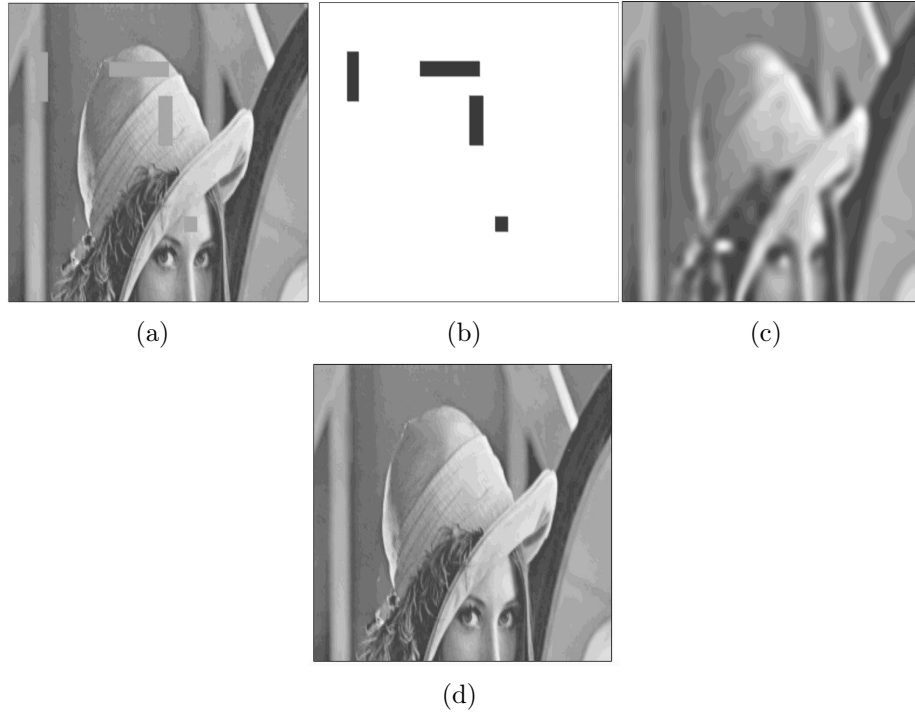


FIGURE 4. (a) Damaged image (256×256 pixels). (b) Masked part. (c) Real part of the solution. (d) Using the information of the image outside D given in (a) in the inpainting result (c).

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