

COMPUTING THE BÉZIER CONTROL POINTS OF THE LAGRANGIAN INTERPOLANT IN ARBITRARY DIMENSION

MARK AINSWORTH [†] AND MANUEL A. SÁNCHEZ [†]

Abstract. The Bernstein-Bézier form of a polynomial is widely used in the fields of computer aided geometric design, spline approximation theory and, more recently, for high order finite element methods for the solution of partial differential equations. However, if one wishes to compute the classical Lagrange interpolant relative to the Bernstein basis, then the resulting Bernstein-Vandermonde matrix is found to be highly ill-conditioned.

In the univariate case of degree n , Marco and Martinez [18] showed that using Neville elimination to solve the system exploits the total positivity of the Bernstein basis and results in an $\mathcal{O}(n^2)$ complexity algorithm. Remarkable as it may be, the Marco-Martinez algorithm has some drawbacks: The derivation of the algorithm is quite technical; the interplay between the ideas of total positivity and Neville elimination are not part of the standard armoury of many non-specialists; and, the algorithm is strongly associated to the univariate case.

The present work addresses these issues. An alternative algorithm for the inversion of the univariate Bernstein-Vandermonde system is presented that has: The same complexity as the Marco-Martinez algorithm and whose stability does not seem to be in any way inferior; a simple derivation using only the basic theory of Lagrange interpolation (at least in the univariate case); and, a natural generalisation to the multivariate case.

Key words. Bernstein polynomials, Bernstein-Vandermonde matrix, total positivity, multivariate polynomial interpolation.

AMS subject classifications. 65D05, 65D17, 65Y20, 68U07

1. Introduction. The classical Lagrangian interpolation problem consists of finding a polynomial p of degree at most n , such that

$$(1.1) \quad p(x_j) = f_j, \quad j = 0, \dots, n.$$

where $\{x_j\}_{j=0}^n \subseteq [0, 1]$ are distinct $(n + 1)$ nodes and $\{f_j\}_{j=0}^n$ are given data. The existence of a unique solution to this problem is well-known from one's first course in numerical analysis. The seasoned reader might also recall that, despite the uniqueness of the interpolant itself, the choice of basis for constructing the Lagrange interpolant is less clear-cut [6]—possibilities include the Lagrangian or the Newton bases, with the much maligned monomial basis even having a shout [14].

The Bernstein basis functions for the space $\mathbb{P}^n([0, 1])$ of polynomials of degree at most n on $[0, 1]$, is given by

$$(1.2) \quad B_k^n(x) = \binom{n}{k} (1-x)^{n-k} x^k, \quad k = 0, \dots, n, \quad x \in [0, 1].$$

The Bernstein functions extend naturally to give a basis for polynomials of total degree at most n on simplices in arbitrary numbers of spatial dimensions (see Section 4).

The basis has many unexpected and attractive properties that have led to it being: almost ubiquitous in the computer aided geometric design (CAGD) community [10, 12] for the representation of curves and surfaces; as an important theoretical tool in the spline approximation theory literature [17]; and, more recently, a practical tool for the

[†]Division of Applied Mathematics, Brown University, Providence, RI 02912, USA (mark.ainsworth@brown.edu, manuel.sanchez_uribe@brown.edu). MA gratefully acknowledges the partial support of this work under AFOSR contract FA9550-12-1-0399.

efficient implementation of high order finite element methods for the solution of partial differential equations [1–3, 15, 16]. For further information on these, and numerous other applications of Bernstein polynomials, we refer the reader to the survey article of Farouki [12].

The Bernstein-Bézier (BB) representation of a polynomial $p \in \mathbb{P}^n([0, 1])$ takes the form $p = \sum_{k=0}^n c_k B_k^n$ in which the coefficients $\{c_k\}$ are referred to as *control points* [10, 12, 17], and which may be associated with the $(n + 1)$ uniformly spaced points on $[0, 1]$ (even if the interpolation nodes are non-uniformly spaced). However, while the polynomial p satisfies $p(0) = c_0$ and $p(1) = c_n$, this property does not hold at the remaining control points. On one hand, this property does not hinder the typical workflow of a CAGD engineer whereby the locations of $\{c_k\}$ are adjusted until a curve of the desired shape is obtained. In effect, the control points are used to *define* the curve directly. On the other hand, the typical usage of polynomials in scientific computing is rather different in that one generally wishes to *fit* a polynomial to a given function. For example, in applying the finite element method to approximate the solution of a partial differential equation, one might have boundary or initial data and require to choose an appropriate (piecewise) polynomial approximation of the data. The approximation is often chosen to be an interpolant, leading to what we shall term the Bernstein-Bézier interpolation problem, which consists of computing the control points $\{c_k\}_{k=0}^n$ such that the associated Bernstein-Bézier polynomial interpolates the data:

$$(1.3) \quad p(x_j) = \sum_{k=0}^n c_k B_k^n(x_j) = f_j, \quad j = 0, \dots, n.$$

Conditions (1.3) may be expressed as a system of linear equations involving the *Bernstein-Vandermonde* matrix [18]. If the monomial basis were to be used, then the standard Vandermonde matrix would emerge. The highly ill-conditioned nature of the Vandermonde matrix is well-documented [9]. Notwithstanding, the inversion of the Vandermonde matrix to compute the Lagrangian interpolant is in some ways preferable to more direct methods [8, 14].

The Bernstein-Vandermonde matrix is likewise found to be highly ill-conditioned [18] suggesting that its inversion may not be the wisest approach. Nevertheless, *structure* of the matrix arising from the *total positivity* of the Bernstein basis means that using Neville elimination [13] to solve the system obviates some of the issues due to ill-conditioning. Marco and Martinez [18] exploit this fact and derive an algorithm for the inversion of the matrix that has $\mathcal{O}(n^2)$ complexity—the same as multiplying by the inverse of the matrix.

Remarkable though the Marco-Martinez algorithm may be, it does have its drawbacks:

- the derivation of the algorithm is highly technical involving non-trivial identities for the minors of Bernstein-Vandermonde matrices;
- the interplay between the ideas of total positivity and Neville elimination are not part of the standard armoury of many non-specialists;
- the algorithm seems to be firmly rooted to the solution of the Bernstein interpolation problem in the univariate case (indeed, total positivity is essentially a univariate concept).

The purpose of the present work is to address these issues. Specifically, we shall present an alternative algorithm for the inversion of the univariate Bernstein-Vandermonde system that has:

- the same complexity as the Marco-Martinez algorithm and whose stability does not seem to be in any way inferior;
- a simple derivation that could be taught to undergraduates familiar with only the basic theory of Lagrange interpolation (at least in the univariate case);
- a natural generalisation to the multivariate case (essential for applications such as finite element analysis in two and three dimensions).

The remainder of this article is organised as follows. The next section deals with deriving the new algorithm in the univariate case using only elementary facts about univariate interpolation and basic properties of Bernstein polynomials. Section 3 shows how the univariate algorithm is easily extended to solve the multivariate Bernstein interpolation problem in the case of tensor product polynomials.

In Section 4, we tackle the much harder task of solving the Bernstein interpolation problem on simplices in higher dimensions in which even the existence of the Lagrangian interpolant is less obvious [8, 19]. Using the standard hypothesis under which the Lagrange interpolant exists, we develop an algorithm for the solution of the multivariate Bernstein interpolation problem on simplices in two dimensions, and indicate how it may be extended to arbitrary dimensions. The ideas used in Section 4 are, modulo technical issues, essentially the same to those used in Section 2 to handle the univariate case. Sections 2-4 are accompanied by some numerical examples illustrating the behaviour and stability of the resulting algorithms.

In summary, the algorithms developed in the present work form a key step towards a more widespread use of Bernstein-Bézier techniques in scientific computing in general, and in finite element analysis in particular by enabling the use of non-homogeneous initial and boundary data. More generally, the problem of how to extend the Marco-Martinez algorithm to the solution of Bernstein-Vandermonde systems to arbitrary dimension is addressed.

2. Bernstein polynomial interpolation in 1D.

2.1. Analytical solution. The *improved Lagrange formula* [6] for the Lagrangian interpolant is given by

$$(2.1) \quad p(x) = \sum_{j=0}^n \mu_j f_j \frac{\ell(x)}{x - x_j}, \quad x \in [0, 1].$$

where $\ell \in \mathbb{P}^{n+1}([0, 1])$ and the barycentric weights μ_j , $j = 0, \dots, n$ are given by

$$(2.2) \quad \ell(x) = \prod_{k=0}^n (x - x_k), \quad \mu_j = \frac{1}{\ell'(x_j)} \text{ for } j = 0, \dots, n.$$

Our first result gives a closed form for the control points satisfying (1.3):

THEOREM 2.1. *For $k = 0, \dots, n$ define the control points $\{c_k\}_{k=0}^n$ by the rule*

$$(2.3) \quad c_k = \sum_{j=0}^n \mu_j f_j \tilde{w}_k(x_j), \quad k = 0, \dots, n,$$

where

$$(2.4) \quad \tilde{w}_k(z) = \frac{w_k(z)}{z(1-z)B_k^n(z)} \text{ and } w_k(z) = - \sum_{i=0}^k a_i B_i^{n+1}(z)$$

for z any node x_j , with $\{a_i\}$ control points of ℓ . Then, $\{c_k\}_{k=0}^n$ are the control points for the Bernstein interpolant (1.3).

Proof. We claim that $\{\tilde{w}_k(x_j)\}_{k=0}^n$ are the control points of the polynomial $\ell(x)/(x - x_j)$ for $j = 0, \dots, n$. Assuming the claim holds, then

$$p(x) = \sum_{k=0}^n c_k B_k^n(x) = \sum_{j=0}^n \mu_j f_j \sum_{k=0}^n w_k(x_j) B_k^n(x) = \sum_{j=0}^n \mu_j f_j \frac{\ell(x)}{x - x_j},$$

and the improved Lagrange formula (2.1) then shows that p is the interpolant.

It remains to prove the claim holds. Let z be any node x_j , then there exist constants \tilde{w}_k such that $\ell(x) = (x - z) \sum_{k=0}^n \tilde{w}_k B_k^n(x)$. Now, using the definition of the Bernstein polynomials we obtain $x - z = (1 - z)B_1^1(x) - zB_0^1(x)$, and

$$(2.5) \quad B_1^1 B_k^n = \frac{k+1}{n+1} B_{k+1}^{n+1} \text{ and } B_0^1 B_k^n = \left(1 - \frac{k}{n+1}\right) B_k^{n+1}.$$

Hence,

$$\ell(x) = \sum_{k=0}^{n+1} \left((1-z) \frac{k}{n+1} \tilde{w}_{k-1} - z \left(1 - \frac{k}{n+1}\right) \tilde{w}_k \right) B_k^{n+1}(x)$$

where we define $\tilde{w}_{-1} = \tilde{w}_{n+1} = 0$. Hence, denoting by $\{a_i\}_{i=0}^{n+1}$ the Bézier control points of ℓ

$$(2.6) \quad a_k = (1-z) \frac{k}{n+1} \tilde{w}_{k-1} - z \left(1 - \frac{k}{n+1}\right) \tilde{w}_k, \quad k = 0, \dots, n+1.$$

Multiplying equation (2.6) by $B_k^{n+1}(z)$ and using the definition of the Bernstein polynomials gives

$$(2.7) \quad a_k B_k^{n+1}(z) = z(1-z) \left(\tilde{w}_{k-1} B_{k-1}^n(z) - \tilde{w}_k B_k^n(z) \right)$$

$$(2.8) \quad = w_{k-1} - w_k, \quad k = 0, \dots, n+1,$$

where $w_k = z(1-z) \tilde{w}_k B_k^n(z)$. The overdetermined system of equations (2.7) is consistent since $\ell(z) = \sum_{i=0}^{n+1} a_i B_i^{n+1}(z) = 0$. Hence,

$$(2.9) \quad w_k = - \sum_{i=0}^k a_i B_i^{n+1}(z), \quad k = 0, \dots, n$$

and the result follows as claimed. \square

Theorem 2.1 gives a new explicit formula for the solution of the Bernstein interpolation problem based on the Lagrangian form of the interpolant. However, direct use of Theorem 2.1 for computation of the actual solution would cost $\mathcal{O}(n^3)$ operations.

2.2. A simple algorithm for computing the univariate control points.

Many elementary numerical analysis textbooks extol the virtues of using the Newton formula [9] for the polynomial interpolant satisfying (1.1):

$$(2.10) \quad p(x) = \sum_{j=0}^n f[x_0, \dots, x_j] w_j(x)$$

where $w_0(x) = 1$ and $w_j(x) = \prod_{k=0}^{j-1} (x - x_k)$, for $j = 1, \dots, n$, and $f[x_0, \dots, x_j]$ are the divided differences defined recursively as follows:

$$f[x_j, \dots, x_k] = \frac{f[x_{j+1}, \dots, x_k] - f[x_j, \dots, x_{k-1}]}{x_k - x_j}, \quad k = j+1, \dots, n \text{ and } j = 0, \dots, n,$$

whilst if $k = j$, then the value is simply f_j .

Is there any advantage to using the Newton form of the interpolant for Bernstein interpolation?

2.2.1. Newton-Bernstein Algorithm. Let $p_k \in \mathbb{P}^k([0, 1])$ be the Newton form of the interpolant at the nodes $\{x_j\}_{j=0}^k$; that is

$$(2.11) \quad p_k(x) = \sum_{j=0}^k f[x_0, \dots, x_j] w_j(x), \quad k = 0, \dots, n.$$

THEOREM 2.2. For $k = 0, \dots, n$ define $\{w_j^{(k)}\}_{j=0}^k$ and $\{c_j^{(k)}\}_{j=0}^k$ by the rules $w_0^{(0)} = 1$, $c_0^{(0)} = f[x_0]$ and

$$\begin{aligned} w_j^{(k)} &= \frac{j}{k} w_{j-1}^{(k-1)} (1 - x_{k-1}) - \frac{k-j}{k} w_j^{(k-1)} x_{k-1}, \\ c_j^{(k)} &= \frac{j}{k} c_{j-1}^{(k-1)} + \frac{k-j}{k} c_j^{(k-1)} + w_j^{(k)} f[x_0, \dots, x_k], \end{aligned}$$

for $j = 0, \dots, k$, where we use the convention $c_{-1}^{(k-1)} = w_{-1}^{(k-1)} = 0$ and $c_k^{(k-1)} = w_k^{(k-1)} = 0$. Then, $\{w_j^{(k)}\}_{j=0}^k$ are the control points of w_k and $\{c_j^{(k)}\}_{j=0}^k$ are the control points of p_k .

Proof. The case $k = 0$ is trivially satisfied. We proceed by induction and suppose that the Bernstein forms of the polynomials, p_{k-1} and w_{k-1} are given by

$$w_{k-1}(x) = \sum_{j=0}^{k-1} w_j^{(k-1)} B_j^{k-1}(x) \text{ and } p_{k-1}(x) = \sum_{j=0}^{k-1} c_j^{(k-1)} B_j^{k-1}(x).$$

Firstly, we derive the Bernstein coefficients of the k -th degree polynomial w_k as follows:

$$\begin{aligned} w_k(x) &= (x - x_k) \sum_{j=0}^{k-1} w_j^{(k-1)} B_j^{k-1}(x) \\ &= (x(1 - x_{k-1}) - x_{k-1}(1 - x)) \sum_{j=0}^{k-1} w_j^{(k-1)} B_j^{k-1}(x). \end{aligned}$$

Using (2.5) with $n = k - 1$ we conclude that the Bernstein coefficients of w_k are given by

$$(2.12) \quad w_j^{(k)} = \frac{j}{k} w_{j-1}^{(k-1)} (1 - x_{k-1}) - \frac{k-j}{k} w_j^{(k-1)} x_{k-1}.$$

Secondly, we use degree raising property

$$(2.13) \quad B_k^{n-1} = \frac{n-k}{n} B_k^n + \frac{k+1}{n} B_{k+1}^n, \quad k = 0, \dots, n-1.$$

to write the $(k-1)$ -th degree polynomial p_{k-1} , in terms of the Bernstein basis of polynomials of degree k .

$$(2.14) \quad p_{k-1}(x) = \sum_{j=0}^k c_j^{(k-1)} B_j^{k-1}(x) = \sum_{j=0}^k \left(\frac{j}{k} c_{j-1}^{(k-1)} + \frac{k-j}{k} c_j^{(k-1)} \right) B_j^k(x)$$

where we again use the convention $c_{-1}^{(k-1)} = 0$ and $c_k^{(k-1)} = 0$. Observe from (2.11) that $p_k(x) = p_{k-1}(x) + w_k(x)f[x_0, \dots, x_k]$. Hence, by (2.14) we have that

$$c_j^{(k)} = \frac{j}{k} c_{j-1}^{(k-1)} + \frac{k-j}{k} c_j^{(k-1)} + w_j^{(k)} f[x_0, \dots, x_k], \quad j = 0, \dots, k$$

are the control points of p_k . \square

Algorithm 1 provides an implementation of the result obtained in Theorem 2.2 in which the simplicity of the procedure is immediately apparent along with an overall complexity of $\mathcal{O}(n^2)$:

Algorithm 1: `NewtonBernstein` ($\{x_j\}_{j=0}^n, \{f_j\}_{j=0}^n$)

Input : Interpolation nodes and data $\{x_j\}_{j=0}^n, \{f_j\}_{j=0}^n$.
Output : Control points $\{c_j\}_{j=0}^n$.

```

1  $c_0 \leftarrow f_0$ ;
2  $w_0 \leftarrow 1$ ;
3 for  $k \leftarrow n, s$  do
4   for  $k \leftarrow n, s$  do
5      $f_k \leftarrow (f_k - f_{k-1}) / (x_k - x_{k-s})$ ;           // Divided differences
6   end
7   for  $k \leftarrow s, 1$  do
8      $w_k \leftarrow \frac{k}{s} w_{k-1} (1 - x_{s-1}) - (1 - \frac{k}{s}) w_k x_{s-1}$ ;
9      $c_k \leftarrow \frac{k}{s} c_{k-1} + (1 - \frac{k}{s}) c_k + f_s w_k$ ;
10  end
11   $w_0 \leftarrow -w_0 x_{s-1}$ ;
12   $c_0 \leftarrow c_0 + f_s w_0$ ;
13 end
14 return  $\{c_j\}_{j=0}^n$ ;
```

Readers who have studied the derivation of the Marco-Martinez algorithm [18], for solving the same problem may be rather surprised by the comparative ease with which the Newton-Bernstein algorithm is derived. The complexity of both algorithms is $\mathcal{O}(n^2)$. In the next section, we compare the performance of the Newton-Bernstein algorithm and the more sophisticated Marco-Martinez algorithm.

2.3. Numerical examples. In this section we compare the performance of Algorithm 1 (`NewtonBernstein`) with Marco-Martinez algorithm [18] (MM) and, the command “\” in `Matlab` for three examples. Examples 2.1 and 2.2 are taken directly from [18]. The true solutions of all of these problems are computed using the command `linsolve` in `Maple 18`. In each case, we present the relative error defined by

$$(2.15) \quad \text{Relative error} = \frac{\|c_{\text{exact}} - c_{\text{approx}}\|_2}{\|c_{\text{exact}}\|_2},$$

where c_{exact} and c_{approx} denote the exact and approximate Bézier control points.

EXAMPLE 2.1. Let $n = 15$ and choose uniformly distributed nodes given by $x_i = (i + 1)/(17)$, $i = 0, \dots, n$. Let A be the Bernstein-Vandermonde matrix of degree n generated by a given set of interpolation nodes $\{x_j\}_{j=0}^n$,

$$(2.16) \quad a_{i,j} = b_i^n(x_j) = \binom{n}{i} (1 - x_j)^{n-i} x_j^i, \quad i, j = 0, \dots, n.$$

In this case the condition number of the Bernstein-Vandermonde matrix is $\kappa(A) = 2.3e + 06$. The right hand sides f_1 , f_2 and f_3 are given by

$$\begin{aligned} (f_1)_j &= (1 - x_j)^n, \\ f_2 &= (2, 1, 2, 3, -1, 0, 1, -2, 4, 1, 1, -3, 0, -1, -1, 2)^T, \\ f_3 &= (1, -2, 1, -1, 3, -1, 2, -1, 4, -1, 2, -1, 1, -3, 1, -4)^T. \end{aligned}$$

Relative errors for each algorithm are displayed in Table 1.

f_i	$A \backslash f_i$	NewtonBernstein	MM
f_1	7.2e-13	7.9e-14	9.2e-13
f_2	7.1e-11	5.9e-16	1.0e-15
f_3	7.1e-11	5.2e-16	4.9e-16

TABLE 1
Example 2.1: Relative errors in L^2 norm.

The results in Table 1 indicate that Newton-Bernstein and Marco-Martinez algorithm give comparable stability and accuracy, with the “\” solver performing poorly.

EXAMPLE 2.2. Let $n = 15$ and A be the Bernstein-Vandermonde matrix generated by the data

$$(2.17) \quad x = \left(\frac{1}{18}, \frac{1}{16}, \frac{1}{14}, \frac{1}{12}, \frac{1}{10}, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{11}{20}, \frac{19}{34}, \frac{17}{30}, \frac{15}{26}, \frac{11}{18}, \frac{9}{14}, \frac{7}{10}, \frac{5}{6}\right)^T.$$

The condition number of the Bernstein-Vandermonde matrix is $\kappa(A) = 3.5e + 09$. Consider the singular value decomposition of the Bernstein-Vandermonde matrix $A = U\Sigma V^T$. We will solve the linear systems $Ac = f_j$, with $f_j = u_j$, $j = 0, \dots, n$, where u_j denotes the j -th column of U , the left singular vectors of A . Relative errors for each algorithm are displayed in Table 2.

This example illustrates the effective well-conditioning introduced by Chan and Foulser in [7]. In particular, the test confirms that the Newton-Bernstein algorithm increases in accuracy as the Chan-Foulser number decreases which is the same (positive) behaviour exhibited by the Marco-Martinez algorithm.

EXAMPLE 2.3. Let $n = 25$ and take the interpolation nodes to be the zeros of the Chebyshev polynomial of the first kind. In this case the condition number of the Bernstein-Vandermonde matrix is $\kappa(A) = 2.1e + 07$. The right hand sides are taken to be f_1 , f_2 and f_3 given by

$$\begin{aligned} (f_1)_j &= (1 - x_j)^n \quad \text{for } j = 1, \dots, 25, \\ f_2 &= (-3, -1, 2, -1, 2, -1, 1, -3, 2, -3, 1, 2, -1, -2, 1, -2, -1, -2, 1, -2, 3, -2, -3, 2, 1, -2)^T, \\ f_3 &= (-1, 2, 1, -1, -2, -3, 2, 3, -2, -1, 2, 1, 3, -2, 1, -1, -1, 2, -2, -3, 1, -1, 1, -3, 2, -1)^T. \end{aligned}$$

Relative errors for each algorithm are displayed in Table 3.

f_i	$\gamma(A, f_i)$	$A \setminus f_i$	NewtonBernstein	MM
f_1	3.5e+09	3.3e-07	1.9e-08	3.1e-07
f_2	2.6e+09	1.1e-07	6.2e-08	2.1e-08
f_3	1.3e+09	8.4e-09	5.6e-09	2.7e-08
f_4	1.2e+09	2.1e-08	1.1e-08	1.4e-08
f_5	5.3e+08	3.8e-08	2.6e-09	1.7e-08
f_6	4.0e+08	4.3e-08	1.0e-08	2.7e-09
f_7	1.1e+08	3.7e-08	1.8e-09	7.4e-09
f_8	5.8e+07	3.5e-08	6.5e-10	2.5e-09
f_9	1.1e+07	1.5e-08	8.7e-10	3.2e-10
f_{10}	3.7e+06	1.2e-08	1.5e-10	3.3e-10
f_{11}	4.8e+05	1.4e-08	4.5e-12	3.6e-12
f_{12}	1.2e+05	3.5e-09	1.3e-11	1.7e-11
f_{13}	6.2e+03	8.2e-09	3.0e-12	2.3e-12
f_{14}	9.3e+02	1.1e-08	7.6e-13	7.8e-13
f_{15}	1.4e+01	1.2e-08	4.2e-14	4.2e-14
f_{16}	1	5.0e-09	7.1e-15	7.9e-15

TABLE 2
Example 2.2: Relative errors in L^2 norm.

f_i	$A \setminus f_i$	NewtonBernstein _{Leja}	NewtonBernstein	MM
f_1	7.7e-11	4.2e-11	4.2e-11	1.0e-09
f_2	1.1e-10	3.2e-16	7.9e-13	1.4e-15
f_3	1.1e-10	4.8e-16	1.6e-13	1.6e-15

TABLE 3
Example 2.3: Relative errors in L^2 norm.

It is well-known that there can be advantages in re-ordering the interpolation nodes. One feature of the Newton-Bernstein Algorithm (not shared by the Marco-Martinez Algorithm MM) is the flexibility to re-order the interpolation nodes. Table 2 compares the relative errors obtained for the interpolation problems using ascending ordering (NewtonBernstein) and using Leja ordering NewtonBernstein_{Leja}, see [14, 20]. In this example the Leja ordering of the interpolation nodes produces better results, and as in the previous example the accuracy of our algorithms increases with the alternating sign pattern of the right hand side.

3. Tensor product polynomial interpolation. In this section we briefly show how the Newton-Bernstein Algorithm 1 can be used to interpolate on tensor product grids in higher dimensions. Consider the two dimensional case. Let $\{x_i\}_{i=0}^n \subseteq [0, 1]$ and $\{y_j\}_{j=0}^m \subseteq [0, 1]$ be given nodes and let $\{f_{i,j}\}_{i,j=0}^{n,m}$ be given data. The polynomial interpolation problem consists of finding the polynomial $p \in \mathbb{P}^n([0, 1]) \times \mathbb{P}^m([0, 1])$ satisfying the conditions:

$$(3.1) \quad p(x_i, y_j) = f_{i,j}, \quad i = 0, \dots, n, j = 0, \dots, m.$$

The associated Bernstein interpolation problem consists of finding the control points $\{c_{k,\ell}\}_{k=0,\ell=0}^{n,m}$ such that the tensor product Bernstein polynomial

$$(3.2) \quad p(x, y) = \sum_{\ell=0}^m \sum_{k=0}^n c_{k,\ell} B_k^n(x) B_\ell^m(y), \quad (x, y) \in [0, 1]^2$$

satisfies (3.1).

The tensor product nature of the problem means that we can exploit the Newton-Bernstein algorithm for the univariate case to solve problem (3.2). The basic idea is to: a) first construct the control points for the univariate Bernstein interpolant $p^{(j)}$ on the lines $y = y_j$ for each j , i.e. for each $j = 0, \dots, n$:

$$(3.3) \quad p^{(j)}(x) = \sum_{k=0}^n c_k^{(j)} B_k^n(x); \quad p^{(j)}(x_i) = f_{i,j}, \quad i = 0, \dots, n;$$

and, b) solve a univariate interpolation problem for the y -variable in which the data are the univariate polynomials obtained in step a), i.e. find the univariate polynomial p satisfying

$$(3.4) \quad p(x, y_j) = p^{(j)}(x), \quad j = 0, \dots, m.$$

The problem in step b) is nothing more than a univariate interpolation problem with the only difference being that the data is now polynomial valued. The derivation of the Newton-Bernstein algorithm presented in the previous section applies equally well to the more general problem of interpolating values from a vector space X . In particular, choosing X to be polynomials shows that the Newton-Bernstein Algorithm 1 can be brought to bear at each of stages a) and b). This idea forms the foundation for Algorithm 2.

Algorithm 2: TensorProduct2D ($\{x_i\}_{i=0}^n, \{y_j\}_{j=0}^m, \{f_{i,j}\}_{i,j=0}^{n,m}$)

Input : Interpolation nodes and data $\{x_i\}_{i=0}^n, \{y_j\}_{j=0}^m, \{f_{i,j}\}_{i,j=0}^{n,m}$.

Output : Control points $\{c_j\}_{j=0}^n$.

```

1 for  $j \leftarrow 0, m$  do
2   |  $\{\tilde{c}_k^{(j)}\}_{k=0}^n \leftarrow \text{NewtonBernstein}(\{x_i\}_{i=0}^n, \{f_{i,j}\}_{i=0}^n);$ 
3 end
4 for  $i \leftarrow 0, n$  do
5   |  $\{c_{i,j}\}_{j=0}^m \leftarrow \text{NewtonBernstein}(\{y_j\}_{j=0}^m, \{\tilde{c}_i^{(j)}\}_{j=0}^m);$ 
6 end
7 return  $\{c_{i,j}\}_{i,j=0}^{n,m};$ 

```

The basic idea used in the two dimensional algorithm is easily extended to higher dimensions resulting in Algorithm 3 which computes the solution of the three dimensional version of the interpolation problem (3.2).

3.1. Numerical examples: Tensor product. In this section we consider examples of the two and three dimensional Bernstein-Bézier interpolation problems, illustrating the performance of Algorithms 2 and 3.

Algorithm 3: TensorProduct3D ($\{x_i\}_{i=0}^n, \{y_j\}_{j=0}^m, \{z_k\}_{k=0}^l, \{f_{i,j,k}\}_{i,j,k=0}^{n,m,l}$)

Input : Interpolation nodes and data $\{x_i\}_{i=0}^n, \{y_j\}_{j=0}^m, \{z_k\}_{k=0}^l, \{f_{i,j,k}\}_{i,j,k=0}^{n,m,l}$.

Output : Control points $\{c_{i,j,k}\}_{i,j,k=0}^{n,m,l}$.

```

1 for  $j \leftarrow 0, m$  do
2    $\{\tilde{c}_{i,j,k}\}_{i=0}^n \leftarrow \text{NewtonBernstein}(\{x_i\}_{i=0}^n, \{f_{i,j,k}\}_{i=0}^n)$ ;
3 end
4 for  $i \leftarrow 0, n$  do
5    $\{\hat{c}_{i,j,k}\}_{j=0}^m \leftarrow \text{NewtonBernstein}(\{y_j\}_{j=0}^m, \{\tilde{c}_{i,j,k}\}_{j=0}^m)$ ;
6 end
7 for  $i \leftarrow 0, n$  do
8    $\{c_{i,j,k}\}_{k=0}^l \leftarrow \text{NewtonBernstein}(\{z_k\}_{k=0}^l, \{\hat{c}_{i,j,k}\}_{k=0}^l)$ ;
9 end
10 return  $\{c_{i,j,k}\}_{i,j,k=0}^{n,m,l}$ ;

```

EXAMPLE 3.1. Consider $n = 15$ and the two dimensional interpolation problem with grid induced by the following nodes

$$x_i = \frac{i+1}{n+2}, \quad y_j = \frac{j+1}{n+3} \quad i, j = 0, \dots, n.$$

Note that in this case the condition number of the two dimensional Bernstein-Vandermonde matrix A_2 (size 256×256) is $\kappa(A_2) = 1.4e + 13$. As load vectors we consider f_1 and f_2 randomly generated, taking integer values between -3 and 3 for each component. Relative errors are displayed in Table 4.

f_i	$A_2 \backslash f_i$	SOLVER2D		
		NewtonBernstein	$A \backslash f_i$	MM
f_1	4.2e-5	2.5e-15	2.3e-11	1.6e-15
f_2	2.6e-5	9.7e-16	3.5e-11	2.8e-15

TABLE 4

Relative errors in L^2 norm for solutions of Example 3.1 using Matlab “\” and TensorProduct2D. The results obtained when NewtonBernstein is replaced in TensorProduct2D by the univariate solvers “\” and MM are also presented.

Observe the significantly higher accuracy of the Newton-Bernstein algorithm in comparison with the results of “\”.

EXAMPLE 3.2. Consider $n = 10$ and the three dimensional interpolation problem with grid induced for the following nodes

$$x_i = \frac{i+1}{n+2}, \quad y_j = \frac{j+1}{n+3}, \quad z_k = \frac{k+2}{n+4} \quad i, j = 0, \dots, n.$$

Note that in this case the condition number of the three dimensional Bernstein-Vandermonde matrix (size 1331×1331) is $\kappa(A_3) = 7.6e + 13$. As load vectors we consider f_1 and f_2 randomly generated, taking integer values between -3 and 3 for each component. Relative errors are displayed in Table 5.

f_i	$A_3 \setminus f_i$	SOLVER3D		
		NewtonBernstein	$A \setminus f_i$	MM
f_1	8.4e-6	6.0e-16	2.1e-12	1.1e-15
f_2	8.2e-6	5.2e-16	2.2e-12	1.3e-15

TABLE 5

Relative errors in L^2 norm for solutions of Example 3.2 using *Matlab* “\” and *TensorProduct3D*. The results obtained when *NewtonBernstein* is replaced in *TensorProduct3D* by the univariate solvers “\” and *MM* are also presented.

The three dimensional case also shows a great accuracy of the Newton-Bernstein algorithm in comparison with the *Matlab* solver, see Table 5.

4. Control points for polynomial interpolation on a simplex. The computation of the control points of the Lagrange interpolant on a simplex in \mathbb{R}^d , $d \in \mathbb{N}$ is rather more problematic than the tensor product case. Nevertheless, in this section, we show how the univariate Newton-Bernstein algorithm can be generalised to solve the problem.

4.1. Preliminaries. For $n \in \mathbb{N}$ define the indexing set $\mathcal{I}_d^n = \{\alpha = (\alpha_1, \dots, \alpha_{d+1}) \in \mathbb{Z}_*^{d+1} : |\alpha| = n\}$ where $|\alpha| = \sum_{j=1}^{d+1} \alpha_j$, $\mathbb{Z}_* = \{0\} \cup \mathbb{N}$ and $|\mathcal{I}_d^n| = \text{card}(\mathcal{I}_d^n) = \binom{n+d}{d}$. For $n, m \in \mathbb{N}$, $\alpha \in \mathcal{I}_d^n$, $\beta \in \mathcal{I}_d^m$ and $\lambda \in \mathbb{R}^{d+1}$, define

$$\binom{\alpha}{\beta} = \prod_{j=1}^{d+1} \binom{\alpha_j}{\beta_j}, \quad \binom{n}{\alpha} = n! \left(\prod_{j=1}^{d+1} \alpha_j! \right)^{-1}, \quad \lambda^\alpha = \prod_{j=1}^{d+1} \lambda_j^{\alpha_j}.$$

Let $T = \text{conv}\{\mathbf{v}_1, \dots, \mathbf{v}_{d+1}\}$ be a non-degenerate d -simplex in \mathbb{R}^d . The following property of a non-degenerate simplices will prove useful [1]:

LEMMA 4.1. *The following conditions are equivalent:*

1. T is a non-degenerate d -simplex;
2. for all $\mathbf{x} \in T$, there exists a unique set of nonnegative scalars $\lambda_1, \lambda_2, \dots, \lambda_{d+1}$ such that $\sum_{\ell=1}^{d+1} \lambda_\ell \mathbf{v}_\ell = \mathbf{x}$ and $\sum_{\ell=1}^{d+1} \lambda_\ell = 1$.

Let \mathcal{D}_d^n be the set of domain points of T defined by $\mathbf{x}_\alpha = \frac{1}{n} \sum_{k=1}^{d+1} \alpha_k \mathbf{v}_k$, $\alpha \in \mathcal{I}_d^n$, where $\lambda \in \mathbb{R}^{d+1}$ are defined as in Lemma 4.1. The Bernstein polynomials of degree $n \in \mathbb{Z}_+$ associated with the simplex T are defined by

$$(4.1) \quad B_\alpha^{T,n}(\mathbf{x}) = B_\alpha^n(\mathbf{x}) = \binom{n}{\alpha} \lambda(\mathbf{x})^\alpha, \quad \alpha \in \mathcal{I}_d^n$$

and satisfy

$$(4.2) \quad B_\alpha^m B_\beta^n = \frac{\binom{\alpha+\beta}{\alpha}}{\binom{m+n}{n}} B_{\alpha+\beta}^{m+n} \quad \alpha \in \mathcal{I}_d^m, \beta \in \mathcal{I}_d^n.$$

The Bernstein polynomial interpolation problem on the simplex T reads: given a set of distinct interpolation nodes $\{\mathbf{x}_j\}_{j=1}^{|\mathcal{I}_d^n|}$ and interpolation data $\{f_j\}_{j=1}^{|\mathcal{I}_d^n|}$, find control points $\{c_\alpha\}_{\alpha \in \mathcal{I}_d^n}$ such that

$$(4.3) \quad p(\mathbf{x}) = \sum_{\alpha \in \mathcal{I}_d^n} c_\alpha B_\alpha^n(\mathbf{x}) : \quad p(\mathbf{x}_j) = f_j, \quad \text{for } j = 1, \dots, |\mathcal{I}_d^n|.$$

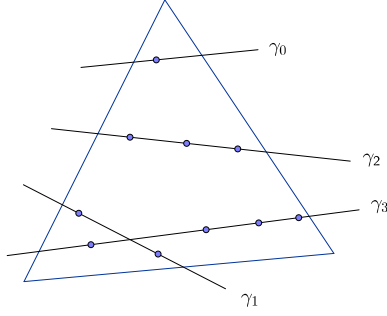


FIG. 1. Illustration of configuration of interpolation nodes under the Solvability Condition (S)

4.2. Two dimensional case. The polynomial interpolation problem in higher dimensions requires that the interpolation nodes appearing in (4.3) satisfy additional conditions beyond simply being distinct in order for the interpolation problem to be well-posed [8]:

Solvability Condition (S) *There exist distinct lines $\gamma_0, \gamma_1, \dots, \gamma_n$ such that the interpolation nodes can be partitioned into (non-overlapping) sets $\mathcal{A}_n, \mathcal{A}_{n-1}, \dots, \mathcal{A}_0$ where \mathcal{A}_j contains $j + 1$ nodes located on $\gamma_j \setminus (\gamma_{j+1}, \dots, \gamma_n)$.*

We will assume that Condition (S) is satisfied throughout. In particular, Condition (S) implies that there is a line γ_n on which $n + 1$ distinct interpolation nodes lie. This means that there exists a (non-unique) polynomial $q_n \in \mathbb{P}^n(T)$ satisfying the *univariate* polynomial interpolation problem on the line γ_n :

$$(4.4) \quad q_n(\mathbf{x}_i) = f_i \quad \forall \mathbf{x}_i \in \mathcal{A}_n.$$

Equally well, Condition (S) implies that the line γ_{n-1} contains n distinct interpolation nodes, none of which lie on γ_n , meaning that the univariate polynomial interpolation problem for these nodes is well-posed. The data for this interpolation problem is chosen as follows. Let Γ_n be an affine polynomial describing the line γ_n , i.e. $\mathbf{x} \in \gamma_n$ iff $\Gamma_n(\mathbf{x}) = 0$. There exists a (non-unique) polynomial $q_{n-1} \in \mathbb{P}^{n-1}(T)$ satisfying the following *univariate* polynomial interpolation problem on the line γ_{n-1} :

$$(4.5) \quad q_{n-1}(\mathbf{x}_i) = \frac{f_i - q_n(\mathbf{x}_i)\Gamma_n(\mathbf{x}_i)}{\Gamma_n(\mathbf{x}_i)} \quad \forall \mathbf{x}_i \in \mathcal{A}_{n-1}.$$

Observe that this data is well-defined thanks to Condition (S). The non-uniqueness of the polynomial stems from the fact that while the values of q_n on the line γ_n are uniquely defined, there are many ways to extend q_n to the simplex—a canonical approach for defining the extension will be presented in Section 4.2.1.

The foregoing arguments can be applied repeatedly to define a sequence of polynomials $q_j \in \mathbb{P}^j(T)$ satisfying a univariate interpolation problem on the line γ_j :

$$(4.6) \quad q_j(\mathbf{x}_i) = \frac{f_i - \sum_{k=j+1}^n q_k(\mathbf{x}_i) \prod_{l=k+1}^n \Gamma_l(\mathbf{x}_i)}{\prod_{l=j+1}^n \Gamma_l(\mathbf{x}_i)} \quad \forall \mathbf{x}_i \in \mathcal{A}_j$$

where Γ_k is an affine polynomial satisfying $\mathbf{x} \in \gamma_k$ iff $\Gamma_k(\mathbf{x}) = 0$, $k = 0, \dots, n$.

The following result presents a general construction for the solution of the full interpolation problem (4.3) in terms of solutions q_j of univariate interpolation problems on the lines γ_j :

THEOREM 4.2. Let $\{q_j\}_{j=0}^n$ be defined as above and define $p \in \mathbb{P}^n(T)$ to be

$$(4.7) \quad p(\mathbf{x}) = \sum_{j=0}^n q_j(\mathbf{x}) \prod_{i=j+1}^n \Gamma_i(\mathbf{x}), \quad \mathbf{x} \in T.$$

Then p solves the interpolation problem (4.3),

Proof. The polynomial p defined in (4.7) clearly belongs to $\mathbb{P}^n(T)$. Let $j \in \{0, \dots, n\}$ and $\mathbf{x}_i \in \mathcal{A}_j$, $j \in \{0, \dots, n\}$. Inserting \mathbf{x}_i into formula (4.7) gives

$$\begin{aligned} p(\mathbf{x}_i) &= \sum_{k=0}^n q_k(\mathbf{x}_i) \prod_{l=k+1}^n \Gamma_l(\mathbf{x}_i) = \sum_{k=j}^n q_k(\mathbf{x}_i) \prod_{l=k+1}^n \Gamma_l(\mathbf{x}_i) \\ &= q_j(\mathbf{x}_i) \prod_{l=j+1}^n \Gamma_l(\mathbf{x}_i) + \sum_{k=j+1}^n q_k(\mathbf{x}_i) \prod_{l=k+1}^n \Gamma_l(\mathbf{x}_i), \end{aligned}$$

where we have used the fact that $\Gamma_j(\mathbf{x}_i) = 0$ due to Condition (S). The result follows thanks to (4.6). \square

Theorem 4.2 reduces the solution of the multivariate interpolation problem to the solution of a sequence of univariate interpolation problems. This feature may be used in conjunction with the univariate Newton-Bernstein algorithm to construct an algorithm for solving the multivariate Bernstein interpolation problem (4.3) as follows:

Algorithm 4: NewtonBernstein2D($\{\mathbf{x}_j, f_j\}_{j=0}^{|\mathcal{I}_2^n|}$)

```

Input      : Interpolation nodes and data  $\{\mathbf{x}_j, f_j\}_{j=0}^{|\mathcal{I}_2^n|}$ .
Output     : Control points  $c^p$ .
1 Initialise  $c^p$ ;
2 for  $j \leftarrow n, 1$  do
3    $G^j \leftarrow \text{BBAffine}(\mathcal{A}_j)$ ;                                     // BB-form of  $\Gamma_j$ 
4    $[z_1, z_2, \kappa] \leftarrow \text{GcapT}(G^j)$ ;                             //  $\Gamma_j \cap T$ 
5    $\bar{\mathcal{A}}_j \leftarrow \text{Transform1D}(\mathcal{A}_j, z_1, z_2)$ ;                       // Map  $[z_1, z_2] \mapsto [0, 1]$ 
6    $c^{\gamma_j} \leftarrow \text{NewtonBernstein}(\bar{\mathcal{A}}_j)$ ;
7    $c^{q_j} \leftarrow \text{BBExtension}(c^{\gamma_j}, z_1, z_2, \kappa)$ ;               //  $q_j$  solves (4.6)
8    $a \leftarrow c^{q_j}$ ;
   /* Computes BB-form of  $q_j \prod_{i=j+1}^n \Gamma_i$                                      */
9   for  $i \leftarrow j+1, n$  do
10     $a \leftarrow \text{BBProduct}(a, G^i)$ ;
11  end
12   $c^p \leftarrow c^p + a$ ;
13  for  $i \leftarrow 1, |\mathcal{I}_2^{j-1}|$  do
14    /* Divided differences                                                         */
     $f_i \leftarrow (f_i - \text{DeCasteljau}(c^{q_j}, \mathbf{x}_i)) / \text{DeCasteljau}(G^j, \mathbf{x}_i)$ ;
15  end
16 end
17 return  $c^p$ ;

```

Algorithm 4 calls five subroutines. The subroutine `DeCasteljau` refers to the well-known *de Casteljau* algorithm [10] for the evaluation of a polynomial written in

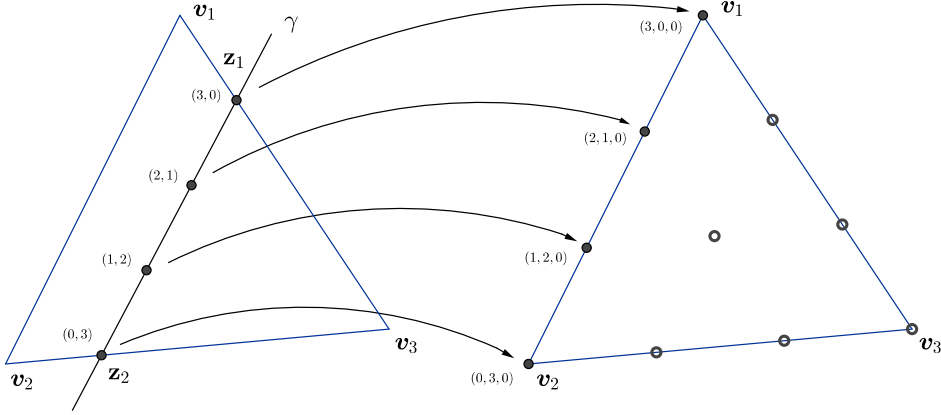


FIG. 2. Illustration of labeling for extension procedure and correspondence of control points $\{c_{\alpha}^{\gamma}\}_{\alpha \in \mathcal{I}_1^j}$ and $\{c_{\alpha}^T\}_{\alpha \in \mathcal{I}_2^j}$ (filled circles), with $j = 3$.

Bernstein form at a cost of $\mathcal{O}(n^{d+1})$ operations in d -dimensions. The four remaining subroutines are the subject of the following sections. It will transpire that the overall complexity of Algorithm 4 is $\mathcal{O}(n^{d+1})$, where $d = 2$ in the current case. A numerical example illustrating the performance of the algorithm is presented in Section 4.3.

Algorithm 5: BBAffine(\mathcal{A}_j)

Input : Interpolation nodes and data \mathcal{A}_j .

Output : BB-form of Γ , G .

- 1 Compute coefficients a, b, c of line $ax + by + c = 0$ fitting the interpolation nodes \mathcal{A}_j ;
 - 2 $d = \max\{|c|, |a + c|, |b + c|\}$;
 - 3 $G_{(1,0,0)} = c/d$;
 - 4 $G_{(0,1,0)} = (a + c)/d$;
 - 5 $G_{(0,0,1)} = (b + c)/d$;
 - 6 return G ;
-

4.2.1. BBExtension. The subroutine BBExtension extends the domain of a solution of the univariate interpolation problem (4.6) from the line γ_j to the whole simplex T . In order to accomplish this task, it is necessary to obtain points z_1 and z_2 satisfying $\gamma_j \cap T = \text{conv}\{z_1, z_2\}$. Roughly speaking, z_1 and z_2 are the points at which the line γ_j intersects the boundary of the simplex. Without loss of generality, assume that γ_j separates the vertices of T into sets $\{v_1, v_2\}$ and $\{v_3\}$ such that $\lambda_1(z_1) > 0$, $\lambda_1(z_2) = 0$ and $\lambda_2(z_1) = 0$, $\lambda_2(z_2) > 0$, where λ_1 and λ_2 are the barycentric coordinates on the simplex T .

In particular, $\text{conv}\{z_1, z_2\}$ is a non-degenerate 1-simplex and hence, in view of Lemma 4.1, we can define barycentric coordinates on the 1-simplex $\text{conv}\{z_1, z_2\}$. In view of the theory presented in Section 2.2, we can find control points $\{c_{\alpha}^{\gamma_j}\}$ of the univariate Bernstein polynomial interpolant of the data \mathcal{A}_j on $\text{conv}\{z_1, z_2\}$. These control points correspond to the Bernstein form of the interpolant relative to the barycentric coordinates on the 1-simplex $\text{conv}\{z_1, z_2\}$. It therefore only remains to

transform these control points to control points for the barycentric coordinates on the simplex T , and thereby implicitly extending the univariate polynomial to the whole simplex:

LEMMA 4.3. *Let $\{c_{\alpha}^{\gamma_j}\}_{\alpha \in \mathcal{I}_1^j}$ be the control points of a polynomial $q^{\gamma_j} \in \mathbb{P}^j(\text{conv}\{z_1, z_2\})$ defined on the 1-simplex $\text{conv}\{z_1, z_2\}$. Define control points $\{c_{\alpha}^T\}_{\alpha \in \mathcal{I}_2^j}$ of a polynomial $q^T \in \mathbb{P}^j(T)$ on the simplex T by the rule: for $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{I}_2^j$*

$$(4.8) \quad c_{(\alpha_1, \alpha_2, \alpha_3)}^T = \begin{cases} c_{(\alpha_1, \alpha_2)}^{\gamma_j} (\lambda_1(z_1))^{-\alpha_1} (\lambda_2(z_2))^{-\alpha_2}, & \text{if } \alpha_3 = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then, $q^T \in \mathbb{P}^j(T)$ coincides with q^{γ_j} on γ_j .

Proof. Consider the Bernstein form of q^T and apply definition (4.8)

$$(4.9) \quad \begin{aligned} q^T(\mathbf{x}) &= \sum_{\alpha \in \mathcal{I}_2^j: \alpha_3=0} c_{\alpha}^T B_{\alpha}^{T,j}(\mathbf{x}) \\ &= \sum_{\alpha \in \mathcal{I}_2^j: \alpha_3=0} c_{(\alpha_1, \alpha_2)}^{\gamma_j} \binom{j}{(\alpha_1, \alpha_2)} \left(\frac{\lambda_1(\mathbf{x})}{\lambda_1(z_1)} \right)^{\alpha_1} \left(\frac{\lambda_2(\mathbf{x})}{\lambda_2(z_2)} \right)^{\alpha_2}, \quad \mathbf{x} \in T. \end{aligned}$$

Observe the factor $\frac{\lambda_1(\mathbf{x})}{\lambda_1(z_1)}$ is linear, and takes values 1 at $\mathbf{x} = z_1$ and 0 at $\mathbf{x} = z_2$, and as such corresponds to the barycentric coordinates on $\text{conv}\{z_1, z_2\}$. The same consideration apply to the other factor in (4.9). Hence, if $\mathbf{x} \in \gamma_j$, then

$$q^T(\mathbf{x}) = \sum_{\alpha \in \mathcal{I}_1^j} c_{\alpha}^{\gamma_j} B_{\alpha}^{\gamma_j,j}(\mathbf{x}) = q^{\gamma_j}(\mathbf{x}),$$

and the result follows. \square

The algorithm for **BBExtension** corresponding to Lemma 4.3 is presented in Algorithm 6.

4.2.2. GcapT. The practical implementation of Lemma 4.3 requires the identification of z_1 and z_2 satisfying $\gamma_j \cap T = \text{conv}(z_1, z_2)$. We observe that γ_j separates the vertices of T in one of the following: $\{v_1, v_2\} \cup \{v_3\}$; $\{v_2, v_3\} \cup \{v_1\}$ and $\{v_3, v_1\} \cup \{v_2\}$. Suppose that the isolated node is given by v_k , $k \in \{1, 2, 3\}$. Let z denote either z_1 or z_2 , and let the barycentric coordinates of z be λ_1, λ_2 and λ_3 , then

$$(4.10) \quad \left. \begin{aligned} \lambda_k &= 0 \\ \sum_{i=1}^3 G_{e_i} \lambda_i &= 0 \\ \sum_{i=1}^3 \lambda_i &= 1 \end{aligned} \right\} \quad \text{subject to} \quad 0 \leq \lambda_i \leq 1, \quad \text{for } i = 1, 2, 3$$

where $e_k \in \mathcal{I}_2^1$ is the multi-index $(e_k)_i = \delta_{k,i}$ for $i = 1, 2, 3$. The first condition holds because z is on the edge opposite to vertex v_k , the second because $\Gamma_j(z) = 0$ and the third by the property of barycentric coordinates. Since there exist only *two* such points z , we know that the system (4.10) is solvable for precisely *two* of the cases $k \in \{1, 2, 3\}$. These observations provide the simple approach to the identification of z_1 and z_2 implemented in Algorithm 7.

4.2.3. Transform1D. In order to use the one dimensional Newton-Bernstein algorithm we need to transform the two dimensional nodes lying on γ_j to one dimensional. Subroutine **Transform1D** transforms the nodes $\{x_j\}$ on the segment $[z_1, z_2]$ to the nodes $\{x_i\}$ in $[0, 1]$. This is implemented in Algorithm 8.

Algorithm 6: BBExtension(\mathcal{A}_j)

Input : $c^{\gamma_j}, \mathbf{z}_1, \mathbf{z}_2, \kappa$.
Output : BB-form of polynomial q^T, c_{α}^T .

```
1 for  $\alpha \in \mathcal{I}_2^j$  do
2   if  $\alpha_{\kappa} = 0$  then
3     if  $\kappa = 3$  then
4        $c_{\alpha}^T \leftarrow c_{(\alpha_1, \alpha_2)}^{\gamma_j} \lambda_1(\mathbf{z}_1)^{-\alpha_1} \lambda_2(\mathbf{z}_2)^{-\alpha_2}$ ;
5     end
6     if  $\kappa = 2$  then
7        $c_{\alpha}^T \leftarrow c_{(\alpha_3, \alpha_1)}^{\gamma_j} \lambda_3(\mathbf{z}_1)^{-\alpha_3} \lambda_1(\mathbf{z}_1)^{-\alpha_1}$ ;
8     end
9     if  $\kappa = 1$  then
10       $c_{\alpha}^T \leftarrow c_{(\alpha_2, \alpha_3)}^{\gamma_j} \lambda_2(\mathbf{z}_1)^{-\alpha_2} \lambda_3(\mathbf{z}_1)^{-\alpha_3}$ ;
11    end
12  end
13 end
14 return  $c_{\alpha}^T$ ;
```

Algorithm 7: GcapT(G)

Input : Control points G of Γ .
Output : Computation of intersection points $\mathbf{z}_1, \mathbf{z}_2$ and index of isolated vertex κ .

```
1 for  $k = 1, 2, 3$  do
2   if (4.10) is solvable for  $k$  then
3      $\mathbf{z}_{temp}^k = \sum_{i=1}^3 \mathbf{v}_i \lambda_i$ ;
4   else
5      $\kappa = k$ ;
6   end
7 end
8  $\kappa_1 = \kappa + 1 \bmod 3$ ;
9  $\kappa_2 = \kappa + 2 \bmod 3$ ;
10  $\mathbf{z}_1 \leftarrow \mathbf{z}_{temp}^{\kappa_1}$ ;
11  $\mathbf{z}_2 \leftarrow \mathbf{z}_{temp}^{\kappa_2}$ ;
12 return  $\mathbf{z}_1, \mathbf{z}_2, \kappa$ ;
```

Algorithm 8: Transform1D(\mathcal{A}_j)

Input : Two dimensional interpolation nodes and data $(\mathbf{x}_i, f_i)_{i=1}^{j+1} = \mathcal{A}_j$.
Output : One dimensional interpolation data $\bar{\mathcal{A}}_j$.

```
1 for  $k = 1, j+1$  do
2    $x_k \leftarrow \|\mathbf{x}_k - \mathbf{z}_1\|_2 / \|\mathbf{z}_2 - \mathbf{z}_1\|_2$ ;
3 end
4  $\bar{\mathcal{A}}_j \leftarrow (x_i, f_i)_{i=1}^{j+1}$ ;
5 return  $\bar{\mathcal{A}}_j$ ;
```

4.2.4. BBProduct - Bernstein form of the product. It remains to define a procedure for computing the BB-form of the product $q_j \Gamma$ in terms of the BB-form of $q_j \in \mathbb{P}^j(T)$ and $\Gamma \in \mathbb{P}^1(T)$. This can be accomplished by means of formula (4.2). Let $\{c_\alpha\}_{\alpha \in \mathcal{I}_2^j}$ and $\{G_\alpha\}_{\alpha \in \mathcal{I}_2^1}$ be the control points of q_j and Γ , respectively. Then the control points of the product are given by

$$(4.11) \quad a_\alpha = \sum_{k=1}^3 c_{\alpha - e_k} G_{e_k} \frac{\alpha_k}{j+1}, \quad \alpha \in \mathcal{I}_2^{j+1}$$

where terms involving negative multi-indices are treated as being zero. This expression forms the basis for Algorithm 9 which computes the desired product for the general d -dimensional case.

Algorithm 9: BBProduct($\{c_\alpha\}_{\alpha \in \mathcal{I}_d^j}, \{G_\alpha\}_{\alpha \in \mathcal{I}_d^1}$)

Input : BB-form of $q \in \mathbb{P}^j(T)$, $\{c_\alpha\}_{\alpha \in \mathcal{I}_d^j}$ and
BB-form of $\Gamma \in \mathbb{P}^1(T)$, $\{G_\alpha\}_{\alpha \in \mathcal{I}_d^1}$.
Output : BB-form of the product $q_j \Gamma$, $\{a_\alpha\}_{\alpha \in \mathcal{I}_d^{j+1}}$.

```

1 for  $\alpha \in \mathcal{I}_d^{j+1}$  do
2   for  $k \leftarrow 1, d+1$  do
3     if  $\alpha_k > 0$  then
4        $a_\alpha \leftarrow a_\alpha + c_{\alpha - e_k} G_{e_k} \frac{\alpha_k}{j+1}$ ;
5     end
6   end
7 end
8 return  $\{a_\alpha\}_{\alpha \in \mathcal{I}_d^{j+1}}$ ;
```

4.3. Numerical example: 2D simplex. In this section we consider the two dimensional Bernstein-Bézier interpolation problem over a simplex with non-trivial interpolation data. We compare performance of Algorithm 4 with command “\” in Matlab.

EXAMPLE 4.1. We consider $n = 10$ and a distribution of points satisfying the Solvability Condition (S). We plot the interpolation points in Figure 3. Similarly to Example 2.1 we consider as a load vector f_1 and f_2 with the alternating sign property. We present and compare our results in Table 6.

f_i	$A \setminus f_i$	S2D-NewtonBernstein
f_1	4.9e-11	4.9e-13
f_2	3.1e-11	3.3e-13

TABLE 6
Example 4.1: Relative errors in L^2 norm.

We observe in Table 6 the superior accuracy of Algorithm 4 with respect to the Matlab solver.

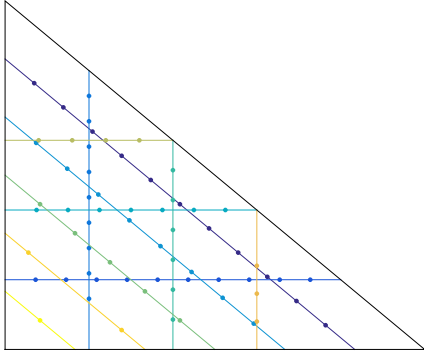


FIG. 3. *Example 4.1. Distribution of interpolation points.*

4.4. Generalisation to d -dimensions. The extension of the procedure described in Algorithm 4 to the general case is not difficult. Firstly, the Solvability Condition (S) is generalised to d -dimensions recursively through requiring the existence of a non-overlapping partition of the interpolation data into sets $\mathcal{A}_n, \mathcal{A}_{n-1}, \dots, \mathcal{A}_0$ of nodes of appropriate dimension on a sub-simplex augmented by the solvability condition on each of the $(d-1)$ -subsimplices for the interpolation problems associated with the sets \mathcal{A}_j , $j = n, n-1, \dots, 0$. Formula (4.7) can likewise be extended to higher dimensions with the functions Γ_j replaced by affine functions vanishing on the appropriate sub-simplex. Thirdly, by utilising Algorithm 4 to solve the $(d-1)$ -dimensional sub-interpolation problems along with an obvious extension of Lemma 4.3, we obtain a subroutine equivalent to **BBExtension** which extends the polynomial on the $(d-1)$ -simplex to the d -simplex. Finally, a subroutine extending **BBAffine** to d -dimension is easily obtained and subroutine **BBProduct** is written in terms of d -dimensions.

REFERENCES

- [1] Mark Ainsworth. Pyramid algorithms for Bernstein-Bézier finite elements of high, nonuniform order in any dimension. *SIAM J. Sci. Comput.*, 36(2):A543–A569, 2014.
- [2] Mark Ainsworth, Gaelle Andriamaro, and Oleg Davydov. Bernstein-Bézier finite elements of arbitrary order and optimal assembly procedures. *SIAM J. Sci. Comput.*, 33(6):3087–3109, 2011.
- [3] Mark Ainsworth, Gaelle Andriamaro, and Oleg Davydov. A Bernstein-Bézier basis for arbitrary order Raviart-Thomas finite elements. *Constr. Approx.*, 41(1):1–22, 2015.
- [4] Douglas N. Arnold, Richard S. Falk, and Ragnar Winther. Geometric decompositions and local bases for spaces of finite element differential forms. *Comput. Methods Appl. Mech. Engrg.*, 198(21-26):1660–1672, 2009.
- [5] J. Baglama, D. Calvetti, and L. Reichel. Fast Leja points. *Electron. Trans. Numer. Anal.*, 7:124–140, 1998. Large scale eigenvalue problems (Argonne, IL, 1997).
- [6] Jean-Paul Berrut and Lloyd N. Trefethen. Barycentric Lagrange interpolation. *SIAM Rev.*, 46(3):501–517 (electronic), 2004.
- [7] Tony F. Chan and David E. Foulser. Effectively well-conditioned linear systems. *SIAM J. Sci. Statist. Comput.*, 9(6):963–969, 1988.
- [8] Charles K. Chui and Hang-Chin Lai. Vandermonde determinant and Lagrange interpolation in \mathbf{R}^s . In *Nonlinear and convex analysis (Santa Barbara, Calif., 1985)*, volume 107 of *Lecture Notes in Pure and Appl. Math.*, pages 23–35. Dekker, New York, 1987.
- [9] Philip J. Davis. *Interpolation and approximation*. Dover Publications, Inc., New York, 1975. Reproduction, with minor corrections, of the 1963 original, with a new preface and bibliography.
- [10] Gerald Farin. *Curves and Surfaces for CAGD: A Practical Guide*. Morgan Kaufmann Pub-

- lishers Inc., San Francisco, CA, USA, 5th edition, 2002.
- [11] Rida T. Farouki. Legendre-Bernstein basis transformations. *J. Comput. Appl. Math.*, 119(1-2):145–160, 2000. Dedicated to Professor Larry L. Schumaker on the occasion of his 60th birthday.
 - [12] Rida T. Farouki. The Bernstein polynomial basis: a centennial retrospective. *Comput. Aided Geom. Design*, 29(6):379–419, 2012.
 - [13] M. Gasca and J. M. Peña. Total positivity and Neville elimination. *Linear Algebra Appl.*, 165:25–44, 1992.
 - [14] Nicholas J. Higham. *Accuracy and stability of numerical algorithms*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 2002.
 - [15] Robert C. Kirby. Fast simplicial finite element algorithms using Bernstein polynomials. *Numer. Math.*, 117(4):631–652, 2011.
 - [16] Robert C. Kirby and Kieu Tri Thinh. Fast simplicial quadrature-based finite element operators using Bernstein polynomials. *Numer. Math.*, 121(2):261–279, 2012.
 - [17] Ming-Jun Lai and Larry L. Schumaker. *Spline functions on triangulations*, volume 110 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2007.
 - [18] Ana Marco and José-Javier Martínez. A fast and accurate algorithm for solving Bernstein-Vandermonde linear systems. *Linear Algebra Appl.*, 422(2-3):616–628, 2007.
 - [19] Peter J. Olver. On multivariate interpolation. *Stud. Appl. Math.*, 116(2):201–240, 2006.
 - [20] Lothar Reichel. Newton interpolation at Leja points. *BIT*, 30(2):332–346, 1990.