

SYMMETRY REDUCTION, CONTACT GEOMETRY, AND PARTIAL FEEDBACK LINEARIZATION*

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Abstract. Let Pfaffian system ω define an intrinsically nonlinear control system which is invariant under a Lie group of symmetries G . Using the contact geometry of Brunovsky normal forms and symmetry reduction, this paper solves the problem of constructing subsystems $\alpha \subset \omega$ such that α defines a static feedback linearizable control system. A method for representing the trajectories of ω from those of α using reduction by a distinguished class G of Lie symmetries is described. A control system will often have a number of inequivalent linearizable subsystems depending upon the subgroup structure of G . This can be used to obtain a variety of representations of the system trajectories. In particular, if G is solvable, the construction of trajectories can be reduced to quadrature. It is shown that the identification of linearizable subsystems in any given problem can be carried out algorithmically once the explicit Lie algebra of G is known. All the constructions have been automated using the Maple package *DifferentialGeometry*. A number of illustrative examples are given.

Key words. feedback linearizable subsystem, Pfaffian system, control symmetries, reconstruction

AMS subject classifications. 93C10, 58A30, 58A17, 53A55

DOI. 10.1137/15M1046538

1. Introduction. Given a control system

$$(1) \quad \dot{x} = f(t, x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^q,$$

it is significant when there exists a static feedback linearization or, more generally, when the system is flat. For those nonlinear systems which are not feedback linearizable or flat there has been a question in the literature (for instance, [23], [25], [26]) as to the existence of *subsystems* of (1) which are feedback linearizable and then to determine how such structures may be useful in describing the trajectories of (1). In this paper a complete solution to this problem is given for control systems with symmetry.

To introduce the ideas, consider the control system (see [1])

$$(2) \quad \dot{x} = u_1 \cos \theta - v \sin \theta, \quad \dot{y} = u_1 \sin \theta + v \cos \theta, \quad \dot{\theta} = u_2, \quad \dot{v} = -\gamma u_1 u_2 - \beta v,$$

representing a simplified model for the guidance of a ship, such as a tanker, at location (x, y) with orientation θ . Parameter $\gamma \neq 0$ is related to the shape of the vessel while β quantifies hydrodynamic drag. Control u_1 represents the surge while u_2 controls the ship's angular velocity about the point (x, y) and v denotes sway. For all $\gamma \neq 0$, system (2) can be shown to be nonlinearizable by *any* local change of variables. A question of interest for this system is *trajectory planning* [1]. That is, we wish to prescribe an arbitrary surface path $\Gamma(t) = (x(t), y(t))$ and then determine the controls

*Received by the editors November 2, 2015; accepted for publication (in revised form) September 5, 2017; published electronically January 18, 2018.

<http://www.siam.org/journals/sicon/56-1/M104653.html>

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$t \mapsto (u_1(t), u_2(t))$ that “realize” Γ . If the system were flat¹ with *flat outputs* x and y , this could be very easily resolved, for in that case, the full system trajectory could be obtained by differentiation of the path Γ or some concomitants of it. However, in this case linearization and flatness are not available. Nevertheless, we are able to make a simple observation: expressing u_1 and v in terms of \dot{x}, \dot{y} , and θ from the first two equations, substituting these into the fourth equation and then, using the third equation, leads to

$$(3) \quad \frac{d\theta}{dt} = \frac{(\ddot{y} + \beta\dot{y}) \cos \theta - (\ddot{x} + \beta\dot{x}) \sin \theta}{(1 - \gamma)(\dot{x} \cos \theta + \dot{y} \sin \theta)},$$

provided $\gamma \neq 1$. While for general choices of path $\Gamma(t)$ this equation has no solution in terms of *known* functions, we have nevertheless made progress. Prescribing $\Gamma(t)$ gives a differential equation for θ alone whose (generally numerical) solution determines the required controls $u_1(t), u_2(t)$ as well as the rest of the system dynamics. Notice too that if we are content to approximate a given $\Gamma(t)$ by a sequence of interpolating straight line segments of the form $t \mapsto (\xi_0 + t\xi, \eta_0 + t\eta)$, where ξ_0, ξ, η_0, η are constants, then for each such segment, (3) becomes

$$(4) \quad \frac{d\theta}{dt} = \frac{\beta}{1 - \gamma} \left(\frac{\eta - \xi \tan \theta}{\xi + \eta \tan \theta} \right).$$

Equation (4) has easily obtained elementary function solutions which can be used to plan the trajectory to arbitrary accuracy by varying the number of straight line interpolating segments. The main point we wish to make in relation to this example is that while (2) is not flat (at least for $\gamma \neq 1$), we can nevertheless generate the full system dynamics from the solution of a single ODE for one of the states, namely, θ . Momentarily, we would like to view this circumstance as a type of generalization of flatness, which we may temporarily call *quasi-flatness*: $\Gamma(t)$ and θ are “quasi-flat outputs” in that they fully determine the system trajectories but, unlike flat outputs, a nontrivial ODE must be solved for one of the states (θ).

To bring this point into sharper focus, let us now consider the case $\gamma = 1$. Here the calculations described above lead to an *algebraic* equation for θ (as opposed to a *differential* equation), namely,

$$(5) \quad \tan \theta = \frac{\ddot{y} + \beta\dot{y}}{\ddot{x} + \beta\dot{x}}.$$

In other words, the formula (5) proves that (2) is flat in case $\gamma = 1$ in that there is a simple “universal” relationship between Γ and θ , while for $\gamma \neq 1$, the relationship continues to exist and continues to fully determine the system trajectories but is mediated by the solution of a nonlinear first order ODE.²

Actually, for $\gamma \neq 1$, it is not known whether or not (2) is flat though its flatness seems unlikely. Hence, the reduction of the trajectory planning problem to the differential equation (3) is a useful simplification—both theoretically and practically—in the face of the system’s purported lack of flatness.

An important point to be made here is that (2) is simple enough, having a kind of “triangular” structure, so that these results can be spotted without recourse to

¹For information on flat control systems, see [21].

²The existence of formula (5) can be explained by the fact that a fourfold partial prolongation of system (2) is static feedback linearizable when $\gamma = 1$.

any deep theory. The purpose of this paper is to systematically explore the above informally described notion of quasi-flatness using differential geometric tools, particularly Lie symmetry. The main motivation for the study of quasi-flatness is that while flatness is an interesting and useful property of a nonlinear control system, it is hampered by being rare. Quasi-flatness is a weaker requirement than flatness and therefore likely to be more prevalent among control systems of interest. We will argue in this paper that control systems that are invariant under certain types of Lie group actions are candidates for quasi-flatness.

We do not want to place too much emphasis on the term “quasi-flatness” and feel that it is premature to provide a definition in this paper. Instead our approach will be to identify a distinguished class of Lie group actions, appearing as symmetries of nonlinear control systems, that we believe capture features of the phenomena that we have attempted to informally describe in the foregoing discussion. Indeed, we will show that intrinsically nonlinear control systems that have static feedback linearizable subsystems have “the quasi-flat property” and this paper provides a solution to the problem of identifying and constructing such subsystems and showing how they may be used to describe the trajectories of the original control system.

We now describe our approach and specific results. This paper studies control systems (1) that are invariant under Lie groups of transformations, that is, control systems with Lie symmetries. A given abstract control system is unlikely to possess any nontrivial symmetry. By contrast, among control systems of interest in real applications, a great many do possess such symmetries due to the fact that they often arise from physical or geometrical considerations which come with inherent symmetries such as, for instance, Galilean, Euclidean, or other invariances. Additionally, left-invariant control systems on Lie groups are common in both theory and practice. Therefore it is reasonable to ask, what can we discover about trajectory generation and the structure of a control system from its Lie symmetries and Lie symmetry reductions? These questions have been considered in numerous works over the last few decades beginning with pioneering paper [17]. We refer to [29] and its references where an overview of these developments is given. We also refer to [37], a companion to the present paper.

To describe our own approach in slightly more detail, let (M, ω) be a control system, where M is the manifold of states, controls, and time while ω is the control system. As intimated in the previous paragraph, our approach to addressing the question of quasi-flatness first involves the study of control systems that admit a certain kind of Lie transformation group G acting on M that leaves ω invariant. We call this Lie transformation group the *control admissible symmetries*. These, in particular, generalize the *state-space* symmetries of [17] and [12]. Second, we shall require that the quotient (symmetry reduction) $\bar{\omega}$ of the control system ω on M be a *static feedback linearizable* control system on the *quotient* M/G by the G -action. Subsequently, we make use of the reconstruction theorem of Anderson and Fels [2] which permits us to construct, for each solution of the feedback linearizable system $\bar{\omega}$ on M/G , a solution of ω . Due to its feedback linearizability, the trajectories of $\bar{\omega}$ are explicitly constructible and this very often permits an effective simplification in representing all the trajectories of the original system ω .

We pay particular attention in this paper to showing how to efficiently identify quotient control systems ω/G on M/G which are static feedback linearizable and whose solution space has the same cardinality as that of ω . This solution space can then be lifted to M so as to describe all the trajectories of ω . In this way, one is able to factor the problem of representing the trajectories of ω into constructing the

linearization of $\bar{\omega} = \omega/G$ and then solving a system of *Lie type* [6], [11], [8] “over” each integral manifold of $\bar{\omega}$. For a given control system with symmetry one usually has some choice about the quotient $\bar{\omega}$ because the Lie group of control symmetries will have Lie subgroups and each subgroup will lead to a choice of linearizable quotient. This gives us a degree of control over the extent to which we must “solve differential equations” in order to describe the trajectories of ω . In general we regard the integration of $\bar{\omega}$ to be trivial, like the prescription of $\Gamma(t)$ in our opening example. The result of the integration of $\bar{\omega}$ then feeds into the system of Lie type as a set of parameters which is analogous to the integration of the θ -equation (3). Picturesquely, one can say that the quotients $\bar{\omega}$ define subsystems $\alpha \subset \omega$, whose properties can be designed via control admissible symmetries.

The abstract notion of decomposing a nonlinear control system S into subsystems S_1, S_2, \dots, S_k in which the set of trajectories of S is the “cascade” of those of the S_i appears to go back to works of Krener [22], Respondek [33], and Nijmeijer [28], among others. In [17], Grizzle and Marcus, formulated an authoritative differential geometric structure theory for nonlinear control systems that built on and improved the general notion of trajectory decomposition of earlier authors. We also here refer to the textbook [12]. The present paper develops the theory of these pioneering works in a number of ways, particularly that of [17]. This development is continued in [37].

The amount of integration required to elucidate all the trajectories of ω is controlled by the size and structure of G . It will be seen that the number of components of the Lie system is equal to the dimension of G and hence one ordinarily requires $\dim G$ to be small. On the other hand, one requires $\dim G$ to be large enough so that ω/G is static feedback linearizable. Hence there is a type of inverse relation between the amount of quadrature and the requirement that the quotient be static feedback linearizable. However, a further consideration is that even if $\dim G$ is small, the parametrization of the trajectories of ω/G may thereby be complicated making the parameters in the Lie system more complex. These are practical considerations that vary from one problem to the next depending upon the precise control theoretic question at hand.

We now give an outline of the contents of this paper. Section 2 is mainly concerned with a brief review of Lie transformation group, Lie symmetry, exterior differential systems, and an account of aspects of the Anderson–Fels reconstruction theorem, as described in [2], that we require.

One of the chief goals of this paper is that of identifying and integrating static feedback linearizable subsystems of intrinsically nonlinear control systems as algorithmically and efficiently as possible in order that any nontrivial integration be confined to representing the solutions of the system of Lie type arising from the reconstruction theorem. To this end we begin in section 3 to reformulate the theory of static feedback linearization in terms of contact geometry. Apart from achieving the above stated goals, the reformulation allows us to unify and elucidate numerous results on the theme of linearization of control systems and more generally Pfaffian systems. For instance in Theorem 3.8 we give a simple, geometric characterization of static feedback linearization of a control system regardless of its local form.

Section 4 solves the aforementioned problem on linearizable subsystems in terms of the theory given in section 3 and the notion of control admissible symmetries (Definition 4.8). These generalize the classical state-space symmetries of [17] and [12] and we will see that the generalization is essential to the capture of the important invariance properties of control systems. These symmetries, special cases of which also implicitly appear in [29], may be described as the maximal class of static feedback

self-equivalences. The main practical outcome of section 4 is that the existence and structure of feedback linearizable subsystems can be determined literally in a matter of minutes once one is in possession of the Lie algebra of infinitesimal control admissible symmetries.

Section 5 is devoted to giving illustrative examples of the theory developed earlier in the paper. We begin with a simple example of a provably nonflat control system with two inputs and prove that it has a flat subsystem using a subalgebra of its Lie algebra of control admissible symmetries. We demonstrate what this implies for the representation of the system trajectories. Our next example, the Heisenberg control system discussed in [4, p. 30], illustrates how trajectory generation in a control system which has a complicated local form can be simplified using symmetry reduction and reconstruction as developed in sections 2–4. Our final example discusses a more sophisticated model of ship guidance than the one we described above and explains the relationship between linearizable subsystems and quasi-flatness.

In this paper we use the language and methods of differential geometry to carry out our analyses. We assume the reader is familiar with the basic elements of smooth manifolds, Lie groups, tangent vectors, and differential forms although they will be briefly recalled in section 2. For those who require it, a rapid, excellent introduction to most of the relevant notions is given in Chapter 1 of [30]. Many parts of [31], [10], [13] are also helpful in relation to providing background on geometric methods.

2. Exterior differential systems with symmetry. In this section we briefly recall the geometric notions used in this paper as well as the results that we shall require from Anderson and Fels [2]. Let M be a smooth manifold and G a Lie group. A (left) *Lie group action* on M is a map $\mu : G \times M \rightarrow M$ such that for all $g, h \in G$ and $x \in M$,

1. $\mu(e, x) = x$,
2. $\mu(g, \mu(h, x)) = \mu(gh, x)$,

where e is the identity element of G . We sometimes write $g \cdot x$ for $\mu(g, x)$. If we want to draw attention to the transformation μ for a fixed $g \in G$ then we sometimes write $\mu_g(x)$ for $\mu(g, x)$. That is, μ_g is a diffeomorphism from M to itself. A geometric object such as a differential form ω on M is said to be *invariant* under the G -action μ if $\mu_g^* \omega = \lambda \omega$ for all $g \in G$ and some real-valued function λ on $M \times G$, where μ_g^* denotes the pullback by μ_g . In this case μ_g or, briefly, g is said to be a *symmetry* of ω . We sometimes say that ω is G -invariant if ω is invariant for all $g \in G$.

We shall use some elements of the theory of exterior differential systems; see [31], [19]. An *exterior differential system* (EDS) on manifold M is a graded differential ideal in the ring of all differential forms on M , denoted $\Omega^*(M)$. Let $\Omega^k(M)$ denote the set of all differential k -forms on M . The EDS \mathcal{I} consists of a direct sum of subsets $\mathcal{I}^k \subset \Omega^k(M)$,

$$\mathcal{I} = \mathcal{I}^1 \oplus \mathcal{I}^2 \oplus \cdots \oplus \mathcal{I}^n,$$

where $n = \dim M$. Not all the subsets \mathcal{I}^j need be nonempty. The qualification “differential” in “differential ideal” implies that if $\theta_j \in \mathcal{I}^j$ then $d\theta_j \in \mathcal{I}^{j+1}$. Given an EDS \mathcal{I} , a local basis $\mathcal{B}(U)$ on an open set $U \subset M$ is a subset $\mathcal{B}(U) \subset \mathcal{I}(U)$ given by

$$\mathcal{B}(U) = \left(\theta_1^{j_1}, \theta_2^{j_2}, \dots, \theta_n^{j_n} \right),$$

where $\theta_k^{j_k} \in \mathcal{I}^{j_k}(U)$.

In this paper we are only concerned with Pfaffian systems. Given a subbundle of $I \subset T^*U$ with U an open subset of manifold M , then there is a differential ideal

\mathcal{I} generated by the sections of I given in a local basis by 1-forms $\omega^1, \dots, \omega^r$ defined over U . That is,

$$\mathcal{I} = \{\alpha^j \wedge \omega^j, \beta^j \wedge d\omega^j \mid \alpha^j, \beta^j \in \Omega^*(U)\},$$

where $\Omega^*(U)$ is the set of all differential forms on U . We will generally not be interested in this object because we are focused on the integral submanifolds of the 1-form equations

$$(6) \quad \omega^1 = 0, \omega^2 = 0, \dots, \omega^r = 0,$$

which is called a *Pfaffian system* [5]. For a Pfaffian system the important associated object as far as its integral manifolds are concerned are its *structure equations* which can be roughly described as $dI \bmod \mathcal{I}^1$. We will illustrate this in section 2.1.

Because we shall only consider Pfaffian systems in this paper, we need only be concerned with I and dI modulo \mathcal{I}^1 that generate the first two components of the differential ideal \mathcal{I} , that is, the degree 1 and degree 2 components $\mathcal{I}^1 \oplus \mathcal{I}^2$. We will not usually explicitly mention \mathcal{I}^2 and refer to the Pfaffian system by a convenient generating set $\{\omega^j\}_{j=1}^r$ of its degree 1 component. We will often denote a Pfaffian system by the symbol ω . For background on differential forms and their applications, see [13], [10]. For background on exterior differential systems, see [31], [19].

DEFINITION 2.1. Let $\mu : G \times M \rightarrow M$ be an action of a Lie group G on a smooth manifold M . Let ω be a Pfaffian system on M .

1. Pfaffian system ω is said to be G -invariant if $\mu_g^* \omega \subseteq \omega$ for all $g \in G$.
2. If \mathcal{V} is a smooth vector field distribution on M then we say that it is G -invariant or that G constitutes the symmetries of \mathcal{V} if $\mu_{g*} \mathcal{V} \subseteq \mathcal{V}$ for all $g \in G$, where μ_{g*} denotes the pushforward or induced tangent map of μ_g .
3. The G -action is said to be free if whenever $x \in M$ satisfies $\mu_g(x) = x$ then $g = e := id_G$.

In practice the Lie symmetry group of a geometric structure on a manifold M (such as \mathcal{V} or ω) is constructed *infinitesimally* by seeking vector fields X on M such that $\mathcal{L}_X \mathcal{V} \subseteq \mathcal{V}$, $\mathcal{L}_X \omega \subseteq \omega$, where \mathcal{L}_X denotes the *Lie derivative* with respect to X . Composing the flows of such vector fields then gives a local Lie transformation group. The vector field X is then called an *infinitesimal symmetry*.

DEFINITION 2.2. A solution or integral submanifold of ω on M is a submanifold $S \subset M$ each of whose tangent spaces is annihilated by ω . In this paper submanifolds will be images of immersions $s : U \subset \mathbb{R}^p \rightarrow M$ and the requirement that $\text{Image}(U) = S \subset M$ be an integral submanifold is that $s^* \theta = 0$ for all $\theta \in \omega$; U is an open subset. The domain of s being p -dimensional implies that the dimension of the image of s is (at most) p .

It follows from these definitions that if s is an integral manifold of ω then so is $\mu_g \circ s$. All actions of Lie group G on smooth manifold M will be assumed to be *regular*, ([30, p. 23, pp. 213–218]), such that the quotient of M by the action of G is a smooth manifold denoted M/G together with a smooth surjection $\mathbf{q} : M \rightarrow M/G$ which assigns each point of M to its G -orbit. We will always assume that M/G has the Hausdorff separation property possibly by restricting it to appropriate open sets. From now on G will always denote a Lie group acting *smoothly*, *freely*, and *regularly* on smooth manifold M .

Note that throughout this paper we use braces to denote the linear span of the enclosed geometric objects. Thus if X_1, X_2, \dots, X_r are vector fields on a manifold M

then $\{X_1, X_2, \dots, X_r\}$ means the linear span of the X_i over the smooth functions on M and denotes the distribution whose sections are given by such linear combinations of the X_i . If the X_i form a basis for a (real) Lie algebra then the *same* notation means the linear span over the *real numbers* of the enclosed vector fields. It should be clear from the context when the notation refers to a distribution and when it refers to a Lie algebra. As in the case of vector field distributions, if $\omega^1, \omega^2, \dots, \omega^r$ are differential p -forms then $\{\omega^1, \omega^2, \dots, \omega^r\}$ denotes the linear span of the enclosed ω^j over the smooth functions on M .

To motivate the next definition, suppose that $\mathbf{q} : M \rightarrow M/G$ is the quotient map for the action of Lie group G on M . It induces the pullback $\mathbf{q}^* : \Omega^*(M/G) \rightarrow \Omega^*(M)$ from forms on M/G to those on M . Hence if ω is a Pfaffian system on M , we can ask about the existence of a Pfaffian system $\bar{\omega}$ on M/G such that $\mathbf{q}^*\bar{\omega} \subseteq \omega$. We shall see that such a Pfaffian system on the quotient of M by G holds significance for applications to control theory.

DEFINITION 2.3. *Let $\mathbf{q} : M \rightarrow M/G$ be the quotient map for the smooth, regular Lie group action on M by G . Let ω be a Pfaffian system on M that is invariant under the action of G . The quotient of ω by the given G -action is the maximal Pfaffian system $\bar{\omega}$ on M/G such that for all $\bar{\theta} \in \bar{\omega}$, $\mathbf{q}^*\bar{\theta} \in \omega$.*

It is instructive to explore this definition further. The following definition is central.

DEFINITION 2.4 (see [2]). *Let M be a smooth manifold with a smooth, regular, and free action of Lie group G . For definiteness, we assume that G acts on the left. Let $\mathbf{q} : M \rightarrow M/G$ be the quotient map.*

1. *A map $\sigma : \bar{U} \subseteq M/G \rightarrow U \subset M$ is said to be a local section (to the action of G) if $\mathbf{q} \circ \sigma : \bar{U} \rightarrow \bar{U}$ is the identity map on \bar{U} , where \bar{U} and U denote open subsets.*
2. *Let $\mathcal{V} \subset TM$ be a distribution and Γ the Lie algebra of infinitesimal generators of the action of G . We say that G is transverse to \mathcal{V} if $\Gamma \cap \mathcal{V} = \{0\}$ and strongly transverse to \mathcal{V} if $\Gamma \cap \mathcal{V}^{(1)} = \{0\}$, where $\mathcal{V}^{(1)} = \mathcal{V} + [\mathcal{V}, \mathcal{V}]$, the first derived bundle of \mathcal{V} . If $\mathcal{V} = \ker \omega$, we say that G is transverse/strongly transverse to ω .*
3. *Let \mathcal{I} be an exterior differential system on M . The semibasic k -forms $\mathcal{I}_{\text{sb}}^k$ satisfy $\theta(\Gamma) = 0$ for all $\theta \in \mathcal{I}^k$.*

Given the hypotheses imposed on the G -manifold M , then $\mathbf{q} : M \rightarrow M/G$ is a left-principal G -bundle. A section of $\mathbf{q} : M/G \rightarrow M$ exists if and only if it is trivial. However, every principal bundle is locally trivial and local sections always exist. In more detail, if $\bar{U} \subset \bar{M} := M/G$ is open then there is a local trivialization $\Phi : \mathbf{q}^{-1}(\bar{U}) \rightarrow \bar{U} \times G$ and there are local cross sections $\sigma : \bar{U} \rightarrow \mathbf{q}^{-1}(\bar{U})$.

As proven in [2], many properties of the quotient $\bar{\omega}$ can be deduced from the semibasic forms alone. In particular, if σ is a local section for the action of G then the quotient $\bar{\omega} := \omega/G$ is equal to $\sigma^*\omega_{\text{sb}}$, where ω_{sb} denotes the semibasic 1-forms. In this paper we are not concerned with exterior differential systems in general but only with Pfaffian systems. The quotient of Pfaffian systems need not be a Pfaffian system. We will assume throughout that the quotient of a Pfaffian system is a Pfaffian system. In [2, Theorem 5.1], conditions are given such that this is the case, namely, that G is strongly transverse to \mathcal{V} .

Throughout this paper we are dealing exclusively with the situation where G acts on M so that the quotient of M by G , denoted M/G , is a smooth manifold. Hence

the differential of the quotient map

$$d\mathbf{q} : T_x M \rightarrow T_{\mathbf{q}(x)}(M/G)$$

is *onto* with $d\mathbf{q} = \mathbf{\Gamma}$ —the Lie algebra of infinitesimal generators of the action of G ; see [30, pp. 213–216]. So

$$d\mathbf{q} : T_x M / \mathbf{\Gamma}_x \rightarrow T_{\mathbf{q}(x)}(M/G)$$

is an isomorphism. Hence

$$d\mathbf{q} : \mathcal{V} / \mathbf{\Gamma} \rightarrow d\mathbf{q}(\mathcal{V})$$

is an isomorphism of vector bundles.

LEMMA 2.5. *Let ω be a Pfaffian system on M invariant under the smooth, regular, and free action of a Lie group G and $\mathbf{q} : M \rightarrow M/G$ the quotient map. Let distribution $\mathcal{V} = \ker \omega$. Then $d\mathbf{q}(\mathcal{V}) = \ker \bar{\omega}$.*

Proof. Suppose $\bar{X}_{\bar{x}} \in d\mathbf{q}(\mathcal{V}_x)$, where $\bar{x} = \mathbf{q}(x)$. Then there exist $X_x \in \mathcal{V}_x$ with $\bar{X}_{\bar{x}} = d\mathbf{q}(X_x)$ and we have for all $\bar{\theta}_{\bar{x}} \in \bar{\omega}_{\bar{x}}$, $\bar{\theta}_{\bar{x}}(\bar{X}_{\bar{x}}) = \bar{\theta}_{\bar{x}}(d\mathbf{q}(X_x)) = (\mathbf{q}^* \bar{\theta})_x(X_x) = \theta_x(X_x) = 0$ for some $\theta \in \omega$ by Definition 2.3; so $d\mathbf{q}(\mathcal{V}_x) \subseteq \ker \bar{\omega}_{\bar{x}}$ and this holds for each $x \in M$. Hence $d\mathbf{q}(\mathcal{V}) \subseteq \ker \bar{\omega}$.

Suppose $\bar{Y} \in \ker \bar{\omega}$. Then $\bar{\theta}(\bar{Y}) = 0$ for all $\bar{\theta} \in \bar{\omega}$. With local section σ we have $\bar{\theta} = \sigma^* \theta_{\text{sb}}$, some $\theta_{\text{sb}} \in \omega_{\text{sb}}$ and, hence, $0 = (\sigma^* \theta_{\text{sb}})(\bar{Y}) = \theta_{\text{sb}}(\sigma_* \bar{Y})$. Now it is possible to show (see proof of Theorem 4.5) that $\ker \omega_{\text{sb}} = \mathcal{V} \oplus \mathbf{\Gamma}$ and, hence, $\sigma_* \bar{Y} \in \mathcal{V} \oplus \mathbf{\Gamma}$ whence $\bar{Y} = d\mathbf{q}(\sigma_* \bar{Y}) \in d\mathbf{q}(\mathcal{V} \oplus \mathbf{\Gamma}) = d\mathbf{q}(\mathcal{V})$. Hence $\ker \bar{\omega} \subseteq d\mathbf{q}(\mathcal{V})$. \square

2.1. An example. We illustrate all of the notions introduced so far via a simple example. The Pfaffian system ω is spanned by the 1-forms

$$\omega^1 = dx_1 - x_2 dt, \quad \omega^2 = dx_2 - x_4 dt, \quad \omega^3 = dx_3 - x_4^2 dt$$

on \mathbb{R}^5 with coordinates t, x_1, x_2, x_3, x_4 . This is a generating set for the degree 1 component \mathcal{I}^1 . The degree 2 component consists of the exterior derivative of ω modulo \mathcal{I}^1 . We have $d\omega^1 = dt \wedge dx_2$, $d\omega^2 = dt \wedge dx_4$, $d\omega^3 = 2x_4 dt \wedge dx_4$, and, hence,

$$d\omega \mod \mathcal{I}^1 = \{dt \wedge dx_4\}.$$

The vector field $X = (t^2/2)\partial_{x_1} + t\partial_{x_2} + 2x_2\partial_{x_3} + \partial_{x_4}$ is an infinitesimal symmetry of ω in that

$$\mathcal{L}_X \omega^1 = 0, \quad \mathcal{L}_X \omega^2 = 0, \quad \mathcal{L}_X \omega^3 = 2\omega^2,$$

where \mathcal{L}_X denotes the Lie derivative with respect to X . The Lie transformation group acting on \mathbb{R}^5 that is, *generated* by X is the \mathbb{R} -action

$$\mu_\varepsilon(t, x_1, x_2, x_3, x_4) = \left(t, x_1 + \frac{1}{2}\varepsilon t^2, x_2 + t\varepsilon, x_3 + 2\varepsilon x_2 + t^2\varepsilon, x_4 + \varepsilon \right)$$

and we have $\mu_\varepsilon^* \omega^1 = \omega^1$, $\mu_\varepsilon^* \omega^2 = \omega^2$, $\mu_\varepsilon^* \omega^3 = 2\varepsilon \omega^2 + \omega^3$. The local first integrals of X are spanned by the functions

$$\text{Inv} = \{tx_2 - 2x_1, \quad 4x_1^2 - 4tx_1x_2 + t^3x_3, \quad t^2x_4 - 2x_1\}$$

in the the sense that if ψ is a real-valued smooth function such that $X\psi = 0$ then ψ is a smooth function of the elements of Inv . It follows that the functions Inv can

be taken to be local coordinates on the quotient of \mathbb{R}^5 by the \mathbb{R} -action μ . That is, $\mathbf{q} : \mathbb{R}^5 \rightarrow \mathbb{R}^5/\mathbb{R}$ is defined locally by

$$\mathbf{q}(t, x_1, x_2, x_3, x_4) = (t, \quad 2x_1 - tx_2, \quad 4x_1^2 - 4tx_1x_2 + t^3x_3, \quad t^2x_4 - 2x_1) = (t, \quad y_1, \quad y_2, \quad y_3).$$

It can be checked that

$$\sigma(t, y_1, y_2, y_3) = \left(t = t, \quad x_1 = \frac{1}{2}y_1, \quad x_2 = 0, \quad x_3 = \frac{y_2 - y_1^2}{t^3}, \quad x_4 = \frac{y_1 + y_3}{t^2} \right)$$

is a (local) section of the action: $(\mathbf{q} \circ \sigma)(t, y_1, y_2, y_3) = (t, y_1, y_2, y_3)$. It is often convenient to work with the dual vector field distribution $\mathcal{V} = \ker \omega$. We have

$$\mathcal{V} = \{ \partial_t + x_2 \partial_{x_1} + x_4 \partial_{x_2} + x_4^2 \partial_{x_3}, \quad \partial_{x_4} \}.$$

It is easily verified that \mathcal{V} is invariant under the \mathbb{R} -action: $(\mu_\varepsilon)_* \mathcal{V} = \mathcal{V}$ which is reflected in the infinitesimal condition $[X, \mathcal{V}] \subset \mathcal{V}$. If the quotient of a Pfaffian system is also a Pfaffian system (which we assume throughout this paper) then we can compute the quotient of \mathcal{V} by the \mathbb{R} -action as follows. Construct a local trivialization $\varphi : \mathbf{q}^{-1}(\bar{U}) \rightarrow G \times \bar{U}$, where $\bar{U} \subset \mathbb{R}^5/\mathbb{R}$ is an open subset:

$$\begin{aligned} \varphi(t, x_1, x_2, x_3, x_4) &= (x_2/t, \quad t, \quad 2x_1 - tx_2, \quad 4x_1^2 - 4tx_1x_2 + x_3, \quad t^2x_4 - 2x_1) \\ &= (z, \quad t, \quad y_1, \quad y_2, \quad y_3) \in G \times \bar{U}. \end{aligned}$$

Then $\varphi_* X = \partial_z$ and

$$d\mathbf{q}\mathcal{V} = \varphi_* \mathcal{V} \mod \varphi_* X = \left\{ \partial_t - \frac{1}{t}(y_1 + y_3)\partial_{y_1} - \frac{1}{t}(4y_1^2 - y_3^2 - 3y_2)\partial_{y_2}, \quad \partial_{y_3} \right\} = \bar{\mathcal{V}}.$$

The quotient Pfaffian system is $\bar{\omega} = \text{ann } \bar{\mathcal{V}}$, where

$$\bar{\omega} = \left\{ dy_2 + \frac{1}{t}(4y_1^2 - y_3^2 - 3y_2)dt, \quad dy_1 + \frac{1}{t}(y_1 + y_3)dt \right\}.$$

As proven in [2], this can also be constructed by pulling back the semibasic forms by the local section σ . In this case we calculate that the semibasic 1-forms are spanned by

$$\omega_{\text{sb}}^1 = \omega^1 - \frac{t}{2}\omega^2, \quad \omega_{\text{sb}}^2 = \omega^3 - \frac{2x_2}{t}\omega^2.$$

That is, letting ι_X denote the interior product by X , if θ is a 1-form on \mathbb{R}^5 and $\iota_X \theta = 0$ then there are functions a_1, a_2 on \mathbb{R}^5 such that $\theta = a_1 \omega_{\text{sb}}^1 + a_2 \omega_{\text{sb}}^2$. It can be verified that $\{\sigma^* \omega_{\text{sb}}^1, \sigma^* \omega_{\text{sb}}^2\} = \bar{\omega}$. We note that there are no nontrivial semibasic 2-forms as $\iota_X dt \wedge dx_4 \neq 0$. Because it can be shown that G is strongly transverse to \mathcal{V} , Theorem 5.1 of [2] guarantees that this quotient $\bar{\omega}$ of ω is a Pfaffian system, as verified. That is, the quotient of ω by this \mathbb{R} -action is determined by its degree 1 component. An instance of an integral submanifold $s : \mathbb{R} \rightarrow \mathbb{R}^5$ is $s(t) = (t, t + \sin t, \cos t, (t - \sin t \cos t)/2, -\sin t)$, meaning $s^* \omega = 0$.

We will often refer to the quotient by Lie group G of distribution \mathcal{V} on manifold M by \mathcal{V}/G or by $\bar{\mathcal{V}}$, a distribution on quotient manifold M/G . The quotient $\bar{\omega}$ of ω by the action of G will sometimes be denoted ω/G . Frequently, we say that the distribution $\{\partial_t + \mathbf{f}(t, \mathbf{x}, \mathbf{u})\partial_{\mathbf{x}}, \partial_{\mathbf{u}}\}$ whose integral submanifolds are the trajectories of (1) is a control system.

Remark 2.6. While symmetry reduction can produce a control system on a lower dimensional manifold, this doesn't imply that the quotient system is more tractable than the original system; in fact, the opposite is very often the case. Therefore for effective use of symmetry reduction in control we focus on the situation in which while ω is *not* static feedback linearizable, we seek a quotient $\bar{\omega}$ that is static feedback linearizable. We then go on to investigate the consequences of this for trajectory generation.

2.2. Reduction and reconstruction. It is shown in [2] that under commonly encountered circumstances the integral manifolds of an EDS \mathcal{I} can be constructed from those of its symmetry reduction \mathcal{I}/G by solving an ODE of Lie type. Such ODE systems form a particularly nice class since they themselves have a reduction theory which aids their solution [6], [11], [8]. We present this in the context of control systems.

THEOREM 2.7 (see [2]). *Let ω be a Pfaffian system on manifold M that is invariant under the smooth, free, regular action of a Lie group G and such that the G -action is transverse to ω . Let $\bar{\omega}$ be the quotient of ω on M/G by the G -action and let $\bar{s} : \mathcal{U} \subseteq \mathbb{R}^k \rightarrow \bar{U} \subset M/G$ be an integral manifold of $\bar{\omega}$, where \bar{U}, \mathcal{U} are open sets. Then*

1. *there exists a local section $\sigma : \bar{U} \subset M/G \rightarrow U \subset M$ to the G -action, where U is an open set;*
2. *there exists a map $g : \mathcal{U} \rightarrow G$ such that $s(x) = \mu(g(x), (\sigma \circ \bar{s})(x))$ is an integral submanifold of ω for all $x \in \mathcal{U}$, where $\mu : G \times M \rightarrow M$ is the G -action;*
3. *the function g is constructed by solving a Frobenius integrable system of Lie type.*

Theorem 2.7 is a specialization of the corresponding theorem in [2], adapted to our situation. Another important specialization we make is that in this paper we are dealing with the symmetry reduction of control systems (1) in which certain manifold coordinates have special inviolable status. In particular we will always require our group actions to *preserve the independent variable*, usually *time t* that appears in (1); see section 4.

3. Contact geometry and static feedback linearization. This section and the next are devoted to the task of characterizing static feedback linearizable quotients (or subsystems) of control systems (1) based on the contact geometry of Brunovsky normal forms and symmetry reduction. The main result of section 4 is a very simple method for determining when a given G -invariant control system has a static feedback G -symmetry reduction based only on knowledge of the Lie algebra of the infinitesimal generators of the G -action. In this we will be required to reformulate the theory of static feedback linearization in terms of contact geometry and give new easily computable criteria for the static feedback linearization of arbitrary smooth control systems, including time-varying systems. This reformulation is described in this section and is preparation for the characterization of static feedback linearizable quotients described in section 4.

The classical problem of linearization is to determine when a (usually) time-invariant nonlinear control system

$$(7) \quad \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^q,$$

can be transformed by a static feedback transformation

$$(8) \quad t \mapsto t, \quad x \mapsto \bar{x} = \alpha(x), \quad u \mapsto \bar{u} = \beta(x, u)$$

to some member of the family of *Brunovsky normal forms* which can be viewed as the family of Pfaffian systems $B(\kappa_1, \dots, \kappa_q)$, defined locally in standard *jet coordinates* as

$$(9) \quad \begin{aligned} \omega_0^1 &= dx_0^1 - x_1^1 dt, & \omega_1^1 &= dx_1^1 - x_2^1 dt, & \dots, & & \omega_{\kappa_1}^1 &= dx_{\kappa_1}^1 - v_1 dt, \\ \omega_0^2 &= dx_0^2 - x_1^2 dt, & \omega_1^2 &= dx_1^2 - x_2^2 dt, & \dots, & & \omega_{\kappa_2}^2 &= dx_{\kappa_2}^2 - v_2 dt, \\ & \dots\dots\dots & & \dots\dots\dots & & & & \\ & \dots\dots\dots & & \dots\dots\dots & & & & \\ \omega_0^q &= dx_0^q - x_1^q dt, & \omega_1^q &= dx_1^q - x_2^q dt, & \dots, & & \omega_{\kappa_q}^q &= dx_{\kappa_q}^q - v_q dt, \end{aligned}$$

each labeled by the sequence of positive integers $\kappa_1, \kappa_2, \dots, \kappa_q$. Relabeling controls v_j as $x_{\kappa_j+1}^j$, one notices immediately that the Brunovsky normal forms are identical to the *partial prolongations* of the contact system

$$(10) \quad \{\omega_0^j = dx_0^j - x_1^j dt\}_{j=1}^q$$

on the jet space $J^1(\mathbb{R}, \mathbb{R}^q)$ in which ω_0^j is prolonged to order $\kappa_j + 1$. For instance, $B(1, 2, 2, 5)$ is the Pfaffian system

$$(11) \quad \begin{aligned} &dx_0^1 - x_1^1 dt, \\ &dx_0^2 - x_1^2 dt, \quad dx_1^2 - x_2^2 dt, \\ &dx_0^3 - x_1^3 dt, \quad dx_1^3 - x_2^3 dt, \\ &dx_0^4 - x_1^4 dt, \quad dx_1^4 - x_2^4 dt, \quad dx_2^4 - x_3^4 dt, \quad dx_3^4 - x_4^4 dt, \quad dx_4^4 - x_5^4 dt. \end{aligned}$$

We say that ω_0^1 is unprolonged; ω_0^2, ω_0^3 are each prolonged to order 2 and ω_0^4 is prolonged to order 5. Some authors refer to this “prolongation” process as “adding integrators.” A *total prolongation* of the ω_0^j implies that for each j , we prolong the same number of times, that is, we add the same number of integrators for each j . This is contrasted with a partial prolongation in which we permit prolongation to different orders. Pfaffian system (9) is also known as a *chained form*.

The Pfaffian system (10) is the *contact system* on jet space $J^1(\mathbb{R}, \mathbb{R}^q)$. The contact system and its k th-order prolongations are responsible for the geometry of the ambient jet space $J^k(\mathbb{R}, \mathbb{R}^q)$.

Dually, we have the *contact distribution* on $J^1(\mathbb{R}, \mathbb{R}^q)$,

$$(12) \quad \mathcal{C}\langle q \rangle = \left\{ \partial_t + x_1^1 \partial_{x_0^1} + x_1^2 \partial_{x_0^2} + \dots + x_1^q \partial_{x_0^q}, \partial_{x_1^1}, \partial_{x_1^2}, \dots, \partial_{x_1^q} \right\}.$$

Partially prolonging to arbitrary order in any or all “directions” $\partial_{x_0^j}$ produces a controllable linear control system in Brunovsky normal form. We shall adopt the notation $\mathcal{C}\langle \rho_1, \rho_2, \dots, \rho_k \rangle$ for the Brunovsky normal form in which ρ_ℓ denotes the “number of variables of order ℓ .” For instance $\ker B(1, 2, 2, 5) = \mathcal{C}\langle 1, 2, 0, 1 \rangle$. Note that a partial prolongation of the contact distribution on $J^1(\mathbb{R}, \mathbb{R}^q)$, the contact distribution on a partial prolongation of jet space $J^1(\mathbb{R}, \mathbb{R}^q)$, and a Brunovsky normal form are different ways of referring to the same object.³ These terms are used interchangeably in this paper. Brunovsky proved the important result that a linear controllable control system $\dot{\mathbf{x}} = A(t)\mathbf{x} + B(t)\mathbf{u}$ can be transformed to a Brunovsky normal form $B(\kappa_1, \kappa_2, \dots, \kappa_q)$ for some collection of indices $\kappa_1, \kappa_2, \dots, \kappa_q$. This spawned an active area of research in which the key problem was that of static feedback linearization.

³The integer q in $J^1(\mathbb{R}, \mathbb{R}^q)$ agrees with the number of entries in $B(\kappa_1, \kappa_2, \dots, \kappa_q)$ and with the sum of entries in $\mathcal{C}\langle \rho_1, \rho_2, \dots, \rho_k \rangle$.

3.1. Geometry of Brunovsky normal forms. While the problem of static feedback linearization is considered by control theorists to be solved, it appears that the field has by and large not emphasized its intimate relation to contact geometry. A number of important practical and conceptual benefits accrue when the underlying contact geometry of linearizable nonlinear control systems is given greater prominence as we point out below. A key classical theorem that animates the direction of a good deal of the work in linearization of control systems is the Goursat normal form and its generalizations. In due course, it raised a general question in differential geometry as to how much of the local structure of a subbundle $\mathcal{V} \subset TM$ of the tangent bundle to smooth manifold M can be encoded by *numerical invariants* associated with its *derived flag*. This is the filtration of the tangent bundle TM of the ambient manifold M defined by $\mathcal{V}^{(j+1)} = \mathcal{V}^{(j)} + [\mathcal{V}^{(j)}, \mathcal{V}^{(j)}]$, $j \geq 0$, and $\mathcal{V}^{(0)} := \mathcal{V}$. Distribution $\mathcal{V}^{(i)}$ is called the *i*th *derived distribution*. Such a set of numerical invariants is referred to as the *derived type* of \mathcal{V} . It is rarely the case that the local structure of \mathcal{V} is determined by its derived type. However the situations for which the derived type *is* a complete local invariant include the most basic theorems of differential geometry such as the Frobenius theorem, the Pfaff theorem, and the Goursat normal form. In recent years several important new cases have been added to this short list, effectively providing a geometric characterization of the contact systems on jet spaces $J^k(\mathbb{R}^n, \mathbb{R}^m)$. The simplest example of a differential system for which the derived type *does not* determine its local structure is the family of generic 2-plane fields on any 5-manifold [9]. In this section we discuss one further case in which the derived type is a complete local invariant, the *generalized Goursat normal form* [34], [35]. This theorem provides a geometric characterization of the partial prolongations of the jet space $J^1(\mathbb{R}, \mathbb{R}^q)$ and inter alia a geometric formulation of linearization in nonlinear control systems, that is to say, a geometric formulation of Brunovsky normal forms. We explain here that the problem of static feedback linearization of control systems (regardless of their local form) can be viewed as a refinement of this general research program in relation to derived type. This reformulation unifies and extends a number of the standard results in control theory such as the Gardner–Shadwick algorithm [16], the extended Goursat normal form [7], the Hunt–Su–Meyer [18] and Respondek–Jakubczyk [24] linearization theorems, and related results. Of immediate relevance for this paper are the benefits it confers on the question of finding and analyzing *linearizable symmetry reductions* of control systems.

To explain this, let us first recall the classical Goursat normal form. Let $\mathcal{V} \subset TM$ be a smooth, rank 2 subbundle over smooth manifold M such that \mathcal{V} is *bracket generating* and $\dim \mathcal{V}^{(i)} = 2 + i$ while $\mathcal{V}^{(i)} \neq TM$. Then there is a generic subset $\widehat{M} \subseteq M$ such that in a neighborhood of each point of \widehat{M} there are local coordinates $t, x_0, x_1, x_2, \dots, x_k$ such that \mathcal{V} has local expression

$$(13) \quad \mathcal{C}\langle 0, 0, \dots, 0, 1 \rangle = \left\{ \partial_t + \sum_{j=1}^k x_j \partial_{x_{j-1}}, \partial_{x_k} \right\},$$

where $k = \dim M - 2$. That is, \mathcal{V} is locally equivalent to $\mathcal{C}\langle 0, 0, \dots, 0, 1 \rangle$ on \widehat{M} , where $k - 1$ zeros precede the final entry, 1.

The Goursat normal form solves the recognition problem of when differential system \mathcal{V} can be identified with the contact distribution (13) via a local diffeomorphism of M in terms of a property of its derived flag. The generalized Goursat normal form does the same job in the case distribution (13) is replaced by the partial prolongations of the contact distribution on jet space $J^1(\mathbb{R}, \mathbb{R}^q)$ with $q > 1$, defined by (12).

This is a much more delicate task involving more subtle invariants but the end result is an analogous theorem. An example of a partial prolongation of (12) is given by

$$(14) \quad \mathcal{C}\langle 1, 2, 0, 0, 1 \rangle = \left\{ \partial_t + x_1^1 \partial_{x_0^1} + x_1^2 \partial_{x_0^2} + x_2^2 \partial_{x_1^2} + x_1^3 \partial_{x_1^3} + x_2^3 \partial_{x_1^3} \right. \\ \left. + x_1^4 \partial_{x_0^4} + x_2^4 \partial_{x_1^4} + x_3^4 \partial_{x_2^4} + x_4^4 \partial_{x_3^4} + x_5^4 \partial_{x_4^4}, \partial_{x_1^1}, \partial_{x_2^2}, \partial_{x_3^3}, \partial_{x_5^5} \right\}$$

in which there is one “dependent variable of order 1,” two of order 2, and one of order 5 (so $q = 4$). The notation $\mathcal{C}\langle 1, 2, 0, 0, 1 \rangle$ denotes one dependent variable of order 1 (1st element), two dependent variables of order 2 (2nd element), zero dependent variables of orders 3 and 4 (3rd and 4th elements), and one variable of order 5 (5th element). Recall that $\ker B(1, 2, 2, 5) = \mathcal{C}\langle 1, 2, 0, 0, 1 \rangle$; cf. (14) and (11).

We are now ready to describe the aforementioned generalized Goursat normal form. This leads to a procedure based on the *refined* derived type [35] of a subbundle for recognizing subbundles as partial prolongations in terms of natural numerical invariants associated to the derived flag. It permits us to construct equivalences that put no restriction on the local form of the subbundle under consideration. This is important for our subsequent work. We begin with an introduction to the basic tools required.

Suppose M is a smooth manifold and $\mathcal{V} \subset TM$ a smooth subbundle of its tangent bundle. The *structure tensor* is the map $\delta : \Lambda^2 \mathcal{V} \rightarrow TM/\mathcal{V}$ defined by

$$\delta(X, Y) = [X, Y] \mod \mathcal{V} \quad \text{for all } X, Y \in \mathcal{V}.$$

In more detail, suppose X_1, \dots, X_r is a basis for \mathcal{V} and $\omega^1, \dots, \omega^r$ is the dual basis for its dual, \mathcal{V}^* . Suppose Z_1, \dots, Z_s is a basis for TM/\mathcal{V} such that $[X_i, X_j] \equiv c_{ij}^k Z_k \mod \mathcal{V}$ for some functions c_{ij}^k on M . Then $\delta = c_{ij}^k \omega^i \wedge \omega^j \otimes Z_k$, that is, a section of $\Lambda^2 \mathcal{V}^* \otimes TM/\mathcal{V}$. The structure tensor encodes important information about a subbundle, the most obvious of which is the extent to which it fails to be Frobenius integrable. Let us define the map $\zeta : \mathcal{V} \rightarrow \text{Hom}(\mathcal{V}, TM/\mathcal{V})$ by $\zeta(X)(Y) = \delta(X, Y)$.

For each $x \in M$, let $\mathcal{S}_x = \{v \in \mathcal{V}_x \setminus 0 \mid \zeta(v) \text{ has less than generic rank}\}$. Then \mathcal{S}_x is the zero set of homogeneous polynomials and so defines a subvariety of the projectivization $\mathbb{P}\mathcal{V}_x$ of \mathcal{V}_x . We call $\text{Sing}(\mathcal{V}) = \cup_{x \in M} \mathcal{S}_x$ the *singular variety* of \mathcal{V} . For $X \in \mathcal{V}$ the matrix of the homomorphism $\zeta(X)$ will be called the *polar matrix* of $[X] \in \mathbb{P}\mathcal{V}$. There is a map $\deg_{\mathcal{V}} : \mathbb{P}\mathcal{V} \rightarrow \mathbb{N}$ well defined by $\deg_{\mathcal{V}}([X]) = \text{rank } \zeta(X)$ for $[X] \in \mathbb{P}\mathcal{V}$. We shall call $\deg_{\mathcal{V}}([X])$ the *degree* of $[X]$ (relative to \mathcal{V}). Function $\deg_{\mathcal{V}}([X])$ is a diffeomorphism invariant: $\deg_{\phi_* \mathcal{V}}([\phi_* X]) = \deg_{\mathcal{V}}([X])$. Hence the singular variety $\text{Sing}(\mathcal{V})$ is also a diffeomorphism invariant.

The computation of the singular variety for any given subbundle $\mathcal{V} \subset TM$ is algorithmic. It involves only differentiation and commutative algebra operations. One computes the *determinantal variety* of the polar matrix for generic $[X]$; see [34], [35] for more details and examples.

Recall that an important invariant object associated with any distribution is its Cauchy bundle. For $\mathcal{V} \subseteq TM$, by $\text{Char } \mathcal{V}$ we denote the Cauchy characteristics of \mathcal{V} , that is, $\text{Char } \mathcal{V} = \{X \in \mathcal{V} \mid [X, \mathcal{V}] \subseteq \mathcal{V}\}$. If \mathcal{V} is such that all derived bundles $\mathcal{V}^{(j)}$ and all their Cauchy characteristics $\text{Char } \mathcal{V}^{(j)}$ have constant rank on M then we say that \mathcal{V} is *totally regular*. In this case we refer to $\text{Char } \mathcal{V}^{(j)}$ as the *Cauchy bundle* of $\mathcal{V}^{(j)}$.

The aforementioned singular bundle has a natural counterpart in which the quotient $\mathcal{V}/\text{Char } \mathcal{V}$ replaces \mathcal{V} . In this case, if $\text{Sing}(\mathcal{V}/\text{Char } \mathcal{V})$ is not empty then each of its points has positive degree.

DEFINITION 3.1 (resolvent bundle). Suppose $\mathcal{V} \subset TM$ is totally regular of rank $c + q + 1$, $q \geq 2, c \geq 0$, $\dim M = c + 2q + 1$. Let $\tilde{\mathcal{V}} = \mathcal{V}/\text{Char } \mathcal{V}$ and suppose further that \mathcal{V} satisfies

- (a) $\dim \text{Char } \mathcal{V} = c$, $\mathcal{V}^{(1)} = TM$;
- (b) $\text{Sing}(\tilde{\mathcal{V}})|_x = \mathbb{P}\tilde{\mathcal{B}}|_x \approx \mathbb{RP}^{q-1}$ for each $x \in M$ and some rank q subbundle $\tilde{\mathcal{B}} \subset \tilde{\mathcal{V}}$.

Then we call $(\mathcal{V}, \mathbb{P}\tilde{\mathcal{B}})$ a Weber structure of rank q on M .

Given a Weber structure $(\mathcal{V}, \mathbb{P}\tilde{\mathcal{B}})$, let $R(\mathcal{V}) \subset \mathcal{V}$ denote the largest subbundle such that

$$\pi(R(\mathcal{V})) = \tilde{\mathcal{B}},$$

where $\pi : \mathcal{V} \rightarrow \mathcal{V}/\text{Char } \mathcal{V}$ is the natural projection. We call the rank $q + c$ bundle $R(\mathcal{V})$ the resolvent bundle associated with the Weber structure $(\mathcal{V}, \mathbb{P}\tilde{\mathcal{B}})$. The bundle $\tilde{\mathcal{B}}$ determined by the singular variety of $\tilde{\mathcal{V}}$ will be called the singular subbundle of the Weber structure. A Weber structure will be said to be integrable if its resolvent bundle is integrable.

An integrable Weber structure descends to the quotient of M by the leaves of $\text{Char } \mathcal{V}$ to be the contact bundle on $J^1(\mathbb{R}, \mathbb{R}^q)$.

PROPOSITION 3.2 (see [34]). If $(\mathcal{V}, \mathbb{P}\tilde{\mathcal{B}})$ is an integrable Weber structure then its resolvent $R(\mathcal{V}) \subset \mathcal{V}$ is the unique maximal integrable subbundle.

It is important to relate a given partial prolongation to its derived type. For this it is convenient to introduce the notions of *velocity* and *deceleration* of a subbundle.

DEFINITION 3.3. Let $\mathcal{V} \subset TM$ be a totally regular subbundle. The velocity of \mathcal{V} is the ordered list of k integers

$$\text{vel}(\mathcal{V}) = \langle \Delta_1, \Delta_2, \dots, \Delta_k \rangle, \quad \text{where } \Delta_j = m_j - m_{j-1}, \quad 1 \leq j \leq k,$$

where $m_j = \dim \mathcal{V}^{(j)}$.

The deceleration of \mathcal{V} is the ordered list of k integers

$$\text{decel}(\mathcal{V}) = \langle -\Delta_2^2, -\Delta_3^2, \dots, -\Delta_k^2, \Delta_k \rangle, \quad \text{where } \Delta_j^2 = \Delta_j - \Delta_{j-1}.$$

The notions of velocity and deceleration are refinements of the well-known *growth vector* of a subbundle. If we think of the growth vector as a type of “displacement” or “distance” then the notions of velocity and deceleration acquire a natural meaning. We will see that the deceleration vector is a complete invariant of a partial prolongation except when $\Delta_k > 1$, in which case one must also add that the resolvent bundle be integrable.

To recognize when a given subbundle has or has not the derived type of a partial prolongation we introduce one further canonically associated subbundle that plays a crucial role.

DEFINITION 3.4. If $\mathcal{V} \subset TM$ is a totally regular subbundle of derived length k we let $\text{Char } \mathcal{V}_{j-1}^{(j)}$ denote the intersections

$$\text{Char } \mathcal{V}_{j-1}^{(j)} = \mathcal{V}^{(j-1)} \cap \text{Char } \mathcal{V}^{(j)}, \quad 1 \leq j \leq k-1.$$

Let

$$\chi_{j-1}^j = \dim \text{Char } \mathcal{V}_{j-1}^{(j)}, \quad 1 \leq j \leq k-1.$$

We shall call the integers $\{m_j, \chi^0, \chi^j, \chi_{j-1}^j\}_{j=1}^{k-1}$ the type numbers of $\mathcal{V} \subset TM$ and the list of lists

$$\mathfrak{d}_r(\mathcal{V}) = \left[[m_0, \chi^0], [m_1, \chi_0^1, \chi^1], [m_2, \chi_1^2, \chi^2], \dots, [m_{k-1}, \chi_{k-2}^{k-1}, \chi^{k-1}], [m_k, \chi^k] \right]$$

as the refined derived type of \mathcal{V} .

It is easy to see that in every partial prolongation expressed in canonical contact coordinates, subbundles $\text{Char } \mathcal{V}_{j-1}^{(j)}$ are nontrivial and integrable, an invariant property of \mathcal{V} . Furthermore, there are simple relationships between the type numbers in any partial prolongation thereby providing further invariants for the local equivalence problem.

PROPOSITION 3.5 (see [34]). *Suppose \mathcal{V} is a partial prolongation of the contact distribution of $\mathcal{C}\langle q \rangle$ on $J^1(\mathbb{R}, \mathbb{R}^q)$. Then the type numbers $m_j, \chi^j, \chi_{j-1}^j$ comprising the refined derived type $\mathfrak{d}_r(\mathcal{V})$ satisfy*

$$\begin{aligned} \chi^j &= 2m_j - m_{j+1} - 1, \quad 0 \leq j \leq k-1, \\ \chi_{i-1}^i &= m_{i-1} - 1, \quad 1 \leq i \leq k-1, \end{aligned}$$

where k is the derived length of \mathcal{V} .

DEFINITION 3.6. *Say that $\mathcal{V} \subset TM$ has the refined derived type of a partial prolongation (of the contact distribution on $J^1(\mathbb{R}, \mathbb{R}^q)$) if its type numbers $m_j, \chi^j, \chi_{i-1}^i$ are those of some partial prolongation, which then necessarily satisfy the equalities in Proposition 3.5.*

DEFINITION 3.7. *A totally regular subbundle $\mathcal{V} \subset TM$ of derived length k will be called a Goursat bundle with deceleration $\sigma = \langle \rho_1, \rho_2, \dots, \rho_k \rangle$ if*

1. \mathcal{V} has the refined derived type of a partial prolongation with signature $\sigma = \text{decel}(\mathcal{V})$;
2. each intersection $\text{Char } \mathcal{V}_{i-1}^{(i)}$ is an integrable subbundle;
3. in the case $\Delta_k > 1$, then $\mathcal{V}^{(k-1)}$ determines an integrable Weber structure of rank Δ_k .

The recognition problem for partial prolongations is solved by the generalized Goursat normal form.

THEOREM 3.8 (generalized Goursat normal form [34]). *Let $\mathcal{V} \subset TM$ be a Goursat bundle over manifold M of derived length $k > 1$, and signature $\sigma = \text{decel}(\mathcal{V})$. Then there is an open, dense subset $\widehat{M} \subseteq M$ such that the restriction of \mathcal{V} to \widehat{M} is locally equivalent to $\mathcal{C}(\sigma)$. Conversely any partial prolongation of $\mathcal{C}\langle q \rangle$ is a Goursat bundle.*

A partial prolongation is generically classified, up to a local diffeomorphism of the ambient manifold, by its deceleration vector. For this reason the deceleration of a Goursat bundle \mathcal{V} will sometimes be called its *signature*, a unique identifier of its local diffeomorphism class. If \mathcal{V} is a Goursat bundle and nonnegative integer ρ_j is the j th component of its signature, then \mathcal{V} is locally diffeomorphic to a partial prolongation with ρ_j dependent variables of order j . The theorem has a counterpart which provides an efficient procedure, **Contact**, for constructing an equivalence to $\mathcal{C}(\sigma)$, where $\sigma = \text{decel}(\mathcal{V})$ is the signature of \mathcal{V} . Procedure **Contact** is described in detail in [35], where its proof of correctness is given, and will be used in this paper. The basic result is that one characterizes the total differential operators and the function

spaces that are used to generate the equivalences by differentiation. The first integrals of the resolvent bundle and those of the *fundamental bundles* $\Xi_{j-1}^{(j)}(\mathcal{V})/\Xi^{(j)}(\mathcal{V})$ [35] are the required functions, where $\Xi^{(j)}(\mathcal{V}) = \text{ann Char } \mathcal{V}^{(j)}$ and $\Xi_{j-1}^{(j)}(\mathcal{V}) = \text{ann Char } \mathcal{V}_{j-1}^{(j)}$. If $\Delta_k = 1$ then the resolvent is replaced by another integrable distribution, Π^k . It is defined in [34], and illustrated in [35] and [36]. Further details, proofs of all results, and illustrative examples can be found in the previously named references.

3.2. Extended static feedback transformations. The generalized Goursat normal form primarily solves the problem of *general equivalence* of a differential system (M, \mathcal{V}) to a partial prolongation $\mathcal{C}(\sigma)$ of the contact distribution $\mathcal{C}\langle q \rangle$, that is, the existence of *some* local diffeomorphism $\varphi : M \rightarrow J^\sigma(\mathbb{R}, \mathbb{R}^q)$ such that $\varphi_* \mathcal{V} = \mathcal{C}\langle \sigma \rangle$. If such an equivalence between (M, \mathcal{V}) and $(J^\sigma(\mathbb{R}, \mathbb{R}^q), \mathcal{C}\langle \sigma \rangle)$ exists then there also exists an infinite dimensional family of equivalences since the contact transformations form an infinite Lie pseudogroup. If (M, \mathcal{V}) is a control system then it is of great importance to know that within the infinite dimensional family of equivalences at least one can be chosen to be an extended static feedback transformation, which is a natural and simple generalization of static feedback transformation to the case of *time-varying* control.

DEFINITION 3.9. *If (M, \mathcal{V}) is the distribution associated with control system (1) then locally there are submanifolds $X(M)$, the submanifold of states and $U(M)$, the submanifold of controls such that locally $M = \mathbb{R} \times X(M) \times U(M)$, where the factor \mathbb{R} is the time coordinate space.*

From now on, when distribution \mathcal{V} or Pfaffian system ω define a control system on manifold M then M is to be viewed as the local product of the manifold of states, controls, and time and we use the notation $M = \mathbb{R} \times X(M) \times U(M)$, where $\dim X(M) = n$, $\dim U(M) = q$.

DEFINITION 3.10 (extended static feedback transformations). *A local diffeomorphism of the manifold $M = \mathbb{R} \times X(M) \times U(M)$ of states, controls, and time, \mathbf{x} , \mathbf{u} , t of the form*

$$t \mapsto t, \quad \mathbf{x} \mapsto \bar{\mathbf{x}} = \boldsymbol{\alpha}(t, \mathbf{x}), \quad \mathbf{u} \mapsto \bar{\mathbf{u}} = \boldsymbol{\beta}(t, \mathbf{x}, \mathbf{u})$$

identifying a pair of control systems

$$\{\partial_t + \mathbf{f}(t, \mathbf{x}, \mathbf{u})\partial_{\mathbf{x}}, \partial_{\mathbf{u}}\} \quad \text{and} \quad \{\partial_t + \bar{\mathbf{f}}(t, \bar{\mathbf{x}}, \bar{\mathbf{u}})\partial_{\bar{\mathbf{x}}}, \partial_{\bar{\mathbf{u}}}\}$$

will be called an extended static feedback transformation (ESFT).

The existence of an *ESFT* identifying a control system to a Brunovsky normal form can be usefully established in terms of the generalized Goursat normal form.

THEOREM 3.11. *Let $\mathcal{V} = \{\partial_t + \mathbf{f}(t, \mathbf{x}, \mathbf{u})\partial_{\mathbf{x}}, \partial_{\mathbf{u}}\}$ be a control system defining a totally regular subbundle of TM , where $M = \mathbb{R} \times X(M) \times U(M)$, $\dim X(M) = n$, $\dim U(M) = q$. Suppose (M, \mathcal{V}) is a Goursat bundle. Then it is equivalent to a Brunovsky normal form $\mathcal{C}\langle \sigma \rangle$ via local diffeomorphism $\varphi : M \rightarrow J^\sigma(\mathbb{R}, \mathbb{R}^q)$, $\varphi_* \mathcal{V} = \mathcal{C}\langle \sigma \rangle$, with derived length $k > 1$. Furthermore, φ can be chosen to be an extended static feedback transformation if and only if*

- (1) $\{\partial_{\mathbf{u}}\} \subseteq \text{Char } \mathcal{V}_0^{(1)}$,
- (2) $dt \in \text{ann Char } \mathcal{V}^{(k-1)}$.

Proof. Suppose φ is an *ESFT* identifying (M, \mathcal{V}) with Brunovsky normal form $(J^\sigma(\mathbb{R}, \mathbb{R}^q), \mathcal{C}\langle \sigma \rangle)$. Then φ has the form

$$(15) \quad (t, \mathbf{x}, \mathbf{u}) \mapsto (t, \boldsymbol{\alpha}(t, \mathbf{x}), \boldsymbol{\beta}(t, \mathbf{x}, \mathbf{u})) = (x, z_{j_p}^p, z_{k_p}^p), \quad 0 \leq j_p \leq k_p - 1,$$

where $\mathbf{z}_{j_p}^p, \mathbf{z}_{k_p}^p$ are the standard coordinates on $J^\sigma(\mathbb{R}, \mathbb{R}^q)$. Here p is an index for the contact coordinates of order k_p ; the largest of these is equal to the derived length of \mathcal{V} . In contact coordinates, x can be taken to be the parameter along the trajectories of $\mathcal{C}\langle\sigma\rangle$ and by explicit computation one sees that it is a first integral of $\text{Char } \mathcal{C}\langle\sigma\rangle^{(k-1)}$. Hence $\varphi^*x = t$ is a first integral of $\text{Char } \mathcal{V}^{(k-1)}$ since φ identifies Cauchy bundles, hence (2) holds.

Again by an explicit computation, it can be verified that $\mathbf{z}_{j_p}^p, 0 \leq j_p \leq k_p - 1$, are first integrals of $\text{Char } \mathcal{C}\langle\sigma\rangle_0^{(1)}$ and hence $\varphi^*\mathbf{z}_{j_p}^p = \alpha(t, \mathbf{x})$ span the first integrals of $\text{Char } \mathcal{V}^{(1)}$. The elements of $\text{Char } \mathcal{V}_0^{(1)}$ are spanned by vector fields

$$Y = T\partial_t + \sum_{i=1}^n A^i \partial_{x_i} + \sum_{\ell=1}^q B_\ell \partial_{u_\ell}.$$

We have $(d\alpha)(Y) = 0$, and we deduce that $T = 0$ (since t is a first integral of $\text{Char } \mathcal{V}^{(1)} \subset \text{Char } \mathcal{V}^{(k-1)}$) and

$$\frac{\partial(\alpha^1, \alpha^2, \dots, \alpha^n)}{\partial(x_1, x_2, \dots, x_n)} \mathbf{A} = 0,$$

where $\mathbf{A} = (A^1 \ A^2 \ \dots \ A^n)^T$. It follows that $\mathbf{A} = 0$ since the components of α are functionally independent. Hence $\text{Char } \mathcal{V}_0^{(1)}$ contains vector fields of the form $Y = \sum_{\ell=1}^q B_\ell \partial_{u_\ell}$ only. Let H be the set of all vector fields of the form $\{Y_s = B_s^\ell \partial_{u_\ell}\} \subseteq \text{Char } \mathcal{V}_0^{(1)}$ which have the $n+1$ functions $(t, \alpha(t, \mathbf{x}))$ as functionally independent first integrals. Because $\text{Char } \mathcal{V}_0^{(1)}$ is Frobenius integrable, we have $H^{(\infty)} \subseteq \text{Char } \mathcal{V}_0^{(1)}$. Suppose $H^{(\infty)} \neq \{\partial_{u_1}, \partial_{u_2}, \dots, \partial_{u_q}\}$. Then there is a first integral of H which has u -dependence which contradicts the functional form of φ . Hence (1) follows.

Conversely suppose hypotheses (1) and (2) of the theorem statement hold with (M, \mathcal{V}) a Goursat manifold so that local diffeomorphism φ exists which identifies \mathcal{V} with partial prolongation $\mathcal{C}\langle\sigma\rangle$. By the proof of correctness of procedure **Contact** [35, pp. 286–287], hypothesis (2) implies that $\varphi^*x = t$ can be taken to be an independent variable in the image system $\mathcal{C}\langle\sigma\rangle$, that is, a parameter along the trajectories of \mathcal{V} . Further, according to procedure **Contact**, the components of the transformation to Brunovsky normal are constructed by differentiating the fundamental functions of order j , $\psi_0^{\ell_j, j}, 1 \leq \ell_j \leq \rho_j$, by the total differential operator \mathbf{Z} :

$$\psi_0^{\ell_j, j}, \psi_1^{\ell_j, j} = \mathbf{Z}\psi_0^{\ell_j, j}, \dots, \psi_j^{\ell_j, j} = \mathbf{Z}\psi_{j-1}^{\ell_j, j}.$$

The proof of Theorem 4.2 in [35] shows that the functions $\{\psi_s^{\ell_j, j}\}_{s=0}^{j-1}$ are first integrals of $\text{Char } \mathcal{V}_0^{(1)}$. Hypothesis (1) in the theorem statement therefore allows us to conclude that φ has the form (15). Hence φ is an ESFT. \square

4. Linearizable quotients. Flat quotients. If an invariant control system is not static feedback linearizable or even flat, it is desirable to know of the existence of static feedback linearizable *subsystems*. Otherwise, it is desirable to know of the existence of flat subsystems. In either case it would be useful to have an a priori algorithmic test for the existence of such quotients, that is, without the necessity of constructing the quotient first.

In this section we use the results of section 3 to give such an a priori check for the existence of static feedback linearizable subsystems only using as data our knowledge of the Lie algebra of infinitesimal generators of the symmetries of the control system. To establish this, we introduce a slight refinement of the notion of Goursat bundle.

DEFINITION 4.1. Let (M, \mathcal{V}) be a totally regular distribution over manifold M . We say that \mathcal{V} is a relative Goursat bundle if its type numbers satisfy Proposition 3.5 and items 2 and 3 of Definition 3.7 are satisfied.

If $\Delta \subset TM$ is a (constant rank) integrable, smooth distribution then we have the following equivalence relation. For all $x, x' \in M$ we say $x \sim x'$ if and only if x and x' lie on the same maximal, connected, integral submanifold of Δ . The set of equivalence classes M/Δ may not have the structure of a smooth manifold. Call Δ regular if M/Δ , endowed with the quotient topology, can be given the structure of a smooth manifold such that the natural projection $\pi : M \rightarrow M/\Delta$ is a smooth submersion [32, Chapter 1].

If $\mathcal{V} \subset TM$ is a distribution we shall henceforth assume that $\text{Char } \mathcal{V}$ is regular in the above sense and denote the quotient map $\pi_{\mathcal{V}} : M \rightarrow M/\text{Char } \mathcal{V}$. There is a Pfaffian system $\bar{\mathcal{V}}$ on $M/\text{Char } \mathcal{V}$ which is the quotient of $\bar{\mathcal{V}} = \text{ann } \mathcal{V}$ by the equivalence relation \sim ; see also [19, pp. 209–212]. Let us denote $\ker \bar{\mathcal{V}}$ by $\bar{\mathcal{V}}$.

THEOREM 4.2. Let (M, \mathcal{V}) be a totally regular, bracket generating distribution with type numbers $m_j, \chi^j, \chi_{i-1}^i$ and suppose $\text{Char } \mathcal{V}$ is regular with $\dim \text{Char } \mathcal{V} = c > 0$. Then the quotient distribution $\bar{\mathcal{V}}$ has type numbers $\bar{m}_j, \bar{\chi}^j, \bar{\chi}_{i-1}^i$ satisfying

$$(16) \quad m_j - \bar{m}_j = \chi^j - \bar{\chi}^j = \chi_{i-1}^i - \bar{\chi}_{i-1}^i = c, \quad 0 \leq j \leq k-1, \quad 1 \leq i \leq k-1.$$

If \mathcal{V} is a relative Goursat bundle on M then $\bar{\mathcal{V}}$ will be a Goursat bundle on $M/\text{Char } \mathcal{V}$.

Proof. Since $\text{Char } \mathcal{V}$ is regular, we have a smooth submersion $\pi_{\mathcal{V}}$ and hence for each $p \in M$,

$$d\pi_{\mathcal{V}} : T_p M \rightarrow T_{\pi_{\mathcal{V}}(p)}(M/\text{Char } \mathcal{V})$$

is onto with $\ker(d\pi_{\mathcal{V}}) = \text{Char } \mathcal{V}$. Hence,

$$d\pi_{\mathcal{V}} : \mathcal{V}/\text{Char } \mathcal{V} \rightarrow d\pi_{\mathcal{V}}(\mathcal{V})$$

is an isomorphism of vector bundles. Let (U, x^1, \dots, x^m) be a chart containing $p \in M$, where $m = \dim M$ and $U \subset M$ is open. By the Frobenius theorem, [30], there is a chart $(V, y^1, \dots, y^{m-c}, z^1, \dots, z^c)$, $V \subset M$ an open set with $U \cap V \neq \emptyset$, where the transition function

$$\tau(x) = (y^1(x), \dots, y^{m-c}(x), z^1(x), \dots, z^c(x))$$

is such that the y^j are independent first integrals of $\text{Char } \mathcal{V}$. We can change basis in \mathcal{V} such that there is a distribution \mathcal{P} satisfying $\mathcal{V} = \mathcal{P} \oplus \text{Char } \mathcal{V}$, that is $\mathcal{P} \cap \text{Char } \mathcal{V} = \{0\}$. Expressing \mathcal{V} in the coordinates (y, z) , it is not difficult to show, by recalling that the y^a are first integrals of $\text{Char } \mathcal{V}$, that $\text{Char } \mathcal{V} = \{\partial_{z^1}, \dots, \partial_{z^c}\}$ and hence there are functions η, ζ such that locally

$$\mathcal{V} = \mathcal{P} \oplus \text{Char } \mathcal{V} = \left\{ \sum_{b=1}^{m-c} \eta_{\ell}^b(y, z) \partial_{y^b} + \sum_{j=1}^c \zeta_{\ell}^j(y, z) \partial_{z^j} \right\}_{\ell=1}^r \oplus \{\partial_{z^1}, \dots, \partial_{z^c}\},$$

where r is defined by $\dim \mathcal{V} = r + c$. We can further refine the basis in \mathcal{V} so that there are no components in \mathcal{P} in directions ∂_{z^j} and that furthermore the basis of \mathcal{P} is resolved,

$$\mathcal{P} = \left\{ Y_1 = \partial_{y^1} + \sum_{\ell=h}^{m-c} \bar{\eta}_1^{\ell}(y, z) \partial_{y^{\ell}}, \dots, Y_r = \partial_{y^r} + \sum_{\ell=h}^{m-c} \bar{\eta}_r^{\ell}(y, z) \partial_{y^{\ell}} \right\}, \quad h = m - c - r + 1.$$

Since $[\text{Char } \mathcal{V}, \mathcal{V}] \subseteq \mathcal{V}$, it follows that $\partial_{z^j} \bar{\eta}_\alpha^\ell = 0$. Moreover, in this local coordinate system and basis for \mathcal{V} , we have shown that $[Y_\ell, \partial_{z^j}] = 0$ for all ℓ, j . Since $\{\partial_{z^1}, \dots, \partial_{z^c}\} = \ker(d\pi_\mathcal{V})$ we have $d\pi_\mathcal{V}(\mathcal{P}) = d\pi_\mathcal{V}(\mathcal{V})$ and similarly to Lemma 2.5, we have $\ker \bar{\vartheta} = d\pi_\mathcal{V}(\mathcal{V})$. But in the given local coordinates on M , $d\pi_\mathcal{V}$ acts trivially on \mathcal{P} and we can write $d\pi_\mathcal{V}(\mathcal{V}) = \mathcal{P}$. Letting $\mathcal{Z} = \text{Char } \mathcal{V}$, we have $[\mathcal{P}^{(i)}, \mathcal{Z}^{(i)}] = 0$ and $\mathcal{P}^{(i)} \cap \mathcal{Z}^{(i)} = 0$ for all $i \geq 0$. Then using the constructed basis for \mathcal{V} , we deduce that

$$(17) \quad \begin{aligned} \dim \mathcal{V}^{(i)} &= \dim \mathcal{P}^{(i)} + c, \quad \dim \text{Char } \mathcal{V}^{(j)} = \dim \text{Char } \mathcal{P}^{(j)} + c, \\ \dim \text{Char } \mathcal{V}_{i-1}^{(i)} &= \dim \text{Char } \mathcal{P}_{i-1}^{(i)} + c. \end{aligned}$$

Distribution $\bar{\mathcal{V}} = \mathcal{P}$ has type numbers $\bar{m}_j, \bar{\chi}^j, \bar{\chi}_{i-1}^i$ while $\mathcal{V} = \mathcal{P} \oplus \mathcal{Z}$ has type numbers $m_j, \chi^j, \chi_{i-1}^i$. The relation between the two sets of type numbers follows from (17).

If \mathcal{V} is a relative Goursat bundle then it follows from (17) that the type numbers of $\bar{\mathcal{V}}$ will satisfy Proposition 3.5. This together with the fact that $\text{Char } \bar{\mathcal{V}} = \{0\}$ implies that $\bar{\mathcal{V}}$ will have the type numbers of a partial prolongation and hence will be a Goursat bundle on $M/\text{Char } \mathcal{V}$ according to Definitions 3.7 and 4.1. \square

PROPOSITION 4.3. *If (M, \mathcal{V}) is a relative Goursat bundle then*

$$\text{decel } \mathcal{V} = \text{decel } ((d\pi_\mathcal{V}) \mathcal{V}).$$

That is, the signature of \mathcal{V} and that of its quotient by $\text{Char } \mathcal{V}$ both agree.

Proof. From the previous theorem, we have $\bar{m}_j = m_j - c$. Since the deceleration of any bundle is a first or second difference of the derived flag bundle ranks, \bar{m}_j, m_j , we deduce that the decelerations of \mathcal{V} and $\mathcal{V}/\text{Char } \mathcal{V}$ are identical. \square

LEMMA 4.4. *Let $\mathcal{V} \subset TM$ be a smooth distribution invariant under the free action of Lie group G with Lie algebra Γ . If G is strongly transverse to \mathcal{V} and $\text{Char } \mathcal{V} = \{0\}$ then $\text{Char } (\mathcal{V} \oplus \Gamma) = \Gamma$.*

Proof. Denote $\mathcal{V} \oplus \Gamma$ by $\hat{\mathcal{V}}$. Certainly $\Gamma \subseteq \text{Char } \hat{\mathcal{V}}$ but suppose $\text{Char } \hat{\mathcal{V}}$ contains an element $\xi \notin \Gamma$. Then $\xi = \eta + \gamma$, where $0 \neq \eta \in \mathcal{V}$ and $\gamma \in \Gamma$. Let $X \in \mathcal{V}$ be an arbitrary nonzero element. Then $[X, \xi] = \eta_1 + \gamma_1$ for some $\eta_1 \in \mathcal{V}$ and $\gamma_1 \in \Gamma$. Now $[X, \xi] = [X, \eta] + [X, \gamma]$ and it is easy to see that the right-hand side must be an element of $\mathcal{V}^{(1)}$. It then follows that $\gamma_1 \in \mathcal{V}^{(1)}$ and by strong transversality we deduce that $\gamma_1 = 0$. Since X is arbitrary this implies that $\xi \in \text{Char } \mathcal{V}$ and hence $\xi = 0$, contradicting $\xi \notin \Gamma$. \square

THEOREM 4.5 (existence of linearizable quotients). *Let $\mathcal{V} \subset TM$ be a subbundle over smooth manifold M that is invariant under the smooth, free, regular action of a Lie group G with Lie algebra Γ , where G is strongly transverse to \mathcal{V} and suppose $\text{Char } \mathcal{V} = \{0\}$. Then*

- (1) *the semibasic 1-forms for the G -action satisfy $\ker \omega_{\text{sb}} = \mathcal{V} \oplus \Gamma$;*
- (2) *if $(M, \mathcal{V} \oplus \Gamma)$ is a relative Goursat bundle then the quotient $(M/G, \mathcal{V}/G)$ of (M, \mathcal{V}) is locally equivalent to a partial prolongation of the contact distribution on $J^1(\mathbb{R}, \mathbb{R}^q)$, some $q \geq 1$;*
- (3) *if $(M, \mathcal{V} \oplus \Gamma)$ is a relative Goursat bundle then signature $\sigma = \text{decel}(\mathcal{V} \oplus \Gamma) = \text{decel}(\mathcal{V}/G)$, whence $(M/G, \mathcal{V}/G) \simeq \mathcal{C}(\sigma)$.*

Proof. Let $\omega = \text{ann } \mathcal{V}$, $\mathbf{q} : M \rightarrow M/G$ the quotient map, and $\bar{\omega} = \omega/G$ the quotient of ω by the action of G . Recall that $\mathbf{q}^* \bar{\omega} \subset \omega$; in particular, $\mathcal{V} \subset \ker \mathbf{q}^* \bar{\omega}$. We also have $\Gamma \subset \ker \mathbf{q}^* \bar{\omega}$ since $(\mathbf{q}^* \bar{\omega})(v) = \bar{\omega}(\mathbf{q}_* v) = \bar{\omega}(0) = 0$ for all $v \in \Gamma$. Hence $\mathcal{V} \oplus \Gamma \subseteq \ker \mathbf{q}^* \bar{\omega}$.

LEMMA 4.6. $\mathbf{q}^*\bar{\omega} = \omega_{\text{sb}}$.

Proof of Lemma 4.6. Since $(\mathbf{q}^*\bar{\omega})(\Gamma) = 0$ we have $\mathbf{q}^*\bar{\omega} \subseteq \omega_{\text{sb}}$. We invoke the following elementary fact: If $f : M \rightarrow N$ is a smooth surjective submersion with $\dim N = n$ and $\Psi \subset T^*N$ is a rank $k \leq n$ subbundle then $f^*\Psi \subset T^*M$ has rank k . Then from [2, Theorem 5.1], we have that $\dim \bar{\omega} = \dim \omega - \dim \Gamma = \dim \omega_{\text{sb}}$ and using the above fact, Lemma 4.6 is proven. \square

Returning now to the proof of Theorem 4.5, we have $\dim \ker \mathbf{q}^*\bar{\omega} = \dim M - \dim \mathbf{q}^*\bar{\omega} = \dim M - \dim \omega_{\text{sb}} = \dim \mathcal{V} + \dim \Gamma = \dim (\mathcal{V} \oplus \Gamma)$. We have therefore proven that $\ker \mathbf{q}^*\bar{\omega} = \ker \omega_{\text{sb}} = \mathcal{V} \oplus \Gamma$, which is item (1). Now suppose that $\mathcal{V} \oplus \Gamma$ is a relative Goursat bundle. By Lemma 4.4 we have $\text{Char}(\mathcal{V} \oplus \Gamma) = \Gamma$ and hence by application of Theorem 4.2, $d\mathbf{q}(\mathcal{V} \oplus \Gamma) = d\mathbf{q}(\mathcal{V})$ has the refined derived type of a partial prolongation and hence is a Goursat bundle. By Theorem 3.8, this proves item (2). From this and Proposition 4.3, the quotient $d\mathbf{q}(\mathcal{V})/\text{Char}(d\mathbf{q}(\mathcal{V}))$ is locally equivalent to $\mathcal{C}\langle\sigma\rangle$, where $\sigma = \text{decel}(\mathcal{V} \oplus \Gamma)$, proving item (3). This concludes the proof of Theorem 4.5. \square

Remark 4.7. An important conclusion we draw from Theorem 4.5 is that the existence of a static feedback linearizable quotient can be checked algorithmically from the refined derived type of $\mathcal{V} \oplus \Gamma$: the kernel of the semibasic 1-forms. In particular, explicit construction of the quotient $d\mathbf{q}(\mathcal{V})$ is unnecessary. Ordinarily integration is required in order to construct ω/G if the action is not known or else only known infinitesimally.

4.1. Control morphisms and linearizable quotients. We next investigate the extent to which the quotient of a control system (M, \mathcal{V}) by its Lie symmetry group G is also a control system on the quotient M/G . Ultimately, this leads to the following.

DEFINITION 4.8 (control symmetries). *Let $\mu : G \times M \rightarrow M$ be a Lie transformation group with Lie algebra Γ of infinitesimal generators leaving control system (1) invariant and acting regularly, and freely on M . We say that G is a control admissible or simply admissible symmetry group if the function t is invariant: $\mu_g^*t = t$ for all $g \in G$ and the rank of the distribution $d\pi(\Gamma)$ is equal to $\dim G$, where π is the projection $\pi : M \rightarrow \mathbb{R} \times X(M)$, satisfying $\pi(t, \mathbf{x}, \mathbf{u}) = (t, \mathbf{x})$.*

We will now show that if G is an admissible transformation group acting on $M =_{\text{loc}} \mathbb{R} \times X(M) \times U(M)$ then its elements are extended static feedback transformations.

THEOREM 4.9. *Let $\mu : G \times M \rightarrow M$ be an admissible Lie transformation group acting smoothly, regularly, and freely on M and leaving invariant the control system (M, \mathcal{V}) defined by (1) such that G is strongly transverse to \mathcal{V} . Suppose $\dim G < \dim X(M)$. Then locally the quotient $(M/G, \mathcal{V}/G)$ is a control system in which*

$$\dim X(M/G) = \dim X(M) - \dim G \quad \text{and} \quad \dim U(M/G) = \dim U(M).$$

Proof. The distribution \mathcal{V} has the form $\mathcal{V} = \{\partial_t + \mathbf{f}(t, \mathbf{x}, \mathbf{u})\partial_{\mathbf{x}}, \partial_{\mathbf{u}}\}$ and any admissible symmetry of \mathcal{V} must preserve the subdistribution $\{\partial_{\mathbf{u}}\}$. For if $v \in \Gamma$ is admissible then $v(t) = 0$ and hence $[v, \partial_{u_a}](t) = 0$. Since v is an infinitesimal symmetry, $[v, \partial_{u_a}] = \alpha T + \beta_a \partial_{u_a}$ for some functions α, β_a , where $T = \partial_t + \mathbf{f}(t, \mathbf{x}, \mathbf{u})\partial_{\mathbf{x}}$. We deduce that $\alpha = 0$.

Next if $v = \xi^i \partial_{x_i} + \eta^a \partial_{u_a}$ is an infinitesimal generator of an admissible symmetry of \mathcal{V} then $\{\partial_{\mathbf{u}}\} \ni [\partial_{u_a}, v] = \frac{\partial \xi^i}{\partial u_a} \partial_{x_i} + \frac{\partial \eta^b}{\partial u_a} \partial_{u_b}$ which implies that $\frac{\partial \xi^i}{\partial u_a} = 0$. Hence, the

corresponding infinitesimal generators of such an admissible symmetry group consists of vector fields of the form $v = \xi(t, \mathbf{x})\partial_{\mathbf{x}} + \eta(t, \mathbf{x}, \mathbf{u})\partial_{\mathbf{u}}$. If $\mu : G \times M \rightarrow M$ is the Lie transformation group with infinitesimal generators of the form v then μ has the general local form

$$\mu_g(t, \mathbf{x}, \mathbf{u}) = (t, \mathbf{a}(t, \mathbf{x}, g), \mathbf{b}(t, \mathbf{x}, \mathbf{u}, g)) = (\bar{t}, \bar{\mathbf{x}}, \bar{\mathbf{u}}).$$

Recall that $r = \dim G < \dim X(M) = n$. The action being admissible means that $d\pi(\Gamma)$ is a rank r subbundle of TM , $r = \dim G$. If $\Gamma = \{X_1, \dots, X_r\}$, then

$$d\pi(X_i) = \frac{\partial a_\ell}{\partial g_i} \partial_{x_\ell} \Big|_{g=\text{id}}, \quad 1 \leq i \leq r.$$

Since the $d\pi(X_i)$ span a rank r subbundle, there is a subset $a_{i_1}, a_{i_2}, \dots, a_{i_r}$ such that

$$\det \frac{\partial(a_{i_1}, a_{i_2}, \dots, a_{i_r})}{\partial(g_1, g_2, \dots, g_r)} \Big|_{g=\text{id}} \neq 0.$$

By the implicit function theorem, in a neighborhood of a point $(p, \text{id}) \in \mathbb{R} \times X \times G$ there are functions $g_j = \gamma_j(t, \mathbf{x})$ such that $a_{i_s}(t, \mathbf{x}, \gamma(t, \mathbf{x})) = c_s$, $1 \leq s \leq r$, for some constants c_s . By the theory of equivariant moving frames [14, 15], the $n - r$ nonconstant functions that remain among the components of $\mathbf{a}(t, \mathbf{x}, \gamma(t, \mathbf{x}))$ and the q nonconstant functions $\mathbf{b}(t, \mathbf{x}, \mathbf{u}, \gamma(t, \mathbf{x}))$ together with t span the $n + q + 1 - r$ invariants of the G -action. Setting y_ℓ , $1 \leq \ell \leq n - r$, equal to the nonconstant functions among the $\mathbf{a}(t, \mathbf{x}, \gamma(t, \mathbf{x}))$ and v_a , $1 \leq a \leq q$, equal to the functions $\mathbf{b}(t, \mathbf{x}, \mathbf{u}, \gamma(t, \mathbf{x}))$ produces the quotient map $\mathbf{q} : M \rightarrow M/G$ in which local coordinates on M/G have the form

$$(18) \quad t, \quad y_\ell = y_\ell(t, \mathbf{x}), \quad v_a = v_a(t, \mathbf{x}, \mathbf{u}), \quad 1 \leq \ell \leq n - r, \quad 1 \leq a \leq q,$$

and are components of the quotient map \mathbf{q} . It follows that the quotient $d\mathbf{q}(\mathcal{V})$ has the local form of control system

$$(19) \quad d\mathbf{q}(\mathcal{V}) = \left\{ \partial_t + \sum_{\ell=1}^{n-r} \bar{f}^\ell(t, \mathbf{y}, \mathbf{v}) \partial_{y_\ell}, \partial_{v_1}, \dots, \partial_{v_q} \right\}$$

for some functions \bar{f}^ℓ with the claimed dimensions of $X(M/G)$ and $U(M/G)$. \square

COROLLARY 4.10. *The control admissible symmetries of a control system form a Lie transformation group of extended static feedback self-equivalences.*

DEFINITION 4.11. *If $\mathbf{q} : M \rightarrow M/G$ is such that $d\mathbf{q}(\mathcal{V})$ is a control system then we will say that \mathbf{q} is a control morphism.*

Not only do we wish to know when a symmetry group induces a control morphism \mathbf{q} but also when $d\mathbf{q}(\mathcal{V})$ is locally equivalent to a Brunovsky normal form by an (extended) static feedback transformation, directly from knowledge of the Lie algebra Γ .

THEOREM 4.12. *Let $(M, \mathcal{V} = \{\partial_t + \mathbf{f}(t, \mathbf{x}, \mathbf{u})\partial_{\mathbf{x}}, \partial_{\mathbf{u}}\})$ determine a control system and be a totally regular subbundle of TM , where $M = \mathbb{R} \times_{\text{loc}} X(M) \times U(M)$, $\dim X(M) = n$, $\dim U(M) = q$. Suppose \mathcal{V} is invariant under the free, regular, and admissible action of Lie group G on M with Lie algebra Γ . Suppose $(M, \mathcal{V} \oplus \Gamma)$ is a relative Goursat bundle of derived length $k > 1$ and that G is strongly transverse to \mathcal{V} with $\text{Char } \mathcal{V} = \{0\}$. Then $\mathbf{q} : M \rightarrow M/G$ is a control morphism and $d\mathbf{q}(\mathcal{V})$ is locally equivalent to a Brunovsky normal form $\mathcal{C}\langle\sigma\rangle$ via local diffeomorphisms $\varphi : M/G \rightarrow J^\sigma(\mathbb{R}, \mathbb{R}^q)$, $\varphi_* d\mathbf{q}(\mathcal{V}) = \mathcal{C}\langle\sigma\rangle$. A local equivalence φ identifying $d\mathbf{q}(\mathcal{V})$ and $\mathcal{C}\langle\sigma\rangle$ can be*

chosen to be an (extended) static feedback transformation if and only if

- (1) $\{\partial_u\} \subseteq \text{Char } \widehat{\mathcal{V}}_0^{(1)}$,
- (2) $d\bar{t} \in \text{ann Char } \widehat{\mathcal{V}}^{(k-1)}$,

where $\widehat{\mathcal{V}} = \mathcal{V} \oplus \Gamma$.

Proof. By Lemma 4.4 we have $\text{Char } \widehat{\mathcal{V}} = \Gamma$. Let us construct a distribution \mathcal{P} as in the proof of Theorem 4.2 but replacing $\text{Char } \mathcal{V}$ by Γ . This defines a local trivialization of the principal bundle $\mathbf{q} : M \rightarrow M/G$, $\tau : \mathbf{q}^{-1}(\bar{U}) \rightarrow \bar{U} \times G$, where $\bar{U} \subset M/G$ is an open subset. Using this coordinate system on M , we have $\widehat{\mathcal{V}} = \mathcal{V} \oplus \Gamma = \mathcal{P} \oplus \Gamma$ and hence $d\mathbf{q}(\widehat{\mathcal{V}}) = d\mathbf{q}(\mathcal{V}) = d\mathbf{q}(\mathcal{P}) = \mathcal{P}$, since $d\mathbf{q}$ acts as the identity on \mathcal{P} and hence $\mathcal{P} = \widehat{\mathcal{V}}$. Thus, we can write $\widehat{\mathcal{V}} = \bar{\mathcal{V}} \oplus \Gamma$. From Theorem 4.9, $\bar{\mathcal{V}}$ is a control system of the form (19) with quotient map \mathbf{q} of the form (18). Suppose there is an extended static feedback transformation $\varphi : M/G \rightarrow J^\sigma(\mathbb{R}, \mathbb{R}^s)$ such that $\varphi_* \bar{\mathcal{V}} = \mathcal{C}(\sigma)$. Then by Theorem 3.11, we conclude that

$$\{\partial_v\} \subseteq \text{Char } \bar{\mathcal{V}}_0^{(1)} \quad \text{and} \quad d\bar{t} \in \text{ann Char } \bar{\mathcal{V}}^{(k-1)},$$

where $\bar{t} = (\pi_1 \circ \tau)(\mathbf{p})$ and k is the derived length of $\bar{\mathcal{V}}$ which agrees with the derived length of $\widehat{\mathcal{V}}$ by Theorem 4.3. Here $\pi_1 : \mathbb{R} \times X \times U \rightarrow \mathbb{R}$ is a projection onto the first factor and $\mathbf{p} \in \mathbb{R} \times X \times U$ is a typical point. Thinking of the local trivialization τ from the active point of view, we have $d\bar{t}(\tau_* \Gamma) = d(\tau^* \bar{t})(\Gamma) = dt(\Gamma) = 0$ and hence $d\bar{t} \in \text{ann}(\text{Char } \bar{\mathcal{V}}^{(k-1)} \oplus \tau_* \Gamma) = \text{ann}(\tau_* \text{Char } \widehat{\mathcal{V}}^{(k-1)})$ and thus $dt \in \text{ann Char } \widehat{\mathcal{V}}^{(k-1)}$ which is item (2).

We deduce $\{\partial_v\} = \tau_* \{\partial_u\} \subseteq \text{Char } \bar{\mathcal{V}}_0^{(1)} \oplus \tau_* \Gamma = \tau_* \text{Char } \widehat{\mathcal{V}}_0^{(1)}$ from which item (1) follows.

Conversely, suppose (1) and (2) hold. Since $\widehat{\mathcal{V}}$ is a relative Goursat bundle, by Theorem 4.9, there is a local diffeomorphism $\varphi : M/G \rightarrow J^\sigma(\mathbb{R}, \mathbb{R}^q)$ such that $\varphi_*(\mathcal{V}/G) = \mathcal{C}(\sigma)$, some integer q , and signature σ . With τ the local trivialization and hypothesis (1), we have $\tau_* \{\partial_u\} \subseteq \tau_* \text{Char } \widehat{\mathcal{V}}_0^{(1)}$ which implies $\{\partial_v\} \subseteq \text{Char } \bar{\mathcal{V}}_0^{(1)}$. Since $(\pi_1 \circ \tau)(\mathbf{p}) = \bar{t}$, from (2) we deduce that $d\bar{t} \in \text{ann } \tau_* \text{Char } \widehat{\mathcal{V}}^{(k-1)}$ which implies that $dt \in \text{ann Char } \widehat{\mathcal{V}}^{(k-1)}$. By Theorem 3.11, we conclude that φ can be chosen to be an extended static feedback transformation. \square

5. Examples. In this section we give three examples to illustrate the implementation of the foregoing results and highlight particular features.

5.1. Example 1: Nonflat control systems with flat subsystems. We begin with a simple example that illustrates most of the constructions. Furthermore, it exemplifies a nonflat control system that has a flat quotient and we comment on its role in trajectory generation. We work through the constructions in this example in some detail. For background on flat control systems, see, [21].

While the generic control system is not flat, there are a few results that provide us with classes of *provably* nonflat control systems with more than a single input. One source of 2-input nonflat control systems is described by the following result.

THEOREM 5.1 (see [27]). *A driftless system $\dot{x} = f_1(x)u_1 + f_2(x)u_2$ in n states x and two inputs u is flat if and only if the elements of the derived flag of $E = \{f_1, f_2\}$ satisfy*

$$\dim E^{(k)} = \dim E^{(k-1)} + 1, \quad E^{(0)} = E, \quad k = 1, \dots, n-2.$$

A particularly elegant class of provably nonflat control systems in 5 states and 2 controls can be constructed by taking E to be a generic 2-plane distribution on

\mathbb{R}^5 , characterized by growth vector [2, 3, 5]. Such distributions have been classified [9]—the *Cartan systems*. The most symmetric of these has local realization $E = \{\partial_{x_1} + x_3\partial_{x_2} + x_5\partial_{x_3} + x_5^2\partial_{x_4}, \partial_{x_5}\} = \{f_1, f_2\}$. The control system corresponding to the driftless system of Theorem 5.1 is therefore

$$(20) \quad \mathcal{V} = \left\{ \partial_t + u_1 f_1 + u_2 f_2, \partial_{u_1}, \partial_{u_2} \right\}.$$

Because of its nonflatness it is guaranteed that the trajectories of \mathcal{V} cannot be expressed in integration-free terms. Thus, we investigate the role that symmetry reduction may play in identifying *flat subsystems* of (20). A goal will be to describe all the trajectories of \mathcal{V} in a way that involves the “least” quadrature.

It turns out that the Lie algebra of control admissible symmetries Γ of \mathcal{V} is 14-dimensional acting on the space of states and controls. Lie algebra Γ is isomorphic to the symmetry algebra $\text{Sym}(E)$ of E which acts on \mathbb{R}^5 . For instance, one of the 14 elements forming a basis of Γ is

$$\frac{1}{3}x_1^3\partial_{x_2} + x_1^2\partial_{x_3} + (-x_2 + x_3x_1)\partial_{x_4} + x_1\partial_{x_5} + u_1\partial_{u_2}.$$

In this case there is a 5-dimensional subalgebra of state-space symmetries $\mathfrak{g} \subset \Gamma$ given by $\mathfrak{g} = \{X_1, \dots, X_5\}$, where

$$X_1 = \frac{1}{2}x_1^2\partial_{x_2} + x_1\partial_{x_3} + 2x_3\partial_{x_4} + \partial_{x_5}, \\ X_2 = x_1\partial_{x_2} + \partial_{x_3}, \quad X_3 = \partial_{x_4}, \quad X_4 = \partial_{x_1}, \quad X_5 = \partial_{x_2}.$$

Let us, for instance, consider the quotient of $(\mathbb{R}^8, \mathcal{V})$ by the local Lie transformation group G_0 generated by the abelian subalgebra $\Gamma_0 = \{X_1, X_3\}$.

Invoking Theorem 4.5, we check that the refined derived type of $\widehat{\mathcal{V}} := \mathcal{V} \oplus \Gamma_0$ is $\mathfrak{d}_r(\widehat{\mathcal{V}}) = [[5, 2], [7, 4, 5], [8, 8]]$ which satisfies the constraints of Proposition 3.5 with signature $\sigma = \text{decel } \widehat{\mathcal{V}} = \langle 1, 1 \rangle$. The only nontrivial fundamental bundle is $\Xi_0^{(1)}/\Xi^{(1)}$ while calculation shows that this is equal to $\{dt\}$ and $\Xi_0^{(1)}$ is integrable. These observations guarantee, by Definition 4.1, that $\widehat{\mathcal{V}}$ is a relative Goursat bundle of signature $\sigma = \langle 1, 1 \rangle$. By Theorem 4.5, the quotient \mathcal{V}/G_0 is locally diffeomorphic to the partial prolongation $\mathcal{C}(1, 1)$. That is, the control system defined by (20) can be put into the Brunovsky normal form

$$\dot{x}_0^1 = v_1, \\ \dot{x}_0^2 = x_1^2, \quad \dot{x}_1^2 = v_2,$$

by some local diffeomorphism of the ambient manifold. However, this local diffeomorphism may not be a static feedback transformation. We turn to Theorem 4.12 to check for the existence of a static feedback linearization taking \mathcal{V}/G_0 to $\mathcal{C}(1, 1)$. We find that (with derived length $k = 2$), $dt \notin \Xi^{(1)}(\widehat{\mathcal{V}}) := \text{ann Char } \widehat{\mathcal{V}}^{(1)}$ and hence no static feedback transformation exists; cf. Theorem 4.12, item (2). In summary, we have the following facts, so far, concerning control system (20):

- System (20) is not flat by Theorem 5.1.
- The quotient \mathcal{V}/G_0 of (20) is linearizable but not via a static feedback transformation according to Theorem 4.12.

It turns out that \mathcal{V}/G_0 is flat. To see this, prolong \mathcal{V} to get $\text{pr } \mathcal{V} = \{\partial_t + u_1 f_1 + u_2 f_2 + v\partial_{u_1}, \partial_v, \partial_{u_2}\}$. This is equivalent to augmenting the control system by the equation $\dot{u}_1 = v$. Again, as per Theorem 4.12, we study the augmented distribution $\text{pr } \widehat{\mathcal{V}} :=$

$\text{pr } \mathcal{V} \oplus \mathbf{\Gamma}_0$ and calculate the refined derived type to be $\mathfrak{d}_r(\text{pr } \widehat{\mathcal{V}}) = [[5, 2], [7, 4, 4], [9, 9]]$. This satisfies item 1 of Definition 3.7 with signature $\sigma = \text{decel}(\text{pr } \widehat{\mathcal{V}}) = \langle 0, 2 \rangle$. Since the final entry of σ is greater than 1, we complete the check that $\text{pr } \widehat{\mathcal{V}}$ is a relative Goursat bundle by verifying that its resolvent bundle is integrable. Indeed, we get $\text{Char pr } \widehat{\mathcal{V}}^{(1)} = \{\partial_{x_4}, \partial_{u_2}, \partial_{v_1}, X_1\}$ and calculate that the rank 6 resolvent bundle is integrable. Therefore we deduce that the prolongation $\text{pr } \mathcal{V}$ of \mathcal{V} has a static feedback linearizable quotient by G_0 . This settles the recognition problem for the existence of a static feedback linearizable quotient by G_0 . Let us now implement the static feedback linearization of $\text{pr } \mathcal{V}/G_0$ and the reconstruction theorem for the solutions of \mathcal{V} . The Lie transformation group G_0 generated by $\mathbf{\Gamma}_0$ acts freely on \mathbb{R}^9 . The first integrals of $\mathbf{\Gamma}_0$ are spanned by $\mathbf{\Gamma}_0 = \{t, u_1, u_2, x_1, x_1x_3 - 2x_2, x_1^2x_5 - 2x_2, v\}$ and determine the quotient map $\mathbf{q} : \mathbb{R}^9 \rightarrow \mathbb{R}^9/G_0$. With $w_1 = x_1$, $w_2 = (x_1x_3 - 2x_2)/x_1$, $w_3 = (x_1^2x_5 - 2x_2)/x_1^2$, we obtain $\bar{\mathcal{V}} := \text{pr } \mathcal{V}/G_0$ with

$$(21) \quad \bar{\mathcal{V}} = \left\{ \partial_t + u_1 \partial_{w_1} - \frac{u_1}{w_1} (w_3 w_1 - 2w_2) \partial_{w_2} + \frac{1}{w_1^2} (u_2 w_1^2 - 2u_1 w_2) \partial_{w_3} + v \partial_{u_1}, \partial_v, \partial_{u_2} \right\}$$

as the quotient by the G_0 -action. We implement procedure **Contact** [35] (see also [36]) to construct the feedback linearization of $\bar{\mathcal{V}}$ (and hence its integral submanifolds). Computing the singular bundle of $\bar{\mathcal{V}}^{(1)}/\text{Char } \bar{\mathcal{V}}^{(1)}$ leads to the integrable resolvent $\{\partial_v, \partial_{u_1}, \partial_{u_2}, \partial_{w_3}\}$ whose annihilator $\{dt, dw_1, dw_2\}$ shows that we can take w_1, w_2 as the fundamental functions of highest order 2. These are the functions that generate the static feedback linearization upon successive differentiation by the total differential operator \mathbf{Z} , the first element of $\bar{\mathcal{V}}$. Procedure **Contact** is described and its proof of correctness is given in [35]. It shows the components of the static feedback linearization are determined by

$$(22) \quad x_0^1 = w_1, \quad x_0^2 = w_2, \quad x_1^1 = \mathbf{Z}x_0^1, \quad x_1^2 = \mathbf{Z}x_0^2, \quad x_2^1 = \mathbf{Z}x_1^1, \quad x_2^2 = \mathbf{Z}x_1^2.$$

Since the x_j^a are contact coordinates then there are arbitrary functions $f^1(t), f^2(t)$ such that

$$x_j^a = \frac{d^j f^a}{dt^j}, \quad j = 0, 1, 2, \quad a = 1, 2,$$

describe all the integral submanifolds of the Brunovsky normal form $\mathcal{C}\langle 0, 2 \rangle$. It follows that solving (22) for $w_1, w_2, w_3, u_1, u_2, v$ determines the integral submanifolds of $\text{pr } \mathcal{V}$. Thereby we obtain the explicit formulas

$$(23) \quad \begin{aligned} w_1 &= f^1, \quad w_2 = f^2, \quad w_3 = \frac{2\dot{f}^1 f^2 + f^1 \dot{f}^2}{f^1 \dot{f}^1}, \quad u_1 = \dot{f}^1, \quad v = \ddot{f}^1, \\ u_2 &= \frac{1}{f^1 (\dot{f}^1)^2} \left(2(\dot{f}^1)^2 \dot{f}^2 - f^1 (\ddot{f}^1 \dot{f}^2 - \dot{f}^1 \ddot{f}^2) \right). \end{aligned}$$

Thus the w_1, w_2, w_3, u_1 , and u_2 components of the above solution of $\bar{\mathcal{V}}$ are the solutions $\bar{s} : \mathbb{R} \rightarrow \mathbb{R}^8/G_0$ of \mathcal{V}/G_0 . Next we wish to implement the reconstruction theorem, Theorem 2.7, in which for each solution of \mathcal{V}/G_0 we determine a solution of \mathcal{V} by solving an ODE of Lie type. Since the symmetry group G_0 that we are dividing by is abelian, we expect these ODEs to reduce to quadrature.

For this we require the explicit action of G_0 and a local section $\sigma : \mathbb{R}^8/G_0 \rightarrow \mathbb{R}^8$. From the quotient map $\mathbf{q} : \mathbb{R}^8 \rightarrow \mathbb{R}^8/G_0$ we easily compute σ :

$$x_1 = w_1, \quad x_2 = -\frac{1}{2} w_1 w_2, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = w_3 - \frac{w_2}{w_1}, \quad u_1 = u_1, \quad u_2 = u_2,$$

and the G_0 -action is

$$\mu_g(t, x, u) = \left(t, x_1, x_2 + \frac{1}{2}\varepsilon_1 x_1^2, x_3 + \varepsilon_1 x_1, x_4 + \varepsilon_2 + \varepsilon_1^2 x_1 + 2\varepsilon_1 x_3, x_5 + \varepsilon_1, u_1, u_2 \right),$$

where $\varepsilon_1, \varepsilon_2$ are coordinates on G_0 . The reconstruction theorem now amounts to letting the ε_i depend upon the parameter t along the integral curves of \mathcal{V}/G_0 , forming the trial solution

$$s(t) = \mu(g(t), \sigma \circ \bar{s}(t))$$

and deducing the ODE for the components of $g(t) = (\varepsilon_1(t), \varepsilon_2(t))$ by forming the equation $s^* \omega = 0$, where $\omega = \text{ann } \mathcal{V}$. We obtain

$$(24) \quad \frac{d\varepsilon_1}{dt} = \frac{\dot{F}_1}{F_2^2}, \quad \frac{d\varepsilon_2}{dt} = \frac{\dot{F}_1^2}{\dot{F}_2 F_2^2},$$

where $F_1 = f^1 f^2$, $F_2 = f^1$. The controls u_a were recorded above. The states x_i are given by

$$x_1 = F_2, \quad x_2 = \frac{1}{2}(\varepsilon_1 F_2^2 - F_1), \quad x_3 = \varepsilon_1 F_2, \quad x_4 = \varepsilon_1^2 F_2 + \varepsilon_2, \quad x_5 = \varepsilon_1 + \frac{\partial_t(F_1 F_2)}{F_2^2 \dot{F}_2}.$$

Hence two quadratures are required to determine all the trajectories. Taking a slightly different approach to the linearization, this can be reduced to a *single* quadrature. Recall that f^1 and f^2 are arbitrary functions of t . This is the minimum possible quadrature for this control system.

Finally let us explicitly identify the feedback linearizable subsystem of the prolongation $\text{pr } \omega$ of ω in this example. Setting $\text{pr } \bar{\omega} = \text{ann } \bar{\mathcal{V}}$, we have

$$\text{pr } \omega \supset \mathbf{q}^* \text{pr } \bar{\omega} = \left\{ \omega^1, \quad \omega^3 - \frac{2}{x_1} \omega^2, \quad \omega^5 - \frac{2}{x_1^2} \omega^2, \quad du_1 - v dt \right\}$$

is a subsystem of $\text{pr } \omega = \{\omega_1, \dots, \omega_5, du_1 - v dt\} = \text{ann } \text{pr } \mathcal{V}$ which, as shown, projects to a static feedback linearizable control system on M/G_0 , $\mathbf{q}: M \rightarrow M/G_0$ being the quotient map.

Much more challenging examples having the same structure as (20) can be obtained by choosing the Cartan system E to be less symmetric. The symmetry group of any Cartan system E is a subgroup of the exceptional simple Lie group G_2 which has dimension 14. For instance, the Cartan system

$$E = \{f_1 = \partial_{x_1} + x_3 \partial_{x_2} + 2v \partial_{x_3} + vx_5 \partial_{x_4}, \quad f_2 = \partial_{x_5}\},$$

where

$$v = x_1 x_2 x_3 x_4 x_5$$

has a 3-dimensional symmetry group. The corresponding driftless control system

$$(25) \quad \mathcal{F} = \left\{ \partial_t + u_1 f_1 + u_2 f_2, \quad \partial_{u_1}, \quad \partial_{u_2} \right\}$$

has a 3-dimensional Lie group of control symmetries whose Lie algebra is $\Delta = \{X_1, X_2, X_3\}$, where

$$\begin{aligned} X_1 &= x_2 \partial_{x_2} + x_3 \partial_{x_3} - x_5 \partial_{x_5} - u_2 \partial_{u_2}, \\ X_2 &= 2x_1 \partial_{x_1} - x_2 \partial_{x_2} - 3x_3 \partial_{x_3} - 3x_4 \partial_{x_4} + 2u_1 \partial_{u_1}, \\ X_3 &= -\frac{1}{x_4} \partial_{x_4} + \frac{x_5}{x_4^2} \partial_{x_5} - \frac{1}{x_4^2} (2x_1 x_2 x_3 x_5^3 u_1 - u_2) \partial_{u_2}. \end{aligned}$$

Explicitly, the trajectories of \mathcal{F} satisfy

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_1 x_3, \quad \dot{x}_3 = 2u_1 v, \quad \dot{x}_4 = u_1 v x_5, \quad \dot{x}_5 = u_2.$$

Interestingly, for the above control system the subalgebra of pure state-space symmetries is trivial. This exemplifies the reason for our introduction of control admissible symmetries (Definition 4.8). The subalgebra $\Delta_0 = \{X_1, X_2\}$ is abelian and we find again that \mathcal{F}/H_0 is locally equivalent to $\mathcal{C}\langle 1, 1 \rangle$, where H_0 is the Lie transformation group generated by Δ_0 . After prolongation we find a static feedback equivalence to $\mathcal{C}\langle 0, 2 \rangle$ which permits us to implement the reconstruction theorem. As before the trajectory generation problem is reducible to two quadratures though in this case the formulas for the solutions of $\bar{\mathcal{V}}$ are much more complicated.

5.2. The Heisenberg system. We refer to the control system

$$(26) \quad \ddot{x} = \frac{1}{\Delta} ((1+x^2)u_1 + xyu_2), \quad \ddot{y} = \frac{1}{\Delta} ((1+y^2) + xy) u_1, \quad \ddot{z} = \frac{1}{\Delta} (yu_1 - xu_2),$$

where $\Delta = 1 + x^2 + y^2$, as described in [4, p. 30, (1.8.3)]. With $x_1 = \dot{x}$, $y_1 = \dot{y}$, $z_1 = \dot{z}$, the distribution corresponding to (26) is

$$(27) \quad \mathcal{V} = \left\{ \partial_t + x_1 \partial_x + y_1 \partial_y + z_1 \partial_z + \frac{1}{\Delta} \left(((1+x^2)u_1 + xyu_2) \partial_{x_1} + u_1 ((1+y^2) + xy) \partial_{y_1} + (yu_1 - xu_2) \partial_{z_1} \right), \partial_{u_1}, \partial_{u_2} \right\}.$$

It's refined derived type is $\mathfrak{d}_r(\mathcal{V}) = [[3, 0], [5, 2, 2], [7, 2, 2], [9, 9]]$, proving that it is not equivalent to any Brunovsky form (Proposition 3.5). Furthermore, after experimenting with various prolongations, we conclude that \mathcal{V} does not obviously prolong to a static feedback linearizable system. Hence, we study its symmetries and symmetry reductions with the goal of reducing the construction of its trajectories to quadrature, if possible. Calculation obtains that the infinitesimal control symmetries of \mathcal{V} consist of the 2-dimensional abelian Lie algebra spanned by

$$\Gamma = \{t\partial_z + \partial_{z_1}, \quad \partial_z\}.$$

We deduce that $\hat{\mathcal{V}} = \mathcal{V} \oplus \Gamma$ is a relative Goursat bundle with signature $\langle 0, 2 \rangle$. It can also be checked that $\hat{\mathcal{V}}$ satisfies the hypotheses of Theorem 4.12 and hence the quotient \mathcal{V}/G is static feedback equivalent to Brunovsky form $\mathcal{C}\langle 0, 2 \rangle$, that is, \mathcal{V}/G is static feedback linearizable. In this case the quotient system has a local expression given by \mathcal{V} with the ∂_z and ∂_{z_1} components omitted. Again invoking procedure **Contact** [35] we obtain the integral submanifolds $\bar{s}: \mathbb{R} \rightarrow M/G$ of $(M/G, \mathcal{V}/G)$,

$$(28) \quad \begin{aligned} x = f, \quad x_1 = \dot{f}, \quad y = g, \quad y_1 = \dot{g}, \quad u_1 &= \frac{\ddot{g}(f^2 + g^2 + 1)}{fg + g^2 + 1}, \\ u_2 &= -\frac{\ddot{g}(f^2 + 1)(f^2 + g^2 + 1)}{fg(fg + g^2 + 1)} + \frac{\ddot{f}}{fg}, \end{aligned}$$

where f, g are arbitrary smooth functions of t . The action on M is easily computed,

$$(29) \quad \mathbf{x}' = \mu(\mathbf{x}, \varepsilon) = (t, x, y, z + t\varepsilon_1 + \varepsilon_2, z_1 + \varepsilon_1, u_1, u_2) \quad \forall (\varepsilon_1, \varepsilon_2) \in G.$$

The quotient map \mathbf{q} and a local cross section σ are conveniently given by

$$\mathbf{q}(\mathbf{x}) = (t, x, x_1, y, y_1, u_1, u_2)$$

and

$$\sigma(t, x, x_1, y, y_1, u_1, u_2) = (t, x, x_1, y, y_1, 0, 0, u_1, u_2).$$

Allowing $t \mapsto \varepsilon(t) = (\varepsilon_1(t), \varepsilon_2(t))$ to be a smooth curve in G , we form (according to Theorem 2.7), the trial solution $s(t) = \mu(\varepsilon(t), \sigma \circ \bar{s}(t))$ and find that $s^* \omega = 0$ if and only if the curve in G satisfies the system of Lie type given by

$$(30) \quad \dot{\varepsilon}_1 = \rho, \quad \dot{\varepsilon}_2 = -t\rho,$$

where

$$\rho = \frac{\ddot{g}(f^2 + g^2 + 1)}{g(fg + g^2 + 1)} - \frac{\ddot{f}}{g}.$$

For any choice of curve $t \mapsto (x(t), y(t)) = (f, g)$ in the xy -plane, the trajectories of \mathcal{V} are

$$\begin{aligned} x = f, \quad y = g, \quad x_1 = \dot{f}, \quad y_1 = \dot{g}, \quad z = t\varepsilon_1 + \varepsilon_2, \quad z_1 = \varepsilon_1, \quad u_1 = \frac{\ddot{g}(f^2 + g^2 + 1)}{fg + g^2 + 1}, \\ u_2 = -\frac{\ddot{g}(f^2 + 1)(f^2 + g^2 + 1)}{fg(fg + g^2 + 1)} + \frac{\ddot{f}}{fg}, \end{aligned}$$

where $\varepsilon(t)$ solves (30) subject to $\varepsilon(0) = (0, 0) = \text{id}_G$. In this way, the representation of the trajectories of the Heisenberg system (27) is reduced to two quadratures. In [37] it is shown how this quadrature can in fact be eliminated.

5.3. The underactuated ship. We briefly comment on the relation between existence of static feedback linearizable subsystems and trajectory planning for a control system closely related to the one we considered in our introduction. A frequently studied control system for the guidance of marine vessels is given by (see [20])

$$(31) \quad \begin{aligned} \dot{x} &= u \cos \theta - v \sin \theta, & \dot{y} &= u \sin \theta + v \cos \theta, & \dot{\theta} &= r, \\ \dot{u} &= \frac{1}{\gamma_1} vr - \beta_1 u + u_1, & \dot{v} &= -\gamma_1 ur - \beta_2 v, & \dot{r} &= \gamma_2 uv - \beta_3 r + u_2, \end{aligned}$$

where

$$\gamma_1 = \frac{m_{11}}{m_{22}}, \quad \gamma_2 = \frac{m_{11} - m_{22}}{I_{33}}.$$

In fact this is a more sophisticated model of ship guidance than the one presented in the introduction but as we shall see, it is closely related to it. The parameters m_{ii} and I_{33} are the components of the diagonal mass-inertia tensor. The β_j quantify hydrodynamic drag. A crucial parameter is γ_2 . It can be shown that if $\gamma_2 = 0$ then this system has the same reduction as we found for (2) when $\gamma = 1$. Namely, we can express θ in the form (5) but with the parameter β replaced by β_2 . In the case $\gamma_2 \neq 0$, we obtain (3) but again with β_2 replacing β . Hence, as in our opening example (2), we can view θ as a quasi-flat output for (31) as we did for (2). The flatness or otherwise of (31), according to whether γ_2 is zero or otherwise is reflected in the structure of its Lie group of control symmetries. If $\gamma_2 \neq 0$ we find that the Lie algebra of control admissible symmetries Γ is 4-dimensional and solvable. Then if ω is the Pfaffian system associated with (31), it can be shown that for all subalgebras $\mathfrak{h} \subset \Gamma$ such that $\dim \mathfrak{h} > 1$ then ω/H is static feedback linearizable, where H is the Lie transformation group generated by Lie algebra \mathfrak{h} . It turns out that the symmetry

reduction by a certain 3-dimensional subalgebra of $\mathbf{\Gamma}$ leads to the aforementioned equation for θ , namely, (3).

For $\gamma_2 = 0$, system (31) is flat and we discover that the Lie algebra of infinitesimal control admissible symmetries $\mathbf{\Gamma}_0$ is 8-dimensional and has quite a different structure from that of $\mathbf{\Gamma}$. Lie algebra $\mathbf{\Gamma}_0$ is isomorphic to $\mathfrak{sl}(2) \oplus_{\mathfrak{s}} \mathfrak{s}$, where \mathfrak{s} is a solvable Lie algebra of dimension 5. This can be compared to $\mathbf{\Gamma}$ which is a solvable Lie algebra of dimension 4. Interestingly, precisely the same dichotomy arises in the control symmetries of (2). Lie algebra $\mathbf{\Gamma}_0$ has a 4-dimensional abelian subalgebra and gives rise to the formula (5) via symmetry reduction. Thus the solvability properties of (2), and reflected in (31), have their counterparts in the control admissible symmetries and corresponding symmetry reductions. The corresponding trajectory planning problem is divided into cases depending upon parameter ranges for the γ_i .

6. Conclusion and open questions. The main contribution of this paper is the introduction of control admissible symmetries and consequent formulation of a coordinate-free approach to the identification and construction of static feedback linearizable subsystems $\alpha \subset \omega$ of a given intrinsically nonlinear control system ω that is invariant under the action of a Lie group, G , and the application of this to trajectory generation. Control symmetries are a natural generalization of state-space symmetries since they constitute the maximal class of static feedback self-equivalences and are essential for apprehending the invariance properties of control systems. We have shown that once a linearizable subsystem α has been identified, then the representations of the trajectories of ω can be expressed as the composition of the trajectories of α together with those of an ODE system of Lie type, the latter fact arising from the reconstruction theorem of Anderson and Fels [2]. A general goal has been to minimize the extent to which the practitioner is required to solve differential equations in order to obtain an explicit representation of the trajectories of intrinsically nonlinear G -invariant systems, ω . In general, if G is *solvable* then any required differential equations can be reduced to quadrature. In [37] the geometry of the concomitant Lie system is analyzed to reveal further properties and applications.

Numerous interesting questions have arisen in the course of our study. For instance, given a control system with symmetry, will there always be a flat or else static feedback linearizable subsystem? We have observed this to be the case in all the examples we have examined to date. We conjecture that if a control system is feedback linearizable then the Lie transformation group of control symmetries will be infinite dimensional and otherwise finite dimensional.⁴ Furthermore, will reconstruction over $(M/G, \bar{\omega})$ always account for *all* trajectories of ω , as we have observed in all examples so far? Can the control symmetries or symmetry reductions of flat or dynamic feedback linearizable systems be characterized? How can the Lie structure of the Lie system evolving along the fibers of the principal bundle $\mathbf{q} : M \rightarrow M/G$ be best exploited, for instance, for building controllers or for motion planning? These matters are, of course, only a sample of those we hope to better understand in due course.

Acknowledgments. The research in this paper was greatly facilitated by the use of the Maple symbolic processing package `DifferentialGeometry` [3]. We thank the anonymous referees for their close reading of the manuscript and suggestions which led to significant improvements.

⁴The set of *all* symmetries of a linearizable control system is infinite dimensional. It is isomorphic to the Lie pseudogroup of contact transformations.

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