

**THE CAUCHY PROBLEM ON LARGE TIME FOR SURFACE
WAVES TYPE BOUSSINESQ SYSTEMS II**

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ABSTRACT. This paper is a continuation of a previous work by two of the Authors [36] on long time existence for Boussinesq systems modeling the propagation of long, weakly nonlinear water waves. We provide proofs on examples not considered in [36] in particular we prove a long time well-posedness result for a delicate "strongly dispersive" Boussinesq system.

1. INTRODUCTION

One aim of this paper is to complete the results obtained in a previous paper [36] on the Cauchy theory for some (a,b,c,d) Boussinesq systems for surface water waves

$$(1.1) \quad \begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon \nabla \cdot (\eta \mathbf{u}) + \mu [a \nabla \cdot \Delta \mathbf{u} - b \Delta \eta_t] = 0 \\ \mathbf{u}_t + \nabla \eta + \epsilon \frac{1}{2} \nabla |\mathbf{u}|^2 + \mu [c \nabla \Delta \eta - d \Delta \mathbf{u}_t] = 0. \end{cases}$$

Here μ and ϵ are the small parameters (shallowness and nonlinearity parameters respectively) defined as

$$\mu = \frac{h^2}{\lambda^2}, \quad \epsilon = \frac{\alpha}{h}$$

where α is a typical amplitude of the wave, h a typical depth and λ a typical horizontal wavelength.

In the Boussinesq regime, ϵ and μ are supposed to be of same order, $\epsilon \sim \mu \ll 1$, and we will take for simplicity $\epsilon = \mu$, writing (1.1) as

$$(1.2) \quad \begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon [\nabla \cdot (\eta \mathbf{u}) + a \nabla \cdot \Delta \mathbf{u} - b \Delta \eta_t] = 0 \\ \mathbf{u}_t + \nabla \eta + \epsilon [\frac{1}{2} \nabla |\mathbf{u}|^2 + c \nabla \Delta \eta - d \Delta \mathbf{u}_t] = 0, \end{cases}$$

The class of systems (1.1), (1.2) models water waves on a flat bottom propagating in both directions in the aforementioned regime (see [6, 7, 5]). We will focus here on the *strongly dispersive* case, corresponding to particular choices of the modeling parameters (a,b,c,d) (see below).

One could also derive similar systems with a non trivial bathymetry (non flat bottom), see [13], and one has then to distinguish between the case when the bottom varies slowly and the case where it is strongly varying. In the former case, (1.2) has to be slightly modified and becomes

$$(1.3) \quad \begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon [\nabla \cdot ((\eta - \beta) \mathbf{u}) + a \nabla \cdot \Delta \mathbf{u} - b \Delta \eta_t] = 0 \\ \mathbf{u}_t + \nabla \eta + \epsilon [\frac{1}{2} \nabla |\mathbf{u}|^2 + c \nabla \Delta \eta - d \Delta \mathbf{u}_t] = 0, \end{cases}$$

where β is a smooth function on \mathbb{R}^d , $d = 1, 2$, bounded together with its derivatives. In this case, the results in [36] and those of the present paper extend easily. In the

second case one gets much more complicated systems [13]. We refer to [32] for long time existence results in this case.

Recall (see [6, 15]) that the modeling parameters are constrained by the relation

$$a + b + c + d = \frac{1}{3} - \tau,$$

where $\tau \geq 0$ is the surface tension parameter (Bond number).

Recall also [6] that (1.2) is linearly well-posed when

$$a \leq 0, c \leq 0, b \geq 0, d \geq 0,$$

and when

$$a = c, b \geq 0, d \geq 0.$$

An important step to justify rigorously (1.2) as an asymptotic model for water waves is to establish the well-posedness of the Cauchy problem on time scales of order $1/\epsilon$, with uniform bounds in suitable Sobolev spaces, the error estimate being then (see [5, 29]).

$$\|U_{\text{Boussinesq}} - U_{\text{Euler}}\| = O(\epsilon^2 t)$$

in suitable Sobolev norms.

This step has been established in [36] (see also [34]) for most of Boussinesq systems with and without surface tension. The idea in [36] is to find an appropriate symmetrization of the system and this is not a straightforward task since one cannot obviously use the classical symmetrizer of the underlying Saint-Venant (shallow water) hyperbolic system. This will be reviewed in the first section of this paper. A complete proof of cases that were not fully developed in [36] will be given in Section 4.¹

Introducing surface tension enlarges the range of physically admissible parameters (a,b,c,d) and so even a local theory² for a few linearly well-posed systems is still missing, for instance the cases $b = d = 0, a < 0, c = 0$ and $b = d = 0, a = 0, c < 0$. Both cases will be considered here but the later leads to serious difficulties and the long time existence for it is the main result of the present paper.

Note also that the (linearly well-posed) "exceptional KdV-KdV" case $b = d = 0, a = c > 0$ which is studied in [30] leading to well-posedness on time scales of order $1/\sqrt{\epsilon}$ in Sobolev spaces $H^s(\mathbb{R}^2), s > 3/2$ which are larger than the "hyperbolic" one $H^s(\mathbb{R}^2), s > 2$ is not covered neither in [36] nor in the present paper so that a long time existence is still open in this case³.

An important mathematical issue concerning Boussinesq systems (1.2) is that despite they describe the same dynamics of water waves, their *mathematical* properties are rather different, due essentially to their different linear dispersion relations. Of course those dispersion relations all coincide in the long wave limit but there are quite different in the short wave limit. A convenient way to classify the system is according to the order of the Fourier multiplier operator given by the eigenvalues of the linearized operator (see [6]). The order can be $-1, 0, 1, 2$ or 3 . The two last cases are referred to as the *strongly dispersive* ones.

¹Due to the large number of cases to be considered, we chose in [36] to give complete proofs for a limited number of cases.

²That is not taking care of the dependence of the lifespan of the solution with respect to ϵ

³However, the case $b = d = 0, a < 0, c < 0$ that can only occur with a strong surface tension is covered by the theory in [36].

After a brief review of our previous results, we will consider in the third section the ("local") Cauchy problem for two strongly dispersive Boussinesq systems of *Schrödinger type*, namely $b = d = c = 0, a < 0$ and $a = b = d = 0, c < 0$, two situations that are admissible in case of strong surface tension and that have not been considered before. In the first case, it turns out that the local (that is on time scales of order $1/\sqrt{\epsilon}$) Cauchy theory can be obtained by "elementary" energy methods on the original formulation, as in the purely gravity waves cases $a < 0, c < 0, b = 0, d > 0$ or $a < 0, c < 0, b > 0, d = 0$ considered in [30]. On the other hand, the second case $a = b = d = 0, c < 0$, leads to serious difficulties that are explained in this section.

In the fourth section we provide detailed proofs (not given in [36]) for the long time well-posedness of the strongly dispersive case $b = d = c = 0, a < 0$ and for two systems which can be viewed as *weakly dispersive*, namely $b > 0, a = c = d = 0$ and $d > 0, a = b = c = 0$. We conclude this section by establishing long time existence for the difficult case $a = b = d = 0, c < 0$ by a quasilinearization method quite different from the other cases. As was aforementioned this is the main result of the present paper (see Theorems 4.6 and 4.7). We explain first how to get the needed a priori estimates, the complete proof being given in the next section.

Finally we show in Section 6 that the symmetrization method can be used to obtain long time existence results for a fifth order Boussinesq system and we briefly allude to possible extensions to nonlocal *Full dispersion* Boussinesq type systems.

During the completion of the present paper we were informed of the very interesting paper [11] where an alternative proof of long time existence for most of the Boussinesq systems is provided (excluding the "strongly dispersive" ones $b = d = 0$, thus the "difficult case" $a = b = d = 0, c < 0$). This proof also relaxes the non-cavitation condition on the initial data η_0 .

We were also informed by Vincent Duchêne of the article [18] which contains in the one-dimensional case (see Appendix A) results related to ours in Subsection 4.4.

Notations. We will denote $|\cdot|_p$ the norm in the Lebesgue space $L^p(\mathbb{R})$, $1 \leq p \leq \infty$ and $\|\cdot\|_s$ the norm in the Sobolev space $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$. $(\cdot|\cdot)_2$ denotes the scalar product in L^2 . We will denote \hat{f} or $\mathcal{F}(f)$ the Fourier transform of a tempered distribution f . For any $s \in \mathbb{R}$, we define $|D|^s f$ by its Fourier transform $\widehat{|D|^s f}(\xi) = |\xi|^s \hat{f}(\xi)$. We also denote $|D_x|^s f = \mathcal{F}^{-1}(|\xi_1|^s \hat{f})$ and $|D_y|^s f = \mathcal{F}^{-1}(|\xi_2|^s \hat{f})$. Finally we will denote $\Lambda = (I - \Delta)^{1/2}$ and $J_\epsilon = (I - \epsilon\Delta)^{1/2}$.

2. A REVIEW OF LONG TIME WELL-POSED BOUSSINESQ SYSTEMS

As recalled previously, in order to fully justify the Boussinesq systems, one needs to prove the well-posedness of the Cauchy problem on time scales of order at least $O(1/\epsilon)$ (together with the relevant uniform bounds). This would be achieved of course if one could obtain the *global well-posedness* (also with uniform bounds). This is only known however for a very limited number of Boussinesq systems in one-dimension. A first idea would be to use appropriate conservation laws, but contrary to the *one-directional or quasi one-directional* equations such as the Korteweg-de Vries or the Kadomtsev-Petviashvili equations which are derived in the same regime, the Boussinesq systems do not possess the two invariants (L^2 norm and energy) that provide useful a priori bounds.

Nevertheless, when $b = d$, the Boussinesq systems are endowed with an Hamiltonian structure. More precisely, denoting by J the skew adjoint matrix operator

$$J = \begin{pmatrix} 0 & \partial_x(I - \epsilon b \Delta)^{-1} & \partial_y(I - \epsilon b \Delta)^{-1} \\ \partial_x(I - \epsilon b \Delta)^{-1} & 0 & 0 \\ \partial_y(I - \epsilon b \Delta)^{-1} & 0 & 0 \end{pmatrix},$$

and

$$U = \begin{pmatrix} \eta \\ \mathbf{u} \end{pmatrix},$$

the Boussinesq systems write in this case

$$\partial_t U = -J(\text{grad } H_\epsilon)(U),$$

where $H_\epsilon(U)$ is the Hamiltonian given by

$$H_\epsilon(U) = \frac{1}{2} \int_{\mathbb{R}^2} (-c\epsilon |\nabla \eta|^2 - a\epsilon |\nabla \mathbf{u}|^2 + \eta^2 + |\mathbf{u}|^2 + \epsilon \eta |\mathbf{u}|^2) dx dy,$$

so that $H_\epsilon(U)$ is conserved by the flow. This can be used (see [7]) in the one dimensional case where $b = d > 0, a \leq 0, c \leq 0$ or $b = d > 0, a = 0, c < 0$ to establish the global well-posedness of the corresponding Boussinesq systems provided $H_\epsilon(\eta_0, u_0)$ is small enough and the non cavitation condition $\inf_x(1 + \epsilon \eta_0(x)) > 0$ is satisfied. The proof uses in a crucial way the fact that $b = d > 0$ and the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ thus it does not work in two dimensions.

Another one-dimensional situation leading to global well-posedness is when $a = b = c = 0, d > 0$. Then Amick and Schonbeck [4, 38] use the underlying hyperbolic structure of the shallow-water (Saint-Venant) system to get a priori bounds stemming from an entropy functional. This allows to prove the global well-posedness under the condition $\inf_{x \in \mathbb{R}}(1 + \epsilon \eta_0(x)) > 0$ ⁴ but again the extension of this result to the two-dimensional case is unclear. We will prove in this case the large time existence in Section 4.

As far as long time results are concerned, it has been claimed in [36] that the Boussinesq systems (1.2) are well-posed in a suitable Sobolev setting (with uniform bounds) on time scales of order $1/\epsilon$ in the following cases:

- (1) $b > 0, d = 0, a, c < 0$;
- (2) $b > 0, d = 0, a = 0, c < 0$;
- (3) $b = 0, d > 0, a, c < 0$;
- (4) $b \neq d, b, d > 0, a, c < 0$ or $b = 0, d > 0, a = 0, c < 0$;
- (5) $b \neq d, b, d > 0, a = 0, c < 0$;
- (6) $b = d > 0, a, c < 0$ or $b > 0, d = 0, a < 0, c = 0$;
- (7) $b > 0, d = 0, a = c = 0$ or $b = d > 0, a = 0, c < 0$;
- (8) $b, d > 0, a < 0, c = 0$ or $b = 0, d > 0, a = c = 0$;
- (9) $b = 0, d > 0, a < 0, c = 0$;
- (10) $b, d > 0, a = c = 0$;
- (11) $b = d = 0, a, c < 0$.

Note that the last case can occur only in case of a strong surface tension, as the two following that were not considered in [36] :

- (12) $b = d = 0, a = 0, c < 0$;

⁴Contrary to what was claimed in [36], the results in [4, 38] do not need a smallness assumption on the initial data.

$$(13) \quad b = d = 0, a < 0, c = 0.$$

Actually, the same scheme of proof (by symmetrization) is used in [36] but because of the many different cases to be dealt with (the technical details cannot be treated in an unified way), we only provided a complete proof in [36] for cases (4) ("generic case"), (1) and (11), which are "strongly dispersive". The other cases can be dealt with by similar symmetrization techniques but the proofs for some of them need more explanations that we detail below.

3. SOME STRONGLY DISPERSIVE BOUSSINESQ SYSTEMS

We will study here the local well-posedness (that is on time scales of order $1/\sqrt{\epsilon}$) of the Cauchy problem for two strongly dispersive Boussinesq systems having an order two dispersion. They occur only for capillary-gravity waves when the surface tension parameter is greater than $1/3$. Two (purely gravity waves) systems having also an order two dispersion corresponding respectively to $a < 0, c < 0, b = 0, d > 0$ and $a < 0, c < 0, d = 0, b > 0$ have been studied in [30] under a curl free condition in the later case. The local well-posedness on time scale of order $1/\sqrt{\epsilon}$ was proven there while well-posedness on time scales of order $1/\epsilon$ is established in [36] together with the appropriate uniform bounds. As in [30] we will use somehow the dispersive properties of the systems which allows to enlarge the space of resolution but will not provide existence on the "long" time scale $1/\epsilon$ which will be considered in Section 4.

Those systems will be referred to as "Schrödinger type" since in space dimension two their dispersion relations for large frequencies are reminiscent of the Schrödinger one (in one dimension, the analogy is with the Benjamin-Ono equation). This will be made clear when rewriting the systems in an equivalent form after diagonalizing the linear part.

3.1. A first Schrödinger type system. We consider the Boussinesq systems when $a < 0, b = c = d = 0$, a case which occurs for capillary surface waves with strong enough surface tension, $\tau > 1/3$ and which was not considered in [36]. One can obviously restrict to the case where $a = -1$ and we consider first the one-dimensional system

$$(3.1) \quad \begin{cases} \eta_t + u_x + \epsilon(u\eta)_x - \epsilon u_{xxx} = 0, \\ u_t + \eta_x + \epsilon u u_x = 0. \end{cases}$$

Note that this system has the hamiltonian structure

$$\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} + J \text{grad } H_\epsilon(\eta, u) = 0$$

where

$$J = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}$$

and

$$H_\epsilon(\eta, u) = \frac{1}{2} \int_{\mathbb{R}} (\epsilon u_x^2 + \eta^2 + u^2 + \epsilon u^2 \eta) dx.$$

Unfortunately the formal conservation of the Hamiltonian cannot be used to get a global $L^2 \times H^1$ bound.

As for other order two Boussinesq systems (see [7, 30]) one can solve the local Cauchy problem for (3.1) by "elementary" energy methods.

For $U = (\eta, u)^T$ we define

$$\|U\|_{X_\epsilon^s} = \left(\int_{\mathbb{R}} (\eta^2 + u^2 + |D_x^s \eta|^2 + |D_x^s u|^2 + \epsilon |D_x^{s+1} u|^2) dx \right)^{1/2}.$$

Theorem 3.1. *Let $s > 1/2$ and $(\eta_0, u_0) \in X_\epsilon^s$. There exists $T_\epsilon = O(1/\sqrt{\epsilon})$ and a unique solution $(\eta, u) \in C([0, T_\epsilon]; X_\epsilon^s)$ of (3.1) with initial data (η_0, u_0) .*

Proof. In order to restrict the technicalities, we consider only the case $s = 1$ and we derive only the suitable a priori estimates. The complete proof would use various Kato-Ponce type commutator estimates and an approximation argument.⁵

We take successively the L^2 scalar product of the first equation in (3.1) by $\eta - \eta_{xx}$ and of the second equation by $(I - \epsilon \partial_x^2)(u - u_{xx})$. After several integrations by parts we obtain by adding the resulting equations :

$$(3.2) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\eta^2 + u^2 + \eta_x^2 + (1 + \epsilon)u_x^2 + \epsilon u_{xx}^2) dx \\ = -\epsilon \int_{\mathbb{R}} \left[\frac{1}{2} u_x \eta^2 + \frac{3}{2} u_x \eta_x^2 + \eta \eta_x u_{xx} + \frac{1 + \epsilon}{2} u_x^3 + \frac{5\epsilon}{2} u_x u_{xx}^2 \right] dx. \end{aligned}$$

We now use Hölder inequality, the standard inequality $|u|_\infty \lesssim |u|_2^{1/2} |u_x|_2^{1/2}$ and that

$$|\eta|_\infty, |\eta|_2, |\eta_x|_2, |u_x|_2, \sqrt{\epsilon} |u_x|_\infty, \sqrt{\epsilon} |u_{xx}|_2 \lesssim \|U\|_{X_\epsilon^1}$$

to obtain from (3.2) that $\|U\|_{X_\epsilon^1} \leq C$ on the maximal existence time interval $[0, T_\epsilon]$ of the ODE

$$y' \leq C\sqrt{\epsilon} y^2$$

and one readily checks that $T_\epsilon = O(\frac{1}{\sqrt{\epsilon}})$.

This leads to the existence of a weak solution $U \in L^\infty(0, T_\epsilon; X_\epsilon^1)$ (with an uniform H^1 bound).

To prove uniqueness, we set $N = \eta_1 - \eta_2$, $V = u_1 - u_2$ where (η_1, u_1) , (η_2, u_2) are two solutions in $C([0, T_\epsilon]; X_\epsilon^s)$. Thus

$$(3.3) \quad \begin{cases} N_t + V_x + \epsilon[(V\eta_1)_x + (u_2 N)_x] - \epsilon V_{xxx} = 0, \\ V_t + N_x + \epsilon[Vu_{1x} + u_2 V_x] = 0. \end{cases}$$

One takes the L^2 product scalar of the first equation by N and successively the L^2 scalar product of the second equation by V and $-\epsilon V_{xx}$. Adding the resulting equalities we obtain

$$(3.4) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (N^2 + V^2 + \epsilon V_x^2) dx \\ = \int_{\mathbb{R}} \{ -\epsilon[(V\eta_1)_x N + (u_2 N)_x N] - \epsilon[V^2 u_{1x} + u_2 V V_x] \\ + \epsilon^2(V V_{xx} u_{1x} + u_2 V_x V_{xx}) \} dx, \end{aligned}$$

⁵We will treat the general situation in the two-dimensional case.

from which we deduce

$$\begin{aligned}
(3.5) \quad & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (N^2 + V^2 + \epsilon V_x^2) dx \\
& \leq C \epsilon [|\eta_1|_{\infty} |V_x|_2 |N|_2 + |V|_2^{1/2} |V_x|_2^{1/2} |N|_2 |\eta_{1x}|_2 \\
& \quad + |u_{2x}|_{\infty} |N|_2^2 + |u_{1x}|_{\infty} |V|_2^2 + |u_{2x}|_{\infty} |V|_2^2 \\
& \quad + \epsilon ((|u_{1x}|_{\infty} + |u_{2x}|_{\infty}) |V_x|_2^2 + |V|_2^{1/2} |V_x|_2^{3/2} |u_{1xx}|_2)],
\end{aligned}$$

so that

$$(3.6) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (N^2 + V^2 + \epsilon V_x^2) dx \leq C (|N|_2^2 + |V|_2^2 + \epsilon |V_x|_2^2)$$

and $N = V = 0$ by Gronwall's lemma.

It remains to prove the strong continuity in time of the solution with value in X_{ϵ}^1 and the continuity of the flow, but this results from the Bona-Smith approximation procedure [9]. \square

The local well-posedness of the Cauchy problem (3.1) for data of low regularity weighted Sobolev spaces was proven in [25] by analogy with the DNLS equation. We indicate now another possible method to obtain the local well-posedness of (3.1) in a different functional setting by reducing it to a system of Benjamin-Ono type equations. A natural idea is to transform (3.1) by diagonalizing the dispersive part. We denote

$$\widehat{A}(\xi) = i\xi \begin{pmatrix} 0 & 1 + \epsilon|\xi|^2 \\ 1 & 0 \end{pmatrix}.$$

the Fourier transform of the dispersion matrix with eigenvalues $\pm i\xi(1 + \epsilon|\xi|^2)^{1/2}$. In what follows we will denote $J_{\epsilon} = (I - \epsilon \partial_x^2)^{1/2}$.

Setting

$$U = \begin{pmatrix} \eta \\ u \end{pmatrix}$$

and

$$W = \begin{pmatrix} \zeta \\ v \end{pmatrix} = P^{-1}U, \quad P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & J_{\epsilon} \\ 1 & -J_{\epsilon} \end{pmatrix},$$

the linear part of (3.1) is diagonalized as

$$W_t + \partial_x DW = 0,$$

where

$$D = \begin{pmatrix} J_{\epsilon} & 0 \\ 0 & -J_{\epsilon} \end{pmatrix}.$$

Since $U = PW$, where

$$P = \begin{pmatrix} 1 & 1 \\ J_{\epsilon}^{-1} & -J_{\epsilon}^{-1} \end{pmatrix}.$$

one can therefore reduce (3.1) to the equivalent form

$$(3.7) \quad \begin{cases} \zeta_t + J_{\epsilon} \zeta_x + \frac{\epsilon}{2} N_1(\zeta, v) = 0, \\ v_t - J_{\epsilon} v_x + \frac{\epsilon}{2} N_2(\zeta, v) = 0. \end{cases}$$

where

$$N_1(\zeta, v) = \partial_x [(\zeta + v) J_{\epsilon}^{-1} (\zeta - v)] + J_{\epsilon} [J_{\epsilon}^{-1} (\zeta - v) J_{\epsilon}^{-1} (\zeta_x - v_x)]$$

and

$$N_2(\zeta, v) = \partial_x[(\zeta + v)J_\epsilon^{-1}(\zeta - v)] - J_\epsilon[J_\epsilon^{-1}(\zeta - v)J_\epsilon^{-1}(\zeta_x - v_x)]$$

Since

$$(1 + \epsilon\xi^2)^{1/2} - \epsilon^{1/2}|\xi| = \frac{1}{(1 + \epsilon\xi^2)^{1/2} + \epsilon^{1/2}|\xi|},$$

(3.7) writes

$$(3.8) \quad \begin{cases} \zeta_t + \epsilon^{1/2}\mathcal{H}\zeta_{xx} + R_\epsilon\zeta + \frac{\epsilon}{2}N_1(\zeta, v) = 0, \\ v_t - \epsilon^{1/2}\mathcal{H}v_{xx} - R_\epsilon v + \frac{\epsilon}{2}N_2(\zeta, v) = 0. \end{cases}$$

where R_ϵ is the (order zero) skew-adjoint operator with symbol $\frac{i\xi}{(1 + \epsilon\xi^2)^{1/2} + \epsilon^{1/2}|\xi|}$.

Note that the nonlinear term are similar but in a sense nicer than the quadratic term uu_x of the Benjamin-Ono equation and thus we should apply for instance the method Ponce [35] used to solve the Cauchy problem for the Benjamin-Ono equation, that is the dispersive estimates on the group $e^{it\partial_x\mathcal{H}}$, since this method does not used the specific structure of the nonlinear term in the Benjamin-Ono equation. This would imply local well-posedness for $(\zeta_0, v_0) \in H^s(\mathbb{R})^2$, $s \geq 3/2$, which corresponds to $(\eta_0, u_0) = (\zeta_0 + v_0, J_\epsilon^{-1}(\zeta_0 - v_0)) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$. Note the difference with the functional setting of Theorem 3.1. Similarly, it is likely that the method in [22] which leads to a local well-posedness theory in $H^s(\mathbb{R})$, $s > 9/8$ for the Benjamin-Ono equation can be applied to (3.1) leading to a $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 9/8$ theory. Also the new method in [33] could lead to the resolution of the Cauchy problem in the energy space $H^{1/2}(\mathbb{R})$ in the (ζ, v) variables. Those methods would however not enlarge the $O(1/\sqrt{\epsilon})$ lifespan.⁶

In the two-dimensional case, the system writes

$$(3.9) \quad \begin{cases} \eta_t + \nabla \cdot u + \epsilon \nabla \cdot (\eta \mathbf{u}) - \epsilon \nabla \cdot \Delta \mathbf{u} = 0, \\ \mathbf{u}_t + \nabla \eta + \frac{\epsilon}{2} \nabla |\mathbf{u}|^2 = 0. \end{cases}$$

This system has also the Hamiltonian structure

$$\partial_t \begin{pmatrix} \eta \\ \mathbf{u} \end{pmatrix} + J \text{grad } H_\epsilon(\eta, \mathbf{u}) = 0$$

where

$$J = \begin{pmatrix} 0 & \partial_x & \partial_y \\ \partial_x & 0 & 0 \\ \partial_y & 0 & 0 \end{pmatrix}.$$

and

$$H(\eta, \mathbf{u}) = \frac{1}{2} \int_{\mathbb{R}^2} (\epsilon |\nabla \mathbf{u}|^2 + \eta^2 + |\mathbf{u}|^2 + \epsilon |\mathbf{u}|^2 \eta) dx dy.$$

Under a curl free assumption on \mathbf{u} (which is natural since the Boussinesq systems are derived for potential flows), one can obtain the local well-posedness of (3.9) by

⁶On the other hand, Tao's method [39] which leads to a $H^1(\mathbb{R})$ well-posedness theory for the Benjamin-Ono equation uses a gauge transform which strongly relies on the specific structure of the Benjamin-Ono equation and its generalization to (3.1) is problematic.

"elementary" methods. Actually, when $\text{curl } \mathbf{u} = 0$, (3.9) becomes

$$(3.10) \quad \begin{cases} \partial_t \eta + \nabla \cdot \mathbf{u} + \epsilon \nabla \cdot (\eta \mathbf{u}) - \epsilon \nabla \cdot \Delta \mathbf{u} = 0, \\ \partial_t \mathbf{u} + \nabla \eta + \epsilon \mathbf{u} \cdot \nabla \mathbf{u} = 0. \end{cases}$$

Before going further, we present the following commutator estimates (see Theorems 3 and 6 in [28]).

Lemma 3.2. *Let $t_0 > \frac{n}{2}$, $-t_0 < r \leq t_0 + 1$. Then for all $s \geq 0$, $f \in H^{t_0+1} \cap H^{s+r}(\mathbb{R}^n)$ and $u \in H^{s+r-1}(\mathbb{R}^n)$, there holds:*

$$(3.11) \quad |[\Lambda^s, f]u|_{H^r} \leq C(|\nabla f|_{H^{t_0}} |u|_{H^{s+r-1}} + \langle |\nabla f|_{H^{s+r-1}} |u|_{H^{t_0}} \rangle_{s > t_0+1-r}),$$

where $a + \langle b \rangle_{s > s_0}$ equals a if $s \leq s_0$ while equals $a + b$ if $s > s_0$.

Consequently, taking $t_0 = s > 1$ and $r = 0$ in (3.11), we have the following corollary.

Corollary 3.3. *For $s > 1$, $f \in H^{s+1}(\mathbb{R}^2)$, $g \in H^{s-1}(\mathbb{R}^2)$, then*

$$(3.12) \quad |[\Lambda^s, f]g|_2 \lesssim |\nabla f|_{H^s} |g|_{H^{s-1}}.$$

Going back to (3.10), similarly to the one-dimensional case, we denote by $U = (\eta, \mathbf{u})^T$, and then define

$$(3.13) \quad \|U\|_{X_\epsilon^s} = (|\Lambda^s \eta|_2^2 + |\Lambda^s \mathbf{u}|_2^2 + \epsilon |\Lambda^s \nabla \mathbf{u}|_2^2)^{\frac{1}{2}},$$

and we obtain the following theorem.

Theorem 3.4. *Let $s > 1$ and $(\eta_0, \mathbf{u}_0) \in X_\epsilon^s$. Then there exists $T_\epsilon = O(1/\sqrt{\epsilon})$ and a unique solution $(\eta, \mathbf{u}) \in C([0, T_\epsilon]; X_\epsilon^s)$ of (3.10) with initial data (η_0, \mathbf{u}_0) . Moreover,*

$$\sup_{t \in [0, T_\epsilon]} \|(\eta(\cdot, t), \mathbf{u}(\cdot, t))\|_{X_\epsilon^s} < c \|(\eta_0, \mathbf{u}_0)\|_{X_\epsilon^s}.$$

Proof. As in the one-dimensional case we will only provide the suitable a priori estimate.

Taking the L^2 inner product of the first equation in (3.10) by $\Lambda^{2s} \eta$ and of the second equation by $(1 - \epsilon \Delta) \Lambda^{2s} \mathbf{u}$, and then integrating by parts, it results

$$(3.14) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (|\Lambda^s \eta|_2^2 + |\Lambda^s \mathbf{u}|_2^2 + \epsilon |\Lambda^s \nabla \mathbf{u}|_2^2) \\ & = -\epsilon (\Lambda^s \nabla \cdot (\eta \mathbf{u}) | \Lambda^s \eta)_2 - \epsilon (\Lambda^s (\mathbf{u} \cdot \nabla \mathbf{u}) | (1 - \epsilon \Delta) \Lambda^s \mathbf{u})_2, \end{aligned}$$

Now, we deal with the r.h.s terms in (3.14). We first get that

$$(3.15) \quad \begin{aligned} & (\Lambda^s \nabla \cdot (\eta \mathbf{u}) | \Lambda^s \eta)_2 = (\Lambda^s (\mathbf{u} \cdot \nabla \eta) | \Lambda^s \eta)_2 + (\Lambda^s (\eta \nabla \cdot \mathbf{u}) | \Lambda^s \eta)_2 \\ & = ([\Lambda^s, \mathbf{u}] \cdot \nabla \eta | \Lambda^s \eta)_2 - \frac{1}{2} (\nabla \cdot \mathbf{u} \Lambda^s \eta | \Lambda^s \eta)_2 + (\Lambda^s (\eta \nabla \cdot \mathbf{u}) | \Lambda^s \eta)_2, \end{aligned}$$

which together with (3.12) implies that

$$(3.16) \quad \begin{aligned} & |(\Lambda^s \nabla \cdot (\eta \mathbf{u}) | \Lambda^s \eta)_2| \\ & \lesssim |[\Lambda^s, \mathbf{u}] \cdot \nabla \eta|_2 |\Lambda^s \eta|_2 + \|\nabla \cdot \mathbf{u}\|_\infty |\Lambda^s \eta|_2^2 + |\Lambda^s (\eta \nabla \cdot \mathbf{u})|_2 |\Lambda^s \eta|_2 \\ & \lesssim |\nabla \mathbf{u}|_{H^s} |\nabla \eta|_{H^{s-1}} |\eta|_{H^s} + \|\nabla \mathbf{u}\|_{H^s} |\eta|_{H^s}^2 \lesssim |\nabla \mathbf{u}|_{H^s} |\eta|_{H^s}^2. \end{aligned}$$

For the second term on the r.h.s of (3.14), we have

$$\begin{aligned}
& (\Lambda^s(\mathbf{u} \cdot \nabla \mathbf{u}) | (1 - \epsilon \Delta) \Lambda^s \mathbf{u})_2 = ([\Lambda^s, \mathbf{u}] \cdot \nabla \mathbf{u} | \Lambda^s \mathbf{u})_2 \\
& \quad - \frac{1}{2} (\nabla \cdot \mathbf{u} \Lambda^s \mathbf{u} | \Lambda^s \mathbf{u})_2 + \epsilon \sum_{i=1}^2 (\Lambda^s \nabla (u_i \partial_i \mathbf{u}) | \Lambda^s \nabla \mathbf{u})_2 \\
(3.17) \quad & = ([\Lambda^s, \mathbf{u}] \cdot \nabla \mathbf{u} | \Lambda^s \mathbf{u})_2 - \frac{1}{2} (\nabla \cdot \mathbf{u} \Lambda^s \mathbf{u} | \Lambda^s \mathbf{u})_2 \\
& \quad + \epsilon \sum_{i=1}^2 ([\Lambda^s, u_i] \nabla \partial_i \mathbf{u} | \Lambda^s \nabla \mathbf{u})_2 - \frac{1}{2} \epsilon (\nabla \cdot \mathbf{u} \Lambda^s \nabla \mathbf{u} | \Lambda^s \nabla \mathbf{u})_2 \\
& \quad + \epsilon \sum_{i=1}^2 (\Lambda^s (\nabla u_i \partial_i \mathbf{u}) | \Lambda^s \nabla \mathbf{u})_2
\end{aligned}$$

which along with (3.12) implies that

$$(3.18) \quad |(\Lambda^s(\mathbf{u} \cdot \nabla \mathbf{u}) | (1 - \epsilon \Delta) \Lambda^s \mathbf{u})_2| \lesssim |\nabla \mathbf{u}|_{H^s} (|\mathbf{u}|_{H^s}^2 + \epsilon |\nabla \mathbf{u}|_{H^s}^2).$$

Denoting again $U = (\eta, \mathbf{u})^T$, we deduce from (3.14), (3.16) and (3.18) that

$$(3.19) \quad \frac{d}{dt} \|U(t)\|_{X_\epsilon^s} \leq C \sqrt{\epsilon} \|U(t)\|_{X_\epsilon^s}^2,$$

from which, we infer that the maximal existence time interval is $[0, T_\epsilon]$ with $T_\epsilon = O(1/\sqrt{\epsilon})$.

As in the one-dimensional case one justifies the a priori estimates by a suitable approximation of the system (for instance by adding $(-\delta \Delta \eta_t, -\delta \Delta \mathbf{u}_t)^T$, $\delta > 0$). Uniqueness is obtained again by a Gronwall type argument and the strong continuity in time and the continuity of the flow map result from the Bona-Smith trick. \square

As in the one-dimensional case, one has a better insight on the system by diagonalizing the linear part. The dispersion matrix is in Fourier variables

$$\widehat{A}(\xi_1, \xi_2) = i \begin{pmatrix} 0 & \xi_1(1 + \epsilon|\xi|^2) & \xi_2(1 + \epsilon|\xi|^2) \\ \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \end{pmatrix}.$$

The corresponding eigenvalues are zero and

$$\lambda_\pm = \pm i |\xi| (1 + \epsilon |\xi|^2)^{1/2}$$

with corresponding eigenvectors

$$\begin{aligned}
E_0 &= \begin{pmatrix} 0 \\ -\frac{\xi_2}{|\xi|} (1 + \epsilon |\xi|^2)^{-1/2} \\ \frac{\xi_1}{|\xi|} (1 + \epsilon |\xi|^2)^{-1/2} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 \\ \frac{\xi_1}{|\xi|} (1 + \epsilon |\xi|^2)^{-1/2} \\ \frac{\xi_2}{|\xi|} (1 + \epsilon |\xi|^2)^{-1/2} \end{pmatrix}, \\
\text{and } E_2 &= \begin{pmatrix} -1 \\ \frac{\xi_1}{|\xi|} (1 + \epsilon |\xi|^2)^{-1/2} \\ \frac{\xi_2}{|\xi|} (1 + \epsilon |\xi|^2)^{-1/2} \end{pmatrix}.
\end{aligned}$$

Now we set $J_\epsilon = (I - \epsilon\Delta)^{1/2}$ and R_1, R_2 the Fourier multiplier operator with respective symbols $i\xi_1/|\xi|, i\xi_2/|\xi|$.

We also denote

$$P = \begin{pmatrix} 0 & i & -i \\ -R_2J_\epsilon^{-1} & R_1J_\epsilon^{-1} & R_1J_\epsilon^{-1} \\ R_1J_\epsilon^{-1} & R_2J_\epsilon^{-1} & R_2J_\epsilon^{-1} \end{pmatrix}$$

and

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 0 & 2R_2J_\epsilon & -2R_1J_\epsilon \\ -i & -R_1J_\epsilon & -R_2J_\epsilon \\ i & -R_1J_\epsilon & -R_2J_\epsilon \end{pmatrix}.$$

Setting $U = (\eta, \mathbf{u})^T$ and $V = (\zeta, \mathbf{v})^T = P^{-1}U$, (3.9) writes as

$$U_t + AU + \epsilon N(U) = 0,$$

which is transformed after diagonalizing the linear part,

$$V_t + DV + \epsilon P^{-1}N(PV) = 0,$$

or, setting $P^{-1}N(PV) = \tilde{N}(V)$,

$$(3.20) \quad V_t + DV + \epsilon \tilde{N}(V) = 0,$$

where

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i(-\Delta)^{1/2}J_\epsilon & 0 \\ 0 & 0 & -i(-\Delta)^{1/2}J_\epsilon \end{pmatrix}$$

We turn now to the nonlinear part. N is given as a function of U by

$$N(U) = \begin{pmatrix} \nabla \cdot (\eta \mathbf{u}) \\ \frac{1}{2} \partial_x (|\mathbf{u}|^2) \\ \frac{1}{2} \partial_y (|\mathbf{u}|^2) \end{pmatrix} = \begin{pmatrix} \nabla \cdot (\eta \mathbf{u}) \\ \mathbf{u} \cdot \nabla u_1 \\ \mathbf{u} \cdot \nabla u_2 \end{pmatrix},$$

where we used the condition $\text{curl } \mathbf{u} = 0$ in the second equality.

On the other hand, $P^{-1}N(U)$ is given by

$$-\frac{1}{2} \begin{pmatrix} 0 \\ i\nabla \cdot (\eta \mathbf{u}) + R_1J_\epsilon(\mathbf{u} \cdot \nabla u_1) + R_2J_\epsilon(\mathbf{u} \cdot \nabla u_2) \\ -i\nabla \cdot (\eta \mathbf{u}) + R_1J_\epsilon(\mathbf{u} \cdot \nabla u_1) + R_2J_\epsilon(\mathbf{u} \cdot \nabla u_2) \end{pmatrix}.$$

To obtain the expression of $\tilde{N}(V)$, we should express (η, u_1, u_2) as

$$\eta = i(v_1 - v_2), \quad u_1 = -R_2J_\epsilon^{-1}\zeta + R_1J_\epsilon^{-1}(v_1 + v_2),$$

$$u_2 = R_1J_\epsilon^{-1}\zeta + R_2J_\epsilon^{-1}(v_1 + v_2),$$

and the nonlinearity is of the same type as in the one-dimensional case.

Remark 3.1. As in the case of the "KdV-KdV" system ($a = c = 1/6, b = d = 0$) studied in [30], it follows from our analysis that $\zeta = 0$ if ζ_0 is smooth enough, since $\partial_t \zeta = 0$. Moreover,

$$\zeta = 0 \quad \iff \quad R_2u_1 = R_1u_2 \quad \iff \quad \text{curl } \mathbf{u} = 0.$$

We observe that this condition makes sense, since our system is derived from the water waves equations in the irrotational case. Note that \mathbf{u} is the horizontal velocity at a certain height and it differs from the horizontal velocity at the free surface by an $O(\epsilon^2)$ term. Also, since the equation for \mathbf{u} writes $\partial_t \mathbf{u} = \nabla F$, the condition $\text{curl } \mathbf{u} = 0$ is preserved by the evolution.

Remark 3.2. The linear part in (3.20) is "Schrödinger like" for large frequencies (the symbol behaves as $\pm i\epsilon^{1/2}|\xi|^2$ as $|\xi| \rightarrow +\infty$), and "wave like" for small frequencies (the symbol behaves as $\pm i|\xi|$ when $|\xi| \rightarrow 0$).

The quadratic terms however involves order one operators and this one could a priori think of applying the results on the Cauchy problem for quasilinear Schrödinger type equations (see for instance [23]). Those methods however necessitate a high regularity on the data and it is unlikely that they could improve our local result.

3.2. A second Schrödinger type system. We consider here the situation where $a = b = d = 0, c < 0$, say $c = -1$ which again may occur only in the case of strong surface tension. It turns out that this system leads to serious difficulties, which are not present in the other Boussinesq systems, and that we will describe below, even to obtain the local well-posedness by "elementary" or more sophisticated methods using dispersion. We refer to Section 4 and 5 for the long time existence issues.

We consider first the one-dimensional system

$$(3.21) \quad \begin{cases} \eta_t + u_x + \epsilon(u\eta)_x = 0, \\ u_t + \eta_x + \epsilon uu_x - \epsilon\eta_{xxx} = 0. \end{cases}$$

The hamiltonian structure is now

$$\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} + J \text{grad } H_\epsilon(\eta, u) = 0$$

where

$$J = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}$$

and

$$H_\epsilon(\eta, u) = \frac{1}{2} \int_{\mathbb{R}} (\epsilon|\eta_x|^2 + \eta^2 + u^2 + \epsilon u^2 \eta) dx.$$

Similarly to the case $b = d > 0, a = 0, c = -1$ considered in [7] and for a related system in [10], one can use the formal conservation of H_ϵ to derive a global a priori estimate when $H_\epsilon(\eta_0, u_0)$ is small enough and $\inf_{x \in \mathbb{R}} (1 + \epsilon\eta_0(x)) > 0$. First we derive as in [7, 9] a L^∞ bound on η for solutions $(\eta, u) \in C([0, T]; H^1(\mathbb{R})) \times C([0, T]; L^2(\mathbb{R}))$ satisfying the above non-cavitation condition. Actually, one writes

$$(3.22) \quad \begin{aligned} \eta^2(x, t) &\leq \int_{\mathbb{R}} |\eta||\eta_x| dx = \frac{1}{\sqrt{\epsilon}} \int_{\mathbb{R}} \sqrt{\epsilon} |\eta||\eta_x| dx \leq \frac{1}{2\sqrt{\epsilon}} \int_{\mathbb{R}} (\eta^2 + \epsilon|\eta_x|^2) dx \\ &\leq \frac{1}{\sqrt{\epsilon}} |H_\epsilon(\eta, u)| = \frac{1}{\sqrt{\epsilon}} |H_\epsilon(\eta_0, u_0)| \quad t \in [0, T]. \end{aligned}$$

Using (3.22) and the conservation of H_ϵ imply a (formal) $H^1 \times L^2$ bound on (η, u) provided $H_\epsilon(\eta_0, u_0)$ is small enough, that is

$$(3.23) \quad H_\epsilon(\eta_0, u_0) < \epsilon^{-3/2}.$$

Unfortunately, and contrary to the case $b = d > 0, a = 0, c < 0$ (see [7]), one cannot use the above bounds to get a global well-posedness result, say by a compactness method applied to a regularization of the system. The obstruction is that one cannot pass to the limit of the term $\frac{1}{2}\partial_x(u^2)$ in the second equation only from a L^2 bound on u .

To obtain an equivalent "diagonal" system, we proceed as in the other "Schrödinger type system", setting now

$$\widehat{A}(\xi) = i\xi \begin{pmatrix} 0 & 1 \\ 1 + \epsilon|\xi|^2 & 0 \end{pmatrix}.$$

with the same eigenvalues $\pm i\xi(1 + \epsilon|\xi|^2)^{1/2}$.

In the Notation of the previous section, one has still

$$D = \begin{pmatrix} J_\epsilon & 0 \\ 0 & -J_\epsilon \end{pmatrix}.$$

and now

$$P = \begin{pmatrix} J_\epsilon^{-1} & -J_\epsilon^{-1} \\ 1 & 1 \end{pmatrix},$$

$$P^{-1} = \frac{1}{2} \begin{pmatrix} J_\epsilon & 1 \\ -J_\epsilon & 1 \end{pmatrix},$$

Setting again

$$U = \begin{pmatrix} \eta \\ u \end{pmatrix}$$

and

$$W = \begin{pmatrix} \zeta \\ v \end{pmatrix} = P^{-1}U,$$

one can therefore reduce (3.21) to the equivalent form

$$(3.24) \quad \begin{cases} \zeta_t + J_\epsilon \zeta_x + \frac{\epsilon}{2} N_1(\zeta, v) = 0, \\ v_t - J_\epsilon v_x + \frac{\epsilon}{2} N_2(\zeta, v) = 0. \end{cases}$$

where

$$N_1(\zeta, v) = \partial_x J_\epsilon [(\zeta + v) J_\epsilon^{-1} (\zeta - v)] + (\zeta + v)(\zeta + v)_x$$

and

$$N_2(\zeta, v) = -\partial_x J_\epsilon [(\zeta + v) J_\epsilon^{-1} (\zeta - v)] + (\zeta + v)(\zeta + v)_x.$$

We can also write (3.24) as

$$(3.25) \quad \begin{cases} \zeta_t + \epsilon^{1/2} \mathcal{H} \zeta_{xx} + R_\epsilon \zeta + \frac{\epsilon}{2} N_1(\zeta, v) = 0, \\ v_t - \epsilon^{1/2} \mathcal{H} v_{xx} - R_\epsilon v + \frac{\epsilon}{2} N_2(\zeta, v) = 0. \end{cases}$$

where again R_ϵ is the order zero skew-adjoint operator with symbol $\frac{i\xi}{(1+\epsilon\xi^2)^{1/2} + \epsilon^{1/2}|\xi|}$.

Note that the nonlinearity is worse than in the case $a = -1$ and even the local theory does not seem to be straightforward using this formulation.

We turn now to the two-dimensional case that is

$$(3.26) \quad \begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon \nabla \cdot (\eta \mathbf{u}) = 0, \\ \mathbf{u}_t + \nabla \eta + \frac{\epsilon}{2} \nabla |\mathbf{u}|^2 - \epsilon \nabla \Delta \eta = 0. \end{cases}$$

The Hamiltonian structure is now

$$\partial_t \begin{pmatrix} \eta \\ \mathbf{u} \end{pmatrix} + J \text{grad } H_\epsilon(\eta, \mathbf{u}) = 0$$

where

$$J = \begin{pmatrix} 0 & \partial_x & \partial_y \\ \partial_x & 0 & 0 \\ \partial_y & 0 & 0 \end{pmatrix}.$$

and

$$H(\eta, \mathbf{u}) = \frac{1}{2} \int_{\mathbb{R}^2} (\epsilon |\nabla \eta|^2 + \eta^2 + |\mathbf{u}|^2 + \epsilon |\mathbf{u}|^2 \eta) dx dy.$$

As in the one-dimensional case, one can express (3.21) on the equivalent form

$$(3.27) \quad V_t + DV + \tilde{N}(V) = 0,$$

where again $U = (\eta, \mathbf{u})^T$, $V = (\zeta, \mathbf{v})^T$,

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i(-\Delta)^{1/2} J_\epsilon & 0 \\ 0 & 0 & -i(-\Delta)^{1/2} J_\epsilon \end{pmatrix}$$

and $\tilde{N}(V)$ is expressed as

$$\frac{1}{2} \begin{pmatrix} 0 \\ -iJ_\epsilon[\partial_x(u_1\eta) + \partial_y(u_2\eta)] - \frac{1}{2}[R_1\partial_x|\mathbf{u}|^2 + R_2\partial_y|\mathbf{u}|^2] \\ iJ_\epsilon[\partial_x(u_1\eta) + \partial_y(u_2\eta)] - \frac{1}{2}[R_1\partial_x|\mathbf{u}|^2 + R_2\partial_y|\mathbf{u}|^2] \end{pmatrix}$$

with

$$\eta = iJ_\epsilon^{-1}(v_1 - v_2), \quad u_1 = -R_2\zeta + R_1(v_1 + v_2), \quad u_2 = R_1\zeta + R_2(v_1 + v_2).$$

3.3. Comparison between the two Schrödinger type systems. The previous considerations display the difficulties of the Cauchy problem in the case $a = b = d = 0, c < 0$. We indicate here how to reduce it to the case $b = d = c = 0, a < 0$ modulo $O(\epsilon^2)$ terms.

Let us consider for instance the one-dimensional case

$$(3.28) \quad \begin{cases} \eta_t + u_x + \epsilon(\eta u)_x = 0, \\ u_t + \eta_x + \epsilon u u_x - \epsilon \eta_{xxx} = 0. \end{cases}$$

Setting

$$\tilde{\eta} = (1 - \epsilon \partial_x^2) \eta = J_\epsilon^2 \eta,$$

(3.28) can be rewritten as follows :

$$(3.29) \quad \begin{cases} \tilde{\eta}_t + (1 - \epsilon \partial_x^2) u_x + \epsilon(\tilde{\eta} u)_x = \epsilon^2(2\eta_{xx} u_x + 3\eta_x u_{xx} + \eta u_{xxx}), \\ u_t + \tilde{\eta}_x + \epsilon u u_x = 0, \end{cases}$$

that is

$$(3.30) \quad \begin{cases} \tilde{\eta}_t + (1 - \epsilon \partial_x^2) u_x + \epsilon(\tilde{\eta} u)_x \\ = \epsilon^2(2(J_\epsilon^{-2} \tilde{\eta}_{xx}) u_x + 3(J_\epsilon^{-2} \tilde{\eta}_x) u_{xx} + (J_\epsilon^{-2} \tilde{\eta}) u_{xxx}), \\ u_t + \tilde{\eta}_x + \epsilon u u_x = 0, \end{cases}$$

Discarding the $O(\epsilon^2)$ terms, (3.29) reduces to

$$(3.31) \quad \begin{cases} \tilde{\eta}_t + (1 - \epsilon \partial_x^2) u_x + \epsilon(\tilde{\eta} u)_x = 0, \\ u_t + \epsilon \tilde{\eta}_x + \epsilon u u_x = 0. \end{cases}$$

which is exactly the case $b = c = d = 0, a = -1$.

Similarly, we can consider the two-dimensional case

$$(3.32) \quad \begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \epsilon \nabla \cdot (\eta \mathbf{u}) = 0, \\ \mathbf{u}_t + \nabla \eta + \frac{\epsilon}{2} \nabla (|\mathbf{u}|^2) - \epsilon \nabla \Delta \eta = \mathbf{0}. \end{cases}$$

Setting

$$\tilde{\eta} = (1 - \epsilon\Delta)\eta = J_\epsilon^2\eta,$$

(3.32) can be rewritten as follows :

$$(3.33) \quad \begin{cases} \tilde{\eta}_t + (1 - \epsilon\Delta)\nabla \cdot \mathbf{u} + \epsilon\nabla \cdot (\tilde{\eta}\mathbf{u}) = \epsilon^2(\Delta\nabla \cdot (\eta\mathbf{u}) - \nabla \cdot (\Delta\eta\mathbf{u})), \\ \mathbf{u}_t + \nabla\tilde{\eta} + \frac{\epsilon}{2}\nabla(|\mathbf{u}|^2) = 0, \end{cases}$$

that is

$$(3.34) \quad \begin{cases} \tilde{\eta}_t + (1 - \epsilon\Delta)\nabla \cdot \mathbf{u} + \epsilon\nabla \cdot (\tilde{\eta}\mathbf{u}) = \epsilon^2(\Delta\nabla \cdot (\mathbf{u}J_\epsilon^{-2}\tilde{\eta}) - \nabla \cdot (\mathbf{u}\Delta J_\epsilon^{-2}\tilde{\eta})), \\ \mathbf{u}_t + \nabla\tilde{\eta} + \frac{\epsilon}{2}\nabla(|\mathbf{u}|^2) = 0, \end{cases}$$

Discarding the $O(\epsilon^2)$ terms, (3.33) reduces to

$$(3.35) \quad \begin{cases} \tilde{\eta}_t + (1 - \epsilon\Delta)\nabla \cdot \mathbf{u} + \epsilon\nabla \cdot (\tilde{\eta}\mathbf{u}) = 0, \\ \mathbf{u}_t + \nabla\tilde{\eta} + \frac{\epsilon}{2}\nabla(|\mathbf{u}|^2) = \mathbf{0} \end{cases}$$

which is exactly the case $b = c = d = 0, a = -1$.

The bad structure of the nonlinear terms in (3.30), (3.34) (or (3.25), (3.27)) explain why solving the Cauchy problem for systems (3.21) or (3.26) is so difficult, despite their apparent simplicity. One could notice that we always lose one derivative for η or \mathbf{u} . Thus, to solve the case $a = b = d = 0, c < 0$, we turn to quasilinearize the system by applying ∂_t^k instead of the usual ∂_x^α . We shall discuss details in the following section.

4. LONG TIME EXISTENCE FOR SOME BOUSSINESQ SYSTEMS

We first give a complete proof for some systems considered in [36] (in particular the two-dimensional version of the system considered in [4, 38]) and apply the same symmetrization techniques to study the long time existence of solutions (in a smaller Sobolev space) of one of the "Schrödinger type systems" described in the previous section. We then consider the more delicate case $a = b = d = 0, c < 0$.

We associate to (1.2) the initial data

$$(4.1) \quad \eta|_{t=0} = \eta_0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0.$$

4.1. The case $a = c = d = 0, b > 0$ with condition $\text{curl } \mathbf{u} = 0$. Before going further, we state some technical lemmas and definitions.

Definition 4.1. For any $s \in \mathbb{R}, k \in \mathbb{N}, \epsilon \in (0, 1)$, the Banach space $X_{\epsilon^k}^s(\mathbb{R}^n)$ is defined as $H^{s+k}(\mathbb{R}^n)$ equipped with the norm:

$$|u|_{X_{\epsilon^k}^s}^2 = |u|_{H^s}^2 + \epsilon^k |u|_{H^{s+k}}^2$$

Lemma 4.2. For any $i, k \in \mathbb{N}$ and $0 < i < k$, there holds the following interpolation inequality:

$$(4.2) \quad \epsilon^{\frac{i}{2}} |f|_{H^{s+i}} \lesssim |f|_{H^s}^{1-\frac{i}{k}} (\epsilon^{\frac{k}{2}} |f|_{H^{s+k}})^{\frac{i}{k}} \lesssim |f|_{X_{\epsilon^k}^s}.$$

Theorem 4.3. *Let $b > 0, a = c = d = 0$. $n = 1, 2$, $s > 1 + \frac{n}{2}$. Assume that $\eta_0 \in X_{\epsilon^2}^s(\mathbb{R}^n), \mathbf{u}_0 \in X_{\epsilon}^s(\mathbb{R}^n)$ with $\text{curl } \mathbf{u}_0 = 0$ when $n = 2$, satisfy the (non-cavitation) condition*

$$(4.3) \quad 1 + \epsilon\eta_0 \geq H > 0, \quad H \in (0, 1),$$

Then there exists a constant \tilde{c}_0 such that for any $\epsilon \leq \epsilon_0 = \frac{1-H}{\tilde{c}_0(|\eta_0|_{X_{\epsilon^2}^s} + |\mathbf{u}_0|_{X_{\epsilon}^s})}$, there exists $T > 0$ independent of ϵ , such that (1.2)-(4.1) has a unique solution $(\eta, \mathbf{u})^T$ with $\eta \in C([0, T/\epsilon]; X_{\epsilon^2}^s(\mathbb{R}^n))$ and $\mathbf{u} \in C([0, T/\epsilon]; X_{\epsilon}^s(\mathbb{R}^n))$. Moreover,

$$(4.4) \quad \max_{t \in [0, T/\epsilon]} (|\eta|_{X_{\epsilon^2}^s} + |\mathbf{u}|_{X_{\epsilon}^s}) \leq \tilde{c} (|\eta_0|_{X_{\epsilon^2}^s} + |\mathbf{u}_0|_{X_{\epsilon}^s}).$$

Here $\tilde{c} = C(H^{-1})$ and $\tilde{c}_0 = C(H^{-1})$ are nondecreasing functions of their argument. And in what follows, without confusion, we denote $\tilde{c} = C(H^{-1})$ a nondecreasing constant depending on H^{-1} . Otherwise, we denote \tilde{c}_i ($i=0, 1, 2, \dots$) constants having the same properties as \tilde{c} .

Proof. The proof follows the same method used in [36], that is to obtain energy estimates on a suitable symmetrized linearized system followed by an iterative scheme. Here we only give the *a priori estimates* on the full nonlinear system and in the two-dimensional case. Since $c = d = 0$ and $\text{curl } \mathbf{u}_0 = 0$, we deduce from the second equation of (1.2) that

$$(4.5) \quad \text{curl } \mathbf{u} = 0, \quad \text{for } t > 0.$$

Then using (4.5), (1.2) becomes

$$(4.6) \quad \begin{cases} \partial_t \eta + \nabla \cdot \mathbf{u} + \epsilon \nabla \cdot (\eta \mathbf{u}) - b\epsilon \Delta \partial_t \eta = 0, \\ \partial_t \mathbf{u} + \nabla \eta + \epsilon \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{0}. \end{cases}$$

Denoting by $U = (\eta, \mathbf{u})$, (4.6) is rewritten in the condensed form as

$$(4.7) \quad (1 - b\epsilon \Delta) \partial_t U + M(U, D)U = \mathbf{0},$$

where

$$M(U, D) = \begin{pmatrix} \epsilon \mathbf{u} \cdot \nabla & (1 + \epsilon \eta) \partial_1 & (1 + \epsilon \eta) \partial_2 \\ (1 - b\epsilon \Delta) \partial_1 & (1 - b\epsilon \Delta) (\epsilon \mathbf{u} \cdot \nabla) & 0 \\ (1 - b\epsilon \Delta) \partial_2 & 0 & (1 - b\epsilon \Delta) (\epsilon \mathbf{u} \cdot \nabla) \end{pmatrix}.$$

The symmetrizer of $M(U, D)$ is

$$S_U(D) = \begin{pmatrix} 1 - b\epsilon \Delta & 0 & 0 \\ 0 & 1 + \epsilon \eta & 0 \\ 0 & 0 & 1 + \epsilon \eta \end{pmatrix}.$$

We define the energy functional associated to (4.7) as

$$(4.8) \quad \begin{aligned} E_s(U) &= ((1 - b\epsilon \Delta) \Lambda^s U | S_U(D) \Lambda^s U)_2 \\ &= ((1 - b\epsilon \Delta) \Lambda^s \eta | (1 - b\epsilon \Delta) \Lambda^s \eta)_2 + ((1 - b\epsilon \Delta) \Lambda^s \mathbf{u} | (1 + \epsilon \eta) \Lambda^s \mathbf{u})_2. \end{aligned}$$

Assume that

$$(4.9) \quad 1 + \epsilon \eta \geq H > 0, \quad \epsilon |\eta|_{W^{1, \infty}} \leq \kappa_H \quad \text{for } t \in [0, \bar{T}]$$

with κ_H sufficiently small, and

$$(4.10) \quad \max_{0 \leq t \leq \bar{T}} E_s(U) \leq C_0,$$

for some constant C_0 . The assumptions (4.9) and (4.10) hold provided that (4.3) holds and $\epsilon \leq \epsilon_0 \ll 1$ (one can refer to [36]).

Under the conditions (4.9), it is easy to check that

$$(4.11) \quad E_s(U) \sim |\eta|_{X_\epsilon^s}^2 + |\mathbf{u}|_{X_\epsilon^s}^2.$$

The proof of (4.11) is similar to that in [36] and we omit it.

A standard energy estimate leads to

$$(4.12) \quad \begin{aligned} \frac{d}{dt} E_s(U) &= 2((1 - b\epsilon\Delta)\Lambda^s \partial_t U | S_U(D)\Lambda^s U)_2 \\ &\quad + ((1 - b\epsilon\Delta)\Lambda^s U | \partial_t S_U(D)\Lambda^s U)_2 - b\epsilon([S_U(D), \Delta]\Lambda^s U | \Lambda^s U_t)_2 \\ &= -2(\Lambda^s(M(U, D)U) | S_U(D)\Lambda^s U)_2 + ((1 - b\epsilon\Delta)\Lambda^s \mathbf{u} | \epsilon \partial_t \eta \Lambda^s \mathbf{u})_2 \\ &\quad - b\epsilon^2([\eta, \Delta]\Lambda^s \mathbf{u} | \Lambda^s \partial_t \mathbf{u})_2 \\ &\stackrel{\text{def}}{=} I + II + III. \end{aligned}$$

Estimate for I . Firstly, one gets

$$I = -2([\Lambda^s, M(U, D)]U | S_U(D)\Lambda^s U)_2 - 2(M(U, D)\Lambda^s U | S_U(D)\Lambda^s U)_2 \stackrel{\text{def}}{=} I_1 + I_2.$$

For I_1 , one has

$$\begin{aligned} I_1 &= -2([\Lambda^s, \epsilon \mathbf{u}] \cdot \nabla \eta + [\Lambda^s, \epsilon \eta] \nabla \cdot \mathbf{u} | (1 - b\epsilon\Delta)\Lambda^s \eta)_2 \\ &\quad - 2((1 - b\epsilon\Delta)([\Lambda^s, \epsilon \mathbf{u}] \cdot \nabla \mathbf{u}) | (1 + \epsilon \eta)\Lambda^s \mathbf{u})_2 \\ &\stackrel{\text{def}}{=} I_{11} + I_{12}. \end{aligned}$$

Thanks to Lemma 3.2, it is easy to get that for $s > 2$,

$$(4.13) \quad \begin{aligned} |I_{11}| &\lesssim (|[\Lambda^s, \epsilon \mathbf{u}] \cdot \nabla \eta|_2 + |[\Lambda^s, \epsilon \eta] \nabla \cdot \mathbf{u}|_2) |(1 - b\epsilon\Delta)\Lambda^s \eta|_2 \\ &\lesssim \epsilon |\mathbf{u}|_{H^s} |\eta|_{H^s} (|\eta|_{H^s} + \epsilon |\eta|_{H^{s+2}}) \lesssim \epsilon |\mathbf{u}|_{X_\epsilon^s} |\eta|_{X_\epsilon^s}^2. \end{aligned}$$

For I_{12} , integrating by parts, there holds

$$I_{12} = -2([\Lambda^s, \epsilon \mathbf{u}] \cdot \nabla \mathbf{u} | (1 + \epsilon \eta)\Lambda^s \mathbf{u})_2 - 2b\epsilon(\nabla([\Lambda^s, \epsilon \mathbf{u}] \cdot \nabla \mathbf{u}) | \nabla((1 + \epsilon \eta)\Lambda^s \mathbf{u}))_2$$

which along with (4.9) and Lemma 3.2 implies that

$$(4.14) \quad \begin{aligned} |I_{12}| &\lesssim (1 + \epsilon |\eta|_\infty) \epsilon |\mathbf{u}|_{H^s}^3 + \epsilon^2 |\mathbf{u}|_{X_\epsilon^s}^2 ((1 + \epsilon |\eta|_\infty) |\nabla \Lambda^s \mathbf{u}|_2 + \epsilon |\nabla \eta|_\infty |\Lambda^s \mathbf{u}|_2) \\ &\lesssim \epsilon |\mathbf{u}|_{X_\epsilon^s}^3. \end{aligned}$$

Then we get by (4.13) and (4.14) that

$$(4.15) \quad |I_1| \lesssim \epsilon |\mathbf{u}|_{X_\epsilon^s} (|\eta|_{X_\epsilon^s}^2 + |\mathbf{u}|_{X_\epsilon^s}^2).$$

For I_2 , due to the expressions of $M(U, D)$ and $S_U(D)$, we get that

$$\begin{aligned} I_2 &= -2(\epsilon \mathbf{u} \cdot \nabla \Lambda^s \eta | (1 - b\epsilon\Delta)\Lambda^s \eta)_2 - 2((1 - b\epsilon\Delta)(\epsilon \mathbf{u} \cdot \nabla \Lambda^s \mathbf{u}) | (1 + \epsilon \eta)\Lambda^s \mathbf{u})_2 \\ &\quad - 2\{((1 + \epsilon \eta) \nabla \cdot \Lambda^s \mathbf{u} | (1 - b\epsilon\Delta)\Lambda^s \eta)_2 + ((1 - b\epsilon\Delta) \nabla \Lambda^s \eta | (1 + \epsilon \eta)\Lambda^s \mathbf{u})_2\} \\ &\stackrel{\text{def}}{=} I_{21} + I_{22} + I_{23}. \end{aligned}$$

Integrating by parts, one gets that

$$I_{21} = \epsilon(\nabla \cdot \mathbf{u} \Lambda^s \eta | \Lambda^s \eta)_2 + b\epsilon^2(\nabla \cdot \mathbf{u} \nabla \Lambda^s \eta | \nabla \Lambda^s \eta)_2 - 2b\epsilon^2 \sum_{i=1}^2 (\partial_i \mathbf{u} \cdot \nabla \Lambda^s \eta | \partial_i \Lambda^s \eta)_2$$

$$I_{22} = \epsilon(\nabla \cdot ((1 + \epsilon\eta)\mathbf{u}) \Lambda^s \mathbf{u} | \Lambda^s \mathbf{u})_2 + 2b\epsilon^3((\mathbf{u} \cdot \nabla) \Lambda^s \mathbf{u} | \sum_{i=1}^2 \partial_i (\partial_i \eta \Lambda^s \mathbf{u}))_2 \\ - 2b\epsilon^2 \sum_{i=1}^2 (\partial_i \mathbf{u} \cdot \nabla \Lambda^s \mathbf{u} | (1 + \epsilon\eta) \partial_i \Lambda^s \mathbf{u})_2 + b\epsilon^2(\nabla \cdot ((1 + \epsilon\eta)\mathbf{u}) \nabla \Lambda^s \mathbf{u} | \nabla \Lambda^s \mathbf{u})_2,$$

$$I_{23} = 2\epsilon(\nabla \eta \cdot \Lambda^s \mathbf{u} | (1 - b\epsilon\Delta) \Lambda^s \eta)_2.$$

Then thanks to (4.9), (4.10), (4.11) and (4.2), there holds

$$(4.16) \quad |I_2| \lesssim \epsilon |\mathbf{u}|_{X_\epsilon^s} (|\eta|_{X_{\epsilon^2}^s}^2 + |\mathbf{u}|_{X_\epsilon^s}^2).$$

Thanks to (4.15) and (4.16), we obtain

$$(4.17) \quad |I| \lesssim \epsilon |\mathbf{u}|_{X_\epsilon^s} (|\eta|_{X_{\epsilon^2}^s}^2 + |\mathbf{u}|_{X_\epsilon^s}^2).$$

Estimate for II. Integrating by parts, we have

$$II = \epsilon(\Lambda^s \mathbf{u} | \partial_t \eta \Lambda^s \mathbf{u})_2 + b\epsilon^2(\nabla \Lambda^s \mathbf{u} | \nabla (\partial_t \eta \Lambda^s \mathbf{u}))_2$$

which along with (4.2) implies that

$$(4.18) \quad |II| \lesssim \epsilon |\partial_t \eta|_{X_{\epsilon^2}^{s-1}} |\mathbf{u}|_{X_\epsilon^s}^2.$$

Estimate for III. Thanks to Lemma 3.2, we get that

$$(4.19) \quad |III| \lesssim \epsilon^2 |\nabla \eta|_{H^{s-1}} |\mathbf{u}|_{H^{s+1}} |\Lambda^s \partial_t \mathbf{u}|_2 \lesssim \epsilon |\eta|_{X_{\epsilon^2}^s} |\mathbf{u}|_{X_\epsilon^s} |\partial_t \mathbf{u}|_{X_{\epsilon^2}^{s-1}}.$$

Combining (4.12), (4.17), (4.18) and (4.19), we obtain that

$$(4.20) \quad \frac{d}{dt} E_s(U) \lesssim \epsilon (|\eta|_{X_{\epsilon^2}^s} + |\mathbf{u}|_{X_\epsilon^s}) (|\eta|_{X_{\epsilon^2}^s}^2 + |\mathbf{u}|_{X_\epsilon^s}^2 + |\partial_t \eta|_{X_{\epsilon^2}^{s-1}}^2 + |\partial_t \mathbf{u}|_{X_{\epsilon^2}^{s-1}}^2).$$

Thanks to the equations of (4.6), one gets by using (4.9) and (4.10) that

$$(4.21) \quad |\partial_t \eta|_{X_{\epsilon^2}^{s-1}} + |\partial_t \mathbf{u}|_{X_\epsilon^{s-1}} \lesssim (1 + \epsilon |\eta|_\infty) |\mathbf{u}|_{X_\epsilon^s} + \epsilon |\mathbf{u}|_\infty |\eta|_{H^s} + |\eta|_{X_{\epsilon^2}^s} + |\mathbf{u}|_{X_\epsilon^s}^2 \\ \lesssim |\eta|_{X_{\epsilon^2}^s} + |\mathbf{u}|_{X_\epsilon^s},$$

which along with (4.20) implies that

$$(4.22) \quad \frac{d}{dt} E_s(U) \lesssim \epsilon (|\eta|_{X_{\epsilon^2}^s} + |\mathbf{u}|_{X_\epsilon^s}) (|\eta|_{X_{\epsilon^2}^s}^2 + |\mathbf{u}|_{X_\epsilon^s}^2).$$

Then due to (4.11), there holds

$$\frac{d}{dt} (E_s(U))^{\frac{1}{2}} \leq C_1 \epsilon E_s(U),$$

which gives rise to

$$(4.23) \quad (E_s(U))^{\frac{1}{2}} \leq \frac{(E_s(U_0))^{\frac{1}{2}}}{1 - C_1 \epsilon t (E_s(U_0))^{\frac{1}{2}}} \leq 2(E_s(U_0))^{\frac{1}{2}},$$

for any $t \leq \frac{1}{2C_1 (E_s(U_0))^{\frac{1}{2}} \epsilon}$ with $T = \frac{1}{2C_1 (E_s(U_0))^{\frac{1}{2}}}$. This completes the proof of Theorem 4.3. \square

4.2. **Case** $d > 0, a = b = c = 0$.

Theorem 4.4. *Let $d > 0, a = b = c = 0$. $n = 1, 2, s > 1 + \frac{n}{2}$. Assume that $\eta_0 \in X_\epsilon^s(\mathbb{R}^n), \mathbf{u}_0 \in X_{\epsilon^2}^s(\mathbb{R}^n)$ satisfy the (non-cavitation) condition*

$$(4.24) \quad 1 + \epsilon\eta_0 \geq H > 0, \quad H \in (0, 1),$$

Then there exists a constant \tilde{c}_0 such that for any $\epsilon \leq \epsilon_0 = \frac{1-H}{\tilde{c}_0(|\eta_0|_{X_\epsilon^s} + |\mathbf{u}_0|_{X_{\epsilon^2}^s})}$, there exists $T > 0$ independent of ϵ , such that (1.2)-(4.1) has a unique solution $(\eta, \mathbf{u})^T$ with $\eta \in C([0, T/\epsilon]; X_\epsilon^s(\mathbb{R}^n))$ and $\mathbf{u} \in C([0, T/\epsilon]; X_{\epsilon^2}^s(\mathbb{R}^n))$. Moreover,

$$(4.25) \quad \max_{t \in [0, T/\epsilon]} (|\eta|_{X_\epsilon^s} + |\mathbf{u}|_{X_{\epsilon^2}^s}) \leq \tilde{c}(|\eta_0|_{X_\epsilon^s} + |\mathbf{u}_0|_{X_{\epsilon^2}^s}).$$

Remark 4.1. As was previously mentioned, one gets global well-posedness in the one-dimensional case ([4, 38]) in a different functional setting though but the method of proof in [4, 38] does not seem to adapt to the two-dimensional case since it relies strongly on properties of the *one-dimensional* hyperbolic Saint-Venant (shallow water) system.

Proof. The proof also follows the same method used in [36]. Here we only give the *a priori estimates*. For $d > 0, a = b = c = 0$, we rewrite (1.2) in two-dimensional space as follows:

$$(4.26) \quad \begin{cases} \partial_t \eta + \nabla \cdot \mathbf{u} + \epsilon \nabla \cdot (\eta \mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \nabla \eta + \frac{\epsilon}{2} \nabla (|\mathbf{u}|^2) - d\epsilon \Delta \partial_t \mathbf{u} = \mathbf{0}, \end{cases}$$

Denoting by $U = (\eta, \mathbf{u})$, (4.26) is equivalent to the following condensed system

$$(4.27) \quad (1 - d\epsilon \Delta) \partial_t U + M(U, D)U = 0,$$

where

$$M(U, D) = \begin{pmatrix} \epsilon(1 - d\epsilon \Delta)(\mathbf{u} \cdot \nabla) & (1 - d\epsilon \Delta)((1 + \epsilon\eta)\partial_1) & (1 - d\epsilon \Delta)((1 + \epsilon\eta)\partial_2) \\ \partial_1 & \epsilon u_1 \partial_1 & \epsilon u_2 \partial_1 \\ \partial_2 & \epsilon u_1 \partial_2 & \epsilon u_2 \partial_2 \end{pmatrix}.$$

The symmetrizer $S_U(D)$ for $M(U, D)$ is defined by

$$\begin{pmatrix} 1 & \epsilon u_1 & \epsilon u_2 \\ \epsilon u_1 & (1 + \epsilon\eta)(1 - d\epsilon \Delta) & 0 \\ \epsilon u_2 & 0 & (1 + \epsilon\eta)(1 - d\epsilon \Delta) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & d\epsilon^3 u_1 u_1 \Delta & d\epsilon^3 u_1 u_2 \Delta \\ 0 & d\epsilon^3 u_1 u_2 \Delta & d\epsilon^3 u_2 u_2 \Delta \end{pmatrix}.$$

We define the energy functional associated to (4.27) as

$$(4.28) \quad E_s(U) = ((1 - d\epsilon \Delta) \Lambda^s U | S_U(D) \Lambda^s U)_2$$

Assume that

$$(4.29) \quad 1 + \epsilon\eta \geq H > 0, \quad \epsilon |\mathbf{u}|_{W^{1, \infty}} \leq \kappa_H \quad \text{for } t \in [0, \bar{T}]$$

with κ_H sufficiently small, and

$$(4.30) \quad \max_{0 \leq t \leq \bar{T}} E_s(U) \leq C_0,$$

for some constants C_0 . The assumptions (4.29) and (4.30) also hold provided that (4.24) holds and $\epsilon \leq \epsilon_0 \ll 1$ (one can refer to [36]). Under the assumption (4.29), it is not difficult to check that

$$(4.31) \quad E_s(U) \sim |\eta|_{X_\epsilon^s}^2 + |\mathbf{u}|_{X_{\epsilon^2}^s}^2.$$

As usual, a standard computation shows that

$$\begin{aligned}
(4.32) \quad \frac{d}{dt} E_s(U) &= -(\Lambda^s(M(U, D)U) | (S_U(D) + S_U(D)^*) \Lambda^s U)_2 \\
&\quad - d\epsilon([S_U(D)^*, \Delta] \Lambda^s U | \Lambda^s \partial_t U)_2 + ((1 - d\epsilon\Delta) \Lambda^s U | \partial_t S_U(D) \Lambda^s U)_2 \\
&\stackrel{\text{def}}{=} I + II + III,
\end{aligned}$$

where $S_U(D)^*$ is the adjoint matrix of $S_U(D)$.

Estimate for I . One has that

$$\begin{aligned}
I &= -([\Lambda^s, M(U, D)]U | (S_U(D) + S_U(D)^*) \Lambda^s U)_2 \\
&\quad - (\Lambda^s U | (S_U(D) + S_U(D)^*)(M(U, D) \Lambda^s U))_2 \\
&\stackrel{\text{def}}{=} I_1 + I_2
\end{aligned}$$

Estimate for I_1 . Using the expressions of $M(U, D)$ and $S_U(D)$, one gets that

$$\begin{aligned}
([\Lambda^s, M(U, D)]U | S_U(D) \Lambda^s U)_2 &= ([\Lambda^s, \epsilon(1 - d\epsilon\Delta)(\mathbf{u} \cdot \nabla)]\eta | \Lambda^s \eta + \epsilon \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 \\
&\quad + ([\Lambda^s, \epsilon(1 - d\epsilon\Delta)(\eta \nabla)] \cdot \mathbf{u} | \Lambda^s \eta + \epsilon \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 \\
&\quad + \sum_{i,j=1}^2 ([\Lambda^s, \epsilon u_j] \partial_i u_j | \epsilon u_i \Lambda^s \eta + (1 + \epsilon\eta)(1 - d\epsilon\Delta) \Lambda^s u_i + d\epsilon^3 u_i \mathbf{u} \cdot \Delta \Lambda^s \mathbf{u})_2 \\
&\stackrel{\text{def}}{=} I_{11} + I_{12} + I_{13}.
\end{aligned}$$

Integrating by parts, there hold

$$\begin{aligned}
I_{11} &= \epsilon([\Lambda^s, \mathbf{u}] \cdot \nabla \eta | \Lambda^s \eta + \epsilon \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 + d\epsilon^2(\nabla([\Lambda^s, \mathbf{u}] \cdot \nabla \eta) | \nabla(\Lambda^s \eta + \epsilon \mathbf{u} \cdot \Lambda^s \mathbf{u}))_2 \\
I_{12} &= \epsilon([\Lambda^s, \eta] \nabla \cdot \mathbf{u} | \Lambda^s \eta + \epsilon \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 + d\epsilon^2(\nabla([\Lambda^s, \eta] \nabla \cdot \mathbf{u}) | \nabla(\Lambda^s \eta + \epsilon \mathbf{u} \cdot \Lambda^s \mathbf{u}))_2,
\end{aligned}$$

which along with (4.29), (4.30), (3.11) and (4.2) imply that

$$\begin{aligned}
(4.33) \quad |I_{11}| + |I_{12}| &\lesssim \epsilon |\mathbf{u}|_{H^s} |\eta|_{H^s} (|\eta|_{H^s} + \epsilon |\mathbf{u}|_{X_{\epsilon^2}^s}) \\
&\quad + \epsilon^2 (|\nabla \mathbf{u}|_{H^{s-1}} |\nabla \eta|_{H^s} + |\nabla \mathbf{u}|_{H^s} |\nabla \eta|_{H^{s-1}}) (|\eta|_{H^{s+1}} + \epsilon |\mathbf{u}|_{H^s} |\mathbf{u}|_{H^{s+1}}) \\
&\lesssim \epsilon |\mathbf{u}|_{X_{\epsilon^2}^s} (|\eta|_{X_{\epsilon^2}^s}^2 + |\mathbf{u}|_{X_{\epsilon^2}^s}^2).
\end{aligned}$$

Thanks to (3.11), (4.2), (4.30) and (4.31), there holds

$$|I_{13}| \lesssim \epsilon |\mathbf{u}|_{H^s}^2 (\epsilon |\mathbf{u}|_{H^s} |\eta|_{H^s} + |\mathbf{u}|_{X_{\epsilon^2}^s} + d\epsilon^3 |\mathbf{u}|_{X_{\epsilon^2}^s}^3) \lesssim \epsilon |\mathbf{u}|_{X_{\epsilon^2}^s}^2 (|\eta|_{X_{\epsilon^2}^s} + |\mathbf{u}|_{X_{\epsilon^2}^s}),$$

which along with (4.33) shows that

$$|([\Lambda^s, M(U, D)]U | S_U(D) \Lambda^s U)_2| \lesssim \epsilon |\mathbf{u}|_{X_{\epsilon^2}^s} (|\eta|_{X_{\epsilon^2}^s}^2 + |\mathbf{u}|_{X_{\epsilon^2}^s}^2).$$

The same estimate holds for term $([\Lambda^s, M(U, D)]U | S_U(D)^* \Lambda^s U)_2$. Then we obtain that

$$(4.34) \quad |I_1| \lesssim \epsilon |\mathbf{u}|_{X_{\epsilon^2}^s} (|\eta|_{X_{\epsilon^2}^s}^2 + |\mathbf{u}|_{X_{\epsilon^2}^s}^2).$$

Estimate for I_2 . For I_2 , we first calculate $S_U(D)(M(U, D)) = A(U, D) = (a_{ij})$ as follows

$$\begin{aligned}
a_{11} &= \epsilon(1 - d\epsilon\Delta)(\mathbf{u} \cdot \nabla) + \epsilon\mathbf{u} \cdot \nabla = 2\epsilon\mathbf{u} \cdot \nabla - d\epsilon^2\Delta(\mathbf{u} \cdot \nabla), \\
a_{12} &= (1 - d\epsilon\Delta)((1 + \epsilon\eta)\partial_1) + \epsilon^2u_1\mathbf{u} \cdot \nabla, \\
a_{13} &= (1 - d\epsilon\Delta)((1 + \epsilon\eta)\partial_2) + \epsilon^2u_2\mathbf{u} \cdot \nabla, \\
a_{21} &= (1 + \epsilon\eta)(1 - d\epsilon\Delta)\partial_1 + \epsilon^2u_1\mathbf{u} \cdot -d\epsilon^3u_1[\Delta, \mathbf{u}] \cdot \nabla, \\
a_{22} &= \epsilon u_1(1 - d\epsilon\Delta)((1 + \epsilon\eta)\partial_1) + \epsilon(1 + \epsilon\eta)(1 - d\epsilon\Delta)(u_1\partial_1) + d\epsilon^4u_1\mathbf{u} \cdot \Delta(u_1\nabla), \\
a_{23} &= \epsilon u_1(1 - d\epsilon\Delta)((1 + \epsilon\eta)\partial_2) + \epsilon(1 + \epsilon\eta)(1 - d\epsilon\Delta)(u_2\partial_1) + d\epsilon^4u_1\mathbf{u} \cdot \Delta(u_2\nabla), \\
a_{31} &= (1 + \epsilon\eta)(1 - d\epsilon\Delta)\partial_2 + \epsilon^2u_2(1 - d\epsilon\Delta)(\mathbf{u} \cdot \nabla) + d\epsilon^3u_2\mathbf{u} \cdot \nabla\Delta, \\
a_{32} &= \epsilon u_2(1 - d\epsilon\Delta)((1 + \epsilon\eta)\partial_1) + \epsilon(1 + \epsilon\eta)(1 - d\epsilon\Delta)(u_1\partial_2) + d\epsilon^4u_2\mathbf{u} \cdot \Delta(u_1\nabla), \\
a_{33} &= \epsilon u_2(1 - d\epsilon\Delta)((1 + \epsilon\eta)\partial_2) + \epsilon(1 + \epsilon\eta)(1 - d\epsilon\Delta)(u_2\partial_2) + d\epsilon^4u_2\mathbf{u} \cdot \Delta(u_2\nabla).
\end{aligned}$$

Now, we calculate $(S_U(D)(M(U, D)\Lambda^s U) | \Lambda^s U)_2 = (A(U, D)\Lambda^s U | \Lambda^s U)_2$.

For a_{11} , one has

$$\begin{aligned}
(a_{11}\Lambda^s\eta | \Lambda^s\eta)_2 &= 2\epsilon(\mathbf{u} \cdot \nabla\Lambda^s\eta | \Lambda^s\eta)_2 - d\epsilon^2(\Delta(\mathbf{u} \cdot \nabla\Lambda^s\eta) | \Lambda^s\eta)_2 \\
&= -\epsilon(\nabla \cdot \mathbf{u}\Lambda^s\eta | \Lambda^s\eta)_2 - \frac{1}{2}d\epsilon^2(\nabla \cdot \mathbf{u}\nabla\Lambda^s\eta | \nabla\Lambda^s\eta)_2 + d\epsilon^2\sum_{i=1}^2(\partial_i\mathbf{u} \cdot \nabla\Lambda^s\eta | \partial_i\Lambda^s\eta)_2,
\end{aligned}$$

which shows that

$$(4.35) \quad |(a_{11}\Lambda^s\eta | \Lambda^s\eta)_2| \lesssim \epsilon|\mathbf{u}|_{H^s}|\eta|_{X_\epsilon^s}^2.$$

For a_{22} , one gets

$$\begin{aligned}
(a_{22}\Lambda^s u_1 | \Lambda^s u_1)_2 &= \epsilon\{(u_1(1 - d\epsilon\Delta)((1 + \epsilon\eta)\partial_1\Lambda^s u_1) | \Lambda^s u_1)_2 \\
&\quad + ((1 + \epsilon\eta)(1 - d\epsilon\Delta)(u_1\partial_1\Lambda^s u_1) | \Lambda^s u_1)_2\} + d\epsilon^4(u_1\mathbf{u} \cdot \Delta(u_1\nabla\Lambda^s u_1) | \Lambda^s u_1)_2 \\
&= -\epsilon(\Lambda^s u_1 | \epsilon\partial_1\eta(1 - d\epsilon\Delta)(u_1\Lambda^s u_1) + (1 + \epsilon\eta)(1 - d\epsilon\Delta)(\partial_1 u_1\Lambda^s u_1))_2 \\
&\quad - d\epsilon^4\sum_{i=1}^2(\partial_i(u_1\nabla\Lambda^s u_1) | \partial_i(u_1\mathbf{u}\Lambda^s u_1))_2
\end{aligned}$$

which along with (4.29), (4.30) and (4.2) gives rise to

$$(4.36) \quad |(a_{22}\Lambda^s u_1 | \Lambda^s u_1)_2| \lesssim \epsilon|\mathbf{u}|_{H^s}|\mathbf{u}|_{X_\epsilon^s}^2.$$

The same estimate holds for term $(a_{33}\Lambda^s u_2 | \Lambda^s u_2)_2$.

For a_{12} and a_{21} , one calculates that

$$\begin{aligned}
&(a_{12}\Lambda^s u_1 | \Lambda^s\eta)_2 + (a_{21}\Lambda^s\eta | \Lambda^s u_1)_2 \\
&= \{((1 - d\epsilon\Delta)((1 + \epsilon\eta)\partial_1\Lambda^s u_1) | \Lambda^s\eta)_2 + ((1 + \epsilon\eta)(1 - d\epsilon\Delta)\partial_1\Lambda^s\eta | \Lambda^s u_1)_2\} \\
&\quad + \{\epsilon^2(u_1\mathbf{u} \cdot \nabla\Lambda^s u_1 | \Lambda^s\eta)_2 + \epsilon^2(u_1\mathbf{u} \cdot \nabla\Lambda^s\eta | \Lambda^s u_1)_2 - d\epsilon^3(u_1[\Delta, \mathbf{u}] \cdot \nabla\Lambda^s\eta | \Lambda^s u_1)_2\} \\
&= -\epsilon((1 - d\epsilon\Delta)(\partial_1\eta\Lambda^s u_1) | \Lambda^s\eta)_2 - \epsilon^2(\nabla \cdot (u_1\mathbf{u})\Lambda^s u_1 | \Lambda^s\eta)_2 \\
&\quad + \epsilon^3(\nabla\Lambda^s\eta | [\Delta, \mathbf{u}](u_1\Lambda^s u_1))_2,
\end{aligned}$$

which along with (4.29), (4.30) and (4.2) implies

$$\begin{aligned}
(4.37) \quad & |(a_{12}\Lambda^s u_1 | \Lambda^s \eta)_2 + (a_{21}\Lambda^s \eta | \Lambda^s u_1)_2| \\
& \lesssim \epsilon |\eta|_{X_\epsilon^s}^2 |\mathbf{u}|_{X_{\epsilon^2}^s} + \epsilon^2 |\mathbf{u}|_{X_{\epsilon^2}^s}^3 |\eta|_{X_\epsilon^s} \\
& \lesssim \epsilon |\mathbf{u}|_{X_{\epsilon^2}^s} (|\eta|_{X_\epsilon^s}^2 + |\mathbf{u}|_{X_{\epsilon^2}^s}^2).
\end{aligned}$$

The same estimate holds for $(a_{13}\Lambda^s u_2 | \Lambda^s \eta)_2 + (a_{31}\Lambda^s \eta | \Lambda^s u_2)_2$.

At last, for a_{23} and a_{32} , one estimates that

$$\begin{aligned}
& (a_{23}\Lambda^s u_2 | \Lambda^s u_1)_2 + (a_{32}\Lambda^s u_1 | \Lambda^s u_2)_2 \\
& = \epsilon \{ (u_1(1 - d\epsilon\Delta)((1 + \epsilon\eta)\partial_2 \Lambda^s u_2) | \Lambda^s u_1)_2 + ((1 + \epsilon\eta)(1 - d\epsilon\Delta)(u_1 \partial_2 \Lambda^s u_1) | \Lambda^s u_2)_2 \} \\
& \quad + \epsilon \{ ((1 + \epsilon\eta)(1 - d\epsilon\Delta)(u_2 \partial_1 \Lambda^s u_2) | \Lambda^s u_1)_2 + (u_2(1 - d\epsilon\Delta)((1 + \epsilon\eta)\partial_1 \Lambda^s u_1) | \Lambda^s u_2)_2 \} \\
& \quad + d\epsilon^4 \{ (u_1 \mathbf{u} \cdot \Delta(u_2 \nabla \Lambda^s u_2) | \Lambda^s u_1)_2 + (u_2 \mathbf{u} \cdot \Delta(u_1 \nabla \Lambda^s u_1) | \Lambda^s u_2)_2 \} \\
& = -\epsilon (\epsilon \partial_2 \eta (1 - d\epsilon\Delta)(u_1 \Lambda^s u_1) + (1 + \epsilon\eta)(1 - d\epsilon\Delta)(\partial_2 u_1 \Lambda^s u_1) | \Lambda^s u_2)_2 \\
& \quad - \epsilon (\epsilon \partial_1 \eta (1 - d\epsilon\Delta)(u_2 \Lambda^s u_2) + (1 + \epsilon\eta)(1 - d\epsilon\Delta)(\partial_1 u_2 \Lambda^s u_2) | \Lambda^s u_1)_2 \\
& \quad - d\epsilon^4 \{ (\nabla u_2 \cdot \Delta(u_1 \mathbf{u} \Lambda^s u_1) + u_2 \Delta(\nabla \cdot (u_1 \mathbf{u}) \Lambda^s u_1) | \Lambda^s u_2)_2 \\
& \quad - (2u_2 \sum_{i=1}^2 \partial_i \mathbf{u} \cdot \partial_i (u_1 \nabla \Lambda^s u_1) + u_2 \Delta \mathbf{u} \cdot u_1 \nabla \Lambda^s u_1 | \Lambda^s u_2)_2 \},
\end{aligned}$$

which together with (4.29), (4.30) and (4.2) leads to

$$\begin{aligned}
(4.38) \quad & |(a_{23}\Lambda^s u_2 | \Lambda^s u_1)_2 + (a_{32}\Lambda^s u_1 | \Lambda^s u_2)_2| \\
& \lesssim \epsilon (1 + \epsilon |\eta|_{X_\epsilon^s}) |\mathbf{u}|_{X_{\epsilon^2}^s}^3 + \epsilon^3 |\mathbf{u}|_{X_{\epsilon^2}^s}^5 \lesssim \epsilon |\mathbf{u}|_{X_{\epsilon^2}^s}^3.
\end{aligned}$$

Thanks to (4.35), (4.36), (4.37) and (4.38), we obtain that

$$|(S_U(D)(M(U, D)\Lambda^s U) | \Lambda^s U)_2| \lesssim \epsilon (|\eta|_{X_\epsilon^s} + |\mathbf{u}|_{X_{\epsilon^2}^s})^3,$$

provided that there hold (4.29) and (4.30). The same estimate holds for $(S_U(D)^*(M(U, D)\Lambda^s U) | \Lambda^s U)_2$. Then we obtain that

$$(4.39) \quad |I_2| \lesssim \epsilon (|\eta|_{X_\epsilon^s} + |\mathbf{u}|_{X_{\epsilon^2}^s})^3.$$

Due to (4.34) and (4.39), we get that

$$(4.40) \quad |I| \lesssim \epsilon (|\eta|_{X_\epsilon^s} + |\mathbf{u}|_{X_{\epsilon^2}^s})^3.$$

Estimate for II . Using the expression of $S_U(D)$, one obtains that

$$\begin{aligned}
II & = -d\epsilon \left(\epsilon ([\mathbf{u}, \Delta] \cdot \Lambda^s \mathbf{u} | \Lambda^s \partial_t \eta)_2 + \epsilon ([\mathbf{u}, \Delta] \Lambda^s \eta | \Lambda^s \partial_t \mathbf{u})_2 \right. \\
& \quad \left. + \epsilon ((1 - d\epsilon\Delta)([\eta, \Delta] \Lambda^s \mathbf{u}) | \Lambda^s \partial_t \mathbf{u})_2 + d\epsilon^3 \sum_{i,j=1}^2 (\Delta([u_i u_j, \Delta] \Lambda^s u_j) | \Lambda^s \partial_t u_i)_2 \right).
\end{aligned}$$

Along the same line as previous work, by virtue of (3.11), (4.2), (4.29), (4.30) and integrating by parts, we finally get that

$$(4.41) \quad |II| \lesssim \epsilon (|\eta|_{X_\epsilon^s}^2 + |\mathbf{u}|_{X_{\epsilon^2}^s}^2) (|\partial_t \eta|_{X_\epsilon^{s-1}} + |\partial_t \mathbf{u}|_{X_{\epsilon^2}^{s-1}}).$$

Estimate for III. Using the expression of $S_U(D)$ again, one gets that

$$\begin{aligned} III &= \epsilon((1 - d\epsilon\Delta)\Lambda^s\eta | \partial_t \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 + \epsilon((1 - d\epsilon\Delta)\Lambda^s \mathbf{u} | \partial_t \mathbf{u} \Lambda^s \eta)_2 \\ &\quad + \epsilon((1 - d\epsilon\Delta)\Lambda^s \mathbf{u} | \partial_t \eta (1 - d\epsilon\Delta)\Lambda^s \mathbf{u})_2 + d\epsilon^3 \sum_{j=1}^2 ((1 - d\epsilon\Delta)\Lambda^s u_j | \partial_t (u_j \mathbf{u}) \cdot \Delta \Lambda^s \mathbf{u})_2. \end{aligned}$$

Note that

$$((1 - d\epsilon\Delta)\Lambda^s\eta | \partial_t \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 = (\Lambda^s\eta | \partial_t \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 + d\epsilon(\nabla \Lambda^s\eta | \nabla(\partial_t \mathbf{u} \cdot \Lambda^s \mathbf{u}))_2$$

Then (4.29), (4.30) and (4.2) leads to

$$(4.42) \quad |III| \lesssim \epsilon(|\eta|_{X_\epsilon^s}^2 + |\mathbf{u}|_{X_{\epsilon^2}^s}) (|\partial_t \eta|_{X_\epsilon^{s-1}} + |\partial_t \mathbf{u}|_{X_{\epsilon^2}^{s-1}})$$

Combining (4.32), (4.40), (4.41) and (4.42), we obtain that

$$(4.43) \quad \frac{d}{dt} E_s(U) \lesssim \epsilon(|\eta|_{X_\epsilon^s} + |\mathbf{u}|_{X_{\epsilon^2}^s}) (|\eta|_{X_\epsilon^s}^2 + |\mathbf{u}|_{X_{\epsilon^2}^s}^2 + |\partial_t \eta|_{X_\epsilon^{s-1}}^2 + |\partial_t \mathbf{u}|_{X_{\epsilon^2}^{s-1}}^2).$$

Thanks to (4.26), we get by using (4.29) and (4.30) that

$$(4.44) \quad |\partial_t \eta|_{X_\epsilon^{s-1}} + |\partial_t \mathbf{u}|_{X_{\epsilon^2}^{s-1}} \lesssim |\eta|_{X_\epsilon^s} + |\mathbf{u}|_{X_{\epsilon^2}^s},$$

which along with (4.43) implies

$$(4.45) \quad \frac{d}{dt} E_s(U) \lesssim \epsilon(|\eta|_{X_\epsilon^s} + |\mathbf{u}|_{X_{\epsilon^2}^s}) (|\eta|_{X_\epsilon^s}^2 + |\mathbf{u}|_{X_{\epsilon^2}^s}^2).$$

Then due to (4.31), there holds

$$\frac{d}{dt} (E_s(U))^{\frac{1}{2}} \leq C_1 \epsilon E_s(U).$$

Similarly as the proof to Theorem 4.3, there exists $T = \frac{1}{2C_1 (E_s(U_0))^{\frac{1}{2}}}$ such that (4.25) holds. This completes the proof of Theorem 4.4. \square

We now turn to the "Schrödinger like" Boussinesq systems.

4.3. The case $b = d = c = 0, a < 0$. This case can be treated by following the lines developed in [36]. For the sake of completeness we provide some details now.

Theorem 4.5. *Let $b = c = d = 0, a = -1, n = 1, 2, s > 2 + \frac{n}{2}$. Assume that $\eta_0 \in H^s(\mathbb{R}^n), \mathbf{u}_0 \in X_\epsilon^s(\mathbb{R}^n)$ satisfy the (non-cavitation) condition*

$$(4.46) \quad 1 + \epsilon\eta_0 \geq H > 0, \quad H \in (0, 1),$$

Then there exists a constant \tilde{c}_0 such that for any $\epsilon \leq \epsilon_0 = \frac{1-H}{\tilde{c}_0(|\eta_0|_{H^s} + |\mathbf{u}_0|_{X_\epsilon^s})}$, there exists $T > 0$ independent of ϵ , such that (1.2)-(4.1) has a unique solution $(\eta, \mathbf{u})^T$ with $\eta \in C([0, T/\epsilon]; H^s(\mathbb{R}^n))$ and $\mathbf{u} \in C([0, T/\epsilon]; X_\epsilon^s(\mathbb{R}^n))$. Moreover,

$$(4.47) \quad \max_{t \in [0, T/\epsilon]} (|\eta|_{H^s} + |\mathbf{u}|_{X_\epsilon^s}) \leq \tilde{c}(|\eta_0|_{H^s} + |\mathbf{u}_0|_{X_\epsilon^s}).$$

Proof. We only sketch the proof of the two-dimensional case. For $b = c = d = 0, a = -1$, we firstly rewrite the two-dimensional version of (1.2) in the following condensed system

$$(4.48) \quad \partial_t U + M(U, D)U = 0,$$

where $U = (\eta, \mathbf{u})^T$, and

$$M(U, D) = \begin{pmatrix} \epsilon \mathbf{u} \cdot \nabla & (1 + \epsilon \eta - \epsilon \Delta) \partial_1 & (1 + \epsilon \eta - \epsilon \Delta) \partial_2 \\ \partial_1 & \epsilon u_1 \partial_1 & \epsilon u_2 \partial_1 \\ \partial_2 & \epsilon u_1 \partial_2 & \epsilon u_2 \partial_2 \end{pmatrix}.$$

The symmetrizer $S_U(D)$ for $M(U, D)$ is defined by

$$S_U(D) = \begin{pmatrix} 1 & \epsilon u_1 & \epsilon u_2 \\ \epsilon u_1 & 1 + \epsilon \eta - \epsilon \Delta & 0 \\ \epsilon u_2 & 0 & 1 + \epsilon \eta - \epsilon \Delta. \end{pmatrix}$$

We define the energy functional associated to (4.48) as

$$(4.49) \quad E_s(U) = (\Lambda^s U | S_U(D) \Lambda^s U)_2$$

Assume that

$$(4.50) \quad 1 + \epsilon \eta \geq H > 0, \quad \epsilon |\eta|_{W^{1,\infty}} + \epsilon |\mathbf{u}|_{W^{1,\infty}} \leq \kappa_H \quad \text{for } t \in [0, \bar{T}]$$

with κ_H sufficiently small, and

$$(4.51) \quad \max_{0 \leq t \leq \bar{T}} E_s(U) \leq C_0,$$

for some constants C_0 . The assumptions (4.50) and (4.51) also hold provided that (4.46) holds and $\epsilon \leq \epsilon_0 \ll 1$ (one can refer to [36]). Under the assumption (4.50), it is not difficult to check that

$$(4.52) \quad E_s(U) \sim |\eta|_{H^s}^2 + |\mathbf{u}|_{X_\epsilon^s}^2.$$

As usual, we get by a standard energy estimate that

$$(4.53) \quad \begin{aligned} \frac{d}{dt} E_s(U) &= 2(\Lambda^s \partial_t U | S_U(D) \Lambda^s U)_2 + (\Lambda^s U | \partial_t S_U(D) \Lambda^s U)_2 \\ &= -2(\Lambda^s (M(U, D)U) | S_U(D) \Lambda^s U)_2 + (\Lambda^s U | \partial_t S_U(D) \Lambda^s U)_2 \\ &\stackrel{\text{def}}{=} I + II. \end{aligned}$$

Estimate for II. Using the expression of $S_U(D)$ yields that

$$II = 2\epsilon(\Lambda^s \eta | \partial_t \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 + \epsilon(\Lambda^s \mathbf{u} | \partial_t \eta \Lambda^s \mathbf{u})_2,$$

which implies that for $s > 3$,

$$(4.54) \quad |II| \lesssim \epsilon |\partial_t \mathbf{u}|_{H^{s-1}} |\eta|_{H^s} |\mathbf{u}|_{H^s} + \epsilon |\partial_t \eta|_{H^{s-2}} |\mathbf{u}|_{H^s}^2.$$

Estimate for I. We first have that

$$I = -2([\Lambda^s, M(U, D)]U | S_U(D) \Lambda^s U)_2 - 2(M(U, D) \Lambda^s U | S_U(D) \Lambda^s U)_2 \stackrel{\text{def}}{=} I_1 + I_2.$$

Estimate for I_1 . For I_1 , we get that

$$\begin{aligned} I_1 &= -2\epsilon([\Lambda^s, \mathbf{u}] \cdot \nabla \eta + [\Lambda^s, \eta] \nabla \cdot \mathbf{u} | \Lambda^s \eta + \epsilon \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 \\ &\quad - 2\epsilon \sum_{j=1}^2 ([\Lambda^s, \mathbf{u}] \cdot \partial_j \mathbf{u} | u_j \Lambda^s \eta + (1 + \epsilon \eta - \epsilon \Delta) \Lambda^s u_j)_2 \\ &\stackrel{\text{def}}{=} I_{11} + I_{12}. \end{aligned}$$

Thanks to Lemma 3.2 and (4.50), it is easy to get that for $s > 2$,

$$\begin{aligned}
(4.55) \quad |I_{11}| &\lesssim \epsilon (|[\Lambda^s, \mathbf{u}] \cdot \nabla \eta|_2 + |[\Lambda^s, \eta] \nabla \cdot \mathbf{u}|_2) (|\Lambda^s \eta|_2 + |\epsilon \mathbf{u} \cdot \Lambda^s \mathbf{u}|_2) \\
&\lesssim \epsilon |\mathbf{u}|_{H^s} |\eta|_{H^s} (|\eta|_{H^s} + \epsilon |\mathbf{u}|_{H^s}^2) \\
&\lesssim \epsilon |\mathbf{u}|_{H^s} (|\eta|_{H^s}^2 + |\mathbf{u}|_{H^s}^2).
\end{aligned}$$

For I_{12} , integrating by parts, there holds

$$\begin{aligned}
I_{12} &= -2\epsilon \sum_{j=1}^2 ([\Lambda^s, \mathbf{u}] \cdot \partial_j \mathbf{u} | \epsilon u_j \Lambda^s \eta + (1 + \epsilon \eta) \Lambda^s u_j)_2 \\
&\quad - 2\epsilon^2 \sum_{j=1}^2 (\nabla ([\Lambda^s, \mathbf{u}] \cdot \partial_j \mathbf{u}) | \nabla \Lambda^s u_j)_2
\end{aligned}$$

which along with (4.50) and Lemma 3.2 implies that

$$\begin{aligned}
(4.56) \quad |I_{12}| &\lesssim \epsilon |\mathbf{u}|_{H^s}^2 (\epsilon |\mathbf{u}|_\infty |\eta|_{H^s} + (1 + \epsilon |\eta|_\infty) |\mathbf{u}|_{H^s}) + \epsilon^2 |\mathbf{u}|_{H^s} |\mathbf{u}|_{H^{s+1}}^2 \\
&\lesssim \epsilon |\mathbf{u}|_{H^s} |\mathbf{u}|_{X_\epsilon^s}^2.
\end{aligned}$$

Thanks to (4.55) and (4.56), we get that

$$(4.57) \quad |I_1| \lesssim \epsilon |\mathbf{u}|_{H^s} (|\eta|_{H^s}^2 + |\mathbf{u}|_{X_\epsilon^s}^2).$$

Estimate for I_2 . For I_2 , using the expressions of $M(U, D)$ and $S_U(D)$, we obtain that

$$\begin{aligned}
(4.58) \quad I_2 &= -4\epsilon (\mathbf{u} \cdot \nabla \Lambda^s \eta | \Lambda^s \eta)_2 - 2\{((1 + \epsilon \eta - \epsilon \Delta) \nabla \cdot \Lambda^s \mathbf{u} | \Lambda^s \eta)_2 \\
&\quad + (\nabla \Lambda^s \eta | (1 + \epsilon \eta - \epsilon \Delta) \Lambda^s \mathbf{u})_2\} \\
&\quad - 2\epsilon^2 \{(\mathbf{u} \cdot \nabla \Lambda^s \eta | \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 + \sum_{i=1}^2 (\mathbf{u} \cdot \partial_i \Lambda^s \mathbf{u} | u_i \Lambda^s \eta)_2\} \\
&\quad - 2\epsilon \{((1 + \epsilon \eta - \epsilon \Delta) \nabla \cdot \Lambda^s \mathbf{u} | \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 \\
&\quad + \sum_{i=1}^2 (\mathbf{u} \cdot \partial_i \Lambda^s \mathbf{u} | (1 + \epsilon \eta - \epsilon \Delta) \Lambda^s u_i)_2\} \\
&\stackrel{\text{def}}{=} I_{21} + I_{22} + I_{23} + I_{24}.
\end{aligned}$$

Integrating by parts, we get that

$$\begin{aligned}
I_{21} &= 2\epsilon (\nabla \cdot \mathbf{u} \Lambda^s \eta | \Lambda^s \eta)_2, \quad I_{22} = 2\epsilon (\nabla \eta \cdot \Lambda^s \mathbf{u} | \Lambda^s \eta)_2, \\
I_{23} &= 2\epsilon^2 (\nabla \cdot \mathbf{u} \Lambda^s \eta | \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 + 2\epsilon^2 (\Lambda^s \mathbf{u} | (\mathbf{u} \cdot \nabla) \mathbf{u} \Lambda^s \eta)_2, \\
I_{24} &= 2\epsilon^2 (\nabla \eta \cdot \Lambda^s \mathbf{u} | \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 + 2\epsilon \sum_{i=1}^2 ((1 + \epsilon \eta) \Lambda^s u_i | \partial_i \mathbf{u} \cdot \Lambda^s \mathbf{u})_2 \\
&\quad + 2\epsilon^2 \sum_{i=1}^2 (\nabla \Lambda^s u_i | \nabla (\partial_i \mathbf{u} \cdot \Lambda^s \mathbf{u}))_2,
\end{aligned}$$

which along with (4.50) and (4.51) implies that

$$(4.59) \quad |I_2| \lesssim \epsilon |\mathbf{u}|_{H^s} (|\eta|_{H^s}^2 + |\mathbf{u}|_{X_\epsilon^s}^2).$$

Thanks to (4.57) and (4.59), we obtain that

$$(4.60) \quad |I| \lesssim \epsilon |\mathbf{u}|_{H^s} (|\eta|_{H^s}^2 + |\mathbf{u}|_{X_\epsilon^s}^2).$$

Due to (1.2) with $b = c = d = 0, a = -1$, we deduce by using (4.50) and (4.51) that

$$(4.61) \quad |\partial_t \eta|_{H^{s-2}} + |\partial_t \mathbf{u}|_{H^{s-1}} \lesssim |\eta|_{H^s} + |\mathbf{u}|_{X_\epsilon^s}.$$

Combining (4.53), (4.54), (4.60) and (4.61), we finally get that

$$(4.62) \quad \frac{d}{dt} E_s(U) \lesssim \epsilon (|\eta|_{H^s} + |\mathbf{u}|_{X_\epsilon^s}) (|\eta|_{H^s}^2 + |\mathbf{u}|_{X_\epsilon^s}^2).$$

Due to (4.52), there holds

$$(4.63) \quad \frac{d}{dt} (E_s(U))^{\frac{1}{2}} \leq C_1 \epsilon E_s(U).$$

Then following the same line as the proofs of Theorems 4.3 and 4.4, one obtains that there exists $T > 0$ independent of ϵ such that (1.2)-(4.1) has a unique solution on time interval $[0, T/\epsilon]$. Moreover, (4.47) holds and Theorem 4.5 is proved. \square

Remark 4.2. The modelling of internal waves at the interface of a two-fluid system with different densities and in the presence of a rigid top leads, in an appropriate regime, to Boussinesq systems that are similar to those studied in the previous sections (see [8], section 3.1.3) and for which one can obtain the same results as in [36] or in the present paper. The same regime for a two-fluid system but with a free upper surface has been considered in [16], section 2.3.2. One gets a system of four equations for which the methods of [36] and of the present paper are likely to work, including the case of a slowly varying bottom. We also refer to [17] for further investigations on those extended Boussinesq systems, in particular for a construction of symmetrizable ones (modulo ϵ^2 terms).

4.4. The difficult case $a = b = d = 0, c < 0$. The method to solve the long time existence for this case is quite different from the other cases we dealt in the previous subsections and in the paper [36]. We will now quasilinearize the system by applying time together with space derivatives. The key point here is that we improve the regularity in space by improving the regularity in time (applying space derivatives to the system would cause a loss of derivatives).

We first state the long time existence result in the one dimensional case :

Theorem 4.6. *Let $a = b = d = 0, c = -1$. Assume that $\eta_0 \in X_{\epsilon^3}^2(\mathbb{R}), u_0 \in X_{\epsilon^2}^2(\mathbb{R})$ satisfy the (non-cavitation) condition*

$$(4.64) \quad 1 + \epsilon \eta_0 \geq H > 0, \quad H \in (0, 1),$$

Then there exists a constant \tilde{c}_0 such that for any $\epsilon \leq \epsilon_0 = \frac{1}{\tilde{c}_0(|\eta_0|_{X_{\epsilon^3}^2} + |u_0|_{X_{\epsilon^2}^2})}$, there exists $T > 0$ independent of ϵ , such that (3.21)-(4.1) has a unique solution (η, u) with $\eta \in C([0, T/\epsilon]; X_{\epsilon^3}^2(\mathbb{R}))$ and $u \in C([0, T/\epsilon]; X_{\epsilon^2}^2(\mathbb{R}))$. Moreover,

$$(4.65) \quad \begin{aligned} & \sup_{t \in [0, T/\epsilon]} (|\eta|_{X_{\epsilon^3}^2}^2 + |\eta_t|_{X_{\epsilon^2}^1}^2 + |\eta_{tt}|_{X_\epsilon^0}^2 + |u|_{X_{\epsilon^2}^2}^2 + |u_t|_{X_\epsilon^1}^2 + |u_{tt}|_2^2) \\ & \leq C (|\eta_0|_{X_{\epsilon^3}^2}^2 + |u_0|_{X_{\epsilon^2}^2}^2). \end{aligned}$$

Remark 4.3. System (3.21) can be viewed as the Saint-Venant (shallow water) system with surface tension and corresponds to system (A.1) in [18] with $\mu = 0, \delta = 1$. Thus the previous theorem can be compared to Theorem A.3 in [18].

Remark 4.4. It will be clear in the following proof that the regularity we choose for the initial data is the lowest possible one. One could also impose higher regularity on the initial data such as $\eta_0 \in X_{\epsilon^{3+k}}^{2+k}(\mathbb{R})$, $u_0 \in X_{\epsilon^{2+k}}^{2+k}(\mathbb{R})$ for $k \in \mathbb{N}$. In that case, one has to apply ∂_t to (1.2) for $k + 2$ times. For simplicity, we only consider the case $k = 0$.

Proof. We shall divide the proof into several steps. We first indicate how to obtain the a priori energy estimates. We shall use the a priori estimates to prove the existence of the solutions, in Section 5.

Step 1. Reduction of the system. Since $a = b = d = 0, c = -1$, we rewrite (1.2) in the form (3.21). Setting

$$v \stackrel{\text{def}}{=} (1 + \epsilon\eta)u,$$

the first equation of (3.21) becomes

$$\eta_t + v_x = 0.$$

Elementary calculations and the use of (3.21) yield the evolution equation for v :

$$v_t + (1 + \epsilon\eta)\eta_x - \epsilon(1 + \epsilon\eta)\eta_{xxx} + \epsilon\left(\frac{v^2}{1 + \epsilon\eta}\right)_x = 0.$$

Indeed, we have

$$\begin{aligned} \partial_t v &= (1 + \epsilon\eta)u_t + \epsilon u \eta_t \\ &= -(1 + \epsilon\eta)(\eta_x - \epsilon\eta_{xxx}) - \epsilon(1 + \epsilon\eta)uu_x - \epsilon uv_x \\ &= -(1 + \epsilon\eta)(\eta_x - \epsilon\eta_{xxx}) - \epsilon\left(\frac{v^2}{1 + \epsilon\eta}\right)_x. \end{aligned}$$

Then (3.21) is rewritten in terms of (η, v) as follows

$$(4.66) \quad \begin{cases} \eta_t + v_x = 0, \\ \frac{1}{1 + \epsilon\eta}v_t + \eta_x - \epsilon\eta_{xxx} + \frac{\epsilon}{1 + \epsilon\eta}\left(\frac{v^2}{1 + \epsilon\eta}\right)_x = 0. \end{cases}$$

We shall derive energy estimates for this system.

Step 2. Quasilinearization of (4.66). In this step, we shall quasilinearize the system (4.66) by applying to it ∂_t and ∂_t^2 . Applying ∂_t to the first equation of (4.66) leads to

$$\begin{aligned} \partial_t^2 \eta &= -\partial_x v_t = \left((1 + \epsilon\eta)\eta_x\right)_x - \epsilon\left((1 + \epsilon\eta)\eta_{xxx}\right)_x + \epsilon\left(\frac{v^2}{1 + \epsilon\eta}\right)_{xx} \\ &= \left((1 + \epsilon\eta)\eta_x\right)_x - \epsilon\left((1 + \epsilon\eta)\eta_{xxx}\right)_x + \frac{2\epsilon v}{1 + \epsilon\eta}v_{xx} \\ &\quad + 2\epsilon\frac{v_x^2}{1 + \epsilon\eta} - \frac{2\epsilon^2}{(1 + \epsilon\eta)^2}vv_x\eta_x - \epsilon^2\left(\frac{\eta_x v^2}{(1 + \epsilon\eta)^2}\right)_x. \end{aligned}$$

One notices that the last term $\frac{2\epsilon v}{1 + \epsilon\eta}v_{xx}$ in the second line of the above equality is the higher order term. Since by (4.66) $v_x = -\eta_t$, we rewrite this term as

$$\frac{2\epsilon v}{1 + \epsilon\eta}v_{xx} = -\frac{2\epsilon v}{1 + \epsilon\eta}\partial_x \eta_t.$$

Then we obtain

$$(4.67) \quad \eta_{tt} - ((1 + \epsilon\eta)\eta_x)_x + \epsilon((1 + \epsilon\eta)\eta_{xxx})_x + \frac{2\epsilon v}{1 + \epsilon\eta} \partial_x \eta_t = f,$$

with

$$(4.68) \quad f \stackrel{\text{def}}{=} 2\epsilon \frac{v_x^2}{1 + \epsilon\eta} - \frac{2\epsilon^2}{(1 + \epsilon\eta)^2} v v_x \eta_x - \epsilon^2 \left(\frac{\eta_x v^2}{(1 + \epsilon\eta)^2} \right)_x.$$

Applying ∂_t to the second equation of (4.66) one obtains

$$\begin{aligned} \partial_t^2 v &= -(1 + \epsilon\eta) \partial_x \eta_t + \epsilon(1 + \epsilon\eta) \partial_x^3 \eta_t - \epsilon \eta_t (\eta_x - \epsilon \eta_{xxx}) - \epsilon \left(\frac{v^2}{1 + \epsilon\eta} \right)_{xt} \\ &= (1 + \epsilon\eta) v_{xx} - \epsilon(1 + \epsilon\eta) v_{xxxx} - \epsilon \eta_t (\eta_x - \epsilon \eta_{xxx}) \\ &\quad - 2\epsilon v \partial_x \left(\frac{v_t}{1 + \epsilon\eta} \right) - \frac{2\epsilon v_x v_t}{1 + \epsilon\eta} + \epsilon^2 \left(\frac{v^2 \eta_t}{(1 + \epsilon\eta)^2} \right)_x. \end{aligned}$$

Then we get

$$(4.69) \quad \frac{1}{1 + \epsilon\eta} v_{tt} - v_{xx} + \epsilon v_{xxxx} + \frac{2\epsilon v}{1 + \epsilon\eta} \partial_x \left(\frac{v_t}{1 + \epsilon\eta} \right) = g,$$

with

$$(4.70) \quad g \stackrel{\text{def}}{=} -\frac{\epsilon \eta_t}{1 + \epsilon\eta} (\eta_x - \epsilon \eta_{xxx}) - \frac{2\epsilon v_x v_t}{(1 + \epsilon\eta)^2} + \frac{\epsilon^2}{1 + \epsilon\eta} \left(\frac{v^2 \eta_t}{(1 + \epsilon\eta)^2} \right)_x.$$

Combining (4.67) and (4.69), we obtain

$$(4.71) \quad \begin{cases} \eta_{tt} - ((1 + \epsilon\eta)\eta_x)_x + \epsilon((1 + \epsilon\eta)\eta_{xxx})_x + \frac{2\epsilon v}{1 + \epsilon\eta} \partial_x \eta_t = f, \\ \frac{1}{1 + \epsilon\eta} v_{tt} - v_{xx} + \epsilon v_{xxxx} + \frac{2\epsilon v}{1 + \epsilon\eta} \partial_x \left(\frac{v_t}{1 + \epsilon\eta} \right) = g, \end{cases}$$

with (f, g) being defined in (4.68) and (4.70).

We remark here that (4.71) is a diagonalization of (4.66) and that the principal linear part for both equations of (4.71) is the dispersive wave equation

$$(\partial_t^2 - \partial_x^2 + \epsilon \partial_x^4) \Psi.$$

The source terms (f, g) are of lower order. One can then derive the L^2 energy estimate for (4.71).

However, if we want to derive higher order energy estimates, it is not successful to apply ∂_x to the second equation of (4.71) since when ∂_x acts on the term $\frac{1}{1 + \epsilon\eta} v_{tt}$, it will turn out an uncontrolled term $-\frac{\epsilon \eta_x}{(1 + \epsilon\eta)^2} v_{tt}$. One has to apply instead ∂_t to (4.71). In other words, we shall improve the regularity of the unknowns by applying ∂_t^k (not ∂_x^α) to (4.66).

Denoting by $\eta' = \partial_t \eta$ and $v' = \partial_t v$, applying ∂_t to (4.71), it transpires that (η', v') satisfies the following system

$$(4.72) \quad \begin{cases} \eta'_{tt} - ((1 + \epsilon\eta)\eta'_x)_x + \epsilon((1 + \epsilon\eta)\eta'_{xxx})_x + \frac{2\epsilon v}{1 + \epsilon\eta} \partial_x \eta'_t = f', \\ \frac{1}{1 + \epsilon\eta} v'_{tt} - v'_{xx} + \epsilon v'_{xxxx} + \frac{2\epsilon v}{1 + \epsilon\eta} \partial_x \left(\frac{v'_t}{1 + \epsilon\eta} \right) = g', \end{cases}$$

where

$$(4.73) \quad \begin{aligned} f' &\stackrel{\text{def}}{=} \partial_t f + \epsilon(\eta_t \eta_x)_x - \epsilon^2(\eta_t \eta_{xxx})_x - 2\epsilon \left(\frac{v}{1 + \epsilon\eta} \right)_t \eta_{tx}, \\ g' &\stackrel{\text{def}}{=} \partial_t g + \frac{\epsilon \eta_t}{(1 + \epsilon\eta)^2} v_{tt} + \frac{2\epsilon^2 v}{1 + \epsilon\eta} \left(\frac{\eta_t v_t}{(1 + \epsilon\eta)^2} \right)_x - 2\epsilon \left(\frac{v}{1 + \epsilon\eta} \right)_t \left(\frac{v_t}{1 + \epsilon\eta} \right)_x. \end{aligned}$$

The principal part of (4.72) is the same as that of (4.71).

Step 3. Energy estimates for the quasilinear system (4.66)-(4.71)-(4.72). We shall derive energy estimates for (4.66), (4.71) and (4.72) under the assumptions

$$(4.74) \quad 1 + \epsilon\eta \geq H > 0,$$

and

$$(4.75) \quad |\eta(\cdot, t)|_{W^{1,\infty}} + |v(\cdot, t)|_{W^{1,\infty}} + |\eta(\cdot, t)|_\infty + |v(\cdot, t)|_\infty \leq c, \quad \text{for } t \in [0, T],$$

where the constant c is independent of ϵ but depends on the initial data. We remark that (4.74) and (4.75) are consequences of the assumption (4.64) and the *a priori* estimate (4.107) for (η, v) .

Step 3.1. Estimates for (4.66). We notice that the symmetrizer for the linear part of (4.66) is the matrix $\text{diag}(1 - \epsilon\partial_x^2, 1)$. Then taking the L^2 inner product of (4.66) by $((1 - \epsilon\partial_x^2)\eta, v)^T$ leads to

$$(4.76) \quad \frac{1}{2} \frac{d}{dt} E_0(t) = -\frac{\epsilon}{2} \left(\frac{\eta_t v}{(1 + \epsilon\eta)^2} |v \right)_2 - \epsilon \left(\left(\frac{v^2}{1 + \epsilon\eta} \right)_x \middle| \frac{v}{1 + \epsilon\eta} \right)_2,$$

where

$$E_0(t) \stackrel{\text{def}}{=} |\eta|_2^2 + \epsilon |\eta_x|_2^2 + \left(\frac{v}{1 + \epsilon\eta} |v \right)_2.$$

Thanks to (4.74) and (4.75), we have

$$(4.77) \quad E_0(t) \sim |\eta|_2^2 + \epsilon |\eta_x|_2^2 + |v|_{L^2}^2$$

By (4.74), the first term on the r.h.s of (4.76) is estimated as

$$\left| -\frac{\epsilon}{2} \left(\frac{\eta_t v}{(1 + \epsilon\eta)^2} |v \right)_2 \right| \lesssim \epsilon |v|_\infty |\eta_t|_2 |v|_2 \lesssim \epsilon |v|_\infty (|\eta_t|_2^2 + |v|_2^2),$$

while by integration by parts and (4.74), the second term on the r.h.s of (4.76) is estimated as

$$\begin{aligned} \left| -\epsilon \left(\left(\frac{v^2}{1 + \epsilon\eta} \right)_x \middle| \frac{v}{1 + \epsilon\eta} \right)_2 \right| &= \frac{\epsilon}{2} |(v_x | \frac{v}{1 + \epsilon\eta}|^2)_2| \\ &\lesssim \epsilon |v|_\infty |v|_2 |v_x|_2 \lesssim \epsilon |v|_\infty (|v|_2^2 + |v_x|_2^2). \end{aligned}$$

Then we obtain

$$(4.78) \quad \frac{1}{2} \frac{d}{dt} E_0(t) \lesssim \epsilon |v|_\infty (|\eta_t|_2^2 + |v|_2^2 + |v_x|_2^2).$$

Step 3.2. Estimates for (4.71). Taking the L^2 scalar product of the first equation of (4.71) by $(1 - \epsilon\partial_x^2)\eta_t$, we obtain

$$\begin{aligned} &(\eta_{tt} | (1 - \epsilon\partial_x^2)\eta_t)_2 - (((1 + \epsilon\eta)\eta_x)_x | (1 - \epsilon\partial_x^2)\eta_t)_2 \\ &+ \epsilon(((1 + \epsilon\eta)\eta_{xxx})_x | (1 - \epsilon\partial_x^2)\eta_t)_2 + \left(\frac{2\epsilon v}{1 + \epsilon\eta} \partial_x \eta_t | (1 - \epsilon\partial_x^2)\eta_t \right)_2 = (f | (1 - \epsilon\partial_x^2)\eta_t)_2. \end{aligned}$$

Integration by parts gives

$$\begin{aligned}
& (\eta_{tt} | (1 - \epsilon \partial_x^2) \eta_t)_2 = \frac{1}{2} \frac{d}{dt} (|\eta_t|_2^2 + \epsilon |\eta_{tx}|_2^2), \\
& - ((1 + \epsilon \eta) \eta_x)_x | (1 - \epsilon \partial_x^2) \eta_t)_2 = \frac{1}{2} \frac{d}{dt} ((1 + \epsilon \eta) \eta_x | \eta_x)_2 + \epsilon ((1 + \epsilon \eta) \eta_{xx} | \eta_{xx})_2 \\
& \quad - \frac{\epsilon}{2} (\eta_t \eta_x | \eta_x)_2 - \frac{\epsilon^2}{2} (\eta_t \eta_{xx} | \eta_{xx})_2 - \epsilon^2 (\partial_x (\eta_x \eta_x) | \eta_{tx})_2, \\
& \epsilon ((1 + \epsilon \eta) \eta_{xxx})_x | (1 - \epsilon \partial_x^2) \eta_t)_2 = \frac{\epsilon}{2} \frac{d}{dt} ((1 + \epsilon \eta) \eta_{xx} | \eta_{xx})_2 + \epsilon ((1 + \epsilon \eta) \eta_{xxx} | \eta_{xxx})_2 \\
& \quad - \frac{\epsilon^2}{2} (\eta_t \eta_{xx} | \eta_{xx})_2 - \frac{\epsilon^3}{2} (\eta_t \eta_{xxx} | \eta_{xxx})_2 + \epsilon^2 (\eta_x \eta_{xx} | \eta_{tx})_2.
\end{aligned}$$

Then we obtain that

$$\begin{aligned}
(4.79) \quad & \frac{1}{2} \frac{d}{dt} E_{11}(t) + \left(\frac{2\epsilon v}{1 + \epsilon \eta} \partial_x \eta_t | (1 - \epsilon \partial_x^2) \eta_t \right)_2 \\
& = \frac{\epsilon}{2} (\eta_t \eta_x | \eta_x)_2 + \epsilon^2 (\eta_t \eta_{xx} | \eta_{xx})_2 + \epsilon^2 (\eta_{xx} \eta_x | \eta_{tx})_2 \\
& \quad + \frac{\epsilon^3}{2} (\eta_t \eta_{xxx} | \eta_{xxx})_2 + (f | (1 - \epsilon \partial_x^2) \eta_t)_2
\end{aligned}$$

where

$$\begin{aligned}
E_{11}(t) & \stackrel{\text{def}}{=} |\eta_t|_2^2 + \epsilon |\eta_{tx}|_2^2 + ((1 + \epsilon \eta) \eta_x | \eta_x)_2 + 2\epsilon ((1 + \epsilon \eta) \eta_{xx} | \eta_{xx})_2 \\
& \quad + \epsilon^2 ((1 + \epsilon \eta) \eta_{xxx} | \eta_{xxx})_2.
\end{aligned}$$

By (4.74) and (4.75), we have

$$(4.80) \quad E_{11}(t) \sim |\eta_t|_2^2 + \epsilon |\eta_{tx}|_2^2 + |\eta_x|_2^2 + \epsilon |\eta_{xx}|_2^2 + \epsilon^2 |\eta_{xxx}|_2^2.$$

Now, we estimate the second term on the l.h.s of (4.79). Integrating by parts, we have

$$\begin{aligned}
& - \left(\frac{2\epsilon v}{1 + \epsilon \eta} \partial_x \eta_t | (1 - \epsilon \partial_x^2) \eta_t \right)_2 = -\epsilon \left(\frac{v}{1 + \epsilon \eta} | \partial_x (|\eta_t|^2) - \epsilon \partial_x (|\eta_{tx}|^2) \right)_2 \\
& = \epsilon (\partial_x \left(\frac{v}{1 + \epsilon \eta} \right) | |\eta_t|^2 - \epsilon |\eta_{tx}|^2)_2,
\end{aligned}$$

which along with (4.74) and (4.75) implies that

$$(4.81) \quad \left| \left(\frac{2\epsilon v}{1 + \epsilon \eta} \partial_x \eta_t | (1 - \epsilon \partial_x^2) \eta_t \right)_2 \right| \lesssim \epsilon (|\eta_x|_\infty + |v_x|_\infty) (|\eta_t|_2^2 + \epsilon |\eta_{tx}|_2^2).$$

Due to (4.79) and (4.81), we get

$$\begin{aligned}
(4.82) \quad & \frac{1}{2} \frac{d}{dt} E_{11}(t) \lesssim \epsilon (|\eta_t|_\infty + |\eta_x|_\infty + |v_x|_\infty) (|\eta_x|_2^2 + \epsilon |\eta_{xx}|_2^2 \\
& \quad + \epsilon^2 |\eta_{xxx}|_2^2 + |\eta_t|_2^2 + \epsilon |\eta_{tx}|_2^2) + |f|_2 |\eta_t|_2 + \epsilon |f_x|_2 |\eta_{tx}|_2.
\end{aligned}$$

Taking the L^2 scalar product of the second equation of (4.71) by v_t yields

$$(4.83) \quad \frac{1}{2} \frac{d}{dt} E_{12}(t) + \left(\frac{2\epsilon v}{1 + \epsilon \eta} \partial_x \left(\frac{v_t}{1 + \epsilon \eta} \right) | v_t \right)_2 = -\frac{1}{2} \left(\partial_t \left(\frac{1}{1 + \epsilon \eta} \right) v_t | v_t \right)_2 + (g | v_t)_2$$

where

$$E_{12}(t) \stackrel{\text{def}}{=} \left(\frac{v_t}{1 + \epsilon \eta} | v_t \right)_2 + |v_x|_2^2 + \epsilon |v_{xx}|_2^2.$$

Thanks to (4.74), we have

$$(4.84) \quad E_{12}(t) \sim |v_t|_2^2 + |v_x|_2^2 + \epsilon |v_{xx}|_2^2.$$

Similarly as for (4.81), integration by parts on the second term on the l.h.s of (4.83) leads to

$$(4.85) \quad |(\frac{2\epsilon v}{1+\epsilon\eta} \partial_x (\frac{v_t}{1+\epsilon\eta}) |v_t)_2| = \epsilon |(\partial_x v | \frac{v_t}{1+\epsilon\eta})_2| \lesssim \epsilon |v_x|_\infty |v_t|_2^2.$$

We can also bound the first term on the r.h.s of (4.83) as follows

$$| -\frac{1}{2} (\partial_t (\frac{1}{1+\epsilon\eta}) v_t |v_t)_2 | \lesssim \epsilon |\eta_t|_\infty |v_t|_2^2,$$

which along with (4.83) and (4.85) gives rise to

$$(4.86) \quad \frac{1}{2} \frac{d}{dt} E_{12}(t) \lesssim \epsilon (|v_x|_\infty + |\eta_t|_\infty) |v_t|_2^2 + |g|_2 |v_t|_2.$$

Thanks to the expressions (4.68) and (4.70), using the assumptions (4.74) and (4.75), we estimate the source terms $|f|_2 + \epsilon^{\frac{1}{2}} |f_x|_2$ and $|g|_2$ as follows

$$(4.87) \quad \begin{aligned} |f|_2 + \epsilon^{\frac{1}{2}} |f_x|_2 + |g|_2 &\lesssim \epsilon (|\eta_x|_\infty + |v|_\infty + |v_x|_\infty + |\eta_t|_\infty) \\ &\times (|\eta_x|_2 + \epsilon^{\frac{1}{2}} |\eta_{xx}|_2 + \epsilon |\eta_{xxx}|_2 + |v_t|_2 + |v_x|_2 + \epsilon |v_{xx}|_2). \end{aligned}$$

Now, we define $E_1(t) \stackrel{\text{def}}{=} E_{11}(t) + E_{12}(t)$. Then (4.80) and (4.84) yields

$$(4.88) \quad E_1(t) \sim |\eta_x|_{X_{\epsilon^2}^0}^2 + |\eta_t|_{X_\epsilon^0}^2 + |v_x|_{X_\epsilon^0}^2 + |v_t|_2^2.$$

where $|\cdot|_{X_{\epsilon^k}^s}^2 = |\cdot|_{H^s}^2 + \epsilon^k |\cdot|_{H^{s+k}}^2$.

Combining estimates (4.82), (4.86) and (4.87), using (4.88), we obtain

$$(4.89) \quad \frac{1}{2} \frac{d}{dt} E_1(t) \lesssim \epsilon (|\eta_t|_\infty + |\eta_x|_\infty + |v|_\infty + |v_x|_\infty) E_1(t), \quad t \in [0, T].$$

Step 3.3. Estimates for (4.72). Since (4.72) has the same form as (4.71), we have a similar estimate as (4.86) for the second equation of (4.72), that is,

$$(4.90) \quad \frac{1}{2} \frac{d}{dt} E_{22}(t) \lesssim \epsilon (|v_x|_\infty + |\eta_t|_\infty) |v'_t|_2^2 + |g'|_2 |v'_t|_2,$$

where

$$(4.91) \quad \begin{aligned} E_{22}(t) &\stackrel{\text{def}}{=} (\frac{v'_t}{1+\epsilon\eta} |v'_t)_2 + |v'_x|_2^2 + \epsilon |v'_{xx}|_2^2 \\ &\sim |v'_t|_2^2 + |v'_x|_2^2 + \epsilon |v'_{xx}|_2^2 \sim |v'_t|_2^2 + |v'_x|_{X_\epsilon^0}^2. \end{aligned}$$

Taking the L^2 scalar product of the first equation of (4.72) with $(1 - \epsilon \partial_x^2) \eta'_t$, we obtain (see the similar derivation of (4.79)) that

$$(4.92) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} E_{21}(t) + (\frac{2\epsilon v}{1+\epsilon\eta} \partial_x \eta'_t | (1 - \epsilon \partial_x^2) \eta'_t)_2 \\ &= \frac{\epsilon}{2} (\eta_t \eta'_x | \eta'_x)_2 + \epsilon^2 (\eta_t \eta'_{xx} | \eta'_{xx})_2 + \epsilon^2 (\eta_{xx} \eta'_x | \eta'_{tx})_2 \\ &\quad + \frac{\epsilon^3}{2} (\eta_t \eta'_{xxx} | \eta'_{xxx})_2 + (f' | (1 - \epsilon \partial_x^2) \eta'_t)_2 \end{aligned}$$

where

$$E_{21}(t) \stackrel{\text{def}}{=} |\eta'_t|_2^2 + \epsilon |\eta'_{tx}|_2^2 + ((1 + \epsilon\eta)\eta'_x | \eta'_x)_2 + 2\epsilon((1 + \epsilon\eta)\eta'_{xx} | \eta'_{xx})_2 \\ + \epsilon^2((1 + \epsilon\eta)\eta'_{xxx} | \eta'_{xxx})_2.$$

Thanks to (4.64) and (4.75), we have

$$(4.93) \quad E_{21}(t) \sim |\eta'_t|_2^2 + \epsilon |\eta'_{tx}|_2^2 + |\eta'_x|_2^2 + \epsilon |\eta'_{xx}|_2^2 + \epsilon^2 |\eta'_{xxx}|_2^2 \sim |\eta'_t|_{X_\epsilon^0}^2 + |\eta'_x|_{X_{\epsilon^2}^0}^2.$$

Similarly to the derivation of (4.82), we obtain that

$$(4.94) \quad \frac{1}{2} \frac{d}{dt} E_{21}(t) \lesssim \epsilon (|\eta_t|_\infty + |\eta_x|_\infty + \epsilon^{\frac{1}{2}} |\eta_{xx}|_\infty + |v_x|_\infty) E_{21}(t) \\ + |f'|_2 |\eta'_t|_2 + \epsilon |f'_x|_2 |\eta'_{tx}|_2.$$

In order to get the final estimate on system (4.72), we have to estimate the source terms $|f'|_2 + \epsilon^{\frac{1}{2}} |f'_x|_2$ and $|g'|_2$. Thanks to the expressions of f' and g' in (4.73) and the expressions of f and g in (4.68) and (4.70), using (4.74) and (4.75), after tedious but elementary calculations, we obtain that

$$(4.95) \quad |f'|_2 + \epsilon^{\frac{1}{2}} |f'_x|_2 + |g'|_2 \lesssim \epsilon (|\eta_x|_\infty + \epsilon^{\frac{1}{2}} |\eta_{xx}|_\infty + \epsilon |\eta_{xxx}|_\infty + |\eta_t|_\infty \\ + \epsilon^{\frac{1}{2}} |\eta_{tx}|_\infty + |v_x|_\infty + \epsilon^{\frac{1}{2}} |v_{xx}|_\infty + |v_t|_\infty) (|\eta_{xx}|_2 + \epsilon^{\frac{1}{2}} |\eta_{xxx}|_2 \\ + \epsilon |\eta_{xxxx}|_2 + \epsilon^{\frac{3}{2}} |\eta_{xxxxx}|_2 + |\eta_t|_2 + |\eta_{tx}|_2 + \epsilon^{\frac{1}{2}} |\eta_{ttx}|_2 \\ + \epsilon |\eta_{txxx}|_2 + |v_{xx}|_2 + |v_t|_2 + |v_{tx}|_2 + \epsilon^{\frac{1}{2}} |v_{txx}|_2 + |v_{tt}|_2),$$

where we used $\eta' = \eta_t$ and $v' = v_t$.

Now, we define $E_2(t) \stackrel{\text{def}}{=} E_{21}(t) + E_{22}(t)$. Then (4.91) and (4.93) yields

$$(4.96) \quad E_2(t) \sim |\eta'_t|_{X_\epsilon^0}^2 + |\eta'_x|_{X_{\epsilon^2}^0}^2 + |v'_t|_2^2 + |v'_x|_{X_\epsilon^0}^2.$$

Thanks to (4.90), (4.94) and (4.95), using the interpolation inequality (4.2), we obtain that

$$(4.97) \quad \frac{1}{2} \frac{d}{dt} E_2(t) \lesssim \epsilon (|\eta_x|_\infty + \epsilon^{\frac{1}{2}} |\eta_{xx}|_\infty + \epsilon |\eta_{xxx}|_\infty + |\eta_t|_\infty \\ + \epsilon^{\frac{1}{2}} |\eta_{tx}|_\infty + |v_x|_\infty + \epsilon^{\frac{1}{2}} |v_{xx}|_\infty + |v_t|_\infty) \\ \times (|\eta|_{X_{\epsilon^3}^2}^2 + |\eta_t|_{X_{\epsilon^2}^1}^2 + |\eta_{tt}|_{X_\epsilon^0}^2 + |v|_{H^2}^2 + |v_t|_{X_\epsilon^1}^2 + |v_{tt}|_2^2),$$

where we replaced η' , v' by η_t , v_t respectively in the bound.

Step 4. The final estimate on (4.66). Before closing the *a priori estimates*, we first define the energy functional associated to the quasilinear system (4.66)-(4.71)-(4.72) as

$$(4.98) \quad E(t) \stackrel{\text{def}}{=} E_0(t) + E_1(t) + E_2(t).$$

Notice that $\eta' = \eta_t$ and $v' = v_t$. Then (4.77), (4.88) and (4.96) yield that

$$(4.99) \quad E(t) \sim |\eta|_{X_{\epsilon^2}^2}^2 + |\eta_t|_{X_{\epsilon^2}^1}^2 + |\eta_{tt}|_{X_\epsilon^0}^2 + |v|_{X_\epsilon^1}^2 + |v_t|_{X_\epsilon^1}^2 + |v_{tt}|_2^2.$$

In order to close the energy estimate, we also need to define the total energy functional for (4.66) as follows:

$$(4.100) \quad \mathcal{E}(t) \stackrel{\text{def}}{=} |\eta|_{X_{\epsilon^3}^2}^2 + |\eta_t|_{X_{\epsilon^2}^1}^2 + |\eta_{tt}|_{X_\epsilon^0}^2 + |v|_{X_{\epsilon^2}^2}^2 + |v_t|_{X_\epsilon^1}^2 + |v_{tt}|_2^2.$$

With the definitions (4.98) and (4.100), using the interpolation inequality (4.2) and the Sobolev inequality $|\cdot|_{L^\infty(\mathbb{R})} \lesssim |\cdot|_{H^1(\mathbb{R})}$, the energy estimates (4.78), (4.89) and (4.97) give rise to

$$(4.101) \quad \frac{1}{2} \frac{d}{dt} E(t) \lesssim \epsilon \mathcal{E}(t)^{\frac{3}{2}}.$$

To finish the proof, we have to show that

$$(4.102) \quad \mathcal{E}(t) \sim E(t).$$

Indeed, thanks to (4.99) and (4.100), we have

$$\mathcal{E}(t) \sim E(t) + |\eta|_{X_{\epsilon^3}^2}^2 + |v|_{X_{\epsilon^2}^2}^2.$$

Then we only need to show that

$$|\eta|_{X_{\epsilon^3}^2}^2 + |v|_{X_{\epsilon^2}^2}^2 \lesssim E(t).$$

That is to say, we shall recover the regularity in space through the regularity in time. More precisely, (4.66) yields

$$(4.103) \quad v_x = -\eta_t, \quad (1 - \epsilon \partial_x^2) \eta_x = -\frac{v_t}{1 + \epsilon \eta} - \frac{\epsilon}{1 + \epsilon \eta} \left(\frac{v^2}{1 + \epsilon \eta} \right)_x.$$

The first equation of (4.103) shows

$$(4.104) \quad |v|_{X_{\epsilon^2}^2}^2 = |v|_{H^2}^2 + \epsilon^2 |v|_{H^{2+2}}^2 \lesssim |v|_{H^1}^2 + |\eta_{tx}|_2^2 + \epsilon^2 |\eta_{txxx}|_2^2 \lesssim E(t),$$

where we used (4.75). While the second equation of (4.103), (4.75) and (4.104) imply

$$\begin{aligned} |\eta|_{X_{\epsilon^3}^2}^2 &\sim |\eta|_{H^1}^2 + |(1 - \epsilon \partial_x^2) \eta_{xx}|_2^2 + \epsilon |(1 - \epsilon \partial_x^2) \eta_{xxx}|_2^2 \\ &\lesssim |\eta|_{X_{\epsilon^2}^1}^2 + |v|_{X_{\epsilon^2}^1}^2 + |v_t|_{X_{\epsilon^2}^1}^2 \lesssim E(t), \end{aligned}$$

which achieves the proof of (4.102). Due to (4.102) and (4.101), we have

$$(4.105) \quad \frac{1}{2} \frac{d}{dt} E(t) \lesssim \epsilon E(t)^{\frac{3}{2}}.$$

Step 5. Initial data for the quasilinear system and final estimate. In this step, we have to derive the regularity of the initial data to the quasilinear system through the system (4.66) and the regularity of the initial data (η_0, v_0) . The first equation of (4.66) shows that

$$|\eta'|_{t=0}|_{X_{\epsilon^2}^1} = |\eta_t|_{t=0}|_{X_{\epsilon^2}^1} = |\partial_x v_0|_{X_{\epsilon^2}^1} \lesssim |v_0|_{X_{\epsilon^2}^2},$$

while the second equation of (4.66) shows that

$$\begin{aligned} |v'|_{t=0}|_{X_{\epsilon^2}^1} &= |v_t|_{t=0}|_{X_{\epsilon^2}^1} \\ &\lesssim |(1 + \epsilon \eta_0)(1 - \epsilon \partial_x^2) \partial_x \eta_0|_{X_{\epsilon^2}^1} + \epsilon \left| \left(\frac{v_0^2}{1 + \epsilon \eta_0} \right)_x \right|_{X_{\epsilon^2}^1} \\ &\lesssim |\eta_0|_{X_{\epsilon^3}^2} + |v_0|_{X_{\epsilon^2}^2}, \end{aligned}$$

where we assume that $|\eta_0|_{X_{\epsilon^3}^2} + |v_0|_{X_{\epsilon^2}^2} \leq C$ and $\epsilon \leq \epsilon_0$ with ϵ_0 small enough.

Thanks to (4.72), we can also infer that

$$\begin{aligned} |\eta_t'|_{t=0}|_{X_{\epsilon^2}^0} + |v_t'|_{t=0}|_2 &= |\eta_{tt}|_{t=0}|_{X_{\epsilon^2}^0} + |v_{tt}|_{t=0}|_2 \\ &\lesssim |\eta_0|_{X_{\epsilon^3}^2} + |v_0|_{X_{\epsilon^2}^2} \end{aligned}$$

provided that $|\eta_0|_{X_{\epsilon^3}^2} + |v_0|_{X_{\epsilon^2}^2} \leq C$ and $\epsilon \leq \epsilon_0$ with ϵ_0 small enough.

Thus, we have

$$(4.106) \quad E(0) \sim \mathcal{E}(0) \lesssim |\eta_0|_{X_{\epsilon^3}^2}^2 + |v_0|_{X_{\epsilon^2}^2}^2.$$

Step 6. Existence and uniqueness. The estimates (4.105) and (4.106) are crucial to prove the existence of $T > 0$ independent of ϵ such that (4.66) has a unique solution (η, v) on a time interval $[0, T/\epsilon]$ with initial data $(\eta_0, v_0) \in X_{\epsilon^3}^2 \times X_{\epsilon^2}^2$ satisfying moreover by (4.105) and (4.102) the estimate

$$(4.107) \quad \sup_{t \in [0, T/\epsilon]} \mathcal{E}(t) \lesssim |\eta_0|_{X_{\epsilon^3}^2}^2 + |v_0|_{X_{\epsilon^2}^2}^2.$$

We shall precise the existence proof in the following Section 5.

Notice that $v = (1 + \epsilon\eta)u$. Then we have obtained the long time estimate of solutions to the original Boussinesq system (1.2)-(4.1) with $a = b = d = 0, c = -1$ together to the energy estimate (4.65). \square

Now we state the long time existence result for the two-dimensional case.

Theorem 4.7. *Let $a = b = d = 0, c = -1$. Assume that $\eta_0 \in X_{\epsilon^4}^3(\mathbb{R}^2), \mathbf{u}_0 \in X_{\epsilon^3}^3(\mathbb{R}^2)$ satisfy the curl free condition $\text{curl } \mathbf{u}_0 = 0$ and the (non-cavitation) condition*

$$(4.108) \quad 1 + \epsilon\eta_0 \geq H > 0, \quad H \in (0, 1).$$

Then there exists a constant \tilde{c}_0 such that for any $\epsilon \leq \epsilon_0 = \frac{1}{\tilde{c}_0(|\eta_0|_{X_{\epsilon^4}^3} + |\mathbf{u}_0|_{X_{\epsilon^3}^3})}$, there exists $T > 0$ independent of ϵ , such that (1.2)-(4.1) has a unique solution $(\eta, \mathbf{u})^T$ with $\eta \in C([0, T/\epsilon]; X_{\epsilon^4}^3(\mathbb{R}^2))$ and $\mathbf{u} \in C([0, T/\epsilon]; X_{\epsilon^3}^3(\mathbb{R}^2))$. Moreover,

$$(4.109) \quad \sup_{t \in [0, T/\epsilon]} (|\eta|_{X_{\epsilon^4}^3}^2 + |\eta_t|_{X_{\epsilon^3}^2}^2 + |\eta_{tt}|_{X_{\epsilon^2}^1}^2 + |\eta_{ttt}|_{X_{\epsilon^1}^0}^2 + |\mathbf{u}|_{X_{\epsilon^3}^3}^2 + |\mathbf{u}_t|_{X_{\epsilon^2}^2}^2 + |\mathbf{u}_{tt}|_{X_{\epsilon^1}^1}^2 + |\mathbf{u}_{ttt}|_2^2) \leq C(|\eta_0|_{X_{\epsilon^4}^3}^2 + |\mathbf{u}_0|_{X_{\epsilon^3}^3}^2).$$

Remark 4.5. By simplicity, we assume that $\text{curl } \mathbf{u}_0 = 0$. Actually, the equation of \mathbf{u} shows that $\partial_t \text{curl } \mathbf{u}(t, \cdot) = 0$ so that $\text{curl } \mathbf{u}$ is preserved as time evolves. In fact, as pointed out to us by Vincent Duchêne, considering the term $\nabla(|\mathbf{u}|^2)$ is not physically relevant outside the irrotational case. When $\text{curl } \mathbf{u} \neq 0$, one should use instead the term $\mathbf{u} \cdot \nabla \mathbf{u}$, but then the corresponding system is to our knowledge not rigorously justified (see [12] for Green-Naghdi type systems).

Proof. Since the proof is similar to that of Theorem 4.6, we only sketch it. We also divide the proof into several steps. Again we only indicate how to obtain the *a priori* estimates. The existence proof which is similar to the one-dimensional case is postponed to the following Section 5.

Step 1. Reduction of the system. Since $a = b = d = 0, c = -1$, we first rewrite (1.2) in the form (3.26). Setting

$$\mathbf{v} \stackrel{\text{def}}{=} (1 + \epsilon\eta)\mathbf{u},$$

the first equation of (3.26) becomes

$$\eta_t + \nabla \cdot \mathbf{v} = 0.$$

To get the evolution equation for \mathbf{v} , we first get from the second equation of (3.26) that

$$\partial_t \operatorname{curl} \mathbf{u} = 0,$$

which along with the assumption that $\operatorname{curl} \mathbf{u}_0 = 0$ implies that $\operatorname{curl} \mathbf{u} = 0$. Then $\nabla(|\mathbf{u}|^2) = 2\mathbf{u} \cdot \nabla \mathbf{u}$ and the second equation of (3.26) becomes to

$$\partial_t \mathbf{u} + \nabla \eta - \epsilon \nabla \Delta \eta + \epsilon \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{0}.$$

Similarly as one-dimensional case, elementary calculations and the use of the above equation yield the evolution equation for \mathbf{v} :

$$\mathbf{v}_t + (1 + \epsilon \eta) \nabla \eta - \epsilon (1 + \epsilon \eta) \nabla \Delta \eta + \epsilon \nabla \cdot \left(\frac{\mathbf{v}}{1 + \epsilon \eta} \otimes \mathbf{v} \right) = \mathbf{0},$$

where $\left(\nabla \cdot (\mathbf{u} \otimes \mathbf{v}) \right) \stackrel{\text{def}}{=} \partial_j (u^i v^j)$. Then (3.26) is rewritten in terms of (η, \mathbf{v}) as follows

$$(4.110) \quad \begin{cases} \eta_t + \nabla \cdot \mathbf{v} = 0, \\ \frac{1}{1 + \epsilon \eta} \mathbf{v}_t + \nabla \eta - \epsilon \nabla \Delta \eta + \frac{\epsilon}{1 + \epsilon \eta} \nabla \cdot \left(\frac{\mathbf{v}}{1 + \epsilon \eta} \otimes \mathbf{v} \right) = \mathbf{0}. \end{cases}$$

We shall derive energy estimates for this system.

Step 2. Quasilinearization of (4.110). In this step, we shall quasilinearize the system (4.110) by applying to it ∂_t , ∂_t^2 and ∂_t^3 . Applying ∂_t to the first equation of (4.110) leads to

$$\begin{aligned} \partial_t^2 \eta &= -\nabla \cdot \mathbf{v}_t = \nabla \cdot ((1 + \epsilon \eta) \nabla \eta) - \epsilon \nabla \cdot ((1 + \epsilon \eta) \nabla \Delta \eta) + \epsilon \nabla \cdot \left[\nabla \cdot \left(\frac{\mathbf{v}}{1 + \epsilon \eta} \otimes \mathbf{v} \right) \right] \\ &= \nabla \cdot ((1 + \epsilon \eta) \nabla \eta) - \epsilon \nabla \cdot ((1 + \epsilon \eta) \nabla \Delta \eta) + \frac{2\epsilon \mathbf{v}}{1 + \epsilon \eta} \cdot \nabla (\nabla \cdot \mathbf{v}) + \frac{\epsilon |\nabla \cdot \mathbf{v}|^2}{1 + \epsilon \eta} \\ &\quad + 2\epsilon \mathbf{v} \cdot \nabla \left(\frac{1}{1 + \epsilon \eta} \right) (\nabla \cdot \mathbf{v}) + \epsilon \mathbf{v} \cdot \nabla \left[\mathbf{v} \cdot \nabla \left(\frac{1}{1 + \epsilon \eta} \right) \right] + \sum_{i,j=1,2} \partial_j \left(\frac{v^i}{1 + \epsilon \eta} \right) \partial_i v^j. \end{aligned}$$

One notices that the third term $\frac{2\epsilon \mathbf{v}}{1 + \epsilon \eta} \cdot \nabla (\nabla \cdot \mathbf{v})$ in the second line of the above equality is the higher order term. Since by (4.110) $\nabla \cdot \mathbf{v} = -\eta_t$, we rewrite this term as

$$\frac{2\epsilon \mathbf{v}}{1 + \epsilon \eta} \cdot \nabla (\nabla \cdot \mathbf{v}) = -\frac{2\epsilon \mathbf{v}}{1 + \epsilon \eta} \cdot \nabla \eta_t.$$

Then we obtain

$$(4.111) \quad \eta_{tt} - \nabla \cdot ((1 + \epsilon \eta) \nabla \eta) + \epsilon \nabla \cdot ((1 + \epsilon \eta) \nabla \Delta \eta) + \frac{2\epsilon \mathbf{v}}{1 + \epsilon \eta} \cdot \nabla \eta_t = f,$$

with

$$(4.112) \quad \begin{aligned} f &\stackrel{\text{def}}{=} \frac{\epsilon |\nabla \cdot \mathbf{v}|^2}{1 + \epsilon \eta} + 2\epsilon \mathbf{v} \cdot \nabla \left(\frac{1}{1 + \epsilon \eta} \right) (\nabla \cdot \mathbf{v}) \\ &\quad + \epsilon \mathbf{v} \cdot \nabla \left[\mathbf{v} \cdot \nabla \left(\frac{1}{1 + \epsilon \eta} \right) \right] + \epsilon \sum_{i,j=1,2} \partial_j \left(\frac{v^i}{1 + \epsilon \eta} \right) \partial_i v^j. \end{aligned}$$

Applying ∂_t to the second equation of (4.110) one obtains

$$\partial_t^2 \mathbf{v} = -(1 + \epsilon \eta) \nabla \eta_t + \epsilon (1 + \epsilon \eta) \nabla \Delta \eta_t - \epsilon \eta_t (\nabla \eta - \epsilon \nabla \Delta \eta) - \epsilon \nabla \cdot \left(\frac{\mathbf{v}}{1 + \epsilon \eta} \otimes \mathbf{v} \right)_t.$$

Notice that $\eta_t = -\nabla \cdot \mathbf{v}$, we obtain the reduced equation for \mathbf{v}

$$(4.113) \quad \begin{aligned} & \frac{1}{1+\epsilon\eta} \mathbf{v}_{tt} - \nabla(\nabla \cdot \mathbf{v}) + \epsilon \nabla \Delta(\nabla \cdot \mathbf{v}) + \left(\frac{\epsilon \mathbf{v}}{1+\epsilon\eta} \cdot \nabla \right) \left(\frac{\mathbf{v}_t}{1+\epsilon\eta} \right) \\ & + \frac{\epsilon \mathbf{v}}{(1+\epsilon\eta)^2} (\nabla \cdot \mathbf{v}_t) = \mathbf{g}, \end{aligned}$$

with

$$(4.114) \quad \begin{aligned} \mathbf{g} \stackrel{\text{def}}{=} & -\frac{\epsilon \eta_t}{1+\epsilon\eta} (\nabla \eta - \epsilon \nabla \Delta \eta) + \left(\frac{\epsilon^2 \mathbf{v}}{1+\epsilon\eta} \cdot \nabla \right) \left(\frac{\mathbf{v} \eta_t}{(1+\epsilon\eta)^2} \right) \\ & - \frac{\epsilon}{1+\epsilon\eta} \partial_t \left(\frac{\mathbf{v}}{1+\epsilon\eta} \right) (\nabla \cdot \mathbf{v}). \end{aligned}$$

Combining (4.111) and (4.113), we obtain

$$(4.115) \quad \begin{cases} \eta_{tt} - \nabla \cdot ((1+\epsilon\eta) \nabla \eta) + \epsilon \nabla \cdot ((1+\epsilon\eta) \nabla \Delta \eta) + \frac{2\epsilon \mathbf{v}}{1+\epsilon\eta} \cdot \nabla \eta_t = f, \\ \frac{1}{1+\epsilon\eta} \mathbf{v}_{tt} - \nabla(\nabla \cdot \mathbf{v}) + \epsilon \nabla \Delta(\nabla \cdot \mathbf{v}) \\ \quad + \left(\frac{\epsilon \mathbf{v}}{1+\epsilon\eta} \cdot \nabla \right) \left(\frac{\mathbf{v}_t}{1+\epsilon\eta} \right) + \frac{\epsilon \mathbf{v}}{(1+\epsilon\eta)^2} (\nabla \cdot \mathbf{v}_t) = \mathbf{g}, \end{cases}$$

with (f, \mathbf{g}) being defined in (4.112) and (4.114).

We also remark here that (4.115) is a diagonalization of (4.110) and that the principal linear part for both equations of (4.115) is the dispersive wave

$$(\partial_t^2 - \Delta + \epsilon \Delta^2) \Psi.$$

The source terms (f, \mathbf{g}) are of lower order. One can then derive the L^2 energy estimate for (4.115).

Denoting by $\eta' = \partial_t \eta$ and $\mathbf{v}' = \partial_t \mathbf{v}$, applying ∂_t to (4.115), it transpires that (η', \mathbf{v}') satisfies the following system

$$(4.116) \quad \begin{cases} \eta'_{tt} - \nabla \cdot ((1+\epsilon\eta) \nabla \eta') + \epsilon \nabla \cdot ((1+\epsilon\eta) \nabla \Delta \eta') + \frac{2\epsilon \mathbf{v}}{1+\epsilon\eta} \cdot \nabla \eta'_t = f', \\ \frac{1}{1+\epsilon\eta} \mathbf{v}'_{tt} - \nabla(\nabla \cdot \mathbf{v}') + \epsilon \nabla \Delta(\nabla \cdot \mathbf{v}') \\ \quad + \left(\frac{\epsilon \mathbf{v}}{1+\epsilon\eta} \cdot \nabla \right) \left(\frac{\mathbf{v}'_t}{1+\epsilon\eta} \right) + \frac{\epsilon \mathbf{v}}{(1+\epsilon\eta)^2} (\nabla \cdot \mathbf{v}'_t) = \mathbf{g}', \end{cases}$$

where

$$(4.117) \quad \begin{aligned} f' \stackrel{\text{def}}{=} & \partial_t f + \epsilon \nabla \cdot (\eta_t \nabla \eta) - \epsilon^2 \nabla \cdot (\eta_t \nabla \Delta \eta) - 2\epsilon \left(\frac{\mathbf{v}}{1+\epsilon\eta} \right)_t \cdot \nabla \eta_t \\ \mathbf{g}' \stackrel{\text{def}}{=} & \partial_t \mathbf{g} + \frac{\epsilon \eta_t}{(1+\epsilon\eta)^2} \mathbf{v}'_t - \epsilon \left(\partial_t \left(\frac{\mathbf{v}}{1+\epsilon\eta} \right) \cdot \nabla \right) \left(\frac{\mathbf{v}_t}{1+\epsilon\eta} \right) \\ & + \epsilon^2 \left(\frac{\mathbf{v}}{1+\epsilon\eta} \cdot \nabla \right) \left(\frac{\eta_t \mathbf{v}_t}{(1+\epsilon\eta)^2} \right) - \epsilon \partial_t \left(\frac{\mathbf{v}}{(1+\epsilon\eta)^2} \right) (\nabla \cdot \mathbf{v}_t). \end{aligned}$$

The principal part of (4.116) is the same as that of (4.115).

Similarly, denoting by $\eta'' = \partial_t^2 \eta = \partial_t \eta'$ and $\mathbf{v}'' = \partial_t^2 \mathbf{v} = \partial_t \mathbf{v}'$, applying ∂_t to (4.116), it transpires that (η'', \mathbf{v}'') satisfies the following system

$$(4.118) \quad \begin{cases} \eta''_{tt} - \nabla \cdot ((1 + \epsilon\eta)\nabla\eta'') + \epsilon\nabla \cdot ((1 + \epsilon\eta)\nabla\Delta\eta'') + \frac{2\epsilon\mathbf{v}}{1 + \epsilon\eta} \cdot \nabla\eta'' = f'', \\ \frac{1}{1 + \epsilon\eta}\mathbf{v}''_{tt} - \nabla(\nabla \cdot \mathbf{v}'') + \epsilon\nabla\Delta(\nabla \cdot \mathbf{v}'') \\ \quad + \epsilon\left(\frac{\mathbf{v}}{1 + \epsilon\eta} \cdot \nabla\right)\left(\frac{\mathbf{v}''_t}{1 + \epsilon\eta}\right) + \frac{\epsilon\mathbf{v}}{(1 + \epsilon\eta)^2}(\nabla \cdot \mathbf{v}''_t) = \mathbf{g}'', \end{cases}$$

where

$$(4.119) \quad \begin{aligned} f'' &\stackrel{\text{def}}{=} \partial_t f' + \epsilon\nabla \cdot (\eta_t \nabla \eta') - \epsilon^2 \nabla \cdot (\eta_t \nabla \Delta \eta') - 2\epsilon \left(\frac{\mathbf{v}}{1 + \epsilon\eta}\right)_t \cdot \nabla \eta'_t \\ \mathbf{g}'' &\stackrel{\text{def}}{=} \partial_t \mathbf{g}' + \frac{\epsilon\eta_t}{(1 + \epsilon\eta)^2} \mathbf{v}'_{tt} - \epsilon \left(\partial_t \left(\frac{\mathbf{v}}{1 + \epsilon\eta}\right) \cdot \nabla\right) \left(\frac{\mathbf{v}'_t}{1 + \epsilon\eta}\right) \\ &\quad + \epsilon^2 \left(\frac{\mathbf{v}}{1 + \epsilon\eta} \cdot \nabla\right) \left(\frac{\eta_t \mathbf{v}'_t}{(1 + \epsilon\eta)^2}\right) - \epsilon \partial_t \left(\frac{\mathbf{v}}{(1 + \epsilon\eta)^2}\right) (\nabla \cdot \mathbf{v}'_t). \end{aligned}$$

The principal part of (4.118) is also the same as that of (4.115).

Step 3. Energy estimates for the quasilinear system (4.110)-(4.115)-(4.116)-(4.118). We shall derive energy estimates for (4.110), (4.115), (4.116), (4.118) under the assumptions

$$(4.120) \quad 1 + \epsilon\eta \geq H > 0,$$

and for all $t \in [0, T]$,

$$(4.121) \quad |\eta(\cdot, t)|_{W^{1,\infty}} + |\mathbf{v}(\cdot, t)|_{W^{1,\infty}} + |\eta(\cdot, t)|_\infty + |\mathbf{v}(\cdot, t)|_\infty + |\eta(\cdot, t)|_{X_{\epsilon^4}^3} \leq c.$$

where $|\cdot|_{X_{\epsilon^k}^s} = |\cdot|_{H^s} + \epsilon^k |\cdot|_{H^{s+k}}$ and the constant c independent of ϵ depends on the initial data. We remark that (4.120) and (4.121) are consequence of the assumption (4.108) and the *a priori* estimate (4.150) for (η, \mathbf{v}) .

Step 3.1. Estimates for (4.110). Similarly as (4.66), taking the L^2 inner product of (4.110) by $((1 - \epsilon\Delta)\eta, \mathbf{v})^T$ leads to

$$(4.122) \quad \frac{1}{2} \frac{d}{dt} E_0(t) = -\frac{\epsilon}{2} \left(\frac{\eta_t \mathbf{v}}{(1 + \epsilon\eta)^2} \mid \mathbf{v}\right)_2 - \epsilon \left(\nabla \cdot \left(\frac{\mathbf{v}}{1 + \epsilon\eta} \otimes \mathbf{v}\right) \mid \frac{\mathbf{v}}{1 + \epsilon\eta}\right)_2,$$

where

$$E_0(t) \stackrel{\text{def}}{=} |\eta|_2^2 + \epsilon |\nabla \eta|_2^2 + \left(\frac{\mathbf{v}}{1 + \epsilon\eta} \mid \mathbf{v}\right)_2.$$

Thanks to (4.120) and (4.121), we have

$$(4.123) \quad E_0(t) \sim |\eta|_2^2 + \epsilon |\nabla \eta|_2^2 + |\mathbf{v}|_{L^2}^2.$$

By (4.120), the first term on the r.h.s of (4.122) is estimated as

$$|-\frac{\epsilon}{2} \left(\frac{\eta_t \mathbf{v}}{(1 + \epsilon\eta)^2} \mid \mathbf{v}\right)_2| \lesssim \epsilon |\mathbf{v}|_\infty |\eta_t|_2 |\mathbf{v}|_2 \lesssim \epsilon |\mathbf{v}|_\infty (|\eta_t|_2^2 + |\mathbf{v}|_2^2),$$

while by integration by parts and (4.120), the second term on the r.h.s of (4.122) is estimated as

$$\begin{aligned} |-\epsilon \left(\nabla \cdot \left(\frac{\mathbf{v}}{1 + \epsilon\eta} \otimes \mathbf{v}\right) \mid \frac{\mathbf{v}}{1 + \epsilon\eta}\right)_2| &= \frac{\epsilon}{2} |(\nabla \cdot \mathbf{v} \mid \left|\frac{\mathbf{v}}{1 + \epsilon\eta}\right|^2)_2| \\ &\lesssim \epsilon |\mathbf{v}|_\infty |\mathbf{v}|_2 |\nabla \mathbf{v}|_2 \lesssim \epsilon |\mathbf{v}|_\infty (|\mathbf{v}|_2^2 + |\nabla \mathbf{v}|_2^2). \end{aligned}$$

Then we obtain

$$(4.124) \quad \frac{1}{2} \frac{d}{dt} E_0(t) \lesssim \epsilon |\mathbf{v}|_\infty (|\eta_t|_2^2 + |\mathbf{v}|_2^2 + |\nabla \mathbf{v}|_2^2).$$

Step 3.2. Estimates for (4.118). Taking the L^2 scalar product of the first equation of (4.118) with $(1 - \epsilon \Delta) \eta_t''$, we obtain

$$\begin{aligned} & (\eta_{tt}'' | (1 - \epsilon \Delta) \eta_t'')_2 - (\nabla \cdot ((1 + \epsilon \eta) \nabla \eta'') | (1 - \epsilon \Delta) \eta_t'')_2 \\ & + \epsilon (\nabla \cdot ((1 + \epsilon \eta) \nabla \Delta \eta'') | (1 - \epsilon \Delta) \eta_t'')_2 + \left(\frac{2\epsilon \mathbf{v}}{1 + \epsilon \eta} \cdot \nabla \eta_t'' | (1 - \epsilon \Delta) \eta_t'' \right)_2 \\ & = (f'' | (1 - \epsilon \Delta) \eta_t'')_2. \end{aligned}$$

Similar to the derivation of (4.79), using integration by parts, we obtain

$$(4.125) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} E_{31}(t) + \left(\frac{2\epsilon \mathbf{v}}{1 + \epsilon \eta} \cdot \nabla \eta_t'' | (1 - \epsilon \Delta) \eta_t'' \right)_2 \\ & = \frac{\epsilon}{2} (\eta_t \nabla \eta'' | \nabla \eta'')_2 + \epsilon^2 (\eta_t \Delta \eta'' | \Delta \eta'')_2 + \epsilon^2 (\nabla (\nabla \eta \cdot \nabla \eta'') | \nabla \eta_t'')_2 \\ & \quad - \epsilon^2 (\nabla \eta \Delta \eta'' | \nabla \eta_t'')_2 + \frac{\epsilon^3}{2} (\eta_t \nabla \Delta \eta'' | \nabla \Delta \eta'')_2 + (f'' | (1 - \epsilon \Delta) \eta_t'')_2 \end{aligned}$$

where

$$\begin{aligned} E_{31}(t) & \stackrel{\text{def}}{=} |\eta_t''|_2^2 + \epsilon |\nabla \eta_t''|_2^2 + ((1 + \epsilon \eta) \nabla \eta'' | \nabla \eta'')_2 + 2\epsilon ((1 + \epsilon \eta) \Delta \eta'' | \Delta \eta'')_2 \\ & \quad + \epsilon^2 ((1 + \epsilon \eta) \nabla \Delta \eta'' | \nabla \Delta \eta'')_2. \end{aligned}$$

By (4.120) and (4.121), we have

$$(4.126) \quad E_{31}(t) \sim |\eta_t''|_2^2 + \epsilon |\nabla \eta_t''|_2^2 + |\nabla \eta''|_2^2 + \epsilon |\nabla^2 \eta''|_2^2 + \epsilon^2 |\nabla^3 \eta''|_2^2.$$

Now, we estimate the second term on the l.h.s of (4.125). Integrating by parts, we have

$$(4.127) \quad \begin{aligned} & \left(\frac{2\epsilon \mathbf{v}}{1 + \epsilon \eta} \cdot \nabla \eta_t'' | (1 - \epsilon \Delta) \eta_t'' \right)_2 = -\epsilon (\nabla \cdot \left(\frac{\mathbf{v}}{1 + \epsilon \eta} \right) \eta_t'' | \eta_t'')_2 \\ & \quad + 2\epsilon^2 ((\nabla \eta_t'' \cdot \nabla) \left(\frac{\mathbf{v}}{1 + \epsilon \eta} \right) | \nabla \eta_t'')_2 - \epsilon^2 (\nabla \cdot \left(\frac{\mathbf{v}}{1 + \epsilon \eta} \right) \nabla \eta_t'' | \nabla \eta_t'')_2, \end{aligned}$$

which along with (4.120), (4.121) and (4.125) implies that

$$(4.128) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} E_{31}(t) \lesssim \epsilon (|\eta_t|_\infty + |\nabla \eta|_\infty + |\nabla \mathbf{v}|_\infty + \epsilon^{\frac{1}{2}} |\nabla^2 \eta|_\infty) (|\nabla \eta''|_2^2 + \epsilon |\nabla^2 \eta''|_2^2 \\ & \quad + \epsilon^2 |\nabla^3 \eta''|_2^2 + |\eta_t''|_2^2 + \epsilon |\nabla \eta_t''|_2^2) + |f''|_2 |\eta_t''|_2 + \epsilon |\nabla f''|_2 |\nabla \eta_t''|_2. \end{aligned}$$

Taking the L^2 inner product of the second of (4.118) with \mathbf{v}_t'' yields

$$(4.129) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} E_{32}(t) + \epsilon ((\mathbf{v} \cdot \nabla) \left(\frac{\mathbf{v}_t''}{1 + \epsilon \eta} \right) | \frac{\mathbf{v}_t''}{1 + \epsilon \eta})_2 + \left(\frac{\epsilon \mathbf{v}}{(1 + \epsilon \eta)^2} (\nabla \cdot \mathbf{v}_t'') | \mathbf{v}_t'' \right)_2 \\ & = -\frac{1}{2} (\partial_t \left(\frac{1}{1 + \epsilon \eta} \right) \mathbf{v}_t'' | \mathbf{v}_t'')_2 + (\mathbf{g}'' | \mathbf{v}_t'')_2, \end{aligned}$$

where

$$E_{32}(t) \stackrel{\text{def}}{=} \left(\frac{\mathbf{v}_t''}{1 + \epsilon \eta} | \mathbf{v}_t'' \right)_2 + |\nabla \cdot \mathbf{v}''|_2^2 + \epsilon |\nabla (\nabla \cdot \mathbf{v}'')|_2^2.$$

Since $\operatorname{curl}(\frac{\mathbf{v}''}{1+\epsilon\eta}) = 0$, we have

$$\operatorname{curl} \mathbf{v}'' = (1 + \epsilon\eta) \left(v''^1 \partial_2 \left(\frac{1}{1 + \epsilon\eta} \right) - v''^2 \partial_1 \left(\frac{1}{1 + \epsilon\eta} \right) \right) = \frac{\epsilon(v''^2 \partial_1 \eta - v''^1 \partial_2 \eta)}{1 + \epsilon\eta},$$

which along with (4.120) and (4.121) shows that

$$(4.130) \quad \begin{aligned} |\operatorname{curl} \mathbf{v}''|_2 &\lesssim \epsilon |\mathbf{v}''|_2, \\ |\nabla(\operatorname{curl} \mathbf{v}'')|_2 &\lesssim \epsilon (|\mathbf{v}''|_2 + |\nabla \mathbf{v}''|_2) |\eta|_{H^3} \lesssim \epsilon (|\mathbf{v}''|_2 + |\nabla \mathbf{v}''|_2), \end{aligned}$$

where for the second inequality, we used the fact that $|\nabla \eta|_\infty \lesssim |\nabla \eta|_{H^2}$ and the following estimate

$$|\mathbf{v}'' \nabla^2 \eta|_2 \lesssim |\mathbf{v}''|_4 |\nabla^2 \eta|_4 \stackrel{\text{Sobolev}}{\lesssim} |\mathbf{v}''|_{\dot{H}^{\frac{1}{2}}} |\nabla^2 \eta|_{\dot{H}^{\frac{1}{2}}} \stackrel{\text{interpolation}}{\lesssim} (|\mathbf{v}''|_2 + |\nabla \mathbf{v}''|_2) |\nabla^2 \eta|_{H^1}.$$

Then by virtue of div-curl lemma, we obtain

$$(4.131) \quad E_{32}(t) \sim |\mathbf{v}''_t|_2^2 + |\nabla \mathbf{v}''|_2^2 + \epsilon |\nabla^2 \mathbf{v}''|_2^2 + O(\epsilon |\mathbf{v}''|_2^2).$$

Similar to (4.127), integration by parts on the second term on the l.h.s of (4.129) leads to

$$(4.132) \quad \epsilon \left((\mathbf{v} \cdot \nabla) \left(\frac{\mathbf{v}''_t}{1 + \epsilon\eta} \right) \middle| \frac{\mathbf{v}''_t}{1 + \epsilon\eta} \right)_2 = -\frac{\epsilon}{2} \left(\nabla \cdot \mathbf{v} \frac{\mathbf{v}''_t}{1 + \epsilon\eta} \middle| \frac{\mathbf{v}''_t}{1 + \epsilon\eta} \right)_2.$$

For the third term on the l.h.s of (4.129), by integration by parts, we have

$$\begin{aligned} \left(\frac{\epsilon \mathbf{v}}{(1 + \epsilon\eta)^2} (\nabla \cdot \mathbf{v}''_t) \middle| \mathbf{v}''_t \right)_2 &= \epsilon \sum_{i,j=1,2} \left(\frac{v^i}{(1 + \epsilon\eta)^2} (\partial_j v_t''^j) \middle| v_t''^i \right)_2 \\ &= -\epsilon \sum_{i,j=1,2} \left(\partial_j \left(\frac{v^i}{(1 + \epsilon\eta)^2} \right) v_t''^j \middle| v_t''^i \right)_2 - \epsilon \sum_{i,j=1,2} \left(\frac{v^i}{(1 + \epsilon\eta)^2} v_t''^j \middle| \partial_j v_t''^i \right)_2. \end{aligned}$$

To estimate the second term on the right hand side of the above equality, we first obtain by using $\operatorname{curl}(\frac{\mathbf{v}''}{1+\epsilon\eta}) = 0$ that

$$\partial_j v_t''^i = \left(\partial_i \left(\frac{v''^j}{1 + \epsilon\eta} \right) \cdot (1 + \epsilon\eta) + \frac{\epsilon v''^i \partial_j \eta}{1 + \epsilon\eta} \right)_t = \partial_i v_t''^j + \epsilon \left(\frac{v''^i \partial_j \eta - v''^j \partial_i \eta}{1 + \epsilon\eta} \right)_t.$$

By integration by parts, we have

$$\begin{aligned} &-\epsilon \left(\frac{v^i}{(1 + \epsilon\eta)^2} v_t''^j \middle| \partial_j v_t''^i \right)_2 \\ &= \frac{\epsilon}{2} \left(\partial_i \left(\frac{v^i}{(1 + \epsilon\eta)^2} \right) v_t''^j \middle| v_t''^j \right)_2 - \epsilon^2 \left(\frac{v^i}{(1 + \epsilon\eta)^2} v_t''^j \middle| \left(\frac{v''^i \partial_j \eta - v''^j \partial_i \eta}{1 + \epsilon\eta} \right)_t \right)_2. \end{aligned}$$

Then we obtain

$$\begin{aligned} \left(\frac{\epsilon \mathbf{v}}{(1 + \epsilon\eta)^2} (\nabla \cdot \mathbf{v}''_t) \middle| \mathbf{v}''_t \right)_2 &= \sum_{i,j=1,2} \left(-\epsilon \left(\partial_j \left(\frac{v^i}{(1 + \epsilon\eta)^2} \right) v_t''^j \middle| v_t''^i \right)_2 \right. \\ &\quad \left. + \frac{\epsilon}{2} \left(\partial_i \left(\frac{v^i}{(1 + \epsilon\eta)^2} \right) v_t''^j \middle| v_t''^j \right)_2 - \epsilon^2 \left(\frac{v^i}{(1 + \epsilon\eta)^2} v_t''^j \middle| \left(\frac{v''^i \partial_j \eta - v''^j \partial_i \eta}{1 + \epsilon\eta} \right)_t \right)_2 \right), \end{aligned}$$

which along with (4.120), (4.121), (4.129) and (4.132) implies

$$(4.133) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} E_{32}(t) &\lesssim \epsilon (|\eta_t|_\infty + \epsilon^{\frac{1}{2}} |\nabla \eta_t|_4 + |\nabla \mathbf{v}|_\infty + |\nabla \eta|_\infty) \\ &\quad \times (|\mathbf{v}''|_2^2 + \epsilon |\nabla \mathbf{v}''|_2^2 + \epsilon^2 |\mathbf{v}''|_2^2) + |\mathbf{g}''|_2 |\mathbf{v}''|_2. \end{aligned}$$

Now, we define $E_3(t) \stackrel{\text{def}}{=} E_{31}(t) + E_{32}(t)$. Then (4.126) and (4.131) yields

$$(4.134) \quad \begin{aligned} E_3(t) &\sim |\nabla \eta''|_2^2 + \epsilon |\nabla^2 \eta''|_2^2 + \epsilon^2 |\nabla^3 \eta''|_2^2 + |\eta_t''|_2^2 + \epsilon |\nabla \eta_t''|_2^2 \\ &\quad + |\nabla \mathbf{v}''|_2^2 + \epsilon |\nabla^2 \mathbf{v}''|_2^2 + |\mathbf{v}_t''|_2^2 + O(\epsilon |\mathbf{v}''|_2^2). \end{aligned}$$

Combining estimates (4.128) and (4.133), using (4.134), we obtain

$$(4.135) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} E_3(t) &\lesssim \epsilon (|\eta_t|_\infty + \epsilon^{\frac{1}{2}} |\nabla \eta_t|_4 + |\nabla \eta|_\infty + |\nabla^2 \eta|_3 + |\nabla \mathbf{v}|_\infty) E_3(t) \\ &\quad + |f''|_2 |\eta_t''|_2 + \epsilon |\nabla f''|_2 |\nabla \eta_t''|_2 + |\mathbf{g}''|_2 |\mathbf{v}_t''|_2. \end{aligned}$$

Step 3.3. Estimates for (4.115) and (4.116). Since (4.115) and (4.116) have the same form as (4.118), we have similar estimates as (4.135) only with (η'', \mathbf{v}'') being replaced by (η, \mathbf{v}) and (η', \mathbf{v}') respectively.

In order to get the total estimate for system (4.115), (4.116) and (4.118), we have to estimate the source terms $|f|_2 + \epsilon^{\frac{1}{2}} |\nabla f|_2 + |\mathbf{g}|_2$, $|f'|_2 + \epsilon^{\frac{1}{2}} |\nabla f'|_2 + |\mathbf{g}'|_2$ and $|f''|_2 + \epsilon^{\frac{1}{2}} |\nabla f''|_2 + |\mathbf{g}''|_2$. Thanks to the expressions of $f, \mathbf{g}, f', \mathbf{g}'$ and f'', \mathbf{g}'' , using (4.120) and (4.121), after tedious but elementary calculations, we obtain

$$(4.136) \quad \begin{aligned} &|f|_2 + \epsilon^{\frac{1}{2}} |\nabla f|_2 + |\mathbf{g}|_2 + |f'|_2 + \epsilon^{\frac{1}{2}} |\nabla f'|_2 + |\mathbf{g}'|_2 \\ &+ |f''|_2 + \epsilon^{\frac{1}{2}} |\nabla f''|_2 + |\mathbf{g}''|_2 \lesssim \epsilon \mathcal{E}(t), \end{aligned}$$

where

$$(4.137) \quad \begin{aligned} \mathcal{E}(t) &= |\eta|_{X_{\epsilon^4}^3}^2 + |\eta_t|_{X_{\epsilon^3}^2}^2 + |\eta_{tt}|_{X_{\epsilon^2}^1}^2 + |\eta_{ttt}|_{X_{\epsilon^0}^0}^2 + |\mathbf{v}|_{X_{\epsilon^3}^3}^2 \\ &\quad + |\mathbf{v}_t|_{X_{\epsilon^2}^2}^2 + |\mathbf{v}_{tt}|_{X_{\epsilon^1}^1}^2 + |\mathbf{v}_{ttt}|_2^2. \end{aligned}$$

In the process of derivation of (4.136), we used the fact that $\eta' = \eta_t$, $\eta'' = \eta_{tt}$, $\mathbf{v}' = \mathbf{v}_t$, $\mathbf{v}'' = \mathbf{v}_{tt}$ and used the Hölder inequalities, Sobolev inequalities and interpolation inequalities frequently. We shall not show the details here.

Step 4. The final estimate on (4.110). Before closing the *a priori estimates*, we first define the energy functional associated to the quasilinear system (4.110)-(4.115)-(4.116)-(4.118) as

$$(4.138) \quad E(t) \stackrel{\text{def}}{=} E_0(t) + E_1(t) + E_2(t) + E_3(t),$$

and $E_1(t)$, $E_2(t)$ are defined in the same way as $E_3(t)$ with (η'', \mathbf{v}'') being replaced by (η, \mathbf{v}) and (η', \mathbf{v}') respectively. Notice that $\eta' = \eta_t$, $\eta'' = \eta_{tt}$ and $\mathbf{v}' = \mathbf{v}_t$, $\mathbf{v}'' = \mathbf{v}_{tt}$. Then (4.123) and (4.134) yield

$$(4.139) \quad \begin{aligned} E(t) &\sim |\eta|_{X_{\epsilon^2}^1}^2 + |\eta_t|_{X_{\epsilon^1}^1}^2 + |\eta_{tt}|_{X_{\epsilon^0}^1}^2 + |\eta_{ttt}|_{X_{\epsilon^0}^0}^2 + |\mathbf{v}|_{X_{\epsilon^1}^1}^2 \\ &\quad + |\mathbf{v}_t|_{X_{\epsilon^0}^0}^2 + |\mathbf{v}_{tt}|_{X_{\epsilon^0}^0}^2 + |\mathbf{v}_{ttt}|_2^2. \end{aligned}$$

With the definitions (4.138) and (4.137), using the interpolation inequality (4.2) and the inequalities that $|u|_{L^\infty(\mathbb{R}^2)} \lesssim |u|_{H^2(\mathbb{R}^2)}$ and $|u|_{L^4(\mathbb{R}^2)} \lesssim |u|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} |\nabla u|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}$, the energy estimates (4.124), (4.135) and (4.136) give rise to

$$(4.140) \quad \frac{1}{2} \frac{d}{dt} E(t) \lesssim \epsilon \mathcal{E}(t)^{\frac{3}{2}},$$

where $\mathcal{E}(t)$ is the total energy functional to (4.110) which is defined in (4.137).

To finish the proof, we have to show that

$$(4.141) \quad \mathcal{E}(t) \sim E(t).$$

Indeed, thanks to (4.139) and (4.137), we have

$$\mathcal{E}(t) \sim E(t) + |\eta|_{X_{\epsilon^4}^3}^2 + |\eta_t|_{X_{\epsilon^3}^2}^2 + |\mathbf{v}|_{X_{\epsilon^3}^3}^2 + |\mathbf{v}_t|_{X_{\epsilon^2}^2}^2.$$

Then we only need to show that

$$|\eta|_{X_{\epsilon^4}^3}^2 + |\eta_t|_{X_{\epsilon^3}^2}^2 + |\mathbf{v}|_{X_{\epsilon^3}^3}^2 + |\mathbf{v}_t|_{X_{\epsilon^2}^2}^2 \lesssim E(t).$$

That is to say, we shall recover the regularity in space through the regularity in time. More precisely, (4.110) yields

$$(4.142) \quad \nabla \cdot \mathbf{v} = -\eta_t, \quad (1 - \epsilon\Delta)\nabla\eta = -\frac{\mathbf{v}_t}{1 + \epsilon\eta} - \frac{\epsilon}{1 + \epsilon\eta}\nabla \cdot \left(\frac{\mathbf{v}}{1 + \epsilon\eta} \otimes \mathbf{v} \right).$$

To control $|\mathbf{v}_t|_{X_{\epsilon^2}^2}$, we first have

$$\begin{aligned} |\mathbf{v}_t|_{X_{\epsilon^2}^2}^2 &= |\mathbf{v}_t|_{H^2}^2 + \epsilon^2 |\mathbf{v}_t|_{H^{2+2}}^2 \\ &\lesssim |\mathbf{v}_t|_{H^1}^2 + |\nabla(\nabla \cdot \mathbf{v}_t)|_2^2 + |\nabla(\operatorname{curl} \mathbf{v}_t)|_2^2 + \epsilon^2 |\nabla^3(\nabla \cdot \mathbf{v}_t)|_2^2 + \epsilon^2 |\nabla^3(\operatorname{curl} \mathbf{v}_t)|_2^2. \end{aligned}$$

Since $\operatorname{curl}\left(\frac{\mathbf{v}_t}{1 + \epsilon\eta}\right) = 0$, similar derivation as (4.130) leads to

$$\operatorname{curl} \mathbf{v}_t = \frac{\epsilon(v_t^2 \partial_1 \eta - v_t^1 \partial_2 \eta)}{1 + \epsilon\eta},$$

and

$$\begin{aligned} |\nabla(\operatorname{curl} \mathbf{v}_t)|_2 &\lesssim \epsilon(|\mathbf{v}_t|_2 + |\nabla \mathbf{v}_t|_2) |\eta|_{H^3} \lesssim \epsilon |\mathbf{v}_t|_{H^1}, \\ |\nabla^3(\operatorname{curl} \mathbf{v}_t)|_2 &\lesssim \epsilon^{\frac{1}{2}} (|\mathbf{v}_t|_2 + |\nabla^2 \mathbf{v}_t|_2) |\eta|_{X_{\epsilon}^3} \lesssim \epsilon^{\frac{1}{2}} |\mathbf{v}_t|_{H^2}, \end{aligned}$$

Then we have

$$|\mathbf{v}_t|_{X_{\epsilon^2}^2}^2 \lesssim |\mathbf{v}_t|_{H^1}^2 + |\nabla(\nabla \cdot \mathbf{v}_t)|_2^2 + \epsilon^2 |\nabla^3(\nabla \cdot \mathbf{v}_t)|_2^2 + \epsilon^3 |\mathbf{v}_t|_{H^2}^2,$$

which along with the fact that ϵ is small enough implies

$$|\mathbf{v}_t|_{X_{\epsilon^2}^2}^2 \lesssim |\mathbf{v}_t|_{H^1}^2 + |\nabla(\nabla \cdot \mathbf{v}_t)|_2^2 + \epsilon^2 |\nabla^3(\nabla \cdot \mathbf{v}_t)|_2^2.$$

Now using the first equation of (4.142), we obtain that

$$(4.143) \quad |\mathbf{v}_t|_{X_{\epsilon^2}^2}^2 \lesssim |\mathbf{v}_t|_{H^1}^2 + |\nabla \eta_{tt}|_2^2 + \epsilon^2 |\nabla^3 \eta_{tt}|_2^2 \lesssim E(t).$$

Similarly, we obtain

$$(4.144) \quad |\mathbf{v}|_{X_{\epsilon^2}^2}^2 \lesssim |\mathbf{v}|_{H^1}^2 + |\nabla \eta_t|_2^2 + \epsilon^2 |\nabla^3 \eta_t|_2^2 \lesssim E(t).$$

While the second equation of (4.142), (4.121), (4.143) and (4.144) imply

$$(4.145) \quad \begin{aligned} |\eta_t|_{X_{\epsilon^3}^2}^2 &\sim |\eta_t|_{H^1}^2 + |\nabla[(1 - \epsilon\Delta)\nabla\eta_t]|_2^2 + \epsilon |\nabla^2[(1 - \epsilon\Delta)\nabla\eta_t]|_2^2 \\ &\lesssim |\eta_t|_{X_{\epsilon^2}^1}^2 + |\mathbf{v}_{tt}|_{X_{\epsilon}^1}^2 + |\mathbf{v}_t|_{X_{\epsilon^2}^2}^2 + |\mathbf{v}|_{X_{\epsilon^2}^2}^2 + |\eta|_{X_{\epsilon^1}^1}^2 \lesssim E(t). \end{aligned}$$

To bound $|\mathbf{v}|_{X_{\epsilon^3}^3}$, we first have

$$\begin{aligned} |\mathbf{v}|_{X_{\epsilon^3}^3}^2 &\sim |\mathbf{v}|_{H^2}^2 + |\nabla^3 \mathbf{v}|_2^2 + \epsilon^3 |\mathbf{v}|_{H^{3+3}}^2 \\ &\lesssim |\mathbf{v}|_{H^2}^2 + |\nabla^2(\nabla \cdot \mathbf{v})|_2^2 + |\nabla^2(\operatorname{curl} \mathbf{v})|_2^2 + \epsilon^3 |\nabla^5(\nabla \cdot \mathbf{v})|_2^2 + \epsilon^3 |\nabla^5(\operatorname{curl} \mathbf{v})|_2^2. \end{aligned}$$

Similar derivation as (4.130) yields

$$\begin{aligned} |\nabla^2(\operatorname{curl} \mathbf{v})|_2 &\lesssim \epsilon |\mathbf{v}|_{H^2}, \\ \epsilon^{\frac{1}{2}} |\nabla^5(\operatorname{curl} \mathbf{v}_t)|_2^2 &\lesssim |\mathbf{v}|_{X_{\epsilon^2}^3} |\eta|_{X_{\epsilon^3}^3} (1 + |\eta|_{X_{\epsilon}^3}) \lesssim |\mathbf{v}|_{X_{\epsilon^2}^3}. \end{aligned}$$

Then we obtain

$$|\mathbf{v}|_{X_{\epsilon^3}^3}^2 \lesssim |\mathbf{v}|_{H^2}^2 + |\nabla^2(\nabla \cdot \mathbf{v})|_2^2 + \epsilon^3 |\nabla^5(\nabla \cdot \mathbf{v})|_2^2 + \epsilon^2 |\mathbf{v}|_{X_{\epsilon^2}^3},$$

which gives rise to

$$|\mathbf{v}|_{X_{\epsilon^3}^3}^2 \lesssim |\mathbf{v}|_{H^2}^2 + |\nabla^2(\nabla \cdot \mathbf{v})|_2^2 + \epsilon^3 |\nabla^5(\nabla \cdot \mathbf{v})|_2^2.$$

Then using the first equation of (4.141), (4.145) and (4.139), we obtain

$$(4.146) \quad |\mathbf{v}|_{X_{\epsilon^3}^3}^2 \lesssim E(t).$$

For $|\eta|_{X_{\epsilon^4}^3}^2$, similar to the derivation of (4.145), by using the second equation of (4.141), (4.139), (4.143), (4.145) and (4.146), we finally obtain that

$$(4.147) \quad |\eta|_{X_{\epsilon^4}^3}^2 \sim |\eta|_{H^1}^2 + |\nabla^2[(1 - \epsilon\Delta)\nabla\eta]|_2^2 + \epsilon^2 |\nabla^4[(1 - \epsilon\Delta)\nabla\eta]|_2^2 \lesssim E(t).$$

Due to (4.141) and (4.140), we have

$$(4.148) \quad \frac{1}{2} \frac{d}{dt} E(t) \lesssim \epsilon E(t)^{\frac{3}{2}}.$$

Step 5. Initial data for the quasilinear system and final estimate. In this step, we have to derive the regularity for the initial data to the quasilinear system through the system (4.110) and the regularity for initial data (η_0, \mathbf{v}_0) . The first equation of (4.110) shows that

$$|\eta'|_{t=0}|_{X_{\epsilon^3}^2} = |\eta_t|_{t=0}|_{X_{\epsilon^3}^2} = |\nabla \cdot \mathbf{v}_0|_{X_{\epsilon^3}^2} \lesssim |\mathbf{v}_0|_{X_{\epsilon^3}^3},$$

while the second equation of (4.110) shows that

$$\begin{aligned} |\mathbf{v}'|_{t=0}|_{X_{\epsilon^2}^2} &= |\mathbf{v}_t|_{t=0}|_{X_{\epsilon^2}^2} \\ &\lesssim |(1 + \epsilon\eta_0)(1 - \epsilon\Delta)\nabla\eta_0|_{X_{\epsilon^2}^2} + \epsilon |\nabla \cdot \left(\frac{\mathbf{v}_0}{1 + \epsilon\eta_0} \otimes \mathbf{v}_0 \right)|_{X_{\epsilon^2}^2} \\ &\lesssim |\eta_0|_{X_{\epsilon^4}^3} + |\mathbf{v}_0|_{X_{\epsilon^3}^3}, \end{aligned}$$

where we assume that $|\eta_0|_{X_{\epsilon^4}^3} + |\mathbf{v}_0|_{X_{\epsilon^3}^3} \leq C$ and $\epsilon \leq \epsilon_0$ with ϵ_0 small enough.

Similarly, thanks to (4.116), we can obtain the upper bound of $|\eta'_t|_{t=0}|_{X_{\epsilon^2}^1} + |\mathbf{v}'_t|_{t=0}|_{X_{\epsilon^1}^1}$ (or $|\eta_{tt}|_{t=0}|_{X_{\epsilon^2}^1} + |\mathbf{v}_{tt}|_{t=0}|_{X_{\epsilon^1}^1}$). While by (4.118), we can also derive the upper bound for $|\eta''_t|_{t=0}|_{X_{\epsilon^0}^0} + |\mathbf{v}''_t|_{t=0}|_2$ (or $|\eta_{ttt}|_{t=0}|_{X_{\epsilon^0}^0} + |\mathbf{v}_{ttt}|_{t=0}|_2$). Then we finally obtain that

$$(4.149) \quad E(0) \sim \mathcal{E}(0) \lesssim |\eta_0|_{X_{\epsilon^4}^3}^2 + |\mathbf{v}_0|_{X_{\epsilon^3}^3}^2.$$

Step 6. Existence and uniqueness. The estimates (4.148) and (4.149) are crucial to prove the existence of $T > 0$ independent of ϵ such that (4.110) has a unique solution (η, \mathbf{v}) on a time interval $[0, T/\epsilon]$ with initial data $(\eta_0, \mathbf{v}_0) \in X_{\epsilon^4}^3 \times X_{\epsilon^3}^3$ satisfying moreover, by (4.148) and (4.141) the estimate

$$(4.150) \quad \sup_{t \in [0, T/\epsilon]} \mathcal{E}(t) \lesssim |\eta_0|_{X_{\epsilon^4}^3}^2 + |\mathbf{v}_0|_{X_{\epsilon^3}^3}^2.$$

The proof of the existence and uniqueness is postponed to Section 5.

Notice that $\mathbf{v} = (1 + \epsilon\eta)\mathbf{u}$. Then we have obtained the long time estimate of solutions to the original Boussinesq system (1.2)-(4.1) with $a = b = d = 0, c = -1$ together to the energy estimate (4.109). \square

5. EXISTENCE PROOF OF THEOREMS 4.6 AND 4.7

In this section, we shall complete the proof of existence and uniqueness of solutions to the transformed systems (4.66) and (4.110) so that we could complete the proofs to Theorems 4.6 and 4.7. In order to construct the approximate solutions to (4.66) and (4.110), we introduce the mollifier operator \mathcal{J}_δ as follows (see [27]):

$$\widehat{\mathcal{J}_\delta f}(\xi) = \varphi(\delta\xi)\hat{f}(\xi), \quad \forall \xi \in \mathbb{R}^d, \quad \forall f \in L^2(\mathbb{R}^d),$$

where $\varphi \in C_0^\infty(\mathbb{R}^d)$ and $\varphi(0) = 1$. Then using Fourier transform, we obtain the following properties for \mathcal{J}_δ :

Lemma 5.1. *For any $s, s' \in \mathbb{R}$ and $1 \leq p \leq \infty$, there hold:*

$$(i) \quad |\mathcal{J}_\delta f|_{H^{s'}} \leq C_{s,s',\delta} |f|_{H^s};$$

$$(ii) \quad |\mathcal{J}_\delta f|_p \leq C |f|_p;$$

$$(iii) \quad |\mathcal{J}_\delta f - f|_{H^{s-1}} \leq C\delta |f|_{H^s};$$

$$(iv) \quad |\mathcal{J}_\delta f - f|_{H^s} \rightarrow 0 \text{ as } \delta \rightarrow 0;$$

$$(v) \quad |\mathcal{J}_\delta, a|f|_{H^s} \leq C|a|_{H^{t_0+1}} |f|_{H^{s-1}}, \text{ for any } t_0 \geq \frac{d}{2} \text{ and } -t_0 < s \leq t_0 + 1;$$

where C is an universal constant independent of δ and $C_{s,s',\delta}$ is a constant depending on s, s', δ .

Proof. The statements (i), (iii) and (iv) are verified directly by Fourier analysis. For (ii), denoting by $\check{\varphi}(\cdot)$ is the inverse Fourier transform of φ . Then we have

$$\mathcal{J}_\delta f = \delta^{-d} \check{\varphi}\left(\frac{\cdot}{\delta}\right) * f.$$

Notice that $\delta^{-d} |\check{\varphi}\left(\frac{\cdot}{\delta}\right)|_1 \leq C$. Then (ii) follows by Young inequality.

The statement (v) is a consequence of Theorems 3 and 6 in [28]. Indeed, since \mathcal{J}_δ is a zeroth order Fourier multiplier, by [28], we have

$$[\mathcal{J}_\delta, a]f|_{H^s} \lesssim C(\varphi(\delta\cdot))|a|_{H^{t_0+1}} |f|_{H^s},$$

where

$$C(\varphi(\delta\cdot)) = \sup_{|\beta| \leq 2+d+\left[\frac{d}{2}\right]} \sup_{|\xi| \geq \frac{1}{4}} \langle \xi \rangle^{|\beta|} |\partial_\xi^\beta \varphi(\delta\xi)| + \sup_{|\xi| \leq 1} |\varphi(\delta\xi)| \leq C.$$

Thus, the lemma is proved. \square

We only give the details of the existence proof to Theorem 4.6. The existence proof of Theorem 4.7 follows a similar line.

Now, we divide the proof into several steps.

Step 1. Construction of the approximate solutions to (4.66). We construct an approximate solution sequence $\{(\eta^\delta, v^\delta)\}_{\delta>0}$ satisfying the following regularizing system

$$(5.1) \quad \begin{cases} \eta_t^\delta + \mathcal{J}_\delta v_x^\delta = 0 \\ v_t^\delta + (1 + \epsilon \mathcal{J}_\delta \eta^\delta)(1 - \epsilon \partial_x^2) \mathcal{J}_\delta \eta_x^\delta + \epsilon \mathcal{J}_\delta^2 \left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_x = 0, \end{cases}$$

associated with the initial data $\eta^\delta|_{t=0} = \eta_0$ and $v^\delta|_{t=0} = v_0$.

Denoting by $V^\delta = (\eta^\delta, v^\delta)$, then (5.1) can be reduced to the following ODE in the Banach space $H^{m+1} \times H^m$ with $m \geq 0$:

$$(5.2) \quad \frac{d}{dt} V^\delta(t) = F_\delta(V^\delta), \quad V^\delta(0) = V_0^\delta \stackrel{\text{def}}{=} (\eta_0, v_0),$$

where $F_\delta(V^\delta) = (F_\delta^1(V^\delta), F_\delta^2(V^\delta))$ with

$$\begin{aligned} F_\delta^1(V^\delta) &= -\mathcal{J}_\delta v_x^\delta, \\ F_\delta^2(V^\delta) &= -(1 + \epsilon \mathcal{J}_\delta \eta^\delta)(1 - \epsilon \partial_x^2) \mathcal{J}_\delta \eta_x^\delta - \epsilon \mathcal{J}_\delta^2 \left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_x. \end{aligned}$$

For any V_1^δ, V_2^δ , by virtue of the properties of \mathcal{J}_δ in Lemma 5.1, we have

$$|F_\delta^1(V_1^\delta) - F_\delta^1(V_2^\delta)|_{H^{m+1}} = |\mathcal{J}_\delta \partial_x (v_1^\delta - v_2^\delta)|_{H^{m+1}} \leq C_{\delta, m} |v_1^\delta - v_2^\delta|_{H^m}.$$

Similarly, by Lemma 5.1 and the product estimate, we have

$$\begin{aligned} |F_\delta^2(V_1^\delta) - F_\delta^2(V_2^\delta)|_{H^m} &\leq C_{\delta, m} (|\mathcal{J}_\delta \eta_1^\delta|_{H^m}, |\mathcal{J}_\delta \eta_2^\delta|_{H^m}, |\mathcal{J}_\delta v_1^\delta|_{H^m}, |\mathcal{J}_\delta v_2^\delta|_{H^m}) \\ &\quad \times (|\eta_1^\delta - \eta_2^\delta|_{H^m} + |v_1^\delta - v_2^\delta|_{H^m}) \\ &\leq C_{\delta, m} (|V_1^\delta|_2, |V_2^\delta|_2) |V_1^\delta - V_2^\delta|_{H^{m+1} \times H^m}, \end{aligned}$$

where $C_{\delta, m}(\lambda_1, \lambda_2, \dots)$ is a constant depending on δ, m and $\lambda_1, \lambda_2, \dots$. Then we have

$$|F_\delta(V_1^\delta) - F_\delta(V_2^\delta)|_{H^{m+1} \times H^m} \leq C_{\delta, m} (|V_1^\delta|_2, |V_2^\delta|_2) |V_1^\delta - V_2^\delta|_{H^{m+1} \times H^m}$$

so that $F_\delta(\cdot)$ is locally Lipschitz continuous on any open set

$$\mathcal{O}_M = \{V \in H^{m+1} \times H^m(\mathbb{R}) \mid |V|_{H^{m+1} \times H^m} \leq M\}.$$

Thus, Picard (Cauchy-Lipschitz) existence theorem implies that, given any initial data $V_0 \in H^{m+1} \times H^m(\mathbb{R})$, there exists a unique solution $V^\delta \in C^1([0, T_\delta]; \mathcal{O}_M \cap (H^{m+1} \times H^m))$ for some $T_\delta > 0$, with any integer $m \geq 0$.

Going back to the regularizing system (5.1), since $V^\delta = (\eta^\delta, v^\delta) \in C^1([0, T_\delta]; H^{m+1} \times H^m)$, by virtue of the properties to \mathcal{J}_δ , we have

$$\partial_t V^\delta \in C^1([0, T_\delta]; H^{m+1} \times H^m),$$

which implies

$$V^\delta \in C^2([0, T_\delta]; H^{m+1} \times H^m).$$

Moreover, we could obtain that

$$V^\delta \in C^k([0, T_\delta]; H^{m+1} \times H^m), \quad \text{for any } k \in \mathbb{N}.$$

Thus, we could apply ∂_t many times to (5.1).

Step 2. Uniform energy estimates on the approximate solutions on some time interval $[0, T/\epsilon]$. In this step, we shall prove that there exists a uniform existence time interval $[0, T/\epsilon]$ with T being independent of δ and ϵ . To do so, we have to derive the uniform energy estimates for the approximate solutions V^δ .

Step 2.1. The reduction equations. Motivated by the a priori energy estimates for (4.66), we apply ∂_t to (5.1). Similar derivation as (4.71), we obtain

$$(5.3) \quad \begin{cases} \eta_{tt}^\delta - \mathcal{J}_\delta \partial_x ((1 + \epsilon \mathcal{J}_\delta \eta^\delta) \partial_x \mathcal{J}_\delta \eta^\delta) + \epsilon \mathcal{J}_\delta \partial_x ((1 + \epsilon \mathcal{J}_\delta \eta^\delta) \partial_x^3 \mathcal{J}_\delta \eta^\delta) \\ \quad + 2\epsilon \mathcal{J}_\delta^2 \left(\frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \cdot \partial_x \mathcal{J}_\delta^2 \eta_t^\delta \right) = f^\delta, \\ \frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} v_{tt}^\delta - \partial_x^2 \mathcal{J}_\delta^2 v^\delta + \epsilon \partial_x^4 \mathcal{J}_\delta^2 v^\delta \\ \quad + \frac{2\epsilon}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \mathcal{J}_\delta^2 \left(\mathcal{J}_\delta^2 v^\delta \cdot \partial_x \mathcal{J}_\delta^2 \left(\frac{v_t^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) \right) = g^\delta, \end{cases}$$

where

$$\begin{aligned}
f^\delta &\stackrel{\text{def}}{=} \epsilon \mathcal{J}_\delta^3 \left[\left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_{xx} - \frac{2\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \cdot \mathcal{J}_\delta^2 v_{xx}^\delta \right] \\
&\quad - 2\epsilon \mathcal{J}_\delta^2 \left([\mathcal{J}_\delta, \frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}] \partial_x \mathcal{J}_\delta \eta_t^\delta \right), \\
g^\delta &\stackrel{\text{def}}{=} -\frac{\epsilon \mathcal{J}_\delta \eta_t^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} (1 - \epsilon \partial_x^2) \partial_x \mathcal{J}_\delta \eta^\delta - \frac{\epsilon}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \mathcal{J}_\delta^2 \partial_x \left(|\mathcal{J}_\delta^2 v^\delta|^2 \left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_t \right) \\
&\quad - \frac{2\epsilon}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \mathcal{J}_\delta^2 \left(\frac{\mathcal{J}_\delta^2 v_x^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \cdot \mathcal{J}_\delta^2 v_t^\delta - \mathcal{J}_\delta^2 v^\delta \cdot \mathcal{J}_\delta^2 v_t^\delta \left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_x \right) \\
&\quad + \frac{2\epsilon}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \mathcal{J}_\delta^2 \left(\mathcal{J}_\delta^2 v^\delta \cdot [\partial_x \mathcal{J}_\delta^2, \frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}] v_t^\delta \right).
\end{aligned}$$

Similarly, applying ∂_t to (5.3), denoting by $\eta'^\delta = \partial_t \eta^\delta$, $v'^\delta = \partial_t v^\delta$, we obtain

$$(5.4) \quad \begin{cases} \eta_{tt}^\delta - \mathcal{J}_\delta \partial_x \left((1 + \epsilon \mathcal{J}_\delta \eta^\delta) \partial_x \mathcal{J}_\delta \eta'^\delta \right) + \epsilon \mathcal{J}_\delta \partial_x \left((1 + \epsilon \mathcal{J}_\delta \eta^\delta) \partial_x^3 \mathcal{J}_\delta \eta'^\delta \right) \\ \quad + 2\epsilon \mathcal{J}_\delta^2 \left(\frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \cdot \partial_x \mathcal{J}_\delta^2 \eta_t'^\delta \right) = f'^\delta, \\ \frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} v_{tt}'^\delta - \partial_x^2 \mathcal{J}_\delta^2 v'^\delta + \epsilon \partial_x^4 \mathcal{J}_\delta^2 v'^\delta \\ \quad + \frac{2\epsilon}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \mathcal{J}_\delta^2 \left(\mathcal{J}_\delta^2 v^\delta \cdot \partial_x \mathcal{J}_\delta^2 \left(\frac{v_t'^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) \right) = g'^\delta, \end{cases}$$

where

$$\begin{aligned}
f'^\delta &\stackrel{\text{def}}{=} \partial_t f^\delta + \epsilon \mathcal{J}_\delta \partial_x \left(\mathcal{J}_\delta \eta_t^\delta \cdot \partial_x \mathcal{J}_\delta \eta^\delta \right) - \epsilon^2 \mathcal{J}_\delta \partial_x \left(\mathcal{J}_\delta \eta_t^\delta \cdot \partial_x^3 \mathcal{J}_\delta \eta^\delta \right) \\
&\quad - 2\epsilon \mathcal{J}_\delta^2 \left(\partial_t \left(\frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) \cdot \partial_x \mathcal{J}_\delta^2 \eta_t^\delta \right), \\
g'^\delta &\stackrel{\text{def}}{=} \partial_t g^\delta - \partial_t \left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) v_{tt}^\delta - 2\epsilon \partial_t \left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) \mathcal{J}_\delta^2 \left(\mathcal{J}_\delta^2 v^\delta \cdot \partial_x \mathcal{J}_\delta^2 \left(\frac{v_t^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) \right) \\
&\quad - \frac{2\epsilon}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \mathcal{J}_\delta^2 \left(\mathcal{J}_\delta^2 v_t^\delta \cdot \partial_x \mathcal{J}_\delta^2 \left(\frac{v_t^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) \right) \\
&\quad + \frac{2\epsilon^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \mathcal{J}_\delta^2 \left(\mathcal{J}_\delta^2 v^\delta \cdot \partial_x \mathcal{J}_\delta^2 \left(\frac{\mathcal{J}_\delta \eta_t^\delta v_t^\delta}{(1 + \epsilon \mathcal{J}_\delta \eta^\delta)^2} \right) \right).
\end{aligned}$$

Step 2.2. Definitions of the energy functionals. In this step, we always assume that

$$(5.5) \quad 1 + \epsilon \mathcal{J}_\delta \eta^\delta > H > 0.$$

This assumption is a consequence of the initial assumption $1 + \epsilon \eta_0 > H > 0$ together with the smallness of ϵ and the following uniform energy estimates.

In order to derive the uniform energy estimates for approximate solutions, similar to the a priori energy estimates, we first introduce the energy functionals $E^\delta(t)$ and $\mathcal{E}^\delta(t)$ in the similar way as $E(t)$ and $\mathcal{E}(t)$ in (4.98) and (4.100). We define

$$\begin{aligned}
E^\delta(t) &= E_0^\delta(t) + E_1^\delta(t) + E_2^\delta(t) \\
&= E_0^\delta(t) + (E_{11}^\delta(t) + E_{12}^\delta(t)) + (E_{21}^\delta(t) + E_{22}^\delta(t))
\end{aligned}$$

with

$$\begin{aligned}
E_0^\delta(t) &= \sum_{k=0}^2 E_{0k}^\delta \stackrel{\text{def}}{=} \sum_{k=0}^2 \left(|\partial^k \eta^\delta|_2^2 + \epsilon |\partial^k \eta_x^\delta|_2^2 + \left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \partial^k v^\delta \mid \partial^k v^\delta \right)_2 \right), \\
E_{11}^\delta(t) &= |\eta_t^\delta|_2^2 + \epsilon |\eta_{tx}^\delta|_2^2 + ((1 + \epsilon \mathcal{J}_\delta \eta^\delta) \mathcal{J}_\delta \eta_x^\delta \mid \mathcal{J}_\delta \eta_x^\delta)_2 \\
&\quad + 2\epsilon ((1 + \epsilon \mathcal{J}_\delta \eta^\delta) \mathcal{J}_\delta \eta_{xx}^\delta \mid \mathcal{J}_\delta \eta_{xx}^\delta)_2 + \epsilon^2 ((1 + \epsilon \mathcal{J}_\delta \eta^\delta) \mathcal{J}_\delta \eta_{xxx}^\delta \mid \mathcal{J}_\delta \eta_{xxx}^\delta)_2, \\
E_{12}^\delta(t) &= \left(\frac{v_t^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \mid v_t^\delta \right)_2 + |\mathcal{J}_\delta v_x^\delta|_2^2 + \epsilon |\mathcal{J}_\delta v_{xx}^\delta|_2^2, \\
E_{21}^\delta(t) &= |\eta_{tt}^\delta|_2^2 + \epsilon |\eta_{ttx}^\delta|_2^2 + ((1 + \epsilon \mathcal{J}_\delta \eta^\delta) \mathcal{J}_\delta \eta_{tx}^\delta \mid \mathcal{J}_\delta \eta_{tx}^\delta)_2 \\
&\quad + 2\epsilon ((1 + \epsilon \mathcal{J}_\delta \eta^\delta) \mathcal{J}_\delta \eta_{ttx}^\delta \mid \mathcal{J}_\delta \eta_{ttx}^\delta)_2 + \epsilon^2 ((1 + \epsilon \mathcal{J}_\delta \eta^\delta) \mathcal{J}_\delta \eta_{txxx}^\delta \mid \mathcal{J}_\delta \eta_{txxx}^\delta)_2, \\
E_{22}^\delta(t) &= \left(\frac{v_{tt}^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \mid v_{tt}^\delta \right)_2 + |\mathcal{J}_\delta v_{tx}^\delta|_2^2 + \epsilon |\mathcal{J}_\delta v_{txx}^\delta|_2^2.
\end{aligned}$$

We remark that we used $(\eta_t^\delta, v_t^\delta)$ to replace $(\eta'^\delta, v'^\delta)$ when we defined $E_{21}^\delta(t)$ and $E_{22}^\delta(t)$. Using (5.5) and the properties of \mathcal{J}_δ in Lemma 5.1, we have

$$\begin{aligned}
E^\delta(t) \sim \widetilde{E}^\delta(t) &\stackrel{\text{def}}{=} |\eta^\delta|_{X_\epsilon^2}^2 + |\eta_t^\delta|_{X_\epsilon^0}^2 + |\mathcal{J}_\delta \eta_{tx}^\delta|_{X_{\epsilon^2}^1}^2 + |\eta_{tt}^\delta|_{X_\epsilon^0}^2 \\
&\quad + |v^\delta|_{H^2}^2 + |v_t^\delta|_2^2 + |\mathcal{J}_\delta v_{tx}^\delta|_{X_\epsilon^0}^2 + |v_{tt}^\delta|_2^2.
\end{aligned}$$

We also define the full energy functional as follows

$$\begin{aligned}
\mathcal{E}^\delta(t) &= \widetilde{E}^\delta(t) + |\eta_{tx}^\delta|_2^2 + |v_{tx}^\delta|_2^2 + \epsilon^3 |\mathcal{J}_\delta^2 \eta_{xxxx}^\delta|_2^2 + \epsilon^2 |\mathcal{J}_\delta^2 v_{xxxx}^\delta|_2^2, \\
(5.6) \quad &\sim |\eta^\delta|_{X_\epsilon^2}^2 + |\eta_t^\delta|_{H^1}^2 + |\mathcal{J}_\delta \eta_{tx}^\delta|_{X_{\epsilon^2}^0}^2 + |\eta_{tt}^\delta|_{X_\epsilon^0}^2 + \epsilon^3 |\mathcal{J}_\delta^2 \eta_{xxxx}^\delta|_2^2 \\
&\quad + |v^\delta|_{H^2}^2 + |v_t^\delta|_{H^1}^2 + |\mathcal{J}_\delta v_{tx}^\delta|_{X_\epsilon^0}^2 + |v_{tt}^\delta|_2^2 + \epsilon^2 |\mathcal{J}_\delta^2 v_{xxxx}^\delta|_2^2.
\end{aligned}$$

We remark that the mollifier for the highest order derivatives of η^δ and v^δ is \mathcal{J}_δ^2 not \mathcal{J}_δ .

Now, we prove

$$(5.7) \quad \mathcal{E}^\delta(t) \sim \widetilde{E}^\delta(t) \sim E^\delta(t).$$

To obtain (5.7), we only need to control $|\eta_{tx}^\delta|_2^2$, $|v_{tx}^\delta|_2^2$, $\epsilon^3 |\mathcal{J}_\delta^2 \eta_{xxxx}^\delta|_2^2$, $\epsilon^2 |\mathcal{J}_\delta^2 v_{xxxx}^\delta|_2^2$ by $\widetilde{E}^\delta(t)$.

In what follows, we always assume that on the existence time interval

$$(5.8) \quad \mathcal{E}^\delta(t) \leq C(\mathcal{E}^\delta(0))\mathcal{E}^\delta(0),$$

where $C(\mathcal{E}^\delta(0))$ is a constant depending on $\mathcal{E}^\delta(0)$ and in what follows, we shall use $C(\lambda_1, \lambda_2, \dots)$ to denote constants depending on $\lambda_1, \lambda_2, \dots$.

Firstly, thanks to (5.1), we have

$$|\eta_{tx}^\delta|_2 = |\mathcal{J}_\delta v_{xx}^\delta|_2 \leq |v_{xx}^\delta|_2,$$

and

$$\begin{aligned}
|v_{tx}^\delta|_2 &\lesssim \left| \left((1 + \epsilon \mathcal{J}_\delta \eta^\delta)(1 - \epsilon \partial_x^2) \mathcal{J}_\delta \eta_x^\delta \right)_x \right|_2 + \epsilon \left| \mathcal{J}_\delta^2 \left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_{xx} \right|_2 \\
&\leq C |\mathcal{J}_\delta \eta_{xx}^\delta|_2 + C \epsilon |\mathcal{J}_\delta \eta_{xxxx}^\delta|_2 + C \epsilon |\mathcal{J}_\delta \eta^\delta|_\infty (|\mathcal{J}_\delta \eta_x^\delta|_2 + \epsilon |\mathcal{J}_\delta \eta_{xxxx}^\delta|_2) \\
&\quad + C (|\mathcal{J}_\delta^2 v^\delta|_{W^{1,\infty}}, |\mathcal{J}_\delta \eta_x^\delta|_\infty) (|\mathcal{J}_\delta^2 v_x^\delta|_2 + |\mathcal{J}_\delta^2 v_{xx}^\delta|_2 + |\mathcal{J}_\delta \eta_x^\delta|_2 + |\mathcal{J}_\delta \eta_{xx}^\delta|_2) \\
&\leq C (|\mathcal{J}_\delta^2 v^\delta|_{W^{1,\infty}}, |\mathcal{J}_\delta \eta^\delta|_{W^{1,\infty}}) (|\eta^\delta|_{X_\epsilon^2} + |v^\delta|_{H^2} + \epsilon |\mathcal{J}_\delta \eta_{xxxx}^\delta|_2) \\
&\leq C (|v^\delta|_{H^2}, |\eta^\delta|_{H^2}) (|\eta^\delta|_{X_\epsilon^2} + |v^\delta|_{H^2} + \epsilon |\mathcal{J}_\delta \eta_{xxxx}^\delta|_2).
\end{aligned}$$

For the last term $\epsilon |\mathcal{J}_\delta \eta_{xxxx}^\delta|_2$ in the above inequality, we infer by Plancherel theorem that

$$(5.9) \quad \epsilon |\mathcal{J}_\delta \eta_{xxxx}^\delta|_2 \lesssim \epsilon |\eta_{xxx}^\delta|_{\frac{1}{2}} |\mathcal{J}_\delta^2 \eta_{xxxx}^\delta|_{\frac{1}{2}} \lesssim \epsilon^{\frac{1}{2}} |\eta_{xxx}^\delta|_2 + \epsilon^{\frac{3}{2}} |\mathcal{J}_\delta^2 \eta_{xxxx}^\delta|_2.$$

Then we obtain

$$(5.10) \quad |v_{tx}^\delta|_2 \leq C (|v^\delta|_{H^2}, |\eta^\delta|_{H^2}) (|\eta^\delta|_{X_\epsilon^2} + |v^\delta|_{H^2} + \epsilon^{\frac{3}{2}} |\mathcal{J}_\delta^2 \eta_{xxxx}^\delta|_2).$$

Thus, we only need to control $\epsilon^3 |\mathcal{J}_\delta^2 \eta_{xxxx}^\delta|_2^2$ and $\epsilon^2 |\mathcal{J}_\delta^2 v_{xxxx}^\delta|_2^2$ by $\widetilde{E}^\delta(t)$.

By virtue of (5.1), we have

$$\mathcal{J}_\delta v_x^\delta = -\eta_t^\delta, \quad (1 - \epsilon \partial_x^2) \mathcal{J}_\delta \eta_x^\delta = -\frac{v_t^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} - \frac{\epsilon}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \mathcal{J}_\delta^2 \left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_x.$$

Then we have

$$(5.11) \quad \epsilon |\mathcal{J}_\delta^2 v_{xxxx}^\delta|_2 = \epsilon |\mathcal{J}_\delta \eta_{txxx}^\delta|_2 \leq C (\widetilde{E}^\delta(t))^{\frac{1}{2}},$$

and

$$\begin{aligned}
\epsilon^{\frac{3}{2}} |\mathcal{J}_\delta^2 \eta_{xxxx}^\delta|_2 &\leq \epsilon^{\frac{1}{2}} |\mathcal{J}_\delta^2 \eta_{xxx}^\delta|_2 + C \epsilon^{\frac{1}{2}} \left| \mathcal{J}_\delta \left(\frac{v_t^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_{xx} \right|_2 \\
(5.12) \quad &+ C \epsilon^{\frac{3}{2}} \left| \mathcal{J}_\delta \left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \mathcal{J}_\delta^2 \left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_x \right)_{xx} \right|_2 \\
&\stackrel{\text{def}}{=} \epsilon^{\frac{1}{2}} |\mathcal{J}_\delta^2 \eta_{xxx}^\delta|_2 + C(A_1 + A_2).
\end{aligned}$$

For A_1 , we obtain

$$A_1 \lesssim \epsilon^{\frac{1}{2}} |\mathcal{J}_\delta v_{txx}^\delta|_2 + \epsilon^{\frac{1}{2}} |[\partial_x^2 \mathcal{J}_\delta, \frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}] v_t^\delta|_2.$$

Since

$$[\partial_x^2 \mathcal{J}_\delta, a]f = [\mathcal{J}_\delta, a] \partial_x^2 f + \mathcal{J}_\delta (\partial_x^2 a f + \partial_x a \partial_x f),$$

using Lemma 5.1 and Hölder inequality, we obtain

$$\begin{aligned}
|[\partial_x^2 \mathcal{J}_\delta, \frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}] v_t^\delta|_2 &\leq C \left| \frac{\epsilon \mathcal{J}_\delta \eta^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right|_{H^2} |v_{tx}^\delta|_2 \\
&+ \left| \left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_{xx} \right|_2 |v_t^\delta|_\infty + \left| \left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_x \right|_\infty |v_{tx}^\delta|_2 \\
&\leq C (|\mathcal{J}_\delta \eta^\delta|_\infty) \epsilon |\mathcal{J}_\delta \eta^\delta|_{H^2} (|v_t^\delta|_2 + |v_{tx}^\delta|_2).
\end{aligned}$$

Then we have

$$(5.13) \quad |A_1| \leq C (|\mathcal{J}_\delta \eta^\delta|_{H^2}) (|\mathcal{J}_\delta v_{tx}^\delta|_{X_\epsilon^0} + |v_t^\delta|_2 + \epsilon^{\frac{3}{2}} |v_{tx}^\delta|_2).$$

By virtue of (5.10) and (5.13), we have

$$(5.14) \quad |A_1| \leq C(|v^\delta|_{H^2}, |\eta^\delta|_{H^2}) \widetilde{E}^\delta(t) + \epsilon^{\frac{3}{2}} C(|v^\delta|_{H^2}, |\eta^\delta|_{H^2}) \cdot \epsilon^{\frac{3}{2}} |\mathcal{J}_\delta^2 \eta_{xxxx}^\delta|_2.$$

For A_2 , by the product estimates and interpolation estimates (4.2), we have

$$\begin{aligned} |A_2| &\leq \epsilon^{\frac{3}{2}} C(|\mathcal{J}_\delta^2 v^\delta|_{W^{1,\infty}}, |\mathcal{J}_\delta \eta^\delta|_{W^{1,\infty}}) (\epsilon |\mathcal{J}_\delta \eta_x^\delta|_{H^2} + |\mathcal{J}_\delta^2 v_x^\delta|_{H^2}) \\ &\leq C(|v^\delta|_{H^2}, |\eta^\delta|_{H^2}) (|\eta^\delta|_{X_c^2} + |v^\delta|_{H^2} + \epsilon |\mathcal{J}_\delta^2 v_{xxxx}^\delta|_2), \end{aligned}$$

which along with (5.11) implies that

$$(5.15) \quad |A_2| \leq C(|v^\delta|_{H^2}, |\eta^\delta|_{H^2}) \widetilde{E}^\delta(t).$$

Thanks to (5.12), (5.14) and (5.15), we obtain

$$\epsilon^{\frac{3}{2}} |\mathcal{J}_\delta^2 \eta_{xxxx}^\delta|_2 \leq C(|v^\delta|_{H^2}, |\eta^\delta|_{H^2}) \widetilde{E}^\delta(t) + \epsilon^{\frac{3}{2}} C(|v^\delta|_{H^2}, |\eta^\delta|_{H^2}) \cdot \epsilon^{\frac{3}{2}} |\mathcal{J}_\delta^2 \eta_{xxxx}^\delta|_2.$$

By virtue of (5.8), for ϵ sufficiently small (depending on $\mathcal{E}^\delta(0)$), we have

$$(5.16) \quad \epsilon^{\frac{3}{2}} |\mathcal{J}_\delta^2 \eta_{xxxx}^\delta|_2 \leq C(\mathcal{E}^\delta(0)) \widetilde{E}^\delta(t).$$

Thus, combining (5.11) and (5.16), we obtain the equivalence (5.7).

Step 2.3. Uniform energy estimates for $V^\delta = (\eta^\delta, v^\delta)$.

Motivated by the a priori energy estimates (4.101) for (4.66), we obtain

$$(5.17) \quad \frac{d}{dt} E^\delta(t) \leq C(\mathcal{E}^\delta(t)) \epsilon \mathcal{E}^\delta(t)^{\frac{3}{2}},$$

where $C(\mathcal{E}^\delta(t))$ is a constant only depending on $\mathcal{E}^\delta(t)$. The derivation of (5.17) is a little different from (4.101).

(i) *Estimates for $E_0^\delta(t)$.* We first derive the estimates for $E_0^\delta(t)$. As usual, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_{0k}^\delta &= (\partial^k \eta_t^\delta | \partial^k \eta^\delta)_2 + \epsilon (\partial^{k+1} \eta_t^\delta | \partial^{k+1} \eta^\delta)_2 \\ &\quad + \left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \partial^k v_t^\delta | \partial^k v^\delta \right)_2 + \frac{1}{2} \left(\left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_t \partial^k v^\delta | \partial^k v^\delta \right)_2. \end{aligned}$$

Using the equations in (5.1), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_{0k}^\delta &= ([\partial^k, \frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}] v_t^\delta | \partial^k v^\delta)_2 - \epsilon (\partial^k \left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \mathcal{J}_\delta^2 \left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_x \right) | \partial^k v^\delta)_2 \\ &\quad + \frac{1}{2} \left(\left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_t \partial^k v^\delta | \partial^k v^\delta \right)_2 \stackrel{\text{def}}{=} B_1^k + B_2^k + B_3^k. \end{aligned}$$

For B_1^k , we have $B_1^0 = 0$ and

$$\begin{aligned} |B_1^1| + |B_1^2| &\leq \sum_{k=1}^2 |[\partial^k, \frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}] v_t^\delta|_2 |\partial^k v^\delta|_2 \\ &\leq C \epsilon (1 + |\mathcal{J}_\delta \eta_x^\delta|_\infty) (|\mathcal{J}_\delta \eta_x^\delta|_\infty |v_t^\delta|_{H^1} + |\mathcal{J}_\delta \eta_x^\delta|_{H^1} |v_t^\delta|_\infty) |v^\delta|_{H^2} \\ &\leq C (|\eta^\delta|_{H^2}) \epsilon |\eta^\delta|_{H^2} |v_t^\delta|_{H^1} |v^\delta|_{H^2}. \end{aligned}$$

For B_3^k , we have

$$\sum_{k=0}^2 |B_3^k| \leq C \epsilon |\mathcal{J}_\delta \eta_t^\delta|_\infty |v^\delta|_{H^2}^2 \leq C \epsilon |\eta_t^\delta|_{H^1} |v^\delta|_{H^2}^2.$$

For B_2^k , we have

$$\begin{aligned}
B_2^k &= -\epsilon([\partial^k, \frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}] \mathcal{J}_\delta^2 \left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_x | \partial^k v^\delta)_2 \\
&\quad - \epsilon(\partial^k \left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_x | \mathcal{J}_\delta^2 \left(\frac{\partial^k v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right))_2 \stackrel{\text{def}}{=} B_{21}^k + B_{22}^k
\end{aligned}$$

Similar to B_1^k , we have $B_{21}^0 = 0$ and

$$\begin{aligned}
|B_{21}^1| + |B_{21}^2| &\leq C(|\eta^\delta|_{H^2}) \epsilon |\eta^\delta|_{H^2} |\mathcal{J}_\delta^2 \left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_x |_{H^1} |v^\delta|_{H^2} \\
&\leq C(|\eta^\delta|_{H^2}, |v^\delta|_{H^2}) \epsilon (|\eta^\delta|_{H^2}^2 + |v^\delta|_{H^2}^2) |v^\delta|_{H^2}.
\end{aligned}$$

For B_{22}^k , we have

$$\begin{aligned}
B_{22}^k &= \epsilon(\partial^k \left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) | \partial_x \left([\mathcal{J}_\delta^2, \frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}] \partial^k v^\delta \right))_2 \\
&\quad + \epsilon^2(\partial^k \left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2 \mathcal{J}_\delta \eta_x^\delta}{(1 + \epsilon \mathcal{J}_\delta \eta^\delta)^2} \right) | \frac{\mathcal{J}_\delta^2 \partial^k v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta})_2 \\
&\quad - 2\epsilon([\partial^k, \frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}] \mathcal{J}_\delta^2 v_x^\delta | \frac{\mathcal{J}_\delta^2 \partial^k v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta})_2 \\
&\quad - 2\epsilon \left(\frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \cdot \partial_x \mathcal{J}_\delta^2 \partial^k v^\delta | \frac{\mathcal{J}_\delta^2 \partial^k v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_2.
\end{aligned}$$

Notice that the last term of the above equality equals

$$\epsilon \left(\partial_x \left(\frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) \cdot \mathcal{J}_\delta^2 \partial^k v^\delta | \frac{\mathcal{J}_\delta^2 \partial^k v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right)_2$$

Then, proceeding as for the previous terms, using Lemma 5.1 and product estimates, we have

$$\sum_{k=0}^2 |B_{22}^k| \leq C(|\eta^\delta|_{X_c^2}, |v^\delta|_{H^2}) \epsilon (|\eta^\delta|_{H^2}^2 + |v^\delta|_{H^2}^2) |v^\delta|_{H^2}.$$

Combining all the above estimates, we obtain

$$(5.18) \quad \frac{d}{dt} E_0^\delta(t) \leq C(|\eta^\delta|_{X_c^2}, |v^\delta|_{H^2}) \epsilon (\mathcal{E}^\delta(t))^{\frac{3}{2}}.$$

(ii) *Estimates for $E_2^\delta(t)$.* Now, we derive the energy estimates for (5.4). For the second equation of (5.4), taking L^2 inner product with $v_t'^\delta$ yields

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} E_{22}^\delta + 2\epsilon \underbrace{\left(\mathcal{J}_\delta^2 v^\delta \cdot \partial_x \mathcal{J}_\delta^2 \left(\frac{v_t'^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) | \mathcal{J}_\delta^2 \left(\frac{v_t'^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) \right)_2}_I \\
&= \frac{1}{2} \left(\partial_t \left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) v_t'^\delta | v_t'^\delta \right)_2 + (g'^\delta | v_t'^\delta)_2
\end{aligned}$$

By integration by parts, we have

$$I = -\epsilon \left(\mathcal{J}_\delta^2 v_x^\delta \cdot \mathcal{J}_\delta^2 \left(\frac{v_t'^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) | \mathcal{J}_\delta^2 \left(\frac{v_t'^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \right) \right)_2.$$

Then we obtain

$$(5.19) \quad \begin{aligned} \frac{d}{dt} E_{22}^\delta &\leq C\epsilon(|\mathcal{J}_\delta \eta_t^\delta|_\infty + |\mathcal{J}_\delta^2 v_x^\delta|_\infty)(|v_t'^\delta|_2^2 + |\mathcal{J}_\delta^2(\frac{v_t'^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta})|_2^2) + |g'^\delta|_2 |v_t'^\delta|_2 \\ &\leq C\epsilon(|\eta_t^\delta|_{H^1} + |v^\delta|_{H^2})|v_t'^\delta|_2^2 + |g'^\delta|_2 |v_t'^\delta|_2. \end{aligned}$$

For the first equation of (5.4), taking L^2 inner product by $(1 - \epsilon \partial_x^2) \eta_t'^\delta$ results

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E_{21}^\delta + 2\epsilon \underbrace{\left(\frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} \cdot \partial_x \mathcal{J}_\delta^2 \eta_t'^\delta \mid (1 - \epsilon \partial_x^2) \mathcal{J}_\delta^2 \eta_t'^\delta \right)_2}_{II} \\ = \frac{\epsilon}{2} (\mathcal{J}_\delta \eta_t^\delta \mathcal{J}_\delta \eta_x'^\delta \mid \mathcal{J}_\delta \eta_x'^\delta)_2 + \epsilon^2 (\mathcal{J}_\delta \eta_t^\delta \mathcal{J}_\delta \eta_{xx}^\delta \mid \mathcal{J}_\delta \eta_{xx}^\delta)_2 + \epsilon^2 (\mathcal{J}_\delta \eta_{xx}^\delta \mathcal{J}_\delta \eta_x'^\delta \mid \mathcal{J}_\delta \eta_{tx}^\delta)_2 \\ + \frac{\epsilon^3}{2} (\mathcal{J}_\delta \eta_t^\delta \mathcal{J}_\delta \eta_{xxx}^\delta \mid \mathcal{J}_\delta \eta_{xxx}^\delta)_2 + (f'^\delta \mid (1 - \epsilon \partial_x^2) \eta_t'^\delta)_2. \end{aligned}$$

By integration by parts, we have

$$\begin{aligned} II &= -\epsilon (\partial_x (\frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}) \cdot \mathcal{J}_\delta^2 \eta_t'^\delta \mid (1 - \epsilon \partial_x^2) \mathcal{J}_\delta^2 \eta_t'^\delta)_2 \\ &\quad + \epsilon^2 (\partial_x (\frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}) \cdot \mathcal{J}_\delta^2 \eta_{tx}^\delta \mid (1 - \epsilon \partial_x^2) \mathcal{J}_\delta^2 \eta_{tx}^\delta)_2 \\ &\leq C\epsilon (|\mathcal{J}_\delta^2 v_x^\delta|_\infty + \epsilon |\mathcal{J}_\delta^2 v^\delta|_\infty |\mathcal{J}_\delta \eta_x^\delta|_\infty) (|\mathcal{J}_\delta^2 \eta_t'^\delta|_2^2 + \epsilon |\mathcal{J}_\delta^2 \eta_{tx}^\delta|_2^2). \end{aligned}$$

Then we get

$$(5.20) \quad \begin{aligned} \frac{d}{dt} E_{21}^\delta &\leq C\epsilon (|\mathcal{J}_\delta^2 v_x^\delta|_\infty + \epsilon |\mathcal{J}_\delta^2 v^\delta|_\infty |\mathcal{J}_\delta \eta_x^\delta|_\infty + |\mathcal{J}_\delta \eta_t^\delta|_\infty + \epsilon^{\frac{1}{2}} |\mathcal{J}_\delta \eta_x^\delta|_\infty) \\ &\quad \times (|\mathcal{J}_\delta^2 \eta_t'^\delta|_{X_\epsilon^0}^2 + |\mathcal{J}_\delta \eta_x^\delta|_{X_{\epsilon^2}^0}^2 + \epsilon |\mathcal{J}_\delta \eta_{xx}^\delta|_2^2) + |f'^\delta|_2 |\eta_t'^\delta|_2 + \epsilon |f_x'^\delta|_2 |\eta_{tx}^\delta|_2. \end{aligned}$$

Due to (5.19) and (5.20), noticing that $\eta'^\delta = \eta_t^\delta$, we obtain

$$(5.21) \quad \begin{aligned} \frac{d}{dt} E_2^\delta &\leq C(|\eta^\delta|_{H^2})\epsilon (|v^\delta|_{H^2} + |\eta_t^\delta|_{H^1} + |\mathcal{J}_\delta \eta_{tx}^\delta|_{X_{\epsilon^2}^0}) \\ &\quad \times (|\mathcal{J}_\delta^2 \eta_t'^\delta|_{X_\epsilon^0}^2 + |\mathcal{J}_\delta \eta_x^\delta|_{X_{\epsilon^2}^0}^2 + |\mathcal{J}_\delta \eta_x^\delta|_{X_{\epsilon^2}^0}^2 + |v_t'^\delta|_2^2) \\ &\quad + |f'^\delta|_2 |\eta_t'^\delta|_2 + \epsilon |f_x'^\delta|_2 |\eta_{tx}^\delta|_2 + |g'^\delta|_2 |v_t'^\delta|_2. \end{aligned}$$

Noticing that $\eta'^\delta = \eta_t^\delta$ and $v'^\delta = v_t^\delta$. Then (5.21) implies that

$$(5.22) \quad \frac{d}{dt} E_2^\delta \leq C(\mathcal{E}^\delta(t))\epsilon(\mathcal{E}^\delta(t))^{\frac{3}{2}} + |f'^\delta|_2 |\eta_{tt}^\delta|_2 + \epsilon |f_x'^\delta|_2 |\eta_{ttx}^\delta|_2 + |g'^\delta|_2 |v_{tt}^\delta|_2.$$

(iii) *Estimates for E_1^δ .* Similarly as E_2^δ , we also obtain

$$(5.23) \quad \frac{d}{dt} E_1^\delta \leq C(\mathcal{E}^\delta(t))\epsilon(\mathcal{E}^\delta(t))^{\frac{3}{2}} + |f^\delta|_2 |\eta_t^\delta|_2 + \epsilon |f_x^\delta|_2 |\eta_{tx}^\delta|_2 + |g^\delta|_2 |v_t^\delta|_2.$$

(iv) *Estimates for the source terms.* To achieve (5.17), it remains to derive the bound on f^δ , g^δ , f'^δ and g'^δ . Similarly as in the derivation of (4.95), using the

properties of \mathcal{J}_δ in Lemma 5.1, we obtain

$$\begin{aligned}
& |f^\delta|_2 + \epsilon^{\frac{1}{2}}|f_x^\delta|_2 + |g^\delta|_2 + |f'^\delta|_2 + \epsilon^{\frac{1}{2}}|f'_x{}^\delta|_2 + |g'^\delta|_2 \\
(5.24) \quad & \lesssim C(\mathcal{E}^\delta(t))\epsilon(|\eta^\delta|_{X_\epsilon^2} + |v^\delta|_{H^2} + |\eta_t^\delta|_{H^1} + |v_t^\delta|_{H^1} + |\mathcal{J}_\delta\eta_{tx}^\delta|_{X_\epsilon^0} + |\mathcal{J}_\delta^2 v_{tx}^\delta|_{X_\epsilon^0} \\
& + \epsilon|\mathcal{J}_\delta\eta_{xxxx}^\delta|_2 + \epsilon^{\frac{1}{2}}|\mathcal{J}_\delta^2 v_{xxx}^\delta|_2)(|\eta^\delta|_{X_\epsilon^2} + \epsilon|\mathcal{J}_\delta\eta_{xxxx}^\delta|_2 + \epsilon^{\frac{3}{2}}|\mathcal{J}_\delta^2\eta_{xxxx}^\delta|_2 \\
& + |\eta_t^\delta|_{H^1} + |\mathcal{J}_\delta\eta_{tx}^\delta|_{X_\epsilon^0} + |\eta_{tt}^\delta|_{X_\epsilon^0} + |v^\delta|_{H^2} + |v_t^\delta|_{H^1} + |\mathcal{J}_\delta^2 v_{tx}^\delta|_{X_\epsilon^0} + |v_{tt}^\delta|_2).
\end{aligned}$$

To verify (5.24), we first need to check the estimates on the terms involving the commutators in f^δ and g^δ . Using (v) of Lemma 5.1 with $t_0 = 1$, for the commutator term in f^δ , we have

$$\begin{aligned}
(5.25) \quad & |\epsilon\mathcal{J}_\delta^2([\mathcal{J}_\delta, \frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon\mathcal{J}_\delta\eta^\delta}]\partial_x \mathcal{J}_\delta\eta_t^\delta)|_{X_\epsilon^0} \leq C\epsilon|\frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon\mathcal{J}_\delta\eta^\delta}|_{H^2}|\mathcal{J}_\delta\eta_t^\delta|_{X_\epsilon^0} \\
& \leq C(|\eta^\delta|_{H^2}, |v^\delta|_{H^2})\epsilon(|\eta^\delta|_{H^2} + |v^\delta|_{H^2})|\eta_t^\delta|_{H^1}.
\end{aligned}$$

For the commutator term in f'^δ , we have

$$\begin{aligned}
& |\epsilon\mathcal{J}_\delta^2\partial_t([\mathcal{J}_\delta, \frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon\mathcal{J}_\delta\eta^\delta}]\partial_x \mathcal{J}_\delta\eta_t^\delta)|_{X_\epsilon^0} \\
& \leq \epsilon|[\mathcal{J}_\delta, \partial_t(\frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon\mathcal{J}_\delta\eta^\delta})]\partial_x \mathcal{J}_\delta\eta_t^\delta|_{X_\epsilon^0} + \epsilon|[\mathcal{J}_\delta, \frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon\mathcal{J}_\delta\eta^\delta}]\partial_x \mathcal{J}_\delta\eta_{tt}^\delta|_{X_\epsilon^0} \\
& \stackrel{\text{def}}{=} III_1 + III_2.
\end{aligned}$$

For III_1 , by using the product estimates, we have

$$\begin{aligned}
III_1 & \leq C(|\mathcal{J}_\delta\eta_x^\delta|_\infty, |\mathcal{J}_\delta^2 v^\delta|_\infty)\epsilon(|\mathcal{J}_\delta\eta_t^\delta|_\infty + |\mathcal{J}_\delta^2 v_t^\delta|_\infty + \epsilon^{\frac{1}{2}}|\mathcal{J}_\delta\eta_{tx}^\delta|_\infty \\
& + \epsilon^{\frac{1}{2}}|\mathcal{J}_\delta^2 v_{tx}^\delta|_\infty)|\mathcal{J}_\delta\eta_{tx}^\delta|_{X_\epsilon^0}.
\end{aligned}$$

For III_2 , by similar derivation as (5.25), we have

$$III_2 \leq C(|\eta^\delta|_{H^2}, |v^\delta|_{H^2})\epsilon(|\eta^\delta|_{H^2} + |v^\delta|_{H^2})|\mathcal{J}_\delta\eta_{tt}^\delta|_{X_\epsilon^0}.$$

Then by Sobolev inequality, we have

$$\begin{aligned}
(5.26) \quad & |\epsilon\mathcal{J}_\delta^2\partial_t([\mathcal{J}_\delta, \frac{\mathcal{J}_\delta^2 v^\delta}{1 + \epsilon\mathcal{J}_\delta\eta^\delta}]\partial_x \mathcal{J}_\delta\eta_t^\delta)|_{X_\epsilon^0} \leq C(\mathcal{E}^\delta(t))\epsilon(|\eta^\delta|_{H^2} + |v^\delta|_{H^2} \\
& + |\eta_t^\delta|_{H^1} + |v_t^\delta|_{H^1} + |\mathcal{J}_\delta\eta_{tx}^\delta|_{X_\epsilon^0} + |\mathcal{J}_\delta^2 v_{tx}^\delta|_{X_\epsilon^0})(|\mathcal{J}_\delta\eta_{tx}^\delta|_{X_\epsilon^0} + |\mathcal{J}_\delta\eta_{tt}^\delta|_{X_\epsilon^0}).
\end{aligned}$$

Similarly, for the commutator terms in g^δ and g'^δ , noticing that

$$[\partial_x \mathcal{J}_\delta^2, a]f = \partial_x \mathcal{J}_\delta([\mathcal{J}_\delta, a]f) + \partial_x([\mathcal{J}_\delta, a]\mathcal{J}_\delta f) + \partial_x a \mathcal{J}_\delta^2 f,$$

we have

$$\begin{aligned}
(5.27) \quad & |\frac{2\epsilon}{1 + \epsilon\mathcal{J}_\delta\eta^\delta}\mathcal{J}_\delta^2(\mathcal{J}_\delta^2 v^\delta \cdot [\partial_x \mathcal{J}_\delta^2, \frac{1}{1 + \epsilon\mathcal{J}_\delta\eta^\delta}]v_t^\delta)|_2 \\
& + |\partial_t(\frac{2\epsilon}{1 + \epsilon\mathcal{J}_\delta\eta^\delta}\mathcal{J}_\delta^2(\mathcal{J}_\delta^2 v^\delta \cdot [\partial_x \mathcal{J}_\delta^2, \frac{1}{1 + \epsilon\mathcal{J}_\delta\eta^\delta}]v_t^\delta))|_2 \\
& \leq C(\mathcal{E}^\delta(t))\epsilon(|\eta^\delta|_{H^2} + |\eta_t^\delta|_{H^1} + |v_t^\delta|_{H^1})(|\mathcal{J}_\delta\eta_{tx}^\delta|_2 + |v_t^\delta|_{H^1} + |v_{tt}^\delta|_2).
\end{aligned}$$

Another delicate term we have to check is the third terms of f'^δ . Applying $\epsilon^{\frac{1}{2}}\partial_x$ to the third term, we have

$$\begin{aligned} \epsilon^{\frac{5}{2}}\mathcal{J}_\delta\partial_x^2(\mathcal{J}_\delta\eta_t^\delta \cdot \partial_x^3\mathcal{J}_\delta\eta^\delta) &= \epsilon^{\frac{5}{2}}\mathcal{J}_\delta\eta_t^\delta \cdot \partial_x^5\mathcal{J}_\delta^2\eta^\delta \\ &+ \epsilon^{\frac{5}{2}}[\mathcal{J}_\delta, \mathcal{J}_\delta\eta_t^\delta] \cdot \partial_x^5\mathcal{J}_\delta\eta^\delta + \epsilon^{\frac{5}{2}}\mathcal{J}_\delta(\partial_x^2\mathcal{J}_\delta\eta_t^\delta \cdot \partial_x^3\mathcal{J}_\delta\eta^\delta + 2\partial_x\mathcal{J}_\delta\eta_t^\delta \cdot \partial_x^4\mathcal{J}_\delta\eta^\delta). \end{aligned}$$

Then by virtue of the properties of \mathcal{J}_δ in Lemma 5.1, we have

$$\begin{aligned} \epsilon^{\frac{5}{2}}|\mathcal{J}_\delta\partial_x^2(\mathcal{J}_\delta\eta_t^\delta \cdot \partial_x^3\mathcal{J}_\delta\eta^\delta)|_2 &\leq C\epsilon(|\mathcal{J}_\delta\eta_t^\delta|_\infty + \epsilon|\mathcal{J}_\delta\eta_{txx}^\delta|_\infty + \epsilon^{\frac{1}{2}}|\mathcal{J}_\delta\eta_t^\delta|_{H^2}) \\ &\quad \times (\epsilon^{\frac{1}{2}}|\mathcal{J}_\delta\eta_{xxx}^\delta|_2 + \epsilon|\mathcal{J}_\delta\eta_{xxxx}^\delta|_2 + \epsilon^{\frac{3}{2}}|\mathcal{J}_\delta^2\eta_{xxxx}^\delta|_2) \\ &\leq C\epsilon(|\eta_t^\delta|_{H^1} + |\mathcal{J}_\delta\eta_{tx}^\delta|_{X_\epsilon^0})(|\eta^\delta|_{X_\epsilon^2} + \epsilon|\mathcal{J}_\delta\eta_{xxxx}^\delta|_2 + \epsilon^{\frac{3}{2}}|\mathcal{J}_\delta^2\eta_{xxxx}^\delta|_2). \end{aligned}$$

(v) *Derivation of the energy estimates* (5.17). Similar to (5.9), by interpolation inequality, we have

$$(5.28) \quad \epsilon^{\frac{1}{2}}|\mathcal{J}_\delta^2v_{xxx}^\delta|_2 \lesssim \epsilon^{\frac{1}{2}}|\mathcal{J}_\delta^2v_{xx}^\delta|_2^{\frac{1}{2}}|\mathcal{J}_\delta^2v_{xxxx}^\delta|_2^{\frac{1}{2}} \lesssim |v_{xx}^\delta|_2 + \epsilon|\mathcal{J}_\delta^2v_{xxxx}^\delta|_2.$$

With (5.9) and (5.28), we bound the righthand side terms in (5.24) by $C(\mathcal{E}^\delta)\epsilon\mathcal{E}^\delta(t)$. Thus, by virtue of (5.18), (5.22), (5.23) and (5.24), we achieve the proof of (5.17). Thanks to (5.7), under the assumptions (5.5) and (5.8), we have

$$(5.29) \quad \frac{d}{dt}E^\delta(t) \leq C(\mathcal{E}^\delta(t))\epsilon\mathcal{E}^\delta(t)^{\frac{3}{2}} \leq C(\mathcal{E}^\delta(0))\epsilon E^\delta(t)^{\frac{3}{2}}.$$

(vi) *Bound of the initial energy* $\mathcal{E}^\delta(0)$. Proceeding as in the derivation of (4.106) in Step 5 of the proof to Theorem 4.6, we obtain

$$(5.30) \quad \mathcal{E}^\delta(0) \leq C(|\eta^\delta|_{t=0}|_{X_\epsilon^2}^2 + |v^\delta|_{t=0}|_{X_\epsilon^2}^2) \leq C(|\eta_0|_{X_\epsilon^2}^2 + |v_0|_{X_\epsilon^2}^2).$$

With (5.29) and (5.30), there exists $T > 0$ which depends on $|\eta_0|_{X_\epsilon^2} + |v_0|_{X_\epsilon^2}$, such that

$$(5.31) \quad \sup_{0 \leq t \leq T/\epsilon} \mathcal{E}^\delta(t) \leq C(|\eta_0|_{X_\epsilon^2}^2 + |v_0|_{X_\epsilon^2}^2).$$

Step 3. The family of approximate solutions forms a Cauchy sequence in the lower order space $C([0, T/\epsilon], X_\epsilon^0(\mathbb{R}) \times L^2(\mathbb{R}))$. For any $\delta, \delta' \in (0, \epsilon)$, we shall derive estimates on $(\eta^\delta - \eta^{\delta'}, v^\delta - v^{\delta'})$ in $C([0, T/\epsilon], X_\epsilon^0(\mathbb{R}) \times L^2(\mathbb{R}))$. Denoting by

$$\begin{aligned} E_0^{(\delta, \delta')}(t) &\stackrel{\text{def}}{=} |\eta^\delta - \eta^{\delta'}|_2^2 + \epsilon|(\eta^\delta - \eta^{\delta'})_x|_2^2 + \left(\frac{v^\delta - v^{\delta'}}{1 + \epsilon\mathcal{J}_\delta\eta^\delta} |v^\delta - v^{\delta'}\right)_2, \\ &\sim |\eta^\delta - \eta^{\delta'}|_{X_\epsilon^0}^2 + |v^\delta - v^{\delta'}|_2^2. \end{aligned}$$

By (5.1), we deduce that

$$\frac{1}{2}\frac{d}{dt}E_0^{(\delta, \delta')}(t) = \sum_{i=1}^4 I_i,$$

where

$$\begin{aligned}
I_1 &= -\{((\mathcal{J}_\delta v^\delta - \mathcal{J}_{\delta'} v^{\delta'})_x | (1 - \epsilon \partial_x^2)(\eta^\delta - \eta^{\delta'}))_2 \\
&\quad + ((1 - \epsilon \partial_x^2)(\mathcal{J}_\delta \eta^\delta - \mathcal{J}_{\delta'} \eta^{\delta'})_x | v^\delta - v^{\delta'})_2\} \\
I_2 &= -\left(\left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta} - \frac{1}{1 + \epsilon \mathcal{J}_{\delta'} \eta^{\delta'}}\right) v_t^{\delta'} | v^\delta - v^{\delta'}\right)_2 \\
I_3 &= -\frac{1}{2} \left(\partial_t \left(\frac{1}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}\right) (v^\delta - v^{\delta'}) | v^\delta - v^{\delta'}\right)_2 \\
I_4 &= \epsilon \left(\mathcal{J}_{\delta'}^2 \left(\frac{|\mathcal{J}_{\delta'}^2 v^{\delta'}|^2}{1 + \epsilon \mathcal{J}_{\delta'} \eta^{\delta'}}\right)_x - \mathcal{J}_\delta^2 \left(\frac{|\mathcal{J}_\delta^2 v^\delta|^2}{1 + \epsilon \mathcal{J}_\delta \eta^\delta}\right)_x | v^\delta - v^{\delta'}\right)_2.
\end{aligned}$$

For I_1 , integrating by parts, we obtain

$$I_1 = -\left((1 - \epsilon \partial_x^2)(\eta^\delta - \eta^{\delta'}) | (\mathcal{J}_\delta - \mathcal{J}_{\delta'}) v_x^{\delta'}\right)_2 - \left((\mathcal{J}_\delta - \mathcal{J}_{\delta'}) \eta_x^{\delta'} | (1 - \epsilon \partial_x^2)(v^\delta - v^{\delta'})\right)_2,$$

which along with (iii) in Lemma 5.1 implies

$$\begin{aligned}
|I_1| &\leq C \max\{\delta, \delta'\} (|\eta^\delta - \eta^{\delta'}|_{H^2} |v_x^{\delta'}|_{H^1} + |\eta_x^{\delta'}|_{H^1} |v^\delta - v^{\delta'}|_{H^2}) \\
&\leq C(\mathcal{E}^\delta(t) + \mathcal{E}^{\delta'}(t)) \max\{\delta, \delta'\} \leq CM \max\{\delta, \delta'\},
\end{aligned}$$

where $M = C(|\eta_0|_{X_{\epsilon^3}^2}^2 + |v_0|_{X_{\epsilon^2}^2}^2)$ the uniform bound for $\mathcal{E}^\delta(t)$ in (5.31).

For I_2 , we have

$$\begin{aligned}
|I_2| &= \epsilon \left| \left(\frac{(\mathcal{J}_\delta - \mathcal{J}_{\delta'}) \eta^\delta + \mathcal{J}_{\delta'} (\eta^\delta - \eta^{\delta'})}{(1 + \epsilon \mathcal{J}_\delta \eta^\delta)(1 + \epsilon \mathcal{J}_{\delta'} \eta^{\delta'})} v_t^{\delta'} | v^\delta - v^{\delta'} \right)_2 \right| \\
&\leq C \epsilon (|(\mathcal{J}_\delta - \mathcal{J}_{\delta'}) \eta^\delta|_2 + |\mathcal{J}_{\delta'} (\eta^\delta - \eta^{\delta'})|_2) |v_t^{\delta'}|_\infty |v^\delta - v^{\delta'}|_2 \\
&\leq C \epsilon (\max\{\delta, \delta'\} |\eta^\delta|_{H^1} + |\eta^\delta - \eta^{\delta'}|_2) |v_t^{\delta'}|_{H^1} |v^\delta - v^{\delta'}|_2,
\end{aligned}$$

which implies

$$|I_2| \leq CM^{\frac{3}{2}} \max\{\delta, \delta'\} + C \epsilon M^{\frac{1}{2}} E_0^{(\delta, \delta')}(t).$$

Similarly, for I_3 , we have

$$|I_3| \leq C \epsilon M^{\frac{1}{2}} E_0^{(\delta, \delta')}(t),$$

while for I_4 , we have

$$|I_4| \leq C(M) \max\{\delta, \delta'\} + C(M) \epsilon E_0^{(\delta, \delta')}(t).$$

Thus, we have

$$(5.32) \quad \frac{d}{dt} E_0^{(\delta, \delta')}(t) \leq C(M) \max\{\delta, \delta'\} + C(M) E_0^{(\delta, \delta')}(t).$$

Noticing that $E_0^{(\delta, \delta')}(0) = 0$, applying Gronwall inequality to (5.32) yields

$$(5.33) \quad E_0^{(\delta, \delta')}(t) \leq C e^{C(M)t} \max\{\delta, \delta'\},$$

which implies that the approximate solutions $\{V^\delta = (\eta^\delta, v^\delta)\}_{\delta > 0}$ form a Cauchy sequence in $C([0, T/\epsilon], X_\epsilon^0(\mathbb{R}) \times L^2(\mathbb{R}))$.

Step 4. The limit of the approximate solutions solves (4.66). On the one hand, (5.33) shows that there exists a unique $V = (\eta, v) \in C([0, T/\epsilon], X_\epsilon^0(\mathbb{R}) \times L^2(\mathbb{R}))$ such that when $\delta \rightarrow 0$,

$$(5.34) \quad V^\delta \rightarrow V \quad \text{in} \quad C([0, T/\epsilon], X_\epsilon^0 \times L^2),$$

and

$$(5.35) \quad \sup_{0 \leq t \leq T/\epsilon} |V^\delta - V|_{X_\epsilon^0 \times L^2} \leq C_{M,T,\epsilon} \delta,$$

where $C_{M,T,\epsilon}$ is a constant depending on M , T , ϵ .

On the other hand, by Banach-Alaoglu theorem, the uniform estimate (5.31) and (iv) of Lemma 5.1 imply that there exists a subsequence $\{V^{\delta_j}\}_{j \in \mathbb{N}}$ such that when $j \rightarrow \infty$,

$$(5.36) \quad V^{\delta_j} \rightharpoonup V, \quad \text{weakly,}$$

and the energy $\mathcal{E}(t)$ for V has the same bound as in (5.31). Moreover,

$$(5.37) \quad \begin{aligned} V &\in L^\infty([0, T/\epsilon]; X_{\epsilon^3}^2 \times X_{\epsilon^2}^2) \cap Lip([0, T/\epsilon]; X_{\epsilon^2}^1 \times X_\epsilon^1) \\ V_t &\in L^\infty([0, T/\epsilon]; X_{\epsilon^2}^1 \times X_\epsilon^1) \cap Lip([0, T/\epsilon]; X_\epsilon^0 \times L^2). \end{aligned}$$

Thanks to the interpolation inequality (4.2), we have for any $s \in (0, 2]$,

$$(5.38) \quad \begin{aligned} \sup_{0 \leq t \leq T/\epsilon} |V^\delta - V|_{X_{\epsilon^{3-s}}^{2-s} \times X_{\epsilon^{2-s}}^{2-s}} &\leq C \sup_{0 \leq t \leq T/\epsilon} |V^\delta - V|_{X_\epsilon^0 \times L^2}^{\frac{s}{2}} |V^\delta - V|_{X_{\epsilon^3}^2 \times X_{\epsilon^2}^2}^{1-\frac{s}{2}} \\ &\leq C_{M,T,\epsilon} \delta^{\frac{s}{2}}, \end{aligned}$$

where we used (5.35), (5.31) and the obtained result $\sup_{0 \leq t \leq T/\epsilon} \mathcal{E}(t) \leq CM$. By (5.38), we obtain that when $\delta \rightarrow 0$,

$$(5.39) \quad V^\delta \rightarrow V, \quad \text{in } C([0, T/\epsilon]; X_{\epsilon^{3-s}}^{2-s} \times X_{\epsilon^{2-s}}^{2-s}).$$

If $s \in (0, \frac{1}{4})$, the embedding theorem shows that

$$C([0, T/\epsilon]; X_{\epsilon^{3-s}}^{2-s} \times X_{\epsilon^{2-s}}^{2-s}) \hookrightarrow C([0, T/\epsilon]; C^4(\mathbb{R}) \times C^3(\mathbb{R})).$$

Then as $\delta \rightarrow 0$,

$$V^\delta \rightarrow V, \quad \text{in } C([0, T/\epsilon]; C^4(\mathbb{R}) \times C^3(\mathbb{R})).$$

Similarly, we could verify that as $\delta \rightarrow 0$,

$$V_t^\delta \rightarrow V_t, \quad \text{in } C([0, T/\epsilon]; C^2(\mathbb{R}) \times C^1(\mathbb{R})).$$

Thus, taking $\delta \rightarrow 0$ in (5.1), we obtain that $V = (\eta, v)$ satisfies (4.66) in the classic sense.

Step 5. Continuity in time of the solutions. Firstly, by virtue of (5.39), we have

$$(5.40) \quad \eta^\delta \rightarrow \eta \quad \text{in } C([0, T/\epsilon]; L^\infty(\mathbb{R})).$$

Thanks to (5.29) and (5.31), we obtain that

$$\frac{d}{dt} E^\delta(t) \leq C(M),$$

which implies that

$$E^\delta(t) - E^\delta(0) \leq C(M)t.$$

Taking $\delta \rightarrow 0$ yields

$$E(t) - E^0 \leq \limsup_{\delta \rightarrow 0} (E^\delta(t) - E^\delta(0)) \leq C(M)t,$$

where $E^0 = E^\delta(0)|_{\delta=0}$ is determined only by (η_0, v_0) . Then we have

$$(5.41) \quad \limsup_{t \rightarrow 0} E(t) \leq E^0.$$

On the other hand, thanks to (5.31) and (5.4), we have

$$V_{ttt}^\delta \in L^\infty([0, T/\epsilon]; \dot{H}^{-1} \times \dot{H}^{-1}),$$

which along with (5.37) implies

$$(5.42) \quad \begin{aligned} V &\in C_w([0, T/\epsilon]; X_{\epsilon^3}^2 \times X_{\epsilon^2}^2), \quad V_t \in C_w([0, T/\epsilon]; X_{\epsilon^2}^1 \times X_\epsilon^1), \\ V_{tt} &\in C_w([0, T/\epsilon]; X_\epsilon^0 \times L^2). \end{aligned}$$

Due to (5.40) and (5.42), we have

$$E^0 \leq \liminf_{t \rightarrow 0} E(t),$$

which along with (5.41) implies that

$$(5.43) \quad \lim_{t \rightarrow 0} E(t) = E^0.$$

Then by (5.40), (5.42), the definition of $E(t)$ and the arguments in Step 2.2 for the higher order derivatives in x , we have V are strongly continuous in time $t = 0$ in the corresponding functional spaces.

Consider $T_0 \in (0, T/\epsilon)$ and the solution $V(\cdot, T_0) = (\eta(\cdot, T_0), v(\cdot, T_0))$. For fixed time T_0 , $V^{T_0} \stackrel{\text{def}}{=} V(\cdot, T_0) \in X_{\epsilon^3}^2 \times X_{\epsilon^2}^2$ and by (5.29) and (5.31), there exists a constant c_0 which depends on $M \stackrel{\text{def}}{=} |\eta_0|_{X_{\epsilon^3}^2}^2 + |v_0|_{X_{\epsilon^2}^2}^2$ such that

$$(5.44) \quad (E(T_0)) \leq \frac{(E^0)^{\frac{1}{2}}}{1 - c_0 \epsilon T_0 M^{\frac{1}{2}}}.$$

Now we use V^{T_0} as an initial data and construct a forward and backward in time solutions as in the above steps by solving the approximate system (5.1). We obtain the approximate solutions $V_{T_0}^\delta(\cdot, t)$ which also satisfy (5.29) and the limit \tilde{V} of $V_{T_0}^\delta(\cdot, t)$ also solves (4.66) on some time interval $[T_0 - T', T_0 + T']$. By the uniqueness of the solutions, \tilde{V} must coincide with V on the time interval $[T_0 - T', T_0 + T']$. Similar to (5.44), by using (5.31), there exists a constant c'_0 which depends on $M \stackrel{\text{def}}{=} |\eta_0|_{X_{\epsilon^3}^2}^2 + |v_0|_{X_{\epsilon^2}^2}^2$ such that for any $t \in [T_0 - T', T_0 + T']$,

$$(E(t))^{\frac{1}{2}} \leq \frac{(E(T_0))^{\frac{1}{2}}}{1 - c'_0 \epsilon (t - T_0) M^{\frac{1}{2}}} \leq \frac{(E^0)^{\frac{1}{2}}}{(1 - c'_0 \epsilon (t - T_0) M^{\frac{1}{2}})(1 - c_0 \epsilon T_0 M^{\frac{1}{2}})}.$$

Then we obtain the following restriction for T'

$$0 < T' \leq \frac{1}{2c'_0 M^{\frac{1}{2}}}.$$

Following the same argument as the continuity in time $t = 0$, we obtain that the solution is continuous in time $t = T_0$ in corresponding functional spaces.

Thus, we obtain that

$$(5.45) \quad \begin{aligned} V &\in C([0, T/\epsilon]; X_{\epsilon^3}^2 \times X_{\epsilon^2}^2), \quad V_t \in C([0, T/\epsilon]; X_{\epsilon^2}^1 \times X_\epsilon^1), \\ V_{tt} &\in C([0, T/\epsilon]; X_\epsilon^0 \times L^2). \end{aligned}$$

Combining Steps 1 to 5, the existence proof of Theorem 4.6 is completed.

6. POSSIBLE EXTENSIONS

6.1. A fifth order Boussinesq system. When the expansion with respect to ϵ is performed to the next order, one obtains a class of fifth order Boussinesq systems (see [6]). Those models should lead to an error estimate of order $O(\epsilon^3 t)$ instead of $O(\epsilon^2 t)$ for the usual Boussinesq systems. A rigorous proof of this fact requires in particular to establish that the fifth order Boussinesq systems are well-posed on "long" time scales and thus come the issue of long time existence for those systems. One expects of course a lifespan of at least order $1/\epsilon$, the question, (as for the usual Boussinesq systems), being to see whether or not the dispersive terms allow to enlarge this lifespan. Due to the large number of equivalent (to the sense of consistency) systems, we will focus on a particular case (BBM-type) which is shown to be locally well-posed in [7].

We first recall the fifth order Boussinesq system under study (one could obtain the following system from that stated in [6] by scaling):

$$(6.1) \quad \begin{cases} (1 - b\epsilon\partial_x^2 + b_1\epsilon^2\partial_x^4)\eta_t + (1 + a\epsilon\partial_x^2 + a_1\epsilon^2\partial_x^4)u_x + \epsilon(1 - b\epsilon\partial_x^2)(\eta u)_x \\ \quad + (a + b - \frac{1}{3})\epsilon^2(\eta u_{xx})_x = 0, \\ (1 - d\epsilon\partial_x^2 + d_1\epsilon^2\partial_x^4)u_t + (1 + c\epsilon\partial_x^2 + c_1\epsilon^2\partial_x^4)\eta_x + \epsilon(1 + c\epsilon\partial_x^2)(uu_x) \\ \quad + \epsilon^2(\eta\eta_{xx})_x - (c + d - 1)\epsilon^2u_xu_{xx} - (c + d)\epsilon^2uu_{xxx} = 0. \end{cases}$$

As an example we shall deal with the "BBM-type" case:

$$\begin{aligned} b &\geq 0, & b_1 &> 0, & a &< 0, & a_1 &= 0, \\ d &\geq 0, & d_1 &> 0, & c &< 0, & c_1 &= 0. \end{aligned}$$

We now state the existence result in the BBM-type case.

Theorem 6.1. *Let $s > \frac{3}{2}$. Assume that $(\eta_0, u_0) \in X_{\epsilon^3}^s(\mathbb{R})$ satisfy the (non-cavitation) condition*

$$(6.2) \quad 1 + \epsilon\eta_0 \geq H > 0, \quad H \in (0, 1),$$

Then there exists a constant \tilde{c}_0 such that for any $\epsilon \leq \epsilon_0 = \frac{1-H}{\tilde{c}_0(|\eta_0|_{X_{\epsilon^3}^s} + |u_0|_{X_{\epsilon^3}^s})}$, there exists $T > 0$ independent of ϵ , such that (6.1) (the BBM-type case) with the initial data (η_0, u_0) has a unique solution (η, u) with $(\eta, u) \in C([0, T/\epsilon]; X_{\epsilon^3}^s(\mathbb{R}))$. Moreover,

$$(6.3) \quad \max_{t \in [0, T/\epsilon]} (|\eta|_{X_{\epsilon^3}^s} + |u|_{X_{\epsilon^3}^s}) \leq \tilde{c}(|\eta_0|_{X_{\epsilon^3}^s} + |u_0|_{X_{\epsilon^3}^s}).$$

Proof. The proof of the theorem is similar to that in the previous subsection and we only sketch it. Denoting by $U = (\eta, u)^T$, (6.1) (the BBM-type case) is equivalent to the following condensed system

$$(6.4) \quad M_0(\partial_x)U_t + M(U, \partial_x)U = 0,$$

where

$$M_0(\partial_x) = \text{diag}(1 - b\epsilon\partial_x^2 + b_1\epsilon^2\partial_x^4, 1 - d\epsilon\partial_x^2 + d_1\epsilon^2\partial_x^4),$$

$$M(U, \partial_x) = (m_{ij})_{i,j=1,2} \quad \text{with}$$

$$\begin{aligned}
m_{11} &= \epsilon(1 - b\epsilon\partial_x^2)(u\partial_x) + (a + b - \frac{1}{3})\epsilon^2 u_{xx}\partial_x, \\
m_{12} &= (1 + a\epsilon\partial_x^2)\partial_x + \epsilon(1 - b\epsilon\partial_x^2)(\eta\partial_x) + (a + b - \frac{1}{3})\epsilon^2\eta\partial_x^3, \\
m_{21} &= (1 + c\epsilon\partial_x^2)\partial_x + \epsilon^2\partial_x(\eta\partial_x^2), \\
m_{22} &= \epsilon(1 + c\epsilon\partial_x^2)(u\partial_x) - (c + d - 1)\epsilon^2 u_{xx}\partial_x - (c + d)\epsilon^2 u\partial_x^3.
\end{aligned}$$

We could search the symmetrizer of both M_0 and M as follows:

$$S = \text{diag}(1 + c\epsilon\partial_x^2 + \epsilon^2\eta\partial_x^2, 1 + \epsilon\eta + a\epsilon\partial_x^2 + (a - \frac{1}{3})\epsilon^2\eta\partial_x^2).$$

We define the energy functional associated to (6.4) as

$$E_s(U) = (M_0\Lambda^s U | S\Lambda^s U)_2.$$

It is easy to check that under the assumption (6.2) and the assumption that

$$(6.5) \quad \epsilon|\eta|_\infty + \epsilon|\partial_x\eta|_\infty \ll 1,$$

there holds

$$(6.6) \quad E_s(U) \sim |\eta|_{X_{\epsilon^3}^s}^2 + |u|_{X_{\epsilon^3}^s}^2.$$

As usual, we get by a standard energy estimate that

$$\begin{aligned}
(6.7) \quad \frac{d}{dt}E_s(U) &= (M_0\Lambda^s U_t | (S + S^*)\Lambda^s U)_2 + (\Lambda^s U | [M_0, S]\Lambda^s U_t)_2 \\
&\quad + (M_0\Lambda^s U | \partial_t S\Lambda^s U)_2 \\
&= -(M(U, \partial_x)\Lambda^s U | (S + S^*)\Lambda^s U)_2 - ([\Lambda^s, M(U, \partial_x)]U | (S + S^*)\Lambda^s U)_2 \\
&\quad + (\Lambda^s U | [M_0, S]\Lambda^s U_t)_2 + (M_0\Lambda^s U | \partial_t S\Lambda^s U)_2 \\
&\stackrel{\text{def}}{=} I + II + III + IV.
\end{aligned}$$

Estimate for I . We first compute

$$\begin{aligned}
&- (M(U, \partial_x)\Lambda^s U | S\Lambda^s U)_2 \\
&= -(m_{11}\Lambda^s \eta | (1 + c\epsilon\partial_x^2 + \epsilon^2\eta\partial_x^2)\Lambda^s \eta)_2 \\
&\quad - \{(m_{12}\Lambda^s u | (1 + c\epsilon\partial_x^2 + \epsilon^2\eta\partial_x^2)\Lambda^s \eta)_2 \\
&\quad + (m_{21}\Lambda^s \eta | (1 + \epsilon\eta + a\epsilon\partial_x^2 + (a - \frac{1}{3})\epsilon^2\eta\partial_x^2)\Lambda^s u)_2\} \\
&\quad - (m_{22}\Lambda^s u | (1 + \epsilon\eta + a\epsilon\partial_x^2 + (a - \frac{1}{3})\epsilon^2\eta\partial_x^2)\Lambda^s u)_2 \\
&\stackrel{\text{def}}{=} I_1 + I_2 + I_3.
\end{aligned}$$

For I_1 , integrating by parts yields

$$\begin{aligned}
I_1 &= \frac{\epsilon}{2}(\partial_x(u + (a + b - \frac{1}{3})\epsilon u_{xx})\Lambda^s \eta | \Lambda^s \eta)_2 \\
&\quad + \frac{\epsilon^2}{2}(\partial_x((u + (a + b - \frac{1}{3})\epsilon u_{xx})(c + \epsilon\eta))\partial_x\Lambda^s \eta | \partial_x\Lambda^s \eta)_2 \\
&\quad - \frac{b\epsilon^2}{2}(\partial_x u \partial_x \Lambda^s \eta | \partial_x \Lambda^s \eta)_2 - \frac{b\epsilon^3}{2}(\partial_x((c + \epsilon\eta)u)\partial_x^2 \Lambda^s \eta | \partial_x^2 \Lambda^s \eta)_2 \\
&\quad + b\epsilon^3(\partial_x^2 u \partial_x \Lambda^s \eta | (c + \epsilon\eta)\partial_x^2 \Lambda^s \eta)_2 + 2b\epsilon^3(\partial_x u \partial_x^2 \Lambda^s \eta | (c + \epsilon\eta)\partial_x^2 \Lambda^s \eta)_2
\end{aligned}$$

which along with the interpolation inequality (4.2) and the assumption (6.5) implies that

$$(6.8) \quad |I_1| \lesssim \epsilon |u|_{X_{\epsilon^3}^s} |\eta|_{X_{\epsilon^3}^s}^2.$$

For I_2 , integrating by parts gives rise to

$$\begin{aligned} I_2 = & (\epsilon \partial_x \eta \Lambda^s u + (a - \frac{1}{3}) \epsilon^2 \partial_x \eta \partial_x^2 \Lambda^s u | (1 + c\epsilon \partial_x^2 + \epsilon^2 \eta \partial_x^2) \Lambda^s \eta)_2 \\ & + b\epsilon^2 (\partial_x^2 \eta \partial_x \Lambda^s u + 2\partial_x \eta \partial_x^2 \Lambda^s u | (1 + c\epsilon \partial_x^2 + \epsilon^2 \eta \partial_x^2) \Lambda^s \eta)_2, \end{aligned}$$

which along with (4.2) and (6.5) implies that

$$(6.9) \quad |I_2| \lesssim \epsilon |u|_{X_{\epsilon^3}^s} |\eta|_{X_{\epsilon^3}^s}^2.$$

For I_3 , we could derive as for I_1 that

$$(6.10) \quad |I_3| \lesssim \epsilon |u|_{X_{\epsilon^3}^s}^3.$$

Thanks to (6.8), (6.9) and (6.10), we obtain

$$|(M(U, \partial_x) \Lambda^s U | S \Lambda^s U)_2| \lesssim \epsilon |u|_{X_{\epsilon^3}^s} (|\eta|_{X_{\epsilon^3}^s}^2 + |u|_{X_{\epsilon^3}^s}^2).$$

The same estimate holds for term $(M(U, \partial_x) \Lambda^s U | S^* \Lambda^s U)_2$. Then we obtain

$$(6.11) \quad |I| \lesssim \epsilon |u|_{X_{\epsilon^3}^s} (|\eta|_{X_{\epsilon^3}^s}^2 + |u|_{X_{\epsilon^3}^s}^2).$$

Estimate for II . We first calculate that

$$\begin{aligned} |II| & \lesssim |[\Lambda^s, M(U, \partial_x)]U|_2 |(S + S^*) \Lambda^s U|_2 \\ & \lesssim |[\Lambda^s, M(U, \partial_x)]U|_2 (|\eta|_{X_{\epsilon^3}^s} + |u|_{X_{\epsilon^3}^s}) \end{aligned}$$

provided that

$$(6.12) \quad \epsilon |\eta|_{X_{\epsilon^3}^s} < 1.$$

Thanks to the expression of $M(U, \partial_x)$, (4.2) and Lemma 3.2, we get that

$$\begin{aligned} |[\Lambda^s, M(U, \partial_x)]U|_2 & \lesssim \epsilon |(1 - b\epsilon \partial_x^2)([\Lambda^s, u] \partial_x \eta)|_2 + \epsilon^2 |[\Lambda^s, u_{xx}] \partial_x \eta|_2 \\ & \quad + \epsilon^2 |\partial_x ([\Lambda^s, \eta] \partial_x^2 \eta)|_2 + \epsilon |(1 - b\epsilon \partial_x^2)([\Lambda^s, \eta] \partial_x u)|_2 + \epsilon^2 |[\Lambda^s, \eta] \partial_x^3 u|_2 \\ & \quad + \epsilon |(1 + c\epsilon \partial_x^2)([\Lambda^s, u] \partial_x u)|_2 + \epsilon^2 |[\Lambda^s, u_{xx}] \partial_x u|_2 + \epsilon^2 |[\Lambda^s, u] \partial_x^3 u|_2 \\ & \lesssim \epsilon (|\eta|_{X_{\epsilon^3}^s}^2 + |u|_{X_{\epsilon^3}^s}^2). \end{aligned}$$

Then we obtain that

$$(6.13) \quad |II| \lesssim \epsilon (|\eta|_{X_{\epsilon^3}^s}^2 + |u|_{X_{\epsilon^3}^s}^2)^{\frac{3}{2}}.$$

Estimate for III . Using the expressions of M_0 and S , we have

$$\begin{aligned} III = & \epsilon^2 (\Lambda^s \eta | (1 - b\epsilon \partial_x^2 + b_1 \epsilon^2 \partial_x^4)(\eta \partial_x^2 \Lambda^s \eta_t) - \eta \partial_x^2 (1 - b\epsilon \partial_x^2 + b_1 \epsilon^2 \partial_x^4) \Lambda^s \eta_t)_2 \\ & + \epsilon (\Lambda^s u | (1 - d\epsilon \partial_x^2 + d_1 \epsilon^2 \partial_x^4)(\eta \Lambda^s u_t) - \eta (1 - d\epsilon \partial_x^2 + d_1 \epsilon^2 \partial_x^4) \Lambda^s u_t)_2 \\ & + (a - \frac{1}{3}) \epsilon^2 (\Lambda^s u | (1 - d\epsilon \partial_x^2 + d_1 \epsilon^2 \partial_x^4)(\eta \partial_x^2 \Lambda^s u_t) - \eta \partial_x^2 (1 - d\epsilon \partial_x^2 + d_1 \epsilon^2 \partial_x^4) \Lambda^s u_t)_2 \\ \stackrel{\text{def}}{=} & III_1 + III_2 + III_3. \end{aligned}$$

Now we rewrite III_1 , III_2 , III_3 as follows:

$$\begin{aligned} III_1 &= -\epsilon^2([-b\epsilon\partial_x^2 + b_1\epsilon^2\partial_x^4, \eta]\Lambda^s\eta | \partial_x^2\Lambda^s\eta_t)_2 \\ III_2 &= \epsilon(\Lambda^s u | [-d\epsilon\partial_x^2 + d_1\epsilon^2\partial_x^4, \eta]\Lambda^s u_t)_2 \\ III_3 &= -(a - \frac{1}{3})\epsilon^2([-d\epsilon\partial_x^2 + d_1\epsilon^2\partial_x^4, \eta]\Lambda^s u | \partial_x^2\Lambda^s u_t)_2 \end{aligned}$$

which along with (4.2) and Lemma 3.2, we obtain

$$(6.14) \quad |III| \lesssim \epsilon(|\eta|_{X_{\epsilon^3}^s}^2 + |u|_{X_{\epsilon^3}^s}^2)(|\eta_t|_{X_{\epsilon^4}^{s-1}} + |u_t|_{X_{\epsilon^4}^{s-1}}).$$

Estimate for IV. Using the expressions of M_0 and S , we get that

$$\begin{aligned} IV &= \epsilon^2((1 - b\epsilon\partial_x^2 + b_1\epsilon^2\partial_x^4)\Lambda^s\eta | \eta_t\partial_x^2\Lambda^s\eta)_2 \\ &\quad + ((1 - d\epsilon\partial_x^2 + d_1\epsilon^2\partial_x^4)\Lambda^s u | \epsilon\eta_t\Lambda^s u + (a - \frac{1}{3})\epsilon^2\eta_t\partial_x^2\Lambda^s u)_2, \end{aligned}$$

which along with (4.2) implies that

$$(6.15) \quad \begin{aligned} |IV| &\lesssim \epsilon(|\eta_t|_{\infty} + \epsilon^{\frac{1}{2}}|\partial_x\eta_t|_{\infty})(|\eta|_{X_{\epsilon^3}^s}^2 + |u|_{X_{\epsilon^3}^s}^2), \\ &\lesssim \epsilon|\eta_t|_{X_{\epsilon^4}^{s-1}}(|\eta|_{X_{\epsilon^3}^s}^2 + |u|_{X_{\epsilon^3}^s}^2). \end{aligned}$$

Thanks to (6.7), (6.11), (6.13), (6.14) and (6.15), we get that

$$\frac{d}{dt}E_s(U) \lesssim \epsilon(|\eta|_{X_{\epsilon^3}^s} + |u|_{X_{\epsilon^3}^s} + |\eta_t|_{X_{\epsilon^4}^{s-1}} + |u_t|_{X_{\epsilon^4}^{s-1}})(|\eta|_{X_{\epsilon^3}^s}^2 + |u|_{X_{\epsilon^3}^s}^2).$$

Go back to (6.1), we get that

$$|\eta_t|_{X_{\epsilon^4}^{s-1}} + |u_t|_{X_{\epsilon^4}^{s-1}} \lesssim (|\eta|_{X_{\epsilon^3}^s} + |u|_{X_{\epsilon^3}^s})(1 + \epsilon|\eta|_{X_{\epsilon^3}^s} + \epsilon|u|_{X_{\epsilon^3}^s}) \lesssim |\eta|_{X_{\epsilon^3}^s} + |u|_{X_{\epsilon^3}^s},$$

provided that

$$(6.16) \quad \epsilon|\eta|_{X_{\epsilon^3}^s} < 1.$$

Then we get that

$$\frac{d}{dt}E_s(U) \lesssim \epsilon(|\eta|_{X_{\epsilon^3}^s} + |u|_{X_{\epsilon^3}^s})(|\eta|_{X_{\epsilon^3}^s}^2 + |u|_{X_{\epsilon^3}^s}^2).$$

which along with (6.6) implies that

$$(6.17) \quad \frac{d}{dt}E_s(U) \lesssim \epsilon E_s(U)^{\frac{3}{2}}.$$

Thus, using similar arguments as in the previous subsections, we can deduce from (6.17) that there exists $T > 0$ independent of ϵ such that (6.1)(the BBM-type case) has a unique solution on time interval $[0, T/\epsilon]$ with initial data (η_0, u_0) . Moreover (6.3) holds. Theorem 6.1 is proved. \square

6.2. A Boussinesq-Full dispersion system for internal waves. A systematic derivation of asymptotic internal waves models describing waves at the interface of a two-fluids system with a rigid top is given in [8]. We will consider here a specific regime leading to a Boussinesq-Full dispersion system for which the long time existence result is still open. We recall first the relevant parameters. The index 1 stands for the upper layer and 2 for the lower one.

$\gamma = \frac{\rho_1}{\rho_2} < 1$ is the ratio of densities, $\delta = \frac{d_1}{d_2}$ the ratio of typical heights of the layers, λ a typical wavelength, a a typical amplitude of the wave.

We denote $\epsilon = \frac{a}{d_1}$, $\mu = \frac{d_1^2}{\lambda}$, $\epsilon_2 = \frac{a}{d_2} = \epsilon\delta$, $\mu_2 = \frac{d_2^2}{\lambda^2} = \frac{\mu}{\delta^2}$.

We consider here the regime where $\mu \sim \epsilon \ll 1$, ($\mu = \epsilon \ll 1$ from now on) and $\mu_2 \sim 1$.

It is shown in [8] that in this regime (for which one also has $\delta^2 \sim \epsilon$ and thus $\epsilon_2 \sim \epsilon^{3/2} \ll 1$), and in absence of surface tension, the two-layers system is consistent with the *three-parameter family* of Boussinesq/FD systems

$$(6.18) \quad \begin{cases} (1 - \mu b \Delta) \partial_t \zeta + \frac{1}{\gamma} \nabla \cdot ((1 - \epsilon \zeta) \mathbf{v}_\beta) \\ \quad - \frac{\sqrt{\mu}}{\gamma^2} |D| \coth(\sqrt{\mu_2} |D|) \nabla \cdot \mathbf{v}_\beta + \frac{\mu}{\gamma} \left(a - \frac{1}{\gamma^2} \coth^2(\sqrt{\mu_2} |D|) \right) \Delta \nabla \cdot \mathbf{v}_\beta = 0 \\ (1 - \mu d \Delta) \partial_t \mathbf{v}_\beta + (1 - \gamma) \nabla \zeta - \frac{\epsilon}{2\gamma} \nabla |\mathbf{v}_\beta|^2 + \mu c (1 - \gamma) \Delta \nabla \zeta = 0, \end{cases}$$

where $\mathbf{v}_\beta = (1 - \mu \beta \Delta)^{-1} \mathbf{v}$ and the constants a , b , c and d are defined as

$$a = \frac{1}{3}(1 - \alpha_1 - 3\beta), \quad b = \frac{1}{3}\alpha_1, \quad c = \beta\alpha_2, \quad d = \beta(1 - \alpha_2),$$

with $\alpha_1 \geq 0$, $\beta \geq 0$ and $\alpha_2 \leq 1$. Note that $a + b + c + d = \frac{1}{3}$.

It is easily checked that (6.18) is linearly well posed when

$$a \leq 0, c \leq 0, b \geq 0, d \geq 0.$$

The local well-posedness of the Cauchy problem for (6.18) was considered in [14] in the following cases

- (1) $b > 0, d > 0, a \leq 0, c < 0$;
- (2) $b > 0, d > 0, a \leq 0, c = 0$;
- (3) $b = 0, d > 0, a \leq 0, c = 0$;
- (4) $b = 0, d > 0, a \leq 0, c < 0$;
- (5) $b > 0, d = 0, a \leq 0, c = 0$.

On the other hand, we do not know of any *long time existence* results for (6.18) that is existence on time scales of order $1/\epsilon$. This issue will be considered in a subsequent paper [37].

6.3. A full dispersion Boussinesq system. One obtains a *full dispersion system* when in the Boussinesq regime by keeping the original dispersion of the water waves system (see [29], [19], and [2] where interesting numerical simulations of the propagation of solitary waves are performed).⁷

They read, setting $\mathcal{T}_\epsilon = \frac{\tanh \sqrt{\epsilon} |D|}{\sqrt{\epsilon} |D|}$, $D = -i\nabla$:

$$(6.19) \quad \begin{cases} \eta_t + \mathcal{T}_\epsilon u_x + \epsilon(\eta u)_x = 0 \\ u_t + \eta_x + \epsilon u u_x = 0, \end{cases}$$

when $d = 1$ and

$$(6.20) \quad \begin{cases} \eta_t + \mathcal{T}_\epsilon \nabla \cdot \mathbf{u} + \epsilon \nabla \cdot (\eta \mathbf{u}) = 0 \\ \mathbf{u}_t + \nabla \eta + \frac{\epsilon}{2} \nabla |\mathbf{u}|^2 = 0, \end{cases}$$

when $d = 2$.

Taking the limit $\sqrt{\epsilon} |\xi| \rightarrow 0$ in \mathcal{T}_ϵ , (6.19) reduces formally to

$$(6.21) \quad \begin{cases} \eta_t + u_x + \frac{\epsilon}{3} u_{xxx} + \epsilon(\eta u)_x = 0 \\ u_t + \eta_x + \epsilon u u_x = 0, \end{cases}$$

⁷As noticed in [2] the use of nonlocal models for shallow water waves is also suggested in [41].

while in the two-dimensional case, (6.20) reduces in the same limit to

$$(6.22) \quad \begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \frac{\epsilon}{3} \Delta \nabla \cdot \mathbf{u} + \epsilon \nabla \cdot (\eta \mathbf{u}) = 0 \\ \mathbf{u}_t + \nabla \eta + \frac{\epsilon}{2} \nabla |\mathbf{u}|^2 = 0, \end{cases}$$

that is to the (linearly ill-posed) system one gets first by expanding to first order the Dirichlet to Neumann operator with respect to ϵ in the full water wave system (see [29]).

System (6.21) is also known in the Inverse Scattering community as the Kaup system (see [21, 26]). It is completely integrable though linearly ill-posed since the eigenvalues of the dispersion matrix are $\pm i\xi(1 - \frac{\epsilon}{3}\xi^2)^{1/2}$. The Boussinesq system (6.19) can therefore be seen as a (well-posed) regularization of the Kaup system. Whether or not it is completely integrable is an open question.

The full dispersion Boussinesq systems have the following Hamiltonian structure

$$\partial_t \begin{pmatrix} \eta \\ \mathbf{u} \end{pmatrix} + J \text{grad } H_\epsilon(\eta, \mathbf{u}) = 0$$

where

$$J = \begin{pmatrix} 0 & \partial_x & \partial_y \\ \partial_x & 0 & 0 \\ \partial_y & 0 & 0 \end{pmatrix},$$

$$H_\epsilon(U) = \frac{1}{2} \int_{\mathbb{R}^2} (|\mathcal{T}_\epsilon^{1/2} \mathbf{u}|^2 + \eta^2 + \epsilon \eta |\mathbf{u}|^2) dx dy,$$

$$U = \begin{pmatrix} \eta \\ \mathbf{u} \end{pmatrix},$$

when $d = 2$ and

$$\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} + J \text{grad } H_\epsilon(\eta, u) = 0$$

where

$$J = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}$$

and

$$H_\epsilon(\eta, u) = \frac{1}{2} \int_{\mathbb{R}} (|\mathcal{T}_\epsilon^{1/2} u|^2 + \eta^2 + \epsilon u^2 \eta) dx,$$

when $d = 1$.

Note that the full dispersion Boussinesq system (6.19) can be viewed as the two-way propagation counterpart of the Whitham equation (see [40] and [29] for a rigorous derivation):

$$(6.23) \quad \eta_t + (\mathcal{T}_\epsilon)^{1/2} u_x + \epsilon u u_x = 0$$

which displays a very rich dynamics (see [20, 25] and the references therein).

When surface tension is taken into account, one should replace the operator \mathcal{T}_ϵ by $\mathcal{P}_\epsilon = (I + \beta \epsilon |D|^2)^{1/2} \left(\frac{\tanh(\sqrt{\epsilon} |D|)}{\sqrt{\epsilon} |D|} \right)$ where the parameter $\beta > 0$ measures surface tension (see [29]), yielding a more dispersive full dispersion Boussinesq system. When $\beta > \frac{1}{3}$, this full dispersion Boussinesq system yields, taking the limit $\sqrt{\epsilon} |\xi| \rightarrow 0$ in \mathcal{P}_ϵ , Boussinesq systems of the class $a < 0, b = c = d = 0$ for which long time well-posedness is established in Theorem 4.5.

Again we refer to a future work [37] for the study of the Cauchy problem associated to (6.19), (6.20).

7. CONCLUDING REMARKS

1. So far we have encounter only two possibilities for the lifespan T_ϵ of solutions to Boussinesq systems. Either $T_\epsilon = +\infty$, for a few one-dimensional systems, or $T_\epsilon = O(1/\epsilon)$ for essentially all the admissible (linearly well-posed) systems. One may ask whether another possibility might occur. In view of what happens in the scalar case (the *fractionary KdV equation*, see [31]) one could conjecture that there is no other possibility, at least in the one-dimensional case and when the natural no cavitation condition is imposed on the initial data. Note that no general criteria preventing blow-up in finite time seem to be known for Boussinesq systems except in the one-dimensional "BBM/BBM" system ($a = c = 0, b > 0, d > 0$) for which it is proven in [1] that a uniform control on $|1 + \epsilon\eta(\cdot, t)|_\infty$ prevents finite time blow-up.

2. Coming back to (1.1), we remark that all long time existence results in the present paper hold true for (1.1) if one fixes $\mu > 0$ and let ϵ tends to 0.

Acknowledgments. The first Author acknowledges support from the ANR project GEODISP. He thanks Vincent Duchêne for fruitful discussions. The third author is partially supported by NSF of China under grant 11201455 and by innovation grant from National Center for Mathematics and Interdisciplinary Sciences. The three authors would like to appreciate the hospitality and financial support from Morningside Center of Mathematics, CAS.

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