

Codegree thresholds for covering 3-uniform hypergraphs

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Abstract

Given two 3-uniform hypergraphs F and G , we say that G has an F -covering if we can cover $V(G)$ by copies of F . The *minimum codegree* of G is the largest integer d such that every pair of vertices from $V(G)$ is contained in at least d triples from $E(G)$. Define $c_2(n, F)$ to be the largest minimum codegree among all n -vertex 3-graphs G that contain no F -covering. This is a natural problem intermediate (but distinct) from the well-studied Turán problems and tiling problems. In this paper, we determine $c_2(n, K_4)$ (for $n > 98$) and the associated extremal configurations (for $n > 998$), where K_4 denotes the complete 3-graph on 4 vertices. We also obtain bounds on $c_2(n, F)$ which are apart by at most 2 in the cases where F is K_4^- (K_4 with one edge removed), K_5^- , and the tight cycle C_5 on 5 vertices.

1 Introduction

1.1 Notation

Given a set A and a positive integer k , we write $A^{(k)}$ for the collection of k -element subsets of A . We use $[n]$ as a shorthand for the collection of the first n natural numbers, $[n] = \{1, 2, \dots, n\}$. We shall often consider pairs or triples of vertices; when there is no risk of confusion, we write ab and abc as a shorthand for $\{a, b\}$ and $\{a, b, c\}$ respectively. A k -uniform hypergraph, or k -graph, is a pair $G = (V, E)$, where V is a set of *vertices*, and $E \subseteq V^{(k)}$ is a collection of k -subsets of V , which form the *edges* of G . A *subgraph* of G is a k -graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The *degree* of a vertex $x \in V(G)$, which we denote by $d(x)$, is the number of edges of G containing x . The *minimum degree* $\delta_1(G)$ of G is the minimum of $d(x)$ over all vertices $x \in V(G)$.

In this paper, we will focus on 3-graphs $G = (V, E)$ and another degree-like quantity, and its minimum: the *codegree* of a pair $xy \in V^{(2)}$, denoted by $d(xy)$, is the number of edges of G containing the pair xy . We write $\Gamma(y, z)$ for the *neighbourhood* of the pair xy , i.e. the set of $z \in V \setminus \{x, y\}$ such that $xyz \in E(G)$. The *minimum codegree* of G is $\delta_2(G) = \min_{xy \in V^{(2)}} d(xy)$. The *link graph* of a vertex $x \in V(G)$ is the collection G_x of all pairs uv such that $xuv \in E(G)$. The *degree* of u in G_x is the number of vertices v such that $uv \in G_x$; note this is exactly the codegree of x and u . Finally we define the *edit distance* between two 3-graphs G and G' on the same vertex set to be the minimum number of changes required to make G isomorphic to G' , where a change consists in replacing an edge by a non-edge and vice-versa.

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1.2 The problem

Let F be a fixed 3-graph on t vertices with at least one 3-edge. A 3-graph G has an F -covering if we can cover $V(G)$ with F -subgraphs (subgraphs that are isomorphic to F). For $n \geq t$ and $i = 1, 2$, we define

$$c_i(n, F) = \max\{\delta_i(G) : |V(G)| = n \text{ and } G \text{ does not have an } F\text{-covering}\}.$$

and call $c_1(n, F)$ the *covering degree-threshold* and $c_2(n, F)$ the *covering codegree-threshold* of F .

The covering threshold $c_i(n, F)$ was introduced by Han, Zang and Zhao [14] when they studied the minimum degree that guarantees the existence of a K -tiling, where K is a complete 3-partite 3-graph. It was shown implicitly in [14] that $c_1(n, K) = (6 - 4\sqrt{2} + o(1))\binom{n}{2}$ if K has at least two vertices in each part (in contrast, it is easy to see that $c_1(n, K) = o(n^2)$ if some part of K has only one vertex). It was also noted that $c_1(n, F) = (1 - 1/(\chi(F) - 1) + o(1))n$ for all graphs F , where $\chi(F)$ is the chromatic number of F .

Our objective in this paper is to study the behaviour of the function $c_2(n, F)$ for various 3-graphs F . In other words, we seek to determine what codegree condition is necessary to guarantee that *all* vertices in a 3-graph G are contained in copies of F . When determining the exact value of $c_2(F, n)$ is difficult, we may ask instead for its asymptotic behaviour. It can be shown (see Section ??) that the limit

$$c_2(F) = \lim_{n \rightarrow \infty} \frac{c_2(n, F)}{n - 2}$$

exists.¹ We call $c_2(F)$ the *covering codegree-density* of F .

Let us introduce the 3-graphs relevant to the present work. Let $K_t = ([t], [t]^{(3)})$ denote the complete 3-graph on t vertices, and let K_t^- denote the 3-graph obtained from K_t by removing one 3-edge. The *strong* or *tight* t -cycle is the 3-graph C_t on $[t]$ with 3-edges $\{123, 234, 345, \dots, (t-2)(t-1)t, (t-1)t1, t12\}$. We denote by $F_{3,2}$ the 3-graph $([5], \{123, 124, 125, 345\})$. Finally a *Steiner Triple System* (STS) is a 3-graph in which every pair of vertices is contained in exactly one 3-edge; it is a 168 years old result of Kirkman [19] that a STS on t vertices exists if and only if $t \equiv 1, 3 \pmod{6}$. The *Fano plane* is the unique (up to isomorphism) STS on 7 vertices, which we denote by Fano.

1.3 Motivation and related work in extremal hypergraph theory

Before we state our results, let us give some motivation and background for our problem. Let F be a fixed 3-graph on t vertices with at least one 3-edge. A 3-graph G is F -free if it does not contain a copy of F as a subgraph. Further G has an F -tiling, or F -factor, if we can cover $V(G)$ with *vertex-disjoint* F -subgraphs. There has been much research into the degree and/or codegree conditions needed ensure the existence of an F -subgraph or of an F -factor in a 3-graph G . Determining the degree/codegree condition necessary to guarantee an F -covering is intermediate between these two well-studied problems. As we show in the next subsection, the existence, covering, and tiling problems give rise to different thresholds in their codegree versions, so that our work is novel. It is hoped that studying the properties of the covering codegree threshold function $c_2(n, F)$ — such as

¹This is a direct corollary of the proof of Proposition 6 from [7] on the existence of conditional codegree density (with an uncovered vertex x used as the conditional subgraph H), or can be proved in the same way as the existence of the usual codegree density $\gamma(F)$ in [27].

supersaturation, discussed in Section 4, which could be useful for applying semi-random methods to tiling problems — will lead to insights about both the existence and tiling problems.

The *Turán number* $\text{ex}(n, F)$ of F is the maximum number of 3-edges an F -free 3-graph on n vertices can have. The *codegree threshold* $\text{ex}_2(n, F)$ of F is the maximum of $\delta_2(G)$ over all F -free 3-graphs G on n vertices. It is well-known that $\text{ex}(n, F)/\binom{n}{3}$ tends to a limit $\pi(F)$ as $n \rightarrow \infty$; this limit is known as the *Turán density* of F . Similarly, $\text{ex}_2(n, F)/(n-2)$ tends to a limit $\gamma(F)$ called the *codegree density* or *2-Turán density* of F as $n \rightarrow \infty$. The extremal theory of 3-graphs and within it the study of Turán-type problems have received extensive attention from the combinatorics community since the 1950s, with strenuous efforts devoted in particular to the (in)famous and still-open conjecture of Turán [32] that $\pi(K_4) = 5/9$. See the surveys of Füredi [10] and Keevash [16] for an overview of results. There has been significant interest in other extremal quantities, and in particular in codegree densities for 3-graphs. The first result on codegree density was due to Mubayi [25], who showed $\gamma(\text{Fano}) = \frac{1}{2}$. Keevash and Zhao [18] determined the codegree densities of some projective geometries, which included the Fano plane as a special case. The codegree threshold for the Fano plane was determined by Keevash [15] via hypergraph regularity and later by DeBiasio and Jiang [5] by direct combinatorial means. Mubayi and Zhao [27] studied general properties of codegree density, while Falgas-Ravry [6] gave examples of non-isomorphic lower bound constructions for $\gamma(K_t)$. More recently Falgas-Ravry, Marchant, Pikhurko and Vaughan [7] determined the codegree threshold of $F_{3,2}$, and Falgas-Ravry, Pikhurko and Vaughan [8] showed $\gamma(K_4^-) = \frac{1}{4}$ via a flag algebra computation, resolving a conjecture of Czygrinow and Nagle [29]. Another conjecture of Czygrinow and Nagle [4] remains open, namely that $\gamma(K_4) = \frac{1}{2}$. Certainly $\gamma(F) \leq c_2(F)$ for any 3-graph F , and it may be hoped that giving good upper bounds for the latter may also help bounding the former.

In addition to these Turán-type problems, there has been much research activity on the problem of determining thresholds for the existence of F -factors. The situation for ordinary (2-)graphs is now well-understood: the celebrated Hajnal-Szemerédi theorem [12] gives the exact minimum degree condition guaranteeing the existence of F -factors in an n -vertex graph when F is a clique, while Kühn and Osthus [21] determined the minimum degree condition for general graphs F up to an additive constant. On the other hand, until recently not much was known about tiling for k -graphs when $k \geq 3$. While there has been a spate of results in the last few years, see [2, 3, 11, 13, 17, 20, 22, 23, 28, 31], many more open problems remain. We refer to the surveys of Rödl and Ruciński [30] and Zhao [33] for a more detailed discussion of the area, and briefly mention below four results relevant to the present work. For $i \in \{1, 2\}$ and $n \equiv 0 \pmod{|V(F)|}$, let

$$t_i(n, F) = \max\{\delta_i(G) : |V(G)| = n \text{ and } G \text{ does not have an } F\text{-factor}\}.$$

Trivially $c_i(n, F) \leq t_i(n, F)$ for any 3-graph F with at least one edge. Lo and Markström [23, 22] determined $t_2(n, F)$ asymptotically when $F = K_4$ and $F = K_4^-$. Independently Keevash and Mycroft [17] determined $t_2(n, K_4)$ exactly, and recently Han, Lo, Treglown and Zhao [13] determined $t_2(n, K_4^-)$ exactly as well, in both cases for n sufficiently large. Finally in [14] Han, Zang and Zhao asymptotically determined $t_1(n, K)$ for all complete 3-partite 3-graphs K . In particular, they showed that $t_1(n, K) = c_1(n, K) = (6 - 4\sqrt{2} + o(1))\binom{n}{2}$ for *certain* K . This gives further motivation for the present paper: by determining $c_2(n, F)$ for 3-graphs F , we may hope likewise to shed light on $t_2(n, F)$ and facilitate its (asymptotic) computation.

1.4 Results

In this paper, we determine the codegree covering threshold for K_4 for sufficiently large n .

Theorem 1.1. *For every $n \in \mathbb{N}$, $\lfloor \frac{2n-5}{3} \rfloor \leq c_2(K_4, n) \leq \lfloor \frac{2n-3}{3} \rfloor$. Furthermore, for every $n > 98$,*

$$c_2(n, K_4) = \left\lfloor \frac{2n-5}{3} \right\rfloor.$$

In addition, we determine $c_2(F)$ when F is K_4^- , the strong 5-cycle C_5 , and K_5^- — in fact in each case we give upper and lower bounds on $c_2(n, F)$ differing by at most 2.

Theorem 1.2. *Suppose $n = 6m + r$ for some $r \in \{0, 1, 2, 3, 4, 5\}$ and $m \in \mathbb{N}$, with $n \geq 7$. Then*

$$c_2(n, K_4^-) = \begin{cases} 2m-1 \text{ or } 2m & \text{if } r = 0 \\ 2m & \text{if } r \in \{1, 2\} \\ 2m \text{ or } 2m+1 & \text{if } r \in \{3, 4\} \\ 2m+1 & \text{if } r = 5. \end{cases}$$

In particular, $c_2(K_4^-) = \frac{1}{3}$.

Theorem 1.3. $\lfloor \frac{n-3}{2} \rfloor \leq c_2(n, C_5) \leq \lfloor \frac{n}{2} \rfloor$. *In particular, $c_2(C_5) = \frac{1}{2}$.*

Interestingly, there is no unique stable near-extremal configuration for Theorem 1.3: at least two configurations at edit distance $\Omega(n^3)$ of each other exist, see Remark 3.4.

Theorem 1.4. $\lfloor \frac{2n-5}{3} \rfloor \leq c_2(n, K_5^-) \leq \lfloor \frac{2n-2}{3} \rfloor$. *In particular, $c_2(K_5^-) = \frac{2}{3}$.*

Let us compare the Turán density π , the (existence) codegree density γ , the covering codegree density c_2 , and the tiling codegree density t_2 of K_4 , K_4^- , and C_5 in the following table (for a 3-graph F of order f , define $t_2(F) = \lim_{n=mf \rightarrow \infty} t_2(n, F)/(n-2)$ if this limit exists). In the table question marks indicate conjectures, except for $t_2(C_5)$, for which we are not aware of any conjecture.

	γ	π	c_2	t_2
K_4	$\frac{1}{2}?$ [4]	$\frac{5}{9}?$ [32]	$\frac{2}{3}$	$\frac{3}{4}$ [17, 23]
K_4^-	$\frac{1}{4}$ [8]	$\frac{2}{7}?$ [9]	$\frac{1}{3}$	$\frac{1}{2}$ [22]
C_5	$\frac{1}{3}?$ [24]	$2\sqrt{3} - 3?$ [26]	$\frac{1}{2}$?

Finally we give bounds on $c_2(\text{Fano})$, $c_2(F_{3,2})$ and $c_2(K_t)$ for $t \geq 5$, and pose a number of questions.

Our paper is structured as follows. In Section 2, we determine the codegree covering threshold for K_4 and characterize the extremal configurations. In Section 3, we prove our bounds on $c_2(n, F)$ for the other 3-graphs F mentioned above. We end in Section 4 with some discussion and questions.

2 The covering codegree threshold for K_4

In this section we determine the codegree threshold $c_2(n, K_4)$. We give a lower bound construction in Section 2.1 and prove the upper bound in Section 2.2. Finally, in Section 2.3 we provide other extremal constructions, and state a stability theorem that helps to show that these constructions are all possible extremal configurations; as the proofs of these latter results are similar to the proof of Theorem 1.1 we defer them to the appendix.

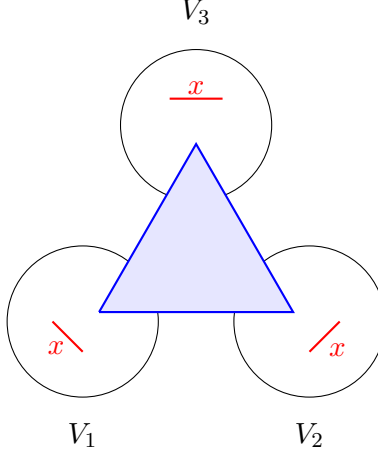


Figure 1: The complement of $F_1(n)$. The red pairs and the blue triples are absent from the link graph of x in F_1 and from $E(F_1)$ respectively.

2.1 Lower bound

Proof of the lower bound in Theorem 1.1. We construct a 3-graph $F_1(n)$ on $V = [n]$. Select a special vertex x . Split the remainder of the vertices into three parts $V_1 \sqcup V_2 \sqcup V_3 = V \setminus \{x\}$ with sizes as equal as possible,

$$|V_3| - 1 \leq |V_1| \leq |V_2| \leq |V_3|.$$

Put in as the link graph of x all pairs between distinct parts, i.e. add in all triples of the form xV_iV_j for $i \neq j$. Further, add in all triples not containing x and meeting at most two of the three parts $(V_i)_{i=1}^3$. Denote the resulting 3-graph by $F_1 = F_1(n)$. The complement of $F_1(n)$ is shown in Figure 1.

Observe that x is contained in no copy of K_4 in F_1 : the only triangles in the link graph of x are tripartite, and thus are not covered by any triple of the 3-graph. Let us now compute the minimum codegree of F_1 . For $v_i, v'_i \in V_i$ and $v_{i+1} \in V_{i+1}$, we have $d(v_i, x) = n - 1 - |V_i|$, $d(v_i, v'_i) = n - 3$ and $d(v_i, v_{i+1}) = n - 2 - |V_{i+2}|$. The minimum codegree $\delta_2(F_1)$ is thus $n - 2 - \lceil \frac{n-1}{3} \rceil$, attained by pairs $(v_1, v_2) \in V_1 \times V_2$. Writing $n = 3m + r$ with $r \in \{0, 1, 2\}$ and $m \in \mathbb{N}$, we have shown that

$$c_2(3m + r, K_4) \geq \delta_2(F_1) = \begin{cases} 2m - 2 & \text{if } r = 0 \\ 2m - 1 & \text{if } r = 1 \text{ or } 2. \end{cases}$$

This lower bound can be expressed more compactly as $c_2(n, K_4) \geq \lfloor \frac{2n-5}{3} \rfloor$. □

2.2 Upper bound

Let us give a general upper bound for $c_2(n, F)$, which turns out to be surprisingly close to the truth in the case of $F = K_4$.

Lemma 2.1. *Given a 3-graph F with at least one 3-edge, let r be the maximum of $\delta_1(F')$ among all subgraphs F' of F . Then $c_2(n, F) \leq \lfloor (1 - 1/r)n + (|V(F)| - 2r - 1)/r \rfloor$.*

Proof. Assume that F contains f vertices. We order the vertices of F as x_1, \dots, x_f such that x_i is a vertex of minimum degree in the subgraph $F - \{x_{i+1}, \dots, x_f\}$. As $r = \max \delta_1(F')$ among all subgraphs F' of F , we know that x_i has at most r neighbours among x_1, \dots, x_i .

Let G be a 3-graph on n vertices such that

$$\delta_2 := \delta_2(G) > \frac{r-1}{r}n + \frac{f-1}{r} - 2.$$

Fix a vertex v_1 of G . We will find a copy of F in G by first mapping x_1 to v_1 , x_2 to any other vertex v_2 , and x_3 to any $v_3 \in \Gamma_G(v_1, v_2)$. Suppose that x_1, \dots, x_i have been embedded to v_1, \dots, v_i . In order to embed x_{i+1} , we consider the neighbours of x_{i+1} among x_1, \dots, x_i . There are $t \leq r$ such neighbours and they are mapped to pairs p_1, \dots, p_t of v_1, \dots, v_i . Each p_j has at least δ_2 neighbours in G and thus at most $n - 2 - \delta_2$ non-neighbours in $V(G) \setminus \{v_1, \dots, v_i\}$. By the definition of δ_2 and $i \leq f - 1$, we have $r(n - 2 - \delta_2) < n - i$. Hence there exists a vertex $v_{i+1} \in V(G) \setminus \{v_1, \dots, v_i\}$ such that v_{i+1} is a common neighbour of p_1, \dots, p_t . Continuing this process, we obtain a copy of F as desired. \square

Remark 2.2. *The proof of Lemma 2.1 actually shows that if $\delta_2(G) > (1 - 1/r)n + (|V(F)| - 2r - 1)/r$ then every triple of $E(G)$ is covered by an F -subgraph.*

Applying Lemma 2.1 with $F = K_4$ and $r = 3$, we obtain that $c_2(n, K_4) \leq \lfloor \frac{2n-3}{3} \rfloor$. When $n \equiv 1 \pmod 3$, this implies that $c_2(n, K_4) \leq \lfloor \frac{2n-5}{3} \rfloor$. Together with the lower bound $c_2(n, K_4) \geq \lfloor \frac{2n-5}{3} \rfloor$, we obtain $c_2(n, K_4) = \lfloor \frac{2n-5}{3} \rfloor$ immediately.

When $n \equiv 0$ or $2 \pmod 3$, more work is required to reduce the upper bound to $\lfloor \frac{2n-5}{3} \rfloor$. In both cases, we shall make use of the following simple observation.

Lemma 2.3. *Let G be a 3-graph on $n \geq 4$ vertices. Suppose that $x \in V(G)$ is not covered by any copy of K_4 and there exists $a, b, c \in V(G)$ such that $abx, bcx, acx \in E(G)$ (thus $abc \notin E(G)$). Let $S = \{a, b, c, x\}$ and for each vertex $y \in V(G) \setminus S$, let S_y consist of all the pairs of S that make a 3-edge with y in G . Then S_y must be a subset of one of the following sets:*

$$S^{1,c} = \{ax, bx, ac, bc\}, \quad S^{1,b} = \{ax, cx, ab, bc\}, \quad S^{1,a} = \{bx, cx, ab, ac\},$$

$$S^{2,a} = \{ab, ac, bc, ax\}, \quad S^{2,b} = \{ab, ac, bc, bx\}, \quad S^{2,c} = \{ab, ac, bc, cx\}, \quad S^3 = \{ax, bx, cx\}.$$

In particular, $|S_y| \leq 4$. \square

Proof of $c_2(n, K_4) \leq \lfloor (2n - 5)/3 \rfloor$ when 3 divides n . Suppose $n = 3m$ for some integer $m \geq 2$ (so that $\lfloor (2n - 5)/3 \rfloor = 2n/3 - 2$). Let $G = (V, E)$ be a 3-graph on n vertices with $\delta_2(G) \geq 2n/3 - 1$. We claim that all vertices of G are covered by copies of K_4 . Suppose instead, that some vertex $x \in V$ is not contained in a copy of K_4 . Since the minimum degree in the link graph G_x of x is at least $2n/3 - 1 > (n - 1)/2$, there exists a triangle $\{ab, bc, ac\}$ in G_x . This implies that $abc \notin E$. Set $S = \{a, b, c, x\}$. For each vertex $y \in V \setminus S$, by Lemma 2.3, at most four pairs of S form edges of G with y . Thus, by the codegree assumption,

$$6 \left(\frac{2n}{3} - 1 \right) \leq d(a, x) + d(b, x) + d(c, x) + d(a, b) + d(b, c) + d(c, a) \leq 4(n - 4) + 9,$$

a contradiction. \square

When $n \equiv 2 \pmod{3}$, we start the proof in the same way. However, since we only have $\delta_2(G) \geq (2n-4)/3$, we will not obtain a contradiction until we prove that G has a similar structure as the 3-graph $F_1(n)$ given in Section 2.1.

Proof of $c_2(n, K_4) \leq \lfloor \frac{2n-5}{3} \rfloor$ when $n \equiv 2 \pmod{3}$. Suppose $n = 3m + 2 > 98$. In order to show that $c_2(n, K_4) \leq \lfloor \frac{2n-5}{3} \rfloor = (2n-7)/3$, consider a 3-graph $G = (V, E)$ on n vertices satisfying $\delta_2(G) \geq (2n-4)/3$.

Suppose that a vertex x of G is not contained in any copy of K_4 . As $(2n-4)/3 > (n-1)/2$, the link graph G_x contains a triangle $\{ab, bc, ac\}$. Set $S = \{a, b, c, x\}$ and for each $y \in V \setminus S$, define S_y as in Lemma 2.3. By Lemma 2.3, S_y is a subset of $S^{1,c}, S^{1,b}, S^{1,a}, S^{2,a}, S^{2,b}, S^{2,c}$ or S^3 . For $i \in \{1, 2\}$ and $j \in \{a, b, c\}$, write $s_{i,j}$ for the number of vertices $y \in V \setminus S$ for which $S_y = S^{i,j}$, and write s_i for the sum $s_{i,a} + s_{i,b} + s_{i,c}$. Finally let s_0 be the number of vertices $y \in V \setminus S$ such that $S_y \neq S^{i,j}$ for any $i \in \{1, 2\}$ and $j \in \{a, b, c\}$. Note that $|S_y| \leq 3$ for such y . We know that $s_1 + s_2 + s_0 = n - 4$. Furthermore, by the codegree assumption,

$$3 \frac{2n-4}{3} \leq d(a, x) + d(b, x) + d(c, x) \leq 2s_1 + s_2 + 3s_0 + 6, \quad (1)$$

$$6 \frac{2n-4}{3} \leq d(a, x) + d(b, x) + d(c, x) + d(a, b) + d(b, c) + d(c, a) \leq 4s_1 + 4s_2 + 3s_0 + 9, \quad (2)$$

Substituting $s_0 = n - 4 - s_1 - s_2$ into (1) and (2) yields that $s_1 + 2s_2 \leq n - 2$ and $s_1 + s_2 \geq n - 5$, respectively. Combining the two inequalities we have just obtained, we get

$$s_2 \leq 3 \quad \text{and} \quad s_1 \geq n - 8.$$

We now show that the weight of s_1 splits almost equally between $s_{1,a}, s_{1,b}, s_{1,c}$. Note that

$$\frac{2n-4}{3} \leq d(b, c) \leq n - 3 - s_{1,a},$$

from which it follows that $s_{1,a} \leq \frac{n-5}{3}$. Similarly we derive that $s_{1,b}, s_{1,c} \leq (n-5)/3$. Consequently

$$s_{1,a} = s_1 - s_{1,b} - s_{1,c} \geq n - 8 - 2 \frac{n-5}{3} = \frac{n-14}{3}.$$

Similarly $s_{1,b}$ and $s_{1,c}$ satisfy the same lower bound. Let $A = \{y \in V \setminus S : S_y = S^{1,a}\} \cup \{a\}$, $B = \{y \in V \setminus S : S_y = S^{1,b}\} \cup \{b\}$ and $C = \{y \in V \setminus S : S_y = S^{1,c}\} \cup \{c\}$. Set $V' = A \cup B \cup C \cup \{x\}$. Then we have just shown the following lemma.

Lemma 2.4.

$$|V'| = 1 + |A| + |B| + |C| \geq n - 4, \quad \text{and} \quad \frac{n-11}{3} \leq |A|, |B|, |C| \leq \frac{n-2}{3}. \quad \square$$

Let \mathcal{B} be the collection of 3-edges of G of the form xAA, xBB, xCC (the ‘bad’ triples). Let \mathcal{M} be the collection of non-edges of G of the form xAB, xAC, xBC (the ‘missing’ triples). Viewing \mathcal{B} and \mathcal{M} as 3-graphs on V' , for two distinct vertices $v_1, v_2 \in V'$, we let $d_{\mathcal{B}}(v_1, v_2)$ denote their codegree in \mathcal{B} and $d_{\mathcal{M}}(v_1, v_2)$ their codegree in \mathcal{M} .

Claim 2.5. *For every $v \in V' \setminus \{x\}$, $d_{\mathcal{B}}(v, x) \leq 4$.*

Proof. Suppose without loss of generality that $v \in A$. If $v = a$, then $d_{\mathcal{B}}(v, x) = 0$ because G contains no 3-edges of the form xaA . We thus assume that $v \neq a$. The bad triples for the pair (v, x) are triples of the form $a'vx$ for $a' \in A \setminus \{a, v\}$. Suppose $a'vx \in \mathcal{B}$. Then since there is no K_4 in G containing x , and since, by the definition of A , $a'bx$, vbx , $a'cx$ and vcx are all in G , it must be the case that both of $a'vb$ and $a'vc$ are missing from G . Further if $c' \in C \cap \Gamma(v, x)$ then all of $c'vx$, bvx , $c'bx$ are in G , whence $bc'v$ is absent from G . Similarly for any $b' \in B$, at most one of $b'cv$, $b'xv$ is in G . Finally since $bcv \notin E(G)$, b and c are contained in exactly one of $\Gamma(b, v)$, $\Gamma(c, v)$, and $\Gamma(x, v)$. To summarize, a vertex y in V' can lie in at most two of $\Gamma(b, v)$, $\Gamma(c, v)$ and $\Gamma(x, v)$ unless y is in $\Gamma_{\mathcal{B}}(x, v)$ (and lies in exactly one of those joint neighbourhoods) or is in $\{b, c, v\}$ (and lies in at most one of those joint neighbourhoods). Together with our codegree assumption, this gives us

$$3 \frac{2n-4}{3} \leq d(b, v) + d(c, v) + d(x, v) \leq 2|V'| - d_{\mathcal{B}}(v, x) - 4 + 3(n - |V'|) \\ = 3n - |V'| - 4 - d_{\mathcal{B}}(v, x) \leq 2n - d_{\mathcal{B}}(v, x),$$

where we apply $|V'| \geq n - 4$ from Lemma 2.4 in the last inequality. It follows that $d_{\mathcal{B}}(v, x) \leq 4$, as claimed. \square

Claim 2.6. *For every $v \in V' \setminus \{x\}$, $d_{\mathcal{M}}(v, x) \leq 8$.*

Proof. Suppose without loss of generality that $v \in A$. Then by the codegree assumption, Claim 2.5 and the bound on $|A|$ from Lemma 2.4 we have

$$\frac{2n-4}{3} \leq d(v, x) \leq n - 1 - |A| + d_{\mathcal{B}}(v, x) - d_{\mathcal{M}}(v, x) \leq n - 1 - \frac{n-11}{3} + 4 - d_{\mathcal{M}}(v, x),$$

which gives that $d_{\mathcal{M}}(v, x) \leq 8$ as claimed. \square

Claim 2.7. *For every $y \in V(G) \setminus \{x\}$, $\Gamma(y, x)$ has a non-empty intersection with at most two of the parts A , B and C .*

Proof. Let $y \in V(G) \setminus \{x\}$. Set $A_y = A \cap \Gamma(x, y)$, $B_y = B \cap \Gamma(x, y)$ and $C_y = C \cap \Gamma(x, y)$. Suppose none of A_y , B_y , C_y is empty. Fix $a' \in A_y$. For $b' \in B_y$, if $b' \in \Gamma(a', x)$, then $a'b'y \notin E(G)$ – otherwise $\{a', b', x, y\}$ spans a copy of K_4 . Similarly, for $c' \in C_y \cap \Gamma(a', x)$, we have $a'c'y \notin E(G)$. Hence,

$$\frac{2n-4}{3} \leq d(a', y) \leq n - 2 - |B_y \cap \Gamma(a', x)| - |C_y \cap \Gamma(a', x)|.$$

Claim 2.6 gives that $d_{\mathcal{M}}(a', x) \leq 8$. Consequently,

$$|B_y \cap \Gamma(a', x)| + |C_y \cap \Gamma(a', x)| = |B_y| + |C_y| - d_{\mathcal{M}}(a', x) \geq |B_y| + |C_y| - 8$$

This implies that

$$\frac{2n-4}{3} \leq n - 2 - |B_y| - |C_y| + 8,$$

which yields $|B_y| + |C_y| \leq (n + 22)/3$. Similarly by considering any vertex $b' \in B_y$ and any vertex $c' \in C_y$ we obtain that

$$|A_y| + |C_y| \leq \frac{n+22}{3} \quad \text{and} \quad |A_y| + |B_y| \leq \frac{n+22}{3}.$$

Summing these three inequalities and dividing by 2, we obtain that

$$|A_y| + |B_y| + |C_y| \leq \frac{n+22}{2}.$$

Furthermore, by the codegree condition,

$$\frac{2n-4}{3} \leq d(x, y) \leq |A_y| + |B_y| + |C_y| + (n - |V'|) \leq \frac{n+22}{2} + 4,$$

where we apply $|V'| \geq n-4$ from Lemma 2.4. Rearranging terms yields $\frac{n}{6} \leq \frac{49}{3}$, which contradicts our assumption $n > 98$. \square

Set $V_1 = \{y \in V \setminus \{x\} : \Gamma(x, y) \cap A = \emptyset\}$, $V_2 = \{y \in V \setminus \{x\} : \Gamma(x, y) \cap B = \emptyset\}$ and $V_3 = \{y \in V \setminus \{x\} : \Gamma(x, y) \cap C = \emptyset\}$. Without loss of generality, assume that

$$|V_1| \leq |V_2| \leq |V_3|. \quad (3)$$

Claim 2.7 shows that $V_1 \cup V_2 \cup V_3$ covers $V(G) \setminus \{x\}$. We now show that in fact V_1, V_2, V_3 are pairwise disjoint, and $A \subseteq V_1$, $B \subseteq V_2$, and $C \subseteq V_3$. Suppose instead, that there exists $y \in V_1 \cap V_2$. Then $\Gamma(x, y) \cap (A \cup B) = \emptyset$. By the codegree condition and Lemma 2.4,

$$\frac{2n-4}{3} \leq d(x, y) \leq |C_y| + (n - |V'|) \leq \frac{n-2}{3} + 4,$$

which implies that $n \leq 14$, a contradiction.

Furthermore, consider $a' \in A$. By Claim 2.6, $a'xv \in E(G)$ for all but at most 8 vertices $v \in B \cup C$. By Lemma 2.4,

$$|B| - 8 \geq \frac{n-11}{3} - 8 > 0$$

which is strictly positive as $n > 35$. Thus we have that $\Gamma(a', x)$ has a non-empty intersection with B ; similarly we have that $\Gamma(a', x) \cap C \neq \emptyset$, from which we can finally deduce by Claim 2.7 that $\Gamma(a', x) \cap A = \emptyset$ and that $A \subseteq V_1$. Similarly we have $B \subseteq V_2$ and $C \subseteq V_3$.

Let $c' \in C$. By the definition of V_3 , we have $\Gamma(c', x) \subseteq V_1 \cup V_2$. By the codegree assumption, it follows that

$$\frac{2n-4}{3} \leq d(c', x) \leq |V_1| + |V_2| = n-1-|V_3|, \quad (4)$$

from which we get that $|V_3| \leq (n+1)/3$. Since $n = 3m+2$, by (3), we derive that $|V_3| = (n+1)/3 = m+1$ and $|V_1| \leq |V_2| \leq (n+1)/3$.

Claim 2.8. *Let $y \in V_i$. Then $\Gamma(y, x)$ contains all but at most 6 vertices from $\bigcup_{j \neq i} V_j$ and no vertex from V_i .*

Proof. Suppose without loss of generality that $y \in V_1$. Then by Claim 2.7, $A \cap \Gamma(y, x) = \emptyset$. Thus

$$\frac{2n-4}{3} \leq d(x, y) \leq |\Gamma(x, y) \cap (V_2 \cup V_3)| + |\Gamma(x, y) \cap (V_1 \setminus A)| \leq |\Gamma(x, y) \cap (V_2 \cup V_3)| + 4$$

because $|V_1 \setminus A| \leq n - |V'| \leq 4$ by Lemma 2.4. Hence $|\Gamma(x, y) \cap (V_2 \cup V_3)| \geq (2n - 16)/3$. Since $|V_i| \leq (n + 1)/3$ for all i ,

$$|(V_2 \cup V_3) \setminus \Gamma(x, y)| \leq 2 \frac{n+1}{3} - \frac{2n-16}{3} = 6.$$

This establishes the first part of our claim.

For the second part of our claim (namely, $\Gamma(y, x) \cap V_1 = \emptyset$), suppose that $yy'x \in E(G)$ for some $y' \in V_1$. Then $\Gamma(y, y') \cap \Gamma(y, x) \cap \Gamma(y', x) = \emptyset$. Consequently,

$$\frac{2n-4}{3} \leq d(y, y') \leq 1 + |V_1| - 2 + |(V_2 \cup V_3) \setminus (\Gamma(y, x) \cap \Gamma(y', x))| \leq 1 + \frac{n+1}{3} - 2 + 2 \cdot 6$$

where in the last inequality we apply $|V_1| \leq (n + 1)/3$ and the first part of the claim. This implies that $n \leq 38$, a contradiction. \square

Claim 2.8 implies that $\Gamma(v_3, x) \subseteq V_1 \cup V_2$ for all $v_3 \in V_3$. Then $d(v_3, v)$ satisfies (4) with two inequalities replaced by equalities. Consequently all triples of the form xvv_3 with $v_3 \in V_3$ and $v \in V_1 \cup V_2$ are in $E(G)$.

Claim 2.8 also implies that most $v_1 \in V_1$ and $v_2 \in V_2$ satisfy $xv_1v_2 \in E(G)$. Fix such v_1 and v_2 . Then $v_1v_2v_3 \notin E(G)$ for any $v_3 \in V_3$ otherwise $xv_1v_2v_3$ induces a copy of K_4 . We thus have

$$2m \leq d(v_1, v_2) \leq |V_1| + |V_2| - 1 = 2m - 1,$$

a contradiction. This completes the proof of Theorem 1.1 in the case $n = 3m + 2$. \square

2.3 Other extremal constructions and stability

Recall the construction $F_1(n)$ described in Section 2.1. There are other extremal families of 3-graphs for K_4 -covering that are not isomorphic to subgraphs of $F_1(n)$.

Case 1: $n = 3m$. We partition $[n] \setminus \{x\}$ into three parts V_1, V_2 and V_3 with sizes $|V_1| = m - 1$ and $|V_2| = |V_3| = m$. A collection \mathcal{E} of pairs of vertices from different parts of $[n] \setminus \{x\}$ is called *admissible* if (i) every vertex $v_1 \in V_1$ is contained in at most two pairs from \mathcal{E} , and (ii) every vertex $v \in V_2 \sqcup V_3$ is contained in at most one pair from \mathcal{E} . Now let $F_1(\mathcal{E}, 3m)$ be the 3-graph obtained from F_1 by deleting all triples xuv and adding all tripartite triples uvw (namely, $w \in V \setminus \{x\}$ is from the part different from the ones containing u or v) for all $uv \in \mathcal{E}$. It is easy to see that $F_1(\mathcal{E}, 3m)$ contains no K_4 covering x and $\delta_2(F_1(\mathcal{E}, 3m)) = \delta_2(F_1(3m)) = 2m - 2$.

Case 2: $n = 3m + 1$. We partition $[n] \setminus \{x\}$ into three parts V_1, V_2 and V_3 with sizes $|V_1| = |V_2| = |V_3| = m$. A collection \mathcal{E} of pairs of vertices from different parts of $[n] \setminus \{x\}$ is called *admissible* if every vertex is contained in at most one pair from \mathcal{E} . Now let $F_1(\mathcal{E}, 3m + 1)$ be the 3-graph obtained from F_1 by deleting all triples xuv and adding all tripartite triples uvw for all $uv \in \mathcal{E}$. It is easy to see that $F_1(\mathcal{E}, 3m + 1)$ contains no K_4 covering x and $\delta_2(F_1(\mathcal{E}, 3m + 1)) = \delta_2(F_1(3m + 1)) = 2m - 1$.

Case 3: $n = 3m + 2$. We partition $[n] \setminus \{x\}$ into three parts V_1, V_2 and V_3 with sizes $|V_1| = |V_2| = m$ and $|V_3| = m + 1$. A collection \mathcal{E} of pairs of vertices from different parts of $[n] \setminus \{x\}$ is called *admissible* if (i) every vertex $v \in V_1 \sqcup V_2$ is contained in at most 2 pairs from \mathcal{E} and (ii) every vertex $v_3 \in V_3$ is contained in at most 1 pair from \mathcal{E} . Now let $F_1(\mathcal{E}, 3m + 2)$ be the 3-graph obtained from F_1 by deleting all triples xuv and adding all tripartite triples uvw for all $uv \in \mathcal{E}$. It is easy to see that $F_1(\mathcal{E}, 3m + 2)$ contains no K_4 covering x and $\delta_2(F_1(\mathcal{E}, 3m + 2)) = \delta_2(F_1(3m + 2)) = 2m - 1$.

There is yet another extremal construction. Partition $[n] \setminus \{x\}$ into three parts V_1 , V_2 and V_3 with sizes $|V_1| = m - 1$ and $|V_2| = |V_3| = m + 1$. In this context, a collection \mathcal{E} of pairs of vertices from different parts of $[n] \setminus \{x\}$ is called *admissible* if (i) every vertex $v_1 \in V_1$ is contained in at most 3 pairs from \mathcal{E} and (ii) every vertex $v \in V_2 \sqcup V_3$ is contained in at most 1 pair from \mathcal{E} . Let F'_1 be the 3-graph on $[n]$ consisting of all triples xuv , where u, v come from different parts, and all triples of $[n] \setminus \{x\}$ that are not tripartite. Now let $F'_1(\mathcal{E}, 3m + 2)$ be the 3-graph obtained from F'_1 by deleting all triples xuv and adding all tripartite triples uvw for all $uv \in \mathcal{E}$. It is easy to see that $F'_1(\mathcal{E}, 3m + 2)$ contains no K_4 covering x and $\delta_2(F'_1(\mathcal{E}, 3m + 2)) = \delta_2(F_1(3m + 2)) = 2m - 1$.

We can show that the above constructions are *all* extremal configurations for n sufficiently large ($n \geq 999$). This can be done by first proving the following stability theorem.

Theorem 2.9 (Stability). *Suppose $n \geq 4$ and $0 < \delta \leq \frac{1}{429}$. Suppose that G is a 3-graph on n vertices with minimum codegree $\delta_2(G) \geq (\frac{2}{3} - \delta)n$ and that there is a vertex $x \in V(G)$ not contained in any copy of K_4 in G . Then there exists a tripartition $V_1 \sqcup V_2 \sqcup V_3$ of $V(G) \setminus \{x\}$ such that the following holds for all $i \in [3]$ and $j \neq i$:*

- (i) *there is no triple in G of the form xV_iV_i ;*
- (ii) *all but at most $9\delta n^2$ triples of the form xV_iV_j are in G ;*
- (iii) *there are at most $4\delta n^3$ triples in G of the form $V_1V_2V_3$;*
- (iv) *all but at most $6\delta n^3$ triples of the form $V_iV_iV_j$ are in G ;*
- (v) $||V_i| - \frac{n-1}{3}| \leq 2\delta n$.

Theorem 2.10. • *For $n \equiv 0 \pmod{3}$ with $n \geq 858$, the extremal configurations for $c_2(n, K_4)$ are isomorphic to a subgraph of $F_1(\mathcal{E}, n)$ for some admissible \mathcal{E} .*

- *For $n \equiv 1 \pmod{3}$ with $n \geq 715$, the extremal configurations for $c_2(n, K_4)$ are isomorphic to a subgraph of $F_1(\mathcal{E}, n)$ for some admissible \mathcal{E} .*
- *For $n \equiv 2 \pmod{3}$ with $n \geq 1001$, the extremal configurations for $c_2(n, K_4)$ are isomorphic to a subgraph of $F_1(\mathcal{E}, n)$ or to a subgraph of $F'_1(\mathcal{E}, n)$ for some admissible \mathcal{E} .*

The proof of Theorem 2.9 is very similar to that of the case $n = 3m + 2$ of Theorem 1.1, while the proof of Theorem 2.10 is a straightforward application of parts (i) and (ii) of Theorem 2.9. We therefore defer these proofs to the appendix.

3 Covering thresholds for other 3-graphs

3.1 K_4^-

Proof of the lower bound in Theorem 1.2. We construct a 3-graph $F_2(n)$ on $V = [n]$. Select a special vertex x . Split the remainder of the vertices into six parts $\sqcup_{i=1}^6 V_i = V \setminus \{x\}$ with sizes as equal as possible, as follows:

$$|V_1| - 1 \leq |V_6| \leq |V_5| \leq |V_4| \leq |V_3| \leq |V_2| \leq |V_1|.$$

Put as the link of x the blow-up of a 6-cycle through the six parts, i.e. add all triples of the form xV_iV_{i+1} for $i \in [6]$, winding round modulo 6 as necessary (identifying V_7 with V_1 , and so on). Finally add those triples not involving x which are not of type $V_iV_iV_{i+1}$, $V_iV_{i+1}V_{i+1}$ or $V_iV_{i+1}V_{i+2}$ for $i \in [6]$ (winding round modulo 6) to form the 3-graph $F_2(n)$.

Observe that the link graph of x in $F_2(n)$ is triangle-free (being the blow-up of a 6-cycle). Thus a putative K_4^- containing x would have to be induced by a 4-set $\{a, b, c, x\}$, with abc , abx and acx all being triples of $F_2(n)$. Since ab is in the link graph of x , we must have that a, b come from different but adjacent parts V_i, V_{i+1} ; by symmetry of $F_2(n)$, we may assume without loss of generality that $a \in V_1$ and $b \in V_2$. Since $acx \in E(F_2(n))$, it follows that $c \in V_2$ or $c \in V_6$. But by construction of $F_2(n)$, there are no triples of type $V_6V_1V_2$ or $V_1V_2V_2$, so that we cannot have in $abc \in E(F_2(n))$. Thus there is no copy of K_4^- in $F_2(n)$ covering x .

Let us now compute the minimum codegree of $F_2(n)$. Consider vertices $a_i, a'_i \in V_i$, $a_{i+1} \in V_{i+1}$, $a_{i+2} \in V_{i+2}$ and $a_{i+3} \in V_{i+3}$. We have that $d(a_i, a'_i) = n - 3 - |V_{i-1}| - |V_{i+1}|$, $d(a_i, a_{i+2}) = n - 3 - |V_{i+1}|$, $d(a_i, a_{i+3}) = n - 3$, and, lastly,

$$d(a_i, x) = |V_{i-1}| + |V_{i+1}| \quad \text{and} \quad d(a_i, a_{i+1}) = 1 + |V_{i+3}| + |V_{i+4}|.$$

Up to the choice of i , this covers all possible pairs in $F_2(n)$. The first three quantities are at least $n - 3 - 2\lceil \frac{n-1}{6} \rceil \geq \frac{2n}{3} - \frac{13}{3}$, which for $n \geq 12$ is greater than $\lfloor \frac{n-1}{3} \rfloor$. The last two quantities are both of order $\frac{n}{3} + O(1)$, however, and we analyse them more closely. Set $n = 6m + r$ for some $r \in \{0, 1, 2, 3, 4, 5\}$. Then

$$d(a_i, x) \geq \min_i (|V_{i-1}| + |V_{i+1}|) = |V_6| + |V_4| = \begin{cases} 2m - 1 & \text{if } r = 0 \\ 2m & \text{if } 0 < r < 5 \\ 2m + 1 & \text{if } r = 5, \end{cases}$$

and

$$d(a_i, a_{i+1}) \geq \min_i (1 + |V_{i+3}| + |V_{i+4}|) = 1 + |V_5| + |V_6| = \begin{cases} 2m & \text{if } r = 0 \\ 2m + 1 & \text{if } 0 < r \leq 5. \end{cases}$$

Thus

$$c_2(n, K_4^-) \geq \delta_2(F_2(n)) = \begin{cases} 2m - 1 & \text{if } r = 0 \\ 2m & \text{if } 0 < r < 5 \\ 2m + 1 & \text{if } r = 5. \end{cases}$$

□

Proof of the upper bound in Theorem 1.2. Let G be a 3-graph on $n \geq 4$ vertices. Suppose $\delta_2(G) > \frac{n}{3}$. Pick an arbitrary vertex $x \in V(G)$. Let abx be any 3-edge containing x . We have $d(a, b) + d(a, x) + d(b, x) - 3 > n - 3$. So by the pigeonhole principle, there exists $c \in V(G) \setminus \{a, b, x\}$ which makes a 3-edge of G with at least two of ab , ax , bx . The 4-set $abcx$ then contains a copy of K_4^- in G covering x , as required. This shows that $c_2(n, K_4^-) \leq \lfloor \frac{n}{3} \rfloor$. □

Remark 3.1. *Again we actually proved something stronger here: our argument establishes that for $\delta_2(G)$ above $\lfloor \frac{n}{3} \rfloor$, every triple of $E(G)$ can be extended to a copy of K_4^- .*

Matching the upper and lower bounds obtained above, we obtain the set of possible values for $c_2(n, K_4^-)$ claimed in Theorem 1.2. □

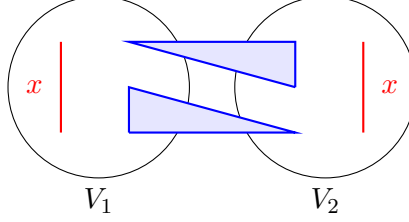


Figure 2: The 3-graph $F_3(n)$. The red pairs and the blue triples make up the link graph of x and the remainder of $E(F_3)$ respectively.

Remark 3.2. We believe the gap between the upper and lower bounds for $c_2(n, K_4^-)$ could be closed using similar (but more involved) stability arguments to those we used on to determine $c_2(n, K_4)$. However since such arguments would be non-trivial (the conjectured extremal configurations in this case are 6-partite) and would greatly increase the length of this paper, we do not pursue them here and leave open the determination of $c_2(n, K_4^-)$ in the case where $n \equiv 0, 3, 4 \pmod 6$.

3.2 C_5

Proof of the lower bound in Theorem 1.3. We construct a 3-graph $F_3(n)$ on $V = [n]$. Select a special vertex x . Split the remainder of the vertices into two parts $V \setminus \{x\} = V_1 \sqcup V_2$ with sizes as equal as possible, $|V_2| - 1 \leq |V_1| \leq |V_2|$. Form the link graph of x by adding in all pairs internal to one of the parts, i.e. all pairs of the form xV_1V_1 or xV_2V_2 . Next, add in all triples not containing x and meeting both of the parts, i.e. all pairs of the form $V_1V_1V_2$ or $V_1V_2V_2$. This yields a 3-graph $F_3(n)$ with minimum codegree $\delta_2(F_3(n)) = |V_1| - 1 = \lfloor \frac{n-3}{2} \rfloor$, attained by x and any vertex $a \in V_1$; see Figure 2.

Now there is no copy of C_5 covering $x \in F_3(n)$. Indeed, let $S = \{a_1, a_2, b_1, b_2\}$ be a set of four distinct vertices in $V \setminus \{x\}$ such that all of a_1a_2x , a_1b_1x and b_1b_2x are triples of $F_3(n)$. Then by construction these four vertices must all lie within the same part of $F_3(n)$. But by construction again we have that S spans no triple of $F_3(n)$, whence $S \cup \{x\}$ does not contain a copy of C_5 . \square

Proof of the upper bound in Theorem 1.3. Let $G = (V, E)$ be a 3-graph on n vertices with minimum codegree $\delta_2(G) > \frac{n}{2}$. Let x be any vertex. Fix an edge ab in the link graph G_x . Since $\delta_1(G_x) > n/2$, a and b each has at least $\frac{n}{2} - 1$ neighbours in $V \setminus \{x, u, v\}$. Hence a and b have a common neighbour c in G_x . We shall use the triangle $\{a, b, c\}$ to find a copy of C_5 covering x . For this purpose, it is convenient to introduce the following notation. Given a 4-set of vertices $\{y_1, y_2, z_1, z_2\}$ from $V \setminus \{x\}$, write $y_1y_1|z_1z_2$ as a shorthand for the statement that all of y_1z_1x , y_1y_2x , y_2z_2x , y_2z_1x and $y_1z_1z_2$ are in $E(G)$ (and in particular that $\{x, y_1, z_1, z_2, y_2\}$ contains a copy of C_5 covering x).

Lemma 3.3. *There is either a copy of C_5 or a copy of K_4 covering x in G .*

Proof. If $abc \in E(G)$ then the claim is immediate since $S = \{a, b, c, x\}$ induces a complete 3-graph. Assume therefore that $abc \notin E(G)$. By our codegree assumption,

$$d(a, b) + d(a, c) + d(b, c) + d(a, x) + d(b, x) + d(c, x) - 9 \geq 6\delta_2(G) - 9 > 3(n - 4).$$

Thus there exists $y \in V \setminus S$ which makes a 3-edge with at least four of the pairs ab, ac, bc, ax, bx, cx . It is now easy to check that $S \cup \{y\}$ contains either a K_4 or a C_5 covering x . Indeed by symmetry we may reduce the case-checking to the following three possibilities:

- if y makes a 3-edge with ab, bc, ac and ax , then $ab|cy$;
- if y makes a 3-edge with ab, ac and at least one of bx or cx , then $bc|ay$;
- if y makes a 3-edge with ab and with both of ax and bx , then $\{a, b, x, y\}$ induces a copy of K_4 . \square

With a view towards proving Theorem 1.3, we may thus assume that there is a copy of K_4 covering x . Let $S = \{a, b, c, x\}$ be a 4-set of vertices inducing such a K_4 . By the codegree assumption,

$$d(a, b) + d(a, c) + d(b, c) + d(a, x) + d(b, x) + d(c, x) - 12 \geq 6\delta_2(G) - 12 > 3(n - 4).$$

Thus there exists $y \in V \setminus S$ which makes a 3-edge with at least four of the pairs ab, ac, bc, ax, bx, cx . It is now easy to check that $S \cup \{y\}$ contains a copy of C_5 covering x . Indeed by symmetry we may reduce the case-checking to the following three possibilities:

- if y makes a 3-edge with ab, ac, bc and ax , then $ab|cy$;
- if y makes a 3-edge with ab, ac and at least one of bx and cx , then $bc|ay$;
- if y makes a 3-edge with ab and with all of ax, bx and cx , then $cy|ab$.

In all three cases we cover x with a copy of C_5 . The claimed upper bound on $c_2(n, C_5)$ follows. \square

Remark 3.4. *Interestingly, as pointed out to us by Jie Han and Allan Lo, another very different construction attains the lower bound in Theorem 1.3. Take a balanced bipartition of $[n]$ into two sets V_1 and V_2 , with $|V_1| \leq |V_2|$. Now take all triples meeting V_1 in an even number of vertices to form a 3-graph $F_4(n)$. Note that $\delta_2(F_4(n)) = \min(|V_1| - 1, |V_2| - 2)$ (attained by pairs from $A \times B$ and $B^{(2)}$ respectively), which is exactly equal to $\lfloor \frac{n-3}{2} \rfloor$. Now, it is an easy exercise to check that every vertex $x \in V_1$ fails to be covered by a C_5 , giving us a second proof that $c_2(n, C_5) \geq \lfloor \frac{n-3}{2} \rfloor$. In particular, we do not have stability for this problem: we have two near-extremal constructions which are easily seen to lie at edit distance $\Omega(n^3)$ from each other. Also we have that just below the codegree threshold for covering by C_5 , we could have as many as $\lfloor \frac{n}{2} \rfloor$ uncovered vertices. This stands in sharp contrast with the situation for K_4 (see the discussion in Section 4).*

3.3 K_5^-

Proof of Theorem 1.4. For the lower bound, note that

$$c_2(n, K_5^-) \geq c_2(n, K_4) \geq \delta_2(F_1(n)) = \left\lfloor \frac{2n-5}{3} \right\rfloor.$$

For the upper bound, let G be a 3-graph on n vertices with $\delta_2(G) > \frac{2n-2}{3}$. By Theorem 1.1, for any vertex $x \in V(G)$ there is a triple a_1, a_2, a_3 such that $S = \{x, a_1, a_2, a_3\}$ induces a copy of K_4 in G . Now

$$d(x, a_1) + d(x, a_2) + d(x, a_3) + d(a_1, a_2) + d(a_1, a_3) + d(a_2, a_3) - 12 > 4(n - 4),$$

whence there exist $a_4 \in V \setminus S$ which makes a 3-edge with at least 5 of the pairs from $S^{(2)}$. Thus $S \cup \{a_4\}$ contains a copy of K_5^- covering x . This shows that $c_2(n, K_5^-) \leq \lfloor \frac{2n-2}{3} \rfloor$. \square

3.4 The Fano plane

Proposition 3.5. $\lfloor \frac{n}{2} \rfloor \leq c_2(n, \text{Fano}) \leq \lfloor \frac{2n}{3} \rfloor$.

Proof. The lower bound is from the codegree threshold of the Fano plane: consider a bipartition of $[n]$ into two sets $V_1 \sqcup V_2$ with $|V_1| = \lfloor \frac{n}{2} \rfloor$ and $|V_2| = \lceil \frac{n}{2} \rceil$, and adding all triples meeting both parts. The resulting 3-graph is easily seen to be Fano-free (it is 2-colourable, whereas the Fano plane is not) and has codegree $\lfloor \frac{n}{2} \rfloor$. For the upper bound, apply Lemma 2.1 with $F = \text{Fano}$ and $r = 3$. \square

3.5 $F_{3,2}$

Theorem 3.6. $1/3 \leq c_2(F_{3,2}) \leq 3/7$.

Proof. The lower bound is from the codegree density of $F_{3,2}$. An $F_{3,2}$ -free construction on n vertices with codegree $\lfloor \frac{n}{3} \rfloor - 1$ is obtained by considering a tripartition of $[n]$ into three parts with sizes as equal as possible, $|V_3| - 1 \leq |V_1| \leq |V_2| \leq |V_3|$ and adding all triples of the form $V_i V_j V_{i+1}$ (this is not actually best possible — see [7] for a determination of the precise codegree threshold and the extremal constructions attaining it).

For the upper bound, let G be a 3-graph on n vertices with $\delta_2(G) = cn$. Suppose there exists $x \in V(G)$ such that there is no copy of $F_{3,2}$ in G covering x . This means that for every vertex $v \in V \setminus \{x\}$, $\Gamma(x, v)$ is an independent (3-edge-free) set in G , and moreover that for every 4-set $\{a, b, c, d\} \subseteq V(G)$, at least one of the triples $\{xab, xcd, abc, abd\}$ is not in $E(G)$. For convenience, we shall write $ab|cde$ as a short-hand for the statement that $\{abc, abd, abe, cde\}$ all are 3-edges of G .

We use the following technical lemma to deduce $c \leq 3/7 + o(1)$.

Lemma 3.7. *If there exist sets $A, B \subseteq V$ such that*

1. *A is a subset of $\Gamma(x, y)$ of size cn for some $y \in V \setminus \{x\}$, and B is a subset of $V \setminus (A \cup \{x\})$ of size cn , and*
2. *B is independent in G and the link graph G_x ,*

then $c \leq 3/7 + o(1)$.

Proof of Lemma 3.7. Let $C = V \setminus (A \cup B)$. We have $|C| = n(1 - 2c)$. By our assumption, A is independent in G . By the codegree assumption, at least $\binom{|A|}{2} (cn - |C|)$ triples of G have two vertices in A and one vertex in B . Consequently, at most $\binom{|A|}{2} |C| = \binom{cn}{2} (1 - 2c)n$ triples of the form AAB are missing from G .

On the other hand, let $b, b' \in B$. Since B is independent in G and G_x , we have $\Gamma(b, b') \subseteq A \cup C$ and $\Gamma(b, x) \subseteq A \cup C$. Consequently $|\Gamma(b', x) \cap A| \geq cn - |C| \geq (3c - 1)n$ and

$$|\Gamma(b, b') \cap \Gamma(b, x) \cap A| \geq 2(cn - |C|) - |A| \geq (5c - 2)n.$$

For any $a \in \Gamma(b, b') \cap \Gamma(b, x) \cap A$ and any $a' \in \Gamma(b', x) \cap A$, the triple $aa'b$ must be absent from G — otherwise $ab|a'bx$. There are at least $\binom{(3c-1)n}{2} - \binom{(1-2c)n}{2}$ such pairs (a, a') because, in general,

there are at least $\binom{|A_1|}{2} - \binom{|A_1| - |A_2|}{2}$ pairs (a_1, a_2) with $a \in A_1$ and $a_2 \in A_2$ for arbitrary sets A_1, A_2 satisfying $|A_1| \geq |A_2|$.

There are thus at least $\binom{(3c-1)n}{2} - \binom{(1-2c)n}{2}$ distinct pairs (a, a') for which $aa'b \notin E(G)$. Summing over all $b \in B$, this gives us a total of at least $\left(\binom{(3c-1)n}{2} - \binom{(1-2c)n}{2}\right) cn$ AAB triples missing from $E(G)$. Combining this together with our upper bound on the number of missing AAB triples yields the inequality

$$\left(\binom{(3c-1)n}{2} - \binom{(1-2c)n}{2}\right) cn \leq \binom{cn}{2} (n(1-2c) - 1),$$

which implies that

$$((3c-1)^2 - (1-2c)^2) \frac{n^2}{2} cn \leq \frac{c^2 n^2}{2} (1-2c)n + O(n^2).$$

This inequality in turn gives $c \leq 3/7 + o(n^{-1})$. \square

We now show that we can find $A, B \subseteq V$ satisfying the properties in Lemma 3.7.

Suppose first of all that G_x is not triangle-free. Let ya_1a_2 vertices spanning a triangle in G_x . Let A be a subset of $\Gamma(x, y)$ of size cn . Then A must be an independent set in G . Let B be a subset of $\Gamma(a_1, a_2)$ in $V \setminus \{x\}$ of size cn . Then B is disjoint from A and is an independent set in G_x — indeed if $b_1b_2 \in G_x$ for some $b_1, b_2 \in B$ then $a_1a_2|xb_1b_2$, a contradiction. We now show that B is an independent set in G . Indeed, for every $b \in B$, $\Gamma(b, x)$ is a subset of $V \setminus B$ of size at least cn . Consider an arbitrary triple $\{b_1, b_2, b_3\}$ of distinct vertices from B . Since

$$d(b_1, x) + d(b_2, x) + d(b_3, x) - 2(n - |B|) \geq (3c - 2(1 - c))n = (5c - 2)n > 0,$$

by the pigeon-hole principle there exists $a \in A \cup C$ with xab_1, xab_2, xab_3 all in $E(G)$. In particular $b_1b_2b_3 \notin E(G)$, as otherwise we would have $ax|b_1b_2b_3$. It follows that B must be an independent set in G . Thus A, B satisfy the two properties in Lemma 3.7, and thus $c \leq 3/7 + o(1)$.

On the other hand, suppose G_x was triangle-free. Let $y \in V \setminus \{x\}$, and let A be a subset of $\Gamma(x, y)$ of size cn . Since G_x is triangle-free and x is not covered by an $F_{3,2}$ -subgraph, A forms an independent set in both G_x and G . Let $a \in A$ be arbitrary, and let B be a subset of $\Gamma(a, x)$ of size cn . Then B is disjoint from A and independent in G_x (since G_x is triangle-free). Thus A, B satisfy the two properties in Lemma 3.7, and $c \leq 3/7 + o(1)$. Thus $c_2(F_{3,2}) \leq 3/7$ as claimed. \square

3.6 K_t , $t \geq 5$

Proposition 3.8. *For all $t \geq 5$, $c_2(K_t) \leq 1 - 1/\binom{t-1}{2}$.*

Proof. Applying Lemma 2.1 with $F = K_t$ and $r = \binom{t-1}{2}$, we get

$$c_2(n, K_t) \leq \left\lfloor \left(1 - \frac{1}{\binom{t-1}{2}}\right) n - \frac{2t-6}{t-2} \right\rfloor. \quad \square$$

We now derive a lower bound for the covering codegree density of K_t by using (small) lower-bound constructions for the codegree threshold of K_{t-1}

Proposition 3.9. *Suppose there exists a K_{t-1} -free 3-graph H on $[m]$ with minimum codegree δ . Then $c_2(K_t) \geq (\delta + 2)/m$.*

Proof. We build a 3-graph G on $n = Nm + 1$ vertices as follows. Set $V = [n]$ and set aside a special vertex x . Partition $V \setminus \{x\}$ into m sets V_1, \dots, V_m , each of size N . Set as the link graph of x all pairs of vertices from distinct parts. For every triple $ijk \in E(H)$, add to G all 3-edges of the form $V_i V_j V_k$. Finally, add all triples of $V \setminus \{x\}$ that contain at least two vertices from one part. The minimum codegree of G is

$$\delta_2(G) = (\delta + 2)N - 1 \geq \frac{\delta + 2}{m}n - 2.$$

Now consider a $(t-1)$ -set $S \subset V \setminus \{x\}$ that induce a t -clique in the link graph of x . By construction, these vertices must come from t different parts of $V \setminus \{x\}$. Since H is K_{t-1} -free, by our construction, some triple of S is absent from G . Thus $S \cup \{x\}$ does not induce a copy of K_t in G . Taking the limit as $n \rightarrow \infty$, the result follows. \square

Corollary 3.10. *Let $t \geq 4$.*

1. $c_2(K_t) \geq \frac{t-2}{t-1}$, and
2. $c_2(K_t) \geq \frac{2t-6}{2t-5}$ if $t \equiv 0, 1 \pmod{3}$.

Proof. For Part 1, we apply Proposition 3.9 with $H = K_{t-1}^-$ (thus $m = t - 1$ and $\delta = t - 4$) and obtain $c_2(K_t) \geq \frac{t-2}{t-1}$.

For Part 2, since $t \equiv 0, 1 \pmod{3}$, we have $2t - 5 \equiv 1, 3 \pmod{6}$, whence there exists a Steiner triple system \mathcal{S} on the vertex set $[2t - 5]$. It is easy to see that every set $T \subset [2t - 5]$ of $t - 1$ vertices spans at least one triple from \mathcal{S} . Indeed, fix a vertex $a \in T$: all the pairs of T containing a must have distinct neighbours under \mathcal{S} in $[2t - 5] \setminus T$. Since $t - 2 > (2t - 5) - (t - 1)$, this is impossible. Therefore the complement 3-graph $\overline{\mathcal{S}}$ is K_{t-1} -free. Applying Proposition 3.9 with $H = \overline{\mathcal{S}}$ (thus $m = 2t - 5$ and $\delta = 2t - 8$), we obtain that $c_2(K_t) \geq \frac{2t-6}{2t-5}$. \square

Remark 3.11. 1. Combining Proposition 3.8 and Corollary 3.10 gives that $\frac{3}{4} \leq c_2(K_5) \leq \frac{5}{6}$ and $\frac{6}{7} \leq c_2(K_6) \leq \frac{9}{10}$.

2. Theorem 1.1 shows that the lower bound in Proposition 3.10 is tight in the case $t = 4$. The bound is also tight in the trivial case $t = 3$, since $c_2(n, K_3) = 1 = o(n)$. If this bound is tight in general, then we do not have stability for the covering codegree-threshold problem: while the 3-edge K_3 and the Fano plane are the unique (up to isomorphism) Steiner triple systems on 3 and 7 vertices respectively, there are for example 11, 084, 874, 829 non-isomorphic Steiner triple systems on 19 vertices (see [1, Section 4.5]).

4 Concluding remarks

There are many questions arising from our work. To begin with, we may ask which of the fundamental properties of Turán density and codegree density does the covering codegree density c_2 share. Explicitly:

1. do we have *supersaturation*? That is, if $\delta_2(G) \geq c_2(n, F) + \varepsilon n$ for some fixed $\varepsilon > 0$, is it the case that every vertex in G is contained in $\Omega(n^{|V(F)|-1})$ copies of F ?

2. do we have *blow-up invariance*? Given a 3-graph F , we define the blow-up $F(t)$ to be the 3-graph on $V(F) \times [t]$ with 3-edges $\{(u, i)(v, j)(w, k) : uvw \in E(F), i, j, k \in [t]\}$. Is it the case that for every F and every fixed t we have $c_2(F) = c_2(F(t))$?
3. is the set of covering codegree densities $\{c_2(F) : F \text{ a 3-graph}\}$ dense in $[0, 1]$, or does it have jumps?

The first two of these questions are addressed in a forthcoming work of the authors. In addition there are some natural variants of the covering codegree threshold $c_2(n, F)$ which may be interesting. What if instead of covering every vertex by a copy of F we wanted to cover every pair? What if we wanted instead to be able to extend every 3-edge to a copy of F ? It is not immediately clear whether the corresponding codegree-extremal functions behave similarly to $c_2(n, F)$ or not.

In a different direction, what if we asked for the threshold for covering all but at most k vertices, for some $k \geq 1$? On the one hand, in the case of C_5 we observed in Remark 3.4 that this does not affect the value of the covering threshold very much. On the other hand, Theorem 2.9 implies that the threshold for covering all but at most 1 vertex with a copy of K_4 is at most $(2/3 - c)n$ for some $c > 0$ (therefore the problem is genuinely different from $c_2(n, K_4)$). Let us sketch a proof. Let G be a 3-graph with $\delta_2(G) \geq (2/3 - c)n$ for some $c > 0$ sufficiently small. Suppose that $x \in V(G)$ is not covered by any copy of K_4 . By Theorem 2.9, there is a partition $V_1 \sqcup V_2 \sqcup V_3$ of $V(G) \setminus \{x\}$ satisfying (i)–(v). If another vertex y is not covered by any copy of K_4 , then there is a partition $V'_1 \sqcup V'_2 \sqcup V'_3$ of $V(G) \setminus \{y\}$ satisfying (i)–(v) as well. Because of (iii) and (iv), these two partitions essentially coincide. Now consider $\Gamma(x, y)$, which has size at least $(2/3 - c)n$. There are about $(2/3 - c)(1/3 - 3c)n^2/2$ pairs $u, v \in \Gamma(x, y)$ coming from different parts of $V_1 \cap V'_1, V_2 \cap V'_2, V_3 \cap V'_3$. Since $(2/3 - c)(1/3 - 3c)n^2/2 > 2 \cdot 10cn^2$ (for c sufficiently small), by (ii), there exists a pair $u, v \in \Gamma(x, y)$ such that both uvx and uvy are edges of G . This implies that $\{u, v, x, y\}$ spans a copy of K_4 , a contradiction. The authors note that the bound on c given by this argument can be significantly improved; this is the subject of future work.

Finally, it would be interesting to determine the value of $c_2(F)$ when F is the Fano plane or $F_{3,2}$, and to have if not a tight result then at least a reasonable guess as to the value of $c_2(K_t)$ for $t \geq 5$. An investigation of $c_1(n, F)$ when $F = K_4^-$ and $F = K_4$ would also be desirable.

We should note here that for such small 3-graphs F the problem of proving upper bounds for c_1 or c_2 should be amenable to flag algebra computations by following the approach of [7] to encode the minimum degree/codegree constraint. Note however that one will need to do computation with non-uniform hypergraphs, containing a mixture of 2-edges (from the link graph of an uncovered vertex x) and 3-edges.

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6 Appendix: proof of Theorem 2.9 and Theorem 2.10

Proof of Theorem 2.9. We run through the proof of the case $n = 3m + 2$ of Theorem 2.9, replacing the codegree assumption $\delta_2(G) = 2m - 2$ by the assumption $\delta_2(G) \geq \frac{2n}{3} - \delta n$. This gives us new versions of our claims and lemmas with error terms involving δn , and conditions on n being sufficiently large replaced by conditions on δ being sufficiently small. For the sake of completeness, we derive them below.

Let G be a 3-graph on n vertices with $\delta_2(G) \geq (\frac{2}{3} - \delta)n$, for some δ : $0 < \delta \leq 1/429$. Suppose there is a vertex x of G not contained in any copy of K_4 . As $2/3 - \delta > 1/2$, the link graph G_x contains a triangle $\{ab, bc, ac\}$. Set $S = \{a, b, c, x\}$ and for each $y \in V \setminus S$, define S_y as in Lemma 2.3. By Lemma 2.3, S_y is a subset of $S^{1,c}, S^{1,b}, S^{1,a}, S^{2,a}, S^{2,b}, S^{2,c}$ or S^3 . For $i \in \{1, 2\}$ and $j \in \{a, b, c\}$, write $s_{i,j}$ for the number of vertices $y \in V \setminus S$ for which $S_y = S^{i,j}$, and write s_i for the sum $s_{i,a} + s_{i,b} + s_{i,c}$. Finally let s_0 be the number of vertices $y \in V \setminus S$ such that $S_y \neq S^{i,j}$ for any $i \in \{1, 2\}$ and $j \in \{a, b, c\}$. Note that $|S_y| \leq 3$ for such y . We know that $s_1 + s_2 + s_0 = n - 4$. Furthermore, by the codegree assumption,

$$3 \left(\frac{2n}{3} - \delta n \right) \leq d(a, x) + d(b, x) + d(c, x) \leq 2s_1 + s_2 + 3s_0 + 6, \quad (5)$$

$$6 \left(\frac{2n}{3} - \delta n \right) \leq d(a, x) + d(b, x) + d(c, x) + d(a, b) + d(b, c) + d(c, a) \leq 4s_1 + 4s_2 + 3s_0 + 9, \quad (6)$$

Substituting $s_0 = n - 4 - s_1 - s_2$ into (5) and (6) yields that $s_1 + 2s_2 \leq n + 3\delta n - 6$ and $s_1 + s_2 \geq n - 6\delta n + 3$, respectively. Combining the two inequalities we have just obtained, we get

$$s_2 \leq 9\delta n - 9 \quad \text{and} \quad s_1 \geq n - 15\delta n + 12.$$

We now show as before that the weight of s_1 splits almost equally between $s_{1,a}$, $s_{1,b}$, $s_{1,c}$. Note that

$$\frac{2n}{3} - \delta n \leq d(b, c) \leq n - 3 - s_{1,a},$$

from which it follows that $s_{1,a} \leq \frac{n}{3} + \delta n - 3$. Similarly we derive that $s_{1,b}, s_{1,c} \leq \frac{n}{3} + \delta n - 3$. Consequently

$$s_{1,a} = s_1 - s_{1,b} - s_{1,c} \geq n - 15\delta n + 12 - 2\left(\frac{n}{3} + \delta n - 3\right) = \frac{n}{3} - 17\delta n + 18.$$

Similarly $s_{1,b}$ and $s_{1,c}$ satisfy the same lower bound. Set $A = \{y \in V \setminus S : S_y = S^{1,a}\} \cup \{a\}$, $B = \{y \in V \setminus S : S_y = S^{1,b}\} \cup \{b\}$ and $C = \{y \in V \setminus S : S_y = S^{1,c}\} \cup \{c\}$. Set $V' = A \cup B \cup C \cup \{x\}$. The calculations above have established the following lemma.

Lemma 6.1 (New version of Lemma 2.4).

$$|V'| \geq n - 15\delta n + 16, \quad \text{and} \quad \frac{n}{3} - 17\delta n + 19 \leq |A|, |B|, |C| \leq \frac{n}{3} + \delta n - 2. \quad \square$$

Let \mathcal{B} be the collection of 3-edges of G of the form xAA, xBB, xCC (the ‘bad’ triples). Let \mathcal{M} be the collection of non-edges of G of the form xAB, xAC, xBC (the ‘missing’ triples). Viewing \mathcal{B} and \mathcal{M} as 3-graphs on V' , for two distinct vertices $v_1, v_2 \in V'$, we let $d_{\mathcal{B}}(v_1, v_2)$ denote their codegree in \mathcal{B} and $d_{\mathcal{M}}(v_1, v_2)$ their codegree in \mathcal{M} .

Claim 6.2 (New version of Claim 2.5). *For every $v \in V' \setminus \{x\}$, $d_{\mathcal{B}}(v, x) \leq 18\delta n - 20$.*

Proof. Suppose without loss of generality that $v \in A$. If $v = a$, then $d_{\mathcal{B}}(v, x) = 0$ because G contains no 3-edges of the form xaA . We thus assume that $v \neq a$. The bad triples for the pair (v, x) are triples of the form $a'vx$ for $a' \in A \setminus \{a, v\}$. Suppose $a'vx \in \mathcal{B}$. Then since there is no K_4 in G containing x , and since, by the definition of A , $a'bx, vbx, a'cx$ and vcx are all in G , it must be the case that both of $a'vb$ and $a'vc$ are missing from G . Further if $c' \in C \cap \Gamma(v, x)$ then all of $c'vx, bvx, c'bx$ are in G , whence $bc'v$ is absent from G . Similarly for any $b' \in B$, at most one of $b'cv, b'xv$ is in G . Finally since $bcv \notin E(G)$, b and c are contained in exactly one of $\Gamma(b, v)$, $\Gamma(c, v)$, and $\Gamma(x, v)$. To summarize, a vertex y in V' can lie in at most two of $\Gamma(b, v)$, $\Gamma(c, v)$ and $\Gamma(x, v)$ unless y is in $\Gamma_{\mathcal{B}}(x, v)$ (and lies in exactly one of those joint neighbourhoods) or is in $\{b, c, v\}$ (and lies in at most one of those joint neighbourhoods). Together with our codegree assumption, this gives us

$$\begin{aligned} 3\left(\frac{2n}{3} - \delta n\right) &\leq d(b, v) + d(c, v) + d(x, v) \leq 2|V'| - d_{\mathcal{B}}(v, x) - 4 + 3(n - |V'|) \\ &= 3n - |V'| - 4 - d_{\mathcal{B}}(v, x) \leq 2n + 15\delta n - 20 - d_{\mathcal{B}}(v, x), \end{aligned}$$

where we apply $|V'| \geq n - 15\delta n + 16$ from Lemma 6.1 in the last inequality. It follows that $d_{\mathcal{B}}(v, x) \leq 18\delta n - 20$, as claimed. \square

Claim 6.3 (New version of claim 2.6). *For every $v \in V' \setminus \{x\}$, $d_{\mathcal{M}}(v, x) \leq 36\delta n - 40$.*

Proof. Suppose without loss of generality that $v \in A$. Then by the codegree assumption, Claim 6.2 and the bound on $|A|$ from Lemma 6.1 we have

$$\begin{aligned} \frac{2n}{3} - \delta n &\leq d(v, x) \leq n - 1 - |A| + d_{\mathcal{B}}(v, x) - d_{\mathcal{M}}(v, x) \\ &\leq n - 1 - \frac{n}{3} + 17\delta n - 19 + 18\delta n - 20 - d_{\mathcal{M}}(v, x), \end{aligned}$$

which gives that $d_{\mathcal{M}}(v, x) \leq 36\delta n - 40$ as claimed. \square

Claim 6.4 (New version of Claim 2.7). *Provided $\delta \leq \frac{1}{429}$, for every $y \in V(G) \setminus \{x\}$, $\Gamma(y, x)$ has a non-empty intersection with at most two of the parts A , B and C .*

Proof. Let $y \in V(G) \setminus \{x\}$. Set $A_y = A \cap \Gamma(x, y)$, $B_y = B \cap \Gamma(x, y)$ and $C_y = C \cap \Gamma(x, y)$. Suppose none of A_y , B_y , C_y is empty. Fix $a' \in A_y$. For $b' \in B_y$, if $b' \in \Gamma(a', x)$, then $a'b'y \notin E(G)$ – otherwise $\{a', b', x, y\}$ spans a copy of K_4 . Similarly, for $c' \in C_y \cap \Gamma(a', x)$, we have $a'c'y \notin E(G)$. Hence,

$$\frac{2n}{3} - \delta n \leq d(a', y) \leq n - 2 - |B_y \cap \Gamma(a', x)| - |C_y \cap \Gamma(a', x)|.$$

Claim 6.3 gives that $d_{\mathcal{M}}(a', x) \leq 36\delta n - 40$. Consequently,

$$|B_y \cap \Gamma(a', x)| + |C_y \cap \Gamma(a', x)| = |B_y| + |C_y| - d_{\mathcal{M}}(a', x) \geq |B_y| + |C_y| - 36\delta n + 40$$

This implies that

$$\frac{2n}{3} - \delta n \leq n - 2 - |B_y| - |C_y| + 36\delta n - 40,$$

which yields $|B_y| + |C_y| \leq \frac{n}{3} + 37\delta n - 42$. Similarly by considering any vertex $b' \in B_y$ and any vertex $c' \in C_y$ we obtain that

$$|A_y| + |C_y| \leq \frac{n}{3} + 37\delta n - 42 \quad \text{and} \quad |A_y| + |B_y| \leq \frac{n}{3} + 37\delta n - 42.$$

Summing these three inequalities and dividing by 2, we obtain that

$$|A_y| + |B_y| + |C_y| \leq \frac{n + 111\delta n - 126}{2}.$$

Furthermore, by the codegree condition,

$$\frac{2n}{3} - \delta n \leq d(x, y) \leq |A_y| + |B_y| + |C_y| + (n - |V'|) \leq \frac{n + 141\delta n - 158}{2},$$

where we apply $|V'| \geq n - 15\delta n + 16$ from Lemma 6.1. Rearranging terms yields $\frac{(1-429\delta)n}{6} \leq -\frac{158}{2}$, which is a contradiction as $\delta \leq 1/429$. \square

Set $V_1 = \{y \in V \setminus \{x\} : \Gamma(x, y) \cap A = \emptyset\}$, $V_2 = \{y \in V \setminus \{x\} : \Gamma(x, y) \cap B = \emptyset\}$ and $V_3 = \{y \in V \setminus \{x\} : \Gamma(x, y) \cap C = \emptyset\}$. Without loss of generality, assume that $|V_1| \leq |V_2| \leq |V_3|$. Claim 6.4 shows that $V_1 \cup V_2 \cup V_3$ covers $V(G) \setminus \{x\}$. We now show that in fact V_1, V_2, V_3 are

pairwise disjoint, and $A \subseteq V_1$, $B \subseteq V_2$, and $C \subseteq V_3$. Suppose instead that there exists $y \in V_1 \cap V_2$. Then $\Gamma(x, y) \cap (A \cup B) = \emptyset$. By the codegree condition and Lemma 6.1,

$$\frac{2n}{3} - \delta n \leq d(x, y) \leq |C_y| + (n - |V'|) \leq \frac{n}{3} + \delta n - 2 + 15\delta n - 16 = \frac{n}{3} + 16\delta n - 18.$$

Rearranging terms yields $\frac{(1-51\delta)n}{3} \leq -18$, which for $\delta \leq 1/51$ is a contradiction.

Furthermore, consider $a' \in A$. By Claim 6.3, $a'xv \in E(G)$ for all but at most $36\delta n - 40$ vertices $v \in B \cup C$. By Lemma 6.1,

$$|B| - 36\delta + 40 \geq \frac{(1 - 159\delta)n}{3} + 59$$

which is strictly positive when $\delta \leq 1/159$. Thus we have that $\Gamma(a', x)$ has a non-empty intersection with B ; similarly we have that $\Gamma(a', x) \cap C \neq \emptyset$, from which we can finally deduce by Claim 6.4 that $\Gamma(a', x) \cap A = \emptyset$ and that $A \subseteq V_1$. Similarly we have $B \subseteq V_2$ and $C \subseteq V_3$.

We claim that

$$\forall i \in \{1, 2, 3\}, \quad |V_i| \leq n/3 + \delta n - 1 \quad (7)$$

Indeed, let $c' \in C$. By the definition of V_3 , we have $\Gamma(c', x) \subseteq V_1 \cup V_2$. By the codegree assumption, it follows that

$$\frac{2n}{3} - \delta n \leq d(c', x) \leq |V_1| + |V_2| = n - 1 - |V_3|,$$

from which we get that $|V_3| \leq n/3 + \delta n - 1$, as claimed. By (7), we have $|V_i| = n - 1 - \bigcup_{j \neq i} |V_j| \geq n/3 - 2\delta n + 1$ and consequently,

$$\frac{4}{3} - 2\delta n \leq |V_i| - \frac{n-1}{3} \leq \delta n - \frac{2}{3}.$$

This gives Part (v) of Theorem 2.9.

Claim 6.5 (New version of Claim 2.8). *Let $y \in V_i$. Then provided $\delta \leq \frac{1}{429}$, $\Gamma(y, x)$ contains all but at most $18\delta n - 18$ vertices from $\bigcup_{j \neq i} V_j$ and no vertex from V_i .*

Proof. Suppose without loss of generality that $y \in V_1$. Then by Claim 6.4, $A \cap \Gamma(y, x) = \emptyset$. Thus

$$\frac{2n}{3} - \delta n \leq d(x, y) \leq |\Gamma(x, y) \cap (V_2 \cup V_3)| + |\Gamma(x, y) \cap (V_1 \setminus A)| \leq |\Gamma(x, y) \cap (V_2 \cup V_3)| + 15\delta n - 16$$

since $|V_1 \setminus A| \leq n - |V'| \leq 15\delta n - 16$ by Lemma 6.1. Hence $|\Gamma(x, y) \cap (V_2 \cup V_3)| \geq \frac{2n}{3} - 16\delta n + 16$. By (7),

$$|(V_2 \cup V_3) \setminus \Gamma(x, y)| \leq 2 \left(\frac{n}{3} + \delta n - 1 \right) - \left(\frac{2n}{3} - 16\delta n + 16 \right) = 18\delta n - 18.$$

This establishes the first part of our claim.

For the second part of our claim (namely that $\Gamma(y, x) \cap V_1 = \emptyset$), suppose that $yy'x \in E(G)$ for some $y' \in V_1$. Then $\Gamma(y, y') \cap \Gamma(y, x) \cap \Gamma(y', x) = \emptyset$. Consequently,

$$\begin{aligned} \frac{2n}{3} - \delta n \leq d(y, y') &\leq 1 + |V_1| - 2 + |(V_2 \cup V_3) \setminus (\Gamma(y, x) \cap \Gamma(y', x))| \\ &\leq 1 + \frac{n}{3} + \delta n - 1 - 2 + 2(18\delta n - 18) \end{aligned}$$

where in the last inequality we applied (7) and the first part of the claim. This implies that $\frac{(1-114\delta)n}{3} \leq -38$, a contradiction when $\delta \leq 1/114$. \square

This establishes Part (i) of Theorem 2.9. By Claim 6.5, the total number of missing xV_iV_j edges, $i \neq j$ is at most

$$\frac{1}{2} \sum_i (18\delta n - 18)|V_i| = \frac{1}{2}(n-1)(18\delta n - 18) < 9\delta n^2,$$

establishing Part (ii) of the theorem. Since a triple $v_1v_2v_3$ with $v_i \in V_i$ is an edge of G only if one of the xv_iv_{i+1} triples is missing, by Part (ii) and Equation (7), there can be at most $9\delta n^2(n/3 + \delta n) < 4\delta n^3$ such triples in total, establishing Part (iii) of the Theorem.

Finally we need to bound the number of non-edges of G intersecting exactly two of V_1, V_2, V_3 . Fix $v_1 \in V_1$ and $v_2 \in V_2$. Given a set $S \subseteq V(G)$, let $d(v_1, v_2, S) = |\Gamma(v_1, v_2) \cap S|$ denote the number of neighbours of v_1 and v_2 in S and $\bar{d}(v_1, v_2, S) = |S \setminus \Gamma(v_1, v_2)|$. By the codegree condition, we have

$$d(v_1, v_2, V_1 \cup V_2) \geq \frac{2}{3}n - \delta n - 1 - d(v_1, v_2, V_3).$$

Together with (7), this implies that

$$\begin{aligned} \bar{d}(v_1, v_2, V_1 \cup V_2) &\leq |V_1| + |V_2| - 2 - \left(\frac{2}{3}n - \delta n - 1 - d(v_1, v_2, V_3) \right) \\ &\leq 2 \left(\frac{n}{3} + \delta n - 1 \right) - 2 - \frac{2}{3}n + \delta n + 1 + d(v_1, v_2, V_3) \\ &= 3\delta n - 3 + d(v_1, v_2, V_3), \end{aligned}$$

The number of non-edges of G in the form of $V_1V_1V_2$ or $V_1V_2V_2$ is thus

$$\frac{1}{2} \sum_{v_1 \in V_1, v_2 \in V_2} \bar{d}(v_1, v_2, V_1 \cup V_2) \leq \frac{1}{2} \left(|V_1||V_2|(3\delta n - 3) + e(V_1, V_2, V_3) \right)$$

where $e(V_1, V_2, V_3)$ denotes the number of tripartite edges. We know that $e(V_1, V_2, V_3) \leq 9\delta n^2(n/3 + \delta n)$. Thus the number of non-edges of G intersecting exactly two of V_1, V_2, V_3 is at most

$$\begin{aligned} \frac{1}{2} \sum_i |V_i||V_{i+1}| (3\delta n - 3) + \frac{3}{2}e(V_1, V_2, V_3) &\leq \frac{n-1}{2} \left(\frac{n}{3} + \delta n - 1 \right) (3\delta n - 3) + \frac{9}{2}\delta n^3(1 + 3\delta) \\ &< \frac{n}{2} \left(\frac{n}{3} + \delta n \right) 3\delta n + \frac{9}{2}\delta n^3(1 + 3\delta) < 6\delta n^3, \end{aligned}$$

where we applied (7) in the second inequality. This establishes Part (iv) of the Theorem. \square

Proof of Theorem 2.10. Case 1: $n = 3m \geq 858$. Let G be a 3-graph on $n = 3m$ vertices with $\delta_2(G) = 2m - 2 = \left(\frac{2}{3} - \frac{2}{n}\right)n$. Suppose $x \in V(G)$ is not covered by any copy of K_4 . Since $\delta = \frac{2}{n} \leq \frac{1}{429}$, we can apply Theorem 2.9 to obtain a tripartition $V_1 \sqcup V_2 \sqcup V_3 = V(G) \setminus \{x\}$ satisfying conditions (i)–(v) from Theorem 2.9. Assume without loss of generality that $|V_1| \leq |V_2| \leq |V_3|$. For any vertex $v_3 \in V_3$, we have (by condition (i))

$$2m - 2 \leq d(x, v_3) \leq |V_1| + |V_2| = 3m - 1 - |V_3|,$$

from which it follows that $|V_3| \leq m + 1$.

Suppose $|V_3| = m + 1$. The condition (i) tells us that all triples of the form xvv_3 with $v \in V_1 \sqcup V_2$ and $v_3 \in V_3$ must be in G , for otherwise $d(x, v_3) < 2m - 2 = \delta_2(G)$. Now consider any pair of vertices $v_1 \in V_1$, $v_2 \in V_2$ for which xv_1v_2 is in G (such pairs must exist by condition (ii), say). For every $v_3 \in V_3$, both of xv_1v_3 and xv_2v_3 are in G , whence the tripartite triple $v_1v_2v_3$ must be absent from G (since otherwise $xv_1v_2v_3$ would induce a copy of K_4 in G). Thus the codegree of v_1, v_2 is at most $|V_1| + |V_2| - 1 = 2m - 3$, a contradiction.

Thus $|V_3| = m$, whence $|V_2| = m$ also and $|V_1| = m - 1$. Now, by condition (i), for every $v_3 \in V_3$ at most one triple vv_3x with $v \in V_1 \sqcup V_2$ can be missing in G , as otherwise $d(v_3, x) < 2m - 2$; similarly for every $v_2 \in V_2$ at most one triple vv_2x with $v \in V_1 \sqcup V_3$ is missing, and for every $v_1 \in V_1$ at most 2 triples vv_1x with $v \in V_2 \sqcup V_3$ are missing. Further a tripartite triple $v_1v_2v_3$ can be included in G only if one of the triples xv_1v_2 , xv_2v_3 , xv_1v_3 is missing from G . This shows that G must be (isomorphic to) a subgraph of $F_1(\mathcal{E}, 3m)$ for some admissible collection of pairs \mathcal{E} .

Case 2: $n = 3m + 1 \geq 715$. Let G be a 3-graph on $n = 3m + 1$ vertices with minimum codegree $\delta_2(G) = 2m - 1 = (\frac{2}{3} - \frac{5}{3n})n$. Suppose $x \in V(G)$ is not covered by any copy of K_4 . Since $\delta = \frac{5}{3n} \leq \frac{1}{429}$, we can apply Theorem 2.9 to obtain a tripartition of $V(G) \setminus \{x\} = V_1 \sqcup V_2 \sqcup V_3$ satisfying conditions (i)–(v) from Theorem 2.9. Assume without loss of generality that $|V_1| \leq |V_2| \leq |V_3|$.

For any vertex $v_3 \in V_3$, we have (by condition (i))

$$2m - 1 \leq d(x, v_3) \leq |V_1| + |V_2| = 3m - |V_3|,$$

from which it follows that $|V_3| \leq m + 1$. If $|V_3| = m + 1$, then by the codegree assumption and condition (i) all triples of the form xvv_3 with $v_3 \in V_3$ and $v \in V_1 \sqcup V_2$ are in $E(G)$. Now consider vertices $v_1 \in V_1$ and $v_2 \in V_2$ for which $xv_1v_2 \in E(G)$ (which must exist by condition (ii), say). The tripartite triple $v_1v_2v_3$ does not lie in $E(G)$ for any $v_3 \in V_3$, since otherwise $xv_1v_2v_3$ would induce a copy of K_4 . Thus $d(v_1, v_2) \leq |V_1| + |V_2| - 1 = 2m - 2 < 2m - 1$, a contradiction. We must thus have $|V_1| = |V_2| = |V_3| = m$.

Now by condition (i) and the codegree assumption, for every vertex $v_i \in V_i$ all but at most 1 of the triples xvv_i with $v \in V(G) \setminus (V_i \cup \{x\})$ must be in $E(G)$. Furthermore a tripartite triple $v_1v_2v_3$ can belong to G only if one of the triples xv_1v_2 , xv_2v_3 , xv_3v_1 is absent from G . This shows that G is (isomorphic to) a subgraph of $F_1(\mathcal{E}, 3m + 1)$ for some admissible collection of pairs \mathcal{E} .

Case 3: $n = 3m + 2 \geq 1001$.

Let G be a 3-graph on $n = 3m + 2$ vertices with minimum codegree $\delta_2(G) = 2m - 1 = (\frac{2}{3} - \frac{7}{3n})n$. Suppose $x \in V(G)$ is not covered by any copy of K_4 . Since $\delta = \frac{7}{3n} \leq \frac{1}{429}$, we find a tripartition of $V(G) \setminus \{x\} = V_1 \sqcup V_2 \sqcup V_3$ satisfying conditions (i)–(v) from Theorem 2.9. Assume without loss of generality that $|V_1| \leq |V_2| \leq |V_3|$.

For any vertex $v_3 \in V_3$, by condition (i), we have

$$2m - 1 \leq d(x, v_3) \leq |V_1| + |V_2| = 3m + 1 - |V_3|,$$

from which it follows that $|V_3| \leq m + 2$. If $|V_3| = m + 2$, then by the codegree assumption and condition (i) all triples of the form xvv_3 with $v_3 \in V_3$ and $v \in V_1 \sqcup V_2$ are in $E(G)$. Now consider vertices $v_1 \in V_1$ and $v_2 \in V_2$ for which $xv_1v_2 \in E(G)$ (which must exist by condition (ii), say). The tripartite triple $v_1v_2v_3$ does not lie in $E(G)$ for any $v_3 \in V_3$, since otherwise $xv_1v_2v_3$ would induce a copy of K_4 . Thus $d(v_1, v_2) \leq |V_1| + |V_2| - 1 = 2m - 2 < 2m - 1$, a contradiction. We must thus have $|V_3| \leq m + 1$. Our assumption that $|V_1| \leq |V_2| \leq |V_3|$ then implies that $|V_3| \leq m + 1$ and that $|V_2| \in \{m, m + 1\}$. We have two subcases to consider.

Case 3a: $|V_1| = |V_2| = m$. Condition (i) and the codegree assumption together imply that for every $v \in V_i$, all but at most 2 of the triples of the form $xv(V \setminus (V_i \cup \{x\}))$ must be in $E(G)$ if $i \in \{1, 2\}$, and all but at most 1 if $i = 3$. Further a tripartite triple $v_1v_2v_3$ can be in $E(G)$ only if one of xv_1v_2 , xv_2v_3 , xv_3v_1 is absent from $E(G)$. This shows that G must be (isomorphic to) a subgraph of $F_1(\mathcal{E}, 3m)$ for some admissible collection of pairs \mathcal{E} .

Case 3b: $|V_1| = m - 1$, $|V_2| = m + 1$. Condition (i) and the codegree assumption together imply that for every $v \in V_i$, all but at most 1 of the triples of the form $xv(V \setminus (V_i \cup \{x\}))$ must be in $E(G)$ if $i \in \{2, 3\}$, and all but at most 3 if $i = 1$. Further a tripartite triple $v_1v_2v_3$ is in $E(G)$ only if one of xv_1v_2 , xv_2v_3 , xv_3v_1 is absent from $E(G)$. This shows that G must be (isomorphic to) a subgraph of $F'_1(\mathcal{E}, 3m)$ for some admissible collection of pairs \mathcal{E} . \square