# The stripping process can be slow: part II 

Pu Gao*<br>University of Waterloo<br>p3gao@uwaterloo.ca


#### Abstract

This paper is a continuation of the previous results on the stripping number of a random uniform hypergraph, and the maximum depth over all non- $k$-core vertices. The previous results focus on the supercritical case, whereas this work analyses these parameters in the subcritical regime and inside the critical window.


## 1 Introduction

Given a hypergraph $H$ and a positive integer $k$, the parallel $k$-stripping process on $H$ is the sequence $H_{0}, H_{1}, H_{2}, \ldots$ such that $H_{0}=H$ and for every $i \geq 1, H_{i}$ is obtained form $H_{i-1}$ by removing all vertices with degree less than $k$ in $H_{i-1}$ together with their incident hyperedges. The process terminates with the $k$-core of $H$ : the maximum subgraph of $H$ with minimum degree at least $k$. Note that the $k$-core of $H$ can be empty. Let $s_{k}(H)$ denote the number of iterations this process takes and we call $s_{k}(H)$ the $k$-stripping number of $H$. As $k$ is fixed in this paper, we often drop $k$ from the above notation.

We will study $s\left(\mathcal{H}_{r}(n, m)\right.$ ), where $\mathcal{H}_{r}(n, m)$ is a uniformly random hypergraph on $n$ vertices and $m$ hyperedges, each of size $r$. The only interesting range of $m$ for this study is $m=\Theta(n)$. For $m$ in other ranges the stripping number can be easily estimated with little effort. We write $m=c n$ throughout this paper, where $c$ is bounded from both above and below by some absolute positive constants. Another closely related random hypergraph model is $\mathcal{H}_{r}(n, p)$ where each hyperedge in $\binom{[n]}{r}$ appears independently with probability $p$. By conditioning on the number of hyperedges in $\mathcal{H}_{r}(n, p)$, properties holding asymptotically almost surely (a.a.s.) in $\mathcal{H}_{r}(n, c n)$ usually translate immediately to $\mathcal{H}_{r}\left(n, r!c / n^{r-1}\right)$.

The stripping number of $\mathcal{H}_{r}(n, c n)$ is known to be small if $c$ is a constant and is not equal to $c_{r, k}$, the $k$-core emergence threshold (given in (1)). It was proved [1] that the stripping number is $O(\log n)$ in this case. For $c>c_{r, k}+\epsilon$, this bound is proved to be tight [14]. For $c<c_{r, k}-\epsilon$, an improved upper bound $O(\log \log n)$ is given in [14, 8]. However, the stripping process can get very slow as $c \rightarrow c_{r, k}$. It was shown [9] that if $c=c_{r, k}+n^{-\delta}(0<\delta<1 / 2)$ then the stripping number becomes $\Theta\left(n^{\delta / 2} \log n\right)$, whereas if $c=c_{r, k}-n^{-\delta}$ then the stripping number is bounded by $\Omega\left(n^{\delta / 2}\right)$ from below. The formal statement is as follows.

[^0]Theorem $1(9)$. Let $r, k \geq 2,(r, k) \neq(2,2)$ be fixed. For any arbitrarily small $\epsilon>0$ :
(a) if $c \geq c_{r, k}+n^{-1 / 2+\epsilon}$, then a.a.s. $s\left(\mathcal{H}_{r}(n, c n)\right)=\Theta(\log n / \sqrt{\xi})$, where $\xi=\left|c-c_{r, k}\right|$.
(b) if $\left|c-c_{r, k}\right| \leq n^{-1 / 2+\epsilon}$, then a.a.s. $s\left(\mathcal{H}_{r}(n, c n)\right)=\Omega\left(n^{1 / 4-\epsilon / 2}\right)$.
(c) if $c \leq c_{r, k}-n^{-1 / 2+\epsilon}$, then a.a.s. $s\left(\mathcal{H}_{r}(n, c n)\right)=\Omega(1 / \sqrt{\xi})$, where $\xi=\left|c-c_{r, k}\right|$.

One of the main contributions in this paper is to prove that the lower bound in Theorem $\mathbb{1}(\mathrm{c})$ is almost tight. We will also provide an upper bound for $s\left(\mathcal{H}_{r}(n, c n)\right)$ when $c$ is inside the critical window $\left|c-c_{r, k}\right| \leq n^{-1 / 2+\epsilon}$.

A $k$-stripping sequence $v_{1}, v_{2}, \ldots$ is a sequence of vertices, which can be deleted from the hypergraph in the order of the sequence, such that each vertex has degree less than $k$ at the moment of its removal. If $v$ is a vertex not contained in the $k$-core, then the depth of $v$ is the minimum integer $i$ such that there is a stripping sequence with $v_{i}=v$. In other words, the depth of $v$ is the minimum number of steps required to remove $v$ from $H$ among all stripping sequences.

For constant $c \neq c_{r, k}$, it is proved [1] that the maximum depth of the non- $k$-core vertices in $\mathcal{H}_{r}\left(n, p=r!c / n^{r-1}\right)$ is bounded by $O(\log n)$ and the same result easily translate to $\mathcal{H}_{r}(n, c n)$. For $c=c_{r, k}+n^{-\delta}$, the maximum depth is raised to $n^{\Theta(\delta)}$, proved by Molloy and the author [9], as follows.

Theorem $2(9])$. Assume $r, k \geq 2,(r, k) \neq(2,2)$ are fixed. There are constants $a=a(r, k)$ and $b=b(r, k)$ such that for any $0<\xi_{n}<1 / \log ^{7} n$ and $c=c_{r, k}+\xi_{n}$, a.a.s. the maximum depth of all non- $k$-core vertices of $\mathcal{H}_{r}(n, c n)$ is between $\xi_{n}^{-a}$ and $\xi_{n}^{-b}$.

In this paper, we will prove that the same statement holds for $c=c_{r, k}-\xi_{n}$.
Before proceeding to the statements of our main results, we briefly discuss the motivation of studying these two parameters, which has been addressed in [9]. Several applications of the parallel stripping process was given in [14, such as parity-check codes and hash-based sketches. However, a major motivation for studying the particular critical case where $c \rightarrow c_{r, k}$ is to investigate the solution clustering in random XORSAT. Research on the solution space of many random constraint satisfaction problems (CSPs) was started in statistical physics, and has received great attention in recent years in many broad areas such as physics, computer science, and combinatorics. Due to the discovery of the physicists, the solution space of a random CSP instance undergoes several phase transitions before its density reaches the satisfiability threshold. These phase transitions include clustering, variable freezing and condensation. See [10] for a brief introduction, and the references in [10] for the literature in this blossoming area. Understanding these phase transitions and the geometric properties of the solution space in each phase has been crucial in several recent achievements in the study of random CSPs, including solving the famous $k$-SAT conjecture (for large $k$ ) [7]. Clustering of random $r$-XORSAT was analysed independently in [3] and [17], with some key arguments missing. The rigorous arguments determining the clustering threshold for random $r$-XORSAT were given independently in [1, 11]. Random $r$-XORSAT clustering coincides with the appearance of a non-empty 2-core of a random $r$-uniform hypergraph $(r \geq 3)$. To bound the connectivity parameter of each cluster, the key arguments in [1] are to link this parameter to the maximum depth of all non-2-core vertices. For constant $c \neq c_{r, 2}$, this
parameter is bounded by $O(\log n)$ [1]. In order to charactersie how XORSAT-clusters are born, and how the cluster connectivity parameter transits around clustering, we need to estimate the maximum depth of the non-2-core vertices of $\mathcal{H}_{r}(n, c n)$ for $c \rightarrow c_{r, 2}$, especially for $c=c_{r, 2}+n^{-\delta}$ and $c=c_{r, 2}-n^{-\delta}$ for some sufficiently small $\delta>0$. The birth of XORSATclusters will be studied in a following paper, whereas a preliminary version has been available in 10] (for $c=c_{r, 2}+n^{-\delta}$ ).

All asymptotics in this paper refers to $n \rightarrow \infty$. For two sequences of real numbers $\left(f_{n}\right)$ and $\left(g_{n}\right)$, we say $f_{n}=O\left(g_{n}\right)$ if there is a constant $C>0$ such that $\left|f_{n}\right| \leq C\left|g_{n}\right|$ for every $n \geq 1$. We write $f_{n}=o\left(g_{n}\right)$ if $\lim _{n \rightarrow \infty} f_{n} / g_{n}=0 ; f_{n}=\Omega\left(g_{n}\right)$ if $f_{n}>0$ and $g_{n}=O\left(f_{n}\right)$. We use $f_{n}=\Theta\left(g_{n}\right)$ if $f_{n}>0, f_{n}=O\left(g_{n}\right)$ and $g_{n}=O\left(f_{n}\right)$.

## 2 Main results

The $k$-core emergence threshold was pursued by several authors [5, 15, 16] before its determination, and was first determined by Pittel, Spencer and Wormald [20] for random graphs $\mathcal{G}(n, m)$. This threshold was further determined in other random graph models and random hypergraphs [18, 13, 12, 6]. Recall that $c_{r, k}$ denotes the $k$-core emergence threshold of $\mathcal{H}_{r}(n, c n)$; then

$$
\begin{equation*}
c_{r, k}=\inf _{\mu>0} \frac{\mu}{r\left[e^{-\mu} \sum_{i=k-1}^{\infty} \mu^{i} / i!\right]^{r-1}} . \tag{1}
\end{equation*}
$$

Our main result of the stripping number of $\mathcal{H}_{r}(n, c n)$ is the following.
Theorem 3. Let $r, k \geq 2,(r, k) \neq(2,2)$ be fixed integers and $\epsilon>0$ be a constant.
(a) If $c \leq c_{r, k}-n^{-1 / 2+\epsilon}$, then a.a.s. $s\left(\mathcal{H}_{r}(n, c n)\right)=O\left(\xi^{-1 / 2} \log (1 / \xi)+\log \log n\right)$, where $\xi=\left|c_{r, k}-c\right|$.
(b) If $\left|c-c_{r, k}\right| \leq n^{-1 / 2+\epsilon}$, then a.a.s. $s\left(\mathcal{H}_{r}(n, c n)\right)=O\left(n^{3 / 4+\epsilon}\right)$.

Remark. Note that if $c_{r, k}-c$ is bounded below by a positive constant, then the upper bound becomes $O(\log \log n)$, which agrees with the upper bound in [14, 8]. If $c \leq c_{r, k}-n^{-1 / 2+\epsilon}$, then the upper bound differs from the lower bound (c.f. Theorem[(c)) by at most a constant factor of $\log n$. For $\xi=O\left(n^{-1 / 2+\epsilon}\right)$, the upper bound does not match the existing lower bound (c.f. Theorem $]_{1}(\mathrm{~b})$ ). The difficulty of obtaining tight bounds inside the critical window lies in the uncertainty of the existence of a non-empty $k$-core.

The next theorem bounds the maximum depth of the non- $k$-core vertices in $H_{r}(n, c n)$ for $c<c_{r, k}$.

Theorem 4. Let $r, k \geq 2,(r, k) \neq(2,2)$ be fixed integers. There exist two constants $a=a(k, r)>0$ and $b=b(k, r)>0$ such that for any $0<\xi_{n}<1 / \log ^{7} n$ and $c=c_{r, k}-\xi_{n}$, a.a.s. the maximum depth of the non-k-core vertices in $\mathcal{H}_{r}(n, c n)$ is between $\xi_{n}^{-a}$ and $\xi_{n}^{-b}$.

Remark. If we write $\xi_{n}=n^{-\delta}$, Theorem 4 states that the maximum depth of the non- $k$-core vertices is $n^{\Theta(\delta)}$. Since the depth of each non- $k$-core vertex is bounded trivially by $n$, the upper bounds in Theorems 4 and 2are non-trivial only for $\xi=n^{-\delta}$ where $\delta>0$ is sufficiently
small. Same as in [9], the condition $\xi_{n}<1 / \log ^{7} n$ can possibly be weakened and we did not try to optimise the power of the logarithm. This is because the most interesting applications of this theorem, e.g. XORSAT clustering, are for $c=c_{r, k}-n^{-\delta}$ with some small constant $\delta>0$.

Our analysis focuses mainly on iterations of the parallel stripping process where the number of vertices is very close to some critical value. To describe this, we start by defining

$$
\begin{aligned}
f_{t}(\lambda) & =e^{-\lambda} \sum_{i \geq t} \frac{\lambda^{i}}{i!} \\
h(\mu)=h_{r, k}(\mu) & =\frac{\mu}{f_{k}(\mu)^{r-1}} .
\end{aligned}
$$

Note that $f_{t}(\lambda)$ is the probability that a Poisson variable with mean $\lambda$ is at least $t$. Now for any $r, k \geq 2,(r, k) \neq(2,2)$, we define $\mu_{r, k}$ to be the value of $\mu$ that minimizes $h(\mu)$; i.e. the (unique) solution to:

$$
\begin{equation*}
c_{r, k}=h\left(\mu_{r, k}\right) / r \tag{2}
\end{equation*}
$$

Define

$$
\begin{align*}
\alpha=\alpha_{r, k} & =f_{k}\left(\mu_{r, k}\right)  \tag{3}\\
\beta=\beta_{r, k} & =\frac{1}{r} \mu_{r, k} f_{k-1}\left(\mu_{r, k}\right) \tag{4}
\end{align*}
$$

For ease of notation, we drop most of the $r, k$ subscripts. For any $c \geq c_{r, k}$, we define $\mu(c)$ to be the larger solution to

$$
c=h(\mu) / r
$$

Then, $\mu_{r, k}=\mu\left(c_{r, k}\right)$. Define

$$
\alpha(c)=f_{k}(\mu(c)), \quad \beta(c)=\frac{1}{r} \mu(c) f_{k-1}(\mu(c))
$$

Let $\mathcal{C}_{k}(H)$ denote the $k$-core of $H$. The following result on the $k$-core emergence threshold can be easily deduced from [13] (see the discussion above [9, Lemma 7]).

Theorem 5. Let $r, k \geq 2,(r, k) \neq(2,2)$ be fixed and $\epsilon>0$ be an arbitrary constant.
(a) If $c \leq c_{r, k}-n^{-1 / 2+\epsilon}$, then a.a.s. $\mathcal{C}_{k}\left(\mathcal{H}_{r}(n, c n)\right)$ is empty.
(b) If $c \geq c_{r, k}+n^{-1 / 2+\epsilon}$, then a.a.s. $\mathcal{C}_{k}\left(\mathcal{H}_{r}(n, c n)\right)$ has $\alpha(c) n+O\left(n^{3 / 4}\right)$ vertices and $\beta(c) n+$ $O\left(n^{3 / 4}\right)$ hyperedges.

When the parallel stripping process is applied to $\mathcal{H}_{r}(n, c n)$ where $c=c_{r, k}+o(1)$, the vertices are stripped off fast in the beginning; in each round there are a linear number of vertices being removed. This continues until the number of vertices gets close to $\alpha n$. Our analysis will start from there.

## 3 The allocation-partition model

It is not easy to analyse random processes such as the parallel stripping process when applied to $\mathcal{H}_{r}(n, m)$, due to the dependency between the hyperedges. Instead we consider the following alternative model, called the allocation-partition model (AP-model). Take rm points and uniformly at random (u.a.r.) allocate the points into a set of $n$ bins. Then, take a uniform partition of the $r m$ points so that each part has size exactly $r$. We call the resulting probability space $A P_{r}(n, m)$ and each element in $A P_{r}(n, m)$ a configuration. Each configuration in $A P_{r}(n, m)$ corresponds to a multi-hypergraph by representing each bin as a vertex and each part in the partition as a hyperedge. In this paper we call bins as vertices for simplicity. Each part in the partition is an $r$-tuple of points, which we may call a hyperedge when there is no confusion. The degree of a vertex $u$ in a configuration is the number of points that $u$ contains. Note that the AP-model is similar to the configuration model of Bollobás [2], except that in the configuration model, the degree sequence is specified initially whereas in the AP-model, the degree sequence is a random variable determined by the allocation of the points into the bins. A simple counting argument shows that each hypergraph in $\mathcal{H}_{r}(n, m)$ corresponds to the same number of configurations in $A P_{r}(n, m)$. Therefore, $A P_{r}(n, m)$ generates the hypergraphs in $\mathcal{H}_{r}(n, m)$ uniformly by conditioning on the resulting hypergraph being simple. For $m=O(n)$, following a result by Chvátal [5] that the probability that a configuration in $A P_{r}(n, m)$ corresponds to a simple hypergraph is bounded away from zero (the proof in [5] is for $r=2$ but easily extends to general $r \geq 2$ ), the following corollary allows one to translate a.a.s. properties of $A P_{r}(n, m)$ to $\mathcal{H}_{r}(n, m)$.

Corollary 1. If $m=O(n)$ and property $Q$ holds a.a.s. in $A P_{r}(n, m)$, then $Q$ holds a.a.s. in $\mathcal{H}_{r}(n, m)$.

Running the parallel stripping process on a configuration of the AP-model is a natural extension. In the rest of the paper, we will study $s(H)$ and the maximum depth of the non-$k$-core vertices of $H$ for $H \in A P_{r}(n, c n)$. We note here that Theorem 5 holds for $A P_{r}(n, c n)$ as well, as claimed in 4].

## 4 Proof of Theorem 3

We bound $s\left(A P_{r}(n, c n)\right)$ in this section. Without loss of generality, we assume $\xi:=$ $\left|c-c_{r, k}\right|=o(1)$ throughout the paper as the case $\xi=\Omega(1)$ has already been verified in previous works, e.g. [1, 8, 14]. Recall that $H_{0}, H_{1}, H_{2}, \ldots$ is the parallel stripping process with $H_{0} \in A P_{r}(n, c n)$. Let $S_{i}(i \geq 0)$ denote the set of light vertices (vertices with degree less than $k)$ in $H_{i}$, i.e. the set of vertices removed during the $(i+1)$-th iteration of the parallel stripping process. Thus, $S_{i}=V\left(H_{i}\right) \backslash V\left(H_{i+1}\right)$.

### 4.1 Proof outline

Rather than starting our analysis with $H_{0}$ in the parallel stripping process, we would start our analysis from some $H_{i}$ (or some configuration "close to" $H_{i}$ ), where $i$ is chosen so that
the number of vertices in $H_{i}$ is very close to $\alpha n$. The choice of $i$ relies on a coupling of two random configurations $A P_{r}(n, c n)$ and $A P_{r}\left(n, c^{\prime} n\right)$. Let $\xi^{\prime}=c^{\prime}-c_{r, k}$. Throughout Section 4 , we always choose $c^{\prime}$ satisfying the following conditions (recall that $\xi=\left|c-c_{r, k}\right|=o(1)$ is assumed):

$$
\begin{equation*}
\xi^{\prime}=O\left(c^{\prime}-c\right)=o(1), c^{\prime}-c=o\left(\sqrt{\xi^{\prime}}\right), c^{\prime} \geq c_{r, k}+n^{-1 / 2+\epsilon} \text { for some constant } \epsilon>0 \tag{5}
\end{equation*}
$$

We do not repeat this assumption in all statements of lemmas. We will describe the coupling in Section 4.2. The value of $c^{\prime}$ is chosen differently in the proofs of part (a) and part (b) of Theorem 3.

We use the coupling to start our analysis from some $H_{i}$, or more precisely some configuration $G_{0}$ (defined in Section 4.2) close to $H_{i}$, such that (a), the number of light vertices in $G_{0}$ is of order $n\left(c^{\prime}-c\right) ;(\mathrm{b}), i=O\left(\log \left(1 / \xi^{\prime}\right) / \sqrt{\xi^{\prime}}\right)$. In Section4.4 we analyse properties of $G_{0}$. In Section 4.8 we prove part (a). We will choose some $c^{\prime}>c_{r, k}$ such that $\xi$ and $\xi^{\prime}$ are of the same asymptotic order. Then we specify two iterations $I_{0}$ and $I_{1}$ in the parallel process such that, by iteration $I_{0},\left|S_{i}\right|$ decreases in each iteration; whereas starting from $I_{1},\left|S_{i}\right|$ increases in each iteration until reaching a linear size; then $\left|S_{i}\right|$ keeps of linear size until the number of remaining vertices in the configuration is at most $\sigma n$, for some small constant $\sigma>0$ ( $\sigma$ is specified in Section 4.7). We will analyse closely the critical iterations from $I_{0}$ to $I_{1}$ during which the growth rate of $\left|S_{i}\right|$ changes from negative to positive, and we will bound $I_{1}-I_{0}$ by $O(1 / \sqrt{\xi})$. Then, we will bound the growth rate of $\left|S_{i}\right|$ from below for each iteration after $I_{1}$, which allows us to bound the number of iterations needed until the number of remaining vertices is at most $\sigma n$. In Section 4.7, we show that it takes $O(\log \log n)$ steps to strip off all vertices when there are at most $\sigma n$ vertices left.

In Section 4.9, we prove part (b). We will choose $c^{\prime}=c_{r, k}+n^{-1 / 2+2 \epsilon}$ in this case. We prove that within $n^{3 / 4+\epsilon}$ iterations, either the parallel stripping process has terminated with a non-empty $k$-core; or $\left|S_{i}\right|$ becomes reasonably large and it will grow in each iteration until reaching a linear size. In the latter case, with a similar argument as for part (a), we can bound the number of remaining iterations until all vertices are removed.

### 4.2 Coupling

We will couple two random configurations $\left(H^{\prime}, H\right)$ as follows. Let $H^{\prime} \in A P_{r}\left(n, c^{\prime} n\right)$ where $c^{\prime}>c$ and $c^{\prime}>c_{r, k}+n^{-\delta^{\prime}}$ for some constant $0<\delta^{\prime}<1 / 2$ (this ensures that a.a.s. $A P_{r}\left(n, c^{\prime} n\right)$ has a non-empty $k$-core by Theorem 54). Generate a random configuration $H$ by uniformly at random removing $\left(c^{\prime}-c\right) n r$-tuples in $H^{\prime}$. This resulting $H$ has the distribution $A P_{r}(n, c n)$ and moreover $H \subseteq H^{\prime}$. Run the parallel stripping process on $H^{\prime}$ which yields a sequence $\left(H_{t}^{\prime}\right)_{t \geq 0}$ with $H_{0}^{\prime}=H^{\prime}$. Recall that $S_{i}^{\prime}=V\left(H_{i}^{\prime}\right) \backslash V\left(H_{i+1}^{\prime}\right)$. Let $B>0$ be a large constant to be specified later. Define

$$
\begin{equation*}
\tau^{\prime}(B)=\min \left\{t \geq B:\left|S_{t}^{\prime}\right| \leq n \xi^{\prime}\right\} . \tag{6}
\end{equation*}
$$

Now we define a coupled process $\left(\widehat{H}_{t}^{\prime}, \widehat{H}_{t}\right)_{t \geq 0}$ as follows. Let $\widehat{H}_{0}^{\prime}=H^{\prime}$ and $\widehat{H}_{0}=H$. For each $1 \leq t \leq \tau^{\prime}(B)$, define $\widehat{H}_{t}^{\prime}=H_{t}^{\prime}$ and $\widehat{H}_{t}$ to be the configuration obtained from $\widehat{H}_{t-1}$ by removing all vertices in $V\left(H_{t-1}^{\prime}\right) \backslash V\left(H_{t}^{\prime}\right)$ together with their incident $r$-tuples. Since
$\widehat{H}_{0} \subseteq \widehat{H}_{0}^{\prime}=H_{0}^{\prime}$, the set of vertices removed in each step of $\left(\widehat{H}_{t}\right)_{t=0}^{\tau^{\prime}(B)}$ has degree less than $k$. Hence, $\left(\widehat{H}_{t}\right)_{t=0}^{\tau^{\prime}(B)}$ can be viewed as a slowed-down version of the parallel stripping process on $\widehat{H}_{0}=H$. In other words, if we let $H^{\prime \prime}$ denote the configuration obtained by removing all light vertices in $\widehat{H}_{\tau^{\prime}(B)}$, then $H^{\prime \prime}$ must be a configuration $H_{i}$ that appears in the parallel stripping process $H_{0}, H_{1}, \ldots$ for some $i \leq \tau^{\prime}(B)+1$. Then,

$$
\begin{equation*}
s(H) \leq \tau^{\prime}(B)+1+s\left(H^{\prime \prime}\right) \leq \tau^{\prime}(B)+s\left(\widehat{H}_{\tau^{\prime}(B)}\right)+1 \tag{7}
\end{equation*}
$$

since $s\left(H^{\prime \prime}\right) \leq s\left(\widehat{H}_{\tau^{\prime}(B)}\right)$ as $H^{\prime \prime} \subseteq \widehat{H}_{\tau^{\prime}(B)}$ by definition. It only remains to specify $B$, and to bound $\tau^{\prime}(B)$ and $s\left(\widehat{H}_{\tau^{\prime}(B)}\right)$.
Specifying $B$ and bounding $\tau^{\prime}(B)$
The behaviour of $\left(\left|S_{i}^{\prime}\right|\right)_{i \geq 0}$ has been well studied in the prior paper 9] (recall that for $H^{\prime}$ we have $c^{\prime} \geq c_{r, k}+n^{-1 / 2+\epsilon}$, assumed in (5)). We cite here the relevant parts of [9, Lemma $49(\mathrm{a}-\mathrm{c})]$, which enables us to bound $\tau^{\prime}(B)$, and will be useful for later use.

Lemma 2. There exist positive constants $B, Y_{1}, Y_{2}$ and $Z_{1}$ dependent only on $r$ and $k$, such that a.a.s. for every $i \geq B$ with $\left|S_{i}^{\prime}\right| \geq\left(1 / \xi^{\prime}\right) \log ^{2} n$, we have
(a) if $\left|S_{i}^{\prime}\right| \geq n \xi^{\prime}$ then $\left(1-Y_{1} \sqrt{\left|S_{i}^{\prime}\right| / n}\right)\left|S_{i}^{\prime}\right| \leq\left|S_{i+1}^{\prime}\right| \leq\left(1-Y_{2} \sqrt{\left|S_{i}^{\prime}\right| / n}\right)\left|S_{i}^{\prime}\right|$;
(b) if $\left|S_{i}^{\prime}\right|<n \xi^{\prime}$ then $\left(1-Y_{1} \sqrt{\xi^{\prime}}\right)\left|S_{i}^{\prime}\right| \leq\left|S_{i+1}^{\prime}\right| \leq\left(1-Y_{2} \sqrt{\xi^{\prime}}\right)\left|S_{i}^{\prime}\right|$;
(c) $\sum_{j \geq i}\left|S_{j}^{\prime}\right| \leq\left(Z_{1} / \sqrt{\xi^{\prime}}\right)\left|S_{i}^{\prime}\right|$.

Let $B$ be a constant chosen to satisfy Lemma 2 this completes the definition of $\tau^{\prime}(B)$ in (6). Since $\left|S_{i}^{\prime}\right|>n \xi^{\prime}$ for all $B \leq i \leq \tau^{\prime}(B)-1$, recursively applying Lemma 2(a), we have

$$
n \xi^{\prime}<\left|S_{\tau^{\prime}(B)-1}^{\prime}\right| \leq\left(1-Y_{2} \sqrt{\xi^{\prime}}\right)^{\tau^{\prime}(B)-1-B}\left|S_{B}^{\prime}\right| \leq n \exp \left(-Y_{2} \sqrt{\xi^{\prime}}\left(\tau^{\prime}(B)-1-B\right)\right) .
$$

This immediately yields that

$$
\begin{equation*}
\tau^{\prime}(B)=O\left(\frac{\log \left(1 / \xi^{\prime}\right)}{\sqrt{\xi^{\prime}}}\right) \tag{8}
\end{equation*}
$$

By (7), It only remains to bound $s\left(\widehat{H}_{\tau^{\prime}(B)}\right)$. By Lemma 2( c ) with $i=\tau^{\prime}(B)$, it follows that

$$
\begin{equation*}
\left|H_{\tau^{\prime}(B)}^{\prime}\right|-\left|\mathcal{C}_{k}\left(H^{\prime}\right)\right|=O\left(n \xi^{\prime} / \sqrt{\xi^{\prime}}\right)=O\left(n \sqrt{\xi^{\prime}}\right) \tag{9}
\end{equation*}
$$

Generation of $\widehat{H}_{\tau^{\prime}(B)}$
Note that $H$ is generated from $H^{\prime}$ by removing u.a.r. $\left(c^{\prime}-c\right) n r$-tuples in $H^{\prime}$. Let $\mathcal{E}=E\left(H^{\prime}\right) \backslash E(H)$ and let $X$ denote the number of $r$-tuples in $\mathcal{E}$ that are in $\widehat{H}_{\tau^{\prime}(B)}^{\prime}=H_{\tau^{\prime}(B)}^{\prime}$. Then, these $X r$-tuples are uniformly distributed among all the $r$-tuples in $H_{\tau^{\prime}(B)}^{\prime}$. Hence, $\widehat{H}_{\tau^{\prime}(B)}$ can be generated by removing $X r$-tuples uniformly at random in $H_{\tau^{\prime}(B)}^{\prime}$.

We bound $X$ in the following lemma (the proof is deferred).
Lemma 3. A.a.s. $X=\Theta\left(\left(c^{\prime}-c\right) n\right)$.

To analyse $s\left(\widehat{H}_{\tau^{\prime}(B)}\right)$, it is more convenient to use a slowed-down version of the parallel stripping process, called SLOW-STRIP (which is also used in [9), defined as follows. It iteratively deletes an $r$-tuple incident with a light vertex. We will use a queue $\mathcal{Q}$ to store all light vertices.

## SLOW-STRIP

Input: A configuration $G$.
Initialize: $t:=0, G_{0}:=G, \mathscr{Q}$ is the set of light vertices in $G$.
While $\mathscr{Q} \neq \emptyset$ :
Let $v$ be the next vertex in $\mathscr{Q}$.
If $v$ contains no points then remove $v$ from $G$ and from $\mathscr{Q}$.
Otherwise:
Remove one point $x$ in $v$, and the $r-1$ points in the same part as $x$. If some other vertex $u$ becomes light then add $u$ to the end of $\mathscr{Q}$.
$G_{t+1}$ is the resulting configuration; $t:=t+1$.
Define

$$
\begin{equation*}
G_{0}=\widehat{H}_{\tau^{\prime}(B)} \tag{10}
\end{equation*}
$$

and $G_{t}$ to be the resulting configuration after $t$ steps of SLOW-STRIP applied to $G_{0}$. We use $L(G)$ to denote the total degree of the light vertices in $G$. For simplicity, let $L_{t}=L\left(G_{t}\right)$. We will analyse $\left(L_{t}\right)_{t \geq 0}$, starting with $L_{0}$.

Lemma 4. A.a.s. $L_{0}=\Theta\left(\left(c^{\prime}-c\right) n\right)$.
The proofs of Lemmas 3 and 4 are simple and are deferred to Section 4.4 where we will have a close study of $G_{0}$. To close this subsection, we discuss several key parameters that will be used throughout the proof. Let $N(G)$ and $D(G)$ denote the number of heavy vertices and the total degree of the heavy vertices of a configuration $G$. Define

$$
\begin{equation*}
\zeta(G)=\frac{D(G)}{N(G)} \tag{11}
\end{equation*}
$$

For simplicity, we use $N_{t}, D_{t}$ and $\zeta_{t}$ for $N\left(G_{t}\right), D\left(G_{t}\right)$ and $\zeta\left(G_{t}\right)$.
When SLOW-STRIP is applied to a random configuration, the algorithm does not expose the partition initially. When a point $x$ in $v$ is chosen to be deleted in a step, SLOW-STRIP exposes the $r$-tuple that contains $x$ by u.a.r. choosing another $r-1$ points from the remaining points. These $r$ points are removed in that step of SLOW-STRIP. The partition of the remaining points remains unexposed and is uniformly distributed.

We use $G_{t}$ to denote the random configuration obtained after $t$ steps of SLOW-STRIP, even though the $r$-tuples in $G_{t}$ have not been exposed yet, nor the allocation of all of the points in $G_{t}$. Let $\mathcal{F}_{t}=\left(L_{t}, N_{t}, D_{t}\right)$. The only information exposed after step $t$ is $\mathcal{F}_{t}$, and the set of vertices in $\mathcal{Q}$ as well as the points they contain. By the definition of the APmodel, conditioning on the exposed information, $G_{t}$ is a random configuration obtained by uniformly allocating the $D_{t}$ points to the $N_{t}$ heavy vertices, subject to each of these vertices receiving at least $k$ points, and u.a.r. partitioning the $L_{t}+D_{t}$ points.

Definition 5. Define $\bar{p}_{t}=\bar{p}_{t}\left(\mathcal{F}_{t}\right)$ to be the probability that a given point is allocated to a bin containing exactly $k$ points, in a uniform allocation of $D_{t}$ points into $N_{t}$ bins, subject to each bin receiving at least $k$ points.

If we choose u.a.r. a point $x$ from the heavy vertices of $G_{t}$, then $\bar{p}_{t}$ is the probability that $x$ is contained in a vertex with degree $k$. Analysing $\bar{p}_{t}$ is a key part of the analysis. We briefly explain why this parameter is important.

In each step of SLOW-STRIP, one point in a given light vertex is removed, together with another $r-1$ points $u_{1}, \ldots, u_{r-1}$ u.a.r. chosen from all the remaining points. For each of these $r-1$ points, if it is contained in a vertex with degree $k$, then the removal of this point results in a new light vertex and $L_{t}$ will increase by $k-1$. If we conditional on that the point is in a heavy vertex, then this probability is approximately $\bar{p}_{t}$.

We will restrict our analysis for steps $t$ such that the number of heavy vertices in $G_{t}$ is at least $\sigma n$ for some constant $\sigma>0$ to be specified in Section 4.7. Let $B_{t}=L_{t}+D_{t}$; i.e. $B_{t}$ denotes the total degree of $G_{t}$. Then, we may assume that $B_{t} \geq k \sigma n$. At step $t+1$, for each point $u_{i}(1 \leq i \leq r-1)$ that are removed in this step, the probability that it lies in a light vertex is $L_{t} / B_{t}+O(1 / n)$. Then, if $L_{t}>0$,

$$
\begin{equation*}
\mathbb{E}\left(L_{t+1}-L_{t} \mid \mathcal{F}_{t}\right)=-1+(r-1)\left(-\frac{L_{t}}{B_{t}}+\left(1-\frac{L_{t}}{B_{t}}\right)(k-1) \bar{p}_{t}\right)+O\left(n^{-1}\right) \tag{12}
\end{equation*}
$$

where $O\left(n^{-1}\right)$ above accounts for errors from two cases: (a), when deleting $u_{i}$, the total degree of the light vertices is $L_{t}+O(1)$ and the total degree is $B_{t}+O(1) ;(\mathrm{b})$, more than one of $u_{i}$ 's are contained in the same vertex that becomes light after step $t+1$.

It is convenient to define

$$
\begin{equation*}
\theta_{t}=-1+(r-1)(k-1) \bar{p}_{t} . \tag{13}
\end{equation*}
$$

Then (12) can be rewritten as

$$
\begin{equation*}
\mathbb{E}\left(L_{t+1}-L_{t} \mid \mathcal{F}_{t}\right)=\theta_{t}-\left(\theta_{t}+r\right) \frac{L_{t}}{B_{t}}+O\left(n^{-1}\right) \tag{14}
\end{equation*}
$$

### 4.3 Useful lemmas from [9] and other related works

In this section, we state several lemmas in the literature that are useful for our analysis. Most of them have appeared in the prior work [9. In Section 4.2 we defined $G_{0}$, which is close to $\mathcal{C}_{k}\left(A P_{r}\left(n, c^{\prime} n\right)\right)$, since $G_{0}$ is obtained by removing a small number of $r$-tuples in $H_{\tau^{\prime}(B)}^{\prime}$, whereas $H_{\tau^{\prime}(B)}^{\prime}$ is close to $\mathcal{C}_{k}\left(A P_{r}\left(n, c^{\prime} n\right)\right)$ by (9).

We mentioned in Section 4.2 that we will keep track of $\bar{p}_{t}$, or correspondingly $\theta_{t}$. Note that $\theta_{t}$ is a function of $\mathcal{F}_{t}$. Next, we specify this function.

Given positive integers $n, m$ and $k \geq 0$ such that $m \geq k n$, define $\operatorname{Multi}(n, m, k)$, the truncated multinomial distribution, to be the probability space consisting of integer vectors $\mathbf{X}=\left(X_{1}, \ldots, X_{n}\right)$ with domain $\mathcal{I}_{k}:=\left\{\mathbf{d}=\left(d_{1}, \ldots, d_{n}\right): \sum_{i=1}^{n} d_{i}=m, d_{i} \geq k, \forall i \in[n]\right\}$, such that for any $\mathbf{d} \in \mathcal{I}_{k}$,

$$
\operatorname{Pr}(\mathbf{X}=\mathbf{d})=\frac{m!}{n^{m} \Psi} \prod_{i \in[n]} \frac{1}{d_{i}!}=\frac{\prod_{i \in[n]} 1 / d_{i}!}{\sum_{\mathbf{d} \in \mathcal{I}_{k}} \prod_{i \in[n]} 1 / d_{i}!},
$$

where

$$
\Psi=\sum_{\mathbf{d} \in \mathcal{I}_{k}} \frac{m!}{n^{m}} \prod_{i \in[n]} \frac{1}{d_{i}!} .
$$

It is well known that the degree distribution of the heavy vertices of $G_{t}$, conditional on $\mathcal{F}_{t}$, follows $\operatorname{Multi}\left(N_{t}, D_{t}, k\right)$ (see e.g. [4] for more details). The truncated multinomial variables can be well approximated by independent truncated Poisson random variables. We formalise this in the following proposition whose proof can be found in [4, Lemma 1].

Proposition 6. Given integers $k, N$ and $D$ with $D>k N$, assume $\mathbf{X} \sim \operatorname{Multi}(N, D, k)$. For any $j \geq k$, let $\rho_{j}$ denote the proportion of components in $X$ that equals $j$. Then, for any $\epsilon>0$, with probability $1-o\left(N^{-1}\right)$,

$$
\begin{equation*}
\rho_{j}=e^{-\lambda} \frac{\lambda^{j}}{f_{k}(\lambda) j!}+O\left(N^{-1 / 2+\epsilon}\right) \tag{15}
\end{equation*}
$$

where $\lambda$ satisfies $\lambda f_{k-1}(\lambda) / f_{k}(\lambda)=D / N$.
Let $\lambda(x)$ be the root of $\lambda f_{k-1}(\lambda)=x f_{k}(\lambda)$, and define

$$
\begin{equation*}
\psi(x)=\frac{e^{-\lambda(x)} \lambda(x)^{k-1}}{f_{k-1}(\lambda(x))(k-1)!} \tag{16}
\end{equation*}
$$

Recall from (11) that $\zeta_{t}$ is the average degree of the heavy vertices of $G_{t}$. By definition, $\bar{p}_{t}=k \rho_{k} / \zeta_{t}$, where $\rho_{k}$ is the proportion of vertices with degree $k$. Thus $\bar{p}_{t}$ is approximately $\psi\left(\zeta_{t}\right)$ by (16) and Proposition 6. With some elementary calculations (e.g. see [9, Lemma 30]) it is easy to show that $\psi(x)$ is decreasing on $(k, \infty)$. Thus, we have the following lemma of the approximation of $\bar{p}_{t}$ and $\theta_{t}$.

Lemma 7. Assume $N_{t}=\Omega(n)$. With probability $1-o\left(n^{-1}\right)$,

$$
\begin{aligned}
\bar{p}_{t} & =\left(1+O\left(n^{-1 / 2} \log n\right)\right) \psi\left(\zeta_{t}\right) \\
\theta_{t} & =-1+(k-1)(r-1) \psi\left(\zeta_{t}\right)+O\left(n^{-1 / 2} \log n\right)
\end{aligned}
$$

Moreover, $\psi(x)$ is a strictly decreasing function on $x>k$; i.e. $\psi^{\prime}(x)<0$ for every $x>k$.
In the analysis for $\left(L_{t}\right)_{t \geq 0}$ and several other sequences of parameters, we frequently apply the following lemma, which is a simple application of the Hoeffding-Azuma inequality.

Lemma 8. Let $a_{n}$ and $c_{n} \geq 0$ be real numbers and $\left(X_{n, i}\right)_{i \geq 0}$ be random variables with respect to a random process $\left(G_{n, i}\right)_{i \geq 0}$ such that

$$
\mathbb{E}\left(X_{n, i+1} \mid G_{n, i}\right) \leq X_{n, i}+a_{n}
$$

and $\left|X_{n, i+1}-X_{n, i}\right| \leq c_{n}$, for every $i \geq 0$ and all (sufficiently large) $n$. Then, for any real number $j \geq 0$,

$$
\operatorname{Pr}\left(X_{n, t}-X_{n, 0} \geq t a_{n}+j\right) \leq \exp \left(-\frac{j^{2}}{2 t\left(c_{n}+\left|a_{n}\right|\right)^{2}}\right)
$$

Proof. Let $Y_{n, i}=X_{n, i}-i a_{n}$. Then,

$$
\mathbb{E}\left(Y_{n, i+1} \mid Y_{n, i}\right)=\mathbb{E}\left(X_{n, i+1} \mid X_{n, i}\right)-(i+1) a_{n} \leq X_{n, i}-i a_{n}=Y_{n, i} .
$$

Thus, $\left(Y_{n, i}\right)_{0 \leq i \leq t}$ is a supermartingale. Moreover, $\left|Y_{n, i+1}-Y_{n, i}\right| \leq c_{n}+\left|a_{n}\right|$. By HoeffdingAzuma's inequality,

$$
\operatorname{Pr}\left(Y_{n, t}-Y_{n, 0} \geq j\right) \leq \exp \left(-\frac{j^{2}}{2 t\left(c_{n}+\left|a_{n}\right|\right)^{2}}\right)
$$

This completes the proof of the lemma.
Recall the definition of $\alpha$ and $\beta$ in (3) and (4). As we mentioned before, our analysis focuses on a range of $t$, such that $G_{t}$ contains $(\alpha+o(1)) n$ vertices. Hence, many parameters of $G_{t}$ are very close to certain critical values. For instance, $\zeta_{t}$ will be very close

$$
\begin{equation*}
\zeta:=\frac{r \beta}{\alpha} . \tag{17}
\end{equation*}
$$

By Lemma 7 and (17), $\bar{p}_{t}$ will be very close to $\bar{p}:=\psi(\zeta)$. A simple calculation (see 9, Lemmas 24 and 25] for a detailed proof) leads to

$$
\begin{equation*}
\psi(\zeta)=1 /(r-1)(k-1) \text { and therefore } \bar{p}=\frac{1}{(r-1)(k-1)} \tag{18}
\end{equation*}
$$

A non-trivial inequality with $\zeta$ is stated below, which will be useful in our proof. The proof can be found in [9, Lemmas 34].

$$
\begin{equation*}
k<\zeta<r(k-1) \tag{19}
\end{equation*}
$$

In order to analyse $G_{0}=\widehat{H}_{\tau^{\prime}(B)}$, which is obtained by removing a few $r$-tuples in $H_{\tau^{\prime}(B)}^{\prime}$, we need information on how much $\zeta\left(H_{\tau^{\prime}(B)}^{\prime}\right)$ deviates from $\zeta$. The following lemma, proved in [9, Corollary 9], estimates how much $\zeta\left(\mathcal{C}_{k}\left(H^{\prime}\right)\right)$ deviates from $\zeta$.

Lemma 9. For fixed $r, k \geq 2,(r, k) \neq(2,2)$, there exist three positive constants $K_{1}=$ $K_{1}(r, k), K_{2}=K_{2}(r, k)$ and $K_{3}=K_{3}(r, k)$ such that: if $c \geq c_{r, k}+n^{-\delta}$ for some constant $0<\delta<1 / 2$ and $\xi:=c-c_{r, k}=o(1)$, then a.a.s. $\mathcal{C}_{k}\left(A P_{r}(n, c n)\right)$ has $\alpha n+K_{1} \sqrt{\xi} n+O\left(\xi n+n^{3 / 4}\right)$ vertices, $\beta n+K_{2} \sqrt{\xi} n+O\left(\xi n+n^{3 / 4}\right) r$-tuples, and average degree $r \beta / \alpha+K_{3} \sqrt{\xi}+O\left(\xi+n^{-1 / 4}\right)$.

By Lemma 9, a.a.s. $\zeta\left(\mathcal{C}_{k}\left(H^{\prime}\right)\right)=\zeta+\Theta\left(\sqrt{\xi^{\prime}}\right)$. Then, by [9, Lemma 52(a)], which states that a.a.s. $\zeta\left(H_{\tau^{\prime}(B)}^{\prime}\right) \geq \zeta\left(\mathcal{C}_{k}\left(H^{\prime}\right)\right)+O(\log n / n) \geq \zeta+\Theta\left(\sqrt{\xi^{\prime}}\right)$. On the other hand, by (9), $H_{\tau^{\prime}(B)}^{\prime}$ differs from $\mathcal{C}_{k}\left(H^{\prime}\right)$ by $O\left(n \sqrt{\xi^{\prime}}\right)$ vertices, which affects the average degree of the heavy vertices (there are a.a.s. $\Omega(n)$ of them) by $O\left(\sqrt{\xi^{\prime}}\right)$. So we must have $\zeta\left(H_{\tau^{\prime}(B)}^{\prime}\right)=$ $\zeta\left(\mathcal{C}_{k}\left(H^{\prime}\right)\right)+O\left(\sqrt{\xi^{\prime}}\right)=\zeta+O\left(\sqrt{\xi^{\prime}}\right)$. It follows then that

$$
\begin{equation*}
\zeta\left(H_{\tau^{\prime}(B)}^{\prime}\right)=\zeta+\Theta\left(\sqrt{\xi^{\prime}}\right) . \tag{20}
\end{equation*}
$$

### 4.4 Properties of $G_{0}$

Recall that $H^{\prime} \in A P_{r}\left(n, c^{\prime} n\right)$. By Theorem 5, a.a.s. $\mathcal{C}_{k}\left(H^{\prime}\right)$ contains approximately $\alpha n$ vertices and $\beta n r$-tuples. We first prove Lemma 3
Proof of Lemma 3. Let $\mathcal{E}_{1}$ denote the set of $r$-tuples in $H_{\tau^{\prime}(B)}^{\prime}$ and $\mathcal{E}_{2}=E\left(H^{\prime}\right) \backslash \mathcal{E}_{1}$. Now $H$ is obtained by u.a.r. removing $\left(c^{\prime}-c\right) n r$-tuples from $E\left(H^{\prime}\right)=\mathcal{E}_{1} \cup \mathcal{E}_{2}$. By definition, $X$ is the number of these deleted $r$-tuples that were in $\mathcal{E}_{1}$. By Theorem 5. a.a.s. $\left|\mathcal{E}_{1}\right|=\Omega(n)$. Conditional on that, $X$ stochastically dominates $\operatorname{Bin}\left(\left(c^{\prime}-c\right) n, C\right)$ for some constant $0<$ $C<1$. So a.a.s. $X=\Theta\left(\left(c^{\prime}-c\right) n\right)$, as $\left(c^{\prime}-c\right) n \rightarrow \infty$ by (5).

Recall that $G_{0}=\widehat{H}_{\tau^{\prime}(B)}$, which is obtained by u.a.r. removing $X r$-tuples from $H_{\tau^{\prime}(B)}^{\prime}$. Equivalently, $G_{0}$ is the configuration obtained by u.a.r. removing $X r$ points from $\widehat{H}_{\tau^{\prime}(B)}$, whereas the remaining vertices are uniformly partitioned. The following proposition follows by the uniformity of the allocation of points to bins in the AP-model.

Proposition 10. Conditional on $N$, the number of heavy vertices in $G_{0}$, and $D$, the total degree of the heavy vertices, and the set $S$ of light vertices, $G_{0}$ is a random configuration uniformly drawn from the space where $D$ points are u.a.r. allocated to $N$ bins subject to each bin receiving at least $k$ points, and all points in the $N+|S|$ bins are uniformly partitioned into parts with size $r$.

Now we bound $L_{0}$, the total degree of the light vertices in $G_{0}$.
Proof of Lemma 4. By the construction of $G_{0}$,

$$
\begin{equation*}
L_{0}=O\left(\left|S_{\tau^{\prime}(B)}^{\prime}\right|+X\right) \tag{21}
\end{equation*}
$$

By (6) and Lemma 2(a), $\left|S_{\tau^{\prime}(B)}^{\prime}\right|=\Theta\left(n \xi^{\prime}\right)$. By Lemma 3, a.a.s. $X=\Theta\left(\left(c^{\prime}-c\right) n\right)$, and thus a.a.s. $L_{0}=O\left(\left(c^{\prime}-c\right) n\right)$ by (21) and (5). It remains to prove the lower bound.

Remove the $X r$-tuples one by one. By Proposition 6, a.a.s. in every step the number of vertices with degree $k$ is $\Theta(n)$. So there is a constant $\gamma_{0}>0$ such that for each $r$ tuple that is removed, the probability that it is incident with a vertex with degree $k$ is at least $\gamma_{0}$. Hence, the number of vertices becoming light after removing $X$ random $r$-tuples stochastically dominates $\operatorname{Bin}\left(X, \gamma_{0}\right)$ and thus is a.a.s. $\Theta\left(X \gamma_{0}\right)=\Theta\left(n\left(c^{\prime}-c\right)\right)$. It only remains to show that not many of these new light vertices are created with degree zero. The expected number of vertices having degree drop from at least $k$ to zero is at most

$$
\sum_{j \geq 0} n\binom{r X}{j+1} \cdot O\left(\left(\frac{k+j}{n}\right)^{j+1}\right)=o\left(\left(c^{\prime}-c\right) n\right)
$$

where $n$ is an upper bound for the number of vertices with degree $k+j,\binom{r X}{j+1}$ is the number of ways to pick $j+1$ points from a set of $r X$ points, and $O\left(((k+j) / n)^{j+1}\right)$ is the probability that $j+1$ u.a.r. chosen points are contained in a given vertex with degree $k+j$. It follows then that a.a.s. $L_{0}=\Theta\left(\left(c^{\prime}-c\right) n\right)$.

Recall that $\theta_{0}=-1+(r-1)(k-1) \bar{p}_{0}$ by definition. We estimate $\theta_{0}$ in the following lemma.

Lemma 11. There are constants $C_{1}>C_{2}>0$ such that a.a.s.

$$
-C_{1} \sqrt{\xi^{\prime}} \leq \theta_{0} \leq-C_{2} \sqrt{\xi^{\prime}} ; \quad C_{2} \sqrt{\xi^{\prime}} \leq \zeta_{0}-\zeta \leq C_{1} \sqrt{\xi^{\prime}}
$$

Proof. By (20), $\zeta\left(H_{\tau^{\prime}(B)}^{\prime}\right)-\zeta=\Theta\left(\sqrt{\xi^{\prime}}\right)$. By Lemma 3, a.a.s. $X=\Theta\left(\left(c^{\prime}-c\right) n\right)$. It follows then that a.a.s.,

$$
\zeta_{0}=\zeta\left(H_{\tau^{\prime}(B)}^{\prime}\right)\left(1+O\left(c^{\prime}-c\right)\right)=\zeta+\Theta\left(\sqrt{\xi^{\prime}}\right)
$$

This is because the number of heavy vertices and their total degree change by $O(X)=$ $O\left(\left(c^{\prime}-c\right) n\right)$ from $H_{\tau^{\prime}(B)}^{\prime}$ to $G_{0}$, whereas a.a.s. the number of heavy vertices in $H_{\tau^{\prime}(B)}^{\prime} \supseteq \mathcal{C}_{k}\left(H^{\prime}\right)$ is $\Omega(n)$. The error $O\left(c^{\prime}-c\right)$ is absorbed by $\Theta\left(\sqrt{\xi^{\prime}}\right)$ as $c^{\prime}-c=o\left(\sqrt{\xi^{\prime}}\right)$ by (5).

Now, by Lemma 7 and (18) and by taking the Taylor expansion of $\psi(x)$ at $x=\zeta$ (note that $\zeta>k$ by (19)), a.a.s.

$$
\begin{aligned}
\bar{p}_{0} & =\left(1+O\left(n^{-1 / 2} \log n\right)\right) \psi\left(\zeta_{0}\right)=\left(1+O\left(n^{-1 / 2} \log n\right)\right)\left(\psi(\zeta)-\Theta\left(\sqrt{\xi^{\prime}}\right)\right) \\
& =\frac{1}{(r-1)(k-1)}-\Theta\left(\sqrt{\xi^{\prime}}\right)
\end{aligned}
$$

where the error $O\left(n^{-1 / 2} \log n\right)$ is absorbed by $-\Theta\left(\sqrt{\xi^{\prime}}\right)$, as $\xi^{\prime} \geq n^{-1 / 2+\epsilon}$. Now, $\theta_{0}=-\Theta\left(\sqrt{\xi^{\prime}}\right)$ follows by (13).

### 4.5 Evolution of $\zeta_{t}$

In order to analyse $\left(L_{t}\right)_{t \geq 0}$, we need to keep track of $\theta_{t}$ that appears in (14). Since $\theta_{t}$ is a function of $\zeta_{t}$ by Lemma 7, we only need to investigate how $\zeta_{t}=D_{t} / N_{t}$ evolves in SLOW-STRIP. Note that $\zeta_{t}$ would change in a step only if a point contained in a heavy vertex is removed. However, in a single step, the number of such points varies between zero and $k(r-1)$. In each step, aside from the point contained in the light vertex in the front of $\mathcal{Q}$, there are $r-1$ other points removed, each u.a.r. chosen from all of the remaining points. We say an occurrence of event $\mathscr{H}$ takes place if such a u.a.r. chosen point is contained in a heavy vertex. In order to trim off the effect of uncertainties, it is convenient to study an auxiliary process, defined below, in which exactly one occurrence of $\mathscr{H}$ takes place in each step.

Let $\mathcal{M}$ be a configuration obtained by u.a.r. allocating $\widehat{D}$ points into $\widehat{N}$ bins, subject to each bin receiving at least $k$ points. Let $\mathcal{M}_{0}=\mathcal{M}$. For every $t \geq 1, \mathcal{M}_{t}$ is obtained from $\mathcal{M}_{t-1}$ by removing a point u.a.r. chosen from all points; if it results in a bin containing less than $k$ points, remove that bin together with all points inside it. Note that if we choose $\widehat{N}$ and $\widehat{D}$ to be the number of heavy vertices and their total degree of $G_{0}$, then $\left(\mathcal{M}_{t}\right)_{t \geq 0}$ encodes the degree sequence of the heavy vertices of $\left(G_{t^{\prime}}\right)_{t^{\prime} \geq 0}$. To distinguish from parameters in $G_{0}, G_{1}, \ldots$, we add a hat to each corresponding parameter (like $\widehat{N_{t}}, \widehat{D_{t}}, \widehat{\theta}_{t}, \widehat{\zeta}_{t}$ etc.) for the sequence $\mathcal{M}_{0}, \mathcal{M}_{1}, \ldots$. We study parameters $\widehat{\zeta}_{t}$ and $\widehat{\theta}_{t}$ in this subsection, and will link them to $\zeta_{t}$ and $\theta_{t}$ in Section 4.6.

Parameters $\widehat{D}$ and $\widehat{N}$ in $\mathcal{M}_{0}$ will eventually be chosen to coincide with $D_{0}$ and $N_{0}$ in $G_{0}$. Thus, by Lemma 11, we will assume the following conditions, which we omit in the statements of lemmas in this subsection.

$$
\begin{equation*}
\widehat{\zeta}_{0}=\zeta+\Theta\left(\sqrt{\xi^{\prime}}\right), \text { where } n^{-1 / 2+\epsilon} \leq \xi^{\prime}=o(1) \tag{22}
\end{equation*}
$$

By Lemma 7 and (18), it is easy to see that

$$
\begin{equation*}
\widehat{\theta}_{0}=-\Theta\left(\sqrt{\xi^{\prime}}\right) \tag{23}
\end{equation*}
$$

which is negative (this is also indicated in Lemma 11).
Fix a constant $0<\sigma<1$ (the value of $\sigma$ will be determined in Section 4.8). Define $\tau_{1}=\tau_{1}(\sigma)$ to be the maximum integer $t$ such that $\widehat{N}_{t} \geq \sigma n$. Obviously, for all $t \leq \tau_{1}(\sigma)$,

$$
\begin{equation*}
\widehat{\zeta}_{t}-\widehat{\zeta}_{0}=O(t / n), \quad \widehat{\theta}_{t}-\widehat{\theta}_{0}=O(t / n) \tag{24}
\end{equation*}
$$

because $\widehat{N}_{t}$ and $\widehat{D}_{t}$ changes by $O(1)$ in each step.
Lemma 12. There exist constants $C_{1}>C_{2}>0$ such that
(a) for all $0 \leq t \leq \tau_{1}(\sigma)$,

$$
\begin{equation*}
-C_{1} / n \leq \mathbb{E}\left(\widehat{\zeta}_{t+1}-\widehat{\zeta}_{t} \mid \mathcal{M}_{t}\right) \leq-C_{2} / n \tag{25}
\end{equation*}
$$

(b) a a.a.s. for all $\log ^{2} n \leq t \leq \tau_{1}(\sigma),-C_{1} t / n \leq \widehat{\zeta}_{t}-\widehat{\zeta}_{0} \leq-C_{2} t / n$;
(c) a.a.s. for all $0 \leq t \leq \tau_{1}(\sigma), C_{2} t / n+O\left(n^{-1 / 2} \log n\right) \leq \widehat{\theta}_{t}-\widehat{\theta}_{0} \leq C_{1} t / n+O\left(n^{-1 / 2} \log n\right)$.

Proof. We first prove parts (a,b) for $0 \leq t \leq \epsilon_{0} n$, for some proper $\epsilon_{0}>0$. It was shown in [9, Lemma 36] (below eq. (30)) that (25) holds as long as $\widehat{\zeta}_{t}<r(k-1)-\epsilon_{1}$ for some constant $\epsilon_{1}$ (and $C_{1}, C_{2}$ in (a) depends only on $\epsilon_{1}$ ). We have $\widehat{\zeta}_{0}=\zeta+o(1)$ by (22) and immediately we have $\widehat{\zeta}_{t}=\zeta+O\left(\epsilon_{0}\right)$ for all $0 \leq t \leq \epsilon_{0} n$ by (24). Therefore, by (19), there exist sufficiently small constants $\epsilon_{1}, \epsilon_{0}>0$ such that $\widehat{\zeta}_{t}<r(k-1)-\epsilon_{1}$ for all $0 \leq t \leq \epsilon_{0} n$.

By Lemma 8 (with $a_{n}=-\Theta(1 / n), c_{n}=\Theta(1 / n)$ and $j=t a_{n}$ ), a.a.s. for all $\log ^{2} n \leq$ $t \leq \epsilon_{0} n, \widehat{\zeta}_{t}-\widehat{\zeta}_{0} \leq-\Theta(t / n)$. Applying Lemma 8 again to $\left(-\widehat{\zeta}_{t}\right)_{t \geq 0}$ (with $a_{n}=\Theta(1 / n)$, $c_{n}=\Theta(1 / n)$ and $\left.j=-t a_{n} / 2\right)$, a.a.s. for all $\log ^{2} n \leq t \leq \epsilon_{0} n, \widehat{\zeta}_{t}-\widehat{\zeta}_{0} \geq-\Theta(t / n)$. It follows then that a.a.s. for all $\log ^{2} n \leq t \leq \epsilon_{0} n, \widehat{\zeta}_{t}-\widehat{\zeta}_{0}=-\Theta(t / n)$.

Next, we discuss $\epsilon_{0} n \leq t \leq \tau_{1}(\sigma)$. We have shown that a.a.s. $\widehat{\zeta}_{\epsilon_{0} n}-\widehat{\zeta}_{0}=-\Theta\left(\epsilon_{0}\right)$, i.e. $\widehat{\zeta}_{\epsilon_{0} n}<\widehat{\zeta}_{0}$ and so the condition $\widehat{\widehat{\zeta}}_{t}<r(k-1)-\epsilon_{1}$ holds for all $\epsilon_{0} n \leq t \leq 2 \epsilon_{0} n$. With the same argument, parts (a,b) hold for all $t$ in this range. Inductively, claims in parts (a,b) hold for all $t \leq \tau_{1}(\sigma)$, as there are only $O\left(1 / \epsilon_{0}\right)=O(1)$ inductive steps.

For part (c), note that $\zeta_{0}>k$ since $\zeta_{0}=\zeta+O\left(\epsilon_{0}\right)$, and $\zeta>k$ by (19) and that $\epsilon_{0}$ can be chosen sufficiently small. By Lemma 7 and by the union bound, we have that a.a.s. for all $0 \leq t \leq \tau_{1}(\sigma)$,

$$
\widehat{\theta}_{t}=-1+(k-1)(r-1) \psi\left(\zeta_{t}\right)+O\left(n^{-1 / 2} \log n\right) .
$$

By Lemma 7 and part (b), and by taking the Taylor expansion of $\psi(x)$ at $x=\zeta_{0}$, we have
$\widehat{\theta}_{t}=-1+(k-1)(r-1)\left(\psi\left(\zeta_{0}\right)+\Theta(t / n)\right)+O\left(n^{-1 / 2} \log n\right)=\widehat{\theta}_{0}+\Theta(t / n)+O\left(n^{-1 / 2} \log n\right)$,
for all $\log ^{2} n \leq t \leq \tau_{1}(\sigma)$. The case $t<\log ^{2} n$ easily follows from (24) by noting that $O\left(\log ^{2} n / n\right)$ is absorbed by $O\left(n^{-1 / 2} \log n\right)$.

Corollary 13. A.a.s. for all $0 \leq t \leq \tau_{1}(\sigma)$, $\widehat{\theta}_{t} \geq 2 \widehat{\theta}_{0}$.
Proof. By Lemma 12(c), for all $t \leq n^{1 / 2} \log ^{2} n$, we have $\widehat{\theta}_{t}=\widehat{\theta}_{0}+O\left(t / n+n^{-1 / 2} \log n\right)=$ $\widehat{\theta}_{0}+O\left(n^{-1 / 2} \log ^{2} n\right)$. By (23) and (22), $\widehat{\theta}_{0}<0$ and $\left|\widehat{\theta}_{0}\right|=\Omega\left(n^{-1 / 4}\right)$. It follows then that $\widehat{\theta}_{t} \geq 2 \widehat{\theta}_{0}$ for all $t \leq n^{1 / 2} \log ^{2} n$. By Lemma 12(c), there is a constant $C>0$ such that a.a.s. $\widehat{\theta}_{t} \geq \widehat{\theta}_{0}+C t / n \geq 2 \widehat{\theta}_{0}$ for all $n^{1 / 2} \log ^{2} n<t \leq \tau_{1}(\sigma)$.

Define $t_{0}(K)=K n \sqrt{\xi^{\prime}}$, where $K>0$ is a large constant. Note that $\xi^{\prime}=o(1)$ implies $t_{0}(K)=o(n)$. The following corollary states when $\widehat{\theta}_{t}$ becomes positive (recall that $\widehat{\theta}_{0}<0$ ).
Corollary 14. Assume that $K>0$ is a sufficiently large constant. A.a.s. there are constants $C_{1}, C_{2}>0$ such that for every $t_{0}(K) \leq t \leq \tau_{1}(\sigma), C_{1} t / n \leq \widehat{\theta}_{t} \leq C_{2} t / n$.

Proof. By Lemma 12(c) and (22), and noting that $n^{-1 / 2} \log n=o\left(t_{0}(K) / n\right)$ by (22), a.a.s. for any $t_{0}(K) \leq t \leq \tau_{1}(\sigma), \widehat{\theta}_{t} \geq-C \sqrt{\xi^{\prime}}+Y t / n$ for some constants $C, Y>0$. Choosing $K \geq 2 C / Y$ we have that a.a.s. $\widehat{\theta}_{t} \geq(Y / 2) t / n$ for all $t_{0}(K) \leq t \leq \tau_{1}(\sigma)$. The upper bound of $\widehat{\theta}_{t}$ follows by (24) and the fact that $\widehat{\theta}_{0}<0$.

This immediately yields the following corollary.
Corollary 15. For any constant $\epsilon>0$, a.a.s. $\widehat{\theta}_{t}=\Omega(\epsilon)$ for all $\epsilon n \leq t \leq \tau_{1}(\sigma)$.

### 4.6 Relating $\hat{\theta}_{t}$ to $\theta_{t}$

Recall that $G_{0}, G_{1}, \ldots$ is the process produced by SLOW-STRIP. To analyse $\theta_{t}$ using $\widehat{\theta}_{t}$ in Section 4.5, let $\widehat{N}$ and $\widehat{D}$ in the definition of $\mathcal{M}_{0}$ in Section 4.5 take the same values as the corresponding parameters in $G_{0}$. Therefore, $\widehat{\zeta}_{0}=\zeta_{0}$ and $\widehat{\theta}_{0}=\theta_{0}$. By Lemma 11, a.a.s. $\theta_{0}=-\Theta\left(\sqrt{\xi^{\prime}}\right)$ and $\zeta_{0}-\zeta=\Theta\left(\sqrt{\xi^{\prime}}\right)$. This verifies the assumption (22)).

Corresponding to $\tau_{1}(\sigma)$, define

$$
\begin{equation*}
\tau_{2}(\sigma)=\max \left\{t: N_{t} \geq \sigma n\right\} \tag{26}
\end{equation*}
$$

The following lemma allows us to establish a relation between $\theta_{t}$ and $\widehat{\theta}_{t}$.
Lemma 16. There is a constant $C>0$ : for any $\log ^{2} n \leq t \leq \tau_{2}(\sigma)$, a.a.s. the number of occurrences of $\mathscr{H}$ by step $t$ is at least $C t$.

Proof. Let $h_{t}$ denote the number of occurrences of $\mathscr{H}$ by step $t$. By our definition of $\tau_{2}(\sigma)$, the number of heavy vertices in every step is at least $\sigma n$ and thus, there is a constant $\sigma^{\prime}>0$ (depending on $\sigma$ ) such that for every point that was u.a.r. chosen, the probability that it was contained in a heavy vertex is at least $\sigma^{\prime}$. In every step, there are $r-1 \geq 1$ such points being chosen. Hence, for all $t \leq \tau_{2}(\sigma)$, we always have

$$
\mathbb{E}\left(h_{t} \mid G_{t-1}\right) \geq h_{t-1}+\sigma^{\prime}
$$

Our claim follows by applying Lemma 8 to $\left(-h_{t}\right)$ (with $a_{n}=-\sigma^{\prime}, c_{n}=r-1$ and $j=$ $\left(\sigma^{\prime} / 2\right) t$ ).

Noting that at most $r-1$ occurrences of $\mathscr{H}$ can take place in a single step, this immediately gives the following corollary.

Corollary 17. There is a constant $0<C<1$ such that a.a.s. for all $\log ^{2} n \leq t \leq \tau_{2}(\sigma)$, $\theta_{t}=\widehat{\theta}_{t^{\prime}}$ for some $C t<t^{\prime} \leq(r-1) t$.

Another corollary follows easily from Corollaries 14 and 17 (recalling that $n \sqrt{\xi^{\prime}}=\Omega\left(n^{3 / 4}\right)$ by (5)).
Corollary 18. Let $K>0$ be a sufficiently large constant. There exist two constants $C_{1}, C_{2}>0$ such that a.a.s. for all $K n \sqrt{\xi^{\prime}} \leq t \leq \tau_{2}(\sigma), C_{1} t / n \leq \theta_{t} \leq C_{2} t / n$.

### 4.7 Specifying $\sigma$

The key lemma we use to specify the constant $\sigma$ is the following.
Lemma 19. Assume $c=c_{r, k}+o(1)$ and consider the parallel $k$-stripping process $H_{0}, H_{1}, H_{2}, \ldots$ with $H_{0} \in A P_{r}(n, c n)$. There is a constant $\sigma_{0}>0$ such that a.a.s. if $H_{i}$ has at most $\sigma_{0} n$ vertices for some $i>0$ then every component of $H_{i+1}$ contains $O(\log n)$ vertices.

We will use the following two lemmas to prove Lemma 19, The first lemma is from [18, Lemma 7].

Lemma 20. Assume $c<c_{r, k}-\epsilon_{0}$ for some constant $\epsilon_{0}>0$ and consider the parallel $k$-stripping process $H_{0}, H_{1}, H_{2}, \ldots$ with $H_{0} \in A P_{r}(n, c n)$. Then, there are positive constants $\left(\gamma_{i}\right)_{i=0}^{\infty}$ with $\lim _{i \rightarrow \infty} \gamma_{i}=0$ such that, a.a.s. for every fixed $i \geq 0,\left|H_{i}\right| \sim \gamma_{i} n$.

Let $\rho_{i}(j)$ denotes the proportion of vertices with degree $j$ in $H_{i}$. Part (a) of the following lemma is from [1, Section 8], whereas part (b) is from [19, Theorem 1].
Lemma 21. Assume $c<c_{r, k}-\epsilon_{0}$ for some constant $\epsilon_{0}>0$ and $H_{0} \in A P_{r}(n, c n)$.
(a) For any constant $K>0$, there is a constant $I>0$ such that for all $i \geq I$,

$$
\begin{equation*}
\rho_{i}(1)>K \sum_{j \geq 2}((k-1) j(j-1)-j) \rho_{i}(j) . \tag{27}
\end{equation*}
$$

(b) A.a.s. a random hypergraph with degree sequence satisfying (27) for some constant $K>1$ has the property that each component has size $O(\log n)$.

Note that Lemmas 20 and 21 were stated for $\mathcal{H}_{r}\left(n, r!c / n^{r-1}\right)$ in the original papers. However, the proofs use the configuration model, which is the AP-model conditioned to "typical" degree sequences. Hence, these results also hold a.a.s. for $A P_{r}(n, c n)$.

Proof of Lemma 19. Fix a small $\epsilon_{0}>0$ and let $c^{\prime \prime}=c_{r, k}-\epsilon_{0}$. Couple $H_{0}^{\prime \prime} \subseteq H_{0}$ in the same way as described in Section 4.2 such that $H_{0}^{\prime \prime} \in A P_{r}\left(n, c^{\prime \prime} n\right)$; i.e. we generate $H_{0}^{\prime \prime}$ by removing u.a.r. $\left(c-c^{\prime \prime}\right) n r$-tuples in $H_{0}$. Let $\mathcal{E}$ denote this set of $r$-tuples. Applying Lemma 21 to $H_{0}^{\prime \prime}$, there is a sufficiently large constant $I$ such that (27) is a.a.s. satisfied with $K=2$. Let $\left(\gamma_{i}\right)$ be the sequence in Lemma 20 for $\left(H_{i}^{\prime \prime}\right)$, and let $\sigma_{0}:=\gamma_{I} / 2$.

Consider the parallel stripping process $H_{0}, H_{1}, \ldots$ Let $i-1$ denote the first iteration after which there are at most $\sigma_{0} n$ vertices remaining, if the process has not terminated by
then. Define $i$ to be $n$ if no such iteration exists. Note that $i$ is not (necessarily) a.a.s. bounded by a constant, since we have $c=c_{r, k}+o(1)$.

Let $\widehat{H}$ be the random configuration obtained by removing all the $r$-tuples in $\mathcal{E}$ from $H_{i-1}$. Since $|\mathcal{E}|=\left(c-c^{\prime \prime}\right) n$, the number of $r$-tuples removed is at most $\left(c-c^{\prime \prime}\right) n=O\left(\epsilon_{0} n\right)$, and so the number of new light vertices created by the removal of the $r$-tuples in $\mathcal{E}$ is $O\left(\epsilon_{0} n\right)$. Let $\widehat{H}^{\prime}$ be the graph obtained from $\widehat{H}$ by removing all light vertices in $\widehat{H}$. Then, $H_{i}$ and $\widehat{H}^{\prime}$ differ by at most $O\left(\epsilon_{0} n\right) r$-tuples. Moreover, as we have discussed before, $\widehat{H}^{\prime}$ would have occurred in the parallel stripping process starting with $H_{0}^{\prime \prime}$, i.e. there is some $j$ such that $H_{j}^{\prime \prime}=\widehat{H}^{\prime}$.

Since $H_{j}^{\prime \prime}=\widehat{H}^{\prime} \subseteq H_{i-1}$, we have $\left|H_{j}^{\prime \prime}\right| \leq \sigma_{0} n$. Hence, we must have $j \geq I$, since $\sigma_{0}=\gamma_{I} / 2$ by definition and $\left|H_{I}^{\prime \prime}\right| \sim \gamma_{I} n$ by Lemma 20,

By Lemma 21(a) and the choice of $I$, a.a.s. $H_{j}^{\prime \prime}=\widehat{H}^{\prime}$ satisfies (27) with $K=2$. But $H_{i}$ and $H_{j}^{\prime \prime}=\widehat{H}^{\prime}$ differ by only $O\left(\epsilon_{0} n\right) r$-tuples as we discussed before. It follows immediately that the degree sequence of $H_{i}$ will satisfy (27) with $K=3 / 2$ as long as we choose $\epsilon_{0}>0$ sufficiently small. By Lemma 21(b), a.a.s. every component in $H_{i}$ has size $O(\log n)$.

In the rest of the paper, we choose $\sigma$ to be the constant that satisfies Lemma 19 and this fixes the definitions of $\tau_{1}(\sigma)$ in Section 4.5 and $\tau_{2}(\sigma)$ in Section 4.6 ,

Recall the definition of $G_{0}$ in Section4.2. Let $\widehat{G}_{0}, \widehat{G}_{1}, \widehat{G}_{2} \ldots$ denote the parallel $k$-stripping process with $\widehat{G}_{0}=G_{0}$. With a slight abuse of notation, we use $\left(S_{i}\right)$ to denote the set of light vertices associated with $\left(\widehat{G}_{i}\right)$; i.e. $S_{i}=V\left(\widehat{G}_{i}\right) \backslash V\left(\widehat{G}_{i+1}\right)$. Recall that $L\left(\widehat{G}_{i}\right)$ is the total degree of $S_{i}$ in $\widehat{G}_{i}$.

Define

$$
\begin{equation*}
I_{\sigma}:=\max \left\{i:\left|\widehat{G}_{i}\right| \geq \sigma n\right\} \tag{28}
\end{equation*}
$$

Then, $\left|\widehat{G}_{I_{\sigma}+1}\right|<\sigma n$ and so by Lemma [19, every component of $\widehat{G}_{I_{\sigma}+2}$ has size $O(\log n)$. It is easy to bound $s\left(\widehat{G}_{I_{\sigma}+2}\right)$ by $O(\log \log n)$ (see the end of Section 4.8). Thus, in order to bound $s\left(G_{0}\right)$, it is sufficient to bound $I_{\sigma}$. We often need to relate an iteration in the parallel stripping process to the step in SLOW-STRIP corresponding to the beginning of that iteration (recall that the process generated by SLOW-STRIP is denoted by $G_{0}, G_{1}, \ldots$ and $L_{t}$ denotes the degree of light vertices in $G_{t}$ ). To do so, we define $t(i)$ to be the step in SLOW-STRIP that the first vertex removed at the $i$-th iteration of the parallel process is pushed to the front of the queue $\mathcal{Q}$. Therefore,

$$
\begin{equation*}
L\left(\widehat{G}_{i}\right)=L_{t(i)} \text { for each } i \geq 0 . \tag{29}
\end{equation*}
$$

In particular, $L\left(\widehat{G}_{0}\right)=L_{0}$.

### 4.8 Subcritical: proof of Theorem 3(a)

Now let $\epsilon>0$ be an arbitrary constant and we assume that $c \leq c_{r, k}-n^{-1 / 2+\epsilon}$. Without loss of generality, we may assume that $\epsilon$ is sufficiently small. Let $0<\varsigma<1$ be a sufficiently small constant to be specified later; define $c^{\prime}=c_{r, k}+\varsigma \xi$; this fixes $c^{\prime}$ introduced in Section4.2, Clearly, all conditions in (5) are satisfied. By Lemma 11, a.a.s. $\theta_{0}=-\Theta(\sqrt{\varsigma \xi})$ and $\zeta_{0}=$ $\zeta+\Theta(\sqrt{\varsigma \bar{\xi}})$.

Recall that $L_{0}$ denotes the total degree of the light vertices in $G_{0}$. By Lemma 4 a.a.s. $L_{0}=\Theta(n \xi)$. Recalling (17) and (10), our goal in this section is to bound $s\left(G_{0}\right)$.
Lemma 22. Suppose $\varsigma>0$ is sufficiently small. There is a constant $C>0$ and an integer $i<C / \sqrt{\xi}$ such that a.a.s. $t(i)=\Theta(n \sqrt{\xi}), \theta_{t(i)}=\Theta(\sqrt{\xi})$ and $L_{t(i)}=\Theta(n \xi)$.

Proof. We have $\theta_{0} \geq-Y_{2} \sqrt{\varsigma \xi}$ for some constant $Y_{2}>0$ (note that $Y_{2}$ is independent of $\varsigma$ ), since $\xi^{\prime}=c^{\prime}-c_{r, k}=\varsigma \xi$. By Corollary 18, there is a constant $Y_{3}>0$ (depending on $Y_{2}$ and the constant in Corollary 18, but not on $\varsigma)$ such that a.a.s. $\theta_{t_{0}} \geq \sqrt{\varsigma \xi}$ for $t_{0}:=Y_{3} n \sqrt{\varsigma \xi}$.

By (14), if $L_{t}>0$,

$$
\begin{equation*}
\mathbb{E}\left(L_{t+1}-L_{t} \mid \mathcal{F}_{t}\right)=\left(1-\frac{L_{t}}{B_{t}}\right) \theta_{t}-r \frac{L_{t}}{B_{t}}+O\left(n^{-1}\right) \tag{30}
\end{equation*}
$$

By Corollary 13 and 17, a.a.s. $\theta_{t} \geq-2 Y_{2} \sqrt{\varsigma \xi}$ for all $0 \leq t \leq t_{0}$. Hence, $\left(1-L_{t} / B_{t}\right) \theta_{t} \geq$ $-2 Y_{2} \sqrt{\varsigma \xi}$. Note that the number of vertices in $G_{0}$ is a.a.s. $(\alpha+o(1)) n$ by the construction of $G_{0}$ and by Theorem 5. This is true for all $G_{t}$ where $0 \leq t \leq t_{0}$ as $t_{0}=o(n)$ by definition. If $L_{t}<x:=\left(\alpha Y_{2} / r\right) n \sqrt{\varsigma \xi}$, then $r L_{t} / B_{t} \leq Y_{2} \sqrt{\varsigma \xi}$ since a.a.s. $B_{t} \geq k(\alpha+o(1)) n \geq \alpha n$. So for all $0 \leq t \leq t_{0}$ with $L_{t}>0$ and $L_{t}<x$,

$$
\mathbb{E}\left(L_{t+1} \mid \mathcal{F}_{t}\right) \geq L_{t}-3 Y_{2} \sqrt{\varsigma \xi}
$$

Applying Lemma 8 to $\left(-L_{t}\right)$ (with $a_{n}=3 Y_{2} \sqrt{\varsigma \xi}, c_{n}=k r$ and $j=3 Y_{2} t \sqrt{\varsigma \delta}$ ), we immediately have that a.a.s. for all $\log ^{2} n \leq t \leq t_{0}$, we have $L_{t} \geq L_{0}-6 Y_{2} t \sqrt{\varsigma \xi}$. Hence, for all $0 \leq t \leq t_{0}$, $L_{t} \geq L_{0}-6 Y_{2} t_{0} \sqrt{\varsigma \xi}=L_{0}-6 Y_{2} Y_{3} n \varsigma \xi=\Omega(n \xi)$ because $L_{0}=\Omega(n \xi)$ and $\varsigma$ can be chosen sufficiently small (noting that the above inequality holds trivially for $t<\log ^{2} n$ ). Thus, we have shown that there is a constant $Y_{4}>0$ such that a.a.s. $L_{t} \geq Y_{4} n \xi$ for all $0 \leq t \leq t_{0}$.

Next, we prove that $L_{t}=O(n \xi)$ for all $0 \leq t \leq t_{0}$, which then will imply that $L_{t}=\Theta(n \xi)$ for all $0 \leq t \leq t_{0}$. By Lemma 12(c), Corollary 17 and (24), a.a.s. there are constants $Y_{5}^{\prime}, Y_{5}>0: \theta_{t} \leq \theta_{0}+Y_{5}^{\prime} t_{0} / n+O\left(n^{-1 / 2} \log n\right) \leq Y_{3} Y_{5} \sqrt{\varsigma \xi}$ for all $0 \leq t \leq t_{0}$. Then by (14),

$$
\mathbb{E}\left(L_{t+1} \mid \mathcal{F}_{t}\right) \leq L_{t}+Y_{3} Y_{5} \sqrt{\varsigma \xi}
$$

and thus by Lemma 8, a.a.s. for all $\log ^{2} n \leq t \leq t_{0}, L_{t} \leq L_{0}+2 Y_{3} Y_{5} t_{0} \sqrt{\varsigma \xi}$ and therefore, a.a.s. for all $0 \leq t \leq t_{0}, L_{t}=O(n \xi)$ (the equation holds trivially for $t \leq \log ^{2} n$ ).

Let $I_{1}$ (which will be the integer $i$ in the statement of this lemma) be the minimum integer that $t\left(I_{1}+1\right) \geq t_{0}$. We have shown that $L_{t} \geq Y_{4} n \xi$ for all $0 \leq t \leq t_{0}$. So, the total degree of vertices in each $S_{i}, i \leq I_{1}$, is at least $Y_{4} n \xi$. Thus, for each $i \leq I_{1}$, the $i$-th iteration of the parallel stripping process is consist of at least $\left(Y_{4} / r\right) n \xi$ steps of SLOW-STRIP, since at most $r$ points contained in $S_{i}$ are deleted in every step of SLOW-STRIP. It follows then that $I_{1}=O\left(t_{0} / Y_{4} n \xi\right)$. Hence, $I_{1}=O(1 / \sqrt{\xi})$ as $t_{0}=Y_{3} n \sqrt{\varsigma \xi}$.

We have shown that a.a.s. $L_{t}=\Theta(n \xi)$ for all $0 \leq t \leq t_{0}$. We also have $t\left(I_{1}\right)<t_{0}$ by our definition of $I_{1}$. So, a.a.s. $L\left(\widehat{G}_{I_{1}}\right)=L_{t\left(I_{1}\right)}=\Theta(n \xi)$.

We have shown that $\theta_{t_{0}}=\Theta(\sqrt{\xi})$. By the definition of $I_{1}, t\left(I_{1}\right) \leq t_{0} \leq t\left(I_{1}+1\right)$. Since a.a.s. $L\left(\widehat{G}_{I_{1}}\right)=L_{t\left(I_{1}\right)}=\Theta(n \xi)$, by (24), we have $\theta_{t\left(I_{1}\right)}=\theta_{t_{0}}+O\left(L\left(\widehat{G}_{I_{1}}\right) / n\right)=\Theta(\sqrt{\xi})$ as $\xi=o(\sqrt{\xi})$.

Finally, $t\left(I_{1}\right) \leq t_{0}=Y_{3} n \sqrt{\varsigma \xi}$ by our definition of $I_{1}$. We also have $t\left(I_{1}\right) \geq t_{0}-L\left(\widehat{G}_{I_{1}}\right)=$ $\Theta\left(t_{0}\right)$ since $L\left(\widehat{G}_{I_{1}}\right)=L_{t\left(I_{1}\right)}=O(n \xi)=o\left(t_{0}\right)$. This shows that $t\left(I_{1}\right)=\Theta(n \sqrt{\xi})$ as required and this completes our proof of the lemma by letting $i=I_{1}$ in the statement of the lemma.

Let $I_{1}$ be the integer specified in Lemma 22, so a.a.s. $\theta_{t\left(I_{1}\right)}=\Theta(\sqrt{\xi}), L_{t\left(I_{1}\right)}=\Theta(n \xi)$ and $I_{1}=O(1 / \sqrt{\xi})$.

Lemma 23. Suppose $\epsilon>0$ is sufficiently small. There are constants $Y_{1}, Y_{2}>0$ such that a.a.s. for all $t\left(I_{1}\right) \leq t \leq \epsilon n, Y_{1} t^{2} / n \leq L_{t} \leq Y_{2} t^{2} / n$.

Proof. Let $t_{0}=t\left(I_{1}\right)$. We will first prove the upper bound. It certainly holds a.a.s. for $t=t_{0}$, as a.a.s. $t_{0}=\Theta(n \sqrt{\xi}), \theta_{t_{0}}=\Theta(\sqrt{\xi})$ and $L_{t_{0}}=\Theta(n \xi)$ by Lemma 22. By Lemma 12(c), Corollary 17 and the fact that a.a.s. $\theta_{t_{0}}=\Theta(\sqrt{\xi})$, a.a.s. $C_{1} t / n \leq \theta_{t} \leq C_{2} t / n$ for some constants $C_{1}, C_{2}>0$ for all $t_{0} \leq t \leq \epsilon n$. By (30), a.a.s. for every $t_{0} \leq t \leq \epsilon n$ (noting that $\theta_{t}>0$ in this range),

$$
\mathbb{E}\left(L_{t+1}-L_{t} \mid \mathcal{F}_{t}\right) \leq \theta_{t}+O\left(n^{-1}\right) \leq \frac{2 C_{2} t}{n}
$$

Then by Lemma 8, for all $t_{0}+\log ^{2} n \leq t \leq \epsilon n$, a.a.s. $L_{t} \leq L_{t_{0}}+4 C_{2} t\left(t-t_{0}\right) / n \leq Y_{2} t^{2} / n$ by choosing sufficiently large $Y_{2}$. For all $t$ between $t_{0}$ and $t_{0}+\log ^{2} n, L_{t}=L_{t_{0}}+O\left(\log ^{2} n\right)$ and thus the upper bound trivially holds for sufficiently large $Y_{2}$, since $Y_{2} t^{2} / n \geq Y_{2} t_{0}^{2} / n=\omega\left(\log ^{2} n\right)$.

Next, we prove the lower bound. Clearly, it holds a.a.s. for $t_{0}$ for some constant $Y_{1}=Y$. Up to step $\epsilon n$, at most $(r-1) \epsilon n$ heavy vertices can be removed, and the number of heavy vertices in $G_{0}$ is $(\alpha+o(1)) n$, as we have discussed before. Hence, the total degree, $B_{t}$, of $G_{t}$, for any $t \leq \epsilon n$, is at least $k(\alpha-r \epsilon) n$. Now, by the upper bound we have just shown, a.a.s. for all $t_{0} \leq t \leq \epsilon n, L_{t} \leq Y_{2} t^{2} / n$ and hence for all $t$ in this range,

$$
\frac{r L_{t}}{B_{t}} \leq \frac{r Y_{2} t^{2} / n}{k(\alpha-r \epsilon) n} \leq\left(r Y_{2} \epsilon / \alpha\right) \frac{t}{n}<\left(C_{1} / 4\right) \frac{t}{n}<\frac{1}{4}
$$

by choosing $\epsilon>0$ sufficiently small. Then, by (30), a.a.s. for every $t_{0} \leq t \leq \epsilon n$,

$$
\begin{equation*}
\mathbb{E}\left(L_{t+1}-L_{t} \mid \mathcal{F}_{t}\right) \geq \frac{3}{4} \theta_{t}-\left(C_{1} / 4\right) \frac{t}{n}+O\left(n^{-1}\right) \geq\left(C_{1} / 3\right) \frac{t}{n}, \tag{31}
\end{equation*}
$$

as $\theta_{t} \geq C_{1} t / n$ for all $t$ in this range.
We split the range $t_{0} \leq t \leq \epsilon n$ into intervals, each with length $t_{0}$ (hence the $j$-th interval is from $(j-1) t_{0}$ to $\left.j t_{0}\right)$ and the last interval is simply the remainder. Similar to our analysis in Lemma 22, for the $j$-th interval, the probability that $L_{t}<L_{(j-1) t_{0}}+\left(C_{1} / 4\right)\left((j-1) t_{0} / n\right)$. $\left(t-(j-1) t_{0}\right)$ is at most $n^{-2}$ for all $(j-1) t_{0}+\log ^{2} n \leq t \leq j t_{0}$, and we always have $L_{t}=L_{(j-1) t_{0}}+O\left(t-(j-1) t_{0}\right)$ for all $(j-1) t_{0} \leq t \leq(j-1) t_{0}+\log ^{2} n$.

Since $t_{0}=\Theta(n \sqrt{\xi})$, the total number of intervals is $O(1 / \sqrt{\xi})$. So, a.a.s. for every interval j,

$$
\begin{aligned}
& L_{(j-1) t_{0}+d} \geq L_{(j-1) t_{0}}+\frac{C_{1}(j-1)}{4 n} t_{0} d, \text { if } d \geq \log ^{2} n \\
& L_{(j-1) t_{0}+d}=L_{(j-1) t_{0}}+O(d), \text { if } d<\log ^{2} n .
\end{aligned}
$$

It is easy to verify that by choosing $Y^{\prime}=\min \left\{Y, C_{1} / 12\right\}$, a.a.s. $L_{t} \geq Y^{\prime} t^{2} / n$ for all $t_{0} \leq t \leq \epsilon n$ except for the first $\log ^{2} n$ numbers in each interval. But then the inequality must hold by choosing $Y_{1}=Y^{\prime} / 2$ since $t_{0}^{2} / n$ is $\omega\left(\log ^{2} n\right)$. This completes the proof for the lower bound.

Suppose $\epsilon>0$ is chosen to satisfy Lemma 23, Define

$$
\begin{equation*}
I_{2}=\max \{i: t(I) \leq \epsilon n\} . \tag{32}
\end{equation*}
$$

In the following lemma, we bound $I_{2}$.
Lemma 24. A.a.s. $I_{2}=O\left(\xi^{-1 / 2} \log (1 / \xi)\right)$.
Proof. We have shown in (31) that a.a.s. for all $t_{0} \leq t \leq t\left(I_{2}\right)$, where $t_{0}=t\left(I_{1}\right)$,

$$
\mathbb{E}\left(L_{t+1}-L_{t} \mid \mathcal{F}_{t}\right) \geq C_{1} t_{0} / n \geq C_{2} \sqrt{\xi}
$$

for some constant $C_{1}, C_{2}>0$. Recalling (29) and by Lemma 8, this immediately gives that a.a.s.

$$
L_{t(i+1)} \geq\left(1+\left(C_{2} / 2\right) \sqrt{\xi}\right) L_{t(i)}, \text { for all } I_{1} \leq i \leq I_{2}-1
$$

Since $L_{t\left(I_{1}\right)}=\Theta(n \xi)$ by Lemma 22 and $L_{t\left(I_{2}\right)}=O(n)$, it follows immediately that $I_{2}-I_{1}=$ $O\left(\xi^{-1 / 2} \log (1 / \xi)\right)$. The lemma follows as $I_{1}=O(1 / \sqrt{\xi})$ by Lemma 22.

Recall from (26) that $\tau_{2}(\sigma)$ is the last step after which $N_{t} \geq \sigma n$ (c.f. $\tau_{1}(\sigma)$ for the sequence $\left(\mathcal{M}_{t}\right)$, defined in Section 4.5).

Lemma 25. There is a constant $\epsilon_{0}>0$ such that a.a.s. $L_{t} \geq \epsilon_{0} n$ for all $t\left(I_{2}\right) \leq t \leq \tau_{2}(\sigma)$.
Proof. We first prove that a.a.s. there is $t_{1} \leq t\left(I_{2}\right)$ such that $L_{t_{1}}=\Theta(n)$. By Lemma (23), for all $t\left(I_{1}\right) \leq t \leq \epsilon n$,

$$
\begin{equation*}
Y_{1} t^{2} / n \leq L_{t} \leq Y_{2} t^{2} / n \tag{33}
\end{equation*}
$$

for some constants $Y_{1}, Y_{2}>0$. Let $\epsilon_{0}=\epsilon /\left(1+Y_{2}\right)$. Then, $\epsilon_{0}<\epsilon$ and so $L_{\epsilon_{0} n} \leq Y_{2} \epsilon_{0}^{2} n$. Let $i_{1}=\max \left\{i: t(i) \leq \epsilon_{0} n\right\}$. Then, $t\left(i_{1}\right) \leq \epsilon_{0} n<t\left(i_{1}+1\right)$. Moreover $t\left(i_{1}+1\right) \leq \epsilon_{0} n+L_{\epsilon_{0} n} \leq$ $\epsilon_{0} n+Y_{2} \epsilon_{0}^{2} n \leq \epsilon_{0}\left(1+Y_{2}\right) n=\epsilon n$. Let $t_{1}=\epsilon_{0} n$. Then $t_{1} \leq t\left(i_{1}+1\right) \leq I_{2}$ by (32); moreover, $L_{t_{1}}$ satisfies (33). So, $L_{t_{1}}=\Theta(n)$.

By Corollaries 15 and 18, there is a constant $\epsilon_{1}>0$ such that a.a.s. $\theta_{t} \geq \epsilon_{1}$ for all $t_{1} \leq t \leq \tau_{2}(\sigma)$. We may assume $\epsilon_{1}<1$. For all $t \leq \tau_{2}(\sigma)$ we have $B_{t} \geq k \sigma n$.

Let $\eta=\min \left\{\left(Y_{1} / 2\right) \epsilon_{0}^{2},\left(\epsilon_{1} k \sigma / 12 r\right)\right\}$. Next, we prove a.a.s. for all $t_{1} \leq t \leq \tau_{2}(\sigma), L_{t} \geq \eta n$. For $t=t_{1}$, this is true since a.a.s. $L_{t_{1}} \geq 2 \eta n$ by (33). Let $A_{t}$ be the event that $L_{t} \geq 2 \eta n-k r$ and $L_{t^{\prime}}<2 \eta n$ for all $t \leq t^{\prime} \leq t+\eta n / k r$. Since $L_{t}$ changes by $k r$ in each step, if $L_{t^{\prime}} \leq \eta n$ for some $t_{1} \leq t^{\prime} \leq \tau_{2}(\sigma)$, then $A_{t}$ must occur for some $t_{1} \leq t \leq t^{\prime}-\eta n / k r$. Next, we bound the probability of $A_{t}$. We may assume $2 \eta n-k r \leq L_{t}<2 \eta n$ since otherwise $A_{t}$ does not hold. Since $L_{t}$ changes by $k r$ in each step, we have $L_{t^{\prime}} \leq 3 \eta n \leq\left(\epsilon_{1} k \sigma / 4 r\right) n$ for all $t \leq t^{\prime} \leq t+\eta n / k r$. Then, $r L_{t} / B_{t} \leq \epsilon_{1} / 4$. Then by (14), for all $t \leq t^{\prime} \leq t+\eta n / k r$,

$$
\mathbb{E}\left(L_{t+1}-L_{t} \mid \mathcal{F}_{t}\right) \geq \epsilon_{1} / 2
$$

By Lemma 8 ,

$$
\operatorname{Pr}\left(A_{t}\right) \leq \operatorname{Pr}\left(L_{t+\eta n / k r}<2 \eta n \mid L_{t} \geq 2 \eta n-k r\right)=o\left(n^{-1}\right) .
$$

By the union bound, the probability that $A_{t}$ occurs for some $t$ is $o(1)$ and thus, a.a.s. $L_{t} \geq \eta n$ for all $t_{1} \leq t \leq \tau_{2}(\sigma)$.

Now we complete the proof of Theorem 3(a). By the definition of $I_{\sigma}$ in (28), $\widehat{G}_{I_{\sigma}+1}$ contains less than $\sigma n$ vertices. By Lemma 19, every component in $\widehat{G}_{I_{\sigma}+2}$ has $O(\log n)$ vertices. It was proved in [8] that each of such components has stripping number $O(\log \log n)$ (the basic idea there is that, a.a.s. every subgraph of $A P_{r}(n, c n)$ (for any $c=\Theta(1)$ ) with size $O(\log n)$ is so sparse that, when the parallel stripping process is applied to it, a positive proportion of vertices are stripped off in each round, and thus a total of $O(\log \log n)$ iterations is sufficient). Hence, a.a.s. $s\left(G_{0}\right) \leq I_{\sigma}+2+O(\log \log n)$. By Lemma [25, a.a.s. $\left|S_{i}\right|=$ $\Omega\left(L_{t(i)}\right)=\Omega(n)$ for every $I_{2} \leq i \leq I_{\sigma}$, and so a.a.s. $I_{\sigma}-I_{2}=O(1)$. By Lemmas 24, a.a.s. $I_{2}=O\left(\xi^{-1 / 2} \log (1 / \xi)\right)$. Recall that $G_{0}=\widehat{H}_{\tau^{\prime}(B)}$ by definition (10). It follows then that a.a.s. $s\left(\widehat{H}_{\tau^{\prime}(B)}\right)=s\left(G_{0}\right)=O\left(\xi^{-1 / 2} \log (1 / \xi)+\log \log n\right)$. Then Theorem 3(a) follows by (7), (8) and Corollary 1 .

### 4.9 Inside the critical window: proof of Theorem 3(b)

Now we fix a constant $\epsilon>0$, and consider $c$ such that $\left|c-c_{r, k}\right| \leq n^{-1 / 2+\epsilon}$. Again, we may assume that $\epsilon$ is sufficiently small. Let $c^{\prime}=c_{r, k}+n^{-1 / 2+2 \epsilon}$. By Theorem (a), a.a.s. $\tau\left(H^{\prime}\right)=O\left(n^{1 / 4-\epsilon} \log n\right)$. Since $c^{\prime}$ satisfies all conditions in (5), all lemmas and corollaries in Sections 4.5 and 4.6 hold. However, we note here that $\xi$ and $\xi^{\prime}$ are no longer of the same order, and so we cannot replace $\xi^{\prime}$ by $\xi$ in any asymptotic expression. By Lemma 11 we have $\zeta_{0}=\zeta+\Theta\left(n^{-1 / 4+\epsilon}\right)$ and $\theta_{0}=\theta-\Theta\left(n^{-1 / 4+\epsilon}\right)$.

Let $G_{0}$ be as defined in (10). Recall that $G_{0}, G_{1}, \ldots$ is the sequence produced by SLOWSTRIP. Let $\widehat{\tau}$ denote the step when SLOW-STRIP terminates. Since $c$ is inside the critical window $c_{r, k}+O\left(n^{-1 / 2+\epsilon}\right)$, whether $G_{\tau}$ is empty or not is not a.a.s. certain.

Let $K>0$ be a constant to be determined later; define $t_{1}=K n^{3 / 4+\epsilon}$ (i.e. $t_{1}=K n \sqrt{\xi^{\prime}}$ ).
Lemma 26. Assume $K>0$ is sufficiently large and $\epsilon$ is sufficiently small. Then, a.a.s. either $\widehat{\tau} \leq t_{1} / 2$; or there is $t \leq t_{1}$ that $L_{t}=\Omega\left(n^{1 / 2+2 \epsilon}\right)$.

Proof. Since $t_{1}=K n \sqrt{\xi^{\prime}}$, by Corollary 18, provided $K>0$ is sufficiently large, we have a.a.s. either $\widehat{\tau} \leq t_{1}$ or $C_{1} t / n \leq \theta_{t} \leq C_{2} t / n$ for some constants $C_{1}, C_{2}>0$ and for all $t_{1} / 2 \leq t \leq \tau_{2}(\sigma)$.

Let $Y_{3}>0$ be a constant to be specified later. Define $\tau_{3}$ to be the minimum integer $t \geq t_{1} / 2$ such that $L_{t+1} \geq Y_{3} n^{3 / 4+\epsilon}$. If no such integer $t$ exists, then define $\tau_{3}=n$. Define $T_{1}=\min \left\{t_{1}, \tau_{3}, \tau_{2}(\sigma)\right\}$. Then, for all $t_{1} / 2 \leq t \leq T_{1}$, we have $L_{t}<Y_{3} n^{3 / 4+\epsilon}$ and $B_{t} \geq k \sigma n$. Therefore, $L_{t} / B_{t} \leq\left(Y_{3} / k \sigma\right) n^{-1 / 4+\epsilon}$. Hence $1-L_{t} / B_{t} \geq 2 / 3$ and $r L_{t} / B_{t} \leq\left(r Y_{3} / k \sigma\right) n^{-1 / 4+\epsilon}$. By (30), for all $t_{1} / 2 \leq t \leq T_{1}$,

$$
\mathbb{E}\left(L_{t+1}-L_{t} \mid \mathcal{F}_{t}\right) \geq\left(2 C_{1} / 3\right) \frac{t}{n}-\left(r Y_{3} / k \sigma\right) n^{-1 / 4+\epsilon}+O\left(n^{-1}\right) \geq\left(C_{1} K / 4\right) n^{-1 / 4+\epsilon},
$$

by choosing $Y_{3}<k \sigma C_{1} K / 12 r$, since $t \geq t_{1} / 2=(K / 2) n^{3 / 4+\epsilon}$.
We may assume that $\tau_{3} \geq t_{1}$ since otherwise there is $t_{1} / 2 \leq t \leq t_{1}$ such that $L_{t} \geq Y_{3} n^{3 / 4+\epsilon}$ and so claim of the lemma is verified. By the definition of $\tau_{2}(\sigma)$, we have $\tau(\sigma)>t_{1}$ always, as $\tau_{2}(\sigma)$ is linear in $n$. Hence, we may assume that $T_{1}=t_{1}$.

Applying Lemma 8 to $\left(L_{t}\right)_{t=t_{1} / 2}^{t_{1}}\left(\right.$ with $a_{n}=-\left(C_{1} K / 4\right) n^{-1 / 4+\epsilon}, c_{n}=k r$ and $\left.j=-t a_{n} / 2\right)$, we have a.a.s. $L_{t_{1}} \geq\left(C_{1} K / 8\right) n^{-1 / 4+\epsilon}\left(t_{1} / 2\right)=\left(C_{1} K^{2} / 16\right) n^{1 / 2+2 \epsilon}$. This means that a.a.s. either $\widehat{\tau} \leq t_{1} / 2$, or there is $t_{1} / 2 \leq t \leq t_{1}$ such that $L_{t} \geq Y_{3} n^{3 / 4+\epsilon}$ (i.e. $\tau_{3} \geq t_{1}$ ), or $L_{t_{1}}=\Omega\left(n^{1 / 2+2 \epsilon}\right)$. In each case, the claim of the lemma is verified (as $Y_{3} n^{3 / 4+\epsilon}>n^{1 / 2+2 \epsilon}$ by taking $\epsilon<1 / 4$ ).

We may assume that there is a $t_{1} / 2 \leq t_{0} \leq t_{1}$ such that $L_{t_{0}} \geq n^{1 / 2+2 \epsilon}$, since otherwise $\widehat{\tau} \leq t_{1}$ by Lemma 26 and so $s\left(G_{0}\right) \leq t_{1}=O\left(n^{3 / 4+\epsilon}\right)$. Next, we prove that the parallel stripping process does not take long from $G_{t_{0}}$.
Lemma 27. Assume $L_{t_{0}} \geq n^{1 / 2+2 \epsilon}$ for some $t_{1} / 2 \leq t_{0} \leq t_{1}$. Then, a.a.s. $s\left(G_{t_{0}}\right)=O\left(n^{1 / 2}\right)$.
Proof. Since $t_{0} \geq(K / 2) n^{3 / 4+\epsilon}=(K / 2) n \sqrt{\xi^{\prime}}$, by Corollary 18, provided that $K>0$ is sufficiently large, we have $\theta_{t} \geq 2 C t / n \geq C K n^{-1 / 4+\epsilon}$ for all $t_{0} \leq t \leq \tau_{2}(\sigma)$, for some constant $C>0$. We will prove that a.a.s. there is no $t_{0} \leq t \leq \tau_{2}(\sigma)$ such that $L_{t}<n^{1 / 2}$. Let $A_{t}$ be the event that $L_{t} \geq 2 n^{1 / 2}-k r$, and $L_{j}<2 n^{1 / 2}$ for all $t \leq j \leq t+n^{1 / 2} / k r$. Since $\left|L_{t+1}-L_{t}\right| \leq k r$ always, $L_{t_{0}} \geq n^{1 / 2+2 \epsilon}$, if $i$ is the first step after $t_{0}$ such that $L_{i}<n^{1 / 2}$, then, there must be some $i^{\prime} \leq i-n^{1 / 2} / k r$ such that $A_{i^{\prime}}$ occurs. Hence, it is sufficient to prove that a.a.s. there is no $t_{0}-n^{1 / 2} / k r \leq i \leq \tau_{2}(\sigma)-n^{1 / 2} / k r$ for which $A_{i}$ occurs.

Since $L_{t_{0}} \geq n^{1 / 2+2 \epsilon}, A_{i}$ cannot occur for any $t_{0}-n^{1 / 2} / k r \leq i \leq t_{0}$ by definition of $A_{i}$. Next, for each $t_{0}<i \leq \tau_{2}(\sigma)-n^{1 / 2} / k r$, we bound the probability of $A_{i}$. We may assume that $2 n^{1 / 2}-k r \leq L_{i}<2 n^{1 / 2}$ since otherwise, $A_{i}$ does not hold by definition. Let $T$ be the minimum integer such that $T>i$ and $L_{T+1} \geq 2 n^{1 / 2}$. Define $T=n$ if no such integer exists. Now, for all $i \leq t \leq T, L_{t}<2 n^{1 / 2}$ and so $L_{t} / B_{t}=o\left(n^{-1 / 4+\epsilon}\right)$. By (30) and the fact that $\theta_{t} \geq C K n^{-1 / 4+\epsilon}$ for all $t_{0} \leq i \leq t \leq \tau_{2}(\sigma)$,

$$
\begin{equation*}
\mathbb{E}\left(L_{t+1}-L_{t} \mid \mathcal{F}_{t}\right) \geq(1 / 2) \theta_{t}+o\left(n^{-1 / 4+\epsilon}\right) \geq(C K / 2) n^{-1 / 4+\epsilon} . \tag{34}
\end{equation*}
$$

Now,

$$
\operatorname{Pr}\left(A_{i}\right) \leq \operatorname{Pr}\left(T \geq i+n^{1 / 2} / k r\right)=\operatorname{Pr}\left(L_{t}<2 n^{1 / 2}, \forall i \leq t \leq i+n^{1 / 2} / k r\right)
$$

However, since $L_{i} \geq n^{1 / 2}$ by assumption, by applying Lemma 8 to (34), with probability $1-o\left(n^{-1}\right)$, we must have $L_{i+n^{1 / 2} / k r} \geq L_{i}+\Omega\left(n^{1 / 4+\epsilon}\right)>2 n^{1 / 2}$. Hence, $\operatorname{Pr}\left(A_{i}\right)=o\left(n^{-1}\right)$ for every $i$. This confirms that a.a.s. $L_{t} \geq n^{1 / 2}$ for all $t_{0} \leq t \leq \tau_{2}(\sigma)$. Therefore, a.a.s. the total degree of the configuration decreases by at least $n^{1 / 2}$ in each iteration of the parallel stripping process applied to $G_{t_{0}}$, until there are at most $\sigma n$ vertices remaining. Now $s\left(G_{t_{0}}\right)=O\left(n / n^{1 / 2}+\log n\right)=O\left(n^{1 / 2}\right)$ by Lemma 19 .

Now we complete the proof of Theorem $3(b)$. If SLOW-STRIP terminates by step $t_{1}$, then $s\left(G_{0}\right)=O\left(n^{3 / 4+\epsilon}\right)$. Hence, $s(H) \leq \tau^{\prime}(B)+1+s\left(G_{0}\right)=O\left(n^{3 / 4+\epsilon}\right)$ by (7), (10) and (8). If not, by Lemma 26 we may assume that there is a step $t_{0} \leq t_{1}$ such that $L_{t_{0}} \geq n^{1 / 2+2 \epsilon}$. By Lemma 27, a.a.s. $s\left(G_{t_{0}}\right)=O\left(n^{1 / 2}\right)$. Hence, a.a.s. $s(H) \leq \tau^{\prime}(B)+1+t_{1}+s\left(G_{t_{0}}\right)=O\left(n^{3 / 4+\epsilon}\right)$. This implies part (b) of Theorem 3.

## 5 Bounding the maximum depth

Let $H_{0} \in A P_{r}(n, c n)$ where $c=c_{r, k}-\xi_{n}$, and let $H_{0}, H_{1}, \ldots$, denote the parallel stripping process. For simplicity, we write $\xi_{n}$ as $n^{\delta}$ ( $\delta$ may depend on $n$ ). Our goal is to show that the maximum depth of all the non- $k$-core vertices of $A P_{r}(n, c n)$ is a.a.s. $n^{\Theta(\delta)}$.

Recall that $S_{i}$ is the set of light vertices in $H_{i}$. Let $I_{\max }$ denote the last iteration of the parallel stripping process; i.e. $I_{\max }=s\left(H_{0}\right)$. Let $v$ be a vertex in $S_{I_{\max }-1}$. Then every stripping sequence ending with $v$ must contain at least one vertex in each $S_{i}, 1 \leq i \leq I_{\max }-1$. Hence, the depth of $v$ is at least $I_{\max }-1$, which is a.a.s. $\Omega\left(n^{\delta / 2}\right)$ by Theorem $1(\mathrm{~b}, \mathrm{c})$. This verifies the lower bound in Theorem 4. The main challenge of this section is to prove the upper bound.

Let $\Psi=v_{1} v_{2} \cdots$ be a stripping sequence that contains all non- $k$-core vertices of a hypergraph $H$. Let $\mathcal{D}(\Psi)$ be a digraph constructed as follows. The set of vertices in $\mathcal{D}(\Psi)$ is $V(H)$. At the moment $v_{i}$ is removed from the hypergraph $H$, consider each hyperedge $x$ that is incident with $v_{i}$ before the removal of $v_{i}$, add a directed edge $u \rightarrow v$ to $\mathcal{D}(\Psi)$, for each of the other $r-1$ vertices $u \in x$. For any vertex $v \in H$, we define $R_{\Psi}^{+}(v)$ to be the set of vertices reachable from $v$ in $\mathcal{D}(\Psi)$. Clearly, $R_{\Psi}^{+}(v)$ forms a stripping sequence ending with $v$ and so $\left|R^{+}(v)\right|$ is an upper bound of the depth of $v$. We will prove the following stronger statement than the upper bound in Theorem 4.

Theorem 6. Let $r, k \geq 2,(r, k) \neq(2,2)$ be fixed and assume $c=c_{r, k}-n^{-\delta}$ where $n^{\delta} \geq \log ^{7} n$ (here $\delta$ can depend on $n$ ). Then, a.a.s. there is a stripping sequence $\Psi$ of $\mathcal{H}_{r}(n, c n)$ containing all non-k-core vertices such that

$$
\begin{equation*}
\left|R_{\Psi}^{+}(v)\right|=n^{O(\delta)} \quad \text { for all } v \in \Psi \tag{35}
\end{equation*}
$$

It was shown in [9] that (35) a.a.s. holds if $c=c_{r, k}+n^{-\delta}$. The analysis for the subcritical case is analogous to the supercritical case. Thus, we summarise the approach in 9 and sketch a proof for Theorem 6 by pointing out how the arguments in [9] should be adapted.

Let $H \in A P_{r}(n, c n)$ and run SLOW-STRIP on $H$. This produces a stripping sequence $\Psi$. Let $\mathcal{D}=\mathcal{D}(\Psi)$ be the digraph associated with $\Psi$. This defines $R_{\Psi}^{+}(v)$ for any non- $k$-core vertex $v$. For simplicity, we drop the subscript $\Psi$ from the notation.

Definition 28. For each $0 \leq j \leq I_{\max }-1$, define $\mathcal{S}_{j}$ to be a graph with vertex set $S_{j}$ and edge set $\left\{f \cap S_{j}: f \in E\left(H_{j}\right)\right\}$.

Take an arbitrary vertex $v \in S_{i}$; set $R_{i}^{\prime}=R_{i}^{\prime}(v):=\{v\}$ and for each $j=i$ to 0 :
(a) We set $R_{j}=R_{j}(v)$ to be the union of the vertex sets of all components of $\mathcal{S}_{j}$ that contain vertices of $R_{j}^{\prime}$.
(b) We set $R_{j-1}^{\prime}$ to be the set of all vertices $v \in S_{j-1}$ that are adjacent to $\cup_{\ell=i}^{j} R_{\ell}$.

Define $R(v)=\cup_{j=0}^{i} R_{j}$.
Note that $\cup_{j \leq i} R_{j}$ contains all vertices reachable from $v$ and therefore $\left|R^{+}(v)\right| \leq \cup_{j \leq i}\left|R_{j}\right|$. The definition of $R_{j}(v)$ makes it easier (compared with $\left|R^{+}(v)\right|$ ) to bound $\left|R_{i}\right|$, given $\left|R_{j}\right|$ $(j>i)$.

### 5.1 Proof outline for the supercritical case [9, Section 5]

Starting with a vertex $v \in S_{i}$, we explore the hyperedges from $v$ to other vertices in $S_{i}$ and $S_{i-1}$, and then onwards; recursively, we will bound $\left|R_{j}\right|$ for each $j \leq i$. We need randomness to allow such analysis; but we can expose $v \in S_{i}$ only after we expose all vertices in $S_{j}, j<i$
and their incident hyperedges, and that has killed all the randomness. A solution to this obstacle is given in [9]. Basically, we will expose all vertices in each $S_{i}, 0 \leq i \leq I_{\max }-1$, as well as some degree information. For instance, we need information on the number of total neighbours $S_{i}$ has in $S_{i-1}$, etc. This procedure is called EXPOSURE. Then we will generate uniformly a random configuration that agrees with the exposed information. This procedure is called EDGE-SELECTION (similar to the configuration model). See [9, Section 5.1] for details. It was proved ([9, Lemma 46]) that EDGE-SELECTION generates the random configuration with the correct distribution, conditional on the information exposed in EXPOSURE.

Now, conditional on the set of parameters exposed in EXPOSURE, we can recursively bound $\left|R_{i}\right|$, given $\left|R_{j}\right|, j>i$, as EDGE-SELECTION defines a probability space that is easy for such analysis. Analysing SLOW-STRIP allows us to prove a.a.s. properties of the set of parameters exposed in EXPOSURE. This gives [9, Lemma 49(a-i)], where the first three of these properties have been stated in Lemma 2. We will state the corresponding properties for the subcritical case in Section 5.2.

We first introduce two lemmas that allow us to focus our analysis on $R_{j}(v)$ for $j$ such that the number of vertices in $H_{j}$ is close to $\alpha n$. The first lemma is proved in [9, Lemma 15 ], with $c=c_{r, k}+n^{-\delta}$ for some $0<\delta<1 / 2$; but exactly the same proof gives the following statement with all $c=c_{r, k}+o(1)$.
Lemma 29. Assume $c=c_{r, k}+o(1)$. Given any constant $\epsilon>0$, there is a constant $B=B(r, k, \epsilon)>0$ such that after $B$ rounds of the parallel stripping process are applied to $A P_{r}(n, c n)$, the number of vertices remaining in $H_{B}$ is $(\alpha+O(\epsilon)) n$.

Given a set of vertices $A$, let $N^{s}(A)$ denote the set of vertices with distance at most $s$ from $A$. The following lemma is from [1, Lemma 34].
Lemma 30. Assume $s, c>0$ are $O(1)$. A.a.s. for every subset of vertices $A$ in $A P_{r}(n, c n)$ such that $A$ induces a connected subgraph, $\left|N^{s}(A)\right|=O(|A|+\log n)$.

In the supercritical case, $H_{0}, H_{1}, \ldots, H_{I_{\max }}$ is the parallel stripping process with $H_{0} \in$ $A P_{r}(n, c n)$, where $c \geq c_{r, k}+n^{-\delta}$. Fix an $\epsilon>0$. By Lemma 29, there is a constant $B>0$ such that the number of vertices in $H_{B}$ is at most $(\alpha+\epsilon) n$. Assume we have successfully bounded $\sum_{B \leq j \leq i}\left|R_{j}(v)\right|$ for some $v \in S_{i}$; then, by Lemma 30, a.a.s.

$$
\begin{equation*}
\left|R^{+}(v)\right| \leq \sum_{0 \leq j \leq i}\left|R_{j}\right|=O\left(\log n+\sum_{B \leq j \leq i}\left|R_{j}\right|\right) \tag{36}
\end{equation*}
$$

Therefore, in order to bound $\left|R^{+}(v)\right|$, it is sufficient to bound $\left|R_{j}(v)\right|$ for every $B \leq j \leq i$.
Let $\mathcal{E}_{i}$ denote the set of $r$-tuples incident with $S_{i}$ in $H_{i}$; i.e. it is the set of hyperedges to be deleted in the $(i+1)$-th iteration. For each $v \in S_{i+1}$, define $d^{-}(v)$ to be the number points in $v$ that are contained in an $r$-tuple in $\mathcal{E}_{i}$. In the case $r=2, d^{-}(v)$ is simply the number of neighbours of $v$ in $S_{i}$ in $H_{i}$. Naturally, we have $d^{-}(v) \geq 1$ for every $v \in S_{i+1}$, since otherwise, $v$ would not be deleted in the $(i+1)$-th iteration. Define $D^{-}(X)=\sum_{v \in X} d^{-}(X)$. Then, $\left|R_{j}(v)\right| \leq D^{-}\left(R_{j}(v)\right)$. A major part of the work in [9, Section 5] is to recursively bound $D^{-}\left(R_{j}(v)\right)$, using the random configuration generated by EDGE-SELECTION, and a set of a.a.s. properties in [9, Lemma 49], as follows.

Lemma 31. If all properties in [9, Lemma $49(a-i)]$ hold then there are constants $B=$ $B(r, k), Z=Z(r, k)>0$ such that for all $j \geq B: \quad$ If $\left|S_{j}\right| \geq n^{\delta} \log ^{2} n$ then:

$$
\begin{equation*}
D^{-}\left(R_{j}(v)\right) \leq D^{-}\left(R_{j+1}(v)\right)+Z \frac{\left|S_{j}\right|}{n} \sum_{\ell=i}^{j+1} D^{-}\left(R_{\ell}(v)\right)+\log ^{14} n \tag{37}
\end{equation*}
$$

Solving this recurrence, together with (36) produces $\left|R^{+}(v)\right|=n^{O(\delta)}$ for all non- $k$-core vertices $v$.

### 5.2 Sketch of the proof of Theorem 6

It only remains to prove the upper bound. We may assume that $c=c_{r, k}-n^{-\delta}$ for some $\delta<1 / 2$ since for the case that $\delta \geq 1 / 2$, the upper bound in the theorem is trivial, as discussed in the remark below Theorem 6. The approach is the same as in the supercritical case. The same set of a.a.s. properties (c.f. [9, Lemma 49(a-i)] for the supercritical case) are exposed in EXPOSURE. Properties in [9, Lemma 49(d-i)] carry to the subcritical case, as the proof of these properties only requires $c=\Theta(1)$. The proof of Lemma 31 depends mainly on these properties, as well as the restriction that $\left|S_{i}\right| / n$ is sufficiently small. Hence, we will have the same recursive bound as in Lemma 31 for most iterations in the subcritical case. However, $\left|S_{i}\right|$ behaves in a different manner in the subcritical case, which results in a different argument in solving the recurrence (37). The behaviour of $\left|S_{i}\right|$ in the supercritical case is characterised in Lemma 2 (which is [9, Lemma 49(a-c)]). This changes significantly in the subcritical case, as below.

Lemma 32. There exist positive constants $B, Y_{1}, Y_{2}$ dependent only on $r, k$, and integers $I_{0}<I_{1}<B^{\prime}$ as growing functions of $n$, such that a.a.s. for every $B \leq i<B^{\prime}$ :
(a) if $i \leq I_{0}$, then $\left(1-Y_{1} \sqrt{\left|S_{i}\right| / n}\right)\left|S_{i}\right| \leq\left|S_{i+1}\right| \leq\left(1-Y_{2} \sqrt{\left|S_{i}\right| / n}\right)\left|S_{i}\right|$ and $\left|S_{i}\right|=\Omega\left(n^{1-\delta}\right)$;
(b) if $i \geq I_{1}$, then $\left(1+Y_{2} \sqrt{\left|S_{i}\right| / n}\right)\left|S_{i}\right| \leq\left|S_{i+1}\right| \leq\left(1+Y_{1} \sqrt{\left|S_{i}\right| / n}\right)\left|S_{i}\right|$ and $\left|S_{i}\right|=\Omega\left(n^{1-\delta}\right)$;
(c) $I_{1}-I_{0}=O\left(n^{\delta / 2}\right)$ and for every $I_{1} \leq i \leq I_{2},\left(1-Y_{1} n^{-\delta / 2}\right)\left|S_{i}\right| \leq\left|S_{i+1}\right| \leq\left(1+Y_{1} n^{-\delta / 2}\right)\left|S_{i}\right|$ and $\left|S_{i}\right|=\Theta\left(n^{1-\delta}\right) ;$

Proof of Lemma 32. The proof of Lemma 12 only requires that $\left|\widehat{\zeta}_{0}-\zeta\right|<\epsilon$ for some sufficiently small constant $\epsilon>0$ (the value of $\epsilon$ depends on the gap between $\zeta$ and $r(k-1)$ in (19)). Fix a small constant $\epsilon^{\prime}>0$. By Lemma 29, there is a sufficiently large constant $B=B\left(r, k, \epsilon^{\prime}\right)>0$, such that the number of vertices in $H_{B}$ is at most $\left(\alpha+\epsilon^{\prime}\right) n$. By Theorem [5) (b), a.a.s. $\left|\zeta\left(H_{B}\right)-\zeta\right|=O\left(\epsilon^{\prime}\right)$. Hence, by choosing $\epsilon^{\prime}>0$ sufficiently small (and thus $B$ sufficiently large), the evolution of $\theta_{t}$ is well depicted by Lemma 12, when SLOW-STRIP is applied to $H_{B}$. This fixes the constant $B$ in the lemma. Comparing with the analysis in Section 4, now we start our analysis from a configuration, $H_{B}$, that appears earlier than $G_{0}$ in SLOW-STRIP applied to $H$. Thus, all results in Section 4 hold, except for a shift of the subscript $t$ in all parameters such as $\theta_{t}$.

Let $I_{\sigma}$ denote the first iteration in the parallel stripping process that the number of vertices becomes at most $\sigma n$. By Lemma 25, there is an integer $B^{\prime}$ (corresponding to $I_{2}$ in

Lemma (25), such that $L_{t\left(B^{\prime}\right)} \geq \epsilon n$ for all $t\left(B^{\prime}\right) \leq t \leq t\left(I_{\sigma}-1\right)$, provided that $\epsilon$ was chosen sufficiently small. This fixes the integer $B^{\prime}$ in the lemma.

By Lemma 22, we can specify two iterations $I_{0}$ and $I_{1}$ (corresponding respectively to iterations 0 and $I_{2}$ in Lemma 22, due to a shift of the subscript) in the parallel stripping process such that the value of $\theta\left(H_{i}\right)$ changes from $-\Theta\left(n^{-\delta / 2}\right)$ to $\Theta\left(n^{-\delta / 2}\right)$. Moreover, Lemma 22 states that a.a.s. $I_{1}-I_{0}=O\left(n^{\delta / 2}\right)$, and both $L_{t\left(I_{0}\right)}$ and $L_{t\left(I_{1}\right)}$ are $\Theta\left(n^{1-\delta}\right)$. This fixes the integers $I_{0}$ and $I_{1}$ in the lemma and verifies the first claim of part (c).

It only remains to prove that $\left|S_{i}\right|$ changes in a rate described in parts (a,b,c). This part of the proof is similar to the approach in [9, Sections 6.2 and 6.3] ( $\theta_{t}$ was denoted by $b r_{t}$ in [9]) so we briefly sketch the idea.

For (b): Consider $t(i) \leq t \leq t(i+1)$ for some $I_{1} \leq i \leq B^{\prime}$. By Lemma 23 and Corollary 18, we may assume that $L_{t}=\Theta\left(t^{2} / n\right)$ and $\theta_{t}=\Theta(t / n)$. Thus, we may assume that $\theta_{t}=\Theta\left(\sqrt{L_{t} / n}\right)$ for all $t$. Hence, by (30)

$$
\mathbb{E}\left(L_{t+1}-L_{t} \mid \mathcal{F}_{t}\right)=\Theta\left(\sqrt{L_{t} / n}\right)+O\left(\frac{L_{t}}{n}\right)
$$

which is $\Theta\left(\sqrt{L_{t} / n}\right)$ as long as $L_{t} / n$ bounded by some sufficiently small constant $\epsilon_{1}>0$. Then, by Lemma 8 we obtain the desired recursion in part (b), with $\left|S_{i}\right|$ replaced by $L_{t(i)}$, until $L_{t(i)}$ reaches $\epsilon_{1} n$. We may choose $B^{\prime}$ appropriately to ensure that a.a.s. $L_{t\left(B^{\prime}\right)}<\epsilon_{1} n$ (e.g. choosing $B^{\prime}$ to be the last step in the parallel stripping process after iteration $I_{1}$, such that the total degree of the light vertices is less than $\epsilon_{1} n$ ). It is easy to show that a.a.s. $\left|S_{i}\right|=\Theta\left(\left|L_{t(i)}\right|\right)$ for every $i$ (e.g. see [9, eq. (98)]), and immediately part (b) follows.

For (a): The analysis in Section 4 only covers iterations in parts (b) and (c), as we started our analysis from iterations close to $I_{0}$ by the choice of $G_{0}$. However, the evolution of $L_{t(i)}$ for $B \leq i \leq I_{0}$ is "symmetric" to iterations from $I_{1}$ to $B^{\prime}$. By Lemma 12, $\theta_{t}$ increases with a linear rate during all steps $t(B) \leq t \leq t\left(B^{\prime}\right)$. Then $L_{t}$ and $\theta_{t}$ for $t(B) \leq t \leq t\left(I_{0}\right)$ can be analysed in the same way as in Lemma 23, except that $L_{t}$ decreases with a certain rate (rather than increases), due to the fact that $\theta_{t}$ is negative for $t(B) \leq t \leq t\left(I_{0}\right)$.

For (c): Between iterations $I_{0}$ and $I_{1}$, we have in the proof of Lemma 22 that $\theta_{t}=$ $O\left(n^{-\delta / 2}\right)$ and $L_{t}=\Omega\left(n^{1-\delta}\right)$. Similar to the argument in part (b), part (c) follows by Lemma 8,

In the subcritical case, a.a.s. the parallel stripping process (and SLOW-STRIP) terminates with an empty graph. Hence, a.a.s. every vertex is a non- $k$-core vertex. Let $I_{\max }$ denote the last iteration of the parallel stripping process. By Theorem 3(a), we may assume that $I_{\max }=O\left(n^{\delta / 2} \log n\right)$.

Let $B$ and $B^{\prime}$ be integers in Lemma 32. We first prove that a.a.s. for every $v \in A P_{r}(n, c n)$, $\left|R^{+}(v) \cap H_{B^{\prime}}\right|=O(\log n)$. Define $R_{j}(v)$ as in Definition [28. We will actually show that $\cup_{j \geq B^{\prime}} R_{j}(v) \supseteq R^{+}(v) \cap H_{B^{\prime}}$ contains $O(\log n)$ vertices. We may assume that $v \in H_{B^{\prime}}$ since otherwise $\cup_{j \geq B^{\prime}} R_{j}(v)=\emptyset$. By Lemma [25, a.a.s. for all $B^{\prime} \leq i \leq I_{\sigma}$, the total degree of $H_{i}$ decreases by $\Omega(n)$ in each iteration of the parallel stripping process. Therefore, a.a.s. $I_{\sigma}-B^{\prime}=O(1)$. By Lemma 19, every component in $H_{I_{\sigma}+2}$ has size $O(\log n)$. It follows then that $\left|\cup_{j \geq I_{\sigma}+2} R_{j}(v)\right|=O(\log n)$. Now by Lemma 30, $\left|\cup_{j \geq B^{\prime}} R_{j}(v)\right| \leq C \log n$ for some constant $C>0$, since all vertices in $\cup_{j \geq B^{\prime}} R_{j}(v)$ are of distance $O(1)$ from $\cup_{j \geq I_{\sigma}+2} R_{j}(v) \subseteq H_{I_{\sigma}+2}$.

This allows us to focus on bounding $R_{j}(v)$ for $j \leq B^{\prime}$. The same proof of Lemma 31 yields the same bound for $D^{-}\left(R_{j}(v)\right)$, as below.

$$
\begin{equation*}
D^{-}\left(R_{j}(v)\right) \leq D^{-}\left(R_{j+1}(v)\right)+Z \frac{\left|S_{j}\right|}{n} \sum_{\ell=i}^{j+1} D^{-}\left(R_{\ell}(v)\right)+\log ^{14} n, \quad \text { for all } B \leq j \leq B^{\prime} \tag{38}
\end{equation*}
$$

Moreover, we have shown that $\left|\cup_{j \geq B^{\prime}} R_{j}(v)\right| \leq C \log n$.
Let $i$ be the integer that $v \in S_{i}$. Next, we will recursively define $r_{j}$ such that

$$
D^{-}\left(R_{i-j}(v)\right) \leq r_{j} \quad \forall 0 \leq j \leq i-B
$$

This approach is similar to the work in [9, Section 5.5]. Note that (44) holds only for $j \leq B^{\prime}$, and thus, we need to specify $r_{j}$ differently depending on whether $i \leq B^{\prime}$ or $i>B^{\prime}$. To cope with that, define

$$
\begin{align*}
& j_{0}=\max \left\{0, i-B^{\prime}\right\}  \tag{39}\\
& r_{j}=n^{2 \delta} \quad \forall 0 \leq j \leq j_{0} \tag{40}
\end{align*}
$$

We first confirm that $D^{-}\left(R_{i-j}(v)\right) \leq r_{j}$ for all $0 \leq j \leq j_{0}$. It is easy to show that a.a.s. the maximum degree of the original configuration $H \in A P_{r}(n, c n)$ is $O(\log n)$ (see, e.g. the proof of [9, Lemma 49(i)]). If $i \leq B^{\prime}$ then $j_{0}=0$, and so $D^{-}\left(R_{i}(v)\right)=O(\log n)$, which is less than $r_{0}$. If $i>B^{\prime}$, then $\left|\cup_{j \geq B^{\prime}} R_{j}(v)\right| \leq C \log n$ and so

$$
\begin{equation*}
\sum_{j=0}^{j_{0}} D^{-}\left(R_{i-j}(v)\right)=O\left(\log n \sum_{j \geq B^{\prime}}\left|R_{j}(v)\right|\right)=O\left(\log ^{2} n\right) \leq r_{j} \tag{41}
\end{equation*}
$$

Now, we have verified that $D^{-}\left(R_{i-j}(v)\right) \leq r_{j}$ for all $0 \leq j \leq j_{0}$. Define

$$
\begin{equation*}
r_{j}=r_{j-1}+Z \frac{\left|S_{i-j}\right|}{n} \sum_{i=0}^{j-1} r_{i}+n^{\delta}, \quad \forall j_{0}+1 \leq j \leq i-B \tag{42}
\end{equation*}
$$

Inductively, we prove that $D^{-}\left(R_{i-j}(v)\right) \leq r_{j}$ for all $j_{0} \leq j \leq i-B$. We have proved the base case. Assume it holds for $j-1$. Then, for $j$, by (44) and induction,

$$
\begin{aligned}
D^{-}\left(R_{i-j}(v)\right) & \leq D^{-}\left(R_{i-j+1}(v)\right)+Z \frac{\left|S_{i-j}\right|}{n} \sum_{\ell=0}^{j-1} D^{-}\left(R_{i-\ell}(v)\right)+\log ^{14} n \\
& \leq r_{j-1}+Z \frac{\left|S_{i-j}\right|}{n} \sum_{\ell=j_{0}}^{j-1} r_{\ell}+Z \frac{\left|S_{i-j}\right|}{n} \sum_{\ell=0}^{j_{0}-1} D^{-}\left(R_{i-\ell}(v)\right)+\log ^{14} n \leq r_{j},
\end{aligned}
$$

by noting that $Z \frac{\left|S_{i-j}\right|}{n} \sum_{\ell=0}^{j_{0}-1} D^{-}\left(R_{i-\ell}(v)\right)+\log ^{14} n=O\left(\log ^{14} n\right) \leq n^{2 \delta}$ by (41).
Now, $r_{j}$ bounds $D^{-}\left(R_{i-j}\right) \geq\left|R_{i-j}\right|$. It is convenient to define

$$
t_{j}=\sum_{j_{0} \leq \ell \leq j} r_{j}
$$

Then, again by (41),

$$
\begin{equation*}
\left|\cup_{j \geq B} R_{j}(v)\right| \leq \sum_{j=0}^{i-B} D^{-}\left(R_{i-j}(v)\right)=\sum_{j=j_{0}}^{i-B} D^{-}\left(R_{i-j}(v)\right)+\sum_{j=0}^{j_{0}-1} D^{-}\left(R_{i-j}(v)\right) \leq t_{i-B}+O\left(\log ^{2} n\right) \tag{43}
\end{equation*}
$$

Noting that $r_{j}=t_{j}-t_{r-1}$, (42) yields the following recurrence for $t_{j}$ :

$$
\begin{equation*}
t_{j}-t_{j-1}=t_{j-1}-t_{j-2}+Z \frac{\left|S_{i-j}\right|}{n} t_{j-1}+n^{2 \delta}, \quad \forall j \geq j_{0}+1 \tag{44}
\end{equation*}
$$

where $t_{j_{0}}=r_{j_{0}}=n^{2 \delta}$ and $t_{j_{0}-1}=0$. Same as in [9], we can find a sequence $\left(a_{j}, b_{j}\right)_{j \geq j_{0}}$ such that $a_{j_{0}}=b_{j_{0}}=1, b_{j} \leq 1$ and $a_{j} \leq 1+D \sqrt{\left|S_{i-j}\right| / n}$ for some constant $D>0$ such that

$$
\begin{equation*}
t_{j}-a_{j} t_{j-1}=b_{j}\left(t_{j-1}-a_{j-1} t_{j-2}\right)+n^{2 \delta}, \quad \forall j \geq j_{0}+1 \tag{45}
\end{equation*}
$$

See [9, eq. (64) and (65)] for the detailed construction of the sequence $\left(a_{j}, b_{j}\right)$.
Let $c_{j}=t_{j}-a_{j} t_{j-1}$. Then (44) becomes

$$
c_{j}=b_{j} c_{j-1}+n^{2 \delta} \leq c_{j-1}+n^{2 \delta} \leq c_{0}+j n^{2 \delta}=r_{0}+j n^{2 \delta} .
$$

Since $r_{0}=n^{2 \delta}$ and $j \leq I_{\max }=O\left(n^{\delta / 2} \log n\right)$, we have

$$
\begin{equation*}
t_{j}-a_{j} t_{j-1} \leq U:=n^{3 \delta}, \quad \forall j \geq j_{0}+1 \tag{46}
\end{equation*}
$$

So far we have deduced a recurrence for $t_{j}$, which is the same as in [9, eq. (66)]. The bound on $t_{j}$ depends on the sequence $\left(a_{j}\right)$, which is a function of $\left|S_{j}\right| / n$. Since the evolution of $\left|S_{j}\right|$ is different in the subcritical case (comparing Lemma 32 with Lemma 2), the analysis is a little different. Our eventual goal is to bound $t_{i-B}$ by $n^{O(\delta)}$ by recursively bounding each $t_{j}$, $j \leq i-B$. We will break the analysis into three different stages and bound $t_{i-I_{1}}, t_{i-I_{0}}$ and $t_{i-B}$ in turn, where $I_{0}$ and $I_{1}$ are the integers stated in Lemma 32 .

Stage 1: bounding $t_{i-I_{1}}$. We may assume that $i>I_{1}$; otherwise we may skip this stage. Recursively applying (46) for all $j_{0}+1 \leq j<i-I_{1}$, we have

$$
\begin{equation*}
t_{i-I_{1}} \leq U\left(1+\sum_{h=j_{0}+2}^{i-I_{1}} \prod_{j=h}^{i-I_{1}} a_{j}\right)+t_{j_{0}} \prod_{j=j_{0}+1}^{i-I_{1}} a_{j} . \tag{47}
\end{equation*}
$$

Our goal is to bound $t_{i-I_{1}}$ by $n^{O(\delta)}$. We first show that $\prod_{j=j_{0}+1}^{i-I_{1}} a_{j}=n^{O(\delta)}$ (this part of the analysis is similar to the work in [9] see [9, eq. (69)-(70)]).

Since $a_{j} \leq 1+D \sqrt{\left|S_{i-j}\right| / n}$ for each $j$, we have

$$
\begin{equation*}
\prod_{j=j_{0}+1}^{i-I_{1}} a_{j} \leq \exp \left(D \sum_{j=j_{0}+1}^{i-I_{1}} \sqrt{\frac{\left|S_{i-j}\right|}{n}}\right)=\exp \left(D \sum_{j=I_{1}}^{i-j_{0}-1} \sqrt{\frac{\left|S_{j}\right|}{n}}\right) \leq \exp \left(D \sum_{j=I_{1}}^{B^{\prime}-1} \sqrt{\frac{\left|S_{j}\right|}{n}}\right) \tag{48}
\end{equation*}
$$

where the last inequality above holds by the definition of $j_{0}$. By Lemma 32(b),

$$
\left|S_{B^{\prime}}\right| \geq\left|S_{I_{1}}\right| \prod_{j=I_{1}}^{B^{\prime}-1}\left(1+Y_{2} \sqrt{\frac{\left|S_{j}\right|}{n}}\right) \geq\left|S_{I_{1}}\right| \exp \left(Y_{3} \sum_{j=I_{1}}^{B^{\prime}-1} \sqrt{\frac{\left|S_{j}\right|}{n}}\right)
$$

for some constant $Y_{3}>0$, as $\left|S_{j}\right| / n$ is small for every $j$ in this range. Again, by Lemma 32(b), $\left|S_{I_{1}}\right| \geq C_{1} n^{1-\delta}$ for some constant $C_{1}>0$. This implies that, using $\left|S_{B^{\prime}}\right| \leq n$,

$$
\exp \left(Y_{3} \sum_{j=I_{1}}^{B^{\prime}-1} \sqrt{\frac{\left|S_{j}\right|}{n}}\right) \leq \frac{\left|S_{B^{\prime}}\right|}{\left|S_{I_{1}}\right|} \leq \frac{n^{\delta}}{C_{1}}
$$

Substituting this into (48), we have

$$
\prod_{j=j_{0}+1}^{i-I_{1}} a_{j} \leq\left(\frac{n^{\delta}}{C_{1}}\right)^{D / Y_{3}}=n^{O(\delta)}
$$

Now (47) gives

$$
\begin{equation*}
t_{i-I_{1}}=O\left(U I_{\max } \prod_{j=j_{0}+1}^{i-I_{1}} a_{j}\right)+t_{j_{0}} \prod_{j=j_{0}+1}^{i-I_{1}} a_{j}=n^{O(\delta)} \tag{49}
\end{equation*}
$$

Stage 2: bounding $t_{i-I_{0}}$. Using the same recursion (46) for $j$ such that $I_{0} \leq i-j<I_{1}$, we have

$$
\begin{equation*}
t_{i-I_{0}} \leq U\left(1+\sum_{h=i-I_{1}+2}^{i-I_{0}} \prod_{j=h}^{i-I_{0}} a_{j}\right)+t_{i-I_{1}} \prod_{j=i-I_{1}+1}^{i-I_{0}} a_{j} \leq\left(U I_{\max }+t_{i-I_{1}}\right) \prod_{j=i-I_{1}+1}^{i-I_{0}} a_{j} . \tag{50}
\end{equation*}
$$

Same as before, we have

$$
\prod_{j=i-I_{1}+1}^{i-I_{0}} a_{j} \leq \exp \left(D \sum_{j=i-I_{1}+1}^{i-I_{0}} \sqrt{\frac{\left|S_{i-j}\right|}{n}}\right)=\exp \left(D \sum_{j=I_{0}}^{I_{1}-1} \sqrt{\frac{\left|S_{j}\right|}{n}}\right)
$$

By Lemma 32 (c), there are constants $C_{2}, C_{3}>0$ such that $\left|S_{j}\right| / n \leq C_{2} n^{-\delta}$ for all $I_{0} \leq j \leq$ $I_{1}-1$ and $I_{1}-I_{0} \leq C_{3} n^{\delta / 2}$. This implies that

$$
\exp \left(D \sum_{j=I_{0}}^{I_{1}-1} \sqrt{\frac{\left|S_{j}\right|}{n}}\right) \leq \exp \left(D C_{3} \sqrt{C_{2}}\right)
$$

and thus, $\prod_{j=i-I_{1}+1}^{i-I_{0}} a_{j}=O(1)$. This together with (50) and (49) implies

$$
\begin{equation*}
t_{i-I_{0}}=n^{O(\delta)} \tag{51}
\end{equation*}
$$

Stage 3: bounding $t_{i-B}$. Using (46) for $j$ such that $B \leq i-j<I_{0}$, we have

$$
\begin{equation*}
t_{i-B} \leq U\left(1+\sum_{h=i-I_{0}+2}^{i-B} \prod_{j=h}^{i-B} a_{j}\right)+t_{i-I_{0}} \prod_{j=i-I_{0}+1}^{i-B} a_{j} \leq\left(U I_{\max }+t_{i-I_{0}}\right) \prod_{j=i-I_{0}+1}^{i-B} a_{j} . \tag{52}
\end{equation*}
$$

By Lemma 32(a),

$$
\left|S_{I_{0}}\right| \leq\left|S_{B}\right| \prod_{j=B}^{I_{0}-1}\left(1-Y_{2} \sqrt{\frac{\left|S_{j}\right|}{n}}\right) \leq\left|S_{B}\right| \exp \left(-Y_{2} \sum_{j=B}^{I_{0}-1} \sqrt{\frac{\left|S_{j}\right|}{n}}\right)
$$

By Lemma 32(a), $\left|S_{I_{0}}\right| \geq C_{4} n^{1-\delta}$ for some constant $C_{4}>0$. This implies that, using $\left|S_{B}\right| \leq n$,

$$
\exp \left(Y_{2} \sum_{j=B}^{I_{0}-1} \sqrt{\frac{\left|S_{j}\right|}{n}}\right) \leq \frac{\left|S_{B}\right|}{\left|S_{I_{0}}\right|} \leq \frac{n^{\delta}}{C_{4}}
$$

This gives

$$
\prod_{j=i-I_{0}+1}^{i-B} a_{j} \leq \exp \left(D \sum_{j=B}^{I_{0}-1} \sqrt{\frac{\left|S_{j}\right|}{n}}\right) \leq\left(\frac{n^{\delta}}{C_{4}}\right)^{D / Y_{2}}=n^{O(\delta)} .
$$

This together with (51) and (52) yields

$$
\begin{equation*}
t_{i-B}=n^{O(\delta)} . \tag{53}
\end{equation*}
$$

Now, we have shown, by (43), that $\sum_{j \geq B}\left|R_{j}(v)\right| \leq t_{i-B}+O\left(\log ^{2} n\right)=n^{O(\delta)}$. It follows immediately that $\sum_{j \geq 0}\left|R_{j}(v)\right|=n^{O(\delta)}$ by Lemma 30. Hence, noting that $\left|R^{+}(v)\right| \leq \cup_{j \geq 0}\left|R_{j}\right|$, we have shown that a.a.s. $\left|R^{+}(v)\right|=n^{O(\delta)}$ for every $v \in A P_{r}(n, c n)$. This proves Theorem 6 and therefore Theorem 4, by Corollary 1 .

## References

[1] D. Achlioptas and M. Molloy, The solution space geometry of random linear equations. Random Structures and Algorithms 46(2): 197-231, (2015).
[2] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, Europ. J. Combinatorics 1: 311-316 (1980).
[3] S. Cocco, O. Dubois, J. Mandler and R. Monasson, Rigorous decimation-based construction of ground pure states for spin-glass models on random lattices. Physical review letters, 90(4): 047205 (2003).
[4] J. Cain and N. Wormald, Encores on cores, Electron. J. Combin., 13(1), Research Paper 81, 13 pp. (2006).
[5] V. Chvátal, Almost all graphs with $1.44 n$ edges are 3-colorable, Random Structures Algorithms 2(1): 11-28, (1991).
[6] C. Cooper, The cores of random hypergraphs with a given degree sequence, Random Structures Algorithms 25: 353-375, (2004).
[7] J. Ding, A. Sly and N. Sun, Proof of the satisfiability conjecture for large $k$, arXiv:1411.0650, (2014).
[8] P. Gao, Analysis of the parallel peeling algorithm: a short proof, arXiv:1402.7326, (2014).
[9] P. Gao and M. Molloy, The stripping process can be slow: part I, arXiv:1501.02695, (2014).
[10] P. Gao and M. Molloy, Inside the clustering threshold for random linear equations, arXiv:1309:6651, (2013).
[11] M. Ibrahimi, Y. Kanoria, M. Kraning and A. Montanari, The set of solutions of random xorsat formulae, Proc. SODA 760-779, 2012.
[12] S. Janson and M. Luczak, A simple solution to the $k$-core problem, Random Struct. Algorithms, 30(1-2): 50-62, (2007).
[13] J. H. Kim, Poisson cloning model for random graphs, International Congress of Mathematicians. Vol. III, 873-897, Eur. Math. Soc., Zürich, (2006).
[14] J. Jiang, M. Mitzenmacher and J. Thaler, Parallel Peeling Algorithms, Proc. SPAA (ACM) 319-330, (2014).
[15] T. Łuczak, Size and connectivity of the $k$-core of a random graph, Discrete Math. 91(1): 61-68, 1991.
[16] T. Łuczak, Sparse random graphs with a given degree sequence, Random graphs, Vol. 2 (Poznań, 1989), Wiley-Intersci. Publ., 165-182, 1992.
[17] M. Mézard, F. Ricci-Tersenghi and R. Zecchina, Two solutions to diluted p-spin models and XORSAT problems, Journal of Statistical Physics, 111(3-4): 505-533, (2003).
[18] M. Molloy, Cores in random hypergraphs and Boolean formulas, Random Structures Algorithms 27(1): 124-135, (2005).
[19] M. Molloy and B. Reed, A critical point for random graphs with a given degree sequence, Random Structures Algorithms 6(2-3): 161-179, (1995).
[20] B. Pittel, J. Spencer and N. Wormald, Sudden emergence of a giant $k$-core in a random graph, J. Comb. Th. B 67: 111-151 (1996).


[^0]:    *Research supported by NSERC. This work started when the author was affiliated with University of Toronto and the research was then supported by NSERC PDF.

