

Induced Matchings in Graphs of Bounded Maximum Degree

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Abstract

For a graph G , let $\nu_s(G)$ be the induced matching number of G . We prove that $\nu_s(G) \geq \frac{n(G)}{(\lceil \frac{\Delta}{2} \rceil + 1)(\lfloor \frac{\Delta}{2} \rfloor + 1)}$ for every graph of sufficiently large maximum degree Δ and without isolated vertices. This bound is sharp. Moreover, there is polynomial-time algorithm which computes induced matchings of size as stated above.

Keywords: induced matching; strong matching; strong chromatic index

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1 Introduction

For a graph G , a set M of edges is an *induced matching* of G if no two edges in M have a common endvertex and no edge of G joins two edges of M . The maximum number of edges that form an induced matching in G is the *strong matching number* $\nu_s(G)$ of G . We denote by $\Delta(G)$ the maximum degree of graph G and let $n(G) = |V(G)|$ and $m(G) = |E(G)|$.

In contrast to the well known matching number $\nu(G)$, which can be computed in polynomial time [4], it is NP-hard to determine the strong matching number even in bipartite subcubic graphs [2, 7, 9]. In fact, the strong matching number is even hard to approximate in restricted graphs classes as for example regular bipartite graphs [3].

To the best of my knowledge, the only known bound in terms of the order and the maximum degree for $\nu_s(G)$ is obtained by the following simple observation [11]. Let G be a graph without isolated vertices. There are at most $2\Delta(G)^2 - 2\Delta(G) + 1$ many edges in distance at most 1 from e including e and $m(G) \geq \frac{1}{2}n(G)$. Thus a simple greedy algorithm implies

$$\nu_s(G) \geq \frac{n(G)}{2(2\Delta(G)^2 - 2\Delta(G) + 1)},$$

which is far away from being sharp if $G \neq K_2$.

It seems that the different behavior of $\nu(G)$ and $\nu_s(G)$ transfers to the corresponding partitioning problems. The chromatic index χ' seems much simpler than the strong chromatic index χ'_s , defined as the minimum number of induced matchings one needs to partition the edge set. While for $\chi'(G)$ Vizing's Theorem always gives $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$ [10], no comparable result holds for the strong chromatic index.

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A trivial greedy algorithm ensures $\chi'_s(G) \leq 2\Delta(G)^2 - 2\Delta(G) + 1$. Erdős and Nešetřil [5] conjectured $\chi'_s(G) \leq \frac{5}{4}\Delta(G)^2$, which would be best possible for even Δ because equality holds for the graph obtained from the 5-cycle by replacing every vertex by an independent set of order $\frac{\Delta}{2}$. The best general result in this direction is due to Molloy and Reed, who proved that $\chi'_s(G) \leq 1.998\Delta(G)^2$ for sufficiently large maximum degree [8]. Thus Erdős and Nešetřil's conjecture is widely open and it is even unknown which technique is suitable to improve Molloy and Reed's result substantially.

In this paper I provide more insight concerning the behavior of induced matchings by improving the known lower bounds on $\nu_s(G)$ to a sharp lower bound provided that the maximum degree is sufficiently large.

Theorem 1. *There is an integer Δ_0 such that for every graph G of maximum degree Δ at least Δ_0 and without isolated vertices,*

$$\nu_s(G) \geq \frac{n(G)}{\left(\lceil \frac{\Delta}{2} \rceil + 1\right) \left(\lfloor \frac{\Delta}{2} \rfloor + 1\right)}$$

holds.

The following construction shows that the bound in Theorem 1 is sharp. Let Δ be an integer at least 3 and let the graph H_1 arise from the complete graph on $\lceil \frac{\Delta}{2} \rceil + 1$ vertices by attaching at each vertex $\lfloor \frac{\Delta}{2} \rfloor$ pendant vertices. Let H_2 arise from the complete graph on $\lfloor \frac{\Delta}{2} \rfloor + 1$ vertices by attaching at each vertex $\lceil \frac{\Delta}{2} \rceil$ pendant vertices. It follows that $\nu_s(H_i) = 1$ and $n(H_i) = (\lceil \frac{\Delta}{2} \rceil + 1) (\lfloor \frac{\Delta}{2} \rfloor + 1)$; that is, the bound of Theorem 1 is sharp. Note that $H_1 = H_2$ if Δ is even.

For the sake of simplicity I do not try to optimize the constant Δ_0 intensively. We show Theorem 1 for $\Delta_0 = 1000$ but with some more effort one can lower the bound down to 200.

In [6] the same bound as in Theorem 1 is already shown by a simple inductive argument for graphs of girth at least 6. Hence one might ask whether the bound in Theorem 1 can be improved for graphs of large girth to $\frac{n(G)}{\Delta^c}$ for some $c < 2$. However, this is not the case. By a result of Bollobás [1], for every $g \geq 3$ and $\Delta \geq 6$, there is a graph H' of maximum degree $\lfloor \frac{\Delta}{2} \rfloor$, girth at least g , and independence number at most $\frac{4 \log \Delta}{\Delta} n(H')$. Let H arise from H' by attaching to each vertex $\lceil \frac{\Delta}{2} \rceil$ many pendant vertices. Note that $\nu_s(H) \leq \frac{4 \log \Delta}{\Delta} n(H')$ and $n(H) = \lceil \frac{\Delta}{2} \rceil n(H')$. Thus $\nu_s(H) \leq \frac{8 \log \Delta}{\Delta^2} n(H)$ and the bound of Theorem 1 can only be improved by a $O(\log \Delta)$ -factor.

Since the proof of Theorem 1 is constructive, it is easy to derive a polynomial-time algorithm, which computes an induced matching of size as guaranteed in Theorem 1.

We use standard notation and terminology. For a graph G , let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. For a vertex v , let $d_G(v)$ be its degree, let $N_G(v)$ be the set of neighbors of v , and let $N_G[v] = N_G(v) \cup \{v\}$. If the corresponding graph is clear from the context, we only write $d(v)$, $N(v)$ and $N[v]$, respectively. A set I of vertices of G is independent if there is no edge joining two vertices in I .

2 Proof of Theorem 1

We prove the theorem for $\Delta_0 = 1000$. Let G be a graph with maximum degree Δ at least Δ_0 and without isolated vertices. For a contradiction, we assume that G is a counterexample such that

- (1) $\nu_s(G)$ is minimum and
- (2) subject to (1), the order of G is maximum.

Since $\nu_s(G) \geq \frac{n(G)}{2\Delta^2}$, the graph G is well-defined.

The choice of G implies that if v is a vertex of G that is adjacent to a vertex of degree 1, then $d(v) = \Delta$ because adding new vertices to G and joining them to v does not increase $\nu_s(G)$ but the order of G .

For some calculations it might help to know that $\frac{\Delta^2}{4} + \Delta + \frac{3}{4} \leq (\lceil \frac{\Delta}{2} \rceil + 1)(\lfloor \frac{\Delta}{2} \rfloor + 1)$.

Claim 1. *For every edge uv of G , we have $d(u) + d(v) > \frac{\Delta}{4}$.*

Proof of Claim 1. For a contradiction, we assume that there is an edge uv such that $d(u) + d(v) \leq \frac{\Delta}{4}$. Let $S = N[u] \cup N[v]$ and let I be the set of all isolated vertices of $G - S$. Let $G' = G - S - I$. Since $\nu_s(G) \geq \nu_s(G') + 1$, the choice of G implies $\nu_s(G') \geq \frac{n(G')}{(\lceil \frac{\Delta}{2} \rceil + 1)(\lfloor \frac{\Delta}{2} \rfloor + 1)}$.

By using the assumption $d(u) + d(v) \leq \frac{\Delta}{4}$, we conclude $|S| + |I| \leq (\frac{\Delta}{4} - 2)\Delta + 2 < (\lceil \frac{\Delta}{2} \rceil + 1)(\lfloor \frac{\Delta}{2} \rfloor + 1)$. Therefore, uv together with a maximum induced matching of G' is an induced matching of G of size at least $\frac{n(G)}{(\lceil \frac{\Delta}{2} \rceil + 1)(\lfloor \frac{\Delta}{2} \rfloor + 1)}$, which contradicts the choice of G . \square

Claim 2. *Every vertex v of G is adjacent to at most $\frac{3}{4}\Delta$ many vertices of degree at most 9.*

Proof of Claim 2. Choose v such that the number of neighbors of degree at most 9 is maximal. Say v has $\alpha\Delta$ many such neighbors. For a contradiction, we assume that $\alpha > \frac{3}{4}$. Let $u \in N(v)$ be of degree at most 9. As above, let $S = N[u] \cup N[v]$ and let I be the set of all isolated vertices of $G - S$. Let $G' = G - S - I$. By Claim 1, every vertex in I that is adjacent to a vertex of degree at most 9, has degree at least 10. Thus there are at most $(1 - \alpha)\Delta + 8$ many vertices in S that are adjacent to vertices in I of degree at most 9. Hence there are at most $\alpha(1 - \alpha)\Delta^2 + 8\Delta$ many vertices in I of degree at most 9. Furthermore, at most $8\alpha\Delta$ edges join vertices in I and vertices in $N(v) \setminus \{u\}$ such that the vertices in $N(v) \setminus \{u\}$ have degree at most 9. Since $\alpha(1 - \alpha) + \frac{1}{10}(1 - \alpha)^2 < 0.22$, this implies

$$\begin{aligned} |I| &\leq \alpha(1 - \alpha)\Delta^2 + 8\Delta + \frac{1}{10}((1 - \alpha)^2\Delta^2 + 8\alpha\Delta) \\ &< 0.22\Delta^2 + 9\Delta \\ &\leq \frac{\Delta^2}{4} - 9. \end{aligned}$$

Since $|S| \leq \Delta + 9$, we obtain

$$|I| + |S| < \left(\left\lceil \frac{\Delta}{2} \right\rceil + 1 \right) \left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1 \right).$$

Again, the edge uv together with a maximum induced matching of G' is an induced matching of G of size at least $\frac{n(G)}{(\lceil \frac{\Delta}{2} \rceil + 1)(\lfloor \frac{\Delta}{2} \rfloor + 1)}$, which contradicts the choice of G . \square

Let $f : V(G) \rightarrow \mathbb{R}$ be such that

$$f(v) = \sum_{w \in N(v): d(w) \neq \Delta} \frac{1}{d(w)}.$$

Claim 3. *If a vertex v of G is not adjacent to a vertex of degree 1, then $f(v) \leq \frac{2}{5}\Delta$.*

Proof of Claim 3. Let v be a vertex that is not adjacent to a vertex of degree 1. By Claim 2, the vertex v has at most $\frac{3}{4}\Delta$ neighbors of degree at most 9, which contribute to $f(v)$ at most $\frac{1}{2}$ each; all remaining neighbors contribute at most $\frac{1}{10}$ each. Thus $f(v) \leq \frac{1}{2} \cdot \frac{3}{4}\Delta + \frac{1}{10} \cdot \frac{1}{4}\Delta = \frac{2}{5}\Delta$. \square

For the rest of the proof, let $v \in V(G)$ be chosen such that $f(v)$ is maximal.

Case 1. v is adjacent to a vertex of degree 1.

Recall that this implies $d(v) = \Delta$. Let $u \in N(v)$ be a vertex of degree 1. As before, we want to combine uv with a maximum induced matching of $G' = G - (N[v] \cup I)$ to obtain a contradiction, where I are the isolated vertices of $G - N[v]$.

If $z \in I$ has degree $d < \Delta$, then z contributes exactly d times exactly $\frac{1}{d}$ to $f(w)$ for some $w \in N(v)$; that is, the total contribution to $\sum_{w \in N(v)} f(w)$ is 1. Since no vertex in I is adjacent to u , there is no vertex $z \in I$ such that $d(z) = \Delta$. This implies that

$$|I| \leq \sum_{w \in N(v)} f(w). \quad (1)$$

Let N_1 and N_Δ be the set of vertices in $N(v)$ of degree 1 and Δ , respectively. Let N_s be the set of vertices in $N(v) \setminus (N_1 \cup N_\Delta)$ of small degree, say such that their degree is between 2 and $\frac{\Delta}{8}$. Let $N_\ell = N(v) \setminus (N_1 \cup N_s \cup N_\Delta)$, and let $n_1 = |N_1|$, $n_s = |N_s|$, $n_\ell = |N_\ell|$, and $n_\Delta = |N_\Delta|$.

Since all vertices in $N_s \cup N_\ell$ do not have degree Δ and by the choice of G , they are not adjacent to a vertex of degree 1. If $w \in N_1$, then $f(w) = 0$ and w contributes 1 to $f(v)$. If $w \in N_s$, then by Claim 1, we conclude $f(w) \leq 1$, and the contribution of w to $f(v)$ is at most $\frac{1}{2}$. If $w \in N_\ell$, then by Claim 3 and the choice of v , we obtain $f(w) \leq \min\{\frac{2}{5}\Delta, f(v)\}$ and the contribution of w to $f(v)$ is at most $\frac{8}{\Delta}$. If $w \in N_\Delta$, then $f(w) \leq f(v)$ and w contributes nothing to $f(v)$. These observations imply both

$$f(v) \leq \frac{8}{\Delta}n_\ell + \frac{1}{2}n_s + n_1$$

and, by using (1),

$$|I| \leq f(v)n_\Delta + \min \left\{ \frac{2}{5}\Delta, f(v) \right\} n_\ell + n_s.$$

In order to prove that $|I| \leq \lceil \frac{\Delta}{2} \rceil \lfloor \frac{\Delta}{2} \rfloor$, we show that

$$f'n_\Delta + \min \left\{ \frac{2}{5}\Delta, f' \right\} n_\ell + n_s \leq \left\lceil \frac{\Delta}{2} \right\rceil \left\lfloor \frac{\Delta}{2} \right\rfloor, \quad (2)$$

under the condition that $n_1, n_s, n_\ell, n_\Delta$ are non-negative integers and $n_1 + n_s + n_\ell + n_\Delta = \Delta$ where

$$f' = \frac{8}{\Delta}n_\ell + \frac{1}{2}n_s + n_1. \quad (3)$$

Let $i(n_1, n_s, n_\ell, n_\Delta) = f'n_\Delta + \min \left\{ \frac{2}{5}\Delta, f' \right\} n_\ell + n_s$. Obviously, $|I| \leq i(n_1, n_s, n_\ell, n_\Delta)$.

Inequality (3) implies $n_s + n_1 \geq f' - 8$. Thus $n_\ell + n_\Delta = \Delta - n_1 - n_s \leq \Delta - f' + 8$ and hence, by (2), we obtain

$$i(n_1, n_s, n_\ell, n_\Delta) \leq f'(\Delta - f' + 8) + \Delta.$$

If $f' \leq \frac{2}{5}\Delta + 8$, then this implies that $i(n_1, n_s, n_\ell, n_\Delta) \leq \frac{6}{25}\Delta^2 + \frac{24}{5}\Delta \leq \frac{\Delta^2}{4} - 1$, which implies the desired result.

Thus we may assume that $f' \geq \frac{2}{5}\Delta + 8$. Suppose $n_\ell \geq 1$ and hence $n_\Delta \leq \Delta - 1$. This implies that

$$\begin{aligned} i(n_1, n_s, n_\ell - 1, n_\Delta + 1) - i(n_1, n_s, n_\ell, n_\Delta) &\geq -\frac{8}{\Delta}n_\Delta - \frac{2}{5}\Delta + \left(f' - \frac{8}{\Delta}\right) \cdot 1 \\ &\geq -\frac{8}{\Delta}(\Delta - 1) - \frac{2}{5}\Delta + \frac{2}{5}\Delta + 8 - \frac{8}{\Delta} \\ &= 0. \end{aligned}$$

Hence, we may assume that $n_\ell = 0$.

Furthermore, we may assume that $n_\Delta \geq 2$; otherwise, by using $f', n_s \leq \Delta$, we conclude $i(n_1, n_s, n_\ell, n_\Delta) \leq 2\Delta$. Suppose $n_s \geq 1$. Thus

$$i(n_1 + 1, n_s - 1, n_\ell, n_\Delta) - i(n_1, n_s, n_\ell, n_\Delta) \geq \frac{1}{2} \cdot 2 - 1 \geq 0.$$

Therefore, we may assume that $n_s = 0$. Thus $n_1 = \Delta - n_\Delta$ and (3) implies that $f' = n_1$. By using (2), we conclude

$$i(n_1, n_s, n_\ell, n_\Delta) = n_\Delta(\Delta - n_\Delta) \leq \left\lceil \frac{\Delta}{2} \right\rceil \left\lfloor \frac{\Delta}{2} \right\rfloor.$$

Therefore, $|N[v]| + |I| \leq (\lceil \frac{\Delta}{2} \rceil + 1)(\lfloor \frac{\Delta}{2} \rfloor + 1)$ and the edge uv together with a maximum induced matching of G' yields $\nu_s(G) \geq \frac{n(G)}{(\lceil \frac{\Delta}{2} \rceil + 1)(\lfloor \frac{\Delta}{2} \rfloor + 1)}$, which is a contradiction to our choice of G .

Case 2. v is not adjacent to a vertex of degree 1.

Let $u \in N(v)$ such that $d(u)$ is minimal. Let $S = N[u] \cup N[v]$ and $G' = G - S - I$ where I is the set of isolated vertices of $G - S$. By double counting the edges between S and I , it is straightforward to see that I contains at most 2Δ vertices of degree Δ . Thus similarly as in (1), we conclude that

$$|I| \leq \sum_{w \in S \setminus \{u, v\}} f(w) + 2\Delta. \quad (4)$$

If $d(u) \geq 10$, then $f(v) \leq \frac{\Delta}{10}$. Thus $|I| \leq \frac{\Delta^2}{5} + 2\Delta$ and hence $|S| + |I| \leq \frac{\Delta^2}{4}$. Therefore, uv together with a maximum induced matching of G' yields $\nu_s(G) > \frac{n(G)}{(\lceil \frac{\Delta}{2} \rceil + 1)(\lfloor \frac{\Delta}{2} \rfloor + 1)}$, which is a contradiction to our choice of G .

Thus we may assume that $d(u) \leq 9$ and hence trivially $\sum_{w \in N(u) \setminus \{v\}} f(w) \leq 8\Delta$ and $|S| \leq \Delta + 9$. Let N_s be the set of neighbors of v of degree at most $\frac{\Delta}{8}$, let $N_\ell = N(v) \setminus N_s$, and let $\alpha = \frac{|N_s|}{\Delta}$ and hence $N_\ell \leq (1 - \alpha)\Delta$.

The contribution of the vertices in N_s to $f(v)$ is at most $\frac{\alpha\Delta}{2}$. Using Claim 1, we conclude that $f(w) \leq 1$ for $w \in N_s$. The contribution of the vertices in N_ℓ to $f(v)$ is at most 8 and $f(w) \leq f(v)$ for $w \in N_\ell$ by the choice of v . This implies that $f(v) \leq \frac{\alpha\Delta}{2} + 8$. Note that $(1 - \alpha)\frac{\alpha}{2} \leq \frac{1}{8}$. Moreover, by (4), we obtain

$$\begin{aligned} |I| &\leq \sum_{w \in N(v) \setminus \{u\}} f(w) + \sum_{w \in N(u) \setminus \{v\}} f(w) + 2\Delta \\ &\leq \sum_{w \in N(v) \setminus \{u\}; w \in N_\ell} f(w) + \sum_{w \in N(v) \setminus \{u\}; w \in N_s} f(w) + 8\Delta + 2\Delta \\ &\leq (1 - \alpha)\Delta f(v) + \alpha\Delta + 10\Delta \\ &\leq (1 - \alpha)\Delta \left(\frac{\alpha\Delta}{2} + 8 \right) + 11\Delta \\ &\leq \frac{\Delta^2}{4} - 2\Delta. \end{aligned}$$

Thus $|I| + |S| \leq \frac{\Delta^2}{4}$. Therefore, uv together with a maximum induced matching of G' yields $\nu_s(G) > \frac{n(G)}{(\lceil \frac{\Delta}{2} \rceil + 1)(\lfloor \frac{\Delta}{2} \rfloor + 1)}$, which is the final contradiction. \square

3 Graphs with Small Maximum Degree

Let C_5^2 be the graph obtained from the 5-cycle by replacing every vertex by an independent set of order 2 and let $K_{3,3}^+$ be the graph obtained from the 5-cycle by replacing the vertices by independent sets of orders 1, 1, 1, 2, and 2, respectively. Note that the graph $K_{3,3}^+$ can also be obtained from a $K_{3,3}$ by subdividing one edge once. The graphs C_5^2 and $K_{3,3}^+$ show

that Theorem 1 is not true for graphs of maximum degree 3 or 4. However, I conjecture that these graphs are the only exceptions.

Conjecture 2. *If connected graph $G \notin \{C_5^2, K_{3,3}^+\}$ with maximum degree $\Delta \geq 3$, then*

$$\nu_s(G) \geq \frac{1}{(\lceil \frac{\Delta}{2} \rceil + 1)(\lfloor \frac{\Delta}{2} \rfloor + 1)} n(G).$$

Note that for $\Delta = 3$, a result in [6], and for $\Delta \geq 1000$, Theorem 1 implies Conjecture 2.

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