

Quasisymmetric functions for nestohedra

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Abstract

For a generalized permutohedron Q the enumerator $F(Q)$ of positive lattice points in interiors of maximal cones of the normal fan Σ_Q is a quasisymmetric function. We describe this function for the class of nestohedra as a Hopf algebra morphism from a combinatorial Hopf algebra of building sets. For the class of graph-associahedra the corresponding quasisymmetric function is a new isomorphism invariant of graphs. The obtained invariant is quite natural as it is the generating function of ordered colorings of graphs and satisfies the recurrence relation with respect to deletions of vertices.

1 Introduction

Let Q be a convex polytope. The *normal fan* Σ_Q is the set of cones over the faces of the polar polytope Q^* . The polytope Q is simple if and only if the normal fan Σ_Q is simplicial. The polytope Q is a *Delzant* polytope if its normal fan Σ_Q is regular, i.e. the generators of the normal cone σ_v at any vertex $v \in Q$ can be chosen to form an integer basis of \mathbb{Z}^n .

The permutohedron Pe^{n-1} is a $(n-1)$ -dimensional polytope which is the convex hull $Pe^{n-1} = \text{Conv}\{x_\omega \mid \omega \in S_n\}$, where $x \in \mathbb{R}^n$ is a point with strictly increasing coordinates $x_1 < \dots < x_n$ and $x_\omega = (x_{\omega(1)}, \dots, x_{\omega(n)})$ for a permutation $\omega \in S_n$. The normal fan $\Sigma_{Pe^{n-1}}$ of the permutohedron Pe^{n-1} is the braid arrangement fan. Postnikov introduced in [16] a class of convex polytopes called generalized permutohedra, which includes some interesting subclasses with a rich combinatorial structure, such as matroid base polytopes, graphic zonotopes, nestohedra and graph-associahedra.

A *generalized permutohedron* Q in \mathbb{R}^n is a convex polytope characterized by equivalent conditions (see [17, Theorem 15.3] for the general statement)

- (i) the normal fan Σ_Q is refined by the braid arrangement fan $\Sigma_{Pe^{n-1}}$
- (ii) any edge lies in a direction $e_i - e_j$ for some $1 \leq i, j \leq n$

(iii) Q is a Minkowski summand of the permutohedron Pe^{n-1} .

We regard a function $f : [n] \rightarrow \mathbb{N}$ as an element of $(\mathbb{R}^n)^*$ by $\langle f, x \rangle = \sum_{i=1}^n f(i)x_i, x \in \mathbb{R}^n$. For a generalized permutohedron Q in \mathbb{R}^n a function $f : [n] \rightarrow \mathbb{N}$ is called Q -generic if it has a unique maximum over Q at a vertex $\max_{x \in Q} \langle f, x \rangle = \langle f, v \rangle$. Thus it lies in the interior of the normal cone σ_v for some vertex $v \in Q$. Let $F(Q)$ be the enumerator function of Q -generic functions

$$F(Q) = \sum_{f: Q\text{-generic}} \mathbf{x}_f = \sum_{v \in Q} \sum_{f \in \sigma_v} \mathbf{x}_f,$$

where $\mathbf{x}_f = x_{f(1)} \cdots x_{f(n)}$. This power series is introduced and its main properties are derived by Billera, Jia and Reiner in ([3], Section 9). It is a homogeneous quasisymmetric function of degree n . Consider its expansion in the monomial basis of quasisymmetric functions

$$F(Q) = \sum_{\alpha \models n} \zeta_\alpha(Q) M_\alpha,$$

where $M_\alpha = \sum_{i_1 < \dots < i_k} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ for a composition $\alpha = (a_1, \dots, a_k) \models n$ of the integer n .

If $Q = Z_\Gamma$ is a graphic zonotope the function $F(Z_\Gamma)$ is easily seen to be Stanley's chromatic symmetric function X_Γ of the graph Γ [20]. For the matroid base polytope $Q = P_M$ the quasisymmetric function $F(P_M)$ is an isomorphism invariant of a matroid M introduced by Billera, Jia and Reiner in [3]. The unifying principle of these two examples is a construction of certain combinatorial Hopf algebras such that prescribed invariants are obtained by the universal morphism to quasisymmetric functions. The theory of combinatorial Hopf algebras is developed by Aguiar, Bergeron and Sottile in [1]. We particularly respond to [3, Problem 9.3] and study the quasisymmetric functions $F(Q)$ for the class of nestohedra.

The nestohedron $Q = P_B$ is a simple polytope obtained from a simplex by a sequence of face truncations. The family of faces by which truncations are performed is encoded by a building set B , which is a subset of the face lattice of the simplex. The ground sets of connected subgraphs of a graph Γ produce the graphical building set $B(\Gamma)$. The class of polytopes $P_{B(\Gamma)}$ is called graph-associahedra. It contains an important series of polytopes such as associahedra or Stasheff polytopes, cyclohedra or Bott-Taubes polytopes, stellohedra and permutohedra. For the class of nestohedra we describe coefficients $\zeta_\alpha(P_B)$ in terms of underlying building set B . We construct a certain combinatorial Hopf algebra of building sets \mathcal{B} and show that the canonical morphism maps a building set B precisely to the generating function $F(P_B)$ of the corresponding nestohedron P_B .

Recently, some of quasisymmetric refinements of Stanley's chromatic symmetric function have appeared, see [11], [12]. We introduce a new quasisymmetric function invariant F_Γ associated to a graph Γ which has independent combinatorial and algebraic descriptions as

- 1) the enumerator function of $P_{B(\Gamma)}$ -generic functions,
- 2) the Hopf morphism from certain combinatorial Hopf algebra of graphs,
- 3) the enumerator function of *ordered colorings* of Γ .

We say a coloring of a graph is ordered if colors are linearly ordered and monochromatic vertices are not connected by paths through vertices colored by smaller colors. In addition the function F_Γ satisfies the recurrence relation with respect to deletions of vertices

$$F_\Gamma = \sum_{v \in V} (F_{\Gamma \setminus v})_1,$$

where $F \mapsto (F)_1$ is a certain shifting operator on quasisymmetric functions.

The paper is organized as follows. In section 2 we review the necessary facts about nestohedra. In section 3 we review weak orders and preorders and their connections with combinatorics of the permutohedron. In section 4 we construct the combinatorial Hopf algebra \mathcal{B} and prove that the assignment $B \mapsto F(P_B)$ comes from the universal Hopf algebra morphism to quasisymmetric functions. In section 5 the function $F(P_B)$ is related with the multiset of unlabeled rooted trees associated to vertices of P_B . In section 6 the theory of P -partitions is used to determine the action of the antipode of quasisymmetric functions on $F(P_B)$. In section 7 we give a graph theoretic interpretation of the invariant $F(P_{B(\Gamma)})$. We prove the recurrence relation for F_Γ with respect to deletions of vertices of a graph which serves as the main computational tool. As an application we compute $F(Q)$ for Q be a permutohedron, associahedron, cyclohedron or stellohedron. As the conclusion some open problems concerning the graph invariant F_Γ and Hopf algebra \mathcal{B} are posed.

2 Nestohedra

In this section we review the necessary definitions and facts about nestohedra. This class of polytopes is introduced and studied in [7], [16], [17], [22].

Let $\Delta_{[n]} = \text{Conv}\{e_1, \dots, e_n\}$ be the standard coordinate simplex in \mathbb{R}^n . To a subset $I \subset [n]$ corresponds the face $\Delta_I = \text{Conv}\{e_i \mid i \in I\} \subset \Delta_{[n]}$. A hypergraph B on the finite set $[n] = \{1, \dots, n\}$ is a collection of nonempty subsets of $[n]$. For convenience we suppose that $\{i\} \in B, i \in [n]$. For a hypergraph B on $[n]$ define the polytope P_B as the Minkowski sum of simplices

$$P_B = \sum_{I \in B} \Delta_I = \sum_{I \in B} \text{Conv}\{e_i \mid i \in I\} = \text{Conv} \sum_{I \in B} \{e_i \mid i \in I\}.$$

The polytope P_B is simple if additionally the hypergraph B satisfies the following condition:

- ◊ If $I, J \in B$ and $I \cap J \neq \emptyset$ then $I \cup J \in B$.

In that case B is called a *building set* and the polytope P_B is called a *nestohedron*.

Example 2.1. Given a simple graph Γ on the vertex set $[n]$, the graphical building set $B(\Gamma)$ is defined as the collection of all $I \subset [n]$ such that

the induced subgraphs $\Gamma|_I$ are connected. Carr and Devadoss studied polytopes $P_{B(\Gamma)}$ in [5] and called them *graph-associahedra*. For instance the series $Pe^{n-1}, As^{n-1}, Cy^{n-1}, St^{n-1}, n > 2$ of permutohedra, associahedra, cyclohedra and stellohedra correspond respectively to complete graphs K_n , path graphs L_n , cycle graphs C_n and star graphs $K_{1,n-1}$ on n vertices.

Let B_{\max} be the collection of maximal by inclusion elements of a building set B . We say that a building set B is *connected* if $[n] \in B$. Since the Minkowski sum is the product for polytopes which are contained in the complementary subspaces, we have

$$P_B = \sum_{I \in B_{\max}} \sum_{J \in B|_I} \Delta_J = \prod_{I \in B_{\max}} P_{B|_I}.$$

Thus we may restrict ourselves to connected building sets. The realization of nestohedra is given by the following proposition.

Proposition 2.2 ([7], Proposition 3.12). *Let B be a connected building set on the finite set $[n]$ and $\mu(B)$ be the number of elements of B . The nestohedron P_B can be described as the intersection of the hyperplane $H_{[n]}$ with the halfspaces $H_{I, \geq}$ corresponding to all $I \in B \setminus \{[n]\}$, where*

$$H_{[n]} = \{x \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \mu(B)\},$$

$$H_{I, \geq} = \{x \in \mathbb{R}^n \mid \sum_{i \in I} x_i \geq \mu(B|_I)\}.$$

As a consequence we have that the nestohedron P_B corresponding to a connected building set B is realized from the dilated standard coordinate simplex $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \dots + x_n = \mu(B), x_1, \dots, x_n \geq 1\}$ by a sequence of face truncations which are encoded by elements $I \in B \setminus \{[n]\}$. For instance the truncations along all faces of the simplex $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \dots + x_n = 2^n - 1, x_1, \dots, x_n \geq 1\}$ realize the permutohedron Pe^{n-1} .

The face lattice of P_B is described by the following proposition.

Proposition 2.3 ([7], Theorem 3.14; [16], Theorem 7.4). *Given a connected building set B on $[n]$, let $\{F_I \mid I \in B \setminus \{[n]\}\}$ be the set of facets of the nestohedron P_B . The intersection $F_{I_1} \cap \dots \cap F_{I_k}, k \geq 2$ is a nonempty face of P_B if and only if*

(N1) $I_i \subset I_j$ or $I_j \subset I_i$ or $I_i \cap I_j = \emptyset$ for any $1 \leq i < j \leq k$.

(N2) $I_{j_1} \cup \dots \cup I_{j_p} \notin B$ for any pairwise disjoint sets I_{j_1}, \dots, I_{j_p} .

A subcollection $\{I_1, \dots, I_k\} \subset B$ that satisfies the conditions (N1) and (N2) is called a *nested set*. The collection N_B of all nested sets form a simplicial complex called the *nested set complex*. The face poset of N_B is opposite to the face poset of P_B . Therefore N_B may be realized as a simplicial polytope which is polar to P_B .

Proposition 2.3 implies that vertices of P_B correspond to maximal nested sets. We denote this correspondence by $v \mapsto N_v$. To a vertex $v \in P_B$ associate the poset $(N_v \cup \{[n]\}, \subset)$. For $I \in N_v \cup \{[n]\}$ let $i_I \in [n]$ be the element such that $\{i_I\} = I \setminus \bigcup\{J \in N_v \mid J \subsetneq I\}$. The correspondence $I \mapsto i_I$ is a well defined bijection by the characterization of maximal nested sets ([16], Proposition 7.6). It defines the partial order \leq_v on $[n]$ by $i_I \leq_v i_J$ if and only if $I \subset J$ in $N_v \cup \{[n]\}$. Denote this poset on $[n]$ by P_v . The Hasse diagram T_v of the poset P_v for $v \in P_B$ is called a *B-tree* [17, Definition 8.1]. So $(i, j) \in T_v$ if and only if $i <_{P_v} j$ is a covering relation in the poset P_v . The root of T_v is the maximal element of P_v .

The following proposition, which is a consequence of Proposition 2.2, describes the coordinates and normal cones at vertices of P_B . Note that any nested set $\{I_1, \dots, I_k\} \subset B$ is ordered by inclusion of sets. The usual covering relations is denoted by $J < I$.

Proposition 2.4. *Let $v \in P_B$ be a vertex of the nestohedron P_B and $N_v \in N_B$ be the corresponding maximal nested set.*

(i) *The coordinates of the vertex v are given by*

$$x_{i_I} = \mu(B \mid_I) - \sum_{J \in N_v : J < I} \mu(B \mid_J), I \in N_v \cup \{[n]\}.$$

(ii) *The interior of the normal cone σ_v at the vertex v is determined by the inequalities*

$$x_i < x_j, \text{ for all } (i, j) \in T_v.$$

3 Preorders, Weak orders and Permutohedra

A binary relation \preceq is called a *preorder* on the finite set $V = \{v_1, \dots, v_n\}$ if it is reflexive and transitive. If it is in addition total, i.e. $u \preceq v$ or $v \preceq u$ for all $u, v \in V$, the preorder \preceq is called a *weak order* or a total preorder. The preorder defines an equivalence relation by $u \sim v$ if and only if $u \preceq v$ and $v \preceq u$. The relation \preceq / \sim is a partial order on the set of equivalence classes V / \sim . If \preceq is a weak order on V then \preceq / \sim is a total order on V / \sim . Any weak order is represented as an ordered partition of V , i.e. as the ordered family (V_1, \dots, V_k) of nonempty disjoint subsets which covers V . The relation is recovered by $u \preceq v$ if and only if $u \in V_i$ and $v \in V_j$ for some $1 \leq i \leq j \leq k$. The type of a weak order \preceq is the corresponding composition $\text{type}(\preceq) = (|V_1|, \dots, |V_k|) \models n$ and k is its length. Any function $f : V \rightarrow \mathbb{N}$ determines a weak order on V by $u \preceq_f v$ if $f(u) \leq f(v)$, for all $u, v \in V$. For any strictly increasing function $g : \mathbb{N} \rightarrow \mathbb{N}$ we have $\preceq_f = \preceq_{g \circ f}$. To a weak order \preceq on V is associated the monomial quasisymmetric function

$$M_{\text{type}(\preceq)} = \sum_{\preceq_f = \preceq} \mathbf{x}_f.$$

Let $\mathbf{WO}(n) = \cup_{k=1}^n \mathbf{WO}_k(n)$ be the set of all weak orders of the set V graded by the lengths. To an ordered partition (V_1, \dots, V_k) is associated the flag $\emptyset \subset V_1 \subset V_1 \cup V_2 \subset \dots \subset V_1 \cup \dots \cup V_{k-1} \subset V$. This is a one-to-one correspondence between ordered partitions and flags on V . Therefore the set of all weak orders $\mathbf{WO}(n)$ is modelled as the simplicial complex $\Delta[n]^{(1)}$ the first barycentric subdivision of the simplex on V . The simplicial complex $\Delta[n]^{(1)}$ is combinatorially equivalent to the convex simplicial polytope whose polar polytope is the permutohedron Pe^{n-1} (see [15]). Thus k -faces of Pe^{n-1} are labeled by ordered partitions (V_1, \dots, V_{n-k}) or equivalently by $(n-k)$ -weak orders on V . Accordingly, to any face $F \subset Pe^{n-1}$ is associated the monomial quasisymmetric function M_F , where

$$M_F = M_{\text{type}(\prec_F)},$$

for the weak order \prec_F on V corresponding to the face F . Specially, facets correspond to pairs $(A, V \setminus A)$, for proper subsets $A \subset V$ and the associated monomial quasisymmetric functions are of the form $M_{(k, n-k)}$ for $1 \leq k \leq n$. Vertices correspond to linear orders $v_{i_1} < \dots < v_{i_n}$ on V with associated monomial quasisymmetric functions equal to $M_{(1, \dots, 1)}$.

By Proposition 2.4 the normal cone at the vertex $v \in Pe^{n-1}$ that corresponds to a permutation $\pi_v = (i_1, \dots, i_n)$ is the Weyl chamber

$$\sigma_v = C_{\pi_v} : x_{i_1} < \dots < x_{i_n}.$$

The braid arrangement \mathcal{A}_n is the arrangement of hyperplanes

$$\mathcal{A}_n : x_i = x_j, 1 \leq i, j \leq n$$

in the quotient space $\mathbb{R}^n / \mathbb{R} \cdot (1, \dots, 1) \cong \mathbb{R}^{n-1}$. The normal fan $\Sigma_{Pe^{n-1}}$ of the permutohedron is the simplicial fan defined by \mathcal{A}_n . A braid cone is the polyhedral cone given by the conjunction of inequalities of the form $x_i \leq x_j$. There is an obvious bijection between preorders \prec on $[n]$ and braid cones determined by equivalency $x_i \leq x_j$ if and only if $i \prec j$. The correspondence and properties of preorders and braid cones are given in [17, Proposition 3.5]. We remark that linear orders on $[n]$ correspond to full-dimensional braid cones. The monomial quasisymmetric function M_F is precisely the enumerator for all positive lattice points in the interior of the normal cone associated to the face $F \subset Pe^{n-1}$.

For each generalized permutohedron Q there is a map $\Psi_Q : S_n \rightarrow \text{Vertices}(Q)$ defined by $\Psi(\pi) = v$ if and only if the normal cone σ_v of Q at v contains the Weyl chamber C_π or equivalently the permutation $\pi \in S_n$ is a linear extension of the poset determined by the normal cone at v [17, Corollary 3.9].

4 Hopf algebra morphism

The goal of this section is to show that the assignment of quasisymmetric function $F(P_B)$ to a building set B is a Hopf algebra morphism. We use the theory of combinatorial Hopf algebras developed in the originating paper by Aguiar,

Bergeron and Sottile [1]. For an extensive survey of the theory see also [8]. A combinatorial Hopf algebra (\mathcal{H}, ζ) over a field \mathbf{k} is a graded, connected Hopf algebra equipped with a multiplicative linear functional $\zeta : \mathcal{H} \rightarrow \mathbf{k}$ called a character.

We construct a graded Hopf algebra associated with the species of building sets in the sense of [18]. Let \mathcal{B} be the graded vector space generated by the set of all isomorphism classes of building sets. The grading is defined by the number of vertices.

For a building set B on $[n]$ and a subset $I \subset [n]$, let $B|_I = \{J \subset I \mid J \in B\}$ be the induced building subset. The contraction of $I \subset [n]$ from B is the building set on $[n] \setminus I$ defined by $B/I = \{J \subset [n] \setminus I \mid J \in B \text{ or } I' \cup J \in B \text{ for some } I' \subset I\}$. Define the multiplication and comultiplication by

$$B_1 \cdot B_2 = B_1 \sqcup B_2 \quad \text{and} \quad \Delta(B) = \sum_{I \subset V} B|_I \otimes B/I.$$

The unit is the building set B_\emptyset on the empty set and the counit is defined by $\epsilon(B_\emptyset) = 1$ and zero otherwise.

Proposition 4.1. *The vector space \mathcal{B} with the above defined operations is a graded commutative and non-cocommutative connected bialgebra.*

Proof. The only nontrivial parts of the statement are the coassociativity and the compatibility of operations, which follows from the straightforward identities $(B/I)|_J = (B|_{I \sqcup J})/I$, $(B/I)/J = B/(I \sqcup J)$ for any disjoint $I, J \subset V$ and $(B_1 \cdot B_2)|_{I_1 \sqcup I_2} = B_1|_{I_1} \cdot B_2|_{I_2}$, $(B_1 \cdot B_2)/(I_1 \sqcup I_2) = B_1/I_1 \cdot B_2/I_2$ for all $I_1 \subset V_1, I_2 \subset V_2$. \square

The antipode of \mathcal{B} is determined by general Takeuchi's formula for the antipode of a graded connected bialgebra ([21, Lemma 14], see also [8, Proposition 1.44])

$$S(B) = \sum_{k \geq 1} (-1)^k \sum_{\mathcal{L}_k} \prod_{j=1}^k (B|_{I_j})/I_{j-1},$$

where the inner sum goes over all chains of subsets $\mathcal{L}_k : \emptyset = I_0 \subset I_1 \subset \dots \subset I_{k-1} \subset I_k = V$.

Define a character $\zeta : \mathcal{B} \rightarrow \mathbf{k}$ by $\zeta(B) = 1$ if B is discrete and zero otherwise. This determines the combinatorial Hopf algebra (\mathcal{B}, ζ) .

Remark 4.2. Another combinatorial Hopf algebra of building set $BSet$, which is a Hopf subalgebra of the chromatic Hopf algebra of hypergraphs is studied in [9], [10]. As algebras \mathcal{B} and $BSet$ are the same but the coalgebra structures are different. This is reflected in the fact that $BSet$ is cocommutative, in opposite to \mathcal{B} .

Remark 4.3. The algebra \mathcal{B} has an additional structure of a differential algebra introduced in [4]. The derivation is determined by

$$d(B) = \sum_{I \in B \setminus \{[n]\}} B|_I \cdot B/I$$

for connected building set on $[n]$ and extended by Leibniz law $d(B_1 B_2) = d(B_1)B_2 + B_1 d(B_2)$.

Definition 4.4. Given a composition $\alpha = (a_1, \dots, a_k) \models n$, we say that the chain $\mathcal{L} : \emptyset = I_0 \subset I_1 \subset \dots \subset I_{k-1} \subset I_k = V$ is a *splitting chain* of the type $\text{type}(\mathcal{L}) = \alpha$ of a building set B if $(B|_{I_j})/I_{j-1}$ is discrete and $|I_j \setminus I_{j-1}| = a_j$ for all $1 \leq j \leq k$. A splitting chain \mathcal{L} determines the weak order $\preceq_{\mathcal{L}} = (I_1, I_2 \setminus I_1, \dots, I_k \setminus I_{k-1})$ on V of the same type.

Theorem 4.5. For a connected building set B the generating function $F(P_B)$ has the following expansion

$$F(P_B) = \sum_{\alpha \models n} \zeta_{\alpha}(B) M_{\alpha},$$

where $\zeta_{\alpha}(B)$ is the total number of splitting chains of the type α .

Proof. We define a map $g : \Lambda \rightarrow \text{Vertices}(P_B)$ from the set Λ of splitting chains of B to the set of vertices of P_B .

Let $\mathcal{L} : \emptyset = I_0 \subset I_1 \subset \dots \subset I_{k-1} \subset I_k = V = [n]$ be a splitting chain of B . Define the level of a vertex $i \in V$ by $l(i) = j$ if $i \in I_j \setminus I_{j-1}$ and a map $S : V \rightarrow B$ by $S(i) = \max\{J \in B \mid i \in J\}$, $i \in V$. The map S is well defined and $S(i) \setminus \{i\} \subset I_{l(i)-1}$ since $(B|_{I_{l(i)}})/I_{l(i)-1}$ is discrete. Particulary $S(i) = \{i\}$ for each $i \in V$ such that $l(i) = 1$ and $S(i) = V$ for the unique $i \in V$. Let $N(\mathcal{L}) = \{S(i) \mid i \in V\} \setminus \{V\}$. We check the conditions of Proposition 2.3 to show that the collection $N(\mathcal{L})$ is a maximal nested set.

- (N1) Suppose that $S(i) \cap S(j) \neq \emptyset$ for some $i, j \in V$. It implies that $S(i) \cup S(j) \in B$. If $l = l(i) = l(j)$ then $\{i, j\} \in (B|_{I_l})/I_{l-1}$ which contradicts the condition that $(B|_{I_l})/I_{l-1}$ is discrete. If $l(j) < l(i)$ then $i \in S(i) \cup S(j) \in B$ which implies $S(j) \subset S(i)$ by definition of $S(i)$.
- (N2) For a collection $S(i_1), \dots, S(i_p)$ such that $S = S(i_1) \cup \dots \cup S(i_p) \in B$ we have by definition that $S = S(i_j)$ for a vertex $i_j \in V$ with the maximal level $l = \max\{l(i_1), \dots, l(i_p)\}$. Thus $S(i_1), \dots, S(i_p)$ can not be a pairwise disjoint collection if $p > 1$.

The map g is given by $g(\mathcal{L}) = v$ if $N(\mathcal{L}) = N_v$. It is well defined since vertices of P_B and maximal nested sets are in one-to-one correspondence. We show the following identity

$$\sum_{f \in \sigma_v} \mathbf{x}_f = \sum_{\mathcal{L} : g(\mathcal{L}) = v} M_{\text{type}(\mathcal{L})}.$$

Let \mathcal{L} be a splitting chain such that $g(\mathcal{L}) = v$. Then the associated level function satisfies $l(i) < l(j)$, for each $S(i) \subset S(j)$ in $N(\mathcal{L})$. By Proposition 2.4 (ii) we have $l \in \sigma_v$ which shows that the monomial quasisymmetric function $M_{\text{type}(\mathcal{L})}$ is a summand of $\sum_{f \in \sigma_v} \mathbf{x}_f$.

On the other hand, for $f \in \sigma_v$ with the set of values $i_1 < \dots < i_k$, let $\mathcal{L}_f : \emptyset \subset I_1 \subset \dots \subset I_k$ be a chain, where $I_j = \{i \in V \mid f(i) \leq i_j\}$, $1 \leq j \leq k$. We can convince that the chain \mathcal{L}_f is a splitting chain of B such that $N(\mathcal{L}_f) = N_v$, which implies that the monomial \mathbf{x}_f appears in $M_{\text{type}(\mathcal{L}_f)}$.

Finally, as

$$F(P_B) = \sum_{v \in P_B} \sum_{f \in \sigma_v} \mathbf{x}_f = \sum_{v \in P_B} \sum_{\mathcal{L} \in g^{-1}(v)} M_{\text{type}(\mathcal{L})} = \sum_{\alpha \models n} \zeta_\alpha(B) M_\alpha,$$

the theorem is proved. \square

Remark 4.6. Definition 4.4 of splitting chains is borrowed from [3] where it appears in the context of matroids. Theorem 4.5 is analogous to [3, Proposition 3.3].

The character $\zeta_Q : QSym \rightarrow \mathbf{k}$, defined on the monomial basis by $\zeta_Q(M_\alpha) = 1$ for either $\alpha = ()$ or $\alpha = (n)$ and zero otherwise, turns the Hopf algebra of quasisymmetric functions $QSym$ into the terminal object in the category of combinatorial Hopf algebras over a field \mathbf{k} [1, Theorem 4.1]. This means that for each combinatorial Hopf algebra (\mathcal{H}, ζ) there is a unique morphism of combinatorial Hopf algebras $\Psi : (\mathcal{H}, \zeta) \rightarrow (QSym, \zeta_Q)$. The explicit definition of this morphism implies the following corollary of Theorem 4.5.

Corollary 4.7. *The map $F : \mathcal{B} \rightarrow QSym$, defined by $F(B) = F(P_B)$, is a morphism of combinatorial Hopf algebras.*

Proof. Let $p_j : \mathcal{B} \rightarrow \mathcal{B}_j$ be the projection on the homogeneous part of degree j . The morphism $\Psi : (\mathcal{B}, \zeta) \rightarrow (QSym, \zeta_Q)$ is defined by

$$\Psi(B) = \sum_{\alpha \models n} p_\alpha(B) M_\alpha,$$

where $p_\alpha = p_{(a_1, \dots, a_k)} = p_{a_1} * \dots * p_{a_k} = m^{(k-1)} \circ (p_{a_1} \otimes \dots \otimes p_{a_k}) \circ \Delta^{(k-1)}$ is the convolution product of projections. It is straightforward to convince that $p_\alpha(B) = \zeta_\alpha(B)$ for any composition $\alpha \models n$, so the morphism Ψ coincides with the map F . \square

As a consequence we obtain the following identities for the function F :

$$\begin{aligned} F(P_{B_1} \times P_{B_2}) &= F(P_{B_1}) F(P_{B_2}), \\ \Delta(F(P_B)) &= \sum_{I \subset V} F(P_{B|_I}) \otimes F(P_{B/I}). \end{aligned}$$

Remark 4.8. The function $F(P_B)$ is not a combinatorial invariant of nestohedra. For example, the building sets $B_1 = \{1, 2, 3, 4, 12, 123\}$ and $B_2 = \{1, 2, 3, 4, 12, 34\}$ on the four element set $V = [4]$ have P_{B_1} and P_{B_2} combinatorially equivalent to the 3-cube, but $F(B_1) \neq F(B_2)$.

5 Unlabeled rooted trees

Let T be an unlabeled rooted tree on the set of vertices $V = \{v_1, \dots, v_n\}$. It defines a poset (V, \leq_T) with $v_i \leq v_j$ if and only if v_j is the node on the unique path from v_i to the root. We do not make a difference between the rooted tree T and the corresponding Hasse diagram of the poset (V, \leq_T) .

Remark 5.1. Let \mathcal{T}_n be the set of all unlabeled rooted trees on n nodes and $r(n)$ be the total number of elements of \mathcal{T}_n . In Neil Sloan's OEIS the sequence $\{r(n)\}_{n \in \mathbb{N}}$ is numerated by A000081.

We need some basic notions from Stanley's theory of P -partitions. A detailed survey of the theory can be found in [19], [8]. Let $f : T \rightarrow \mathbb{N}$ be a function on vertices of a rooted tree T . We call it *natural T -partition* if $f(v_i) \leq f(v_j)$ for $v_i \leq v_j$ in T and *strict T -partition* if in addition $f(v_i) < f(v_j)$ for any pair of vertices with $v_i < v_j$ in T . Write $\mathcal{A}(T)$ for the set of all natural T -partitions and $\mathcal{A}_0(T)$ for its subset of strict T -partitions. Let $F(T)$ be the quasisymmetric enumerator of strict T -partitions

$$F(T) = \sum_{f \in \mathcal{A}_0(T)} \mathbf{x}_f.$$

Example 5.2. There are four unlabeled rooted trees on four vertices whose corresponding enumerators $F(T)$ in the monomial basis are given by

$$\begin{aligned} F\left(\begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}\right) &= M_{(1,1,1,1)}, F\left(\begin{array}{ccc} & \bullet & \\ & | & \\ \bullet & & \bullet \end{array}\right) = 6M_{(1,1,1,1)} + 3M_{(2,1,1)} + 3M_{(1,2,1)} + M_{(3,1)}, \\ F\left(\begin{array}{ccc} & \bullet & \\ & | & \\ \bullet & & \bullet \end{array}\right) &= 3M_{(1,1,1,1)} + M_{(2,1,1)} + M_{(1,2,1)}, F\left(\begin{array}{ccc} & \bullet & \\ & | & \\ \bullet & & \bullet \end{array}\right) = 2M_{(1,1,1,1)} + M_{(2,1,1)}. \end{aligned}$$

The quasisymmetric function $F(T)$ can be determined recursively. To each vertex $v \in V$ define $T_{\leq v}$ as the complete subtree on the set $\{u \in V \mid u \leq v\}$ of predecessors of v . The leaf is a vertex $v \in V$ for which $T_{\leq v} = \{v\}$. For a rooted forest $T = \sqcup_{i=1}^k T_i$ which is a finite collection of rooted trees we extend multiplicatively

$$F(\sqcup_{i=1}^k T_i) = F(T_1) \cdots F(T_k).$$

Definition 5.3. A shifting operator $F \mapsto (F)_1$ on quasisymmetric functions is the linear extension of the map defined on the monomial basis by $(M_\alpha)_1 = M_{(\alpha,1)}$, for each composition α .

Theorem 5.4. Given an unlabeled rooted tree T on the set of vertices V with the root $v_0 \in V$. Let T_1, \dots, T_k be connected components of the forest $T \setminus \{v_0\}$. Then

$$F(T) = \left(\prod_{i=1}^k F(T_i)\right)_1 = F(T \setminus \{v_0\})_1.$$

Proof. Denote by v_1, \dots, v_k the neighbors in T of the root v_0 . Then $T_i = T_{\leq v_i}$ for $i = 1, \dots, k$. A function $f : T \rightarrow \mathbb{N}$ is a T -partition if and only if its restrictions $f|_{T_i} : T_i \rightarrow \mathbb{N}$ are T_i -partitions for all $i = 1, \dots, k$ and $f(v) < f(v_0)$ for each $v \neq v_0$. \square

Each T -partition $f : T \rightarrow \mathbb{N}$ takes the maximal value at the root of T . Therefore each monomial function M_α in the expansion of $F(T)$ in the monomial basis is indexed by the composition α whose last component is 1. Since $r(n) > 2^{n-2} = \dim(QSym_{n-1})$ for $n > 4$, we proved the following

Proposition 5.5. *The quasisymmetric functions $\{F(T)\}_{T \in \mathcal{T}_n}$ are linearly dependent for each $n > 4$.*

Example 5.6. We have $r(5) = 9$ and $\dim(QSym_4)_1 = 8$. The unique linear dependence relation is given by

$$F\left(\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}\right) + 2F\left(\begin{array}{c} \bullet \\ | \\ \bullet \quad \bullet \end{array}\right) = F\left(\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}\right) + F\left(\begin{array}{c} \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array}\right) + F\left(\begin{array}{c} \bullet \\ | \\ \bullet \quad \bullet \end{array}\right).$$

Given a connected building set B , recall that to each vertex $v \in P_B$ is associated the labeled rooted tree T_v , called B -tree, which is the Hasse diagram of the poset P_v . Denote by T_v° the unlabeled rooted tree associated to a B -tree T_v by forgetting the labels. Let $T(B) = \{T_v^\circ \mid v \in P_B\}$ be the multiset of the corresponding unlabeled rooted trees. The following expansion is a special case of [3, Theorem 9.2] which holds for generalized permutohedra.

Theorem 5.7. *For a building set B the quasisymmetric enumerator $F(P_B)$ is the sum of T -partitions enumerators corresponding to vertices of P_B*

$$F(P_B) = \sum_{T \in T(B)} F(T).$$

Proof. It is sufficient to show the identity $F(T_v^\circ) = \sum_{f \in \sigma_v} \mathbf{x}_f$ which follows from the description of the normal cone σ_v at a vertex $v \in P_B$, see Proposition 2.4 (ii). \square

Corollary 5.8. *The quasisymmetric function $F(P_B)$ depends only on the multiset $T(B)$ of unlabeled rooted trees T_v° corresponding to the vertices $v \in P_B$.*

Question 5.9. In what extent the multiset $T(B)$ determines a building set B ?

We say that a weak order \preceq extends a partial order $P = ([n], \leq)$ and write $P \subset \preceq$ if $i < j$ implies $i \prec j$ for each $i, j \in [n]$. Theorem 5.7 implies the following interpretation

$$F(P_B) = \sum_{v \in P_B} \sum_{P_v \subset \preceq} M_{\text{type}(\preceq)}.$$

Since weak orders on the set $[n]$ and faces of the permutohedron Pe^{n-1} are in one-to-one correspondence, we associate to each vertex $v \in P_B$ the collection of faces $F \subset Pe^{n-1}$ such that the weak order \preceq_F corresponding to a face F extends the partial order P_v corresponding to the vertex v . This is exactly the collection of faces that collapses to the vertex $v \in P_B$ by deforming the permutohedron Pe^{n-1} to the nestohedron P_B .

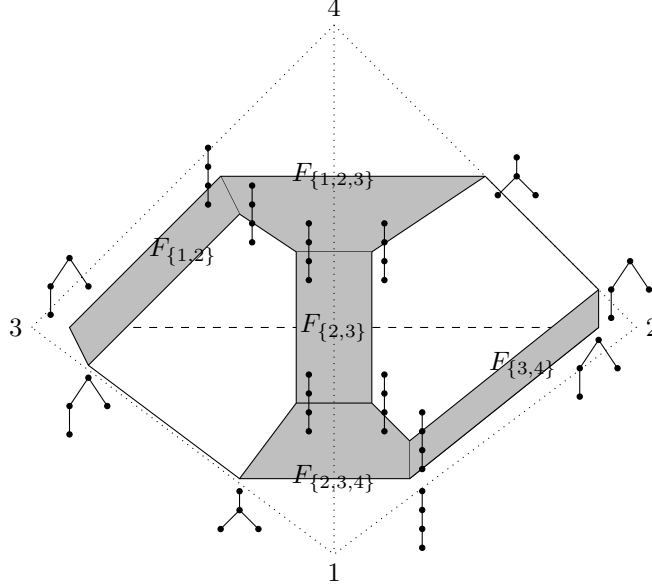


Figure 1: Associahedron As^3

Example 5.10. The 3-dimensional associahedron As^3 is realized as the graph-associahedron $P_{B(L_4)}$ corresponding to the path graph L_4 on the set of vertices $[4]$. The determining building set is $B(L_4) = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$. To illustrate the general construction we describe in more details how the unlabeled rooted trees T_v° are associated to the vertices $v \in As^3$, see Figure 1. The construction depends only on the combinatorial type of a nestohedron P_B . Therefore we can start with a 3-simplex Δ^3 with the faces Δ_I labeled by subsets $I \subset [4]$. Performing the truncations along $\Delta_{[4] \setminus J}$ for $J \in B(L_4) \setminus \{[4]\}$ in nondecreasing order of dimensions, produces As^3 with the facets F_J . Each vertex is an intersection of the form $v = F_{J_1} \cap F_{J_2} \cap F_{J_3}$, where the collection $N_v = \{J_1, J_2, J_3\}$ is a maximal nested set. The rooted tree T_v is the Hasse diagram of the poset P_v and T_v° is the corresponding unlabeled rooted tree.

By Theorem 5.7 and Example 5.2 we find

$$F(As^3) = 4M_{(1,2,1)} + 6M_{(2,1,1)} + 24M_{(1,1,1,1)}.$$

6 The action of the antipode on $F(P_B)$

The action $F^*(Q) = S(F(Q))$ of the antipode S of quasisymmetric functions on the lattice points enumerator $F(Q)$ is determined for a general class of generalized permutohedra in [3, Theorem 9.2]. We consider this formula for a special class of nestohedra.

The formula for the antipode in the monomial basis are obtained independently in [13, Corollary 2.3], [6, Proposition 3.4], see also [8, Theorem 5.11]. The antipode formula on P -partition enumerators (see [14, Theorem 3.1] and [8, Corollary 5.27]) has a particularly nice interpretation for unlabeled rooted trees. Let T^{op} be the unlabeled rooted tree T with the reverse order of vertices. Thus the root is the minimal vertex in T^{op} . Denote by $\widehat{F}(T^{\text{op}})$ the quasisymmetric enumerator of natural T^{op} -partitions

$$\widehat{F}(T^{\text{op}}) = \sum_{f \in \mathcal{A}(T^{\text{op}})} \mathbf{x}_f.$$

Then the following formula holds

$$S(F(T)) = (-1)^n \widehat{F}(T^{\text{op}}).$$

For example $S\left(F\left(\begin{smallmatrix} \bullet & & \bullet \\ & \diagdown & / \\ & \bullet & \end{smallmatrix}\right)\right) = -\widehat{F}\left(\begin{smallmatrix} \bullet & & \bullet \\ & / & \diagdown \\ & \bullet & \end{smallmatrix}\right)$, where $F\left(\begin{smallmatrix} \bullet & & \bullet \\ & \diagdown & / \\ & \bullet & \end{smallmatrix}\right) = 2M_{(1,1,1)} + M_{(2,1)}$ and $\widehat{F}\left(\begin{smallmatrix} \bullet & & \bullet \\ & / & \diagdown \\ & \bullet & \end{smallmatrix}\right) = 2M_{(1,1,1)} + 2M_{(2,1)} + M_{(1,2)} + M_{(3)}$. Consequently Theorem 5.7 implies

$$F^*(P_B) = (-1)^n \sum_{T \in T(B)} \widehat{F}(T^{\text{op}}).$$

Example 6.1. Let $As^3 = P_{B(L_4)}$ as in Example 5.10. We find $F^*(As^3) = 14M_{(4)} + 14M_{(1,3)} + 20M_{(3,1)} + 18M_{(2,2)} + 18M_{(1,1,2)} + 20M_{(1,2,1)} + 24M_{(2,1,1)} + 24M_{(1,1,1,1)}$.

The following proposition, proved in the case of matroid base polytopes [3, Theorem 6.3] and stated in [3, Theorem 9.2] for generalized permutohedra, gives a combinatorial interpretation of the action of the antipode S on $F(P_B)$.

Proposition 6.2. *The quasisymmetric function $F^*(P_B) = S(F(P_B))$ is the enumerator function*

$$F^*(P_B) = (-1)^n \sum_f c(f) \mathbf{x}_f,$$

where the sum is over all $f : [n] \rightarrow \mathbb{N}$ and $c(f) = |\{v \in P_B \mid \min_{x \in P_B} \langle f, x \rangle = \langle f, v \rangle\}|$ is the total number of vertices $v \in P_B$ that minimize a function f .

Proof. Let σ_v^{op} be the opposite cone to the normal cone σ_v at a vertex $v \in P_B$. By Proposition 2.4 (ii) we have that the closure of the opposite cone $\overline{\sigma_v^{\text{op}}}$ is determined by inequalities

$$x_j \leq x_i, \text{ for all } v_i < v_j \in T_v.$$

Therefore a function $f : [n] \rightarrow \mathbb{N}$ is a natural T^{op} -partition $f \in \mathcal{A}(T^{\text{op}})$ if and only if it belongs to the closure of the opposite cone $f \in \overline{\sigma_v^{\text{op}}}$ at some vertex $v \in P_B$. It means that

$$(-1)^n F^*(P_B) = \sum_{T \in T(B)} \widehat{F}(T^{\text{op}}) = \sum_{v \in P_B} \sum_{f \in \overline{\sigma_v^{\text{op}}}} \mathbf{x}_f.$$

It remains to note that $f \in \overline{\sigma_v^{\text{op}}}$ if and only if f is minimized at the vertex v . \square

For $F \in QSym$ let $\chi(F, m) = \text{ps}_m(F)$ be the principal specialization defined by algebraic extension of $\text{ps}_m(x_i) = 1$ for $1 \leq i \leq m$ and $\text{ps}_m(x_i) = 0$ for $i > m$. Since $\text{ps}_m(M_\alpha) = \binom{m}{k(\alpha)}$ we have

$$\chi(P_B, m) = \sum_{\alpha \models n} \zeta_\alpha(B) \binom{m}{k(\alpha)},$$

which counts the number of P_B -generic functions $f : [n] \rightarrow [m]$. It is related with $\chi^*(P_B, m) = \text{ps}_m(F^*(P_B))$ by

$$\chi(P_B, -m) = (-1)^n \chi^*(P_B, m).$$

Specially, for $m = 1$, we obtain the following

Proposition 6.3. *The number of vertices $f_0(P_B)$ of a nestohedron P_B is determined by $f_0(P_B) = (-1)^n \chi(P_B, -1)$.*

Proof. The statement follows from the identity $\text{ps}_1(F^*(P_B)) = c_{(n)}$, where $c_{(n)}$ is the coefficient by $M_{(n)}$ in the expansion of $F^*(P_B)$ in the monomial basis. By Proposition 6.2 this coefficient counts the vertices of P_B that minimize $\langle f, x \rangle$ over P_B for $f = (1, \dots, 1)$. But f is orthogonal to P_B . \square

7 The graph invariant $F(P_{B(\Gamma)})$

In this section we investigate the quasisymmetric function $F(P_{B(\Gamma)})$ associated to a simple graph Γ .

For a graph Γ on the vertex set $[n]$ and a subset $I \subset [n]$ are defined the restriction $\Gamma|_I$ and the contraction Γ/I . The restriction $\Gamma|_I$ is the induced subgraph on the vertex set I and the contraction Γ/I is a graph on $[n] \setminus I$ with two vertices u and v connected by the edge if either $\{u, v\}$ is an edge of Γ or there is a path u, w_1, \dots, w_k, v in Γ with $w_1, \dots, w_k \in I$. To a graph Γ is associated the graphical building set $B(\Gamma)$ as in Example 2.1. It is immediate that $B(\Gamma|_I) = B(\Gamma)|_I$ and $B(\Gamma/I) = B(\Gamma)/I$. An element $I \in B(\Gamma)$ and the corresponding contraction Γ/I are called in [5] the tube and the reconnected complement respectively.

The vector space \mathcal{G} spanned by all isomorphism classes of simple graphs is endowed with the Hopf algebra structure by operations

$$\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \sqcup \Gamma_2 \text{ and } \Delta(\Gamma) = \sum_{I \subset V} \Gamma|_I \otimes \Gamma/I.$$

The map that associates the graphical building set $B(\Gamma)$ to a graph Γ is extended to a Hopf algebra monomorphism $i : \mathcal{G} \rightarrow \mathcal{B}$. The induced character is defined by $\zeta(\Gamma) = 1$ if Γ is discrete and zero otherwise. It follows from Corollary 4.7 that the quasisymmetric function $F(P_{B(\Gamma)})$ is a multiplicative graph invariant. By Theorem 4.5 it may be defined purely in a graph theoretic manner.

Let Γ be a simple graph on n vertices $V = \{v_1, \dots, v_n\}$ and $\lambda : V \rightarrow \mathbb{N}$ be a coloring with the set of colors $\{i_1 < \dots < i_k\}$. Define a flag $\emptyset = I_0 \subset I_1 \subset \dots \subset I_{k-1} \subset I_k = V$ by $I_j = \lambda^{-1}(\{i_1, \dots, i_j\})$ for $1 \leq j \leq k$. We say that λ is a *ordered coloring* of Γ if the graphs $\Gamma|_{I_j/I_{j-1}}$ are discrete for all $1 \leq j \leq k$. This means that each monochromatic set of vertices is discrete and no two vertices of the same color are connected by a path through vertices colored by smaller colors. The type of an ordered coloring λ is the composition $\text{co}(\lambda) = (a_1, \dots, a_k) \models n$, where $a_j = |I_j \setminus I_{j-1}|$ is the number of vertices colored by i_j , for all $1 \leq j \leq k$. Let $\text{Col}^{\leq}(\Gamma)$ be the set of all ordered colorings of the graph Γ and F_{Γ} be the enumerator function

$$F_{\Gamma} = \sum_{\lambda \in \text{Col}^{\leq}(\Gamma)} \mathbf{x}_{\lambda}.$$

By Theorem 4.5 it coincides with the quasisymmetric function of the graph-associahedra $B(\Gamma)$

$$F_{\Gamma} = F(P_{B(\Gamma)}).$$

Thus in the monomial basis it has the expansion $F_{\Gamma} = \sum_{\alpha \models n} \zeta_{\alpha}(\Gamma) M_{\alpha}$, where $\zeta_{\alpha}(\Gamma)$ is the number of ordered colorings $\lambda : V \rightarrow \{1, \dots, k(\alpha)\}$ of the type $\text{co}(\lambda) = \alpha$. The polynomial $\chi(\Gamma, m) = \chi(B(\Gamma), m)$ counts the number of ordered colorings with at most m colors.

Remark 7.1. Stanley's chromatic symmetric function of a graph X_{Γ} introduced in [20] is the enumerator of proper colorings $\lambda : V(\Gamma) \rightarrow \mathbb{N}$. A coloring λ is proper if the graph Γ does not contain a monochromatic edge, i.e. the induced graph on $\lambda^{-1}(\{i\})$ for each color $i \in \mathbb{N}$ is discrete. The sizes of monochromatic parts define the type of the proper coloring which is a partition of the number of vertices of the graph since ordering of colors is inessential. The assignment X_{Γ} is the canonical morphism from the chromatic Hopf algebra of graphs to symmetric functions, see ([1], Example 4.5). The coefficients $c_{\mu}(\Gamma)$ in the expansion in the monomial basis of symmetric functions

$$X_{\Gamma} = \sum_{\mu \vdash n} c_{\mu}(\Gamma) m_{\mu},$$

count the numbers of proper colorings of prescribed types $\mu \vdash n$. Recall that $m_{\mu} = \sum_{s(\alpha)=\mu} M_{\alpha}$, where the sum is over all compositions $\alpha \models n$ that can be rearranged to the partition $\mu \vdash n$.

The coefficients $\zeta_\alpha(\Gamma), \alpha \models n$ satisfy the following properties.

Proposition 7.2. *Given a graph Γ on the set of vertices $V = [n]$.*

- (a) *The coefficients $\zeta_{(k, 1^{n-k})}(\Gamma), 1 \leq k \leq n$ determine the f -vector of the independence complex $\text{Ind}(\Gamma)$ of the graph Γ .*
- (b) *For any pair $\alpha \preceq \beta$ it holds $\zeta_\alpha(\Gamma) \leq \zeta_\beta(\Gamma)$.*
- (c) *$\zeta_\alpha(\Gamma) \leq c_\mu(\Gamma)$ for each composition $\alpha \models n$ such that $s(\alpha) = \mu \vdash n$ and $c_\mu(\Gamma)$ are the coefficients of X_Γ in the monomial basis $\{m_\mu\}_{\mu \vdash n}$ of symmetric functions.*

Proof. Recall that the coefficient $\zeta_\alpha(\Gamma)$ counts the number of ordered colorings $\lambda : V \rightarrow [k(\alpha)]$ of the type $\alpha \models n$.

- (a) The only condition for a coloring $\lambda : V \rightarrow [n - k + 1]$ to be ordered with $\text{type}(\lambda) = (k, 1^{n-k})$ is that the set of vertices colored by 1 is k -element and discrete. Hence $\zeta_{(k, 1^{n-k})}(\Gamma) = (n - k)! f_{k-1}(\text{Ind}(\Gamma))$.
- (b) Suppose that α is obtained from β by combining some of its adjacent parts, i.e. $\alpha = (a_1, \dots, a_i, \dots, a_k)$ and $\beta = (a_1, \dots, a'_i, a''_i, \dots, a_k)$ with $a_i = a'_i + a''_i$. Then any ordered coloring of the type α defines at least $\binom{a_i}{a'_i}$ ordered colorings of the type β .
- (c) It is obvious since any ordered coloring of a type $\alpha \models n$ is the coloring of the type $s(\alpha) \vdash n$.

□

Remark 7.3. The formula similar to Proposition 7.2 (a) for the f -vectors of simplicial complexes is derived in [2, Section 4.5].

The following theorem allows one to define the invariant F_Γ recursively starting with $F_\emptyset = M_\emptyset = 1$. Recall that $F \mapsto (F)_1$ is the shifting operator, see Definition 5.3.

Theorem 7.4. *For a connected graph Γ on the vertex set $[n]$ it holds*

$$F_\Gamma = \sum_{i \in [n]} (F_{\Gamma \setminus \{i\}})_1.$$

Proof. We arrange the vertices $v \in P_{B(\Gamma)}$ according to the maximal elements of corresponding posets P_v . Let $T(B(\Gamma))_i = \{T_v^\circ \mid v \in P_{B(\Gamma)}, \max P_v = i\}$ be the multiset of specified unlabeled $B(\Gamma)$ -trees. Then by Theorem 5.7 we have

$$F_\Gamma = \sum_{i \in [n]} \sum_{T \in T(B(\Gamma))_i} F(T).$$

Denote by v_T the root of a tree T . The formula follows from the recurrence formula for T -partitions enumerators, see Theorem 5.4

$$\sum_{T \in T(B(\Gamma))_i} F(T) = \sum_{T \in T(B(\Gamma))_i} (F(T \setminus \{v_T\}))_1 = (F_{\Gamma \setminus \{i\}})_1.$$

□

Theorem 7.4 provides an effective computational tool for the invariant F_Γ .

Example 7.5. The invariant F_Γ distinguishes five-vertex graphs. In particular, the unique pair of five-vertex graphs with the same chromatic symmetric functions X_Γ given in [20, Figure 1] are distinguished by F_Γ . Figure 2¹ shows a pair of non-isomorphic six-vertex graphs with $F_{\Gamma_1} = F_{\Gamma_2} = 24M_{(1,2,1,1,1)} + 96M_{(2,1,1,1,1)} + 720M_{(1,1,1,1,1,1)}$. On the other hand the chromatic numbers of graphs Γ_1 and Γ_2 are different $\chi(\Gamma_1) = 3$ and $\chi(\Gamma_2) = 4$. Since the chromatic number $\chi(\Gamma)$ can be derived from Stanley's chromatic symmetric function X_Γ we conclude that graph invariants X_Γ and F_Γ are not comparable.

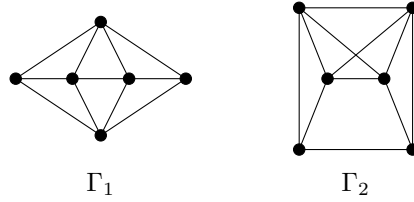


Figure 2: Graphs with $F_{\Gamma_1} = F_{\Gamma_2}$

Note that $\Gamma_1 = \overline{L_2 L_4}$ and $\Gamma_2 = \overline{L_3 L_3}$, where $\overline{\Gamma}$ denotes the complement graph of Γ .

We obtain the recurrence relations satisfied by enumerators $F(Q)$ for $Q = Pe^{n-1}, As^{n-1}, Cy^{n-1}, St^{n-1}$. We assume the realization of Q as a graph-associahedron of the corresponding graph as in Example 2.1. By convention the only (-1) -dimensional polytope is \emptyset .

Corollary 7.6. *For $n \geq 1$ the following recurrence relations hold*

$$\begin{aligned} F(Pe^{n-1}) &= n(F(Pe^{n-2}))_1, \\ F(As^{n-1}) &= \left(\sum_{k=1}^n F(As^{k-2}) F(As^{n-k-1}) \right)_1, \\ F(Cy^{n-1}) &= n(F(As^{n-2}))_1, \\ F(St^{n-1}) &= ((n-1)F(St^{n-2}) + M_{(1)}^{n-1})_1. \end{aligned}$$

¹The author is thankful to Marko Pešović for this example who discovered it by using MathLab program.

From Proposition 6.3 and Corollary 7.6 we recover the recurrence relations satisfied by numbers of vertices of corresponding graph-associahedra. Note the identity $\chi((F)_1, -1) = -\chi(F, -1)$ which is a consequence of $\chi(M_\alpha, -1) = (-1)^{k(\alpha)}$.

Corollary 7.7. *For $n \geq 1$ we have that the numbers $p_n = f_0(Pe^{n-1})$, $a_n = f_0(As^{n-1})$, $c_n = f_0(Cy^{n-1})$ and $s_n = f_0(St^{n-1})$ satisfy*

$$p_n = np_{n-1}, a_n = \sum_{k=1}^n a_{k-1}a_{n-k}, c_n = na_{n-1}, s_n = (n-1)s_{n-1} + 1$$

with $p_1 = a_1 = c_1 = s_1 = 1$. Therefore $p_n = n!$, $a_n = \frac{1}{n+1} \binom{2n}{n}$, $c_n = \binom{2n-2}{n-1}$ and $s_n = (n-1)! \sum_{k=0}^{n-1} \frac{1}{k!}$.

8 Conclusion

We conclude with several natural questions in connection with the Hopf algebra \mathcal{B} and the graph invariant F_Γ .

Problem 8.1. In a combinatorial Hopf algebra are defined the generalized Dehn-Sommerville relations which characterize the odd subalgebra (see [1], Section 5). Find a graph or a building set that satisfies the generalized Dehn-Sommerville relations for \mathcal{B} . The same problem is resolved in [10] for the chromatic Hopf algebra of hypergraphs, where the whole class of solutions called eulerian hypergraphs are found.

Problem 8.2. Stanley asked whether the chromatic symmetric function X_Γ is a complete invariant of trees. This question is still opened. The same question is natural to be posed for the enumerator F_Γ .

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