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ADAPTED NUMERICAL METHODS FOR THE POISSON EQUATION WITH L^2 BOUNDARY DATA IN NONCONVEX DOMAINS*

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Abstract. The very weak solution of the Poisson equation with L^2 boundary data is defined by the method of transposition. The finite element solution with regularized boundary data converges in the $L^2(\Omega)$-norm with order 1/2 in convex domains but has a reduced convergence order in nonconvex domains although the solution remains to be contained in $H^{1/2}(\Omega)$. The reason is a singularity in the dual problem. In this paper we propose and analyze, as a remedy, both a standard finite element method with mesh grading and a dual variant of the singular complement method. The error order 1/2 is retained in both cases, also with nonconvex domains. Numerical experiments confirm the theoretical results.

Key words. elliptic boundary value problem, very weak formulation, finite element method, mesh grading, singular complement method, discretization error estimate

AMS subject classifications. 65N30, 65N15

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1. Introduction.

In this paper we consider the boundary value problem

$$-\Delta y = f \quad \text{in } \Omega, \qquad\qquad y = u \quad \text{on } \Gamma = \partial\Omega, \tag{1}$$

with right-hand side $f \in H^{-1}(\Omega)$ and boundary data $u \in L^2(\Gamma)$. We assume $\Omega \subset \mathbb{R}^2$ to be a bounded polygonal domain with boundary Γ. Such problems arise in optimal control when the Dirichlet boundary control is considered in $L^2(\Gamma)$; see for example [22, 24, 28].

For boundary data $u \in L^2(\Gamma)$ we cannot expect a weak solution $y \in H^1(\Omega)$. Therefore we define a very weak solution by the method of transposition which goes back at least to Lions and Magenes [27, Chapter 2, section 6]: Find

$$y \in L^2(\Omega): \quad (y, \Delta v)_\Omega = (u, \partial_n v)_\Gamma - (f, v)_\Omega \quad \forall v \in V \tag{2}$$

with $(w, v)_G := \int_G wv$ denoting the $L^2(G)$ scalar product or an appropriate duality product. In our previous paper [4] we showed that the appropriate space V for the test functions is

$$V := H^1_\Delta(\Omega) \cap H^1_0(\Omega) \quad \text{with} \quad H^1_\Delta(\Omega) := \{v \in H^1(\Omega) : \Delta v \in L^2(\Omega)\}. \tag{3}$$

Note that from Theorems 4.4.3.7 and 1.4.5.3 of [25] the embedding $V \hookrightarrow H^{3/2+\varepsilon}(\Omega)$ for $0 < \varepsilon < \varepsilon_0$ follows with ε_0 depending on the maximal interior angle of the domain Ω.

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In particular this ensures $\partial_n v \in L^2(\Gamma)$ for $v \in V$ such that the formulation (2) is well defined. We proved the existence of a unique solution $y \in L^2(\Omega)$ for $u \in L^2(\Gamma)$ and $f \in H^{-1}(\Omega)$, and that the solution is even in $H^{1/2}(\Omega)$. The method of transposition is used in different variants also in [24, 9, 15, 14, 22, 28].

Consider now the discretization of the boundary value problem. Let $\mathcal{T}_h$ be a quasi-uniform family of conforming finite element meshes, and introduce the finite element spaces

$$Y_h := \{v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}, \quad Y_{0h} := Y_h \cap H_0^1(\Omega), \quad Y_h^\partial := Y_h|_{\partial\Omega}.$$

Since the boundary datum u is in general not contained in Y_h^∂ we have to approximate it by $u^h \in Y_h^\partial$, e. g., by using $L^2(\Gamma)$-projection or quasi-interpolation. In this way, the boundary datum is even regularized since $u^h \in H^{1/2}(\Gamma)$. Hence we can consider a regularized (weak) solution in $Y_*^h := \{v \in H^1(\Omega) : v|_\Gamma = u^h\}$,

$$y^h \in Y_*^h : \quad (\nabla y^h, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H_0^1(\Omega). \tag{4}$$

The finite element solution y_h is now searched for in $Y_{*h} := Y_*^h \cap Y_h$: Find

$$y_h \in Y_{*h} : \quad (\nabla y_h, \nabla v_h)_\Omega = (f, v_h)_\Omega \quad \forall v_h \in Y_{0h}. \tag{5}$$

The same discretization was derived previously by Berggren [9] from a different point of view. In [4] we showed that the discretization error estimate

$$\|y - y_h\|_{L^2(\Omega)} \leq Ch^s \left(h^{1/2}\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right)$$

holds for $s = 1/2$ if the domain is convex; this is a slight improvement of the result of Berggren, and the convex case is completely treated. In the case of nonconvex domains this convergence order is reduced although the very weak solution y is also in $H^{1/2}(\Omega)$; the finite element method does not lead to the best approximation in $L^2(\Omega)$. In order to describe the result we assume for simplicity that Ω has only one corner with interior angle $\omega \in (\pi, 2\pi)$. We proved in [4] the convergence order $s = \lambda - 1/2 - \varepsilon$, where $\lambda := \pi/\omega$ and $\varepsilon > 0$ arbitrarily small, and showed by numerical experiments that the order of almost $\lambda - 1/2$ is sharp. Note that $s \to 0$ for $\omega \to 2\pi$. This is the state of the art for this kind of problem, and our aim is to devise methods to retain the convergence order $s = 1/2$ in the nonconvex case.

In order to explain the reduction in the convergence order and our first remedy, let us first mention that we have to modify the Aubin–Nitsche method to derive $L^2(\Omega)$-error estimates. The first reason is that our problem has no weak solution, only the dual problem,

$$v_z \in V : \quad (\varphi, \Delta v_z)_\Omega = (z, \varphi)_\Omega \quad \forall \varphi \in L^2(\Omega), \tag{6}$$

has. The second reason is that the solution y has inhomogeneous Dirichlet data such that an estimate of the $L^2(\Gamma)$-interpolation error of $\partial_n v_z$ is needed. The $H^1(\Omega)$-error of a standard finite element method is of order one in convex domains but reduces to $s = \lambda - \varepsilon$ in the case of nonconvex domains; moreover, the order of the $L^2(\Gamma)$-interpolation error of $\partial_n v_z$ reduces from $1/2$ to $\lambda - 1/2 - \varepsilon$. It has been known for a long time that locally refined (graded) meshes and augmenting of the finite element space by singular functions are appropriate to retain the optimal convergence order for such problems; see, e. g., [8, 11, 17, 29, 31, 33]. We use these strategies in this paper.

The novelty is that the adapted methods act now implicitly and occur essentially in the analysis for the dual problem. This sounds particularly simple in the case of mesh grading. However, the convergence proof in [4] contains not only interpolation error estimates for the dual solution and its normal derivative (which are improved now) but also the application of an inverse inequality which gives a too pessimistic result if used unchanged in the case of graded meshes. We prove in section 2 a sharp result by using a weighted norm in intermediate steps. Note we suggest a strong mesh grading with grading parameter $\mu \to 0$ (the parameter is explained in section 2) for $\omega \to 2\pi$ because of the interpolation error estimate of $\partial_n v_z$; the numerical tests show that weaker grading is not sufficient.

The basic idea of the dual singular function method (see [11]), or the singular complement method (see [17]), is to augment the approximation space for the solution by one (or more, if necessary) singular function of type $r^\lambda \sin(\lambda\theta)$ and the space of test functions by a dual function of type $r^{-\lambda}\sin(\lambda\theta)$, where r, θ are polar coordinates at the concave corner. In this paper we do it the other way round and compute an approximate solution

$$z_h \in Y_h \oplus \mathrm{Span}\{r^{-\lambda}\sin(\lambda\theta)\}$$

such that the error estimate

$$\|y - z_h\|_{L^2(\Omega)} \le Ch^{1/2}\left(h^{1/2}\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right)$$

can be shown. Note that the original singular complement method augments the standard finite element space with a function which is part of the representation of the solution. Here, we complement the finite element space with $r^{-\lambda}\sin(\lambda\theta) \notin H^{1/2}(\Omega)$, and, although $y \in H^{1/2}(\Omega)$, this has an effect on the approximation order in the $L^2(\Omega)$-norm. This makes the method different from the original singular complement method, [17], and we call it the *dual singular complement method*. Numerical experiments in section 4 confirm the theoretical results.

Finally in this introduction, we would like to note that higher order finite elements are not useful here since the solution has low regularity. The extension of our methods to three-dimensional domains should be possible in the case of mesh grading (at considerable technical expenses in the analysis) but is not straightforward in the case of the dual singular complement method since the space $V \setminus H^2(\Omega)$ is in general not finite dimensional; see [18] for the Fourier singular complement method to treat special domains. Curved boundaries could be treated at the price of using nonaffine finite elements; see, e. g., [10, 12, 22].

2. Graded meshes. Recall from the introduction that $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain with boundary Γ, and we consider here the case that Ω has exactly one corner (called *singular corner*) with interior angle $\omega \in (\pi, 2\pi)$. The convex case was already treated in [4] and the case of more than one nonconvex corner can be treated similarly since corner singularities are local phenomena.

Without loss of generality we can assume that the singular corner is located at the origin of the coordinate system, and that one boundary edge is contained in the positive x_1-axis. We recall from [25, Theorem 4.4.3.7] or [26, sections 1.5, 2.3, and 2.4] that the weak solution of the boundary value problem (1) with $f \in L^2(\Omega)$ and $u = 0$ is not contained in $H^2(\Omega)$ but in

$$H^1_\Delta(\Omega) \cap H^1_0(\Omega) = \left(H^2(\Omega) \cap H^1_0(\Omega)\right) \oplus \mathrm{Span}\{\xi(r)\, r^\lambda \sin(\lambda\theta)\}, \tag{7}$$

ξ being a cutoff function, while r and θ denote polar coordinates at the singular corner.

Let the finite element mesh $\mathcal{T}_h = \{T\}$ be graded with the mesh grading parameter $\mu \in (0, 1]$, i. e., the element size $h_T = \operatorname{diam} T$ and the distance r_T of the element T to the singular corner are related by

$$\begin{aligned} c_1 h^{1/\mu} &\le h_T \le c_2 h^{1/\mu} && \text{for } r_T = 0, \\ c_1 h r_T^{1-\mu} &\le h_T \le c_2 h r_T^{1-\mu} && \text{for } r_T > 0. \end{aligned} \tag{8}$$

This type of graded mesh was investigated before in [8, 29, 31, 32]; see also the overview and background information in [5, section 2.3] and [1, section 7]. Define the finite element spaces

$$Y_h = \{v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}, \quad Y_{0h} = Y_h \cap H_0^1(\Omega), \quad Y_h^\partial = Y_h|_{\partial\Omega}, \tag{9}$$

and let the regularized boundary datum $u^h \in Y_h^\partial \subset H^{1/2}(\Gamma)$ be defined by the $L^2(\Gamma)$-projection $\Pi_h u$ or by the Carstensen interpolant $C_h u$; see [13]. To define the latter let $\mathcal{N}_\Gamma$ be the set of nodes of the triangulation on the boundary, and set

$$C_h u = \sum_{x \in \mathcal{N}_\Gamma} \pi_x(u) \lambda_x \quad \text{with} \quad \pi_x(u) = \frac{\int_{\omega_x} u \lambda_x}{\int_{\omega_x} \lambda_x} = \frac{(u, \lambda_x)_{\omega_x}}{(1, \lambda_x)_{\omega_x}},$$

where λ_x is the standard hat function related to x and $\omega_x = \operatorname{supp} \lambda_x \subset \Gamma$. As already outlined in [4], the advantages of the interpolant in comparison to the L^2-projection are its local definition and the property

$$u \in [a, b] \quad \Rightarrow \quad C_h u \in [a, b];$$

see [21]; a disadvantage may be that $C_h u_h \neq u_h$ for piecewise linear u_h. With these regularized boundary data we then define the regularized weak solution $y^h \in Y_*^h := \{v \in H^1(\Omega) : v|_\Gamma = u^h\}$ by (4).

Lemma 2.1. *If the mesh is graded with parameter $\mu < 2\lambda - 1$ the effect of the regularization of the boundary datum can be estimated by*

$$\|y - y^h\|_{L^2(\Omega)} \le c h^{1/2} \|u\|_{L^2(\Gamma)}.$$

Proof. In view of

$$\|y - y^h\|_{L^2(\Omega)} = \sup_{z \in L^2(\Omega), z \neq 0} \frac{(y - y^h, z)_\Omega}{\|z\|_{L^2(\Omega)}} \tag{10}$$

we have to estimate $(y - y^h, z)_\Omega$. To this end, let $z \in L^2(\Omega)$ be an arbitrary function and let $v_z \in V$ be defined by (6). Since the weak regularized solution $y^h \in Y_*^h := \{v \in H^1(\Omega) : v|_\Gamma = u^h\}$ defined by (4) is also a very weak solution,

$$(y^h, \Delta v)_\Omega = (u^h, \partial_n v)_\Gamma - (f, v)_\Omega \quad \forall v \in V, \tag{11}$$

we get with (2) and (6)

$$(y - y^h, z)_\Omega = (u - u^h, \partial_n v_z)_\Gamma. \tag{12}$$

If u^h is the $L^2(\Gamma)$-projection $\Pi_h u$ of u we can continue with

$$\begin{aligned}(u-u^h,\partial_n v_z)_\Gamma &= (u-u^h,\partial_n v_z-\Pi_h(\partial_n v_z))_\Gamma = (u,\partial_n v_z-\Pi_h(\partial_n v_z))_\Gamma\\
&\le \|u\|_{L^2(\Gamma)}\,\|\partial_n v_z-\Pi_h(\partial_n v_z)\|_{L^2(\Gamma)}\\
&\le \|u\|_{L^2(\Gamma)}\,\|\partial_n v_z-C_h(\partial_n v_z)\|_{L^2(\Gamma)}\\
&= \|u\|_{L^2(\Gamma)}\Big\|\sum_{x\in\mathcal{N}_\Gamma}\big(\partial_n v_z-\pi_x(\partial_n v_z)\big)\,\lambda_x\Big\|_{L^2(\Gamma)}\\
&\le c\|u\|_{L^2(\Gamma)}\left(\sum_{x\in\mathcal{N}_\Gamma}\|\partial_n v_z-\pi_x(\partial_n v_z)\|^2_{L^2(\omega_x)}\right)^{1/2}.\end{aligned}$$

If u^h is the Carstensen interpolant of u, there holds

$$\begin{aligned}(u-C_h u,\partial_n v_z)_\Gamma &= \left(\sum_{x\in\mathcal{N}_\Gamma}(u-\pi_x u)\lambda_x,\partial_n v_z\right)_\Gamma = \sum_{x\in\mathcal{N}_\Gamma}(u-\pi_x(u),(\partial_n v_z)\lambda_x)_\Gamma\\
&= \sum_{x\in\mathcal{N}_\Gamma}(u-\pi_x(u),(\partial_n v_z-\pi_x(\partial_n v_z))\lambda_x)_\Gamma\\
&\le \sum_{x\in\mathcal{N}_\Gamma}\|u\|_{L^2(\omega_x)}\|\partial_n v_z-\pi_x(\partial_n v_z)\|_{L^2(\omega_x)}\\
&\le c\|u\|_{L^2(\Gamma)}\left(\sum_{x\in\mathcal{N}_\Gamma}\|\partial_n v_z-\pi_x(\partial_n v_z)\|^2_{L^2(\omega_x)}\right)^{1/2},\end{aligned}$$

i. e., in both cases we have to estimate $\sum_{x\in\mathcal{N}_\Gamma}\|\partial_n v_z-\pi_x(\partial_n v_z)\|^2_{L^2(\omega_x)}$.

To this end we notice that

$$v_z\in V=\Big(H^2(\Omega)\cap H^1_0(\Omega)\Big)\oplus\mathrm{Span}\{\xi(r)\,r^\lambda\sin(\lambda\theta)\}$$

and, consequently,

$$\partial_n v_z\in V_\Gamma=\left(\prod_{j=1}^N H^{1/2}_{00}(\Gamma_j)\right)\oplus\mathrm{Span}\{\xi(r)\,r^{\lambda-1}\};$$

see [4, Remark 2.2] or [25, Theorem 1.5.2.8]. This means that we can split $\partial_n v_z = \alpha\xi(r)\,r^{\lambda-1}+\sum_{j=1}^N w_j$ with $w_j\in H^{1/2}_{00}(\Gamma_j)$ and

$$|\alpha|+\sum_{j=1}^N\|w_j\|_{H^{1/2}_{00}(\Gamma_j)} =: \|\partial_n v_z\|_{V_\Gamma}\le c\|v_z\|_V := \|\Delta v_z\|_{L^2(\Omega)} = \|z\|_{L^2(\Omega)}.$$

In the remaining part of the proof we show for $j=1,\dots,N$,

$$\left(\sum_{x\in\mathcal{N}_\Gamma}\|w_j-\pi_x w_j\|^2_{L^2(\omega_x)}\right)^{1/2}\le ch^{1/2}\|w_j\|_{H^{1/2}_{00}(\Gamma_j)}, \tag{13}$$

$$\left(\sum_{x\in\mathcal{N}_\Gamma}\|\xi(r)\,r^{\lambda-1}-\pi_x(\xi(r)\,r^{\lambda-1})\|^2_{L^2(\omega_x)}\right)^{1/2}\le ch^{1/2}, \tag{14}$$

to conclude $\left(\sum_{x\in\mathcal{N}_\Gamma}\|\partial_n v_z-\pi_x(\partial_n v_z)\|^2_{L^2(\omega_x)}\right)^{1/2}\le ch^{1/2}\|z\|_{L^2(\Omega)}$ and, hence,

$$(u-u^h,\partial_n v_z)_\Gamma\le ch^{1/2}\|u\|_{L^2(\Gamma)}\|z\|_{L^2(\Omega)}$$

which, together with (10) and (12), finishes the proof.

We extend w_j to the whole boundary Γ by zero on $\Gamma\setminus\Gamma_j$ and start with the estimate

$$\|w_j-\pi_s w_j\|_{L^2(\omega_x)}\le ch_x^s\|s_j\|_{H^s(\omega_x)},\quad s=0,1,\quad x\in\mathcal{N}_\Gamma. \tag{15}$$

This estimate follows for $s=0$ from the definition of π_x. For $s=1$ it follows from a Bramble–Hilbert-type argument if x is not a corner of Ω. In the case of a corner point x we use instead the zero boundary condition of w_j on one end of ω_x. Adding these estimates and using that $H^{1/2}_{00}(\Gamma_j)$ is an interpolation space of $L^2(\Gamma_j)$ and $H^1_0(\Gamma_j)$ we obtain (13). Note that the local element size h_x is bounded by h from above.

Denote by $\mathcal{N}_{\Gamma,\mathrm{reg}}\subset\mathcal{N}_\Gamma$ the set of nodes where ω_x does not contain the singular corner. Let r_x be the distance of $x\in\mathcal{N}_{\Gamma,\mathrm{reg}}$ to the set of corners of Ω, and note that the local mesh size satisfies both $h_x\le chr_x^{1-\mu}$ and $h_x\le cr_x$. One can estimate by using (15) with $s=1$,

$$\begin{aligned}\sum_{x\in\mathcal{N}_{\Gamma,\mathrm{reg}}}&\|\xi(r)\,r^{\lambda-1}-\pi_x(\xi(r)\,r^{\lambda-1})\|^2_{L^2(\omega_x)}\le c\sum_{x\in\mathcal{N}_{\Gamma,\mathrm{reg}}}h_x^2\|r^{\lambda-2}\|^2_{L^2(\omega_x)}\\&\le ch\sum_{x\in\mathcal{N}_{\Gamma,\mathrm{reg}}}r_x^{1-\mu}r_x\|r^{\lambda-2}\|^2_{L^2(\omega_x)}\le ch\int_0^{\mathrm{diam}\Omega}r^{2-\mu+2(\lambda-2)}\mathrm{d}r=ch\end{aligned}$$

for $\mu<2\lambda-1$. For the three nodes $x\in\mathcal{N}_\Gamma\setminus\mathcal{N}_{\Gamma,\mathrm{reg}}$ we cannot use the $H^1(\omega_x)$-regularity of $r^{\lambda-1}$ but, by using the stability of π_x, the properties of $\xi(\cdot)$, and $h_x\sim h^{1/\mu}$ there holds

$$\|\xi(r)\,r^{\lambda-1}-\pi_x(\xi(r)\,r^{\lambda-1})\|_{L^2(\omega_x)}\le c\|r^{\lambda-1}\|_{L^2(\omega_x)}\sim h_x^{\lambda-1/2}\sim h^{(\lambda-1/2)/\mu}\le ch^{1/2}$$

for $\mu<2\lambda-1$. Note that we computed the norm in the middle step. This finishes the proof. ∎

We consider now a lifting $\tilde B_h u^h\in Y_{*h}:=Y_*^h\cap Y_h$ defined by the nodal values as follows:

$$(\tilde B_h u^h)(x)=\begin{cases}u^h(x) & \text{for all nodes } x\in\Gamma,\\ 0 & \text{for all nodes } x\in\Omega.\end{cases} \tag{16}$$

The function y^h and its finite element approximation $y_h\in Y_{*h}$ are now defined by

$$y^h=y_f+\tilde B_h u^h+\tilde y_0^h\quad\text{as well as}\quad y_h=y_{fh}+\tilde B_h u^h+\tilde y_{0h}, \tag{17}$$

where $y_f,\tilde y_0^h\in H^1_0(\Omega)$ and $y_{fh},\tilde y_{0h}\in Y_{0h}$ satisfy

$$(\nabla y_f,\nabla v)_\Omega=(f,v)_\Omega\quad\forall v\in H^1_0(\Omega), \tag{18}$$

$$(\nabla y_{fh},\nabla v_h)_\Omega=(f,v_h)_\Omega\quad\forall v_h\in Y_{0h}, \tag{19}$$

$$(\nabla\tilde y_0^h,\nabla v)_\Omega=-(\nabla(\tilde B_h u^h),\nabla v)_\Omega\quad\forall v\in H^1_0(\Omega), \tag{20}$$

$$(\nabla\tilde y_{0h},\nabla v_h)_\Omega=-(\nabla(\tilde B_h u^h),\nabla v_h)_\Omega\quad\forall v_h\in Y_{0h}. \tag{21}$$

In order to estimate $\|y^h-y_h\|_{L^2(\Omega)}$ we estimate $\|y_f-y_{fh}\|_{L^2(\Omega)}$ and $\|\tilde y_0^h-\tilde y_{0h}\|_{L^2(\Omega)}$.

LEMMA 2.2. *If the mesh is graded with parameter $\mu < \lambda$ the error in approximating y_f satisfies*

$$\|y_f - y_{fh}\|_{L^2(\Omega)} \le ch\|f\|_{H^{-1}(\Omega)}.$$

Note that the condition $\mu < \lambda$ is weaker than the condition $\mu < 2\lambda - 1$ from Lemma 2.1 since $\lambda < 1$.

Proof. As in the proof of Lemma 2.1, let $z \in L^2(\Omega)$ be an arbitrary function, let $v_z \in V$ be defined via (6), and let $v_{zh} \in Y_{0h}$ be the Ritz projection of v_z. By the definitions (18) and (19) and using the Galerkin orthogonality we get

$$\begin{aligned}(y_f - y_{fh}, z)_\Omega &= (\nabla(y_f - y_{fh}), \nabla v_z)_\Omega = (\nabla(y_f - y_{fh}), \nabla(v_z - v_{zh}))_\Omega \\ &= (\nabla y_f, \nabla(v_z - v_{zh}))_\Omega \le \|\nabla y_f\|_{L^2(\Omega)} \|\nabla(v_z - v_{zh})\|_{L^2(\Omega)}.\end{aligned}$$

By using standard a priori estimates (see, e.g., [7, Theorem 3.2]), we obtain with grading $\mu < \lambda$ the bounds $\|\nabla y_f\|_{L^2(\Omega)} \le \|f\|_{H^{-1}(\Omega)}$, $\|\nabla(v_z - v_{zh})\|_{L^2(\Omega)} \le ch\|z\|_{L^2(\Omega)}$, and, hence, with

$$\|y_f - y_{fh}\|_{L^2(\Omega)} = \sup_{z \in L^2(\Omega), z \ne 0} \frac{(y_f - y_{fh}, z)_\Omega}{\|z\|_{L^2(\Omega)}},$$

the assertion of the lemma. ∎

In the proof of Lemma 2.4 we will employ a regularity result which is proved in [4, section II.C]. Reducing notation for the price of a slightly weaker statement we have the folllowing lemma.

LEMMA 2.3. *If $\omega > \pi$ then the very weak solution y from* (2) *satisfies*

$$\|r^{-\beta} y\|_{L^2(\Omega)} \le c\left(\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right) \quad \text{for all } \beta \in \left(1 - \lambda, \frac{1}{2}\right].$$

Proof. The statement is proved in [4, Lemma 2.8]. Concerning the assumptions on the regularity of the data, note that f and u are from bigger spaces there if $\beta \le \frac{1}{2}$; see [4, Remark 2.7]. Concerning the definition of the solution y in [4, (2.15)] note that the test space there contains V, which is seen by using the splitting (7), and since the solutions of both formulations are unique they must be equal. ∎

In order to estimate $\|\tilde{y}_0^h - \tilde{y}_{0h}\|_{L^2(\Omega)}$, we divide the domain Ω into subsets Ω_J, i.e.,

$$\Omega = \bigcup_{J=0}^{I} \Omega_J,$$

where $\Omega_J := \{x \in \Omega : d_{J+1} \le |x| \le d_J\}$ for $J = 1, \ldots, I-1$, $\Omega_I := \{x \in \Omega : |x| \le d_I\}$, and $\Omega_0 := \Omega \setminus \bigcup_{J=1}^{I} \Omega_J$. The radii d_J are set to 2^{-J} and the index I is chosen such that

$$d_I = 2^{-I} = c_I h^{1/\mu} \tag{22}$$

with a constant $c_I > 1$ exactly specified later on. In addition we define the extended domains Ω'_J and Ω''_J by

$$\Omega'_J := \Omega_{J-1} \cup \Omega_J \cup \Omega_{J+1} \quad \text{and} \quad \Omega''_J := \Omega'_{J-1} \cup \Omega'_J \cup \Omega'_{J+1},$$

respectively, with the obvious modifications for $J = 0, 1$ and $J = I-1, I$.

LEMMA 2.4. *With $\sigma := r + d_I$ there holds the estimate*

$$\|\sigma^{(1-\mu)/2}\nabla\tilde{y}_0^h\|_{L^2(\Omega)} + \|\sigma^{(1-\mu)/2}\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)} \le ch^{-1/2}\|u\|_{L^2(\Gamma)}.$$

Proof. We start by rearranging terms, i.e.,

$$\begin{aligned}\|\sigma^{(1-\mu)/2}\nabla\tilde{y}_0^h\|^2_{L^2(\Omega)} &= \int_\Omega \sigma^{1-\mu}\nabla\tilde{y}_0^h\cdot\nabla\tilde{y}_0^h \\ &= \int_\Omega \nabla\tilde{y}_0^h\cdot\nabla(\tilde{y}_0^h\sigma^{1-\mu}) - \int_\Omega \tilde{y}_0^h\nabla\tilde{y}_0^h\cdot\nabla\sigma^{1-\mu}.\end{aligned}\tag{23}$$

For the first term in (23) we conclude according to (20)

$$\begin{aligned}\int_\Omega \nabla\tilde{y}_0^h\cdot\nabla(\tilde{y}_0^h\sigma^{1-\mu}) &= -\int_\Omega \nabla(\tilde{B}_h u^h)\cdot\nabla(\tilde{y}_0^h\sigma^{1-\mu}) \\ &= -\int_\Omega \sigma^{1-\mu}\nabla(\tilde{B}_h u^h)\cdot\nabla\tilde{y}_0^h - \int_\Omega \tilde{y}_0^h\nabla(\tilde{B}_h u^h)\cdot\nabla\sigma^{1-\mu} \\ &\le \|\sigma^{(1-\mu)/2}\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)}\left(\|\sigma^{(1-\mu)/2}\nabla\tilde{y}_0^h\|_{L^2(\Omega)} + \|\sigma^{(-1-\mu)/2}\tilde{y}_0^h\|_{L^2(\Omega)}\right),\end{aligned}\tag{24}$$

where we used the Cauchy–Schwarz inequality and

$$\nabla\sigma^{1-\mu} = (1-\mu)\sigma^{-\mu}(\cos\theta,\sin\theta)^T.\tag{25}$$

Having in mind the decomposition of the domain in subdomains Ω_J, an application of the Poincaré inequality yields for the latter term in (24)

$$\begin{aligned}\|\sigma^{(-1-\mu)/2}\tilde{y}_0^h\|^2_{L^2(\Omega)} &= \sum_{J=0}^{I}\|\sigma^{(-1-\mu)/2}\tilde{y}_0^h\|_{L^2(\Omega_J)}\|\sigma^{(-1-\mu)/2}\tilde{y}_0^h\|_{L^2(\Omega_J)} \\ &\le \sum_{J=0}^{I} d_J^{(-1-\mu)/2}\|\tilde{y}_0^h\|_{L^2(\Omega_J)}\|\sigma^{(-1-\mu)/2}\tilde{y}_0^h\|_{L^2(\Omega_J)} \\ &\le c\sum_{J=0}^{I} d_J^{(1-\mu)/2}\|\nabla\tilde{y}_0^h\|_{L^2(\Omega'_J)}\|\sigma^{(-1-\mu)/2}\tilde{y}_0^h\|_{L^2(\Omega_J)} \\ &\le c\|\sigma^{(1-\mu)/2}\nabla\tilde{y}_0^h\|_{L^2(\Omega)}\|\sigma^{(-1-\mu)/2}\tilde{y}_0^h\|_{L^2(\Omega)},\end{aligned}$$

where we used $d_J \sim \sigma$ for $x \in \Omega'_J$ twice and the discrete Cauchy–Schwarz inequality. Consequently, we get from (24)

$$\int_\Omega \nabla\tilde{y}_0^h\cdot\nabla(\tilde{y}_0^h\sigma^{1-\mu}) \le c\|\sigma^{(1-\mu)/2}\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)}\|\sigma^{(1-\mu)/2}\nabla\tilde{y}_0^h\|_{L^2(\Omega)}.\tag{26}$$

Similarly to the above steps, we get for the second term in (23) by means of (25)

$$\begin{aligned}\int_\Omega \tilde{y}_0^h\nabla\tilde{y}_0^h\cdot\nabla\sigma^{1-\mu} &\le \|\sigma^{(1-\mu)/2}\nabla\tilde{y}_0^h\|_{L^2(\Omega)}\|\sigma^{(-1-\mu)/2}\tilde{y}_0^h\|_{L^2(\Omega)} \\ &\le \|\sigma^{(1-\mu)/2}\nabla\tilde{y}_0^h\|_{L^2(\Omega)}\left(\|\sigma^{(-1-\mu)/2}(\tilde{y}_0^h + \tilde{B}_h u^h)\|_{L^2(\Omega)} + \|\sigma^{(-1-\mu)/2}\tilde{B}_h u^h\|_{L^2(\Omega)}\right),\end{aligned}\tag{27}$$

such that we infer from (23), (26), and (27) that

$$\begin{aligned}\|\sigma^{(1-\mu)/2}\nabla\tilde{y}_0^h\|_{L^2(\Omega)} \le c\Big(&\|\sigma^{(-1-\mu)/2}\tilde{B}_h u^h\|_{L^2(\Omega)} + \|\sigma^{(1-\mu)/2}\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)} \\ &+ \|\sigma^{(-1-\mu)/2}(\tilde{y}_0^h + \tilde{B}_h u^h)\|_{L^2(\Omega)}\Big).\end{aligned}\tag{28}$$

Due to the definition of $\tilde{B}_h$ and the definition of the element size h_T in the case of graded meshes we easily obtain by means of the norm equivalence in finite dimensional spaces that

$$\begin{aligned}\|\sigma^{(-1-\mu)/2}\tilde{B}_h u^h\|_{L^2(\Omega)} + \|\sigma^{(1-\mu)/2}\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)} &\le ch^{-1/2}\|u^h\|_{L^2(\Gamma)}\\ &\le ch^{-1/2}\|u\|_{L^2(\Gamma)},\end{aligned} \tag{29}$$

where we employed the stability of u^h in $L^2(\Gamma)$ in the last step. Having in mind the definition (22) of d_I and applying Lemma 2.3 with $\beta = \frac{1}{2}$ for the solution $\tilde{y}_0^h + \tilde{B}_h u^h$ of the homogeneous equation with boundary datum u^h we conclude that

$$\begin{aligned}&\|\sigma^{(-1-\mu)/2}(\tilde{y}_0^h + \tilde{B}_h u^h)\|_{L^2(\Omega)} \le d_I^{-\mu/2}\|\sigma^{-1/2}(\tilde{y}_0^h + \tilde{B}_h u^h)\|_{L^2(\Omega)}\\ &\quad\le ch^{-1/2}\|r^{-1/2}(\tilde{y}_0^h + \tilde{B}_h u^h)\|_{L^2(\Omega)} \le ch^{-1/2}\|u^h\|_{L^2(\Gamma)} \le ch^{-1/2}\|u\|_{L^2(\Gamma)},\end{aligned} \tag{30}$$

where we used again the stability of u^h. The estimates (28), (29), and (30) end the proof. ∎

LEMMA 2.5. *Let $\sigma := r + d_I$ and $\mu \in (0, 2\lambda - 1)$. Then there is the estimate*

$$\|\sigma^{-(1-\mu)/2}(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega)} \le ch^{1/2}\|u\|_{L^2(\Gamma)}.$$

Proof. Let $v \in H_0^1(\Omega)$ be the weak solution of

$$-\Delta v = \sigma^{-(1-\mu)}(\tilde{y}_0^h - \tilde{y}_{0h}) \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma,$$

which, according to Theorem 2.15 of [20], has the regularity $v \in V^{2,2}_{(1-\mu)/2}(\Omega)$ (as $\mu < 2\lambda - 1$) and hence $\frac{1}{2}(1-\mu) > 1 - \lambda$) and satisfies the a priori estimate

$$|v|_{V^{2,2}_{(1-\mu)/2}(\Omega)} \le c\|\sigma^{-(1-\mu)}(\tilde{y}_0^h - \tilde{y}_{0h})\|_{V^{0,2}_{(1-\mu)/2}(\Omega)} \le c\|\sigma^{-(1-\mu)/2}(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega)}, \tag{31}$$

where we use the weighted Sobolev space $V^{k,2}_\beta(\Omega) := \{v \in \mathcal{D}' : \|v\|_{V^{k,2}_\beta(\Omega)} < \infty\}$ with

$$\|v\|^2_{V^{k,2}_\beta(\Omega)} := \sum_{j=1}^{k} |v|^2_{V^{j,2}_{\beta-k+j}(\Omega)}, \qquad |v|_{V^{j,2}_\beta(\Omega)} := \|r^\beta \nabla^j v\|_{L^2(\Omega)}.$$

Then we obtain by using integration by parts and the Galerkin orthogonality

$$\begin{aligned}&\|\sigma^{-(1-\mu)/2}(\tilde{y}_0^h - \tilde{y}_{0h})\|^2_{L^2(\Omega)} = (\tilde{y}_0^h - \tilde{y}_{0h}, -\Delta v)_\Omega\\ &= (\nabla(\tilde{y}_0^h - \tilde{y}_{0h}), \nabla(v - I_h v))_\Omega \le \sum_{J=0}^{I} \|\nabla(\tilde{y}_0^h - \tilde{y}_{0h})\|_{L^2(\Omega_J)}\|\nabla(v - I_h v)\|_{L^2(\Omega_J)},\end{aligned} \tag{32}$$

where I_h is the Lagrange interpolant.

By employing standard interpolation error estimates on graded meshes we obtain for any $\mu \in (0, 1]$

$$\|\nabla(v - I_h v)\|_{L^2(\Omega_J)} \le chd_J^{(1-\mu)/2}|v|_{V^{2,2}_{(1-\mu)/2}(\Omega'_J)}, \tag{33}$$

where the constant c is independent of c_I; see, e.g., [6, Lemma 3.7] or [30, Lemma 3.58]. In fact, the constant is essentially the one appearing in the local, elementwise interpolation error estimate. Note that this kind of independence will be crucial when applying a kickback argument further below.

Local finite element error estimates from [23, Theorem 3.4] yield

$$\begin{aligned}\|\nabla(\tilde y_0^h - \tilde y_{0h})\|_{L^2(\Omega_J)} \le c \min_{v_h\in Y_{0h}} \left(\|\nabla(\tilde y_0^h - v_h)\|_{L^2(\Omega'_J)} + \frac{1}{d_J}\|\tilde y_0^h - v_h\|_{L^2(\Omega'_J)}\right) \\ + c\frac{1}{d_J}\|\tilde y_0^h - \tilde y_{0h}\|_{L^2(\Omega'_J)}.\end{aligned}$$

By choosing $v_h \equiv 0$ and by applying the Poincaré inequality, we conclude

$$\begin{aligned}\|\nabla(\tilde y_0^h - \tilde y_{0h})\|_{L^2(\Omega_J)} &\le c\left(\|\nabla \tilde y_0^h\|_{L^2(\Omega'_J)} + \frac{1}{d_J}\|\tilde y_0^h - \tilde y_{0h}\|_{L^2(\Omega'_J)}\right)\\ &\le c\left(\|\nabla \tilde y_0^h\|_{L^2(\Omega''_J)} + d_J^{(-1-\mu)/2}\|\sigma^{-(1-\mu)/2}(\tilde y_0^h - \tilde y_{0h})\|_{L^2(\Omega'_J)}\right), \end{aligned} \tag{34}$$

where we used $d_J \sim \sigma$ for $x \in \Omega'_J$. Consequently, we get from (32)–(34)

$$\begin{aligned}&\|\sigma^{-(1-\mu)/2}(\tilde y_0^h - \tilde y_{0h})\|^2_{L^2(\Omega)}\\ &\le c\sum_{J=0}^{I}\left(h\|\sigma^{(1-\mu)/2}\nabla\tilde y_0^h\|_{L^2(\Omega''_J)} + hd_J^{-\mu}\|\sigma^{-(1-\mu)/2}(\tilde y_0^h - \tilde y_{0h})\|_{L^2(\Omega'_J)}\right)|v|_{V^{2,2}_{(1-\mu)/2}(\Omega'_J)}\\ &\le c\left(h\|\sigma^{(1-\mu)/2}\nabla\tilde y_0^h\|_{L^2(\Omega)} + c_I^{-\mu}\|\sigma^{-(1-\mu)/2}(\tilde y_0^h - \tilde y_{0h})\|_{L^2(\Omega)}\right)|v|_{V^{2,2}_{(1-\mu)/2}(\Omega)},\end{aligned}$$

where we again employed $d_J \sim \sigma$ for $x \in \Omega''_J$, $hd_J^{-\mu} \le c_I^{-\mu}$, which holds due to the definition (22) of d_I, and the discrete Cauchy–Schwarz inequality. For $\mu \in (0, 2\lambda - 1)$ we infer by the a priori estimate (31) that

$$\begin{aligned}&\|\sigma^{-(1-\mu)/2}(\tilde y_0^h - \tilde y_{0h})\|_{L^2(\Omega)} \le\\ &\qquad c\left(h\|\sigma^{(1-\mu)/2}\nabla\tilde y_0^h\|_{L^2(\Omega)} + c_I^{-\mu}\|\sigma^{-(1-\mu)/2}(\tilde y_0^h - \tilde y_{0h})\|_{L^2(\Omega)}\right).\end{aligned}$$

By choosing c_I large enough we can kick back the second term in the above inequality such that Lemma 2.4 yields the desired result. ∎

THEOREM 2.6. *For $\mu \in (0, 2\lambda - 1)$ we get*

$$\|y - y_h\|_{L^2(\Omega)} \le ch^{1/2}\left(\|u\|_{L^2(\Omega)} + h^{1/2}\|f\|_{H^{-1}(\Omega)}\right). \tag{35}$$

Proof. Due to the boundedness of $\sigma^{(1-\mu)/2}$ independent of h for all $\mu \in (0, 1]$ we obtain from Lemma 2.5

$$\|\tilde y_0^h - \tilde y_{0h}\|_{L^2(\Omega)} \le \|\sigma^{(1-\mu)/2}\|_{L^\infty(\Omega)}\|\sigma^{-(1-\mu)/2}(\tilde y_0^h - \tilde y_{0h})\|_{L^2(\Omega)} \le ch^{1/2}\|u\|_{L^2(\Gamma)}. \tag{36}$$

In view of (17) we get by using the triangle inequality

$$\|y - y_h\|_{L^2(\Omega)} \le \|y - y^h\|_{L^2(\Omega)} + \|y_f - y_{fh}\|_{L^2(\Omega)} + \|\tilde y_0^h - \tilde y_{0h}\|_{L^2(\Omega)}.$$

Using Lemmas 2.1 and 2.2 as well as (36) we get (35). ∎

3. The dual singular complement method.

3.1. Analytical background and regularization. Using the notation of the previous section, we recall that the splitting (7) implies that

$$R := \{\Delta v : v \in H^2(\Omega) \cap H_0^1(\Omega)\} \tag{37}$$

is a closed subspace of $L^2(\Omega)$. It is shown in [26, sect. 2.3] that

$$L^2(\Omega) = R \overset{\perp}{\oplus} \mathrm{Span}\{p_s\} \tag{38}$$

with the *dual singular function*

$$p_s = r^{-\lambda} \sin(\lambda\theta) + \tilde{p}_s, \tag{39}$$

where $\tilde{p}_s \in H^1(\Omega)$ is chosen such that the decomposition (38) is orthogonal for the $L^2(\Omega)$ inner product. Therefore, the dual singular function p_s is a solution of

$$w \in L^2(\Omega) : \quad (\Delta v, w) = 0 \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega), \tag{40}$$

which proves the nonuniqueness of the solution of (40).

Due to (38) we can split any $L^2(\Omega)$-function into $L^2(\Omega)$-orthogonal parts. To this end denote by Π_R and Π_{p_s} the orthogonal projections on R and on $\mathrm{Span}\{p_s\}$, respectively, i.e., for $g \in L^2(\Omega)$, it is $g = \Pi_R g + \Pi_{p_s} g$, where

$$\Pi_{p_s} g = \alpha(g)\, p_s \quad \text{with} \quad \alpha(g) = \frac{(g, p_s)_\Omega}{\|p_s\|^2_{L^2(\Omega)}}, \quad \text{and} \quad \Pi_R g = g - \Pi_{p_s} g.$$

Since $p_s \in L^2(\Omega)$ there exists

$$\phi_s \in H^1_\Delta(\Omega) \cap H^1_0(\Omega) : \quad -\Delta\phi_s = p_s; \tag{41}$$

see also section 3.3 for more details on ϕ_s. For the moment we assume that p_s and ϕ_s are explicitly known; the decomposition $g = \Pi_R g + \alpha(g)\, p_s$ can be computed once g is given. Computable approximations of p_s and ϕ_s are discussed in section 3.3.

Now we come back to problem (2) and decompose its solution y in the form

$$y = \Pi_R y + \alpha(y)\, p_s. \tag{42}$$

From the decomposition (38) we see that problem (2) is equivalent to

$$\begin{aligned} (y, p_s)_\Omega &= -(u, \partial_n \phi_s)_\Gamma + (f, \phi_s)_\Omega, \\ (y, \Delta v)_\Omega &= (u, \partial_n v)_\Gamma - (f, v)_\Omega \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega), \end{aligned}$$

and with the orthogonal splitting (42) to

$$\begin{aligned} \alpha(y)\, (p_s, p_s)_\Omega &= -(u, \partial_n \phi_s)_\Gamma + (f, \phi_s)_\Omega, \\ (\Pi_R y, \Delta v)_\Omega &= (u, \partial_n v)_\Gamma - (f, v)_\Omega \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \end{aligned}$$

The first equation directly yields $\alpha(y)$, namely,

$$\alpha(y) = \frac{-(u, \partial_n \phi_s)_\Gamma + (f, \phi_s)_\Omega}{(p_s, p_s)_\Omega}, \tag{43}$$

hence the projection of y on p_s is known. It remains to find an approximation of $\Pi_R y$.

At this point we recall the regularization approach from [4] which we summarized already in the introduction. Let $u^h \in H^{1/2}(\Gamma)$ be a regularized boundary datum (this can be any, for example, $\Pi_h u$ or $C_h u$ from section 2, but we do not assume graded meshes here) such that we can define the regularized (weak) solution in $Y_*^h := \{v \in H^1(\Omega) : v|_\Gamma = u^h\}$,

$$y^h \in Y_*^h : \quad (\nabla y^h, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H_0^1(\Omega). \tag{44}$$

In [4, Remark 2.13] we showed that the regularization error can be estimated by

$$\|y - y^h\|_{L^2(\Omega)} \le c\|u - u^h\|_{H^{-s}(\Gamma)},$$

where $0 < s < \lambda - \frac{1}{2}$ (if Ω was convex we would get $s = \frac{1}{2}$, that means the regularization error is in general bigger in the nonconvex case). With the next lemma we show that $\Pi_R(y - y^h)$ is not affected by nonconvex corners.

Lemma 3.1. *There holds the estimate*

$$\|\Pi_R(y - y^h)\|_{L^2(\Omega)} \le C\|u - u^h\|_{H^{-1/2}(\Gamma)}.$$

Proof. Recall $V = H^1_\Delta(\Omega) \cap H^1_0(\Omega)$ from (3). From (44) and the Green formula, we have for any $v \in V$

$$(f, v)_\Omega = (\nabla y^h, \nabla v)_\Omega = -(y^h, \Delta v)_\Omega + (y^h, \partial_n v)_\Gamma.$$

Note that $v \in V$ is sufficient for the Green formula, and $v \in H^2(\Omega)$ is not required; see [19, Lemma 3.4]. Subtracting this expression from the very weak formulation (2), we get

$$(y - y^h, \Delta v)_\Omega = (u - u^h, \partial_n v)_\Gamma \quad \forall v \in V.$$

Restricting this identity to $v \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$(\Pi_R(y - y^h), \Delta v)_\Omega = (u - u^h, \partial_n v)_\Gamma \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{45}$$

Now for any $z \in R$, we let $v_z \in H^2(\Omega) \cap H_0^1(\Omega)$ be the unique solution of

$$\Delta v_z = z, \tag{46}$$

which satisfies

$$\|\partial_n v_z\|_{H^{1/2}(\Gamma)} \le c\|v_z\|_{H^2(\Omega)} \le c\|z\|_{L^2(\Omega)}. \tag{47}$$

Since for any $g \in L^2(\Omega)$ the equality

$$(\Pi_R(y - y^h), g)_\Omega = (\Pi_R(y - y^h), \Pi_R g)_\Omega = (y - y^h, \Pi_R g)_\Omega$$

holds, we get with (45)–(47)

$$\begin{aligned}\|\Pi_R(y - y^h)\|_{L^2(\Omega)} &= \sup_{z \in R, z \ne 0} \frac{(y - y^h, z)_\Omega}{\|z\|_{L^2(\Omega)}} = \sup_{z \in R, z \ne 0} \frac{(u - u^h, \partial_n v_z)_\Gamma}{\|z\|_{L^2(\Omega)}} \\ &\le \|u - u^h\|_{H^{-1/2}(\Gamma)} \sup_{z \in R, z \ne 0} \frac{\|\partial_n v_z\|_{H^{1/2}(\Gamma)}}{\|z\|_{L^2(\Omega)}} \le c\|u - u^h\|_{H^{-1/2}(\Gamma)}\end{aligned}$$

which is the estimate to be proved. ∎

3.2. Motivation for the dual singular complement method. As already discussed in the introduction, the adapted methods are motivated by the suboptimal convergence rate of the finite element solution on a family of quasi-uniform meshes. In this subsection, we recall these results and extend them by proving an estimate for the projection of the error into the space R from (37) which yields a better convergence rate. The insight into this structure of the discretization error motivates the new method which we call the dual singular complement method.

Recall from (9) the finite element spaces

$$Y_h = \{v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}, \quad Y_{0h} = Y_h \cap H_0^1(\Omega), \quad Y_h^\partial = Y_h|_{\partial\Omega},$$

defined now on a quasi-uniform family $\mathcal{T}_h$ of conforming finite element meshes. Assume that the regularized boundary datum u^h is contained in Y_h^∂ such that the estimates

$$\|u^h\|_{L^2(\Gamma)} \le c\|u\|_{L^2(\Gamma)}, \tag{48}$$

$$\|u - u^h\|_{H^{-1/2}(\Gamma)} \le Ch^{1/2}\|u\|_{L^2(\Gamma)}, \tag{49}$$

hold. It can be derived from [4, Lemma 2.14] that this can be accomplished by using the $L^2(\Gamma)$-projection or by quasi-interpolation: The stability (48) is explicitly stated there, and the error estimate (49) follows from the definition of the $H^{-1/2}(\Gamma)$-norm and the third estimate in [4, Lemma 2.14]. A consequence of Lemma 3.1 and (49) is the estimate

$$\|\Pi_R(y - y^h)\|_{L^2(\Omega)} \le Ch^{1/2}\|u\|_{L^2(\Gamma)}. \tag{50}$$

(In the case of a convex domain the operator Π_R is the identity, and the corresponding error estimates were already proven in [4].)

As already done in the introduction, define further the finite element solution $y_h \in Y_{*h} := Y_*^h \cap Y_h$ via

$$y_h \in Y_{*h} : \quad (\nabla y_h, \nabla v_h)_\Omega = (f, v_h)_\Omega \quad \forall v_h \in Y_{0h}. \tag{51}$$

We proved in [4] that in the case of a quasi-uniform family of meshes $\mathcal{T}_h$

$$\|y - y_h\|_{L^2(\Omega)} \le Ch^s \left(h^{1/2}\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right) \tag{52}$$

holds for $s \in (0, \lambda - \frac{1}{2})$ (again $s = \frac{1}{2}$ for convex domains). As before, in the next lemma we show that $\Pi_R(y - y_h)$ is not affected by the nonconvex corners.

Lemma 3.2. *The following discretization error estimate holds:*

$$\|\Pi_R(y - y_h)\|_{L^2(\Omega)} \le Ch^{1/2} \left(h^{1/2}\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right).$$

Proof. By the triangle inequality we have

$$\|\Pi_R(y - y_h)\|_{L^2(\Omega)} \le \|\Pi_R(y - y^h)\|_{L^2(\Omega)} + \|\Pi_R(y^h - y_h)\|_{L^2(\Omega)}. \tag{53}$$

The first term is estimated in (50). For the second term we first notice that $y^h - y_h \in H_0^1(\Omega)$ satisfies the Galerkin orthogonality

$$(\nabla(y^h - y_h), \nabla v_h)_\Omega = 0 \quad \forall v_h \in Y_{0h}; \tag{54}$$

see (4) and (5). With that, we estimate $\|\Pi_R(y^h - y_h)\|_{L^2(\Omega)}$ by similar arguments as $\|\Pi_R(y - y^h)\|_{L^2(\Omega)}$ in the proof of Lemma 3.1. Recall from (46) and (47) that $v_z \in H^2(\Omega) \cap H_0^1(\Omega)$ is the weak solution of $\Delta v_z = z \in R$. It can be approximated by the Lagrange interpolant $I_h v_z$ satisfying

$$\|\nabla(v_z - I_h v_z)\|_{L^2(\Omega)} \le ch\|v_z\|_{H^2(\Omega)} \le ch\|z\|_{L^2(\Omega)}.$$

We get

$$\begin{aligned} \|\Pi_R(y^h - y_h)\|_{L^2(\Omega)} &= \sup_{z\in R, z\neq 0} \frac{(y^h - y_h, z)_\Omega}{\|z\|_{L^2(\Omega)}} = \sup_{z\in R, z\neq 0} \frac{(\nabla(y^h - y_h), \nabla v_z)_\Omega}{\|z\|_{L^2(\Omega)}} \\ &= \sup_{z\in R, z\neq 0} \frac{(\nabla(y^h - y_h), \nabla(v_z - I_h v_z))_\Omega}{\|z\|_{L^2(\Omega)}} \\ &\le ch\|\nabla(y^h - y_h)\|_{L^2(\Omega)}. \end{aligned} \tag{55}$$

In order to bound $\|\nabla(y^h - y_h)\|_{L^2(\Omega)}$ by the data we consider the lifting $\tilde{B}_h u^h \in Y_{*h}$ defined by (16). The next steps are simpler than in section 2 since we have a quasi-uniform family of meshes and obtain a sharp estimate also by using an inverse inequality below. The homogenized solution $y_0^h = y^h - \tilde{B}_h u^h \in H_0^1(\Omega)$ satisfies

$$(\nabla y_0^h, \nabla v)_\Omega = (f, v)_\Omega - (\nabla(\tilde{B}_h u^h), \nabla v)_\Omega \quad \forall v \in H_0^1(\Omega).$$

By taking $v = y_0^h$ we see that

$$\|\nabla y_0^h\|^2_{L^2(\Omega)} \le \|f\|_{H^{-1}(\Omega)}\|y_0^h\|_{H^1(\Omega)} + \|\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)}\|\nabla y_0^h\|_{L^2(\Omega)}.$$

Using the Poincaré inequality we obtain

$$\|\nabla y_0^h\|_{L^2(\Omega)} \le c\|f\|_{H^{-1}(\Omega)} + \|\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)}, \tag{56}$$

and with the Céa lemma

$$\|\nabla(y^h - y_h)\|_{L^2(\Omega)} \le \|\nabla y_0^h\|_{L^2(\Omega)} \le c\|f\|_{H^{-1}(\Omega)} + \|\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)}.$$

The remaining term $\|\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)}$ is estimated by using the inverse inequality

$$\|\nabla(\tilde{B}_h u^h)\|_{L^2(T)} \le ch^{-1/2}\|u^h\|_{L^2(E)}$$

for $E \subset T \cap \Gamma$, $T \in \mathcal{T}_h$, which can be proved by standard scaling arguments, to get

$$\|\nabla(\tilde{B}_h u^h)\|_{L^2(\Omega)} \le ch^{-1/2}\|u^h\|_{L^2(\Gamma)}. \tag{57}$$

Hence we proved

$$\|\nabla(y^h - y_h)\|_{L^2(\Omega)} \le c\|f\|_{H^{-1}(\Omega)} + ch^{-1/2}\|u^h\|_{L^2(\Gamma)}.$$

With (53), (50), (55), the previous inequality, and (48) we finish the proof. ∎

With (42) we can immediately conclude the following result.

Corollary 3.3. *Let $y_h \in Y_{*h}$ be the solution of* (51)*, then the discretization error estimate*

$$\|y - (\Pi_R y_h + \alpha(y)p_s)\|_{L^2(\Omega)} \le Ch^{1/2}\left(h^{1/2}\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right)$$

holds, noting that p_s and $\alpha(y)$ are given by (39) *and* (43)*, respectively.*

Hence the positive result is that $\Pi_R y_h + \alpha(y)p_s$ is a better approximation of y than y_h. The problem is that p_s and ϕ_s are used explicitly, and in practice they are not known. A remedy of this drawback is the aim of the next section.

3.3. Approximate singular functions. Following [17], we approximate p_s from (39) by

$$
\begin{aligned}
&p_s^h = p_h^* - r_h + r^{-\lambda}\sin(\lambda\theta), \quad r_h = \tilde{B}_h\left(r^{-\lambda}\sin(\lambda\theta)\right),\\
&p_h^* \in Y_{0h}: \quad (\nabla p_h^*, \nabla v_h)_\Omega = (\nabla r_h, \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h}
\end{aligned}
\tag{58}
$$

with $\tilde{B}_h$ from (16). The function ϕ_s from (41) admits the splitting

$$
\phi_s = \tilde{\phi} + \beta r^\lambda \sin(\lambda\theta) \tag{59}
$$

with $\tilde{\phi} \in H^2(\Omega)$ and $\beta = \pi^{-1}\|p_s\|^2_{L^2(\Omega)}$; see again [17]. It is approximated by

$$
\begin{aligned}
&\phi_s^h = \phi_h^* - \beta_h s_h + \beta_h r^\lambda \sin(\lambda\theta), \quad s_h = \tilde{B}_h\left(r^\lambda\sin(\lambda\theta)\right), \quad \beta_h = \frac{1}{\pi}\|p_s^h\|^2_{L^2(\Omega)},\\
&\phi_h^* \in Y_{0h}: \quad (\nabla\phi_h^*, \nabla v_h)_\Omega = (p_s^h, v_h)_\Omega + \beta_h(\nabla s_h, \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h},
\end{aligned}
\tag{60}
$$

that means $\tilde{\phi}$ is approximated by $\tilde{\phi}_h = \phi_h^* - \beta_h s_h \in Y_h$. The approximation errors are bounded by

$$
\|p_s - p_s^h\|_{L^2(\Omega)} \le ch^{2\lambda-\epsilon} \le ch, \tag{61}
$$

$$
|\beta - \beta_h| \le ch^{2\lambda-\varepsilon} \le ch, \tag{62}
$$

$$
\|\phi_s - \phi_s^h\|_{1,\Omega} \le ch; \tag{63}
$$

see [17, Lemmas 3.1–3.3], where (62) and (63) imply

$$
\|\tilde{\phi} - \tilde{\phi}_h\|_{1,\Omega} \le ch. \tag{64}
$$

At the end of section 3.2 we saw that $\Pi_R y_h + \alpha(y)p_s$ is a better approximation of y than y_h. Since this function is not computable we approximate it by

$$
z_h = \Pi_R^h\, y_h + \alpha_h p_s^h \tag{65}
$$

with

$$
\Pi_R^h\, y_h = y_h - \gamma_h p_s^h, \quad \gamma_h = \frac{(y_h, p_s^h)_\Omega}{\|p_s^h\|^2_{L^2(\Omega)}}, \tag{66}
$$

and a suitable approximation α_h of $\alpha(y)$ from (43). To this end we write the problematic term by using (59) as

$$
(u, \partial_n\phi_s)_\Gamma = (u, \partial_n\tilde{\phi})_\Gamma + \beta(u, \partial_n(r^\lambda\sin(\lambda\theta)))_\Gamma,
$$

and replace the term $(u, \partial_n\tilde{\phi})_\Gamma$ by $(u^h, \partial_n\tilde{\phi})_\Gamma$. Since $\tilde{\phi}$ belongs to $H^2(\Omega)$ and u^h is the trace of $\tilde{B}_h u^h$, we get by using the Green formula

$$
(u^h, \partial_n\tilde{\phi})_\Gamma = (\tilde{B}_h u^h, \Delta\tilde{\phi})_\Omega + (\nabla\tilde{B}_h u^h, \nabla\tilde{\phi})_\Omega = -(\tilde{B}_h u^h, p_s)_\Omega + (\nabla\tilde{B}_h u^h, \nabla\tilde{\phi})_\Omega \tag{67}
$$

as $\Delta\tilde{\phi} = \Delta\phi_s = -p_s$. With all these notations and results, we define

$$
\alpha_h = \frac{(\tilde{B}_h u^h, p_s^h)_\Omega - (\nabla\tilde{B}_h u^h, \nabla\tilde{\phi}_h)_\Omega - \beta_h(u, \partial_n(r^\lambda\sin(\lambda\theta)))_\Gamma + (f, \phi_s^h)_\Omega}{(p_s^h, p_s^h)^2_\Omega}. \tag{68}
$$

Note that α_h can be computed explicitly and therefore z_h as well.

Let us estimate the approximation errors made.

LEMMA 3.4. *Let $y_h \in Y_{*h}$ be the solution of* (51). *Then the error estimates*

$$\|\Pi_R y_h - \Pi_R^h y_h\|_{L^2(\Omega)} \le ch \left(\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right), \tag{69}$$

$$|\alpha(y) - \alpha_h| \le ch^{1/2} \left(h^{1/2}\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right) \tag{70}$$

hold.

Proof. With the definitions of Π_R and Π_R^h, with $\gamma := (y_h, p_s)_\Omega/\|p_s\|^2_{L^2(\Omega)}$, and by using the triangle inequality we have

$$\|\Pi_R y_h - \Pi_R^h y_h\|_{L^2(\Omega)} = \|\gamma p_s - \gamma_h p_s^h\|_{L^2(\Omega)} \le |\gamma - \gamma_h|\, \|p_s^h\|_{L^2(\Omega)} + |\gamma|\, \|p_s - p_s^h\|_{L^2(\Omega)}.$$

We write

$$\begin{aligned}
\gamma - \gamma_h &= \frac{(y_h, p_s)_\Omega}{\|p_s\|^2_{L^2(\Omega)}} - \frac{(y_h, p_s^h)_\Omega}{\|p_s^h\|^2_{L^2(\Omega)}} \\
&= \frac{(y_h, p_s - p_s^h)_\Omega}{\|p_s\|^2_{L^2(\Omega)}} + (y_h, p_s^h)_\Omega \left(\frac{1}{\|p_s\|^2_{L^2(\Omega)}} - \frac{1}{\|p_s^h\|^2_{L^2(\Omega)}}\right) \\
&= \frac{(y_h, p_s - p_s^h)_\Omega}{\|p_s\|^2_{L^2(\Omega)}} + (y_h, p_s^h)_\Omega \frac{(p_s^h + p_s, p_s^h - p_s)_\Omega}{\|p_s\|^2_{L^2(\Omega)}\|p_s^h\|^2_{L^2(\Omega)}},
\end{aligned}$$

and by the Cauchy–Schwarz inequality and (61) we get

$$|\gamma - \gamma_h| \le ch\|y_h\|_{L^2(\Omega)}.$$

We have used that $\|p_s\|_{L^2(\Omega)}$ and $\|p_s^h\|_{L^2(\Omega)}$ can be treated as constants due to the definition of p_s and due to (61). We conclude with $|\gamma| \le c\|y_h\|_{L^2(\Omega)}$ and (61) that

$$\|\Pi_R y_h - \Pi_R^h y_h\|_{L^2(\Omega)} \le ch\|y_h\|_{L^2(\Omega)}. \tag{71}$$

In view of the finite element error estimate (52) and the standard a priori estimate for the very weak solution,

$$\|y\|_{L^2(\Omega)} \le c\left(\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right)$$

(see Lemma 2.3 of [4]), we have

$$\|y_h\|_{L^2(\Omega)} \le \|y\|_{L^2(\Omega)} + \|y - y_h\|_{L^2(\Omega)} \le c\left(\|f\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\Gamma)}\right).$$

This estimate together with (71) proves (69).

The proof of the estimate (70) is based on writing the problematic term in the definition of $\alpha(y)$ without approximation as

$$\begin{aligned}
(u, \partial_n\phi_s)_\Gamma &= (u, \partial_n\tilde{\phi})_\Gamma + \beta(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma \\
&= (u - u^h, \partial_n\tilde{\phi})_\Gamma + (u^h, \partial_n\tilde{\phi})_\Gamma + \beta(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma \\
&= (u - u^h, \partial_n\tilde{\phi})_\Gamma - (\tilde{B}_h u^h, p_s)_\Omega + (\nabla\tilde{B}_h u^h, \nabla\tilde{\phi})_\Omega + \beta(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma,
\end{aligned}$$

where we used (67) in the last step. Consequently, we showed that

$$\alpha(y)-\alpha_h = \frac{1}{\|p_s\|^2_{L^2(\Omega)}}\Big(-(u-u^h,\partial_n\tilde{\phi})_\Gamma+(\tilde{B}_h u^h, p_s-p_s^h)_\Omega-(\nabla\tilde{B}_h u^h,\nabla(\tilde{\phi}-\tilde{\phi}_h))_\Omega$$
$$-(\beta-\beta_h)\,(u,\partial_n(r^\lambda\sin(\lambda\theta)))_\Gamma+(f,\phi_s-\phi_s^h)_\Omega\Big).$$

To prove (70), in view of (61), (62), and (63) it remains to show that

$$\begin{aligned}\Big|(u-u^h,\partial_n\tilde{\phi})_\Gamma\Big| &\le ch^{1/2}\|u\|_{L^2(\Gamma)},\\ \Big|(\tilde{B}_h u^h, p_s-p_s^h)_\Omega\Big| &\le ch^{1/2}\|u\|_{L^2(\Gamma)},\\ \Big|(\nabla\tilde{B}_h u^h,\nabla(\tilde{\phi}-\tilde{\phi}^h))_\Omega\Big| &\le ch^{1/2}\|u\|_{L^2(\Gamma)}.\end{aligned}$$

The first estimate follows from the estimate (49) and the fact that $\tilde{\phi}$ belongs to $H^2(\Omega)$. The second one follows from the Cauchy–Schwarz inequality and the estimates (57) and (61). Similarly, the third estimate follows from the Cauchy–Schwarz inequality and the estimates (57) and (64). ∎

COROLLARY 3.5. *Let Ω be a nonconvex domain and let $y_h \in Y_{*h}$ be the solution of* (51) *and let z_h be derived by* (65), (66), *and* (68), *then there holds*

$$\|y-z_h\|_{L^2(\Omega)} \le Ch^{1/2}\left(h^{1/2}\|f\|_{H^{-1}(\Omega)}+\|u\|_{L^2(\Gamma)}\right).$$

Proof. The main ingredients of the proof were already derived. Indeed, it is

$$\begin{aligned}\|y-z_h\|_{L^2(\Omega)} &= \|\Pi_R y+\alpha(y)p_s-\Pi_R^h y_h-\alpha_h p_s^h\|_{L^2(\Omega)}\\ &\le \|\Pi_R y-\Pi_R y_h\|_{L^2(\Omega)}+\|\Pi_R y_h-\Pi_R^h y_h\|_{L^2(\Omega)}+\\ &\qquad |\alpha(y)-\alpha_h|\,\|p_s\|_{L^2(\Omega)}+|\alpha_h|\,\|p_s-p_s^h\|_{L^2(\Omega)}.\end{aligned}$$

The first three terms can be estimated by using Lemmas 3.2 and 3.4. So it remains to treat the fourth term. To bound $|\alpha_h|$ we use the triangle inequality

$$|\alpha_h| \le |\alpha_h-\alpha(y)|+|\alpha(y)|.$$

For the first term we use (70), while for the second term we use (43) noting that ϕ_s belongs to $H^{3/2+\epsilon}(\Omega)$ with some $\epsilon>0$. Altogether we have

$$|\alpha_h| \le C\left(\|f\|_{H^{-1}(\Omega)}+\|u\|_{L^2(\Gamma)}\right)$$

and conclude by using (61). ∎

3.4. The method in the form of an algorithm. Before we describe the numerical experiments, let us summarize the algorithm.

1. Compute the finite element solution $y_h \in Y_{*h}$ via (5) with $u^h \in Y_h^\partial$ being an approximation of the boundary datum u satisfying (48) and (49).

2. Compute the approximate singular functions (compare (58) and (60)):

$$
\begin{aligned}
&r_h = \tilde{B}_h\left(r^{-\lambda}\sin(\lambda\theta)\right),\\
&p_h^* \in Y_{0h}:\ (\nabla p_h^*, \nabla v_h)_\Omega = (\nabla r_h, \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h},\\
&\tilde{p}_h = p_h^* - r_h,\\
&\beta_h = \tfrac{1}{\pi}\|\tilde{p}_h + r^{-\lambda}\sin(\lambda\theta)\|^2_{L^2(\Omega)},\\
&s_h = \tilde{B}_h\left(r^{\lambda}\sin(\lambda\theta)\right),\\
&\phi_h^* \in Y_{0h}:\\
&\quad (\nabla\phi_h^*, \nabla v_h)_\Omega = (\tilde{p}_h + r^{-\lambda}\sin(\lambda\theta), v_h)_\Omega + \beta_h(\nabla s_h, \nabla v_h)_\Omega \quad \forall v_h \in Y_{0h},\\
&\tilde{\phi}_h = \phi_h^* - \beta_h s_h.
\end{aligned}
$$

3. Compute

$$
\begin{aligned}
&\gamma_h = \frac{(y_h, p_s^h)_\Omega}{(p_s^h, p_s^h)_\Omega} \quad \text{with } p_s^h = \tilde{p}_h + r^{-\lambda}\sin(\lambda\theta),\\
&\alpha_h = \frac{(\tilde{B}_h u^h, p_s^h)_\Omega - (\nabla\tilde{B}_h u^h, \nabla\tilde{\phi}_h)_\Omega - \beta_h(u, \partial_n(r^\lambda\sin(\lambda\theta)))_\Gamma + (f, \phi_s^h)_\Omega}{(p_s^h, p_s^h)^2_\Omega},\\
&\delta_h = \alpha_h - \gamma_h,\\
&\tilde{z}_h = y_h + \delta_h\tilde{p}_h
\end{aligned}
$$

(compare (66) and (68)). According to (65), the numerical solution is

$$
z_h = \tilde{z}_h + \delta_h r^{-\lambda}\sin(\lambda\theta).
$$

Note that all integrals with r^λ and $r^{-\lambda}$ must be computed with care.

4. Numerical experiment. This section is devoted to the numerical verification of our theoretical results. For that purpose we present an example with known solution. Furthermore, to examine the influence of the corner singularities, we consider several polygonal domain Ω_ω's depending on an interior angle $\omega \in (0, 2\pi)$; we present here the results for $\omega = 270°$ and $\omega = 355°$. The computational domains are defined by

$$
\Omega_\omega := (-1,1)^2 \cap \{x \in \mathbb{R}^2 : (r(x), \theta(x)) \in (0, \sqrt{2}] \times [0, \omega]\}, \tag{72}
$$

where r and θ stand for the polar coordinates located at the origin. The boundary of Ω_ω is denoted by Γ_ω. We solve the problem $-\Delta y = 0 \quad \text{in } \Omega_\omega,\ y = u \quad \text{on } \Gamma_\omega$, numerically by using a standard finite element method with graded meshes and the proposed dual singular function method with a quasi-uniform family of meshes. The boundary datum u is chosen to be

$$
u := r^{-0.4999}\sin(-0.4999\,\theta) \quad \text{on } \Gamma_\omega.
$$

This function belongs to $L^p(\Gamma)$ for every $p < 2.0004$. The exact solution of our problem is simply $y = r^{-0.4999}\sin(-0.4999\,\theta)$, since y is harmonic.

The quasi-uniform family of finite element meshes is generated from a coarse initial mesh by recursively using a newest vertex bisection algorithm; see [16]. Graded

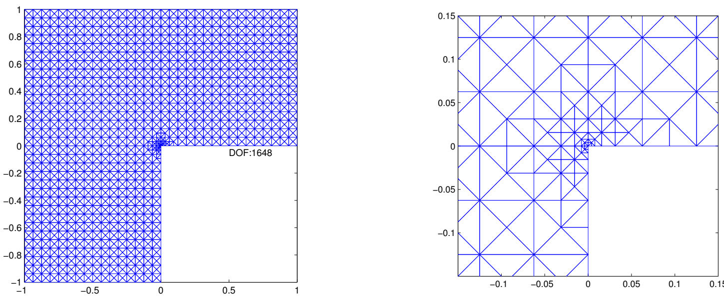

FIG. 1. *Graded mesh with $\mu = 0.3333$, generated by newest vertex bisection; left: whole mesh, right: zoom.*

TABLE 1
Discretization errors for $\omega = 270°$; left: $e_h = y - y_h$ with quasi-uniform meshes (standard) and $e_h = y - z_h$ (DSCM); right: $e_h = y - z_h$ with graded meshes ($\mu = 0.333$).

Unknowns	Standard	eoc	DSCM	eoc
113	0.645		0.587	
417	0.568	0.193	0.423	0.472
1601	0.503	0.181	0.303	0.482
6273	0.447	0.175	0.216	0.489
24833	0.397	0.171	0.154	0.493
98817	0.353	0.169	0.109	0.496
394241	0.314	0.168	0.077	0.498
Expected		0.167		0.5

Unknowns	Error	eoc
113	0.645	
428	0.445	0.559
1648	0.312	0.524
6463	0.220	0.512
25544	0.155	0.508
101563	0.110	0.504
405014	0.077	0.502
Expected		0.5

meshes are generated by marking and bisecting every element $T \in \mathcal{T}_h$ which satisfies $h_T > h$ or $h_T > h\,(r_{T,C}/R)^{1-\mu}$ until the desired global mesh size h is reached, where $r_{T,C}$ denotes the distance between the origin and the centroid of the triangle T; cf. [30, section 3.2.5]. Note that only elements which fulfill $r_T < R$ are gradually refined, where R is a fixed parameter; see Figure 1. As a regularization we have used the $L^2(\Gamma)$-projection. The discretization errors are calculated by adaptive quadrature.

The discretization errors for different mesh sizes and the corresponding experimental orders of convergence (eoc) are given in Table 1 for the interior angle $\omega = 270°$ and in Table 2 for the interior angle $\omega = 355°$. We see that the numerical results confirm the expected convergence rate 1/2 for the dual singular complement method (DSCM) and the finite element method on sufficently graded meshes. Further tests and illustrations of the numerical solutions can be found in the preprint version [3] of the paper.

Concerning the DSCM, we emphasize that the quadrature formula for the numerical evaluation of the integral $(u, \partial_n(r^\lambda \sin(\lambda\theta)))_\Gamma$ has to be adapted in order to get a sufficiently good approximation. Otherwise, the error due to quadrature dominates the overall error. In our implementation, we chose for the numerical integration a graded mesh on the boundary ($h_E \sim h r_E^{1-\mu}$ if the distance r_E of the boundary edge E satisfies $0 < r_E < R$ with R being the radius of the refinement zone and μ being the refinement parameter, and $h_T = h^{1/\mu}$ for $r_E = 0$) combined with a one-point Gauss quadrature rule on each element. The choice $\mu \le 2\pi/\omega - 1$ seems to be the correct grading to achieve a convergence order of 1/2. For the results presented in Tables 1 and 2 we used $R = 0.1$ and $\mu = 2\pi/\omega - 1$.

TABLE 2
Discretization errors for $\omega = 355°$; left: $e_h = y - y_h$ with quasi-uniform meshes (standard) and $e_h = y - z_h$ (DSCM); right: $e_h = y - z_h$ with graded meshes ($\mu = 0.014085$).

Unknowns	Standard	eoc	DSCM	eoc
159	1.069		1.021	
589	1.049	0.029	0.834	0.291
2265	1.036	0.018	0.590	0.500
8881	1.028	0.012	0.417	0.500
35169	1.021	0.010	0.295	0.499
139969	1.015	0.008	0.209	0.497
558465	1.010	0.008	0.148	0.495
Expected		0.007		0.5

Unknowns	Error	eoc
159	1.069	
970	0.854	0.325
4116	0.600	0.509
16154	0.424	0.502
62949	0.298	0.508
247276	0.210	0.505
979316	0.148	0.505
Expected		0.5

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