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REPRESENTATION OF LARGE MATCHINGS IN BIPARTITE GRAPHS

RON AHARONI, DANI KOTLAR, AND RAN ZIV

ABSTRACT. Let f(n) be the smallest number such that every collection of n matchings, each of size at least f(n), in a bipartite graph, has a full rainbow matching. Generalizing famous conjectures of Ryser, Brualdi and Stein, Aharoni and Berger [1] conjectured that f(n) = n + 1 for every n > 1. Clemens and Ehrenmüller [4] proved that $f(n) \leq \frac{3}{2}n + o(n)$. We show that the o(n) term can be reduced to a constant, namely $f(n) \leq \lfloor \frac{3}{2}n \rfloor + 1$.

1. INTRODUCTION

Given sets F_1, F_2, \ldots, F_n of edges in a graph, a *(partial) rainbow matching* is a choice of disjoint edges from some of the F_i s. In other words, it is a partial choice function whose range is a matching. If the rainbow matching represents all F_i s then we say that it is *full*. For a comprehensive survey on rainbow matchings and the related subject of transversals in Latin squares see [7].

As in the abstract, we assume the graph is bipartite and define f(n) to be the least number such that if $|F_i| \ge f(n)$ for all i = 1, ..., n, then there exists a full rainbow matching. A greedy choice of representatives shows that if $|F_i| \ge 2n - 1$ for all i = 1, ..., n then there is a rainbow matching. Thus, $f(n) \le 2n - 1$. On the other hand, for every n > 1 there exists a family $F_1, ..., F_n$ of matchings of size n with no full rainbow matching: for an arbitrary $1 \le k \le n$ let $F_1, ..., F_k$ be all equal to the perfect matching in the cycle C_{2n} consisting of the odd edges, and let $F_{k+1}, ..., F_n$ be all equal to the perfect matching in C_{2n} consisting of the even edges. This shows that $f(n) \ge n+1$ for all n > 1 (in fact, this example can be modified to produce 2n-2 matchings of size n with no rainbow matching of size n). In [1] it was conjectured that this bound is sharp:

Conjecture 1.1. [1] f(n) = n + 1 for all n > 1.

If true, this would easily imply:

Conjecture 1.2. A family of n matchings in a bipartite graph, each of size n, has a rainbow matching of size n - 1.

This strengthens a famous conjecture of Ryser-Brualdi-Stein.

Conjecture 1.3. [3, 9, 10] A partition of the edges of the complete bipartite graph $K_{n,n}$ into n matchings, each of size n, has a rainbow matching of size n - 1.

Another strengthening of the last conjecture is due to Stein:

Conjecture 1.4. [10] A partition of the edges of the complete bipartite graph $K_{n,n}$ into n subsets, each of size n, has a rainbow matching of size n - 1.

In our terminology, the weaker condition that Stein demands on sets F_i is not that they are matchings, but that each has degree at most 1 in one side of the graph, and that jointly their degree at each vertex in the other side is at most n. Possibly the 'right' requirement is even more general: that the degree at each vertex is at most n, and that each F_i is a set, and not a multiset, namely it does not contain repeating edges.

Successive improvements on the trivial bound $f(n) \leq 2n - 1$ were $f(n) \leq \lfloor \frac{7}{4}n \rfloor$ [2], $f(n) \leq \lfloor \frac{5}{3}n \rfloor$ [6] and $f(n) \leq \lfloor \frac{3}{2}n \rfloor + o(n)$ [4]. The latter was extended in [5] to general graphs, and to the more general case in

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which the sets F_i are not assumed to be matchings, but disjoint unions of cliques, each containing 3n + o(n) vertices. Pokrovskiy [8] showed that if we add the requirement that the *n* matchings are edge disjoint, then $|F_i| \ge n + o(n)$ suffices. In this note we prove:

Theorem 1.5. $f(n) \leq \lfloor \frac{3}{2}n \rfloor + 1$.

2. Proof of Theorem 1.5

The following was shown in [6]:

Proposition 2.1. A family $\mathcal{F} = \{F_1, \ldots, F_n\}$ of *n* matchings in a bipartite graph, each of size at least $\lfloor \frac{3}{2}n \rfloor$, has a rainbow matching of size n - 1.

Proof of Theorem 1.5. Let G be the given bipartite graph and let $U, W \subset V(G)$ be the two sides of G. Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a family of matchings in G, each of size at least $\lceil \frac{3}{2}n \rceil + 1$, and let R be a rainbow matching of maximal size. By Proposition 2.1, $|R| \ge n-1$. We assume, for contradiction, that |R| = n-1 and without loss of generality we may assume that $R \cap F_n = \emptyset$. For each $i = 1, \ldots, n-1$ let $F_i \cap R = \{r_i\}$ and let $r_i = \{u_i, w_i\}$, where $u_i \in U$ and $w_i \in W$. Let $X \subset U$ and $Y \subset W$ be the sets of vertices of G not matched by R. We shall use the following notation:

Notation 2.2. For any two sets of vertices $A \subseteq U$ and $B \subseteq W$ we denote by E(A, B) or E(B, A) the set of edges in E(G) with one endpoint in A and the other endpoint in B.

Let F_n^Y be the subset of F_n consisting of edges matching vertices in Y. Since R has maximal size, $F_n^Y \subset E(Y, U \setminus X)$. Let U' be the set of vertices in $U \setminus X$ that are endpoints of the edges in F_n^Y . Let R' be the subset of R that matches the vertices in U', and let W' be the set of vertices in W that are endpoints of edges in R' (the set U' is matched by R' to W'). The main idea of the proof is to replace some edges in R' by edges in $E(X, W \setminus Y)$, thus freeing vertices in U'. This will allow us to add an edge from F_n^Y to the rainbow matching.

Let $\ell = |F_n^Y|$. Since $|W \setminus Y| = n - 1$ and $|F_n| \ge \lceil 3n/2 \rceil + 1$ we have $\ell \ge \lceil n/2 \rceil + 2$. By possibly ignoring some edges of F_n we shall assume that

(1)
$$\ell = \lceil n/2 \rceil + 2.$$

(2) $|U'| = |W'| = |R'| = \lceil n/2 \rceil + 2.$

Define,

$$\mathcal{F}' = \{ F_i \in \mathcal{F} | F_i \cap R' \neq \emptyset \}.$$

That is, \mathcal{F}' consists of the matchings that are represented in the partial rainbow matching R'.

Notation 2.3. For each $F_i \in \mathcal{F}'$ let e_i be the edge of F_n^Y such that $e_i \cap r_i \neq \emptyset$. Let y_i be the endpoint of e_i in Y (Figure 1).

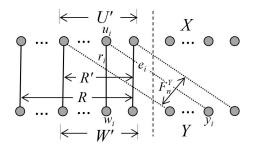


Figure 1

Claim 1. For each $F_i \in \mathcal{F}'$, we have $|F_i \cap E(X, W \setminus Y)| \ge \lfloor n/2 \rfloor + 1$ and $|F_i \cap E(Y, U \setminus X)| \ge \lfloor n/2 \rfloor + 1$.

Proof. We show that each $F_i \in \mathcal{F}'$ has at most one edge between X and Y. Suppose F_i has two edges e and f between X and Y. The edge e_i is disjoint from one of them, say e. Thus, $(R \setminus \{r_i\}) \cup \{e_i, e\}$ is a rainbow matching of size n, contradicting the maximality of R (Figure 2).

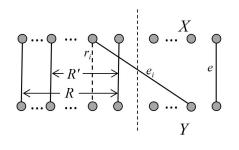


FIGURE 2

Remark 2.4. In all the figures, dashed lines represent edges that are candidates to be removed from the rainbow matching, and solid and dotted lines represent edges that are candidates for being added in.

Notation 2.5. For each $F_i \in \mathcal{F}'$ we denote $F_i^Y = F_i \cap E(Y \setminus \{y_i\}, U \setminus X)$. Let U^* be the union of U' and the set of vertices in $U \setminus X$ that are endpoints of edges in $\bigcup \{F_i^Y \mid F_i \in \mathcal{F}'\}$. Let R^* be the subset of R that matches the elements in U^* and let W^* be the set of vertices in W that are matched by R^* . We define

$$\mathcal{F}^* = \{ F_j \in \mathcal{F} | F_j \cap R^* \neq \emptyset \}.$$

(Note that $U' \subset U^* \subset U \setminus X$, $W' \subset W^* \subset W \setminus Y$, $R' \subset R^* \subset R$ and $\mathcal{F}' \subset \mathcal{F}^* \subset \mathcal{F}$.) Let $\mathcal{F}'' = \mathcal{F}^* \setminus \mathcal{F}'$ and let $d = |\mathcal{F}''|$ (it is possible that d = 0).

Claim 2. For each $F_j \in \mathcal{F}''$, $|F_j \cap E(X, W \setminus Y)| \ge \lceil n/2 \rceil$ and $|F_j \cap E(Y, U \setminus X)| \ge \lceil n/2 \rceil$.

Proof. Let $F_j \in \mathcal{F}''$. We show that F_j has at most two edges between X and Y. By the definition of \mathcal{F}^* , there exists $F_i \in \mathcal{F}'$ and an edge $f \in F_i$ such that $f \cap r_j = \{u_j\} \subset U \setminus X$ and the other endpoint y of f is in $Y \setminus \{y_i\}$. Now suppose F_j has three edges between X and Y. Then one of them, say e, has an endpoint in $Y \setminus \{y_i, y\}$. Now, $R \setminus \{r_i, r_j\} \cup \{f, e_i, e\}$ is a rainbow matching, contradicting the maximality of R (Figure 3).

FIGURE 3

Claim 3. For each $F_i \in \mathcal{F}^*$, $|F_i \cap E(X, W^*)| \ge d+3$.

Proof. Since $|R^*| = |R'| + d$, it follows by (2), that $|R \setminus R^*| = n - 1 - (\lceil n/2 \rceil + 2 + d) = \lfloor n/2 \rfloor - d - 3$. Let $F_i \in \mathcal{F}^* = \mathcal{F}' \cup \mathcal{F}''$ (disjoint union). Since $F_i \cap E(X, W^*) = F_i \cap E(X, W \setminus Y) \cap R^*$, we have by Claims 1 and 2, that $F_i \cap E(X, W^*) \ge \lceil n/2 \rceil - (\lfloor n/2 \rfloor - d - 3) \ge d + 3$.

We shall inductively choose edges $f_1, f_2, \ldots, f_i \in E(X, W^*)$ from distinct F_j s and $r_1, r_2, \ldots, r_i, r_{i+1} \in R^*$ from distinct F_j s, as follows. To start the process, we assume, without loss of generality, that $F_1 \in \mathcal{F}'$. By Claim 3, there exists $f_1 \in F_1 \cap E(X, W^*)$. Let w_2 be the endpoint of f_1 in W^* , and without loss of generality we may assume that $w_2 \in r_2$, where $r_2 \in R^* \cap F_2$. Again, by Claim 3, there exists $f_2 \in F_2 \cap E(X \setminus \{x_1\}, W^*)$. We continue in this manner, choosing at each step an edge $f_i \in E(X, W^*)$, disjoint from all $f_j, j < i$, and belonging to the same matching as r_i , and the edge $r_{i+1} \in R^*$, such that $f_i \cap r_{i+1} \cap W^* \neq \emptyset$. The process ends when we have obtain a set of disjoint edges $F = \{f_1, f_2, \ldots, f_m\} \subseteq E(X, W^*)$ and a set of distinct edges $P = \{r_1, r_2, \ldots, r_m, r_{m+1}\} \subseteq R^*$ such that $f_i \cap r_{i+1} \cap W^* \neq \emptyset$ for $i = 1, \ldots, m$, and for each i, f_i and r_i belong to the same matching (without loss of generality we assume that $f_i, r_i \in F_i$ for $i = 1, \ldots, m$, and $r_{m+1} \in F_{m+1}$), so that one of two options holds:

(1) m < d + 3 and the matching F_{m+1} has an edge $f_{m+1} \in E(X \setminus (f_1 \cup f_2 \cup \ldots \cup f_m), W^*)$ such $f_{m+1} \cap r_t \cap W^* \neq \emptyset$ for some $t \in \{1, \ldots, m\}$, or (2) m = d + 3.

(Note that by Claim 3 one of these two options must hold.)

In Case (1) the partial rainbow matching R can be augmented as follows: If $F_i \in \mathcal{F}'$ for some $i \in \{t, \ldots, m+1\}$, then $(R \setminus \{r_t, \ldots, r_{m+1}\}) \cup \{f_t, \ldots, f_{m+1}, e_i\}$ is a full rainbow matching (Figure 4(a)). If $F_i \in \mathcal{F}''$ for all $i \in \{t, \ldots, i+1\}$, then, by the definition of \mathcal{F}^* , there exists $F_j \in \mathcal{F}'$ and an edge $e \in F_j^Y$ so that $e \cap r_t \in U^*$. In this case $(R \setminus \{r_t, \ldots, r_{m+1}, r_j\}) \cup \{f_t, \ldots, f_{m+1}, e, e_j\}$ is a full rainbow matching (Figure 4(b)). (Note that e and e_j are disjoint by the definition of F_j^Y .)

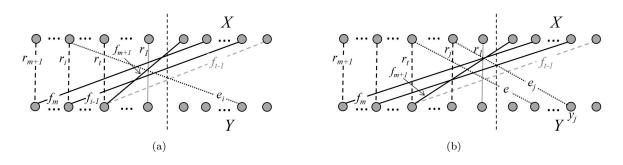


Figure 4

In Case (2) let $Q = (R \setminus P) \cup F$. Then, Q is a partial rainbow matching of size n - 2, since it excludes the matchings F_{m+1} and F_n . We shall augment Q with edges two edges in $E(Y, U^*)$, from F_{m+1} and F_n respectively.

Claim 4. If $F_i \in \mathcal{F}'$, then the size of the set $\{e \in F_i^Y : e \cap (\bigcup_{j=1}^m r_j) \neq \emptyset\}$ is at least 2.

Proof. Let U^i be the set of endpoints in $U \setminus X$ of the edges in F_i^Y . Note that $|U^i| \ge \ell - 1$ (Claim 1 and (1)), $U^i \subset U^*$ (since $F_i \in \mathcal{F}'$), and $|U^*| = \ell + d$. Recall that for each edge $r_j \in R \cap F_j$ its endpoint in $U \setminus X$ was denoted u_j . Since $|U^* \setminus \{u_1, \ldots, u_m\}| = \ell + d - m = \ell + d - (d+3) = \ell - 3$, the claim follows.

There are two sub-cases to consider: (2a) $F_{m+1} \in \mathcal{F}'$, and (2b) $F_{m+1} \in \mathcal{F}''$.

(2a) Assume $F_{m+1} \in \mathcal{F}'$. By Claim 4, there exists and edge $e \in F_{m+1}$ connecting a vertex in $Y \setminus \{y_{m+1}\}$ with some u_t , which is the endpoint in U of some $r_t \in P \setminus \{r_{m+1}\}$. Since m = d+3 and |P| = m+1 = d+4, at least four of the edges in P are in R' (actually, three are enough in this case). For at least one of these four edges, say r_i , its corresponding e_i (the edge of F_n^Y meeting r_i in U) avoids both endpoints of e. Then, $Q \cup \{e, e_i\}$ is a rainbow matching of size n (Figure 5(a)).

(2b) Assume $F_{m+1} \in \mathcal{F}''$ By Claim 2, $|F_{m+1}^Y| \ge \lceil n/2 \rceil$. Since by (2) we have $|R \setminus R'| = n - 1 - (\lceil n/2 \rceil + 2) = \lfloor n/2 \rfloor - 3$, there is an edge $e \in F_{m+1}^Y$ sharing an endpoint with an edge $r_s \in R'$. Assume first that $s \in \{1, \ldots, m\}$. As in the previous paragraph, there exists e_i disjoint from r_s and e, so that $Q \cup \{e, e_i\}$ is a rainbow matching of size n. Now assume that $s \notin \{1, \ldots, m\}$ and let again e be the edge of F_{m+1} sharing an endpoint with r_s . Since $r_s \in R'$, there exists, by Claim 4, an edge $e' \in F_s^Y$, disjoint from e, sharing an

endpoint with some r_t with $t \in \{1, \ldots, m\}$. Since $|P \cap R'| \ge 4$, there exists an edge $e_i \in F_n^Y$, avoiding both endpoints of e' and the endpoint of e in Y, such that $u_i \in \{u_1, \ldots, u_{m+1}\}$. Then, $Q \setminus \{r_t\} \cup \{e, e', e_i\}$ is a rainbow matching of size n (Figure 5(b)). This completes the proof.

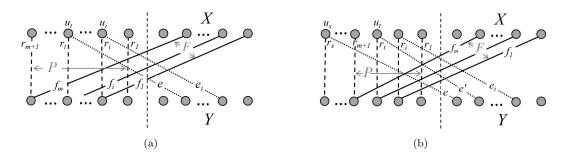


FIGURE 5

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DEPARTMENT OF MATHEMATICS, TECHNION, HAIFA, ISRAEL

E-mail address, Ron Aharoni: raharoni@gmail.com

Computer Science Department, Tel-Hai college, Upper Galilee, Israel

E-mail address, Dani Kotlar: dannykotlar@gmail.com

Computer Science Department, Tel-Hai college, Upper Galilee, Israel

E-mail address, Ran Ziv: ranzivziv@gmail.com