# SEMIALGEBRAIC GEOMETRY OF NONNEGATIVE TENSOR RANK 

YANG QI, PIERRE COMON, AND LEK-HENG LIM


#### Abstract

We study the semialgebraic structure of $D_{r}$, the set of nonnegative tensors of nonnegative rank not more than $r$, and use the results to infer various properties of nonnegative tensor rank. We determine all nonnegative typical ranks for cubical nonnegative tensors and show that the direct sum conjecture is true for nonnegative tensor rank. We show that nonnegative, real, and complex ranks are all equal for a general nonnegative tensor of nonnegative rank strictly less than the complex generic rank. In addition, such nonnegative tensors always have unique nonnegative rank- $r$ decompositions if the real tensor space is $r$-identifiable. We determine conditions under which a best nonnegative rank- $r$ approximation has a unique nonnegative rank- $r$ decomposition: for $r \leq 3$, this is always the case; for general $r$, this is the case when the best nonnegative rank- $r$ approximation does not lie on the boundary of $D_{r}$. Many of our general identifiability results also apply to real tensors and real symmetric tensors.


## 1. Introduction

In many applications, notably algebraic statistics [34, 33, 5, 4, 49, 30, 3, one frequently needs to find (i) the nonnegative rank, (ii) a nonnegative rank- $r$ decomposition, or (iii) a best nonnegative rank-r approximation, of a nonnegative third order tensor. Such problems also arise for instance in chemometrics [45] and hyperspectral imaging [58], where quantities like concentration and intensity can only take on nonnegative values. This article addresses questions pertaining to these three problems using tools from semialgebraic geometry.

Questions regarding nonnegative decompositions of a nonnegative tensor are often regarded as being more difficult than the corresponding questions over the complex numbers. One reason is that the tools of classical algebraic geometry are often at one's disposal in the latter case but not the former. In this article we study nonnegative tensors under the light of semialgebraic geometry. The first main result of our article (cf. Theorem 5.7) is that for a general nonnegative tensor with nonnegative rank strictly less than the complex generic rank, its rank over complex numbers, real numbers, and nonnegative real numbers, are all equal. Furthermore, for such a nonnegative tensor, its nonnegative rank- $r$ decomposition is unique if the real tensor space is $r$-identifiable. We determine the nonnegative typical ranks in

[^0]Propositions 6.5 and 6.6 and show in Lemma 4.1 that the nonnegative direct sum conjecture is true, i.e., the nonnegative rank of the direct sum of two nonnegative tensors equals the sum of the respective nonnegative ranks. In our earlier work [50], we showed that a general nonnegative tensor has a unique best nonnegative rank- $r$ approximation. But it remains to be seen whether this approximation itself has a unique nonnegative rank- $r$ decomposition; we show that this is the case for $r \leq 3$ in Theorem 7.8, and, for general $r$, we show in Corollary 7.6 that uniqueness holds for an open subset of nonnegative tensors under some conditions on the tensor space.

The paper is organized as follows. Section 2 lists some preliminary facts in semialgebraic geometry. The definition of $X$-rank and its basic properties are introduced in Section 3, Lemma 3.4 is necessary to determine nonnegative typical ranks in Propositions 6.5 and 6.6. Our main contributions are then presented in Sections 5. 6, 7. Although we focus on nonnegative tensors, some of our techniques apply almost verbatim to real tensors and real symmetric tensors, and thus we will also derive a few identifiability results for such tensors.

We begin with a short list of standard definitions. Let $V_{1}, \ldots, V_{d}$ be vector spaces over a field $\mathbb{K}$, and denote the dual of $V_{i}$ by $V_{i}^{*}$. The tensor space $V_{1}^{*} \otimes \cdots \otimes V_{d}^{*}$ is the space of multilinear $\mathbb{K}$-valued functions on $V_{1} \times \cdots \times V_{d}$. Its elements are called order- $d$ tensors or $d$-tensors or just tensors if the order is implicit. We will write $\mathbb{K}^{n_{1} \times \cdots \times n_{d}}=\mathbb{K}^{n_{1}} \otimes \cdots \otimes \mathbb{K}^{n_{d}}$ and regard the elements as $d$-dimensional hypermatrices.

A nonzero tensor in $V_{1} \otimes \cdots \otimes V_{d}$ is said to have rank-one if it is of the form $v_{1} \otimes \cdots \otimes v_{d}$, where $v_{i} \in V_{i}$ and $v_{1} \otimes \cdots \otimes v_{d}$ is defined by

$$
v_{1} \otimes \cdots \otimes v_{d}\left(u_{1}, \ldots, u_{d}\right)=v_{1}\left(u_{1}\right) \cdots v_{d}\left(u_{d}\right)
$$

for all $u_{i} \in V_{i}^{*}$. The rank of a nonzero tensor $T$, denoted by $\operatorname{rank}(T)$, is the minimum number $r$ such that $T$ is a sum of $r$ rank-one tensors. In addition, $\operatorname{rank}(T)=0$ iff $T=0$. An expression of $T$ as a sum of $r=\operatorname{rank}(T)$ rank-one tensors is called a rank- $r$ decomposition 1 . A rank- $r$ decomposition

$$
\begin{equation*}
T=\sum_{i=1}^{r} T_{i}, \quad T_{i}=u_{i}^{(1)} \otimes \cdots \otimes u_{i}^{(d)} \tag{1.1}
\end{equation*}
$$

is said to be (essentially) unique if the unordered set $\left\{T_{i}: i=1, \ldots, r\right\}$ is unique [22], i.e., each $u_{i}^{(k)}$ is unique up to permutation and scaling [40, 36, 41, 27, 44]. The tensor space $V_{1} \otimes \cdots \otimes V_{d}$ is said to be $r$-identifiable if a general rank- $r$ tensor has a unique rank- $r$ decomposition [19]. There has been intense research on tensor ranks and uniqueness of rank- $r$ decompositions. See [22] for a review.

We note that the names PARAFAC, CANDECOMP, canonical polyadic, or CP decomposition have often been used in the literature for (1.1). However (1.1) and the corresponding notion of rank were originally proposed by F. L. Hitchcock [39], and it was followed by many subsequent works in mathematics long before the psychometricians [15, 37] coined the names candecomp and Parafac. Hitchcock had used 'polyadic' in a different sense and the terms CP-rank and CP decompositions are better known as something entirely different [7, 14, 46, 51, As such we think it is fair to use a neutral and unambiguous term like 'rank- $r$ decomposition' to describe (1.1).

[^1]In this article, the field $\mathbb{K}$ will be either the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. We will also extend the above to a semiring, denoted by $\mathcal{R}$. Of particular interest to us is the semiring of nonnegative real numbers $\mathbb{R}_{+}:=[0, \infty)$. It is possible that $\mathcal{R}=\mathbb{R}$ or $\mathbb{C}$, i.e., a result stated for semiring would also apply to a field unless stated otherwise. For convenience of notations, all our results are stated for 3-tensors, i.e., $d=3$, although most of them can be generalized to tensors of arbitrary order without difficulties.

## 2. SEMIALGEBRAIC GEOMETRY

In this section we briefly review some well-known facts in semialgebraic geometry, providing in particular a summary of the relevant portions of [13, 24, 48, 31, 25] for our later use.

A semialgebraic subset of $\mathbb{R}^{n}$ is the union of finitely many subsets of the form

$$
\left\{x \in \mathbb{R}^{n}: P(x)=0, Q_{1}(x)>0, \ldots, Q_{m}(x)>0\right\}
$$

where $P, Q_{1}, \ldots, Q_{m} \in \mathbb{R}\left[X_{1}, \ldots, X_{n}\right]$, are polynomials in $n$ variables with real coefficients. Let $S$ and $T$ be semialgebraic sets. A map $f: S \rightarrow T$ is called semialgebraic if its graph $G(f):=\{(s, t) \in S \times T: f(s)=t\}$ is semialgebraic. A semialgebraic set is called nonsingular if it is an open subset of the set of nonsingular points of some algebraic set. A Nash manifold is a semialgebraic analytic submanifold of $\mathbb{R}^{n}$ and a Nash mapping between Nash manifolds is an analytic mapping with a semialgebraic graph.

A point $p$ in a semialgebraic set $S$ is said to be general with respect to some property $\mathscr{P}$ if the points in $S$ that do not have the property $\mathscr{P}$ are all contained in a semialgebraic subset $C$ of $S$ with $\operatorname{dim} C<\operatorname{dim} S$ and $p \notin C$. To aid readers unacquainted with the notion, we give familiar measure theoretic and topological interpretations of a general point but note that these cannot replace its formal definition. Given the Lebesgue measure $\mu$ on $S$, if a point $p \in S$ is general with respect to a property $\mathscr{P}$, then (i) $C:=\{q \in S: q$ does not satisfy $\mathscr{P}\}$ is a measurezero subset of $S$; and (ii) $p \notin C$. Hence in the sense of measure theory, the statement that a general point satisfies $\mathscr{P}$ is equivalent to the statement that almost every point satisfies $\mathscr{P}$. On the other hand, in the sense of topology, the statement that a general point satisfies $\mathscr{P}$ has a stronger connotation - it implies that the subset $C$ lies in a hypersurface of $S$. Take $S=\mathbb{R}$ for example, that a general point satisfies $\mathscr{P}$ implies that at most finitely many points in $\mathbb{R}$ do not satisfy $\mathscr{P}$. Note that this is a stronger conclusion than 'almost every point in $S$ satisfies $\mathscr{P}$ ' in the measure theoretic sense.

Let $f: M \rightarrow N$ be a Nash mapping between Nash manifolds $M$ and $N$. The usual semialgebraic version of Sard's theorem [13] says that the set of critical values of $f$ is a semialgebraic subset of $N$ with smaller dimension. As we focus on polynomial maps in this article, we have the following stronger version of Sard's theorem about critical points of $f$.

Lemma 2.1. Let $f: \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$ be a nonconstant polynomial map. Then the set of critical points of $f$ is a subvariety of $\mathbb{R}^{m}$, with dimension strictly less than $m$.

Proof. Let $d:=\operatorname{dim} \operatorname{Im} f$ and $\nabla f$ be the Jacobian of $f$ (i.e., the matrix of first order partial derivatives if we choose coordinates). Then every $d \times d$ minor of $\nabla f$ must vanish on the points $x \in \mathbb{R}^{m}$ where $\nabla f(x)$ has rank strictly less than $d$. At least one of these minors is not identically zero since there are points $x \in \mathbb{R}^{m}$ where
$\nabla f(x)$ has rank exactly $d$. Thus these minors define a subvariety whose dimension is strictly less than $m$.

Aside from Sard's theorem, we also quote a few selected results and definitions from [13, 31] for the reader's easy reference. These results are somewhat technical and although they logically belong to this section, we will not need them until Section 7 In particular, Sections 3 through 6 do not require any of the following.

Theorem 2.2 (Nash Tubular Neighborhood). Let $N \subset \mathbb{R}^{n}$ be a Nash submanifold. Then there is an open semialgebraic neighborhood $U \subset \mathbb{R}^{n}$ and a Nash retraction $f: U \rightarrow N$ such that $\operatorname{dist}(p, N)=\|p-f(p)\|$ for each $p \in U$. Here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^{n}$.

Definition 2.3. A Whitney stratification of a semialgebraic set $S \subseteq \mathbb{R}^{n}$ is a finite partition of $S$ into semialgebraically connected submanifolds $S=\bigcup_{i} S_{i}$ satisfying the following two conditions, known respectively as the 'frontier condition' and 'Whitney condition (a)'.
(i) For $i \neq j$, if $S_{i} \cap \operatorname{cl}\left(S_{j}\right) \neq \varnothing$, then $S_{i} \subseteq \operatorname{cl}\left(S_{j}\right) \backslash S_{j}$.
(ii) For any sequence of points $\left(x_{k}\right)$ in a stratum $S_{j}$, if $x_{k}$ converges to a point $y$ in a stratum $S_{i}$, and the sequence of tangent ( $\operatorname{dim} S_{j}$ )-planes $\mathrm{T}_{x_{k}} S_{j}$ converges to a $\left(\operatorname{dim} S_{j}\right)$-plane $T$, then $T$ contains the tangent $\left(\operatorname{dim} S_{i}\right)$-plane $\mathrm{T}_{y} S_{i}$.

Given two finite families $\left\{B_{i}\right\}$ and $\left\{C_{j}\right\}$ of subsets of $\mathbb{R}^{n},\left\{B_{i}\right\}$ is said to be compatible with $\left\{C_{j}\right\}$ if $B_{i} \cap C_{j}=\varnothing$ or $B_{i} \subseteq C_{j}$ for all $i$ and $j$.

Theorem 2.4. For semialgebraic subsets $S, C_{1}, \ldots, C_{m}$ of $\mathbb{R}^{n}, S$ admits a Whitney stratification compatible with $C_{1}, \ldots, C_{m}$.

Proposition 2.5. Let $f: S \rightarrow \mathbb{R}^{n}$ be a semialgebraic function on a semialgebraic set. Then $S$ admits a Whitney stratification $S=\bigcup_{i} S_{i}$ such that each graph of $\left.f\right|_{S_{i}}$ is a nonsingular semialgebraic set.

Proposition 2.6. Let $S$ be a nonsingular semialgebraic set, and $f: S \rightarrow \mathbb{R}^{n}$ be a function such that $G(f)$ is nonsingular and semialgebraic. Then the set of points of $S$ where $f$ is not differentiable is contained in a closed lower-dimensional semialgebraic subset of $S$.

## 3. $X$-RANKS

There has been several attempts to describe tensor ranks in different settings in a unified and general way, e.g. [10, 57] but they do not usually include nonnegative rank as a special case. Here we introduce a generalization of $X$-rank 60 to the setting of an arbitrary cone $X$ and coefficients in a semiring $\mathcal{R}$ in order to treat nonnegative, real, and complex tensor ranks in a unified setting.

Definition 3.1. Let $\mathbb{K}$ be a field, and $\mathcal{R} \subseteq \mathbb{K}$ be a semiring. Given a vector space $V$ over $\mathbb{K}$, and a subset $X \subseteq V$, an $\mathcal{R}$-span of $X$, denoted by $\operatorname{span}_{\mathcal{R}}(X)$, is the set of all finite $\mathcal{R}$-linear combinations of elements of $X$, that is,

$$
\operatorname{span}_{\mathcal{R}}(X):=\left\{\sum_{i=1}^{k} \alpha_{i} x_{i}: k>0, \alpha_{i} \in \mathcal{R}, x_{i} \in X\right\}
$$

When $\mathcal{R}=\mathbb{K}$, an $\mathcal{R}$-span is a subspace. When $\mathbb{K}=\mathbb{R}$ and $\mathcal{R}=\mathbb{R}_{+}$, an $\mathcal{R}$-span is a convex cone. We will denote the $\mathbb{R}_{+}$-cone of nonnegative vectors in a vector space $V$ by either $V^{2} V^{+}$or $V_{+}$. Note that in order to specify $V_{+}$, we will need to first specify a choice of basis on $V$. See [50] for further discussions. With this notation, $V_{1}^{+} \otimes \cdots \otimes V_{d}^{+}$is the cone of nonnegative tensors as defined in [50, Definition 2].
Definition 3.2. We say $X$ is an $\mathcal{R}$-cone, if for $x \in X$ we always have $\lambda x \in X$ for any $\lambda \in \mathcal{R}$. Given an $\mathcal{R}$-cone $X$, for any $p \in \operatorname{span}_{\mathcal{R}}(X)$, the $X$-rank of $p, \operatorname{rank}_{X}(p)$, is defined to be

$$
\operatorname{rank}_{X}(p):=\min \left\{r: p=x_{1}+\cdots+x_{r} ; x_{1}, \ldots, x_{r} \in X\right\}
$$

Recall that in algebraic geometry, the affine cone $X \subseteq \mathbb{K}^{n}$ over a projective variety $Y \subseteq \mathbb{K} \mathbb{P}^{n-1}$ is defined as $X:=\pi^{-1}(Y) \cup\{0\}$ where $\pi: \mathbb{K}^{n} \backslash\{0\} \rightarrow \mathbb{K} \mathbb{P}^{n-1}$, $\left(x_{1}, \ldots, x_{n}\right) \mapsto\left[x_{1}: \cdots: x_{n}\right]$ is the canonical projection. Note that an affine cone is a $\mathbb{K}$-cone in the sense of Definition 3.2,
(i) Let $\mathcal{R}=\mathbb{K}=\mathbb{R}, V=V_{1} \otimes \cdots \otimes V_{d}$, and $X$ be the cone of tensors of rank $\leq 1$ (i.e., affine cone over the real projective Segre variety). Then $\operatorname{rank}_{X}(p)$ is the real rank of $p$, usually denoted $\operatorname{rank}_{\mathbb{R}}(p)$. Real tensor rank is invariant under the action of $\mathrm{GL}\left(V_{1}\right) \times \cdots \times \mathrm{GL}\left(V_{d}\right)$, where $\mathrm{GL}(V)$ denotes the general linear group of $V$.
(ii) Let $\mathcal{R}=\mathbb{R}_{+}, \mathbb{K}=\mathbb{R}, V=V_{1} \otimes \cdots \otimes V_{d}$, and $X$ be the $\mathbb{R}_{+}$-cone of nonnegative tensors of rank $\leq 1$. Then $\operatorname{rank}_{X}(p)$ is the nonnegative rank of $p$, usually denoted $\operatorname{rank}_{+}(p)$. Nonnegative tensor rank is invariant under the action of

$$
\left\{\left(g_{1}, \ldots, g_{d}\right) \in \mathrm{GL}\left(V_{1}\right) \times \cdots \times \mathrm{GL}\left(V_{d}\right): g_{i}\left(V_{i}^{+}\right) \subseteq V_{i}^{+}, i=1, \ldots, d\right\}
$$

Note that this set is just a monoid - it does not necessarily contain the inverses of its elements.
(iii) Let $\mathcal{R}=\mathbb{K}$ be an algebraically closed field and $X$ be the affine cone over an irreducible nondegenerate projective variety. Then $\operatorname{rank}_{X}(p)$ is the $X$-rank as defined in [60, 41, 10]. $X$-rank is invariant under the automorphism group of $X$, a subgroup of GL $(V)$.
The discussions above are purely algebraic but subsequent discussions will require topological structures on our vector space and field. Recall that a topological vector space over a topological field is one where the vector addition and scalar multiplication are continuous. We will not require any results regarding topological vector space beyond its definition.
Definition 3.3. Let $V$ be a finite-dimensional topological vector space over a topological field $\mathbb{K}$ of characteristic zero, and $\mathcal{R} \subseteq \mathbb{K}$ be a semiring. Let $X \subseteq V$ be an $\mathcal{R}$-cone such that $\operatorname{span}_{\mathcal{R}}(X)$ contains a nonempty open subset of $V$. If the set $\left\{p \in \operatorname{span}_{\mathcal{R}}(X): \operatorname{rank}_{X}(p)=r\right\}$ contains a nonempty open subset of $V$, then $r$ is called a typical $X$-rank. In particular, when $\mathbb{K}=\mathbb{C}$ and $V$ is endowed with the Zariski topology, $r$ is called a complex generic $X$-rank whenever $\left\{p \in \operatorname{span}_{\mathbb{C}}(X)\right.$ : $\left.\operatorname{rank}_{X}(p)=r\right\}$ contains a nonempty Zariski open subset of $V$. The maximum typical $X$-rank is

$$
\max \left\{r: r \text { is a typical } X \text {-rank of } \operatorname{span}_{\mathcal{R}}(X)\right\}
$$

[^2]whereas the maximum $X$-rank is
$$
\max \left\{\operatorname{rank}_{X}(p): p \in \operatorname{span}_{\mathcal{R}}(X)\right\}
$$

To provide a more familiar perspective, when $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and $V$ is endowed with the Euclidean topology and the Lebesgue measure, then $r$ is a typical $X$-rank whenever $\left\{p \in \operatorname{span}_{\mathcal{R}}(X): \operatorname{rank}_{X}(p)=r\right\}$ has positive measure.

Recall that a variety is called irreducible if it is not the union of two nonempty proper subvarieties. If the ideal of an affine variety $X \subseteq \mathbb{C}^{n}$ is generated by polynomials with real coefficients $f_{1}, \ldots, f_{k}$, we will denote by $X(\mathbb{R})$ the set of real points of $X$, i.e., $X(\mathbb{R})=X \cap \mathbb{R}^{n}$. In fact $X(\mathbb{R})$ equals the zero locus of $f_{1}, \ldots, f_{k}$ in $\mathbb{R}^{n}$. On the other hand, if $Y \subseteq \mathbb{R}^{n}$ is a real variety defined by real polynomials $f_{1}, \ldots, f_{k}$, we will denote by $Y(\mathbb{C})$ the complexification of $Y$, the complex variety defined by $f_{1}, \ldots, f_{k}$ in $\mathbb{C}^{n}$. For an irreducible real affine variety $Y \subseteq \mathbb{R}^{n}$, its complexification $Y(\mathbb{C})$ is also irreducible $[10$. Furthermore $Y$ is Zariski dense in $Y(\mathbb{C})$ if and only if $Y(\mathbb{C})$ has a nonsingular real point [10, 53].

A (projective) variety $X \subseteq V(X \subseteq \mathbb{P} V)$ is said to be nondegenerate if $X$ is not contained in any hyperplane. It is shown in [10, Theorem 2] that when $X$ is an irreducible nondegenerate real projective variety whose complexification $X(\mathbb{C})$ has a real smooth point, there is a unique complex generic $X$-rank, and it is equal to the minimum real typical $X$-rank. For example, the space of $2 \times 2 \times 2$ tensors has the complex generic rank 2 and the real typical ranks 2 and 3 [26].

We deduce the following lemma using an argument in 32, where it is proved for the case $\mathbb{K}=\mathbb{R}, V=V_{1} \otimes V_{2} \otimes V_{3}$, and $X=\left\{A \in V: \operatorname{rank}_{\mathbb{R}}(A) \leq 1\right\}$. See also [8, Theorem 1.1] for the case where $X$ is the affine cone of a nondegenerate irreducible real projective variety.

Lemma 3.4. Let $\mathbb{K}=\mathbb{R}$ and $X$ be a nonempty semialgebraic $\mathcal{R}$-cone whose Zariski closure $\bar{X}$ is a nondegenerate irreducible real variety that is Zariski dense in $\bar{X}(\mathbb{C})$. If $m$ and $M$ are two typical $X$-ranks, then any integer between $m$ and $M$ is also $a$ typical X-rank.
Proof. Let $\operatorname{dim} V=n$. For each $k \in \mathbb{N}$, define the polynomial map $\varphi_{k}$ by

$$
\varphi_{k}: X \times \cdots \times X \rightarrow \operatorname{span}_{\mathcal{R}}(X), \quad\left(x_{1}, \ldots, x_{k}\right) \mapsto x_{1}+\cdots+x_{k}
$$

Assume WLOG that $m \leq M$ and suppose that $r \in\{m, \ldots, M\}$ is the minimum integer which is not a typical $X$-rank. For any fixed $k \in \mathbb{N}$ and for any open subset $\mathcal{W} \subseteq V, \varphi_{k}^{-1}(\mathcal{W})$ is open in $X \times \cdots \times X$; thus it is a union of open subsets of the form $\mathcal{U}_{1} \times \cdots \times \mathcal{U}_{k}$ where each $\mathcal{U}_{i}$ is open in $X$. Since $\bar{X}$ is irreducible, the dimension of each $\mathcal{U}_{i}$ equals $\operatorname{dim} X$. By [38, Exercise II.3.22], the dimension of each $\varphi_{r}\left(\mathcal{U}_{1} \times \cdots \times \mathcal{U}_{r}\right)$ equals $n$. So every nonempty open subset of $\operatorname{Im} \varphi_{r}$ has dimension $n$. Since $r$ is not a typical rank, $\operatorname{Im} \varphi_{r} \backslash \operatorname{Im} \varphi_{r-1}$ does not contain a subset of dimension $n$, and thus $\operatorname{Im} \varphi_{r} \backslash \operatorname{Im} \varphi_{r-1}$ does not contain an open subset of $\operatorname{Im} \varphi_{r}$, which implies that a general $p=x_{1}+\cdots+x_{r} \in \operatorname{Im} \varphi_{r}$ is within $\operatorname{Im} \varphi_{r-1}$, i.e., $p=\widetilde{x}_{1}+\cdots+\widetilde{x}_{r-1}$. Hence a general $q=x_{1}+\cdots+x_{r+1} \in \operatorname{Im} \varphi_{r+1}$ can be written with $r$ summands as $q=\widetilde{x}_{1}+\cdots+\widetilde{x}_{r-1}+x_{r+1}$, which is in $\operatorname{Im} \varphi_{r}$. But we may repeat the same argument to conclude that $q$ is in $\operatorname{Im} \varphi_{r-1}$. So by induction, a general point in $\operatorname{Im} \varphi_{M}$ is in $\operatorname{Im} \varphi_{r-1}$, i.e., $\operatorname{dim} \operatorname{Im} \varphi_{M} \backslash \operatorname{Im} \varphi_{r-1}<\operatorname{dim} V$, contradicting our assumption that $M$ is a typical $X$-rank.

We will require the use of Lemma 3.4 in Propositions 6.5 and 6.6. This simple lemma is surprisingly potent. As an illustration we provide a short proof for the
main result in [9] (see also [8]), that every integer between $\lfloor(d+2) / 2\rfloor$ and $d$ is a typical rank of $S^{d}\left(\mathbb{R}^{2}\right)$, originally conjectured in [23].

Corollary 3.5 (Blekherman). Every $m$ with $\lfloor(d+2) / 2\rfloor \leq m \leq d$ is a typical rank of $S^{d}\left(\mathbb{R}^{2}\right)$.

Proof. The complex generic rank $\lfloor(d+2) / 2\rfloor$ is necessarily the minimum typical rank by [10]. It has been shown in [16] that $f \in \mathrm{~S}^{d}\left(\mathbb{R}^{2}\right)$ has real rank $d$ if and only if $f$ has $d$ distinct real roots when regarded as a degree- $d$ homogeneous polynomial in two variables. Since $d$ is the maximum real rank [23], and having $d$ distinct real roots imposes an open condition on $\mathrm{S}^{d}\left(\mathbb{R}^{2}\right), d$ is therefore the maximum typical rank. The required result then follows from Lemma 3.4

We now introduce a 'semialgebraic version' of Terracini's lemma. First observe that for semialgebraic sets $X, Y \subseteq V$, if we define the semialgebraic map $\varphi$ by

$$
\varphi: X \times Y \rightarrow V, \quad(x, y) \mapsto x+y
$$

then $\operatorname{Im}(\varphi)$ is semialgebraic by the Tarski-Seidenberg Theorem.
Lemma 3.6 (Semialgebraic Terracini's lemma). Let $X$ and $Y$ be nonempty semialgebraic subsets. Suppose their Zariski closures $\bar{X}, \bar{Y}$ are irreducible real varieties and that $\bar{X}(\mathbb{C}), \bar{Y}(\mathbb{C})$ have real smooth points. Then for general points $x \in X$ and $y \in Y$, the tangent space of $\varphi(X \times Y)$ at $x+y$ is the span of the tangent spaces $\mathrm{T}_{x} X$ and $\mathrm{T}_{y} Y$, i.e.,

$$
\mathrm{T}_{x+y} \varphi(X \times Y)=\operatorname{span}\left\{\mathrm{T}_{x} X, \mathrm{~T}_{y} Y\right\}
$$

Proof. Since $\bar{X}$ and $\bar{Y}$ are irreducible and have real smooth points, $\overline{\varphi(X \times Y)}$ is irreducible and its complexification $\overline{\varphi(X \times Y)}(\mathbb{C})$ has real smooth points. Thus the set of smooth points of $\varphi(X \times Y)$ is open dense in $\varphi(X \times Y)$. Then for a general $(x, y) \in X \times Y, \varphi(x, y)=x+y$ is smooth in $\varphi(X \times Y)$. Hence

$$
\begin{aligned}
\mathrm{T}_{x+y} \varphi(X \times Y)=\varphi_{*}\left(\mathrm{~T}_{(x, y)} X \times Y\right) & =\varphi_{*}\left(\mathrm{~T}_{x} X \oplus \mathrm{~T}_{y} Y\right) \\
& =\mathrm{T}_{x} X+\mathrm{T}_{y} Y=\operatorname{span}\left\{\mathrm{T}_{x} X, \mathrm{~T}_{y} Y\right\}
\end{aligned}
$$

The following is also immediate from Tarski-Seidenberg Theorem and our earlier work.

Proposition 3.7. $D_{r}:=\left\{A \in \mathbb{R}_{+}^{n_{1} \times \cdots \times n_{d}}: \operatorname{rank}_{+}(A) \leq r\right\}$ is a closed semialgebraic set, i.e., there exists a finite number of polynomials $P_{1}, \ldots, P_{m}$ with real coefficients that cuts out $D_{r}$ as a set, i.e.,

$$
D_{r}=\left\{A \in \mathbb{R}^{n_{1} \times \cdots \times n_{d}}: P_{1}(A) \geq 0, \ldots, P_{m}(A) \geq 0\right\} .
$$

Furthermore, $C_{r}:=\left\{A \in \mathbb{R}_{+}^{n_{1} \times \cdots \times n_{d}}: \operatorname{rank}_{+}(A)=r\right\}$ is also a semialgebraic set but not closed in general.

Proof. By the Tarski-Seidenberg Theorem [13], $D_{r}$ is a semialgebraic set and thus so is $C_{r}=D_{r} \backslash D_{r-1}$. By [45, Proposition 6.2], $D_{r}$ is closed.

## 4. Direct sum conjecture for nonnegative rank

We now show that the direct sum conjecture is true for nonnegative rank. Given vector spaces $V_{1}, \ldots, V_{d}$, and $W_{1}, \ldots, W_{d}$ over $\mathbb{K}$, for any $A \in V_{1} \otimes \cdots \otimes V_{d}$ and $B \in W_{1} \otimes \cdots \otimes W_{d}$, we have the direct sum $A \oplus B \in\left(V_{1} \oplus W_{1}\right) \otimes \cdots \otimes\left(V_{d} \oplus W_{d}\right)$. For $d=2$, it is obvious that the rank of a block diagonal matrix is the sum of the ranks of the diagonal blocks, i.e., if $A$ and $B$ are matrices, then

$$
\operatorname{rank}(A \oplus B)=\operatorname{rank}\left(\left[\begin{array}{cc}
A & 0 \\
0 & B
\end{array}\right]\right)=\operatorname{rank}(A)+\operatorname{rank}(B)
$$

It has been conjectured by Strassen 55 that the same is true for $d>2$, i.e., $\operatorname{rank}(A \oplus B)=\operatorname{rank}(A)+\operatorname{rank}(B)$ for any $d$-tensors. This has been a long-standing open problem in algebraic computational complexity. We show here that the analogous statement for nonnegative rank is true. The next two results are true for nonnegative tensors of arbitrary order $d$ but we will state and prove them for $d=3$ for notational simplicity.

In the following, let $U_{1}, V_{1}, W_{1}, U_{2}, V_{2}, W_{2}$ be real vector spaces of dimensions $m_{1}, n_{1}, p_{1}, m_{2}, n_{2}, p_{2}$ respectively. Fix a basis for each vector space and choose the bases for $U_{1} \oplus U_{2}, V_{1} \oplus V_{2}$, and $W_{1} \oplus W_{2}$ so that for $a=\left(a_{1}, \ldots, a_{m_{1}}\right) \in U_{1}$ and $b=\left(b_{1}, \ldots, b_{m_{2}}\right) \in U_{2}, a \oplus b$ has coordinates $a \oplus b=\left(a_{1}, \ldots, a_{m_{1}}, b_{1}, \ldots, b_{m_{2}}\right)$ in $U_{1} \oplus U_{2}$; likewise for $V_{1} \oplus V_{2}$ and $W_{1} \oplus W_{2}$.

Lemma 4.1 (Nonnegative direct sum conjecture). For $A \in U_{1}^{+} \otimes V_{1}^{+} \otimes W_{1}^{+}$and $B \in U_{2}^{+} \otimes V_{2}^{+} \otimes W_{2}^{+}$,

$$
\operatorname{rank}_{+}(A \oplus B)=\operatorname{rank}_{+}(A)+\operatorname{rank}_{+}(B)
$$

Proof. Fix a basis for each vector space and let $a_{i j k}$ and $b_{i^{\prime} j^{\prime} k^{\prime}}$ denote the coordinates of $A$ and $B$. Note that $(A \oplus B)_{i j k}=a_{i j k},(A \oplus B)_{i^{\prime} j^{\prime} k^{\prime}}=b_{i^{\prime} j^{\prime} k^{\prime}}$ and other terms are zero. Suppose that $r:=\operatorname{rank}_{+}(A \oplus B)<\operatorname{rank}_{+}(A)+\operatorname{rank}_{+}(B)$. Let $A \oplus B=\sum_{i=1}^{r} u_{i} \otimes v_{i} \otimes w_{i}$. Then at least one of the summands $u_{i} \otimes v_{i} \otimes w_{i}$ is neither in $U_{1}^{+} \otimes V_{1}^{+} \otimes W_{1}^{+}$nor in $U_{2}^{+} \otimes V_{2}^{+} \otimes W_{2}^{+}$. So without loss of generality we may assume that $u_{1} \in\left(U_{1} \oplus U_{2}\right)^{+} \backslash\left(U_{1}^{+} \oplus\{0\} \cup\{0\} \oplus U_{2}^{+}\right)$. Thus at least one of the following indices

$$
\left(i, j^{\prime}, k\right),\left(i, j, k^{\prime}\right),\left(i, j^{\prime}, k^{\prime}\right),\left(i^{\prime}, j, k^{\prime}\right),\left(i^{\prime}, j^{\prime}, k\right),\left(i^{\prime}, j, k\right)
$$

which we denote by $(\alpha, \beta, \gamma)$, will be such that $(A \oplus B)_{\alpha \beta \gamma}$ is positive, a contradiction.

We may also deduce the following, clearly also true for $d>3$, from the above proof.

Corollary 4.2. If $A$ and $B$ have unique nonnegative rank decompositions in $U_{1}^{+} \otimes$ $V_{1}^{+} \otimes W_{1}^{+}$and $U_{2}^{+} \otimes V_{2}^{+} \otimes W_{2}^{+}$respectively, then $A \oplus B$ also has a unique nonnegative rank decomposition.

For a real tensor $A \in \mathbb{R}^{m_{1} \times \cdots \times m_{d}} \subseteq \mathbb{R}^{n_{1} \times \cdots \times n_{d}}$, the real rank of $A$ regarded as a tensor in $\mathbb{R}^{m_{1} \times \cdots \times m_{d}}$ equals the real rank of $A$ regarded as a tensor in $\mathbb{R}^{n_{1} \times \cdots \times n_{d}}$ [26, Proposition 3.1]. As a corollary of Lemma 4.1, we see that this also holds for nonnegative rank.

In the following, let $U_{1} \subseteq U_{2}, V_{1} \subseteq V_{2}$, and $W_{1} \subseteq W_{2}$ be inclusions of real vector spaces. Choose bases for $U_{2}, V_{2}$, and $W_{2}$ such that $u \in U_{1}$ has coordinates $u=\left(u_{1}, \ldots, u_{m_{1}}, 0, \ldots, 0\right)$ as a vector in $U_{2}$; likewise for $V_{2}$ and $W_{2}$. Then we have
the following corollary, which is stated for $d=3$, but can be easily generalized to arbitrary $d>3$.

Corollary 4.3. Let $A \in U_{1}^{+} \otimes V_{1}^{+} \otimes W_{1}^{+} \subseteq U_{2}^{+} \otimes V_{2}^{+} \otimes W_{2}^{+}$. Then the nonnegative rank of $A$ regarded as a nonnegative tensor in $U_{1}^{+} \otimes V_{1}^{+} \otimes W_{1}^{+}$is the same as the nonnegative rank of $A$ regarded as a nonnegative tensor in $U_{2}^{+} \otimes V_{2}^{+} \otimes W_{2}^{+}$.

Proof. Let $U_{1}^{\prime} \subseteq U_{2}$ be a complementary subspace of $U_{1}$, i.e., $U_{2}=U_{1} \oplus U_{1}^{\prime}$. So $u^{\prime} \in$ $U_{1}^{\prime}$ has coordinates $u^{\prime}=\left(0, \ldots, 0, u_{m_{1}+1}^{\prime}, \ldots, u_{m_{2}}^{\prime}\right)$ as a vector in $U_{2}$. Likewise, we let $V_{1}^{\prime} \subseteq V_{2}$ and $W_{1}^{\prime} \subseteq W_{2}$ be complementary subspaces of $V_{1}$ and $W_{1}$. The required statement then follows from applying Lemma 4.1 to the case $A \in U_{1}^{+} \otimes V_{1}^{+} \otimes W_{1}^{+}$ and $B:=0 \in U_{1}^{\prime+} \otimes V_{1}^{\prime+} \otimes W_{1}^{\prime+}$.

The following simple observation is a nonnegative analogue of [26, Corollary 3.3]. We assume that we fix a basis for each $V_{i}$ so that $V_{i}^{+}$is defined, $i=1, \ldots, d$.

Proposition 4.4. For any $k \in\{2, \ldots, d-1\}$, let $A \in V_{1}^{+} \otimes \cdots \otimes V_{k}^{+}$be arbitrary and let $u_{k+1} \in V_{k+1}^{+}, \ldots, u_{d} \in V_{d}^{+}$be nonzero. Then

$$
\operatorname{rank}_{+}(A)=\operatorname{rank}_{+}\left(A \otimes u_{k+1} \otimes \cdots \otimes u_{d}\right)
$$

Proof. The isomorphism of $\mathbb{R}_{+}$-cones,

$$
V_{1}^{+} \otimes \cdots \otimes V_{k}^{+} \cong V_{1}^{+} \otimes \cdots \otimes V_{k}^{+} \otimes \operatorname{span}_{\mathbb{R}_{+}}\left(u_{k+1}\right) \otimes \cdots \otimes \operatorname{span}_{\mathbb{R}_{+}}\left(u_{d}\right)
$$

given by $A \mapsto A \otimes u_{k+1} \otimes \cdots \otimes u_{d}$ implies the required equality.

## 5. General equivalence of complex, real, and nonnegative ranks

It is well-known that a real tensor may have different real and complex ranks. Likewise a nonnegative tensor may also have different nonnegative and real ranks. In fact, strict inequality can also occur for the nonnegative and real ranks of a nonnegative matrix, a well-known example was provided by H. Robbins [22].

For the case of 3 -tensors, two explicit examples are as follows. Let $e_{1}, e_{2} \in \mathbb{R}^{2}$ be the standard basis vectors, i.e., $e_{1}=[1,0]^{\top}$, $e_{2}=[0,1]^{\top}$. Let

$$
\begin{align*}
& A=e_{1} \otimes e_{1} \otimes e_{1}+e_{2} \otimes e_{2} \otimes e_{1}+e_{1} \otimes e_{2} \otimes e_{2}+e_{2} \otimes e_{1} \otimes e_{2}  \tag{5.1}\\
& B=e_{1} \otimes e_{1} \otimes e_{1}-e_{1} \otimes e_{2} \otimes e_{2}+e_{2} \otimes e_{1} \otimes e_{2}+e_{2} \otimes e_{2} \otimes e_{1}
\end{align*}
$$

Then $A \in \mathbb{R}_{+}^{2 \times 2 \times 2} \subseteq \mathbb{R}^{2 \times 2 \times 2}$ and $B \in \mathbb{R}^{2 \times 2 \times 2} \subseteq \mathbb{C}^{2 \times 2 \times 2}$. We have

$$
\begin{aligned}
\operatorname{rank}_{\mathbb{C}}(A)=\operatorname{rank}_{\mathbb{R}}(A)=2<4 & =\operatorname{rank}_{+}(A), \\
\operatorname{rank}_{\mathbb{C}}(B) & =2<3=\operatorname{rank}_{\mathbb{R}}(B) .
\end{aligned}
$$

See Section 6 for the nonnegative, real, and complex ranks of $A$ and [26] for the real and complex ranks of $B$. We will show in this section that this does not happen for a general nonnegative tensor of nonnegative rank strictly less than the complex generic rank - its nonnegative, real, and complex ranks will all be equal.

For notational simplicity we focus on 3-tensors, although many of the statements and proofs in this section can be generalized without difficulty to $d$-tensors for any $d>3$. Let $U, V$ and $W$ be real vector spaces of dimensions $n_{U}, n_{V}$ and $n_{W}$ respectively. Denote by $V_{\mathbb{C}}$ the complexification of $V$, i.e., $V_{\mathbb{C}}=V \otimes_{\mathbb{R}} \mathbb{C}$.

We define the polynomial map

$$
\begin{align*}
& \Sigma_{r}^{\mathbb{C}}:\left(U_{\mathbb{C}} \times V_{\mathbb{C}} \times W_{\mathbb{C}}\right)^{r} \rightarrow U_{\mathbb{C}} \otimes V_{\mathbb{C}} \otimes W_{\mathbb{C}} \\
&\left(u_{1}, v_{1}, w_{1}, \ldots, u_{r}, v_{r}, w_{r}\right) \mapsto \sum_{i=1}^{r} u_{i} \otimes v_{i} \otimes w_{i} \tag{5.2}
\end{align*}
$$

and denote the restriction of $\Sigma_{r}^{\mathbb{C}}$ to $(U \times V \times W)^{r}$ by $\Sigma_{r}^{\mathbb{R}}$, and the restriction to $\left(U_{+} \times V_{+} \times W_{+}\right)^{r}$ by $\Sigma_{r}^{\mathbb{R}_{+}}$. We have the following commutative diagram:


Henceforth, we will use the following abbreviated notation when specifying an element of $(U \times V \times W)^{r}$,

$$
\begin{equation*}
\left(u_{1}, \ldots, w_{r}\right):=\left(u_{1}, v_{1}, w_{1}, \ldots, u_{r}, v_{r}, w_{r}\right) \tag{5.4}
\end{equation*}
$$

Then we have

$$
\operatorname{Im} \Sigma_{r}^{\mathbb{R}_{+}}=D_{r}:=\left\{A \in U_{+} \otimes V_{+} \otimes W_{+}: \operatorname{rank}_{+}(A) \leq r\right\}
$$

The notation is consistent with Proposition 3.7, which also implies that $\operatorname{Im} \Sigma_{r}^{\mathbb{R}_{+}}$is closed. Note that $\operatorname{Im} \Sigma_{r}^{\mathbb{R}}$ and $\operatorname{Im} \Sigma_{r}^{\mathbb{C}}$ are usually not closed.

As in Definition 3.3, if $r_{g}$ is the complex generic rank of $U_{\mathbb{C}} \otimes V_{\mathbb{C}} \otimes W_{\mathbb{C}}$, then the set of rank- $r_{g}$ tensors contains a Zariski open subset. Put in another way, the complex generic rank is the minimum $r$ such that the morphism $\Sigma_{r}^{\mathbb{C}}$ is dominant. As we mentioned earlier, the result [10, Theorem 2] shows that the complex generic rank equals the minimum real typical rank.

The expected dimension of $\operatorname{Im} \Sigma_{r}^{\mathbb{R}}$ is $\min \left\{r\left(n_{U}+n_{V}+n_{W}-2\right), n_{U} n_{V} n_{W}\right\}$ and thus the expected complex generic rank is

$$
\left\lceil\frac{n_{U} n_{V} n_{W}}{n_{U}+n_{V}+n_{W}-2}\right\rceil
$$

which is at least $r_{g}$.
Definition 5.1. If $\operatorname{dim}\left(\operatorname{Im} \Sigma_{r}^{\mathbb{R}}\right)<\min \left\{r\left(n_{U}+n_{V}+n_{W}-2\right), n_{U} n_{V} n_{W}\right\}$, then $U \otimes V \otimes W$ is called $r$-defective over $\mathbb{R}$.

The definition of defectivity over $\mathbb{C}$, i.e., identical to Definition 5.1 but with $U, V, W$ being complex vector spaces, is classical in algebraic geometry [59]. More generally, a complex projective variety $X$ is called $r$-defective [17] if the $r$ th secant variety of $X$ does not have the expected dimension. In our context this is equivalent to $\operatorname{dim}_{\mathbb{C}}\left(\operatorname{Im} \Sigma_{r}^{\mathbb{C}}\right)<\min \left\{r\left(n_{U}+n_{V}+n_{W}-2\right), n_{U} n_{V} n_{W}\right\}$. Note that if $U \otimes V \otimes W$ is $r$-identifiable, then $U \otimes V \otimes W$ is not $r$-defective.

Lemma 5.2. Let $r<r_{g}$. Then a general $A \in D_{r}$ has real rank $r$.

Proof. Let the Jacobian of $\Sigma_{r}^{\mathbb{R}}$ be $\nabla \Sigma_{r}^{\mathbb{R}}$. If $\operatorname{rank}\left(\nabla \Sigma_{r-1}^{\mathbb{R}}\right)=\operatorname{rank}\left(\nabla \Sigma_{r}^{\mathbb{R}}\right)$ at general points, then inductively,

$$
\operatorname{rank}\left(\nabla \Sigma_{r-1}^{\mathbb{R}}\right)=\operatorname{rank}\left(\nabla \Sigma_{r}^{\mathbb{R}}\right)=\operatorname{rank}\left(\nabla \Sigma_{r+1}^{\mathbb{R}}\right)=\cdots
$$

at general points, which implies that

$$
\operatorname{dim}\left(\operatorname{Im} \Sigma_{r-1}^{\mathbb{R}}\right)=\operatorname{dim}\left(\operatorname{Im} \Sigma_{r}^{\mathbb{R}}\right)=\cdots=n_{U} n_{V} n_{W}
$$

Hence if $r<r_{g}, \operatorname{rank}\left(\nabla \Sigma_{r-1}^{\mathbb{R}}\right)<\operatorname{rank}\left(\nabla \Sigma_{r}^{\mathbb{R}}\right)$ at general points, implying that

$$
\operatorname{dim}\left(\operatorname{Im} \Sigma_{r-1}^{\mathbb{R}}\right)<\operatorname{dim}\left(\operatorname{Im} \Sigma_{r}^{\mathbb{R}}\right)
$$

On the other hand, since $\left(U_{+} \times V_{+} \times W_{+}\right)^{r}$ contains an open subset of $(U \times V \times W)^{r}$, by Lemma 2.1, $\nabla \Sigma_{r}^{\mathbb{R}_{+}}=\nabla \Sigma_{r}^{\mathbb{R}}$ at a general point, $\operatorname{Im} \Sigma_{r}^{\mathbb{R}_{+}}$contains an open subset of $\operatorname{Im} \Sigma_{r}^{\mathbb{R}}$, i.e.,

$$
\begin{aligned}
\operatorname{dim}\left(D_{r-1}\right)=\operatorname{dim}\left(\operatorname{Im} \Sigma_{r-1}^{\mathbb{R}_{+}}\right)=\operatorname{dim} & \left(\operatorname{Im} \Sigma_{r-1}^{\mathbb{R}}\right) \\
& <\operatorname{dim}\left(\operatorname{Im} \Sigma_{r}^{\mathbb{R}}\right)=\operatorname{dim}\left(\operatorname{Im} \Sigma_{r}^{\mathbb{R}_{+}}\right)=\operatorname{dim}\left(D_{r}\right)
\end{aligned}
$$

Thus a general $A \in D_{r}$ has nonnegative rank $r$, and the real rank of $A$ is also $r$.
We now relate real rank to complex rank (and later to nonnegative rank) via general relations between real algebraic varieties and their complexifications. For a field of characteristic zero $\mathbb{K}$, we write $\mathbb{K} \mathbb{P}^{n}$ for the projective space of dimension $n$ over $\mathbb{K}$. As we briefly mentioned after Definition 3.2, the affine cone of a projective variety $X \subseteq \mathbb{K} \mathbb{P}^{n}$ is the affine variety

$$
\widehat{X}:=\left\{x \in \mathbb{K}^{n+1}: \pi(x) \in X\right\} \cup\{0\}=\pi^{-1}(X) \cup\{0\}
$$

where $\pi: \mathbb{K}^{n+1} \rightarrow \mathbb{K}^{n}$ is the natural projection that takes a point $x \in \mathbb{K}^{n+1}$ to the equivalence class $\pi(x)=\left\{\lambda x \in \mathbb{K}^{n+1}: \lambda \in \mathbb{K}^{\times}\right\} \in \mathbb{K}^{n}$.

Definition 5.3. Let $X, Y \subseteq \mathbb{K}^{n}$ be projective varieties. Let $\varphi: \widehat{X} \times \widehat{Y} \rightarrow \mathbb{K}^{n+1}$, $(x, y) \mapsto x+y$. The join of $X$ and $Y$ is the projective variety $J(X, Y) \subseteq \mathbb{K}^{n}$ whose affine cone is the Zariski closure of the image $\varphi(\widehat{X} \times \widehat{Y}) \subseteq \mathbb{K}^{n}$. The $k$ th secant variety of $X$ is the projective variety defined by

$$
\sigma_{k}^{\mathbb{K}}(X):= \begin{cases}J(X, X) & \text { if } k=2 \\ J\left(X, \sigma_{k-1}^{\mathbb{K}}(X)\right) & \text { if } k>2\end{cases}
$$

We define
$\operatorname{Var}\left(\mathbb{R}^{n}\right):=\left\{X \subseteq \mathbb{R} \mathbb{P}^{n}: X\right.$ a real projective variety that is
(i) irreducible, (ii) nondegenerate, (iii) Zariski dense in $X(\mathbb{C})\}$

Let $I(X) \subseteq \mathbb{R}\left[x_{0}, \ldots, x_{n}\right]$ be the homogeneous ideal of $X$ and $r_{g}(X)$ be the complex generic $X$-rank. Standard elimination theory (see [52, Section 2.1] and [10, Section 2.2]) yields the following relation between a real secant variety and its complexification.
Lemma 5.4. Let $X \in \operatorname{Var}\left(\mathbb{R P}^{n}\right)$ and $r<r_{g}(X)$. Then there exists a set of homogeneous generators $f_{1}, \ldots, f_{m}$ of the ideal $I\left(\sigma_{r}^{\mathbb{R}}(X)\right)$ that also generates the ideal $I\left(\sigma_{r}^{\mathbb{C}}(X(\mathbb{C}))\right)$. In particular, $\sigma_{r}^{\mathbb{C}}(X(\mathbb{C}))$ is the complexification of $\sigma_{r}^{\mathbb{R}}(X)$.

It is also not difficult to see the following relation between smooth points on a real secant variety and general points on its complexification.

Lemma 5.5. Let $X \in \operatorname{Var}\left(\mathbb{R}^{n}\right)$ and $r<r_{g}(X)$. Then $\sigma_{r}^{\mathbb{R}}(X) \in \operatorname{Var}\left(\mathbb{R P}^{n}\right)$.
Proof. It suffices to show that at least one point in $\sigma_{r}^{\mathbb{R}}(X)$ is a smooth point in $\sigma_{r}^{\mathbb{C}}(X(\mathbb{C}))$. Suppose not. Then $\sigma_{r}^{\mathbb{R}}(X)$ is in the singular locus of $\sigma_{r}^{\mathbb{C}}(X(\mathbb{C}))$. Let $k=\operatorname{dim} \sigma_{r}^{\mathbb{C}}(X(\mathbb{C}))$. Then $\sigma_{r}^{\mathbb{R}}(X)$ satisfies the equations given by the vanishing of the $(n-k) \times(n-k)$ minors of

$$
\left[\begin{array}{ccc}
\partial f_{1} / \partial x_{0} & \cdots & \partial f_{1} / \partial x_{n} \\
\vdots & \ddots & \vdots \\
\partial f_{m} / \partial x_{0} & \cdots & \partial f_{m} / \partial x_{n}
\end{array}\right]
$$

which are defined over $\mathbb{R}$. On the other hand, these minors are not all in $I\left(\sigma_{r}^{\mathbb{R}}(X)\right)$ as $\sigma_{r}^{\mathbb{R}}(X)$ itself has at least one real smooth point - a contradiction. Hence at least one point in $\sigma_{r}^{\mathbb{R}}(X)$ is a smooth point of $\sigma_{r}^{\mathbb{C}}(X(\mathbb{C}))$.

By [2, Corollary 1.8], $\sigma_{r-1}^{\mathbb{C}}(X(\mathbb{C}))$ is in the singular locus of $\sigma_{r}^{\mathbb{C}}(X(\mathbb{C}))$. Applying this to $X=\operatorname{Seg}(\mathbb{P} U \times \mathbb{P} V \times \mathbb{P} W)$, the Segre variety of rank-one tensors, we obtain the following from Lemma 5.5 .

Lemma 5.6. Let $r<r_{g}$. Then a general real tensor $A$ of real rank $r$ has complex rank r.

Theorem 5.7. Let $r<r_{g}$. Then a general $A \in D_{r}$ has both real rank and complex rank equal to $r$. If $U \otimes V \otimes W$ is r-identifiable, then $A$ has a unique nonnegative rank-r decomposition.

Proof. The claims about ranks are just Lemmas 5.2 and 5.6. Since $D_{r}$ contains an open subset of $\operatorname{Im} \Sigma_{r}^{\mathbb{R}}$, a general point in $D_{r}$ has a unique rank- $r$ decomposition.

There has been a significant amount of work on both defectivity [56, 43, 1] and identifiability [40, [54, 19, 27, 28, ,12, 21, 29]. While these focus mainly on complex tensors, some of these methods can be also adapted to real tensors. Two notable examples are [19, Theorem 1.1] and [29, Proposition 1.6], stated below for real tensors.

Theorem 5.8 (Chiantini-Ottaviani). Let $U, V$, and $W$ be real vector spaces with dimensions $\operatorname{dim} U \leq \operatorname{dim} V \leq \operatorname{dim} W$. Let $\alpha, \beta$ be minimum integers such that $2^{\alpha} \leq \operatorname{dim} U$ and $2^{\beta} \leq \operatorname{dim} V$. Then $U \otimes V \otimes W$ is $r$-identifiable if $r \leq 2^{\alpha+\beta-2}$.

Theorem 5.9 (Domanov-De Lathauwer). Let $U, V$, and $W$ be real vector spaces with dimensions $\operatorname{dim} U=m, \operatorname{dim} V=n$, and $\operatorname{dim} W=p$. If

$$
2 \leq m \leq n \leq p \leq r \quad \text { and } \quad 2 r \leq m+n+2 p-2-\sqrt{(m-n)^{2}+4 p}
$$

then $U \otimes V \otimes W$ is $r$-identifiable.
Applying Theorem 5.8 to Theorem 5.7, we obtain explicit examples.
Corollary 5.10. Let $n \geq 4$ and $r \leq\left\lfloor n^{2} / 16\right\rfloor$. A general $A \in \mathbb{R}_{+}^{n \times n \times n}$ with $\operatorname{rank}_{+}(A)=r$ has complex rank $r$ (and therefore real rank $r$ ) and a unique nonnegative rank- $r$ decomposition.

In fact we may also derive identifiability results for real tensors from the identifiability results for complex tensors.

Lemma 5.11. Let $X \in \operatorname{Var}\left(\mathbb{R}^{n}\right)$ and $r<r_{g}(X)$. If a general point in $\sigma_{r}^{\mathbb{C}}(X(\mathbb{C}))$ has a unique rank-r decomposition, then a general point in $\sigma_{r}^{\mathbb{R}}(X)$ has a unique complex rank-r decomposition.
Proof. Suppose not, then there is some nonempty Euclidean open subset $\mathcal{U}$ of $\sigma_{r}^{\mathbb{R}}(X)$ such that any point in $\mathcal{U}$ does not have a unique complex rank- decomposition. By assumption, the set of points in $\sigma_{r}^{\mathbb{C}}(X(\mathbb{C}))$ that do not have unique rank-r decompositions is contained in a subvariety $Y \subseteq \sigma_{r}^{\mathbb{C}}(X(\mathbb{C}))$. Then $\mathcal{U} \subset Y$, and so the Zariski closure of $\mathcal{U}$, i.e., $\sigma_{r}^{\mathbb{R}}(X)$, is contained in $Y$. But by Lemma 5.5, $\sigma_{r}^{\mathbb{R}}(X)$ is Zariski dense in $\sigma_{r}^{\mathbb{C}}(X(\mathbb{C}))$, a contradiction.

Lemma 5.11 does not guarantee that a general point in $\sigma_{r}^{\mathbb{R}}(X)$ has a unique real rank- $r$ decomposition as there may be a Euclidean open subset in $\sigma_{r}^{\mathbb{R}}(X)$ where every point has real rank greater than $r$. We now apply Lemma 5.11 to the case $X=\operatorname{Seg}(\mathbb{P} U \times \mathbb{P} V \times \mathbb{P} W)$.
Theorem 5.12. Let $U, V$, and $W$ be real vector spaces and let $r<r_{g}$. If $U_{\mathbb{C}} \otimes$ $V_{\mathbb{C}} \otimes W_{\mathbb{C}}$ is r-identifiable, then $U \otimes V \otimes W$ is $r$-identifiable.
Proof. If $U_{\mathbb{C}} \otimes V_{\mathbb{C}} \otimes W_{\mathbb{C}}$ is $r$-identifiable, then a general point in $\sigma_{r}^{\mathbb{C}}\left(\operatorname{Seg}\left(\mathbb{P} U_{\mathbb{C}} \times \mathbb{P} V_{\mathbb{C}} \times\right.\right.$ $\left.\mathbb{P} W_{\mathbb{C}}\right)$ ) has a unique complex rank- $r$ decomposition. By Lemma 5.11 a general point in $\sigma_{r}^{\mathbb{R}}(\operatorname{Seg}(\mathbb{P} U \times \mathbb{P} V \times \mathbb{P} W))$ has a unique complex rank- $r$ decomposition. Since $\operatorname{Im} \Sigma_{r}^{\mathbb{R}}$ contains a Euclidean open subset of $\sigma_{r}^{\mathbb{R}}(\operatorname{Seg}(\mathbb{P} U \times \mathbb{P} V \times \mathbb{P} W)$ ), a general point $A \in \operatorname{Im} \Sigma_{r}^{\mathbb{R}}$ has real rank $r$ and a unique complex rank- $r$ decomposition. By Lemma 5.6, $A$ has complex rank $r$; and so the unique complex rank- $r$ decomposition of $A$ is in fact its unique real rank- $r$ decomposition. Therefore $U \otimes V \otimes W$ is $r$ identifiable.

A consequence of Theorem 5.12 is the following corollary of [21, Theorem 1.1].
Corollary 5.13. Let $n_{1} \geq \cdots \geq n_{d}$ and

$$
r_{0}=\left\lceil\frac{\prod_{i=1}^{d} n_{i}}{1+\sum_{i=1}^{d}\left(n_{i}-1\right)}\right\rceil
$$

Then $\mathbb{R}^{n_{1} \times \cdots \times n_{d}}$ is $r$-identifiable for $r<r_{0}$ if $\prod_{i=1}^{d} n_{i} \leq 15000$ and $\left(n_{1}, \ldots, n_{d}, r\right)$ is not one of the following cases:

| $\left(n_{1}, \ldots, n_{d}\right)$ | $r$ |
| :---: | :---: |
| $(4,4,3)$ | 5 |
| $(4,4,4)$ | 6 |
| $(6,6,3)$ | 8 |
| $(n, n, 2,2)$ | $2 n-1$ |
| $(2,2,2,2,2)$ | 5 |
| $n_{1}>\prod_{i=2}^{d} n_{i}-\sum_{i=2}^{d}\left(n_{i}-1\right)$ | $r \geq \prod_{i=2}^{d} n_{i}-\sum_{i=2}^{d}\left(n_{i}-1\right)$ |

By Lemma 5.5] we may also apply the algorithm proposed in [21] for complex tensors to directly test if a general real tensor of real rank-r or a general nonnegative tensor of nonnegative rank- $r$ has a unique complex rank- $r$ decomposition. The sufficient condition to ensure the smoothness of a specific complex tensor in [21, Lemma 5.1] may also be adapted to real tensors.

This discussion would not be complete without examples of non-identifiability cases. As most of the non-identifiability cases in the literature are for the complex case, we provide a result that allows us to translate them to the real case.

Lemma 5.14. Let $V_{1}, \ldots, V_{d}$ be real vector spaces of dimensions $n_{1}, \ldots, n_{d}$ respectively. Let $U_{1}, \ldots, U_{d}$ be their complexifications, i.e., $U_{i}=V_{i} \otimes_{\mathbb{R}} \mathbb{C}, i=1, \ldots, d$. If $U_{1} \otimes \cdots \otimes U_{d}$ is $r$-defective and $r<r_{g}$, then $V_{1} \otimes \cdots \otimes V_{d}$ is also $r$-defective.

Proof. Let $A=\sum_{i=1}^{r} v_{i}^{(1)} \otimes \cdots \otimes v_{i}^{(d)} \in V_{1} \otimes \cdots \otimes V_{d}$ be a general real rank- $r$ tensor. Let

$$
X:=\operatorname{Seg}\left(\mathbb{P} V_{1} \times \cdots \times \mathbb{P} V_{d}\right) \quad \text { and } \quad X(\mathbb{C}):=\operatorname{Seg}\left(\mathbb{P} U_{1} \times \cdots \times \mathbb{P} U_{d}\right)
$$

By our semialgebraic Terracini's lemma, i.e., Lemma 3.6

$$
\mathrm{T}_{A} \widehat{\sigma}_{r}^{\mathbb{R}}(X)=\operatorname{span}_{\mathbb{R}}\left\{V_{1} \otimes v_{1}^{(2)} \otimes \cdots \otimes v_{1}^{(d)}, \ldots, v_{r}^{(1)} \otimes \cdots \otimes v_{r}^{(d-1)} \otimes V_{d}\right\}
$$

By Lemma 5.5. $A$ is a smooth point of $\sigma_{r}^{\mathbb{C}}(X(\mathbb{C}))$, and thus by the usual complex Terracini's lemma,

$$
\mathrm{T}_{A} \widehat{\sigma}_{r}^{\mathbb{C}}(X(\mathbb{C}))=\operatorname{span}_{\mathbb{C}}\left\{U_{1} \otimes v_{1}^{(2)} \otimes \cdots \otimes v_{1}^{(d)}, \ldots, v_{r}^{(1)} \otimes \cdots \otimes v_{r}^{(d-1)} \otimes U_{d}\right\}
$$

By assumption,

$$
\operatorname{dim}_{\mathbb{C}} \mathbf{T}_{A} \widehat{\sigma}_{r}^{\mathbb{C}}(X(\mathbb{C}))<r\left(n_{1}+\cdots+n_{d}-d+1\right)
$$

i.e., there exist $u_{1}^{(k)}, \ldots, u_{r}^{(k)} \in U_{i}$ with $\left[u_{i}^{(k)}\right] \neq\left[v_{i}^{(k)}\right] \in \mathbb{P} U_{i}$ for $k=1, \ldots, d$, $i=1, \ldots, r$, and

$$
u_{1}^{(1)} \otimes v_{1}^{(2)} \otimes \cdots \otimes v_{1}^{(d)}+\cdots+v_{r}^{(1)} \otimes \cdots \otimes v_{r}^{(d-1)} \otimes u_{r}^{(d)}=0
$$

By taking the real part or the imaginary part of each $u_{i}^{(k)}$, we have $\operatorname{dim}_{\mathbb{R}} \mathrm{T}_{A} \widehat{\sigma}_{r}^{\mathbb{R}}(X)<$ $r\left(n_{1}+\cdots+n_{d}-d+1\right)$, i.e., $V_{1} \otimes \cdots \otimes V_{d}$ is $r$-defective.

Using the corresponding results for complex tensors in [1, 12] and Lemma 5.14 we deduce the following nonuniqueness result for real tensors.

Theorem 5.15. (i) $\mathbb{R}^{4 \times 4 \times 3}$ is 5-defective. So a general $4 \times 4 \times 3$ real tensor of real rank 5 does not have a unique rank-5 decomposition over $\mathbb{R}$.
(ii) For any $n \geq 2, \mathbb{R}^{n \times n \times 2 \times 2}$ is ( $2 n-1$ )-defective. So a general $n \times n \times n \times 2$ real tensor of real rank $2 n-1$ does not have a unique rank- $(2 n-1)$ decomposition over $\mathbb{R}$.
(iii) For $n_{1} \geq \cdots \geq n_{d} \geq 2$, $\mathbb{R}^{n_{1} \times \cdots \times n_{d}}$ is $r$-defective if

$$
n_{1}>\prod_{i=2}^{d} n_{i}-\sum_{i=2}^{d}\left(n_{i}-1\right) \quad \text { and } \quad r \geq \prod_{i=2}^{d} n_{i}-\sum_{i=2}^{d}\left(n_{i}-1\right)
$$

So a general $\left(n_{1} \times \cdots \times n_{d}\right)$-real tensor of real rank $r<r_{g}$ does not have a unique rank-r decomposition over $\mathbb{R}$.

A complex analogue of Theorem 5.15 may be found in [21, Theorem 1.1].
We may also apply the techniques in this section to obtain analogous results for real symmetric tensors. We will denote the set of real or complex symmetric $d$ tensors by $S^{d}\left(\mathbb{R}^{n}\right)$ or $S^{d}\left(\mathbb{C}^{n}\right)$ respectively. We say $S^{d}\left(\mathbb{C}^{n}\right)$ is $r$-identifiable if a general symmetric rank- $r$ tensor in $\mathrm{S}^{d}\left(\mathbb{C}^{n}\right)$ has a unique symmetric rank decomposition (also known as Waring decomposition). Applying Lemma5.11 to $X=\nu_{d}\left(\mathbb{R} \mathbb{P}^{n}\right)$, the Veronese variety of symmetric rank-one symmetric tensors, we deduce the following.

Theorem 5.16. Let $r<r_{g}\left(\nu_{d}\left(\mathbb{R P}^{n}\right)\right)$. If $\mathrm{S}^{d}\left(\mathbb{C}^{n+1}\right)$ is $r$-identifiable, then $\mathrm{S}^{d}\left(\mathbb{R}^{n+1}\right)$ is r-identifiable.

When $r<r_{g}\left(\nu_{d}\left(\mathbb{R} \mathbb{P}^{n}\right)\right)$, the $r$-identifiability of $\mathrm{S}^{d}\left(\mathbb{C}^{n+1}\right)$ has been completely determined for all values of $r, d, n$ [20, Theorem 1.1]; this together with Lemma 5.11 gives us the following.
Corollary 5.17. $\mathrm{S}^{d}\left(\mathbb{R}^{n+1}\right)$ is $r$-identifiable when

$$
r<\left\lceil\frac{\binom{n+d}{d}}{n+1}\right\rceil
$$

and if $(d, n, r) \notin\{(6,2,9),(4,3,8),(3,5,9)\}$.
Proof. This follows from [18, [6, Theorem 1.1], 47, Theorem 4.1], and [20, Theorem 1.1].

## 6. Typical and maximum nonnegative ranks

In this section, we investigate typical, maximum, and maximum nonnegative typical ranks, as defined in Definition 3.3. The following rephrases [45, Proposition 6.2] in the context of this article and may be viewed as a generalization of [11, Theorem 3.1].
Proposition 6.1. Let $A \in U_{+} \otimes V_{+} \otimes W_{+}$with $\operatorname{rank}_{+}(A)=r$. Then there is an open ball $B(A, \varepsilon) \subseteq U \otimes V \otimes W$ such that

$$
\operatorname{rank}_{+}\left(A^{\prime}\right) \geq r
$$

for all $A^{\prime} \in B(A, \varepsilon) \cap U_{+} \otimes V_{+} \otimes W_{+}$.
It follows immediately that the maximum nonnegative typical rank and the maximum nonnegative rank always coincide.

Lemma 6.2. If $r$ is the maximum nonnegative rank of $U_{+} \otimes V_{+} \otimes W_{+}$, then $r$ is the maximum nonnegative typical rank.

What about the minimum nonnegative typical rank then? It turns out that it is always equal to the (complex) generic rank.
Lemma 6.3. The minimum nonnegative typical rank of $U_{+} \otimes V_{+} \otimes W_{+}$is the complex generic rank $r_{g}$ of $U_{\mathbb{C}} \otimes V_{\mathbb{C}} \otimes W_{\mathbb{C}}$.

Proof. Since $\left(U_{+} \times V_{+} \times W_{+}\right)^{r}$ contains an open subset of $(U \times V \times W)^{r}$, by Lemma 2.1] $\operatorname{rank}\left(\nabla \Sigma_{r}^{\mathbb{R}_{+}}\right)=\operatorname{rank}\left(\nabla \Sigma_{r}^{\mathbb{R}^{2}}\right)$ at general points. Hence $\operatorname{dim} \operatorname{Im}\left(\Sigma_{r}^{\mathbb{R}_{+}}\right)=$ $\operatorname{dim} \operatorname{Im}\left(\sum_{r}^{\mathbb{R}}\right)$, which implies that $r_{g}$ is the minimum nonnegative typical rank.

We will illustrate these with a $2 \times 2 \times 2$ example. In this case, the complex generic rank of $\mathbb{C}^{2 \times 2 \times 2}$ is 2 and the real typical ranks of $\mathbb{R}^{2 \times 2 \times 2}$ are 2 and 3 [26. By Lemmas 3.4, 6.2, and 6.3, to completely determine the nonnegative typical ranks of $\mathbb{R}_{+}^{2 \times 2 \times 2}$, it remains to find the maximum nonnegative rank. We will construct a nonnegative tensor with maximum nonnegative rank explicitly. Consider the tensor

$$
\begin{equation*}
A=e_{1} \otimes e_{1} \otimes e_{1}+e_{2} \otimes e_{2} \otimes e_{1}+e_{1} \otimes e_{2} \otimes e_{2}+e_{2} \otimes e_{1} \otimes e_{2} \tag{6.1}
\end{equation*}
$$

that we saw earlier in (5.1). A may be represented by a nonnegative hypermatrix

$$
A=\left[\begin{array}{ll|ll}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{array}\right] \in \mathbb{R}_{+}^{2 \times 2 \times 2}
$$

Now let $A=\sum_{k=1}^{r} x_{k} \otimes y_{k} \otimes z_{k}$ be a nonnegative rank- $r$ decomposition. Then we must be able to write $A=\sum_{k=1}^{r^{\prime}} X_{k} \otimes z_{k}$ where each $X_{k}$ is a nonnegative matrix.

Observe that $z_{k}$ cannot be of the form $\alpha e_{1}+\beta e_{2}$ where $\alpha, \beta>0$. Otherwise by the nonnegativity of each $z_{k}$ and $X_{k}$, there is some $i, j \in\{1,2\}$ such that the $(i, j, 1)$ th coordinate and the $(i, j, 2)$ th coordinate of $A$ are both positive, which contradicts the construction of $A$. Hence we must have $z_{k}=e_{1}$ or $e_{2}$ for all $k=1, \ldots, r^{\prime}$. So without loss of generality we may assume that $z_{1}=e_{1}$ and $z_{2}=e_{2}$. Then $X_{1}=e_{1} \otimes e_{1}+e_{2} \otimes e_{2}$ and $X_{2}=e_{1} \otimes e_{2}+e_{2} \otimes e_{1}$. By the uniqueness of the nonnegative decompositions of $X_{1}$ and $X_{2}$, the nonnegative rank- $r$ decomposition of $A$ in (6.1) is unique. Hence $\operatorname{rank}_{+}(A)=4$. Since any $T \in \mathbb{R}_{+}^{2 \times 2 \times 2}$ has the form $T=Y_{1} \otimes e_{1}+Y_{2} \otimes e_{2}$ where $Y_{1}, Y_{2}$ are nonnegative matrices, and the nonnegative rank of a nonnegative $2 \times 2$ matrix is at most 2 , we may conclude that the nonnegative rank of $T$ is at most 4 . Thus the nonnegative typical ranks of $\mathbb{R}_{+}^{2 \times 2 \times 2}$ are 2,3 , and 4 .

Both the real and complex ranks of $A$ are 2 [26]. In fact for any $A^{\prime}$ in a sufficiently small open ball $B(A, \varepsilon)$, both the real and complex ranks of $A^{\prime}$ are also 2 . If in addition, $A^{\prime} \in B(A, \varepsilon) \cap\left(\mathbb{R}_{+}^{2 \times 2 \times 2}\right)$, then the nonnegative rank of $A^{\prime}$ is 4 . This example can be generalized as follows.
Lemma 6.4. Let $P_{1}, \ldots, P_{n} \in \mathbb{R}_{+}^{n \times n} \cong \mathbb{R}_{+}^{n} \otimes \mathbb{R}_{+}^{n}$ be $n$ permutation matrices such that for each $(i, j) \in\{1, \ldots, n\} \times\{1, \ldots, n\}$, there is one and only one $P_{k}$ whose $(i, j)$ th entry is one. Let $e_{1}, \ldots, e_{n} \in \mathbb{R}_{+}^{n}$ be the standard basis of $\mathbb{R}^{n}$. Define

$$
A=P_{1} \otimes e_{1}+\cdots+P_{n} \otimes e_{n} \in \mathbb{R}_{+}^{n \times n \times n}
$$

Then $\operatorname{rank}_{+}(A)=n^{2}$ and $A$ has a unique nonnegative rank- $n^{2}$ decomposition.
Proof. It suffices to show that $A$ has a unique nonnegative rank- $n^{2}$ decomposition. Suppose

$$
A=\sum_{i=1}^{n^{2}}\left[\sum_{j=1}^{n} \alpha_{i}^{j} e_{j}\right] \otimes\left[\sum_{j=1}^{n} \beta_{i}^{j} e_{j}\right] \otimes\left[\sum_{j=1}^{n} \gamma_{i}^{j} e_{j}\right]
$$

for nonnegative $\alpha_{i}^{j}, \beta_{i}^{j}, \gamma_{i}^{j}$. Without loss of generality, we may assume $\alpha_{1}^{1}, \beta_{1}^{1}, \gamma_{1}^{1} \neq 0$. Since there is only one $P_{k}$ whose $(1,1)$ th entry is nonzero, this $P_{k}$ must be $P_{1}$ and $\gamma_{1}^{j}=0$ for all $j>1$. Repeating this procedure we may show that when we regard $A$ as a nonnegative matrix in $\mathbb{R}_{+}^{n^{2} \times n} \cong \mathbb{R}_{+}^{n \times n} \otimes \mathbb{R}_{+}^{n}$, it has a unique nonnegative matrix factorization given by $A=P_{1} \otimes e_{1}+\cdots+P_{n} \otimes e_{n}$. Since each $P_{k}$ has a unique nonnegative matrix factorization [42], $A$ has a unique nonnegative rank- $n^{2}$ decomposition.

A $d$-tensor in $V_{1} \otimes \cdots \otimes V_{d}$ is said to be cubical if $\operatorname{dim} V_{1}=\cdots=\operatorname{dim} V_{d}$. By [43, Theorem 4.4], [56, Theorem 4.6], Lemmas 3.4, 6.3, 6.2, and 6.4, we completely determine the nonnegative typical ranks of cubical nonnegative tensors.
Proposition 6.5. For $n=2$, the nonnegative typical ranks of $\mathbb{R}_{+}^{2 \times 2 \times 2}$ are given by all integers $m$ where

$$
2 \leq m \leq 4
$$

For $n=3$, the nonnegative typical ranks of $\mathbb{R}_{+}^{3 \times 3 \times 3}$ are given by all integers $m$ where

$$
5 \leq m \leq 9
$$

For $n \geq 4$, the nonnegative typical ranks of $\mathbb{R}_{+}^{n \times n \times n}$ are given by all integers $m$ where

$$
\left\lceil\frac{n^{3}}{3 n-2}\right\rceil \leq m \leq n^{2}
$$

For nonnegative tensors that are not cubical, we may determine the maximum nonnegative typical ranks but since the complex generic ranks for 3-tensors are still not known in some instances, we do not have a complete list of nonnegative typical ranks.

Proposition 6.6. Write maxrank $_{+}(m, n, p)$ for the maximum nonnegative typical rank of $\mathbb{R}_{+}^{m \times n \times p}$ and suppose without loss of generality that $m \geq n \geq p$. Then

$$
\operatorname{maxrank}_{+}(m, n, p)= \begin{cases}n p & \text { if } m=n \geq p \\ n^{2} & \text { if } m \geq n=p \\ n p & \text { if } m>n>p\end{cases}
$$

Proof. The required arguments are as in the proof of Lemma 6.4 but 'padded with the appropriate number of zeros,' i.e., applied to matrices of the form

$$
\left[\begin{array}{c}
P_{k} \\
0
\end{array}\right] \quad \text { or } \quad\left[\begin{array}{ll}
P_{k} & 0
\end{array}\right]
$$

where $P_{k}$ is a permutation matrix.

## 7. General uniqueness of decompositions of approximations

In our previous work [50], we established that a general nonnegative tensor has a unique best nonnegative rank- $r$ approximation. Here we investigate whether this best nonnegative rank- $r$ approximation has a unique nonnegative rank- $r$ decomposition.

Let $U, V, W$ be real vector spaces of dimensions $n_{U}, n_{V}, n_{W}$ respectively. We will assume a choice of basis on these vector spaces, so that $U \cong \mathbb{R}^{n_{U}}, V \cong \mathbb{R}^{n_{V}}$, and $W \cong \mathbb{R}^{n_{W}}$. For a vector $u_{i} \in U$, we let $u_{i, j}$ denote the $j$ th coordinate of $u_{i}$. Likewise for $V$ and $W$. For any smooth curve $\gamma(t), t \in[0,1]$, the right derivative at 0 is denoted by

$$
\gamma^{\prime}(0):=\lim _{t \rightarrow 0^{+}} \frac{\gamma(t)-\gamma(0)}{t-0}
$$

Recall the map $\Sigma_{r}^{\mathbb{R}_{+}}:\left(U_{+} \times V_{+} \times W_{+}\right)^{r} \rightarrow U_{+} \otimes V_{+} \otimes W_{+}$defined in (5.2) and (5.3). The pushforward of $\Sigma_{r}^{\mathbb{R}_{+}}$at $\gamma^{\prime}(0)$ is denoted

$$
\Sigma_{r *}^{\mathbb{R}_{+}}\left(\gamma^{\prime}(0)\right):=\lim _{t \rightarrow 0^{+}} \frac{\Sigma_{r}^{\mathbb{R}_{+}}(\gamma(t))-\Sigma_{r}^{\mathbb{R}_{+}}(\gamma(0))}{t-0}
$$

Let $S_{r} \subseteq U_{+} \otimes V_{+} \otimes W_{+}$denote the set of nonnegative tensors on which the distance function $\operatorname{dist}\left(\cdot, D_{r}\right)$ is not smooth. Then $S_{r}$ contains the nonnegative tensors with non-unique best nonnegative rank-r approximations and is a nowhere dense semialgebraic subset [35]. Let $\pi_{r}: U_{+} \otimes V_{+} \otimes W_{+} \backslash S_{r} \rightarrow D_{r}$ be the map sending a nonnegative tensor to its unique best nonnegative rank- $r$ approximation. Since the distance function $\operatorname{dist}\left(\cdot, D_{r}\right)$ is semialgebraic [24, 35], the graph of $\pi_{r}$,

$$
G\left(\pi_{r}\right)=\left\{(p, q) \in\left(U_{+} \otimes V_{+} \otimes W_{+} \backslash S_{r}\right) \times D_{r}: \operatorname{dist}\left(p, D_{r}\right)=\|p-q\|\right\}
$$

is also semialgebraic. By Proposition 2.6, the subset of points in $U_{+} \otimes V_{+} \otimes W_{+} \backslash S_{r}$ where $\pi_{r}$ is not smooth is contained in a hypersurface $H_{r}$. Henceforth we will focus on the restriction of $\pi_{r}$ (also denoted $\pi_{r}$ with a slight abuse of notation) to a subset of smooth points in $U_{+} \otimes V_{+} \otimes W_{+}$,

$$
\pi_{r}: U_{+} \otimes V_{+} \otimes W_{+} \backslash\left(S_{r} \cup H_{r}\right) \rightarrow D_{r}
$$

In the following the support of a vector $u \in U$ is defined to be

$$
\operatorname{supp}(u):=\left\{i \in\left\{1, \ldots, n_{U}\right\}: u_{i} \neq 0\right\}
$$

The next lemma is a slight rephrase of [50, Lemma 13]. We will use it to partition $D_{r}$ into a union of semialgebraic sets later.

Lemma 7.1. Let $p \in U_{+} \otimes V_{+} \otimes W_{+} \backslash\left(S_{r} \cup H_{r}\right)$ where $\pi_{r}(p)$ has a nonnegative rank-r decomposition

$$
\begin{equation*}
\pi_{r}(p)=\sum_{i=1}^{r} u_{i} \otimes v_{i} \otimes w_{i} \tag{7.1}
\end{equation*}
$$

Then for any $x_{i} \in U_{+}, i=1, \ldots, r$, we have

$$
\begin{equation*}
\left\langle p, x_{i} \otimes v_{i} \otimes w_{i}\right\rangle \leq\left\langle\pi_{r}(p), x_{i} \otimes v_{i} \otimes w_{i}\right\rangle \tag{7.2}
\end{equation*}
$$

where $\langle\cdot, \cdot\rangle$ denotes the Euclidean inner product. With respect to the nonnegative vectors $u_{1}, \ldots, u_{r}$ in (7.1), define the subspaces

$$
\begin{equation*}
\widetilde{U}_{i}:=\left\{u \in U: \operatorname{supp}(u) \subseteq \operatorname{supp}\left(u_{i}\right)\right\} \tag{7.3}
\end{equation*}
$$

for $i=1, \ldots, r$, and define $\widetilde{V}_{i}$ and $\widetilde{W}_{i}$ similarly. Then for $x_{i} \in \widetilde{U}_{i}, i=1, \ldots, r$, we have

$$
\begin{equation*}
\left\langle p, x_{i} \otimes v_{i} \otimes w_{i}\right\rangle=\left\langle\pi_{r}(p), x_{i} \otimes v_{i} \otimes w_{i}\right\rangle . \tag{7.4}
\end{equation*}
$$

The analogous statement for $\widetilde{V}_{i}$ or $\widetilde{W}_{i}$ in place of $\widetilde{U}_{i}$ holds true as well.
We first remind the reader of our abbreviated notation in (5.4). Let

$$
\mathrm{T}_{\pi_{r}(p)}\left(u_{1}, \ldots, w_{r}\right):=\operatorname{span}_{\mathbb{R}}\left(\bigcup_{i=1}^{r} \widetilde{U}_{i} \otimes v_{i} \otimes w_{i} \cup u_{i} \otimes \widetilde{V}_{i} \otimes w_{i} \cup u_{i} \otimes v_{i} \otimes \widetilde{W}_{i}\right)
$$

By Lemma 3.6, this is the tangent space of $D_{r}$ at $\pi_{r}(p)$ when $\pi_{r}(p)$ is a smooth point of $D_{r}$. Then (7.4) implies that ${ }^{3}$

$$
\begin{equation*}
\left\langle\mathrm{T}_{\pi_{r}(p)}\left(u_{1}, \ldots, w_{r}\right), p-\pi_{r}(p)\right\rangle=0 \tag{7.5}
\end{equation*}
$$

i.e., $p-\pi_{r}(p)$ is orthogonal to the subspace $\mathrm{T}_{\pi_{r}(p)}\left(u_{1}, \ldots, w_{r}\right)$.

Let $\sigma_{r}$ denote the Euclidean closure of $\operatorname{Im} \Sigma_{r}^{\mathbb{R}}$. Then $D_{r} \subseteq \sigma_{r}$. By the TarskiSeidenberg Theorem, $\sigma_{r}$ is semialgebraic. By 35, Theorem 3.7], a general $A \in$ $U \otimes V \otimes W \backslash \sigma_{r}$ has a unique best approximation $\widetilde{\pi}_{r}(A)$ in $\sigma_{r}$. Note that for a nonnegative $A, \widetilde{\pi}_{r}(A) \in \sigma_{r}$ may be different from $\pi_{r}(A) \in D_{r}$.

In order to study best nonnegative rank approximations, i.e., the image of $\pi_{r}$, we first partition $D_{r}$ into a union of special semialgebraic subsets. For any index set $I_{i} \subseteq\left\{1, \ldots, n_{U}\right\}$, let

$$
U_{+}\left(I_{i}\right):=\left\{u \in U_{+}: \operatorname{supp}(u)=I_{i}^{c}\right\}
$$

and likewise for $V_{+}\left(J_{i}\right)$ and $W_{+}\left(K_{i}\right)$ with index sets $J_{i} \subseteq\left\{1, \ldots, n_{V}\right\}$ and $K_{i} \subseteq$ $\left\{1, \ldots, n_{W}\right\}$. Here $I_{i}^{c}:=\left\{1, \ldots, n_{U}\right\} \backslash I_{i}$ denotes set-theoretic complement. Given tuples of index sets

$$
I=\left(I_{1}, \ldots, I_{r}\right), \quad J=\left(J_{1}, \ldots, J_{r}\right), \quad K=\left(K_{1}, \ldots, K_{r}\right)
$$

[^3]with $I_{i} \subseteq\left\{1, \ldots, n_{U}\right\}, J_{i} \subseteq\left\{1, \ldots, n_{V}\right\}, K_{i} \subseteq\left\{1, \ldots, n_{W}\right\}, i=1, \ldots, r$, we define a cell of $D_{r}$ corresponding to these index sets by
\[

$$
\begin{aligned}
D_{r}(I, J, K):=\left\{A \in D_{r}: A=\right. & \sum_{i=1}^{r} u_{i} \otimes v_{i} \otimes w_{i} \\
& \left.u_{i} \in U_{+}\left(I_{i}\right), v_{i} \in V_{+}\left(J_{i}\right), w_{i} \in W_{+}\left(K_{i}\right), i=1, \ldots, r\right\}
\end{aligned}
$$
\]

The notion of a cell is important for our study of uniqueness because of the following easy observation.
Lemma 7.2. Let $A \in D_{r}$. If $A$ belongs to distinct cells, then the nonnegative $r$-term decomposition of $A$ is not unique.

Clearly, if $I_{i}=J_{i}=K_{i}=\varnothing$ for all $i=1, \ldots, r$, then $\operatorname{dim} D_{r}(I, J, K)=\operatorname{dim} D_{r}$ and we call this the trivial cell. The union of all nontrivial cells is called the boundary of $D_{r}$, and denoted by $\partial D_{r}$.

Lemma 7.3. If $r<r_{g}$ and $U \otimes V \otimes W$ is not $r$-defective, then $\operatorname{dim} \partial D_{r}<\operatorname{dim} D_{r}$.
Proof. We first describe $\partial D_{r}$ explicitly. Let $\alpha \in\left\{1, \ldots, n_{U}\right\}$ and $i \in\{1, \ldots, r\}$. Let $\widetilde{U}_{+}(\alpha)=\left\{u \in U_{+}: \alpha \notin \operatorname{supp}(u)\right\}$. Define
$\partial D_{r, U}^{(i, \alpha)}:=\sum_{r}^{\mathbb{R}_{+}}\left(\left(U_{+} \times V_{+} \times W_{+}\right)^{i-1} \times\left(\widetilde{U}_{+}(\alpha) \times V_{+} \times W_{+}\right) \times\left(U_{+} \times V_{+} \times W_{+}\right)^{r-i}\right)$.
We write

$$
\partial D_{r, U}:=\bigcup_{i=1}^{r} \bigcup_{\alpha=1}^{n_{U}} \partial D_{r, U}^{(i, \alpha)}
$$

and likewise define $\partial D_{r, V}$ and $\partial D_{r, W}$. The boundary is then the union of these three semialgebraic subsets,

$$
\partial D_{r}=\partial D_{r, U} \cup \partial D_{r, V} \cup \partial D_{r, W}
$$

From this description of $\partial D_{r}$, the required result is evident.
We caution our reader that our notion of boundary of $D_{r}$ differs from both its topological boundary and its algebraic boundary as defined in 3.

Let $A \in U_{+} \otimes V_{+} \otimes W_{+}$where $\pi_{r}(A)$ has a nonnegative rank- $r$ decomposition $\pi_{r}(A)=\sum_{i=1}^{r} u_{i} \otimes v_{i} \otimes w_{i}$. If there is some $i \in\{1, \ldots, r\}$ such that strict inequality holds in (7.2), i.e., there is some $x_{i} \in U_{+}$with

$$
\begin{equation*}
\left\langle A, x_{i} \otimes v_{i} \otimes w_{i}\right\rangle<\left\langle\pi_{r}(A), x_{i} \otimes v_{i} \otimes w_{i}\right\rangle, \tag{7.6}
\end{equation*}
$$

then $\widetilde{\pi}_{r}(A) \neq \pi_{r}(A)$ and $\pi_{r}(A) \in \partial D_{r}$ by Lemma 7.1. Similarly, if

$$
\begin{align*}
\quad\left\langle A, u_{i} \otimes y_{i} \otimes w_{i}\right\rangle & <\left\langle\pi_{r}(A), u_{i} \otimes y_{i} \otimes w_{i}\right\rangle  \tag{7.7}\\
\text { or } \quad\left\langle A, u_{i} \otimes v_{i} \otimes z_{i}\right\rangle & <\left\langle\pi_{r}(A), u_{i} \otimes v_{i} \otimes z_{i}\right\rangle \tag{7.8}
\end{align*}
$$

for some $y_{i} \in V_{+}$or $z_{i} \in W_{+}$, then $\tilde{\pi}_{r}(A) \neq \pi_{r}(A)$ and $\pi_{r}(A) \in \partial D_{r}$. We define the following sets:

$$
\begin{align*}
\mathcal{L} & =\left\{\pi_{r}(A) \in \partial D_{r}: \pi_{r}(A) \text { satisfies (7.6), (7.7), or (7.8) }\right\},  \tag{7.9}\\
\mathcal{N} & =\left\{A \in U_{+} \otimes V_{+} \otimes W_{+} \backslash\left(S_{r} \cup H_{r}\right): \pi_{r}(A) \in \mathcal{L}\right\} . \tag{7.10}
\end{align*}
$$

We will next show that every positive tensor (i.e., a tensor whose coordinates are positive) in $\mathcal{N}$ is an interior point.

Proposition 7.4. If $A \in \mathcal{N}$ is positive, then $A$ has an open neighborhood $\mathcal{V}$ such that $\mathcal{V} \subseteq \mathcal{N}$.

Proof. We first describe the structure of an open neighborhood $B(A, \eta)$ of a positive $A \in U_{+} \otimes V_{+} \otimes W_{+}$and its image $\pi_{r}(B(A, \eta))$. By [50, Proposition 15], $\pi_{r}(A)$ always has nonnegative rank- $r$. Since $\pi_{r}$ is smooth, for any $\delta>0$, there is some $\eta>0$ such that $\pi_{r}(B(A, \eta)) \subseteq B\left(\pi_{r}(A), \delta\right) \cap D_{r}$. Observe that $\left(\sum_{r}^{\mathbb{R}_{+}}\right)^{-1}\left(B\left(\pi_{r}(A), \delta\right) \cap D_{r}\right)$ is a union of at most a countable number of products of open balls, say,

$$
\bigcup_{j=1}^{s}\left(B\left(u_{1}^{(j)}, \delta_{1}^{(j)}\right) \cap U_{+}\right) \times \cdots \times\left(B\left(w_{r}^{(j)}, \delta_{r}^{(j)}\right) \cap W_{+}\right) \subseteq\left(U_{+} \times V_{+} \times W_{+}\right)^{r}
$$

where $s \in \mathbb{N} \cup\{\infty\}, u_{i}^{(j)} \in U_{+}, v_{i}^{(j)} \in V_{+}, w_{i}^{(j)} \in W_{+}$, and $\delta_{i}^{(j)}>0$ for $i=1, \ldots, r$, and $j=1, \ldots, s$. By dimension count, there exists some $j$ such that the image of

$$
\mathcal{U}:=\left(B\left(u_{1}^{(j)}, \delta_{1}^{(j)}\right) \cap U_{+}\right) \times \cdots \times\left(B\left(w_{r}^{(j)}, \delta_{r}^{(j)}\right) \cap W_{+}\right)
$$

under $\Sigma_{r}^{\mathbb{R}_{+}}$contains an open subset of $B\left(\pi_{r}(A), \delta\right) \cap D_{r}$. For notational convenience, we drop the superscript on $u_{i}^{(j)}, v_{i}^{(j)}, w_{i}^{(j)}$ and write $u_{i}, v_{i}, w_{i}$ below. By decreasing $\delta$ we may choose $\delta_{1}^{(j)}=\cdots=\delta_{r}^{(j)}=\varepsilon$ for some $\varepsilon>0$ small enough. Furthermore, we may assume that $\pi_{r}(A)=\sum_{i=1}^{r} u_{i} \otimes v_{i} \otimes w_{i}$ is a nonnegative rank- $r$ decomposition. So for any $p \in B(A, \eta), \pi_{r}(p)$ has a nonnegative rank- $r$ decomposition $\pi_{r}(p)=$ $\sum_{i=1}^{r} u_{i}(p) \otimes v_{i}(p) \otimes w_{i}(p)$ where

$$
\left\|u_{i}-u_{i}(p)\right\| \leq \varepsilon, \quad\left\|v_{i}-v_{i}(p)\right\| \leq \varepsilon, \quad\left\|w_{i}-w_{i}(p)\right\| \leq \varepsilon
$$

for $i=1, \ldots, r$. Thus

$$
\begin{equation*}
\operatorname{supp}\left(u_{i}\right) \subseteq \operatorname{supp}\left(u_{i}(p)\right), \quad \operatorname{supp}\left(v_{i}\right) \subseteq \operatorname{supp}\left(v_{i}(p)\right), \quad \operatorname{supp}\left(w_{i}\right) \subseteq \operatorname{supp}\left(w_{i}(p)\right) \tag{7.11}
\end{equation*}
$$

for $i=1, \ldots, r$, and all $u_{i}(p), v_{i}(p)$ and $w_{i}(p)$ depend continuously on $p$. The function defined by

$$
g(p):=\left\langle p-\pi_{r}(p), x_{i} \otimes v_{i}(p) \otimes w_{i}(p)\right\rangle
$$

is therefore continuous on $B(A, \eta)$ for any fixed $x_{i} \in U_{+}$. If there is some $x_{i} \in U_{+}$ such that $\left\langle A-\pi_{r}(A), x_{i} \otimes v_{i} \otimes w_{i}\right\rangle<0$, then by the continuity of $g$, there is an open neighborhood $\mathcal{V} \subseteq B(A, \eta)$ such $g(p)<0$ for all $p \in \mathcal{V}$. Therefore $\mathcal{V} \subseteq \mathcal{N}$.

The following theorem is the main result of this section. It characterizes the relation between the image of $\pi_{r}$ and the cells of $D_{r}$. Its implication on nonnegative tensor decomposition and approximation will be given in Corollary 7.6.
Theorem 7.5. Let $\pi_{r}(A) \in D_{r}(I, J, K)$ for some cell $D_{r}(I, J, K) \neq\{0\}$. Let $\mathcal{V}$ be an open neighborhood of $A$. Then $\pi_{r}(\mathcal{V})$ contains an open subset of $D_{r}(I, J, K)$.
Proof. We consider two cases: If $\pi_{r}(\mathcal{V})$ is zero-dimensional, then we are led to a contradiction and so this case cannot occur. If $\pi_{r}(\mathcal{V})$ is positive-dimensional, then we show that it must have full dimension in $D_{r}(I, J, K)$ and therefore the required result follows.

Case 1. $\pi_{r}(\mathcal{V})=\pi_{r}(A)$ is a point.
Let $\gamma(t)$ be a curve in $\mathcal{V}$ with $\gamma(0)=A$. Then $\pi_{r}(\gamma(t))=\pi_{r}(A)$ for any $t$. By (7.5) we have

$$
\left\langle\mathbf{T}_{\pi_{r}(A)}\left(u_{1}, \ldots, w_{r}\right), \gamma(t)-\pi_{r}(A)\right\rangle=0, \quad\left\langle\mathbf{T}_{\pi_{r}(A)}\left(u_{1}, \ldots, w_{r}\right), A-\pi_{r}(A)\right\rangle=0
$$

implying that

$$
\left\langle\mathrm{T}_{\pi_{r}(A)}\left(u_{1}, \ldots, w_{r}\right), \gamma(t)-A\right\rangle=0
$$

Since the curve $\gamma(t)$ is arbitrary, we are led to the conclusion that

$$
\left\langle\mathrm{T}_{\pi_{r}(A)}\left(u_{1}, \ldots, w_{r}\right), U \otimes V \otimes W\right\rangle=0
$$

contradicting the definition of $\mathrm{T}_{\pi_{r}(A)}\left(u_{1}, \ldots, w_{r}\right)$.
Case 2. $\pi_{r}(\mathcal{V})$ is of positive dimension.
We will show that $\operatorname{dim} \pi_{r}(\mathcal{V})=\operatorname{dim} D_{r}(I, J, K)$. By (7.11), we may assume that $\pi_{r}(A)$ is a smooth point of $\pi_{r}(\mathcal{V})$ without loss of generality. By giving $\pi_{r}(\mathcal{V})$ a finer stratification, we may furthermore assume that $\pi_{r}(\mathcal{V})$ is a Nash manifold. Suppose that $\operatorname{dim} \pi_{r}(\mathcal{V})<\operatorname{dim} D_{r}(I, J, K)$. Then by Theorem 2.2 there is an open semialgebraic neighborhood $\mathcal{R}$ of $\pi_{r}(\mathcal{V})$ in $D_{r}(I, J, K)$ and a Nash retraction $f: \mathcal{R} \rightarrow \pi_{r}(\mathcal{V})$ such that

$$
\operatorname{dist}\left(p, \pi_{r}(\mathcal{V})\right)=\|p-f(p)\|
$$

for any $p \in \mathcal{R}$. So there is a smooth curve $\gamma(t) \subseteq \mathcal{R}$ such that $\gamma(0)=\pi_{r}(A)$ and $f(\gamma(t))=\pi_{r}(A)$. Let $A(t):=A-\pi_{r}(A)+\gamma(t)$ and $X(t):=\pi_{r}(A(t)) \subseteq \pi_{r}(\mathcal{V})$. Note that

$$
\gamma(t), X(t) \subseteq D_{r}(I, J, K), \quad A^{\prime}(0), X^{\prime}(0) \in \mathrm{T}_{\pi_{r}(A)}\left(u_{1}, \ldots, w_{r}\right)
$$

By Lemma 7.1

$$
\lim _{t \rightarrow 0^{+}} \frac{d}{d t}\langle A(t)-X(t), A(t)-X(t)\rangle=2\left\langle A^{\prime}(0)-X^{\prime}(0), A-X(0)\right\rangle=0
$$

In fact, for any $s>0$ small enough, we have

$$
\left.\frac{d}{d t}\langle A(t)-X(t), A(t)-X(t)\rangle\right|_{t=s}=2\left\langle A^{\prime}(s)-X^{\prime}(s), A(s)-X(s)\right\rangle=0
$$

implying that $\|A(t)-X(t)\|$ is constant around $t=0$. On the other hand,

$$
\|A(t)-\gamma(t)\|=\left\|A-\pi_{r}(A)\right\|
$$

So by the uniqueness of $\pi_{r}(A(t)), X(t)=\gamma(t)$, contradicting $\gamma(t) \subseteq \mathcal{R} \backslash \pi_{r}(\mathcal{V})$ for $t>0$. Therefore we must have $\operatorname{dim} \pi_{r}(\mathcal{V})=\operatorname{dim} D_{r}(I, J, K)$.
Corollary 7.6. Let $r<r_{g}, U \otimes V \otimes W$ be $r$-identifiable, and $A \in U_{+} \otimes V_{+} \otimes W_{+}$ be general. If the unique best nonnegative rank- $r$ approximation $\pi_{r}(A)$ of $A$ is not in the boundary $\partial D_{r}$, then $\pi_{r}(A)$ has a unique nonnegative rank- $r$ decomposition.

Proof. Since $r<r_{g}$ and $U \otimes V \otimes W$ is not $r$-defective, by Lemma 7.3

$$
\operatorname{dim} \partial D_{r}<\operatorname{dim} D_{r}<\operatorname{dim} U \otimes V \otimes W
$$

For any smooth point $q \in D_{r}$, there is an open neighborhood $\mathcal{Q} \subseteq D_{r}$ of $q$ such that any point in $\mathcal{Q}$ is also smooth. By Theorem 2.2, there is an open semialgebraic neighborhood $\mathcal{R}$ of $\mathcal{Q}$ in $U_{+} \otimes V_{+} \otimes W_{+}$and a Nash retraction $f: \mathcal{R} \rightarrow \mathcal{Q}$ such that $\operatorname{dist}(p, \mathcal{Q})=\|p-f(p)\|$ for every $p \in \mathcal{R}$. By shrinking $\mathcal{R}$ if necessary, we may assume that

$$
\|p-f(p)\|=\operatorname{dist}(p, \mathcal{Q})=\operatorname{dist}\left(p, D_{r}\right)
$$

for every $p \in \mathcal{R}$, i.e., $\pi_{r}(p)=f(p)$. Thus every smooth point of $D_{r}$ is contained in $\operatorname{Im}\left(\pi_{r}\right)$, i.e., $\operatorname{Im}\left(\pi_{r}\right)$ is a semialgebraic subset of $D_{r}$ with

$$
\begin{equation*}
\operatorname{dim} \operatorname{Im}\left(\pi_{r}\right)=\operatorname{dim} D_{r}>\operatorname{dim} \partial D_{r} \tag{7.12}
\end{equation*}
$$

The required result then follows from Theorem 5.7 and Theorem7.5 with the trivial cell $D_{r}(I, J, K) \supseteq D_{r} \backslash \partial D_{r}$.

A measure theoretic consequence of Corollary [7.6 is that there is a positive measured subset of nonnegative tensors, such that each nonnegative tensor in this subset has a unique best nonnegative rank- $r$ approximation, and furthermore this approximation has a unique nonnegative rank- $r$ decomposition.

In the case of real tensors, it is possible that best rank- $r$ approximations always lie on the boundary of the set of tensors of rank $\leq r$ [26, Section 8]. So one might perhaps wonder whether Corollary 7.6 is vacuous. Fortunately this is not the case for nonnnegative tensors provided that $r<r_{g}$ and $U \otimes V \otimes W$ is not $r$-defective. In fact, the condition (7.12) implies that $\pi_{r}(A)$ is not always in $\partial D_{r}$.

For the special cases $r=2$ and 3 , we can say considerably more than Corollary 7.6 We will first make an observation regarding the case when $\pi_{r}(A) \in \mathcal{L}$ where $\mathcal{L}$ is as defined in (7.9).
Lemma 7.7. Let $\pi_{r}(A) \in \mathcal{L}$. Then

$$
\begin{aligned}
\operatorname{supp}\left(u_{1}\right) \cup \cdots \cup \operatorname{supp}\left(u_{r}\right) & =\left\{1, \ldots, n_{U}\right\} \\
\operatorname{supp}\left(v_{1}\right) \cup \cdots \cup \operatorname{supp}\left(v_{r}\right) & =\left\{1, \ldots, n_{V}\right\} \\
\operatorname{supp}\left(w_{1}\right) \cup \cdots \cup \operatorname{supp}\left(w_{r}\right) & =\left\{1, \ldots, n_{W}\right\}
\end{aligned}
$$

Proof. Suppose $1 \notin \bigcup_{i=1}^{r} \operatorname{supp}\left(u_{i}\right)$. Then by definition

$$
\left\langle A-\pi_{r}(A), e_{1} \otimes v_{1} \otimes w_{1}\right\rangle \leq 0
$$

where $e_{1}=(1,0, \ldots, 0)$. Since the coordinate $\left(\pi_{r}(A)\right)_{1 j k}=0$ for any $j=1, \ldots, n_{V}$, $k=1, \ldots, n_{W}$, and $A$ is positive, we have that $\left(A-\pi_{r}(A)\right)_{1 j k}>0$. On the other hand, $\left(e_{1} \otimes v_{1} \otimes w_{1}\right)_{i j k}=0$ for $i \neq 1$, and $\left(e_{1} \otimes v_{1} \otimes w_{1}\right)_{1 j k} \geq 0$. Hence

$$
\left\langle A-\pi_{r}(A), e_{1} \otimes v_{1} \otimes w_{1}\right\rangle>0
$$

a contradiction.
A cell $D_{r}(I, J, K)$ is called admissible if

$$
\bigcap_{i=1}^{r} I_{i}=\bigcap_{i=1}^{r} J_{i}=\bigcap_{i=1}^{r} K_{i}=\varnothing .
$$

By Proposition 7.4, Theorem 7.5, and Lemma 7.7, if $A \in \mathcal{N}$, then there is an open neighborhood $\mathcal{V}$ of $A$ such that $\pi_{r}(\mathcal{V})$ contains an open subset of some admissible cell $D_{r}(I, J, K)$. For small values of $r$, we may check these admissible cells and possibly obtain uniqueness for nonnegative rank- $r$ decomposition of $\pi_{r}(A)$ for a general $A$. We will do this explicitly for $r=2$ and 3 .
Theorem 7.8. Let $r=2$ or 3 and let $n_{U}, n_{V}, n_{W} \geq 3$. Then for a general $A \in U_{+} \otimes V_{+} \otimes W_{+}$, its unique best nonnegative rank-r approximation $\pi_{r}(A)$ has a unique nonnegative rank-r decomposition.

Proof. By Corollary 7.6, it remains to check the case $\pi_{r}(A) \in \partial D_{r}$ for a general A. Theorem 7.5 and Lemma 7.7 further restrict the remaining case to checking (i) whether $\pi_{r}(A)$ can be contained in an admissible cell, and (ii) whether $\pi_{r}(A)$ contained in an admissible cell (if any) has a unique decomposition.

When $r=2$, for a general $p$ in any admissible cell $D_{r}(I, J, K)$, let $p=u_{1} \otimes$ $v_{1} \otimes w_{1}+u_{2} \otimes v_{2} \otimes w_{2}$ be its nonnegative rank- 2 decomposition. Then each set $\left\{u_{1}, u_{2}\right\},\left\{v_{1}, v_{2}\right\}$, and $\left\{w_{1}, w_{2}\right\}$ consists of a pair of linearly independent vectors.

By [40, $p$ has a unique real rank-2 decomposition and thus the nonnegative rank-2 decomposition is unique.

When $r=3$, we may assume without loss of generality [26, Theorem 5.2] that $n_{U}=n_{V}=n_{W}=3$. The only situation where a general point $p$ of an admissible cell $D_{r}(I, J, K)$ does not have a unique nonnegative rank- $r$ decomposition is if

$$
\begin{gathered}
I_{1}=I_{2}=\{2,3\}, I_{3} \subseteq\{1\}, \quad J_{1}=J_{3}=\{2,3\}, J_{2} \subseteq\{1\} \\
K_{2}=K_{3}=\{2,3\}, K_{1} \subseteq\{1\}
\end{gathered}
$$

up to a permutation of the index set $\{1,2,3\}$. We claim that $\pi_{r}(A)$ cannot be contained in such a cell $D_{r}(I, J, K)$. Suppose not and $\pi_{r}(A) \in D_{r}(I, J, K)$, i.e.,

$$
u_{1}=u_{2}=(1,0, \ldots, 0), \quad v_{1}=v_{3}=(1,0, \ldots, 0), \quad w_{2}=w_{3}=(1,0, \ldots, 0)
$$

Then $\left(\pi_{r}(A)\right)_{1 j k}=0$ for $j=2,3, k=2,3$. Let

$$
p=u_{1} \otimes v_{1} \otimes w_{1}+u_{2} \otimes v_{2} \otimes\left(w_{2}+z\right)+u_{3} \otimes v_{3} \otimes w_{3}
$$

for some $z=(0, \alpha, \beta)$ with $\alpha, \beta>0$ small enough. Then $\|A-p\|<\left\|A-\pi_{r}(A)\right\|$ for a positive $A$, contradicting the definition of $\pi_{r}(A)$. Therefore $\pi_{r}(A) \notin D_{r}(I, J, K)$, a contradiction.

It is possible that a general point in an admissible cell $D_{r}(I, J, K)$ may have nonunique nonnegative rank- $r$ decompositions. To show uniqueness, we need to exclude such a possibility, i.e., check whether $\pi_{r}(A)$ is contained in such a cell for a typical $A$. For small values of $r$, we may test all cells case-by-case but evidently this becomes prohibitive for even moderately large values of $r$. Further results in this direction would require more precise descriptions of $I_{1}, \ldots, K_{r}$ where $D_{r}(I, J, K) \cap \operatorname{Im} \pi_{r} \neq \varnothing$.

## Acknowledgment

The authors would like to thank G. Blekherman, L. Chiantini, I. Domanov, P. Eyssidieux, S. Friedland, J. M. Landsberg, B. Mourrain, Z. Teitler and N. Vannieuwenhoven for useful discussions. The authors are very grateful to the anonymous referees for their suggestions and comments that greatly improved and clarified our manuscript.

## References

[1] H. Abo, G. Ottaviani, and C. Peterson, Induction for secant varieties of Segre varieties, Trans. Amer. Math. Soc., 361 (2009), pp. 767-792.
[2] B. ÅdlandSvik, Joins and higher secant varieties, Math. Scand., 61 (1987), pp. 213-222.
[3] E. Allman, J. Rhodes, B. Sturmfels, and P. Zwiernik, Tensors of nonnegative rank two, Linear Algebra Appl., 473 (2015), pp. 37-53.
[4] E. S. Allman and J. A. Rhodes, Phylogenetic invariants for the general markov model of sequence mutation, Math. Biosci., 186 (2003), pp. 113-144.
[5] E. S. Allman and J. A. Rhodes, Phylogenetic ideals and varieties for the general markov model, Adv. Appl. Math., 40 (2008), pp. 127-148.
[6] E. Ballico, On the weak non-defectivity of veronese embeddings of projective spaces, Central Eur. J. Math., 3 (2005), pp. 183-187.
[7] A. Berman and U. G. Rothblum, A note on the computation of the CP-rank, Linear Algebra Appl., 419 (2006), pp. 1-7.
[8] A. Bernardi, G. Blekherman, and G. Ottaviani, On real typical ranks, (2015), http://arxiv.org/abs/1512.01853
[9] G. Blekherman, Typical real ranks of binary forms, Found. Comput. Math., 15 (2013), pp. 793-798.
[10] G. Blekherman and Z. Teitler, On maximum, typical and generic ranks, Math. Ann., 362 (2015), pp. 1021-1031.
[11] C. Bocci, E. Carlini, and F. Rapallo, Perturbation of matrices and nonnegative rank with a view toward statistical models, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 1500-1512.
[12] C. Bocci, L. Chiantini, and G. Ottaviani, Refined methods for the identifiability of tensors, Ann. Mat. Pur. Appl., 193 (2014), pp. 1691-1702.
[13] J. Bochnak, M. Coste, and M.-F. Roy, Real Algebraic Geometry, Springer, Berlin, 1998.
[14] I. M. Bomze, W. Schachinger, and R. Ullrich, New lower bounds and asymptotics for the CP-rank, SIAM J. Matrix Anal. Appl., 36 (2015), pp. 20-37.
[15] J. D. Carroll and J. J. Chang, Analysis of individual differences in multidimensional scaling via n-way generalization of eckart-young decomposition, Psychometrika, 35 (1970), pp. 283-319.
[16] A. Causa and R. Re, On the maximum rank of a real binary form, Ann. Mat. Pur. Appl., 190 (2011), pp. 55-59.
[17] L. Chiantini and C. Ciliberto, Weakly defective varieties, Trans. Amer. Math. Soc., 354 (2002), pp. 151-178.
[18] L. Chiantini and C. Ciliberto, On the concept of $k$-secant order of a variety, J. Lond. Math. Soc., 73 (2006), pp. 436-454.
[19] L. Chiantini and G. Ottaviani, On generic identifiability of 3-tensors of small rank, SIAM J. Matrix Anal. Appl., 33 (2012), pp. 1018-1037.
[20] L. Chiantini, G. Ottaviani, and N. Vannieuwenhoven, On generic identifiability of symmetric tensors of subgeneric rank, Trans. Amer. Math. Soc. to appear.
[21] L. Chiantini, G. Ottaviani, and N. Vannieuwenhoven, An algorithm for generic and low-rank specific identifiability of complex tensors, SIAM J. Matrix Anal. Appl., 35 (2014), pp. 1265-1287.
[22] P. Comon, Tensors: a brief introduction, IEEE Signal Proc. Mag., 31 (2014), pp. 44-53.
[23] P. Comon and G. Ottaviani, On the typical rank of real binary forms, Linear and Multilinear Algebra, 60 (2012), pp. 657-667.
[24] M. Coste, An introduction to semialgebraic geometry, Rennes, 2002, https://perso.univ-rennes1.fr/michel.coste/polyens/SAG.pdf
[25] M. Coste, Real algebraic sets, Rennes, 2005,https://perso.univ-rennes1.fr/michel.coste/polyens/RASroot.pdf
[26] V. De Silva and L.-H. Lim, Tensor rank and the ill-posedness of the best low-rank approximation problem, SIAM J. Matrix Anal. Appl., 30 (2008), pp. 1084-1127.
[27] I. Domanov and L. De Lathauwer, On the uniqueness of the canonical polyadic decomposition of third-order tensors-part I: Basic results and uniqueness of one factor matrix, SIAM J. Matrix Anal. Appl., 34 (2013), pp. 855-875.
[28] I. Domanov and L. De Lathauwer, On the uniqueness of the canonical polyadic decomposition of third-order tensors-part II: Uniqueness of the overall decomposition, SIAM J. Matrix Anal. Appl., 34 (2013), pp. 876-903.
[29] I. Domanov and L. De Lathauwer, Generic uniqueness conditions for the canonical polyadic decomposition and INDSCAL, SIAM J. Matrix Anal. Appl., 36 (2015), pp. 15671589.
[30] M. Drton, B. Sturmfels, and S. Sullivant, Lectures on algebraic statistics, Birkhäuser, Basel, 2009.
[31] A. Durfee, Neighborhoods of algebraic sets, Trans. Amer. Math. Soc., 276 (1983), pp. 517530.
[32] S. Friedland, On the generic and typical ranks of 3-tensors, Linear Algebra Appl., 436 (2012), pp. 478-497.
[33] S. Friedland, On tensors of border rank $l$ in $\mathbb{C}^{m \times n \times l}$, Linear Algebra Appl., 438 (2013), pp. 713-737.
[34] S. Friedland and E. Gross, A proof of the set-theoretic version of the salmon conjecture, J. Algebra, 356 (2012), pp. 374-379.
[35] S. Friedland and M. Stawiska, Some approximation problems in semi-algebraic geometry, constructive approximation of functions, Banach Cent. Publ., 107 (2016), pp. 129-143.
[36] W. Hackbusch, Tensor Spaces and Numerical Tensor Calculus, Springer, Berlin, 2012.
[37] R. A. Harshman, Foundations of the Parafac procedure: models and conditions for an explanatory multi-modal factor analysis, UCLA Working Papers in Phonetics, 16 (1970), pp. 1-84.
[38] R. Hartshorne, Algebraic Geometry, Springer, New York, NY, 1977.
[39] F. Hitchcock, The expression of a tensor or a polyadic as a sum of products, J. Math. Phys., 6 (1927), pp. 164-189.
[40] J. B. Kruskal, Three-way arrays: Rank and uniqueness of trilinear decompositions, Linear Algebra Appl., 18 (1977), pp. 95-138.
[41] J. M. Landsberg, Tensors: Geometry and Applications, AMS, Providence, RI, 2012.
[42] H. Laurberg, M. Christensen, M. Plumbley, L. Hansen, and S. Jensen, Theorems on positive data: On the uniqueness of NMF, Comput. Intell. Neurosci., 2008 (2008).
[43] T. Lickteig, Typical tensorial rank, Linear Algebra Appl., 69 (1985), pp. 95-120.
[44] L.-H. Lim, Tensors and hypermatrices, in Handbook of Linear Algebra, L. Hogben, ed., CRC Press, Boca Raton, FL, 2 ed., 2013, ch. 15, pp. 15-1-15-30.
[45] L.-H. Lim and P. Comon, Nonnegative approximations of nonnegative tensors, J. Chemometr., 23 (2009), pp. 432-441.
[46] R. Loewy and B.-S. Tam, CP rank of completely positive matrices of order 5, Linear Algebra Appl., 363 (2003), p. 161176.
[47] M. Mella, Singularities of linear systems and the Waring problem, Trans. Amer. Math. Soc., 358 (2006), pp. 5523-5538.
[48] J. Milnor, Singular points of complex hypersurfaces, Princeton University Press, Princeton, NJ, 1968.
[49] L. Pachter and B. Sturmfels, Algebraic statistics for computational biology, Cambridge University Press, New York, NY, 2005.
[50] Y. Qi, P. Comon, And L.-H. Lim, Uniqueness of nonnegative tensor approximations, IEEE Trans. Inform. Theory, 62 (2016), pp. 2170-2183.
[51] N. Shaked-Monderer, I. M. Bomze, F. Jarre, and W. Schachinger, On the CP-rank and minimal CP factorizations of a completely positive matrix, SIAM J. Matrix Anal. Appl., 34 (2013), pp. 355-368.
[52] J. Sidman and P. Vermeire, Equations defining secant varieties: geometry and computation, in Combinatorial aspects of commutative algebra and algebraic geometry, vol. 6, Springer, Berlin Heidelberg, 2011, pp. 155-174.
[53] F. Sottile, Real algebraic geometry for geometric constraints, (2016), http://arxiv.org/abs/1606.03127
[54] A. Stegeman, On uniqueness conditions for Candecomp/Parafac and Indscal with full column rank in one mode, Linear Algebra Appl., 431 (2009), pp. 211-227.
[55] V. Strassen, Vermeidung von divisionen, J. Reine Angew. Math., 264 (1973), pp. 184-202.
[56] V. Strassen, Rank and optimal computation of generic tensors, Linear Algebra Appl., 52 (1983), pp. 645-685.
[57] Z. Teitler, Geometric lower bounds for generalized ranks, (2014), http://arxiv.org/abs/1406.5145.
[58] M. Veganzones, J. E. Cohen, R. Cabral-Farias, J. Chanussot, and P. Comon, Nonnegative tensor CP decomposition of hyperspectral data, IEEE Trans. Geosci. Remote., (2016), doi:10.1109/TGRS.2015.2503737 to appear.
[59] F. L. ZAK, Tangents and secants of algebraic varieties, AMS, Providence, RI, 1993.
[60] F. L. ZAK, Determinants of projective varieties and their degrees, in Algebraic Transformation Groups and Algebraic Varieties, V. L. Popov, ed., Springer, Berlin, 2004, pp. 207-238.

CNRS, Gipsa-Lab, Université Grenoble Alpes, F-38000 Grenoble, France
E-mail address: yang.qi@gipsa-lab.grenoble-inp.fr, pierre.comon@gipsa-lab.grenoble-inp.fr
Computational and Applied Mathematics Initiative, Department of Statistics, University of Chicago, 5734 South University Avenue, Chicago, IL 60637, USA

E-mail address: lekheng@galton.uchicago.edu


[^0]:    2010 Mathematics Subject Classification. 14P10, 15A69, 41A50, 41A52.
    Key words and phrases. nonnegative tensors, nonnegative tensor rank, nonnegative typical ranks, real tensor rank, symmetric tensor rank, best nonnegative rank- $r$ approximations, semialgebraic geometry, uniqueness and identifiability.

    YQ and PC are supported by the ERC under the European Community's Seventh Framework Program FP7/2007-2013 Grant 320594. LHL is supported by AFOSR FA9550-13-1-0133, DARPA D15AP00109, NSF IIS 1546413, DMS 1209136, and DMS 1057064.

[^1]:    ${ }^{1}$ An expression of $T$ as a sum of $s$ rank-one tensors where $s$ is not necessarily $\operatorname{rank}(T)$ will just be called an $s$-term decomposition.

[^2]:    ${ }^{2}$ Allowing both superscript and subscript provides notational flexibility when indices or powers are involved.

[^3]:    ${ }^{3}$ Our convention: $\langle S, u\rangle=\langle u, S\rangle=0$ for $S \subseteq U$ means that every vector in $S$ is orthogonal to $u ;\langle S, T\rangle=0$ for $S, T \subseteq U$ means that any vector in $S$ is orthogonal to any vector in $T$.

