# On the finite convergence of the Douglas-Rachford algorithm for solving (not necessarily convex) feasibility problems in Euclidean spaces 

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#### Abstract

Solving feasibility problems is a central task in mathematics and the applied sciences. One particularly successful method is the Douglas-Rachford algorithm. In this paper, we provide many new conditions sufficient for finite convergence. Numerous examples illustrate our results.


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## 1 Introduction

The Douglas-Rachford algorithm (DRA) was first introduced in [25] as an operator splitting technique to solve partial differential equations arising in heat conduction. As a result of findings by Lions and Mercier [36] in the monotone operator setting, the method has been extended to find solutions of the sum of two maximally monotone operators. When specialized to normal cone operators, the method is very useful in solving feasibility problems. To fix our setting, we assume throughout that

$$
\begin{equation*}
X \text { is a Euclidean space, } \tag{1}
\end{equation*}
$$

i.e, a finite-dimensional real Hilbert space with inner product $\langle\cdot, \cdot\rangle$ and induced norm $\|\cdot\|$. Given closed subsets $A$ and $B$ of $X$ with nonempty intersection, we consider the fundamental feasibility problem

$$
\begin{equation*}
\text { find a point in } A \cap B \tag{2}
\end{equation*}
$$

which frequently arises in science and engineering applications. A common approach for solving (2) is to use projection algorithms that employ projectors onto the underlying sets; see, e.g., [5] [6],

[^0][18], [20], [19], [21], [22], [31], and the references therein. Among those algorithms, the DouglasRachford algorithm applied to (2) has attracted much attention; see, e.g., [2] and [23] and the references therein for further information.

In the convex case, it is known, see, e.g., Lions and Mercier [36] and Svaiter [39], that the sequence generated by the DRA always converges while the "shadow sequence" converges to a point of the intersection. Even when the convex feasibility problem is inconsistent, i.e., $A \cap B=\varnothing$, it was shown in [7] that the "shadow sequence" is bounded and its cluster points solve a best approximation problem; the entire sequence converges if one of the sets is an affine subspace [8].

Although the Douglas-Rachford algorithm has been applied successfully to various problems involving one or more nonconvex sets, the theoretical justification is far from complete. Recently, in the case of a Euclidean sphere and a line, Borwein and Sims [17] have proved local convergence of the DRA at points of the intersection, while Aragón Artacho and Borwein [1] have given a region of convergence for this model in the plane; moreover, Benoist [15] has even shown that the DRA sequence converges in norm to a point of the intersection except when the starting point belongs to the hyperplane of symmetry. In another direction, [13] proved local convergence for finite unions of convex sets.

On the convergence rate, it has been shown by Hesse, Luke and Neumann [33] that the DRA for two subspaces converges linearly. Furthermore, the rate is then actually the cosine of the Friedrichs angle between the subspaces [4]. In the potentially nonconvex case, under transversality assumptions, Hesse and Luke [32] proved local linear convergence of the DRA for a superregular set and an affine subspace, while Phan [38] obtained such a rate for two super-regular sets. Specialized to the convex setting, the result in [38] implies linear convergence of the DRA for two convex sets whose the relative interiors have a nonempty intersection; see also [14]. It is worth mentioning that the linear convergence of the DRA may fail even for simple settings in the Euclidean plane, as shown in [9]. Based on Hölder regularity properties, Borwein, Li, and Tam [16] established sublinear convergence for two convex basic semi-algebraic sets. For the linear convergence of the DRA in the framework of optimization problems involving a sum of two functions, we refer the reader to, e.g., Giselsson's [28], [29], Li and Pong [34], Liang, Faili, Peyré, and Luke [35], Patrinos, Stella, and Bemporad's [37], and the references therein.

Davis and Yin [24] observed that the DRA may converge arbitrarily slowly in infinite dimensions; however, in finite dimensions, it often works extremely well. Very recently, the globally finite convergence of the DRA has been shown in [10] for an affine subspace and a locally polyhedral set, or for a hyperplane and an epigraph, and then by Aragón Artacho, Borwein, and Tam [3] for a finite set and a halfspace.

The goal of this paper is to provide various finite-convergence results. The sufficient conditions we present are new and complementary to existing conditions.

After presenting useful results on projectors and the DRA (Section 2) and on locally identical sets (Section 3), we specifically derive results related to the following five scenarios:

R1 $A$ is a halfspace and $B$ is an epigraph of a convex function; $A$ is either a hyperplane or a halfspace, and $B$ is a halfspace (see Section 4 ).
R2 $A$ and $B$ are supersets or modifications of other sets where the DRA is better understood (see Section 5 ).
R3 $A$ and $B$ are subsets of other sets where the DRA is better understood (see Section6).
R4 $B$ is a finite, hence nonconvex, set (see Section 7 ).

R5 $A$ is an affine subspace and $B$ is a polyhedron in the absence of Slater's condition (see Section (8).

The paper concludes with a list of open problem in Section 9
Before we start our analysis, let us note that our notation and terminology is standard and follows, e.g., [6]. The nonnegative integers are $\mathbb{N}$, and the real numbers are $\mathbb{R}$, while $\mathbb{R}_{+}:=$ $\{\alpha \in \mathbb{R} \mid \alpha \geq 0\}, \mathbb{R}_{++}:=\{\alpha \in \mathbb{R} \mid \alpha>0\}$, and $\mathbb{R}_{-}:=\{\alpha \in \mathbb{R} \mid \alpha \leq 0\}$. Let $C$ be a subset of $X$. Then the closure of $C$ is $\bar{C}$, the interior of $C$ is int $C$, the boundary of $C$ is bdry $C$, and the smallest affine and linear subspaces containing $C$ are, respectively, aff $C$ and span $C$. The relative interior of $C$, ri $C$, is the interior of $C$ relative to aff $C$. The smallest convex cone containing $C$ is cone $C$, the orthogonal complement of $C$ is $C^{\perp}:=\{y \in X \mid(\forall x \in C)\langle x, y\rangle=0\}$, and the dual cone of $C$ is $C^{\oplus}:=\{y \in X \mid(\forall x \in C)\langle x, y\rangle \geq 0\}$. The normal cone operator of $C$ is denoted by $N_{C}$, i.e., $N_{C}(x)=\{y \in X \mid(\forall c \in C)\langle y, c-x\rangle \leq 0\}$ if $x \in C$, and $N_{C}(x)=\varnothing$ otherwise. If $x \in X$ and $\rho \in \mathbb{R}_{++}$, then ball $(x ; \rho):=\{y \in X \mid\|x-y\| \leq \rho\}$ is the closed ball centered at $x$ with radius $\rho$.

## 2 Auxiliary results

For the reader's convenience we recall in this section preliminary concepts and auxiliary results which are mostly well known and which will be useful later.

Let $A$ be a nonempty closed subset of $X$. The distance function of $A$ is

$$
\begin{equation*}
d_{A}: X \rightarrow \mathbb{R}: x \mapsto \min _{a \in A}\|x-a\| . \tag{3}
\end{equation*}
$$

The projector onto $A$ is the mapping

$$
\begin{equation*}
P_{A}: X \rightrightarrows A: x \mapsto \underset{a \in A}{\operatorname{argmin}}\|x-a\|=\left\{a \in A \mid\|x-a\|=d_{A}(x)\right\} \tag{4}
\end{equation*}
$$

and the reflector across $A$ is defined by

$$
\begin{equation*}
R_{A}:=2 P_{A}-\mathrm{Id}, \tag{5}
\end{equation*}
$$

where Id is the identity operator. Note that closedness of the set $A$ is necessary and sufficient for $A$ to be proximinal, i.e., $(\forall x \in X) P_{A} x \neq \varnothing$ (see, e.g., [6, Corollary 3.13]). In the following, we shall write $P_{A} x=a$ if $P_{A} x=\{a\}$ is a singleton.

Fact 2.1 (Projection onto a convex set). Let $A$ be a nonempty closed convex subset of $X$, and let $x$ and $p$ be in X . Then the following hold:
(i) $P_{A}$ is single-valued and

$$
\begin{equation*}
p=P_{A} x \quad \Leftrightarrow \quad[p \in A \text { and }(\forall y \in A)\langle x-p, y-p\rangle \leq 0] \quad \Leftrightarrow \quad x-p \in N_{A}(p) . \tag{6}
\end{equation*}
$$

(ii) $P_{A}$ is firmly nonexpansive, i.e.,

$$
\begin{equation*}
(\forall x \in X)(\forall y \in X) \quad\left\|P_{A} x-P_{A} y\right\|^{2}+\left\|\left(\operatorname{Id}-P_{A}\right) x-\left(\operatorname{Id}-P_{A}\right) y\right\|^{2} \leq\|x-y\|^{2} . \tag{7}
\end{equation*}
$$

(iii) $R_{A}$ is nonexpansive, i.e.,

$$
\begin{equation*}
(\forall x \in X)(\forall y \in X) \quad\left\|R_{A} x-R_{A} y\right\| \leq\|x-y\| \tag{8}
\end{equation*}
$$

In particular, $P_{A}$ and $R_{A}$ are continuous on $X$.
Proof. (i); [6, Theorem 3.14 and Proposition 6.46]. (ii): [6, Proposition 4.8]. (iii); [6, Corollary 4.10].

Lemma 2.2. Let $A$ and $B$ be closed subsets of $X$ such that $A \subseteq B$, and let $x \in X$. Then the following hold:
(i) $A \cap P_{B} x \subseteq P_{A} x$.
(ii) $(\forall p \in A) P_{B}^{-1} p \subseteq P_{A}^{-1} p$.
(iii) If $P_{B} x=p \in A$, then $P_{A} x=P_{B} x$.
(iv) If $B$ is convex and $P_{B} x \in A$, then $P_{A} x=P_{B} x$.

Proof. (i). The conclusion is obvious if $A \cap P_{B} x=\varnothing$. Assume $A \cap P_{B} x \neq \varnothing$, and let $p \in A \cap P_{B} x$. Then $\|x-p\| \leq\|x-y\|$ for all $y \in B$, and so for all $y \in A$ since $A \subseteq B$. This combined with $p \in A$ gives $p \in \operatorname{argmin}_{y \in A}\|x-y\|=P_{A} x$.
(ii): Let $p \in A$. For all $x \in P_{B}^{-1} p$, we have $p \in P_{B} x$, and by (i), $p \in A \cap P_{B} x \subseteq P_{A} x$, which implies $x \in P_{A}^{-1} p$.
(iii). Assume that $P_{B} x=p \in A$. Using (i), we have $p \in P_{A} x$, and so

$$
\begin{equation*}
P_{A} x=\{y \in A \mid\|x-y\|=\|x-p\|\} \subseteq\{y \in B \mid\|x-y\|=\|x-p\|\}=P_{B} x=\{p\} \tag{9}
\end{equation*}
$$

It follows that $P_{A} x=P_{B} x=\{p\}$.
(iv) By Fact 2.1|(i), if $B$ is convex, then $P_{B} x$ is a singleton, and if additionally $P_{B} x \in A$, then by (iii), $P_{A} x=P_{B} x$.

Example 2.3 (Projection onto an affine subspace). Let $Y$ be a real Hilbert space, let $L$ be a linear operator from $X$ to $Y$, let $v \in \operatorname{ran} L$, and set $A=\{x \in X \mid L x=v\}$. Then

$$
\begin{equation*}
(\forall x \in X) \quad P_{A} x=x-L^{\dagger}(L x-v) \tag{10}
\end{equation*}
$$

where $L^{\dagger}$ denotes the Moore-Penrose inverse of $L$.

Proof. This follows from [11, Lemma 4.1], see also [6, Example 28.14].
Example 2.4 (Projection onto a hyperplane or a halfspace). Let $u \in X \backslash\{0\}$, and let $\eta \in \mathbb{R}$. Then the following hold:
(i) If $A=\{x \in X \mid\langle x, u\rangle=\eta\}$, then

$$
\begin{equation*}
(\forall x \in X) \quad P_{A} x=x-\frac{\langle x, u\rangle-\eta}{\|u\|^{2}} u \tag{11}
\end{equation*}
$$

(ii) If $A=\{x \in X \mid\langle x, u\rangle \leq \eta\}$, then

$$
(\forall x \in X) \quad P_{A} x= \begin{cases}x & \text { if }\langle x, u\rangle \leq \eta  \tag{12}\\ x-\frac{\langle x, u\rangle-\eta}{\|u\|^{2}} u & \text { if }\langle x, u\rangle>\eta\end{cases}
$$

Proof. (i): [6, Example 28.15]. (ii), [6, Example 28.16].

Example 2.5 (Projection onto a ball). Let $B=\operatorname{ball}(u ; \rho)$ with $u \in X$ and $\rho \in \mathbb{R}_{++}$. Then

$$
\begin{equation*}
(\forall x \in X) \quad P_{B} x=u+\frac{\rho}{\max \{\|x-u\|, \rho\}}(x-u) . \tag{13}
\end{equation*}
$$

Proof. Let $x \in X$. We have to prove $P_{B} x=x$ if $\|x-u\| \leq \rho$, and $P_{B} x=b:=u+\frac{\rho}{\|x-u\|}(x-u)$ otherwise. Indeed, if $\|x-u\| \leq \rho$, then $x \in B$, and thus $P_{B} x=x$. Assume that $\|x-u\|>\rho$. On the one hand, for all $y \in B$, by using $\|y-u\| \leq \rho$ and the triangle inequality,

$$
\begin{equation*}
\|x-b\|=\|x-u\|-\rho \leq\|x-u\|-\|y-u\| \leq\|x-y\| . \tag{14}
\end{equation*}
$$

On the other hand, $\|b-u\|=\rho$, and so $b \in \operatorname{ball}(u ; \rho)$, then by combining with the convexity of $B$ and the above inequality, $P_{B} x=b$, which completes the formula.

Example 2.6 (Projection onto an epigraph). Let $f: X \rightarrow \mathbb{R}$ be convex and continuous, set $B=$ epi $f:=\{(x, \rho) \in X \times \mathbb{R} \mid f(x) \leq \rho\}$, and let $(x, \rho) \in(X \times \mathbb{R}) \backslash B$. Then there exists $p \in X$ such that $P_{B}(x, \rho)=(p, f(p))$,

$$
\begin{equation*}
x \in p+(f(p)-\rho) \partial f(p) \text { and } \rho<f(p) \leq f(x) \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
(\forall y \in X) \quad\langle y-p, x-p\rangle \leq(f(y)-f(p))(f(p)-\rho) \tag{16}
\end{equation*}
$$

Proof. See [10, Lemma 5.1].
In order to solve the feasibility problem (2), where $A$ and $B$ are closed subsets of $X$ with nonempty intersection, we employ the Douglas-Rachford algorithm (also called averaged alternating reflections) that generates a sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ by

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad x_{n+1} \in T_{A, B} x_{n}, \quad \text { where } x_{0} \in X \tag{17}
\end{equation*}
$$

and where

$$
\begin{equation*}
T_{A, B}:=\frac{1}{2}\left(\operatorname{Id}+R_{B} R_{A}\right) \tag{18}
\end{equation*}
$$

is the Douglas-Rachford operator associated with the ordered pair $(A, B)$. The sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ in (17) is called a DRA sequence with respect to $(A, B)$, with starting point $x_{0}$. By Fact 2.1(i), when $A$ and $B$ are convex, then $P_{A}, P_{B}$ and hence $T_{A, B}$ are single-valued. Notice that

$$
\begin{equation*}
(\forall x \in X) \quad T_{A, B} x=\frac{1}{2}\left(\operatorname{Id}+R_{B} R_{A}\right) x=\left\{x-a+P_{B}(2 a-x) \mid a \in P_{A} x\right\} \tag{19}
\end{equation*}
$$

and if $P_{A}$ is single-valued then

$$
\begin{equation*}
T_{A, B}=\frac{1}{2}\left(\operatorname{Id}+R_{B} R_{A}\right)=\operatorname{Id}-P_{A}+P_{B} R_{A} . \tag{20}
\end{equation*}
$$

In the sequel we adopt the convention that in the case where $P_{A} x$ is not a singleton, $\left(P_{A} x, P_{B} R_{A} x\right)=$ $\left\{\left(a, P_{B}(2 a-x)\right) \mid a \in P_{A} x\right\}$.

The set of fixed points of $T_{A, B}$ is defined by Fix $T_{A, B}:=\left\{x \in X \mid x \in T_{A, B} x\right\}$. It follows from $T_{A, B} x=x-P_{A} x+P_{B} R_{A} x$ that

$$
\begin{equation*}
x \in \operatorname{Fix} T_{A, B} \quad \Leftrightarrow \quad P_{A} x \cap P_{B} R_{A} x \neq \varnothing, \tag{21}
\end{equation*}
$$

and that modified for clarity

$$
\left.\begin{array}{c}
x \in \operatorname{Fix} T_{A, B}  \tag{22}\\
P_{A} x \text { is a singleton }
\end{array}\right\} \quad \Rightarrow \quad P_{A} x \in A \cap B .
$$

For the convex case, the basic convergence result of the DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ and the "shadow sequence" $\left(P_{A} x_{n}\right)_{n \in \mathbb{N}}$ is as follows.

Fact 2.7 (Convergence of DRA in the convex consistent case). Let $A$ and $B$ be closed convex subsets of $X$ with $A \cap B \neq \varnothing$, and let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be a DRA sequence with respect to $(A, B)$. Then the following hold:
(i) $x_{n} \rightarrow x \in \operatorname{Fix} T_{A, B}=(A \cap B)+N_{A-B}(0)$ and $P_{A} x_{n} \rightarrow P_{A} x \in A \cap B$.
(ii) If $0 \in \operatorname{int}(A-B)$, then $x_{n} \rightarrow x \in A \cap B$; the convergence is finite provided that $x \in A \cap \operatorname{int} B$.

Proof. (i): This follows from [36, Theorem 1] and [39, Theorem 1]; see also [7, Corollary 3.9 and Theorem 3.13]. (ii), Clear from [10, Lemma 3.2].

## 3 Locally identical sets

Definition 3.1. Let $A$ and $B$ be subsets of $X$ such that $A \cap B \neq \varnothing$. Then $A$ and $B$ are called locally identical around $c \in A \cap B$ if there exists $\varepsilon \in \mathbb{R}_{++}$such that $A \cap \operatorname{ball}(c ; \varepsilon)=B \cap$ ball $(c ; \varepsilon)$. We say that $A$ and $B$ are locally identical around a set $C \subseteq A \cap B$ if they are locally identical around every point in $C$. When $A$ and $B$ are locally identical around a point $c$ (respectively, a set $C$ ), we also say that $(A, B)$ is locally identical around $c$ (respectively, $C$ ).

Lemma 3.2. Let $A$ and $B$ be subsets of $X$ such that $A \cap B \neq \varnothing$. Then the following hold:
(i) $A$ and $B$ are locally identical around $\operatorname{int}(A \cap B)$.
(ii) If $A$ and $B$ are locally identical around $c \in A \cap B$, then $A, B$ and $A \cap B$ are also locally identical around $c$.
(iii) If $A \subseteq B$, and $c$ is a point in $A$ such that $d_{B \backslash A}(c)>0$, then $A$ and $B$ are locally identical around $c$.
(iv) If $A$ is closed convex, and $C$ is a closed subset of $A$ such that $A$ and $C$ are locally identical around $C$, then $A=C$.
(v) If $A$ and $B$ are closed convex and locally identical around $A \cap B$, then $A=B$.

Proof. (i). Let $c \in \operatorname{int}(A \cap B)$. Then there exists $\varepsilon \in \mathbb{R}_{++}$such that ball $(c ; \varepsilon) \subseteq A \cap B$, which implies $A \cap \operatorname{ball}(c ; \varepsilon)=\operatorname{ball}(c ; \varepsilon)=B \cap \operatorname{ball}(c ; \varepsilon)$, so $A$ and $B$ are locally identical around $c$.
(ii). Note that if $A \cap \operatorname{ball}(c ; \varepsilon)=B \cap \operatorname{ball}(c ; \varepsilon)$ then $A \cap \operatorname{ball}(c ; \varepsilon)=B \cap \operatorname{ball}(c ; \varepsilon)=(A \cap B) \cap$ ball $(c ; \varepsilon)$.
(iii) Since $d_{B \backslash A}(c)>0$, there exists $\varepsilon \in \mathbb{R}_{++}$such that $(B \backslash A) \cap$ ball $(c ; \varepsilon)=\varnothing$. Combining with $A \subseteq B$, we get $A \cap \operatorname{ball}(c ; \varepsilon)=(A \cap \operatorname{ball}(c ; \varepsilon)) \cup((B \backslash A) \cap \operatorname{ball}(c ; \varepsilon))=B \cap \operatorname{ball}(c ; \varepsilon)$.
(iv) Let $c \in C$. It suffices to show that

$$
\begin{equation*}
\left(\forall \varepsilon \in \mathbb{R}_{++}\right) \quad A \cap \operatorname{ball}(c ; \varepsilon)=C \cap \operatorname{ball}(c ; \varepsilon) . \tag{23}
\end{equation*}
$$

Suppose to the contrary that (23) does not hold. Since $A$ and $C$ are locally identical around $C$ which includes $c$,

$$
\begin{equation*}
0<\bar{\varepsilon}:=\sup \left\{\varepsilon \in \mathbb{R}_{++} \mid A \cap \operatorname{ball}(c ; \varepsilon)=C \cap \operatorname{ball}(c ; \varepsilon)\right\}<+\infty . \tag{24}
\end{equation*}
$$

Then $(\forall \varepsilon \in] \bar{\varepsilon},+\infty[) A \cap \operatorname{ball}(c ; \varepsilon) \supsetneqq C \cap \operatorname{ball}(c ; \varepsilon)$. Now let $\varepsilon_{n} \downarrow \bar{\varepsilon}$ and

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad a_{n} \in A \cap \operatorname{ball}\left(c ; \varepsilon_{n}\right) \backslash C . \tag{25}
\end{equation*}
$$

By the boundedness of $\left(a_{n}\right)_{n \in \mathbb{N}}$ and the closedness of $A$, we assume without loss of generality that $a_{n} \rightarrow a \in A$. It follows from $\left\|a_{n}-c\right\| \leq \varepsilon_{n}$ that $\varepsilon:=\|a-c\| \leq \bar{\varepsilon}$. By the convexity of $A,(\forall \lambda \in] 0,1[) a_{\lambda}=\lambda a+(1-\lambda) c \in A$, and $\left\|a_{\lambda}-c\right\|=\lambda\|a-c\|=\lambda \varepsilon<\bar{\varepsilon}$, which yields $a_{\lambda} \in A \cap \operatorname{ball}(c ; \lambda \varepsilon)=C \cap \operatorname{ball}(c ; \lambda \varepsilon)$, using the definition of $\bar{\varepsilon}$. From $a_{\lambda} \in C$ and the closedness of $C$, letting $\lambda \rightarrow 1^{-}$, we obtain $a \in C$, thus $A$ and $C$ are locally identical around $a$, i.e., $A \cap$ ball $(a ; \rho)=$ $C \cap \operatorname{ball}(a ; \rho)$ for some $\rho \in \mathbb{R}_{++}$. Since $a_{n} \rightarrow a$, we find $n_{0} \in \mathbb{N}$ satisfying $a_{n_{0}} \in \operatorname{ball}(a ; \rho)$. Then $a_{n_{0}} \in A \cap \operatorname{ball}(a ; \rho)=C \cap \operatorname{ball}(a ; \rho) \subseteq C$, which contradicts the fact that $(\forall n \in \mathbb{N}) a_{n} \notin C$. Therefore, (23) holds.

Now pick an arbitrary $a \in A$, and let $\varepsilon>\|a-c\|$. By combining with (23), $a \in A \cap$ ball $(c ; \varepsilon)=$ $C \cap$ ball $(c ; \varepsilon)$, and so $a \in C$. It follows that $A \subseteq C \subseteq A$, which gives $A=C$.
(v) Set $C:=A \cap B$. Then $C$ is closed, $C \subseteq A, C \subseteq B$, and by (ii), $A, B$ and $C$ are locally identical around $C$. Now apply (iv).

The following example illustrates that the assumption on convexity of $A$ in Lemma 3.2|(iv) is important.

Example 3.3. Suppose that $X=\mathbb{R}$, that $A=\{0,1\}$ and that $C=\{0\}$. Then $A$ and $C$ are closed and locally identical around $C$, and $C \subseteq A$, but $C \neq A$. This does not contradict Lemma 3.2(iv) because $A$ is not convex.

Lemma 3.4. Let $A$ and $B$ be closed subsets of $X$, and assume that $A$ and $B$ are locally identical around some $c \in A \cap B$, say there exists $\varepsilon \in \mathbb{R}_{++}$such that $A \cap \operatorname{ball}(c ; \varepsilon)=B \cap \operatorname{ball}(c ; \varepsilon)$. Let

$$
\begin{equation*}
p \in A \cap \operatorname{int}(\operatorname{ball}(c ; \varepsilon))=B \cap \operatorname{int}(\operatorname{ball}(c ; \varepsilon)) . \tag{26}
\end{equation*}
$$

Then the following hold:
(i) If $A \subseteq B$, then $(\forall x \in X) P_{B} x \cap$ ball $(c ; \varepsilon) \subseteq P_{A} x$.
(ii) If $A$ and $B$ are convex, then $(\forall x \in X) p=P_{A} x \Leftrightarrow p=P_{B} x$. Equivalently, if $A$ and $B$ are convex then $P_{A}^{-1} p=P_{B}^{-1} p$.
(iii) If $A \subseteq B$ and $B$ is convex, then $(\forall x \in X)$
(a) $P_{B} x \in \operatorname{ball}(c ; \varepsilon) \Rightarrow P_{A} x=P_{B} x$;
(b) $p \in P_{A} x \quad \Leftrightarrow \quad p=P_{B} x$;
(c) $p \in P_{A} x \Rightarrow P_{A} x=P_{B} x=p$.

Proof. (i). Observe that $P_{B} x \cap \operatorname{ball}(c ; \varepsilon)=P_{B} x \cap(B \cap \operatorname{ball}(c ; \varepsilon))=P_{B} x \cap(A \cap \operatorname{ball}(c ; \varepsilon)) \subseteq A \cap P_{B} x$. The conclusion follows Lemma 2.2(i),

To prove (ii) and (iii), note that since $p \in \operatorname{int}($ ball $(c ; \varepsilon))$, there exists $\rho \in \mathbb{R}_{++}$such that ball $(p ; \rho) \subseteq$ ball $(c ; \varepsilon)$, which yields

$$
\begin{equation*}
A \cap \operatorname{ball}(p ; \rho)=B \cap \operatorname{ball}(p ; \rho) . \tag{27}
\end{equation*}
$$

(ii): By [10, Lemma 2.12], it follows from $p \in A \cap B$ and (27) that $N_{A}(p)=N_{B}(p)$. Now using (6), $(\forall x \in X) p=P_{A} x \Leftrightarrow x-p \in N_{A}(p)=N_{B}(p) \Leftrightarrow p=P_{B} x$. Hence, $P_{A}^{-1} p=P_{B}^{-1} p$.
(iii)(a) Let $x \in X$. Assume that $P_{B} x \in \operatorname{ball}(c ; \varepsilon)$. Then $P_{B} x \in B \cap \operatorname{ball}(c ; \varepsilon)=A \cap \operatorname{ball}(c ; \varepsilon) \subseteq A$. By Lemma 2.2((iv), $P_{A} x=P_{B} x$.
(iii)(b) Using (27) and applying (ii) for two convex sets $A \cap$ ball $(p ; \rho)$ and $B$, we obtain $P_{A \cap \text { ball }(p ; \rho)}^{-1} p=P_{B}^{-1} p$. Next applying Lemma 2.2(ii) for $A \cap \operatorname{ball}(p ; \rho) \subseteq A$ and $A \subseteq B$, we have $P_{A}^{-1} p \subseteq P_{A \cap \text { ball }(p ; \rho)}^{-1} p=P_{B}^{-1} p \subseteq P_{A}^{-1} p$, and so $P_{A}^{-1} p=P_{B}^{-1} p$.
(iii)(c) Now assume $p \in P_{A} x$. Then (iii)(b) gives $P_{B} x=p \in A$, and Lemma 2.2(iv) gives $P_{A} x=P_{B} x=p$.

## 4 Cases involving halfspaces

In this section, we assume that

$$
\begin{equation*}
f: X \rightarrow \mathbb{R} \text { is convex and continuous, } \tag{28}
\end{equation*}
$$

and that

$$
\begin{equation*}
\text { epi } f:=\{(x, \rho) \in X \times \mathbb{R} \mid f(x) \leq \rho\} \tag{29}
\end{equation*}
$$

In the space $X \times \mathbb{R}$, we set

$$
\begin{equation*}
H:=X \times\{0\} \quad \text { and } \quad B:=\operatorname{epi} f \tag{30}
\end{equation*}
$$

Then the projection onto $H$ is given by

$$
\begin{equation*}
(\forall(x, \rho) \in X \times \mathbb{R}) \quad P_{H}(x, \rho)=(x, 0) \tag{31}
\end{equation*}
$$

the projection onto $B$ is described as in Example 2.6, and the effect of performing each step of the DRA applied to $H$ and $B$ is characterized in the following result.
Fact 4.1 (One DRA step). Let $z=(x, \rho) \in X \times \mathbb{R}$, and set $z_{+}:=\left(x_{+}, \rho_{+}\right)=T_{H, B}(x, \rho)$. Then the following hold:
(i) If $\rho \leq-f(x)$, then $z_{+}=(x, 0) \in H$. Otherwise, there exists $x_{+}^{*} \in \partial f\left(x_{+}\right)$such that

$$
\begin{equation*}
x_{+}=x-\rho_{+} x_{+}^{*}, f\left(x_{+}\right) \leq f(x), \text { and } \rho_{+}=\rho+f\left(x_{+}\right)>0 ; \tag{32}
\end{equation*}
$$

in which either $\left(\rho \geq 0\right.$ and $\left.z_{+} \in B\right)$ or $\left(\rho<0\right.$ and $\left.T_{H, B} z_{+} \in B\right)$.
(ii) $\operatorname{ran} T_{H, B} \subseteq X \times \mathbb{R}_{+}$, or equivalently, $(\forall z \in X \times \mathbb{R}) z_{+} \in X \times \mathbb{R}_{+}$.

Proof. (i): [10, Corollary 5.3(i)\&(ii)]. (ii), Clear from(i),
We have the following result on convergence of the DRA in the case of a hyperplane and an epigraph.
Fact 4.2 (Finite convergence of DRA in the (hyperplane,epigraph) case). Suppose that

$$
\begin{equation*}
A=H \quad \text { and } \quad B=\operatorname{epi} f \text { with } \inf _{X} f<0 \tag{33}
\end{equation*}
$$

Given a starting point $z_{0}=\left(x_{0}, \rho_{0}\right) \in X \times \mathbb{R}$, generate the $D R A$ sequence $\left(z_{n}\right)_{n \in \mathbb{N}}$ by

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad z_{n+1}=\left(x_{n+1}, \rho_{n+1}\right)=T_{A, B} z_{n} . \tag{34}
\end{equation*}
$$

Then $\left(z_{n}\right)_{n \in \mathbb{N}}$ converges finitely to a point in $A \cap B$.

Proof. See [10, Theorem 5.4].
In view of Fact 4.2 , it is natural to ask about the convergence of the DRA when $A$ is a halfspace instead of a hyperplane.

Theorem 4.3 (Finite convergence of DRA in the (halfspace,epigraph) case). Suppose that either
(i) $A=H_{+}:=X \times \mathbb{R}_{+}$and $B=\operatorname{epi} f$, or
(ii) $A=H_{-}:=X \times \mathbb{R}_{-}$and $B=\operatorname{epi} f$ with $\inf _{X} f<0$.

Then the $D R A$ sequence (34) converges finitely to a point in $A \cap B$.
Proof. (i); Let $z=(x, \rho) \in X \times \mathbb{R}$. If $z \in H_{-}$, then $P_{A} z=P_{H} z$, and so $z_{+}:=T_{A, B} z=T_{H, B} z \in$ $H_{+}$due to Fact 4.1|((ii). If $z \in H_{+} \cap B=A \cap B$, we are done. If $z \in H_{+} \backslash B$, then $P_{A} z=z$, $R_{A} z=z$, and by Example 2.6, $P_{B} R_{A} z=P_{B} z=\left(x_{+}, f\left(x_{+}\right)\right)$with $f\left(x_{+}\right)>\rho \geq 0$, which implies $z_{+}=z-P_{A} z+P_{B} R_{A} z=\left(x_{+}, f\left(x_{+}\right)\right) \in H_{+} \cap B=A \cap B$. We deduce that the DRA sequence (34) converges in at most two steps.
(ii)] If $z_{0} \in H_{-}=A$, then $P_{A} z_{0}=z_{0}, R_{A} z_{0}=z_{0}$, and $z_{1}=P_{B} R_{A} z_{0}=\left(x_{1}, f\left(x_{1}\right)\right) \in B$, which gives $z_{1} \in A \cap B$ if $f\left(x_{1}\right) \leq 0$, and $z_{1} \in H_{+}$otherwise. It is thus sufficient to consider the case $z_{0} \in H_{+}$. Then $P_{A} z_{0}=P_{H} z_{0}$, and so $z_{1}=T_{A, B} z_{0}=T_{H, B} z_{0} \in H_{+}$due to Fact 4.1|(ii). This implies that

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad z_{n} \in H_{+} \quad \text { and } \quad z_{n+1}=T_{H, B} z_{n} \tag{35}
\end{equation*}
$$

Now apply Fact 4.2 .
The following example whose special cases can be found in [9] illustrates that the Slater's condition $\inf _{X} f<0$ in Fact 4.2 and Theorem 4.3(ii) is important.

Example 4.4. Suppose that either $A=H$ or $A=H_{-}:=X \times \mathbb{R}_{-}$, that $B=\operatorname{epi} f$ with $\inf _{X} f \geq 0$, and that $f$ is differentiable at its minimizers (if they exist). Let $z_{0}=\left(x_{0}, \rho_{0}\right) \in B$, where $x_{0}$ is not a minimizer of $f$, and generate the DRA sequence $\left(z_{n}\right)_{n \in \mathbb{N}}$ as in (34). Then $\left(P_{A} z_{n}\right)_{n \in \mathbb{N}}$ and thus also $\left(z_{n}\right)_{n \in \mathbb{N}}$ do not converge finitely.

Proof. Firstly, we claim that if $z=(x, \rho) \in B$, where $x$ is not a minimizer of $f$, then $z_{+}:=T_{A, B} z=$ $T_{H, B} z=\left(x_{+}, \rho_{+}\right) \in B$ and $x_{+}$is not a minimizer of $f$. Indeed, by assumption, $\rho>0$, so $P_{A} z=P_{H} z$, and then $z_{+}=T_{A, B} z=T_{H, B} z$. By using Fact 4.1)(i), $z_{+} \in B$ and

$$
\begin{equation*}
x_{+}=x-\rho_{+} x_{+}^{*} \quad \text { with } \quad x_{+}^{*} \in \partial f\left(x_{+}\right), \quad \text { and } \quad \rho_{+}=\rho+f\left(x_{+}\right)>0 . \tag{36}
\end{equation*}
$$

If $x_{+}$is a minimizer of $f$, then $x_{+}^{*}=\nabla f\left(x_{+}\right)=0$, and by (36), $x=x_{+}$is a minimizer, which is absurd. Hence, the claim holds. As a result,

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad x_{n} \text { is not a minimizer of } f . \tag{37}
\end{equation*}
$$

Now assume that $\left(P_{A} z_{n}\right)_{n \in \mathbb{N}}=\left(x_{n}, 0\right)_{n \in \mathbb{N}}$ converges finitely. Then there exists $n \in \mathbb{N}$ such that $x_{n+1}=x_{n}$. Using again (36), we get $x_{n+1}^{*}=0 \in \partial f\left(x_{n+1}\right)$, which contradicts (37).

Theorem 4.5 (Finite convergence of DRA in (hyperplane or halfspace,halfspace) case). Suppose that $A$ is either a hyperplane or a halfspace, that $B$ is a halfspace of $X$, and that $A \cap B \neq \varnothing$. Then every DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$ converges finitely to a point $x$, where $x \in A \cap B$ or $(\forall n \in \mathbb{N})$ $x_{n}=x \in B$ with $P_{A} x \in A \cap B$.

Proof. If $\operatorname{dim} X=0$, i.e., $X=\{0\}$, then the result is trivial, so we will work in the space $X \times \mathbb{R}$ with $\operatorname{dim} X \geq 0$, and denote by $\left(z_{n}\right)_{n \in \mathbb{N}}$ the DRA sequence. After rotating the sets if necessary, we can and do assume that $A=X \times \mathbb{R}_{-}$, and $B=\{(x, \rho) \in X \times \mathbb{R} \mid\langle(x, \rho),(u, v)\rangle \leq \eta\}$, with $(u, v) \in X \times \mathbb{R} \backslash\{(0,0)\}$ and $\eta \in \mathbb{R}$. Noting that $\langle(x, \rho),(u, v)\rangle=\langle x, u\rangle+\rho v$, we distinguish the following three cases.

Case 1: $v<0$. Then

$$
\begin{equation*}
B=\left\{(x, \rho) \in X \times \mathbb{R} \left\lvert\, \frac{\eta-\langle x, u\rangle}{v} \leq \rho\right.\right\} \tag{38}
\end{equation*}
$$

is the epigraph of the linear function

$$
\begin{equation*}
f: X \rightarrow \mathbb{R}: x \mapsto \frac{\eta-\langle x, u\rangle}{v} . \tag{39}
\end{equation*}
$$

If $\inf _{X} f<0$, we are done due to Theorem4.43(ii), Assume that $\inf _{X} f \geq 0$. Then $u=0 \in X$ since $u \in X \backslash\{0\}$ implies $\inf _{X} f \leq \inf _{\lambda \in \mathbb{R}_{-}} f(\lambda u)=\inf _{\lambda \in \mathbb{R}_{-}} \frac{\eta-\lambda\|u\|^{2}}{v}=-\infty$. Now in turn, $(\forall x \in X)$ $f(x)=\frac{\eta}{v}$, and so $\frac{\eta}{v}=\inf _{X} f \geq 0$, which gives $\eta \leq 0$. By the assumption that $A \cap B \neq \varnothing$, we must have $\eta=0$, and then $B=X \times \mathbb{R}_{+}$. Let $z=(x, \rho) \in X \times \mathbb{R}$. If $z \in B$, then $R_{A} z=(x,-\rho)$, and $R_{B} R_{A} z=(x, \rho)=z$, which gives $T_{A, B} z=z$, i.e., $z \in$ Fix $T_{A, B}$, in which case $P_{A} z=(x, 0) \in A \cap B$. If $z \notin B$, then $z \in A$ and $R_{A} z=z, R_{B} R_{A} z=R_{B} z=(x,-\rho)$, so

$$
\begin{equation*}
T_{A, B} z=\frac{1}{2}\left(z+R_{B} R_{A} z\right)=(x, 0) \in A \cap B . \tag{40}
\end{equation*}
$$

Case 2: $v>0$. Then

$$
\begin{equation*}
B=\left\{(x, \rho) \in X \times \mathbb{R} \left\lvert\, \frac{\eta-\langle x, u\rangle}{v} \geq \rho\right.\right\} . \tag{41}
\end{equation*}
$$

After reflecting the sets across the hyperplane $X \times\{0\}$, we have $A=X \times \mathbb{R}_{+}$, and $B$ is the epigraph of a linear function. Now apply Theorem 4.3)(i),

Case 3: $v=0$. Then $u \in X \backslash\{0\}$ and

$$
\begin{equation*}
B=\{(x, \rho) \in X \times \mathbb{R} \mid\langle x, u\rangle \leq \eta\}=\{x \in X \mid\langle x, u\rangle \leq \eta\} \times \mathbb{R} . \tag{42}
\end{equation*}
$$

Let $z=(x, \rho) \in X \times \mathbb{R}$. If $z \in A \cap B$, we are done. If $z \in A \backslash B$, then $\rho \in \mathbb{R}_{-}, R_{A} z=P_{A} z=z \notin B$, and by Example 2.4(ii),

$$
\begin{equation*}
P_{B} R_{A} z=P_{B} z=\left(x-\frac{\langle x, u\rangle-\eta}{\|u\|^{2}} u, \rho\right) \in B \tag{43}
\end{equation*}
$$

which is also in $A=X \times \mathbb{R}_{-}$and which yields

$$
\begin{equation*}
T_{A, B} z=z-P_{A} z+P_{B} R_{A} z=P_{B} R_{A} z \in A \cap B . \tag{44}
\end{equation*}
$$

Now assume that $z \notin A$. We have $P_{A} z=(x, 0)$ and $R_{A} z=(x,-\rho)$. If $(x,-\rho) \in B$, then $R_{B} R_{A} z=$ $(x,-\rho)$, and $T_{A, B} z=\frac{1}{2}\left(z+R_{B} R_{A} z\right)=(x, 0) \in A \cap B$. Finally, if $(x,-\rho) \notin B$, then again by Example 2.4(ii),

$$
\begin{equation*}
P_{B} R_{A} z=P_{B}(x,-\rho)=(x,-\rho)-\frac{\langle x, u\rangle-\eta}{\|u\|^{2}}(u, 0), \tag{45}
\end{equation*}
$$

and thus,

$$
\begin{equation*}
T_{A, B} z=z-P_{A} z+P_{B} R_{A} z=\left(x-\frac{\langle x, u\rangle-\eta}{\|u\|^{2}} u, 0\right) \in A . \tag{46}
\end{equation*}
$$

Moreover, $\left\langle x-\frac{\langle x, u\rangle-\eta}{\|u\|^{2}} u, u\right\rangle=\eta$, so $T_{A, B} z \in B$, and we get $T_{A, B} z \in A \cap B$.
The proof for the (hyperplane,halfspace) case is similar and uses Fact 4.2

## 5 Expanding and modifying sets

Lemma 5.1 (Expanding sets). Let $A$ and $B$ be closed (not necessarily convex) subsets of $X$ such that $A \cap B \neq \varnothing$, and let $x_{0}$ be in $X$. Suppose that the DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$, with starting point $x_{0}$, converges to $x \in \operatorname{Fix} T_{A, B}$. Suppose further that there exist two closed convex sets $A^{\prime}$ and $B^{\prime}$ in $X$ such that $A \subseteq A^{\prime}, B \subseteq B^{\prime}$, and that both $\left(A, A^{\prime}\right)$ and $\left(B, B^{\prime}\right)$ are locally identical around some $c \in P_{A} x$. Then $P_{A} x=P_{A^{\prime}} x, x \in \operatorname{Fix} T_{A^{\prime}, B^{\prime}}$ and

$$
\begin{equation*}
\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A^{\prime}, B^{\prime}} x_{n} \tag{47}
\end{equation*}
$$

i.e., $\left(\exists n_{0} \in \mathbb{N}\right)(\forall n \in \mathbb{N}) \quad T_{A, B}^{n} x_{n_{0}}=T_{A^{\prime}, B^{\prime}}^{n} x_{n_{0}}$.

Proof. By assumption and Lemma 3.4)(iii)(c), $P_{A} x=P_{A^{\prime}} x=c$, and so $R_{A} x=R_{A^{\prime}} x$. Since $x \in$ Fix $T_{A, B}$, it follows from (21) that $c \in P_{B} R_{A} x=P_{B} R_{A^{\prime}} x$. Using again Lemma 3.4)(iii)(c), $P_{B} R_{A^{\prime}} x=$ $P_{B^{\prime}} R_{A^{\prime}} x=c$. We get $P_{A^{\prime}} x=P_{B^{\prime}} R_{A^{\prime}} x=c$, and again by (21), $x \in$ Fix $T_{A^{\prime}, B^{\prime}}$. Now by the definition of $A^{\prime}$ and $B^{\prime}$, there exists $\varepsilon \in \mathbb{R}_{++}$such that

$$
\begin{equation*}
A \cap \operatorname{ball}(c ; \varepsilon)=A^{\prime} \cap \operatorname{ball}(c ; \varepsilon) \quad \text { and } \quad B \cap \operatorname{ball}(c ; \varepsilon)=B^{\prime} \cap \operatorname{ball}(c ; \varepsilon) . \tag{48}
\end{equation*}
$$

There exists $n_{0} \in \mathbb{N}$ such that

$$
\begin{equation*}
\left(\forall n \geq n_{0}\right) \quad\left\|x_{n}-x\right\|<\varepsilon . \tag{49}
\end{equation*}
$$

Let $n \geq n_{0}$. Since $P_{A^{\prime}}, P_{B^{\prime}}$ are (firmly) nonexpansive and $R_{A^{\prime}}$ is nonexpansive (Fact 2.1|(ii) \&(iii)),

$$
\begin{equation*}
\left\|P_{A^{\prime}} x_{n}-c\right\|=\left\|P_{A^{\prime}} x_{n}-P_{A^{\prime}} x\right\| \leq\left\|x_{n}-x\right\|<\varepsilon, \tag{50}
\end{equation*}
$$

and also

$$
\begin{equation*}
\left\|P_{B^{\prime}} R_{A^{\prime}} x_{n}-c\right\|=\left\|P_{B^{\prime}} R_{A^{\prime}} x_{n}-P_{B^{\prime}} R_{A^{\prime}} x\right\| \leq\left\|x_{n}-x\right\|<\varepsilon . \tag{51}
\end{equation*}
$$

Thus, $P_{A^{\prime}} x_{n} \in \operatorname{ball}(c ; \varepsilon)$ and $P_{B^{\prime}} R_{A^{\prime}} x_{n} \in \operatorname{ball}(c ; \varepsilon)$. By Lemma 3.4(iii)(a), $P_{A} x_{n}=P_{A^{\prime}} x_{n}$ and $P_{B} R_{A^{\prime}} x_{n}=P_{B^{\prime}} R_{A^{\prime}} x_{n}$, which implies $R_{A} x_{n}=R_{A^{\prime}} x_{n}$ and $P_{B} R_{A} x_{n}=P_{B^{\prime}} R_{A^{\prime}} x_{n}$. We deduce that $x_{n+1}=T_{A, B} x_{n}=T_{A^{\prime}, B^{\prime}} x_{n}$.

If the assumption that $A \subseteq A^{\prime}$ and $B \subseteq B^{\prime}$ in Lemma 5.1 is replaced by the assumption on convexity of $A$ and $B$, then (47) still holds, as shown in the following lemma. We shall now look at situations where $\left(A^{\prime}, B^{\prime}\right)$ are modifications of $(A, B)$ that preserve local structure.

Lemma 5.2. Let $A$ and $B$ be closed convex subsets of $X$ such that $A \cap B \neq \varnothing$, and let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be the DRA sequence with respect to $(A, B)$, with starting point $x_{0} \in X$. Suppose that there exist two closed convex sets $A^{\prime}$ and $B^{\prime}$ in $X$ such that both $\left(A, A^{\prime}\right)$ and $\left(B, B^{\prime}\right)$ are locally identical around $P_{A} x \in A \cap B$, where $x \in \operatorname{Fix} T_{A, B}$ is the limit of $\left(x_{n}\right)_{n \in \mathbb{N}}$. Then

$$
\begin{equation*}
\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A^{\prime}, B^{\prime}} x_{n} \tag{52}
\end{equation*}
$$

i.e., $\left(\exists n_{0} \in \mathbb{N}\right)(\forall n \in \mathbb{N}) \quad T_{A, B}^{n} x_{n_{0}}=T_{A^{\prime}, B^{\prime}}^{n} x_{n_{0}}$.

Proof. Recall from Fact 2.7(i) that $x_{n} \rightarrow x \in \operatorname{Fix} T_{A, B}$ with $P_{A} x \in A \cap B$. Setting $c:=P_{A} x=P_{B} R_{A} x$, from the assumption on $A^{\prime}$ and $B^{\prime}$, there is $\varepsilon \in \mathbb{R}_{++}$such that

$$
\begin{equation*}
A \cap \operatorname{ball}(c ; \varepsilon)=A^{\prime} \cap \operatorname{ball}(c ; \varepsilon) \quad \text { and } \quad B \cap \operatorname{ball}(c ; \varepsilon)=B^{\prime} \cap \operatorname{ball}(c ; \varepsilon) . \tag{53}
\end{equation*}
$$

Furthermore, there exists $n_{0} \in \mathbb{N}$ such that

$$
\begin{equation*}
\left(\forall n \geq n_{0}\right) \quad\left\|x_{n}-x\right\|<\varepsilon . \tag{54}
\end{equation*}
$$

Let $n \geq n_{0}$. According to Fact 2.1 (ii) \& (iii), $P_{A}, P_{B}$ are (firmly) nonexpansive and $R_{A}$ is nonexpansive, so

$$
\begin{equation*}
\left\|P_{A} x_{n}-c\right\|=\left\|P_{A} x_{n}-P_{A} x\right\| \leq\left\|x_{n}-x\right\|<\varepsilon, \tag{55}
\end{equation*}
$$

and also

$$
\begin{equation*}
\left\|P_{B} R_{A} x_{n}-c\right\|=\left\|P_{B} R_{A} x_{n}-P_{B} R_{A} x\right\| \leq\left\|x_{n}-x\right\|<\varepsilon . \tag{56}
\end{equation*}
$$

Therefore, $P_{A} x_{n} \in A \cap \operatorname{int}$ ball $(c ; \varepsilon)$ and $P_{B} R_{A} x_{n} \in B \cap \operatorname{intball}(c ; \varepsilon)$. Using Lemma 3.4(ii), $P_{A} x_{n}=$ $P_{A^{\prime}} x_{n}$ and $P_{B} R_{A} x_{n}=P_{B^{\prime}} R_{A} x_{n}$. Hence $R_{A} x_{n}=R_{A^{\prime}} x_{n}$ and $P_{B} R_{A} x_{n}=P_{B^{\prime}} R_{A^{\prime}} x_{n}$. We obtain that $x_{n+1}=T_{A, B} x_{n}=T_{A^{\prime}, B^{\prime}} x_{n}$.

Theorem 5.3 (Modifying sets). Let $A$ and $B$ be closed convex subsets of $X$ such that $A \cap B \neq \varnothing$. Suppose that there exist two closed convex sets $A^{\prime}$ and $B^{\prime}$ in $X$ such that both $\left(A, A^{\prime}\right)$ and $\left(B, B^{\prime}\right)$ are locally identical around $A \cap B$. Then for any $D R A$ sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$ in (17),

$$
\begin{equation*}
\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A^{\prime}, B^{\prime}} x_{n} \tag{57}
\end{equation*}
$$

and this is still true when exchanging the roles of $T_{A, B}$ and $T_{A^{\prime}, B^{\prime}}$ in (17) and (57).
Proof. By Fact [2.7(i), $x_{n} \rightarrow x \in$ Fix $T_{A, B}$ with $P_{A} x \in A \cap B$. Now apply Lemma 5.2 .
Let us exchange the roles of $T_{A, B}$ and $T_{A^{\prime}, B^{\prime}}$ in (17) and (57), i.e., $(\forall n \in \mathbb{N}) x_{n+1}=T_{A^{\prime}, B^{\prime}} x_{n}$, and we shall prove that

$$
\begin{equation*}
\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A, B} x_{n} . \tag{58}
\end{equation*}
$$

By the assumption on $A^{\prime}$ and $B^{\prime}$, we have $A \cap B \subseteq A^{\prime}, A \cap B \subseteq B^{\prime}$, and for all $c \in A \cap B$, there exists $\varepsilon \in \mathbb{R}_{++}$such that

$$
\begin{equation*}
A \cap \operatorname{ball}(c ; \varepsilon)=A^{\prime} \cap \operatorname{ball}(c ; \varepsilon) \quad \text { and } \quad B \cap \operatorname{ball}(c ; \varepsilon)=B^{\prime} \cap \operatorname{ball}(c ; \varepsilon) . \tag{59}
\end{equation*}
$$

Then

$$
\begin{equation*}
(A \cap B) \cap \operatorname{ball}(c ; \varepsilon)=\left(A^{\prime} \cap B^{\prime}\right) \cap \operatorname{ball}(c ; \varepsilon) . \tag{60}
\end{equation*}
$$

Therefore, $A \cap B$ and $A^{\prime} \cap B^{\prime}$ are locally identical around $A \cap B$. Noting that $A \cap B$ and $A^{\prime} \cap B^{\prime}$ are closed convex, and $A \cap B \subseteq A^{\prime} \cap B^{\prime}$, Lemma 3.2(iv) gives $A \cap B=A^{\prime} \cap B^{\prime}$. Next again by Fact 2.7 (i), $x_{n} \rightarrow x \in \operatorname{Fix} T_{A^{\prime}, B^{\prime}}$ with $P_{A^{\prime}} x \in A^{\prime} \cap B^{\prime}=A \cap B$. By assumption, both $\left(A, A^{\prime}\right)$ and $\left(B, B^{\prime}\right)$ are locally identical around $P_{A^{\prime}} x$, and hence the proof is completed by applying Lemma 5.2 .

In the following, we say that the DRA applied to $(A, B)$ converges finitely globally if the sequence $\left(T_{A, B}^{n} x\right)_{n \in \mathbb{N}}$ converges finitely for all $x \in X$.

Theorem 5.4. Let $A$ and $B$ be nonempty closed convex subsets of $X$. Then the DRA applied to $(A, B)$ converges finitely globally provided one of the following holds:
(i) $A \cap B \neq \varnothing$ and $A \cap$ bdry $B=\varnothing$; equivalently, $A \subseteq \operatorname{int} B$.
(ii) $A \cap$ bdry $B \neq \varnothing$ and there exist two closed convex sets $A^{\prime}$ and $B^{\prime}$ in $X$ such that both $\left(A, A^{\prime}\right)$ and $\left(B, B^{\prime}\right)$ are locally identical around $A \cap$ bdry $B$, and that the DRA applied to $\left(A^{\prime}, B^{\prime}\right)$ converges finitely globally when $A^{\prime} \cap B^{\prime} \neq \varnothing$.
(iii) $A \cap \operatorname{int} B \neq \varnothing, A \cap$ bdry $B \neq \varnothing$ and there exist two closed convex sets $A^{\prime}$ and $B^{\prime}$ in $X$ such that both $\left(A, A^{\prime}\right)$ and $\left(B, B^{\prime}\right)$ are locally identical around $A \cap \mathrm{bdry} B$, and that the $D R A$ applied to $\left(A^{\prime}, B^{\prime}\right)$ converges finitely globally when $A^{\prime} \cap$ int $B^{\prime} \neq \varnothing$.

Proof. Let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be a DRA sequence with respect to $(A, B)$.
(i). It follows from $A \cap B \neq \varnothing, A \cap \operatorname{bdry} B=\varnothing$ and the closedness of $B$ that $A \cap \operatorname{int} B=$ $A \cap B \neq \varnothing$, and so $0 \in \operatorname{int}(A-B)$. By Fact 2.7 (fii), $x_{n} \rightarrow x \in A \cap B=A \cap \operatorname{int} B$ finitely.

Now if $A \subseteq$ int $B$, then $A \cap B=A \neq \varnothing$, and $A \cap$ bdry $B \subseteq$ int $B \cap$ bdry $B=\varnothing$, which implies $A \cap$ bdry $B=\varnothing$. Conversely, assume that $A \cap B \neq \varnothing$ and $A \cap$ bdry $B=\varnothing$. Let $a \in A$. Then $a \notin$ bdry $B$. We have to show $a \in \operatorname{int} B$. Suppose to the contrary that $a \notin \operatorname{int} B$. Pick $b \in A \cap B$. By convexity, $[a, b]:=\{\lambda a+(1-\lambda) b \mid 0 \leq \lambda \leq 1\} \subseteq A$, and so $[a, b] \cap$ bdry $B=\varnothing$, which is impossible since $a \notin B$ and $b \in B$. Hence, $a \in \operatorname{int} B$ for all $a \in A$. This means $A \subseteq \operatorname{int} B$.
(ii): By assumption, $A \cap$ bdry $B \subseteq A^{\prime} \cap B^{\prime}$, and so the DRA applied to ( $A^{\prime}, B^{\prime}$ ) converges finitely globally. If $A \cap \operatorname{int} B=\varnothing$, then both $\left(A, A^{\prime}\right)$ and $\left(B, B^{\prime}\right)$ are locally identical around $A \cap \mathrm{bdry} B=$ $A \cap B$ (using the closedness of $B$ ), and using Theorem 5.3.

$$
\begin{equation*}
\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A^{\prime}, B^{\prime}} x_{n} \tag{61}
\end{equation*}
$$

which implies the finite convergence of $\left(x_{n}\right)_{n \in \mathbb{N}}$ due to the finite convergence of the DRA applied to $\left(A^{\prime}, B^{\prime}\right)$.

Next assume that $A \cap \operatorname{int} B \neq \varnothing$. Then $\operatorname{int}(A-B) \neq \varnothing$. By Fact 2.7(ii), $x_{n} \rightarrow x \in A \cap B$, and this convergence is finite when $x \in A \cap \operatorname{int} B$. It thus suffices to consider the case when $x \in A \cap \mathrm{bdry} B$. Then $\left(A, A^{\prime}\right)$ and $\left(B, B^{\prime}\right)$ are locally identical around $x=P_{A} x$, and by Lemma 5.2, (61) holds. Using again the finite convergence of the DRA applied to $\left(A^{\prime}, B^{\prime}\right)$, we are done.
(iii); First, we show that $A^{\prime} \cap \operatorname{int} B^{\prime} \neq \varnothing$. Let $c \in A \cap \operatorname{bdry} B$. By the assumption on $A^{\prime}$ and $B^{\prime}$, there is $\varepsilon \in \mathbb{R}_{++}$such that

$$
\begin{equation*}
A \cap \operatorname{ball}(c ; \varepsilon)=A^{\prime} \cap \operatorname{ball}(c ; \varepsilon) \quad \text { and } \quad B \cap \operatorname{ball}(c ; \varepsilon)=B^{\prime} \cap \operatorname{ball}(c ; \varepsilon) . \tag{62}
\end{equation*}
$$

Now let $d \in A \cap \operatorname{int} B$. Then $c \neq d$, and by the convexity of $A$ and $B$, [6, Proposition 3.35] implies $] c, d]:=\{\lambda c+(1-\lambda) d \mid 0 \leq \lambda<1\} \subseteq A \cap \operatorname{int} B$. Therefore, $] c, d] \cap \operatorname{int}$ ball $(c ; \varepsilon) \subseteq A \cap \operatorname{ball}(c ; \varepsilon)=$ $A^{\prime} \cap \operatorname{ball}(c ; \varepsilon) \subseteq A^{\prime}$ and $\left.] c, d\right] \cap \operatorname{int} \operatorname{ball}(c ; \varepsilon) \subseteq \operatorname{int}(B \cap \operatorname{ball}(c ; \varepsilon))=\operatorname{int}\left(B^{\prime} \cap \operatorname{ball}(c ; \varepsilon)\right) \subseteq \operatorname{int} B^{\prime}$. We deduce that $A^{\prime} \cap \operatorname{int} B^{\prime} \neq \varnothing$. By assumption, the DRA applied to ( $A^{\prime}, B^{\prime}$ ) converges finitely globally. Now argue as the case where $A \cap \operatorname{int} B \neq \varnothing$ in the proof of part (ii)

Corollary 5.5. Let $A$ and $B$ be closed convex subsets of $X$ such that $A \cap B \neq \varnothing$. Suppose that both $(A, \operatorname{aff} A)$ and $(B, \operatorname{aff} B)$ are locally identical around $A \cap \operatorname{bdry} B$ when $A \cap \operatorname{bdry} B \neq \varnothing$. Then every $D R A$ sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$ converges linearly with rate $c_{F}(\operatorname{aff} A-\operatorname{aff} A, \operatorname{aff} B-\operatorname{aff} B)$ to a point $x \in \operatorname{Fix} T_{A, B}$ with $P_{A} x \in A \cap B$, where $c_{F}(U, V)$ is the cosine of the Friedrichs angle between two subspaces $U$ and $V$ defined by

$$
\begin{equation*}
c_{F}(U, V):=\sup \left\{|\langle u, v\rangle|\left\|u \in U \cap(U \cap V)^{\perp}, v \in V \cap(U \cap V)^{\perp},\right\| u\|\leq 1,\| v \| \leq 1\right\} . \tag{63}
\end{equation*}
$$

Proof. If $A \cap$ bdry $B=\varnothing$, then by Theorem [5.4|(i), we are done. Now assume that $A \cap \mathrm{bdry} B \neq \varnothing$. By assumption and Theorem 5.3.

$$
\begin{equation*}
\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{\text {aff } A, \text { aff } B} x_{n} . \tag{64}
\end{equation*}
$$

Since we work with a finite-dimensional space, [4, Corollary 4.5] completes the proof.

Example 5.6. Suppose that $X=\mathbb{R}^{3}$, that $A=[(2,1,2),(-2,1,-2)]$, and that $B=$ $\left\{(\alpha, \beta, \gamma) \in \mathbb{R}^{3}| | \alpha|\leq 2,|\beta| \leq 2, \gamma=1\}\right.$. Then $A \cap B=\{(1,1,1)\} \in$ ri $A \cap$ ri $B$. By [38, Theorem 4.14], every DRA sequence with respect to $(A, B)$ converges linearly. Furthermore, aff $A=\left\{(\alpha, \beta, \gamma) \in \mathbb{R}^{3} \mid \alpha-\gamma=0, \beta=1\right\}$, aff $B=\left\{(\alpha, \beta, \gamma) \in \mathbb{R}^{3} \mid \gamma=1\right\}$, aff $A-\operatorname{aff} A=$ $\left\{(\alpha, \beta, \gamma) \in \mathbb{R}^{3} \mid \alpha-\gamma=0, \beta=0\right\}$, aff $B-\operatorname{aff} B=\left\{(\alpha, \beta, \gamma) \in \mathbb{R}^{3} \mid \gamma=0\right\}$, and both $(A$, aff $A)$ and ( $B, \operatorname{aff} B$ ) are locally identical around $A \cap$ bdry $B=A \cap B$. By applying Corollary 5.5, the linearly rate is $c_{F}(\operatorname{aff} A-\operatorname{aff} A, \operatorname{aff} B-\operatorname{aff} B)=1 / \sqrt{2}$.

Proposition 5.7 (Finite convergence of the DRA in the (hyperplane or halfspace,ball) case). Let $A$ be either a hyperplane or a halfspace, and $B$ be a closed ball of $X$ such that $A \cap \operatorname{int} B \neq \varnothing$. Then every $D R A$ sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$ converges in finitely many steps to a point in $A \cap B$.

Proof. If $\operatorname{dim} X=0$, i.e., $X=\{0\}$, then the result is trivial, so we will work in the space $X \times \mathbb{R}$ with $\operatorname{dim} X \geq 0$, and denote by $\left(z_{n}\right)_{n \in \mathbb{N}}$ the DRA sequence. We just prove the the result for the case when $A$ is a hyperplane because the case when $A$ is a halfspace is similar. Without loss of generality, we assume that $A=X \times\{0\}$ and that $B=\operatorname{ball}((0, \theta) ; 1)$ is the closed ball of radius 1 and center $(0, \theta) \in X \times \mathbb{R}$ with $0 \leq \theta<1$. Nothing that

$$
\begin{equation*}
B=\left\{(x, \rho) \in X \times \mathbb{R} \mid \theta-\sqrt{1-\|x\|^{2}} \leq \rho \leq \theta+\sqrt{1-\|x\|^{2}}\right\} \tag{65}
\end{equation*}
$$

we write $B=B_{-} \cup B_{+}$, where

$$
\begin{align*}
& B_{-}=\left\{(x, \rho) \in X \times \mathbb{R} \mid \theta-\sqrt{1-\|x\|^{2}} \leq \rho \leq \theta\right\}  \tag{66a}\\
& B_{+}=\left\{(x, \rho) \in X \times \mathbb{R} \mid \theta \leq \rho \leq \theta+\sqrt{1-\|x\|^{2}}\right\} \tag{66b}
\end{align*}
$$

We distinguish two cases.
Case 1: $\theta=0$. Then the two halves $B_{-} \subseteq X \times \mathbb{R}_{-}$and $B_{+} \subseteq X \times \mathbb{R}_{+}$of the ball $B$ are symmetric with respect to the hyperplane $A$. By symmetry, we can and do assume that $z_{0}=\left(x_{0}, \rho_{0}\right) \in$ $X \times \mathbb{R}_{+}$. Now for any $z=(x, \rho) \in X \times \mathbb{R}_{+}$, we have $P_{A} z=(x, 0), R_{A} z=(x,-\rho)$, and by Example 2.5.

$$
\begin{equation*}
P_{B} R_{A} z=\delta(x,-\rho) \quad \text { with } \quad \delta:=\frac{1}{\max \left\{\sqrt{\|x\|^{2}+\rho^{2}}, 1\right\}} \leq 1 \tag{67}
\end{equation*}
$$

which gives

$$
\begin{equation*}
T_{A, B} z=z-P_{A} z+P_{B} R_{A} z=(x, \rho)-(x, 0)+\delta(x,-\rho)=(\delta x,(1-\delta) \rho) \in X \times \mathbb{R}_{+} \tag{68}
\end{equation*}
$$

Hence, $(\forall n \in \mathbb{N}) z_{n} \in X \times \mathbb{R}_{+}$. From $B_{-}=\left\{(x, \rho) \in X \times \mathbb{R} \mid-\sqrt{1-\|x\|^{2}} \leq \rho \leq 0\right\}$, we have

$$
\begin{equation*}
B_{-} \subseteq B^{\prime}:=\operatorname{epi} f:=\{(x, \rho) \in X \times \mathbb{R} \mid f(x) \leq \rho\} \tag{69}
\end{equation*}
$$

where $f: X \rightarrow \mathbb{R}: x \mapsto-\sqrt{1-\|x\|^{2}}$. Since $R_{A} z_{n} \in X \times \mathbb{R}_{-}, P_{B} R_{A} z_{n}=P_{B_{-}} R_{A} z_{n}=P_{B^{\prime}} R_{A} z_{n}$, and so

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad z_{n+1}:=T_{A, B} z_{n}=T_{A, B^{\prime}} z_{n} . \tag{70}
\end{equation*}
$$

According to Fact 4.2, $\left(z_{n}\right)_{n \in \mathbb{N}}$ converges finitely to a point in $A \cap B^{\prime}=A \cap B$.
Case 2: $0<\theta<1$. Let $B^{\prime}:=\operatorname{epi} f$, where $f: X \rightarrow \mathbb{R}: x \mapsto \theta-\sqrt{1-\|x\|^{2}}$. Then $B \subseteq$ $B^{\prime}, A \cap B=A \cap B^{\prime}=\left\{(x, 0) \in X \times \mathbb{R} \mid \theta-\sqrt{1-\|x\|^{2}} \leq 0\right\} \subseteq A=X \times\{0\}$, and $B^{\prime} \backslash B=$
$\left\{(x, \rho) \in X \times \mathbb{R} \mid \theta+\sqrt{1-\|x\|^{2}}<\rho\right\} \subseteq X \times \mathbb{R}_{++}$, which implies $(\forall c \in A \cap B) d_{B^{\prime} \backslash B}(c)>0$. Following Lemma 3.2|(iii), $B$ and $B^{\prime}$ are locally identical around $A \cap B$. By using Theorem 5.3,

$$
\begin{equation*}
\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad z_{n+1}=T_{A, B^{\prime}} z_{n} \tag{71}
\end{equation*}
$$

and again by Fact 4.2, we are done.
Remark 5.8. It follows from Example 4.4 that the conclusion of Proposition 5.7 no longer holds without Slater's condition $A \cap \operatorname{int} B \neq \varnothing$.

Proposition 5.9. Let $A=\bigcap_{i \in I} A_{i}$ and $B=\bigcap_{j \in J} B_{j}$ be finite intersections of closed convex sets in $X$ such that $A \cap B \neq \varnothing$. Suppose that $\left(\forall x \in \operatorname{Fix} T_{A, B}\right)(\exists i \in I)(\exists j \in J)$ both $\left(A, A_{i}\right)$ and $\left(B, B_{j}\right)$ are locally identical around $P_{A} x$. Then the following holds for any $D R A$ sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$ :

$$
\begin{equation*}
(\exists i \in I)(\exists j \in J)\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A_{i}, B_{j}} x_{n} . \tag{72}
\end{equation*}
$$

Proof. Since $A$ and $B$ are closed convex, Fact 2.7(i) gives $x_{n} \rightarrow x \in \operatorname{Fix} T_{A, B}$ with $P_{A} x \in A \cap B$. By assumption, $(\exists i \in I)(\exists j \in J)$ both $\left(A, A_{i}\right)$ and $\left(B, B_{j}\right)$ are locally identical around $P_{A} x$. Noting that $A \subseteq A_{i}, B \subseteq B_{j}$, the conclusion follows from Lemma 5.1.

Corollary 5.10. Let $A=\bigcap_{i \in I} A_{i}$ and $B=\bigcap_{j \in J} B_{j}$ be finite intersections of closed convex sets in $X$ such that $0 \in \operatorname{int}(A-B)$. Suppose that $(\forall x \in A \cap B)(\exists i \in I)(\exists j \in J)$ both $\left(A, A_{i}\right)$ and $\left(B, B_{j}\right)$ are locally identical around $x$. Then (72) holds for any DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$.

Proof. Since $0 \in \operatorname{int}(A-B)$, Fact 2.7 (ii) implies $x_{n} \rightarrow x \in A \cap B$. Then $P_{A} x=x$, and Proposition 5.9 completes the proof.

Corollary 5.11. Let $A$ be a hyperplane or a halfspace, and $B=\bigcap_{j \in J} B_{j}$ be a finite intersection of closed balls in $X$. Suppose that $A \cap \operatorname{int} B \neq \varnothing$, and for all $x \in A \cap \operatorname{bdry} B$, there exists a unique $j \in J$ such that $x \in \operatorname{bdry} B_{j}$. Then every DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$ converges finitely to a point in $A \cap B$.

Proof. From $A \cap \operatorname{int} B \neq \varnothing$, we immediately have $0 \in \operatorname{int}(A-B)$. Let $x \in A \cap B$. If $x \in \operatorname{int} B$, then $(\forall j \in J) x \in \operatorname{int} B_{j}$, and so $B$ and $B_{j}$ are locally identical around $x$, following Lemma 3.2(i), If $x \in \operatorname{bdry} B$, then by assumption, there exists a unique $j \in J$ such that $x \in \operatorname{bdry} B_{j}$, which implies that $B$ and $B_{j}$ are locally identical around $x$. Now using Corollary 5.10 ,

$$
\begin{equation*}
(\exists j \in J)\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A, B_{j}} x_{n} . \tag{73}
\end{equation*}
$$

Since $B \subseteq B_{j}$, we also have $A \cap \operatorname{int} B_{j} \neq \varnothing$, and so $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges finitely due to Proposition 5.7

Corollary 5.12. Let $A$ be a closed convex set, and $B$ be a closed ball in $\mathbb{R}^{2}$ such that $A \cap \operatorname{int} B \neq \varnothing$. Suppose that $A$ is locally identical with some polyhedral set around $A \cap \mathrm{bdry} B$, and that no vertex of $A$ lies in bdry $B$. Then every DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$ converges finitely to a point in $A \cap B$.


Figure 1: A GeoGebra snapshot that illustrates Corollary 5.11 .
Proof. By Theorem 5.4|(i) \& (iii), it is sufficient to consider the case where $A$ is a polyhedral set in $\mathbb{R}^{2}$ satisfying $A \cap \operatorname{int} B \neq \varnothing$. Then $0 \in \operatorname{int}(A-B)$, and using Fact 2.7(rii), $x_{n} \rightarrow x \in A \cap B$, and this convergence is finite if $x \in A \cap \operatorname{int} B$. It thus suffices to consider the case where $x \in A \cap \mathrm{bdry} B$. We can write $A=\bigcap_{i=1}^{m} A_{i}$, where each $A_{i}$ is a halfplane in $\mathbb{R}^{2}$. Since all vertices of $A$ are not in bdry $B$, we deduce that $x$ is not a vertex of $A$. Hence, $A$ and $A_{i}$ are locally identical around $x$ for some $i$. Now using Lemma 5.1.

$$
\begin{equation*}
\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A_{i}, B} x_{n} . \tag{74}
\end{equation*}
$$

Moreover, $A_{i} \cap \operatorname{int} B \neq \varnothing$, and by Proposition 5.7, $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges finitely.

## 6 Shrinking sets

In this section we focus on cases where we use information of the DRA for $(A, B)$ to understand the DRA for $\left(A^{\prime}, B^{\prime}\right)$ where $A^{\prime} \subseteq A$ and $B^{\prime} \subseteq B$.

Lemma 6.1 (Shrinking sets). Let A be a closed convex subset and B be a closed (not necessarily convex) subset of $X$ such that $A \cap B \neq \varnothing$, and let $x_{0}$ be in $X$. Suppose that the DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$, with starting point $x_{0}$, converges to $x \in X$. Suppose further that there exist two closed sets $A^{\prime}$
and $B^{\prime}$ in $X$ such that $A^{\prime} \subseteq A, B^{\prime} \subseteq B$, and that both $\left(A^{\prime}, A\right)$ and $\left(B^{\prime}, B\right)$ are locally identical around $c:=P_{A} x \in A^{\prime} \cap B^{\prime}$. Then

$$
\begin{equation*}
\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1} \in T_{A^{\prime}, B^{\prime}} x_{n} . \tag{75}
\end{equation*}
$$

Proof. By assumption, there exists $\varepsilon \in \mathbb{R}_{++}$such that

$$
\begin{equation*}
A \cap \operatorname{ball}(c ; \varepsilon)=A^{\prime} \cap \operatorname{ball}(c ; \varepsilon) \quad \text { and } \quad B \cap \operatorname{ball}(c ; \varepsilon)=B^{\prime} \cap \operatorname{ball}(c ; \varepsilon) . \tag{76}
\end{equation*}
$$

Then, there is $n_{0} \in \mathbb{N}$ such that

$$
\begin{equation*}
\left(\forall n \geq n_{0}\right) \quad\left\|x_{n}-x\right\|<\varepsilon / 3 . \tag{77}
\end{equation*}
$$

Let $n \geq n_{0}$. Since $P_{A}$ is (firmly) nonexpansive (Fact [2.1](ii)),

$$
\begin{equation*}
\left\|P_{A} x_{n}-c\right\|=\left\|P_{A} x_{n}-P_{A} x\right\| \leq\left\|x_{n}-x\right\|<\varepsilon / 3 \tag{78}
\end{equation*}
$$

which implies $P_{A} x_{n} \in$ ball $(c ; \varepsilon)$. Using the convexity of $A$ and applying Lemma 3.4(iii)(a) for $A^{\prime} \subseteq A$, we have $P_{A^{\prime}} x_{n}=P_{A} x_{n}$, and also $R_{A^{\prime}} x_{n}=R_{A} x_{n}$. Noting that $x_{n+1}-x_{n}+P_{A} x_{n} \in P_{B} R_{A} x_{n}$ and

$$
\begin{equation*}
\left\|x_{n+1}-x_{n}+P_{A} x_{n}-c\right\| \leq\left\|x_{n+1}-x\right\|+\left\|x_{n}-x\right\|+\left\|P_{A} x_{n}-c\right\|<\varepsilon, \tag{79}
\end{equation*}
$$

we get $x_{n+1}-x_{n}+P_{A} x_{n} \in P_{B} R_{A} x_{n} \cap$ ball $(c ; \varepsilon)$, and then applying Lemma 3.4(i) for $B^{\prime} \subseteq B$ yields $x_{n+1}-x_{n}+P_{A} x_{n} \in P_{B^{\prime}} R_{A} x_{n}=P_{B^{\prime}} R_{A^{\prime}} x_{n}$. Hence, $x_{n+1} \in x_{n}-P_{A^{\prime}} x_{n}+P_{B^{\prime}} R_{A^{\prime}} x_{n}=T_{A^{\prime}, B^{\prime}} x_{n}$.

Remark 6.2. If $A^{\prime}$ and $B^{\prime}$ in Lemma 6.1 are convex, then $T_{A^{\prime}, B^{\prime}}$ is single-valued, and we have the conclusion that

$$
\begin{equation*}
\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A^{\prime}, B^{\prime}} x_{n} \tag{80}
\end{equation*}
$$

i.e., $\left(\exists n_{0} \in \mathbb{N}\right)(\forall n \in \mathbb{N}) \quad T_{A, B}^{n} x_{n_{0}}=T_{A^{\prime}, B^{\prime}}^{n} x_{n_{0}}$.

Corollary 6.3. Let $A$ be a closed convex subset and $B=\bigcup_{j \in J} B_{j}$ be a finite union of disjoint closed convex sets in $X$ such that $A \cap B \neq \varnothing$, and let $x_{0}$ be in $X$. Suppose that the DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$, with starting point $x_{0}$, is bounded and asymptotically regular, i.e., $x_{n}-x_{n+1} \rightarrow 0$. Then $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges to a point $x \in \operatorname{Fix} T_{A, B}$, and there exists $j \in J$ such that

$$
\begin{equation*}
P_{A} x \in A \cap B_{j} \quad \text { and } \quad\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A, B_{j}} x_{n} . \tag{81}
\end{equation*}
$$

Proof. According to [13, Theorem 2], $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges to a point $x \in \operatorname{Fix} T_{A, B}$. Since $A$ is convex, $P_{A} x$ is a singleton, and by (22), $P_{A} x \in A \cap B$. Then there exists $j \in J$ such that $P_{A} x \in A \cap B_{j}$. By assumption, there exists $\varepsilon \in \mathbb{R}_{++}$such that $(\forall k \in J \backslash\{j\}) B_{k} \cap$ ball $\left(P_{A} x ; \varepsilon\right)=\varnothing$. This implies $B \cap \operatorname{ball}\left(P_{A} x ; \varepsilon\right)=B_{j} \cap \operatorname{ball}\left(P_{A} x ; \varepsilon\right)$, so $B$ and $B_{j}$ are locally identical around $P_{A} x$. Now apply Lemma6.1.

Corollary 6.4. Let $A$ be a hyperplane or a halfspace, and $B=\bigcup_{j \in J} B_{j}$ be a finite union of disjoint closed balls in $X$ such that $A \cap B \neq \varnothing$, and $A \cap \operatorname{int} B_{j} \neq \varnothing$ whenever $A \cap B_{j} \neq \varnothing$. Let $x_{0}$ be in $X$. Suppose that the DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$, with starting point $x_{0}$, is bounded and asymptotically regular, i.e., $x_{n}-x_{n+1} \rightarrow 0$. Then $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges finitely to a point $x \in A \cap B$.

Proof. Using Corollary 6.3, $x_{n} \rightarrow x \in \operatorname{Fix} T_{A, B}$, and there is $j \in J$ such that

$$
\begin{equation*}
P_{A} x \in A \cap B_{j} \quad \text { and } \quad\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A, B_{j}} x_{n} . \tag{82}
\end{equation*}
$$

Then $A \cap B_{j} \neq \varnothing$, and by assumption, $A \cap \operatorname{int} B_{j} \neq \varnothing$. Now by Proposition 5.7, the convergence of $\left(x_{n}\right)_{n \in \mathbb{N}}$ to $x$ is finite, and $x \in A \cap B_{j} \subseteq A \cap B$.

## 7 When one set is finite

If the $B_{j}$ in Corollary 6.3 are singletons and $A$ is either an affine subspace or a halfspace, then it is possible to obtain stronger conclusions.
Theorem 7.1. Let $A$ be an affine subspace or a halfspace, and $B$ be a finite subset of $X$ such that $A \cap B \neq \varnothing$, and let $x_{0}$ be in $X$. Suppose that the DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$, with starting point $x_{0}$, is asymptotically regular, i.e., $x_{n}-x_{n+1} \rightarrow 0$. Then $\left(x_{n}\right)_{n \in \mathbb{N}}$ converges finitely to a point $x \in \operatorname{Fix} T_{A, B}$ with $P_{A} x \in A \cap B$.

Proof. Observe that $P_{A}$ is single-valued as $A$ is convex. According to (22), it suffices to show that $x_{n} \rightarrow x \in \operatorname{Fix} T_{A, B}$ finitely. Set

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad b_{n}:=x_{n+1}-x_{n}+P_{A} x_{n} \in P_{B} R_{A} x_{n} \subseteq B . \tag{83}
\end{equation*}
$$

Let us first consider the case when $A$ is an affine subspace. Then we can represent $A=$ $\{x \in X \mid L x=v\}$, where $L$ is a linear operator from $X$ to a real Hilbert space $Y$, and $v \in \operatorname{ran} L$. Denoting by $L^{\dagger}$ the Moore-Penrose inverse of $L$, Example 2.3 gives

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad P_{A} x_{n}=x_{n}-L^{\dagger}\left(L x_{n}-v\right) \tag{84}
\end{equation*}
$$

and so

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad x_{n+1}=x_{n}-P_{A} x_{n}+b_{n}=L^{\dagger}\left(L x_{n}-v\right)+b_{n} . \tag{85}
\end{equation*}
$$

Since $L^{\dagger} L L^{\dagger}=L^{\dagger}$ (see [30, Chapter II, Section 2]), we get

$$
\begin{align*}
(\forall n \in \mathbb{N}) \quad L^{\dagger}\left(L x_{n+1}-v\right) & =L^{\dagger} L\left(L^{\dagger}\left(L x_{n}-v\right)+b_{n}\right)-L^{\dagger} v  \tag{86a}\\
& =L^{\dagger}\left(L x_{n}-v\right)+L^{\dagger}\left(L b_{n}-v\right), \tag{86b}
\end{align*}
$$

and then (84) gives

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad P_{A} x_{n+1}=x_{n+1}-L^{\dagger}\left(L x_{n+1}-v\right)=-L^{\dagger}\left(L b_{n}-v\right)+b_{n} . \tag{87}
\end{equation*}
$$

Now in turn,

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad x_{n+2}=x_{n+1}-P_{A} x_{n+1}+b_{n+1}=x_{n+1}+L^{\dagger}\left(L b_{n}-v\right)-b_{n}+b_{n+1} . \tag{88}
\end{equation*}
$$

Using the asymptotic regularity of $\left(x_{n}\right)_{n \in \mathbb{N}}$, (88) and (86) yield

$$
\begin{align*}
L^{\dagger}\left(L b_{n}-v\right) & =L^{\dagger} L\left(x_{n+1}-x_{n}\right) \rightarrow 0  \tag{89a}\\
b_{n+1}-b_{n} & =x_{n+2}-x_{n+1}-L^{\dagger}\left(L b_{n}-v\right) \rightarrow 0 . \tag{89b}
\end{align*}
$$

Since $\left(b_{n}\right)_{n \in \mathbb{N}}$ lies in $B$ and $B$ is finite, there exists $n_{0} \in \mathbb{N}$ such that $\left(\forall n \geq n_{0}\right) b_{n+1}=b_{n}=b \in B$. Then by (89a), $L^{\dagger}(L b-v)=0$, which together with (88) gives

$$
\begin{equation*}
\left(\forall n \geq n_{0}\right) \quad x_{n+2}=x_{n+1}+L^{\dagger}(L b-v)=x_{n+1} \tag{90}
\end{equation*}
$$

and $\left(x_{n}\right)_{n \in \mathbb{N}}$ thus converges finitely.
Now consider the case when $A$ is a halfspace. Without loss of generality, we assume that $A=\{x \in X \mid\langle x, u\rangle \leq 0\}$, where $u \in X$ and $\|u\|=1$. Using Example 2.4((ii), we have

$$
(\forall n \in \mathbb{N}) \quad P_{A} x_{n}= \begin{cases}x_{n} & \text { if } x_{n} \in A  \tag{91}\\ x_{n}-\left\langle x_{n}, u\right\rangle u & \text { if } x_{n} \notin A\end{cases}
$$

and by (83),

$$
(\forall n \in \mathbb{N}) \quad x_{n+1}= \begin{cases}b_{n} & \text { if } x_{n} \in A,  \tag{92}\\ \left\langle x_{n}, u\right\rangle u+b_{n} & \text { if } x_{n} \notin A .\end{cases}
$$

If $(\exists n \in \mathbb{N}) x_{n} \in A$ and $b_{n} \in A$, then (92) gives $x_{n+1}=b_{n} \in A \cap B$, and we are done. Assume that $(\forall n \in \mathbb{N}) x_{n} \notin A$ or $b_{n} \notin A$. By using (92), $(\forall n \in \mathbb{N}) x_{n} \in A \Rightarrow x_{n+1}=b_{n} \notin A$. Thus, the set $\left\{n \in \mathbb{N} \mid x_{n} \notin A\right\}$ is infinite, and denoted by $\left(n_{k}\right)_{k \in \mathbb{N}}$ the enumeration of that set, we have

$$
\begin{equation*}
(\forall k \in \mathbb{N}) \quad x_{n_{k}} \notin A \text {, i.e., }\left\langle x_{n_{k}}, u\right\rangle>0, \quad \text { and } \quad n_{k+1}-n_{k} \in\{1,2\} . \tag{93}
\end{equation*}
$$

Then $x_{n_{k+1}}-x_{n_{k}}=x_{n_{k}+1}-x_{n_{k}}$ or $x_{n_{k+1}}-x_{n_{k}}=\left(x_{n_{k}+2}-x_{n_{k}+1}\right)+\left(x_{n_{k}+1}-x_{n_{k}}\right)$, and the asymptotic regularity of $\left(x_{n}\right)_{n \in \mathbb{N}}$ implies the one of $\left(x_{n_{k}}\right)_{k \in \mathbb{N}}$ and also of $\left(x_{n_{k}+1}\right)_{k \in \mathbb{N}}$. Since $x_{n_{k}} \notin A$, (92) gives

$$
\begin{equation*}
x_{n_{k}+1}=\left\langle x_{n_{k}}, u\right\rangle u+b_{n_{k^{\prime}}} \tag{94}
\end{equation*}
$$

and so

$$
\begin{equation*}
b_{n_{k+1}}-b_{n_{k}}=\left(x_{n_{k+1}+1}-x_{n_{k}+1}\right)-\left\langle x_{n_{k+1}}-x_{n_{k}}, u\right\rangle u \rightarrow 0 . \tag{95}
\end{equation*}
$$

But $\left(b_{n_{k}}\right)_{k \in \mathbb{N}}$ is in the finite set $B$, there exists $k_{0} \in \mathbb{N}$ such that

$$
\begin{equation*}
\left(\forall k \geq k_{0}\right) \quad b_{n_{k+1}}=b_{n_{k}}=: b \in B . \tag{96}
\end{equation*}
$$

On the other hand, (94) implies

$$
\begin{equation*}
(\forall k \in \mathbb{N}) \quad\left\langle x_{n_{k}+1}, u\right\rangle=\left\langle x_{n_{k}}, u\right\rangle+\left\langle b_{n_{k}}, u\right\rangle, \tag{97}
\end{equation*}
$$

and then

$$
\begin{equation*}
\left\langle b_{n_{k}}, u\right\rangle=\left\langle x_{n_{k}+1}-x_{n_{k}}, u\right\rangle \rightarrow 0, \tag{98}
\end{equation*}
$$

which yields $\langle b, u\rangle=0$, and thus $b \in A \cap B$. Let $k \geq k_{0}$. It follows from (96) and (97) that

$$
\begin{equation*}
\left\langle x_{n_{k}+1}, u\right\rangle=\left\langle x_{n_{k}}, u\right\rangle+\langle b, u\rangle=\left\langle x_{n_{k}}, u\right\rangle . \tag{99}
\end{equation*}
$$

Hence $x_{n_{k}+1} \notin A$ as $x_{n_{k}} \notin A$. We obtain $n_{k+1}=n_{k}+1$, and by combining with (94) and (96),

$$
\begin{equation*}
x_{n_{k}+2}=\left\langle x_{n_{k}+1}, u\right\rangle u+b=\left\langle x_{n_{k}}, u\right\rangle+b=x_{n_{k}+1}, \tag{100}
\end{equation*}
$$

which completes the proof.
The following examples illustrate that without asymptotic regularity a DRA sequence with respect to $(A, B)$ may fail to converge.

Example 7.2. Suppose that $X=\mathbb{R}^{2}, A=\mathbb{R} \times\{0\}$ and $B=\{(0,-2),(1,2),(-2,0)\}$. Then $A \cap B \neq$ $\varnothing$ but the DRA sequence with respect to $(A, B)$ with starting point $x_{0}=(0,-1)$ does not converge since it cycles between two points $x_{0}=(0,-1)$ and $x_{1}=(1,1)$.

Example 7.3. Suppose that $X=\mathbb{R}^{2}$, that $A=\mathbb{R} \times \mathbb{R}_{-}$is a halfspace, and that $B=$ $\{(2,5),(20,-20),(8,7),(-20,0)\}$ is a finite set. Then $A \cap B \neq \varnothing$ but when started at $x_{0}=(2,17)$, the DRA cycles between four points $x_{0}=(2,17), x_{1}=(20,-3), x_{2}=(8,7)$ and $x_{3}=(2,12)$,as shown in Figure 2 which was created by GeoGebra [27].


Figure 2: A 4-cycle of the DRA for a halfspace and a finite set.

Remark 7.4 (Order matters). Notice that if $A$ is a halfspace and $B$ is a finite subset of $X$ such that $A \cap B \neq \varnothing$, then every DRA sequence with respect to $(B, A)$ converges finitely due to [3, Theorem 4.2]. Recall from [12] that if we work with an affine subspace instead of a halfspace, then the quality of convergence of the DRA sequence with respect to $(A, B)$ is the same as the one with respect to $(B, A)$.

Theorem 7.5. Let A be either a hyperplane or a halfspace of $X$, and $B$ be a finite subset of one in two halfspaces generated by $A$, and let $x_{0}$ be in $X$. Then either: (i) the DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$, with starting point $x_{0}$, converges finitely to a point $x \in \operatorname{Fix} T_{A, B}$ with $P_{A} x \in A \cap B$, or (ii) $A \cap B=\varnothing$ and $\left\|x_{n}\right\| \rightarrow+\infty$ in which case $\left(P_{A} x_{n}\right)_{n \in \mathbb{N}}$ converges finitely to a best approximation solution $a \in A$ relative to $A$ and $B$ in the sense that $d_{B}(a)=\min _{a^{\prime} \in A} d_{B}\left(a^{\prime}\right)$.

Proof. Case 1: A is a hyperplane. Without loss of generality, we assume that

$$
\begin{equation*}
A=H:=\{x \in X \mid\langle x, u\rangle=0\} \quad \text { with } \quad u \in X,\|u\|=1, \tag{101a}
\end{equation*}
$$

and that

$$
\begin{equation*}
(\forall b \in B) \quad\langle b, u\rangle \geq 0 . \tag{101b}
\end{equation*}
$$

By Example 2.4(i),

$$
\begin{equation*}
(\forall x \in X) \quad P_{A} x=x-\langle x, u\rangle u \tag{102}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
(\forall x \in X) \quad R_{A} x=2 P_{A} x-x=x-2\langle x, u\rangle u, \tag{103}
\end{equation*}
$$

and also

$$
\begin{equation*}
(\forall x \in X) \quad d_{A}(x)=\left\|x-P_{A} x\right\|=|\langle x, u\rangle| . \tag{104}
\end{equation*}
$$

Now setting

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad b_{n}:=x_{n+1}-x_{n}+P_{A} x_{n} \in P_{B} R_{A} x_{n} \subseteq B, \tag{105}
\end{equation*}
$$

we have

$$
\begin{align*}
& (\forall n \in \mathbb{N}) \quad x_{n+1}=T_{A, B} x_{n}=x_{n}-P_{A} x_{n}+P_{B} R_{A} x_{n}=\left\langle x_{n}, u\right\rangle u+b_{n},  \tag{106a}\\
& \left\langle x_{n+1}, u\right\rangle=\left\langle\left\langle x_{n}, u\right\rangle u+b_{n}, u\right\rangle=\left\langle x_{n}, u\right\rangle+\left\langle b_{n}, u\right\rangle \geq\left\langle x_{n}, u\right\rangle,  \tag{106b}\\
& P_{A} x_{n+1}=x_{n+1}-\left\langle x_{n+1}, u\right\rangle u=b_{n}-\left\langle b_{n}, u\right\rangle u,  \tag{106c}\\
& R_{A} x_{n+1}=x_{n+1}-2\left\langle x_{n+1}, u\right\rangle u=b_{n}-\left(\left\langle x_{n}, u\right\rangle+2\left\langle b_{n}, u\right\rangle\right) u \text {, } \tag{106d}
\end{align*}
$$

and so

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad x_{n+2}=\left(\left\langle x_{n}, u\right\rangle+\left\langle b_{n}, u\right\rangle\right) u+b_{n+1}=x_{n+1}+\left\langle b_{n}, u\right\rangle u+b_{n+1}-b_{n} . \tag{107}
\end{equation*}
$$

It follows that $b_{n}-R_{A} x_{n+1}=\left(\left\langle x_{n}, u\right\rangle+2\left\langle b_{n}, u\right\rangle\right) u$, and

$$
\begin{align*}
& \left\|b_{n+1}-R_{A} x_{n+1}\right\|^{2}=\left\|\left(b_{n+1}-b_{n}\right)+\left(b_{n}-R_{A} x_{n+1}\right)\right\|^{2}  \tag{108a}\\
& =\left\|b_{n+1}-b_{n}\right\|^{2}+2\left(\left\langle x_{n}, u\right\rangle+2\left\langle b_{n}, u\right\rangle\right)\left\langle b_{n+1}-b_{n}, u\right\rangle+\left\|b_{n}-R_{A} x_{n+1}\right\|^{2} . \tag{108b}
\end{align*}
$$

From $b_{n+1}=P_{B} R_{A} x_{n+1}$ and $b_{n} \in B$, we have $\left\|b_{n+1}-R_{A} x_{n+1}\right\| \leq\left\|b_{n}-R_{A} x_{n+1}\right\|$, which yields

$$
\begin{align*}
0 \leq\left\|b_{n+1}-b_{n}\right\|^{2} & \leq 2\left(\left\langle x_{n}, u\right\rangle+2\left\langle b_{n}, u\right\rangle\right)\left\langle b_{n}-b_{n+1}, u\right\rangle  \tag{109a}\\
& =2\left(\left\langle x_{n}, u\right\rangle+2\left\langle b_{n}, u\right\rangle\right)\left(\left\langle b_{n}, u\right\rangle-\left\langle b_{n+1}, u\right\rangle\right) . \tag{109b}
\end{align*}
$$

Case 1.1: $(\forall n \in \mathbb{N})\left\langle x_{n}, u\right\rangle \leq 0$. By combining with (106b), the sequence $\left(\left\langle x_{n}, u\right\rangle\right)_{n \in \mathbb{N}}$ converges, and so

$$
\begin{equation*}
\left\langle b_{n}, u\right\rangle=\left\langle x_{n+1}, u\right\rangle-\left\langle x_{n}, u\right\rangle \rightarrow 0 . \tag{110}
\end{equation*}
$$

But $\left(b_{n}\right)_{n \in \mathbb{N}}$ lies in the finite set $B$; hence, there exists $n_{0} \in \mathbb{N}$ such that $\left(\forall n \geq n_{0}\right)\left\langle b_{n}, u\right\rangle=0$, equivalently, $b_{n} \in A$. Then (109) implies $\left(\forall n \geq n_{0}\right) b_{n+1}=b_{n}$, and by (107), $x_{n+2}=x_{n+1} \in \operatorname{Fix} T_{A, B}$.

Case 1.2: $\left(\exists n_{0} \in \mathbb{N}\right)\left\langle x_{n_{0}}, u\right\rangle>0$. Then (106b) and 101b) give

$$
\begin{equation*}
\left(\forall n \geq n_{0}\right) \quad\left\langle x_{n}, u\right\rangle+2\left\langle b_{n}, u\right\rangle>0 . \tag{111}
\end{equation*}
$$

Combining with (109), this implies

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad 0 \leq\left\langle b_{n+1}, u\right\rangle \leq\left\langle b_{n}, u\right\rangle, \tag{112}
\end{equation*}
$$

and the sequence $\left(\left\langle b_{n}, u\right\rangle\right)_{n \in \mathbb{N}} \subseteq B$ thus converges. Since again $B$ is finite, there exists $n_{1} \in \mathbb{N}$, $n_{1} \geq n_{0}$ such that $\left(\forall n \geq n_{1}\right)\left\langle b_{n+1}, u\right\rangle=\left\langle b_{n}, u\right\rangle$, which yields $b_{n+1}=b_{n}=: b \in B$ due to (109). By combining with (106c),

$$
\begin{equation*}
\left(\forall n \geq n_{1}\right) \quad P_{A} x_{n+1}=b-\langle b, u\rangle u \quad \text { and } \quad\left\|P_{A} x_{n+1}-b\right\|=|\langle b, u\rangle|=\langle b, u\rangle, \tag{113}
\end{equation*}
$$

so $\left(P_{A} x_{n}\right)_{n \in \mathbb{N}}$ converges finitely. Furthermore, if $\langle b, u\rangle=0$, i.e., $b \in A$, then $b \in A \cap B$, in which case $A \cap B \neq \varnothing$ and by (107), $\left(\forall n \geq n_{1}\right) x_{n+2}=x_{n+1} \in \operatorname{Fix} T_{A, B}$.

Now assume that $\langle b, u\rangle \neq 0$. Then $\langle b, u\rangle>0$ due to (101b). It follows from 106b) and (106d) that

$$
\begin{equation*}
\left(\forall n \geq n_{1}\right) \quad R_{A} x_{n+1}=b-\left(\left\langle x_{n_{1}}, u\right\rangle+\left(n-n_{1}+2\right)\langle b, u\rangle\right) u . \tag{114}
\end{equation*}
$$

Let $n \geq n_{1}$, and let $b^{\prime} \in B$. Since $b=b_{n+1}=P_{B} R_{A} x_{n+1}$, we have $\left\|b-R_{A} x_{n+1}\right\| \leq\left\|b^{\prime}-R_{A} x_{n+1}\right\|$, and so

$$
\begin{equation*}
\left\|b-R_{A} x_{n+1}\right\|^{2} \leq\left\|b^{\prime}-b\right\|^{2}+2\left\langle b^{\prime}-b, b-R_{A} x_{n+1}\right\rangle+\left\|b-R_{A} x_{n+1}\right\|^{2}, \tag{115}
\end{equation*}
$$

which implies

$$
\begin{align*}
\left\|b^{\prime}-b\right\|^{2} & \geq 2\left\langle b-b^{\prime},\left(\left\langle x_{n_{1}}, u\right\rangle+\left(n-n_{1}+2\right)\langle b, u\rangle\right) u\right\rangle  \tag{116a}\\
& =2\left(\left\langle x_{n_{1}}, u\right\rangle+\left(n-n_{1}+2\right)\langle b, u\rangle\right)\left(\langle b, u\rangle-\left\langle b^{\prime}, u\right\rangle\right) . \tag{116b}
\end{align*}
$$

Noting that $\left\langle x_{n_{1}}, u\right\rangle+\left(n-n_{1}+2\right)\langle b, u\rangle \rightarrow+\infty$, we deduce $\langle b, u\rangle \leq\left\langle b^{\prime}, u\right\rangle$. Hence

$$
\begin{equation*}
0<\langle b, u\rangle=\min _{b^{\prime} \in B}\left\langle b^{\prime}, u\right\rangle=\min _{b^{\prime} \in B} d_{A}\left(b^{\prime}\right) \tag{117}
\end{equation*}
$$

This yields $A \cap B=\varnothing$, and by (106b ,

$$
\begin{equation*}
\left\|x_{n}\right\| \geq\left\langle x_{n}, u\right\rangle=\left\langle x_{n_{1}}, u\right\rangle+\left(n-n_{1}\right)\langle b, u\rangle \rightarrow+\infty \quad \text { as } \quad n \rightarrow+\infty, \tag{118}
\end{equation*}
$$

while by (113), $\left(\forall n \geq n_{1}\right)\left(P_{A} x_{n+1}, b\right)$ is a best approximation pair relative to $A$ and $B$.
Case 2: $A$ is a halfspace. By assumption, we assume without loss of generality that either

$$
\begin{equation*}
A=H_{+}:=\{x \in X \mid\langle x, u\rangle \geq 0\} \quad \text { and } \quad B \subseteq H_{+}, \tag{119a}
\end{equation*}
$$

or

$$
\begin{equation*}
A=H_{-}:=\{x \in X \mid\langle x, u\rangle \leq 0\} \quad \text { and } \quad B \subseteq H_{+}, \tag{119b}
\end{equation*}
$$

where $u \in X$ and $\|u\|=1$.
Case 2.1: 119a) holds. If $(\forall n \in \mathbb{N})\left\langle x_{n}, u\right\rangle \leq 0$, i.e. $x_{n} \in H_{-}$, then $P_{A} x_{n}=P_{H} x_{n}$, so

$$
\begin{equation*}
x_{n+1}=T_{A, B} x_{n}=T_{H, B} x_{n} \tag{120}
\end{equation*}
$$

and according to Case 1.1, we must have $H \cap B \neq \varnothing$ and the finite convergence of $\left(x_{n}\right)_{n \in \mathbb{N}}$. If $\left(\exists n_{0} \in \mathbb{N}\right)\left\langle x_{n_{0}}, u\right\rangle \geq 0$, i.e. $x_{n_{0}} \in H_{+}$, then $R_{A} x_{n_{0}}=P_{A} x_{n_{0}}=x_{n_{0}}$, which yields $x_{n_{0}+1}=x_{n_{0}}-$ $P_{A} x_{n_{0}}+P_{B} R_{A} x_{n_{0}}=P_{B} x_{n_{0}} \in B=A \cap B$, and we are done.

Case 2.2: 119b holds. If $\left\langle x_{0}, u\right\rangle \leq 0$, i.e. $x_{0} \in H_{-}$, then $R_{A} x_{0}=P_{A} x_{0}=x_{0}$, and thus $x_{1}=$ $x_{0}-P_{A} x_{0}+P_{B} R_{A} x_{0}=P_{B} x_{0} \in B \subseteq H_{+}$. It is therefore sufficient to consider $\left\langle x_{0}, u\right\rangle \geq 0$, i.e. $x_{0} \in H_{+}$. Then $P_{A} x_{0}=P_{H} x_{0}, x_{1}=T_{A, B} x_{0}=T_{H, B} x_{0}$, and by (106b), $\left\langle x_{1}, u\right\rangle \geq\left\langle x_{0}, u\right\rangle \geq 0$. This yields

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad x_{n} \in H_{+} \quad \text { and } \quad x_{n+1}=T_{H, B} x_{n} . \tag{121}
\end{equation*}
$$

Now apply Case 1.
Example 7.6. Suppose that $X=\mathbb{R}^{2}, A=\mathbb{R} \times\{0\}$ and $B=\{(0,1),(1,2)\}$. Then $A \cap B=\varnothing$, and for starting point $\left.x_{0} \in\right] 1,+\infty\left[\times\{-1\}\right.$, the DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$ satisfies $(\forall n \in\{2,3, \ldots\}) x_{n}=(0, n)$ and $P_{A} x_{n}=(0,0)$. See Figure 3 for an illustration, created with GeoGebra [27].


Figure 3: An illustration for Example 7.6 with the starting point $x_{0}=(2,-1)$.

## 8 When $A$ is an affine subspace and $B$ is a polyhedron

In view of Definition 3.1. we recall a result on finite convergence of the Douglas-Rachford algorithm under Slater's condition.

Fact 8.1 (Finite convergence of DRA in the affine-polyhedral case). Let $A$ be an affine subspace and $B$ be a closed convex subset of $X$ such that Slater's condition

$$
\begin{equation*}
A \cap \operatorname{int} B \neq \varnothing \tag{122}
\end{equation*}
$$

holds. Suppose that $B$ is locally identical with some polyhedral set around $A \cap$ bdry $B$. Then every $D R A$ sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$ converges finitely to a point in $A \cap B$.

Proof. Combine [10, Theorem 3.7 and Definition 2.7] with Definition 3.1.

A natural question is whether the conclusion of Fact 8.1 holds when the Slater's condition $A \cap \operatorname{int} B \neq \varnothing$ is replaced by $A \cap B \neq \varnothing$ and int $B \neq \varnothing$. In the sequel, we shall provide a positive answer in $\mathbb{R}^{2}$ (Theorem 8.7) and a negative answer in $\mathbb{R}^{3}$ (Example 8.8). For the next little while, we work with

$$
\begin{equation*}
X=\mathbb{R}^{2} \quad \text { and } \quad A=\mathbb{R} \times\{0\} \tag{123}
\end{equation*}
$$

and consider the (counter-clockwise) rotator defined by

$$
(\forall \theta \in \mathbb{R}) \quad \mathcal{R}_{\theta}:=\left[\begin{array}{cc}
\cos \theta & -\sin \theta  \tag{124}\\
\sin \theta & \cos \theta
\end{array}\right] .
$$

Let $\theta \in[0, \pi]$, and set

$$
\begin{equation*}
e_{0}:=(1,0), \quad e_{\pi / 2}:=(0,1), \quad e_{\theta}:=(\cos \theta) e_{0}+(\sin \theta) e_{\pi / 2} . \tag{125}
\end{equation*}
$$

Then $\mathbb{R}_{+} \times\{0\}=\mathbb{R}_{+} \cdot e_{0}$ is the positive $x$-axis, and $\mathcal{R}_{\theta}\left(\mathbb{R}_{+} \times\{0\}\right)=\mathbb{R}_{+} \cdot e_{\theta}$ is the ray starting at $0 \in X$ and making an angle of $\theta$ with respect to $\mathbb{R}_{+} \times\{0\}$ in counter-clockwise direction.

For $x, y \in X$, we write $\angle(x, y):=\theta$ if $y \in \mathbb{R}_{+} \mathcal{R}_{\theta}(x)$, and $\angle(x, y)=\theta-\pi$ if $y \in \mathbb{R}_{-} \mathcal{R}_{\theta}(x)$.
Fact 8.2. Let $\theta \in[0, \pi]$. Then

$$
\begin{equation*}
T_{A, \mathcal{R}_{\theta}(A)}=\mathrm{Id}-P_{A}+P_{\mathcal{R}_{\theta}(A)} R_{A}=(\cos \theta) \mathcal{R}_{\theta} . \tag{126}
\end{equation*}
$$

Proof. This follows from [4, Section 5].
Lemma 8.3. Assume that $\theta \in[0, \pi], B=\mathcal{R}_{\theta}\left(\mathbb{R}_{+} \times\{0\}\right), H=B^{\oplus}$, and $H^{\prime}=R_{A}(H)$. Let $x=(\alpha, \beta) \in$ $X$, and set $x_{+}=T_{A, B} x$. Then $x_{+}=(0, \beta)$ if $x \notin H^{\prime}$, and $x_{+}=(\cos \theta) \mathcal{R}_{\theta}(z)$ otherwise. In the latter case, $x_{+}=0$ if $\theta=\pi / 2$, and

$$
\angle\left(x, x_{+}\right)= \begin{cases}\theta, & \text { if } \theta<\pi / 2 ;  \tag{127}\\ \theta-\pi, & \text { if } \theta>\pi / 2 .\end{cases}
$$

Furthermore,

$$
\text { Fix } T_{A, B}= \begin{cases}\mathbb{R}_{+} \times \mathbb{R}^{\prime}, & \text { if } \theta=0 ;  \tag{128}\\ \{0\} \times \mathbb{R}_{+}, & \text {if } 0<\theta<\pi ; \\ \mathbb{R}_{-} \times \mathbb{R}, & \text { if } \theta=\pi .\end{cases}
$$

Proof. We have $P_{A} x=(\alpha, 0)$ and $R_{A} x=(\alpha,-\beta)$. If $x=(\alpha, \beta) \notin H^{\prime}$, then $R_{A} x \notin H$, and so $P_{B} R_{A} x=(0,0)$, which yields

$$
\begin{equation*}
x_{+}=\left(\operatorname{Id}-P_{A}+P_{B} R_{A}\right) x=(\alpha, \beta)-(\alpha, 0)+(0,0)=(0, \beta) . \tag{129}
\end{equation*}
$$

Now we consider the case $x \in H^{\prime}$. Then $R_{A} x \in H$, so $P_{B} R_{A} x=P_{\mathcal{R}_{\theta}(A)} R_{A} x$, and by applying Fact 8.2,

$$
\begin{equation*}
x_{+}=(\cos \theta) \mathcal{R}_{\theta}(x) . \tag{130}
\end{equation*}
$$

The rest is clear.
Lemma 8.4. Let

$$
\begin{equation*}
A=\mathbb{R} \times\{0\} \quad \text { and } \quad B=\mathcal{R}_{\theta}\left(\mathbb{R}_{+} \times\{0\}\right), \tag{131}
\end{equation*}
$$

where $\theta \in[0, \pi]$. Then every DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$ converges to a point $x \in$ Fix $T_{A, B}$, and the "shadow sequence" $\left(P_{A} x_{n}\right)_{n \in \mathbb{N}}$ converges to $P_{A} x \in A \cap B$ in at most $N$ iterations, where

$$
N= \begin{cases}\left\lfloor\frac{\pi}{\theta}\right\rfloor+3, & \text { if } \theta \leq \pi / 2 ;  \tag{132}\\ \left\lfloor\frac{\pi}{\pi-\theta}\right\rfloor+3, & \text { if } \theta>\pi / 2 .\end{cases}
$$

Proof. Set $H=B^{\oplus}$, and $H^{\prime}=R_{A}(H)$. We will study the behavior of the iterations in regions

$$
\begin{align*}
& R_{1}=\left\{(\alpha, \beta) \in X \mid(\alpha, \beta) \notin H^{\prime}, \beta<0\right\}  \tag{133a}\\
& R_{2}=H^{\prime},  \tag{133b}\\
& R_{3}=\left\{(\alpha, \beta) \in X \mid(\alpha, \beta) \notin H^{\prime}, \beta \geq 0\right\} \tag{133c}
\end{align*}
$$

as shown in Figure 4.



Figure 4: The DRA for the case of a line and a ray in the Euclidean plane

Since $\theta \in[0, \pi]$, we have $0 \times \mathbb{R}_{+} \subseteq H$, and so $\{0\} \times \mathbb{R}_{-} \subseteq H^{\prime}$. Set $x_{0}:=\left(\alpha_{0}, \beta_{0}\right) \in X$. According to Lemma 8.3, if $x_{0} \in R_{1}$, then $x_{1}=\left(0, \beta_{0}\right) \in\{0\} \times \mathbb{R}_{-} \subseteq H^{\prime}$; if $x_{0} \in R_{3}$, then $x_{1}=\left(0, \beta_{0}\right) \in 0 \times \mathbb{R}_{+} \subseteq$ Fix $T_{A, B}$. So it is sufficient to consider the case $x_{0} \in H^{\prime}=R_{2}$. If $\theta=\pi / 2$, we have immediately $x_{1}=0 \in A \cap B$. Now we assume without loss of generality that $\theta<\pi / 2$. Then, (127) yields the implication

$$
\begin{equation*}
x_{0}, \ldots, x_{n-1} \in R_{2} \quad \Rightarrow \quad \angle\left(x_{0}, x_{n}\right)=n \theta \tag{134}
\end{equation*}
$$

There thus exists $n_{0} \in \mathbb{N}$ such that

$$
\begin{equation*}
x_{0}, \ldots, x_{n_{0}-1} \in R_{2}, \quad \text { and } \quad x_{n_{0}} \notin R_{2} \tag{135}
\end{equation*}
$$

which yields $x_{n_{0}} \in R_{3}$. Using again Lemma 8.3, $x_{n_{0}+1}=\left(0, \beta_{n_{0}}\right) \in 0 \times \mathbb{R}_{+} \subseteq$ Fix $T$. Noting that

$$
\begin{equation*}
\angle\left(x_{0}, x_{n_{0}}\right)=n_{0} \theta \leq \pi+\theta, \tag{136}
\end{equation*}
$$

we get $n_{0} \leq\lfloor\pi / \theta\rfloor+1$. Hence, $x_{n}=x \in \operatorname{Fix} T_{A, B}$ and $P_{A} x_{n}=P_{A} x \in A \cap B$ for all $n \geq\lfloor\pi / \theta\rfloor+3$ iterations.

Lemma 8.5. Let either $A=\mathbb{R} \times\{0\}$ or $A=\mathbb{R} \times \mathbb{R}_{-}$, and let $B$ be the convex cone generated by the union of the rays

$$
\begin{equation*}
B_{1}=\mathcal{R}_{\theta_{1}}\left(\mathbb{R}_{+} \times\{0\}\right) \text { and } B_{2}=\mathcal{R}_{\theta_{2}}\left(\mathbb{R}_{+} \times\{0\}\right) \tag{137}
\end{equation*}
$$

with $\theta_{1}, \theta_{2} \in[0, \pi]$. Then the DRA applied to $(A, B)$ converges finitely globally uniformly in the sense that there exists $N \in \mathbb{N}$ such that $(\forall x \in X)$ the sequence $\left(T_{A, B}^{n} x\right)_{n \in \mathbb{N}}$ converges to a point in Fix $T_{A, B}$ in at most $N$ iterations.


Figure 5: The DRA for the case of a line and a cone in the Euclidean plane

Proof. We shall prove this for the case $A=\mathbb{R} \times\{0\}$, the other case being similar. For $i \in\{1,2\}$, set $H_{i}=B_{i}^{\oplus}, H_{i}^{\prime}=R_{A}\left(H_{i}\right), B_{i}^{\prime}=R_{A}\left(B_{i}\right)$, and let $B_{1}^{\prime \prime}=\mathcal{R}_{\pi / 2}\left(B_{1}^{\prime}\right), B_{2}^{\prime \prime}=\mathcal{R}_{\pi / 2}^{-1}\left(B_{2}^{\prime}\right)$. Without loss of generality, we distinguish two cases: $0 \leq \theta_{1}<\pi / 2<\theta_{2} \leq \pi$ or $0 \leq \theta_{1}<\theta_{2} \leq \pi / 2$.

Case 1: $0 \leq \theta_{1}<\pi / 2<\theta_{2} \leq \pi$. As shown in the left image in Figure 4, we study the behavior of the iterations in regions

$$
\begin{align*}
& R_{1}=\operatorname{cone}\left(\{0\} \times \mathbb{R}_{-} \cup B_{1}^{\prime}\right)=R_{A}(B) \cap\left(\mathbb{R}_{+} \times \mathbb{R}\right),  \tag{138a}\\
& R_{2}=\operatorname{cone}\left(B_{1}^{\prime} \cup B_{1}^{\prime \prime}\right) \subseteq H_{1}^{\prime} \backslash R_{A}(B),  \tag{138b}\\
& R_{3}=\operatorname{cone}\left(B_{1}^{\prime \prime} \cup B_{2}^{\prime \prime}\right),  \tag{138c}\\
& R_{4}=\operatorname{cone}\left(B_{2}^{\prime \prime} \cup B_{2}^{\prime}\right) \subseteq H_{2}^{\prime} \backslash R_{A}(B),  \tag{138d}\\
& R_{5}=\operatorname{cone}\left(B_{2}^{\prime} \cup\{0\} \times \mathbb{R}_{-}\right)=R_{A}(B) \cap\left(\mathbb{R}_{-} \times \mathbb{R}\right) . \tag{138e}
\end{align*}
$$

Set $x_{0}:=\left(\alpha_{0}, \beta_{0}\right)$.
Case 1.1: $x_{0} \in R_{1} \cup R_{5}$. Then $P_{A} x_{0}=\left(\alpha_{0}, 0\right)$, and $R_{A} x_{0}=\left(\alpha_{0},-\beta_{0}\right) \in B=R_{A}\left(R_{1} \cup R_{5}\right)$, so

$$
\begin{equation*}
x_{1}=\left(\operatorname{Id}-P_{A}+P_{B} R_{A}\right) x_{0}=\left(\alpha_{0}, \beta_{0}\right)-\left(\alpha_{0}, 0\right)+\left(\alpha_{0},-\beta_{0}\right)=\left(\alpha_{0}, 0\right) \in R_{2} \cup R_{4} . \tag{139}
\end{equation*}
$$

Case 1.2: $x_{0} \in R_{2}$. Then $x_{0} \in H_{1}^{\prime} \backslash R_{A}(B)$, and $R_{A} x_{0} \in H_{1} \backslash B$. We also see that $R_{A} x_{0}$ belongs to the halfspace with boundary span $B_{1}$ and not containing $B_{2}$. Thus, $P_{B} R_{A} x_{0}=P_{B_{1}} R_{A} x_{0}$, and

$$
\begin{equation*}
x_{1}=T_{A, B} x_{0}=T_{A, B_{1}} x_{0} . \tag{140}
\end{equation*}
$$

Using Lemma 8.3, this implies

$$
\begin{equation*}
x_{0}, \ldots, x_{n-1} \in R_{2} \quad \Rightarrow \quad \angle\left(x_{0}, x_{n}\right)=n \theta_{1} . \tag{141}
\end{equation*}
$$

Therefore, as in the proof of Lemma 8.4, there exists $n_{0} \in \mathbb{N}, n_{0} \leq\left\lfloor\pi /\left(2 \theta_{1}\right)\right\rfloor+1$ such that $x_{n_{0}} \in R_{3}$.
Case 1.3: $x_{0} \in R_{4}$. By an argument similar to the above, we have $x_{n_{0}} \in R_{3}$ for some $n_{0} \in \mathbb{N}$, $n_{0} \leq\left\lfloor\pi /\left(2 \pi-2 \theta_{2}\right)\right\rfloor+1$.

Case 1.4: $x_{0}=\left(\alpha_{0}, \beta_{0}\right) \in R_{3}$. Then $\beta_{0} \geq 0$ and $R_{A} x_{0} \notin H_{1} \cup H_{2}$ since $R_{3} \nsubseteq H_{1}^{\prime} \cup H_{2}^{\prime}$. Therefore, $P_{B} R_{A} x_{0}=(0,0)$, and

$$
\begin{equation*}
x_{1}=\left(\alpha_{0}, \beta_{0}\right)-\left(\alpha_{0}, 0\right)+(0,0)=\left(0, \beta_{0}\right) \in\{0\} \times \mathbb{R}_{+} \subseteq \operatorname{Fix} T_{A, B} \tag{142}
\end{equation*}
$$

Hence, in all cases, there exists $n_{1} \in \mathbb{N}$ such that

$$
\begin{equation*}
n_{1} \leq N:=\max \left\{\left\lfloor\frac{\pi}{2 \theta_{1}}\right\rfloor,\left\lfloor\frac{\pi}{2\left(\pi-\theta_{2}\right)}\right\rfloor\right\}+3, \tag{143}
\end{equation*}
$$

and $x_{n_{1}} \in \operatorname{Fix} T_{A, B}$. This shows that $x_{n} \rightarrow x_{n_{1}} \in \operatorname{Fix} T_{A, B}$ in at most $N$ iterations.
Case 2: $0 \leq \theta_{1}<\theta_{2} \leq \pi / 2$. Partitioning

$$
\begin{align*}
& R_{1}=\operatorname{cone}\left(B_{1}^{\prime \prime} \cup B_{2}^{\prime \prime}\right) \cap\left(\mathbb{R} \times \mathbb{R}_{-}\right),  \tag{144a}\\
& R_{2}=\operatorname{cone}\left(B_{2}^{\prime \prime} \cup B_{2}^{\prime}\right) \subseteq H_{2}^{\prime} \backslash R_{A}(B),  \tag{144b}\\
& R_{3}=\operatorname{cone}\left(B_{2}^{\prime} \cup B_{1}^{\prime}\right)=R_{A}(B),  \tag{144c}\\
& R_{4}=\operatorname{cone}\left(B_{1}^{\prime} \cup B_{1}^{\prime \prime}\right) \subseteq H_{1}^{\prime} \backslash R_{A}(B),  \tag{144d}\\
& R_{5}=\operatorname{cone}\left(B_{1}^{\prime \prime} \cup B_{2}^{\prime \prime}\right) \cap\left(\mathbb{R} \times \mathbb{R}_{+}\right) \tag{144e}
\end{align*}
$$

(see the right image in Figure 5) and arguing as in the above case, we obtain that $x_{n} \rightarrow x \in \operatorname{Fix} T_{A, B}$ in at most $N$ iterations, where

$$
\begin{equation*}
N:=\left\lfloor\frac{\pi}{2 \theta_{1}}\right\rfloor+\left\lfloor\frac{\pi}{2 \theta_{2}}\right\rfloor+5 . \tag{145}
\end{equation*}
$$

The proof is complete.
Remark 8.6. By the same argument, Lemma 8.5 also remains true when $\theta_{1}, \theta_{2} \in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
Theorem 8.7. Let A be either a line or a halfplane, and B be a closed convex set in the Euclidean plane $\mathbb{R}^{2}$. Suppose that $A \cap B \neq \varnothing$, and that $B$ is locally identical with some polyhedral set around $A \cap$ bdry $B$. Then every DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$ converges finitely to a point $x \in \operatorname{Fix} T_{A, B}$ with $P_{A} x \in A \cap B$.

Proof. Using Theorem 5.4, it suffices to prove for the case where $B$ is a polyhedral set in $\mathbb{R}^{2}$ satisfying $A \cap B \neq \varnothing$. Then $B=\bigcap_{j \in J} B_{j}$ is a finite intersection of halfplanes $B_{j}$. Now by Fact $2.7(\mathrm{i})$. $x_{n} \rightarrow x \in \operatorname{Fix} T_{A, B}$ with $P_{A} x \in A \cap B=A \cap\left(\bigcap_{j \in J} B_{j}\right)$.

Case 1: $P_{A} x$ is not a vertex of $B$. Then there exists $j \in J$ such that $B$ and $B_{j}$ are locally identical around $P_{A} x$. Applying Lemma 5.1 for $A^{\prime}=A$ and $B^{\prime}=B_{j}$, we have

$$
\begin{equation*}
\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A, B_{j}} x_{n} . \tag{146}
\end{equation*}
$$

Since $A$ is either a line or a halfplane, and $B_{j}$ is a halfplane in $\mathbb{R}^{2}$, Theorem 4.5 implies that $x_{n} \rightarrow x$ finitely.

Case 2: $P_{A} x$ is a vertex of $B$. Noting that there are exactly two of halfplanes $B_{j}$ through each vertex of $B$, it can also represent $B=\bigcap_{j \in J} C_{j}$, where each $C_{j}$ is a closed convex cone in $\mathbb{R}^{2}$. We then find $j \in J$ such that $B$ and $C_{j}$ are locally identical around $P_{A} x$. By using again Lemma 5.1.

$$
\begin{equation*}
\left(\exists n_{0} \in \mathbb{N}\right)\left(\forall n \geq n_{0}\right) \quad x_{n+1}=T_{A, C_{j}} x_{n} . \tag{147}
\end{equation*}
$$

Here $A$ is either a line or a halfplane through vertex $P_{A} x$ of the cone $C_{j}$. Now apply Lemma 8.5 and Remark 8.6

Example 8.8. Suppose that $X=\mathbb{R}^{3}$, that $A=\{x \in X \mid L x=a\}$, and that $B=\mathbb{R}_{+}^{3}$, where

$$
L=\left[\begin{array}{lll}
1 & 1 & 0  \tag{148}\\
1 & 0 & 1
\end{array}\right] \quad \text { and } \quad a=\left[\begin{array}{l}
1 \\
0
\end{array}\right] .
$$

Then for starting point $x_{0}=(1 / 3,2 / 3,1 / 3) \in X$, the DRA sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ with respect to $(A, B)$ converges $x_{\infty}=(1 / 3,1,1 / 3)$ with $P_{A} x_{\infty}=(0,1,0) \in A \cap B$, but this convergence is not finite.

Proof. It is easy to see that $A=\{(-\lambda, \lambda+1, \lambda) \mid \lambda \in \mathbb{R}\}$, and so

$$
\begin{equation*}
A \cap B=\{(0,1,0)\} . \tag{149}
\end{equation*}
$$

Let $x=(\alpha, \beta, \gamma) \in X$. Noting that the Moore-Penrose inverse of $L$ is given by

$$
L^{\dagger}=\frac{1}{3}\left[\begin{array}{cc}
1 & 1  \tag{150}\\
2 & -1 \\
-1 & 2
\end{array}\right],
$$

we learn from Example 2.3 that $P_{A} x=x-L^{\dagger}(L x-a)$, and so

$$
R_{A} x=2 P_{A} x-x=x-2 L^{\dagger}(L x-a)=\frac{1}{3}\left(\left[\begin{array}{ccc}
-1 & -2 & -2  \tag{151}\\
-2 & -1 & 2 \\
-2 & 2 & -1
\end{array}\right] x+\left[\begin{array}{c}
2 \\
4 \\
-2
\end{array}\right]\right) .
$$

By, e.g., [6, Example 6.28], $P_{B} x=(\max \{\alpha, 0\}, \max \{\beta, 0\}, \max \{\gamma, 0\})$, and thus

$$
\begin{equation*}
R_{B} x=(|\alpha|,|\beta|,|\gamma|) . \tag{152}
\end{equation*}
$$

Setting $x_{+}:=\left(\alpha_{+}, \beta_{+}, \gamma_{+}\right)=T_{A, B} x$, we claim that if

$$
\begin{align*}
\frac{2}{3} & \leq \alpha+\gamma  \tag{153a}\\
-\frac{2}{3} & \leq \alpha-\gamma \leq \frac{2}{3},  \tag{153b}\\
\frac{2}{3} & \leq \beta \leq \frac{4}{3} \tag{153c}
\end{align*}
$$

then $x_{+}=\frac{1}{3}(M x+b)$, where

$$
M:=\left[\begin{array}{ccc}
2 & 1 & 1  \tag{154}\\
-1 & 1 & 1 \\
1 & -1 & 2
\end{array}\right] \quad \text { and } \quad b:=\left[\begin{array}{c}
-1 \\
2 \\
1
\end{array}\right]
$$

and (153) also holds for $\alpha_{+}, \beta_{+}$and $\gamma_{+}$. Indeed, recall that

$$
\begin{equation*}
R_{A} x=\frac{1}{3}(-\alpha-2 \beta-2 \gamma+2,-2 \alpha-\beta+2 \gamma+4,-2 \alpha+2 \beta-\gamma-2) . \tag{155}
\end{equation*}
$$

It follows from (153) that $\alpha \geq 0, \gamma \geq 0$, and

$$
\begin{align*}
& -\alpha-2 \beta-2 \gamma+2 \leq-(\alpha+\gamma)-2 \beta+2 \leq-\frac{2}{3}-2 \cdot \frac{2}{3}+2=0  \tag{156a}\\
& -2 \alpha-\beta+2 \gamma+4=-2(\alpha-\gamma)-\beta+4 \geq-2 \cdot \frac{2}{3}-\frac{4}{3}+4=\frac{4}{3}>0  \tag{156b}\\
& -2 \alpha+2 \beta-\gamma-2 \leq-(\alpha+\gamma)+2 \beta-2 \leq-\frac{2}{3}+2 \cdot \frac{4}{3}-2=0 \tag{156c}
\end{align*}
$$

By (152) and a direct computation,

$$
\begin{equation*}
x_{+}=\frac{1}{2}\left(x+R_{B} R_{A} x\right)=\frac{1}{3}(M x+b), \tag{157}
\end{equation*}
$$

which means

$$
\begin{align*}
& \alpha_{+}=\frac{1}{3}(2 \alpha+\beta+\gamma-1),  \tag{158a}\\
& \beta_{+}=\frac{1}{3}(-\alpha+\beta+\gamma+2),  \tag{158b}\\
& \gamma_{+}=\frac{1}{3}(\alpha-\beta+2 \gamma+1) . \tag{158c}
\end{align*}
$$

Using again (153) we get

$$
\begin{align*}
\alpha_{+}+\gamma_{+} & =\alpha+\gamma \geq \frac{2}{3}  \tag{159a}\\
-\frac{2}{3}<-\frac{4}{9} \leq \alpha_{+}-\gamma_{+} & =\frac{1}{3}((\alpha-\gamma)+2 \beta-2) \leq \frac{4}{9}<\frac{2}{3}  \tag{159b}\\
\frac{2}{3} \leq \beta_{+} & =\frac{1}{3}(-(\alpha-\gamma)+\beta+2) \leq \frac{4}{3}, \tag{159c}
\end{align*}
$$

as claimed. Now let $x_{0}=(1 / 3,2 / 3,1 / 3)$, the above claim implies that

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad x_{n+1}=T_{A, B} x_{n}=\frac{1}{3}\left(M x_{n}+b\right) . \tag{160}
\end{equation*}
$$

A direct argument yields

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad x_{n+3}=\frac{5}{3} x_{n+2}-x_{n+1}+\frac{1}{3} x_{n} \tag{161}
\end{equation*}
$$

and then

$$
\begin{equation*}
x_{n}=\left(\frac{1}{3}-\frac{\sqrt{2}}{2} \frac{\sin (n \arctan \sqrt{2})}{3^{\frac{n}{2}+1}}, 1-\frac{\cos (n \arctan \sqrt{2})}{3^{\frac{n}{2}+1}}, \frac{1}{3}+\frac{\sqrt{2}}{2} \frac{\sin (n \arctan \sqrt{2})}{3^{\frac{n}{2}+1}}\right) . \tag{162}
\end{equation*}
$$

Therefore, $x_{n} \rightarrow x_{\infty}=(1 / 3,1,1 / 3)$ linearly with rate $1 / \sqrt{3}$, but not finitely.

## 9 Open problems

We conclude with a list of specific open problems.
P1 Do the conclusions of Fact 4.2 and Theorem 4.3 hold when $A$ is any hyperplane or halfspace?
P2 Does Proposition 5.7 remain true if $A$ is an affine subspace or a polyhedron?
P3 Does Corollary 5.11 remain true without assumption on the uniqueness?
P4 Do Corollary 5.12 and Theorem 8.7 remain true in $\mathbb{R}^{n}$ with $n>2$ ?
P5 Does Fact 8.1 remain true if we replace "affine subspace" by "halfspace"?
P6 What can be said about convergence of the DRA for two polyhedrons or for two balls?

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