

A NOVEL ITERATIVE METHOD TO APPROXIMATE STRUCTURED SINGULAR VALUES

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Abstract. A novel method for approximating structured singular values (also known as μ -values) is proposed and investigated. These quantities constitute an important tool in the stability analysis of uncertain linear control systems as well as in structured eigenvalue perturbation theory. Our approach consists of an inner-outer iteration. In the outer iteration, a Newton method is used to adjust the perturbation level. The inner iteration solves a gradient system associated with an optimization problem on the manifold induced by the structure. Numerical results and comparison with the well-known Matlab function `mussv`, implemented in the MATLAB Control Toolbox, illustrate the behavior of the method.

Key words. Structured singular value, μ -value, spectral value set, block diagonal perturbations, stability radius, differential equation, low-rank matrix manifold.

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1. Introduction. The structured singular value (SSV) [14] is an important and versatile tool in control, as it allows to address a central problem in the analysis and synthesis of control systems: To quantify the stability of a closed-loop linear time-invariant systems subject to structured perturbations. The class of structures addressed by the SSV is very general and allows to cover all types of parametric uncertainties that can be incorporated into the control system via real or complex linear fractional transformations. We refer to [1, 3, 4, 8, 9, 10, 14, 17, 20] and the references therein for examples and applications of the SSV.

The versatility of the SSV comes at the expense of being notoriously hard, in fact NP hard [2], to compute. Algorithms used in practice thus aim at providing upper and lower bounds, often resulting in a coarse estimate of the exact value. An upper bound of the SSV provides sufficient conditions to guarantee robust stability, while a lower bound provides sufficient conditions for instability and often also allows to determine structured perturbations that destabilize the closed loop linear system.

The widely used function `mussv` in the MATLAB Control Toolbox computes an upper bound of the SSV using diagonal balancing / LMI techniques [19, 5]. The lower bound is computed by a generalization of the power method developed in [18, 15]. This algorithm *resembles a mixture of the power methods for computing the spectral radius and the largest singular value*, which is not surprising, since the SSV can be viewed as a generalization of both. When the algorithm converges, a lower bound of the SSV results and this is always an equilibrium point of the iteration. However, in contrast to the standard power method, there are, in general, several stable equilibrium points and not all of them correspond to the SSV. In turn, one cannot guarantee convergence to the exact value but only to a lower bound. We remark that, despite this drawback, `mussv` is a very reliable and powerful routine, which reflects the state of the art in the approximation of the SSV.

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In this paper, we present a new approach to computing a lower bound of the SSV associated with general mixed real/complex perturbations. The main ingredient of our new algorithm is a gradient system that evolves perturbations on a certain matrix manifold towards critical perturbations. Among the theoretical properties established for this gradient system, we prove a monotonicity property that indicates robustness and can also be exploited in the numerical discretization. We show several numerical examples for which our algorithm provides tighter bounds than those computed by `muSSV`.

1.1. Overview of the article. Section 2 provides the basic framework for the proposed methodology. In particular, we explain how the computation of the SSV can be addressed by an inner-outer algorithm, where the outer algorithm determines the perturbation level ε and the inner algorithm determines a (local) extremizer of the structured spectral value set. Moreover, an example illustrates that the output `muSSV` may fail to satisfy a necessary condition for optimality.

In Section 3 we develop the inner algorithm for the case of complex structured perturbations. An important characterization of extremizers shows that we can restrict ourselves to a manifold of structured perturbations with normalized and low-rank blocks. A gradient system for finding extremizers on this manifold is established and analyzed.

Section 4 extends the results of Section 3 to perturbations with complex full blocks alternated and mixed complex/real repeated scalar blocks.

The outer algorithm is addressed in Section 5, where a Newton method for determining the correct perturbation level ε is developed. The algorithm proposed in this work is presented in Section 5.3.

Finally, in Section 6, we present a range of numerical experiments to compare the quality of the lower bounds obtained with our algorithm to those obtained with `muSSV`.

2. Framework. We consider a matrix $M \in \mathbb{C}^{n \times n}$ and an underlying perturbation set with prescribed block diagonal structure,

$$\mathbb{B} = \left\{ \text{diag}(\delta_1 I_{r_1}, \dots, \delta_S I_{r_S}, \Delta_1, \dots, \Delta_F), \delta_i \in \mathbb{C}(\mathbb{R}), \Delta_j \in \mathbb{C}^{m_j \times m_j}(\mathbb{R}^{m_j \times m_j}) \right\}, \quad (2.1)$$

where I_{r_i} denotes the $r_i \times r_i$ identity matrix. Each of the scalars δ_i and the $m_j \times m_j$ matrices Δ_j may be constrained to stay real in the definition of \mathbb{B} . The integer S denotes the number of repeated *scalar* blocks (that is, scalar multiples of the identity) and F denotes the number of *full* blocks. This implies $\sum_{i=1}^S r_i + \sum_{j=1}^F m_j = n$. In order to distinguish complex and real scalar blocks, we assume that the first $S' \leq S$ blocks are complex while the (possibly) remaining $S - S'$ blocks are real. Similarly we assume that the first $F' \leq F$ full blocks are complex and the (possibly) remaining $F - F'$ blocks are real. The literature (see, e.g., [14]) usually does not consider real full blocks, that is, $F' = F$. In fact, in control theory, full blocks arise from uncertainties associated to the frequency response of a system, which is complex-valued.

For simplicity, we assume that all full blocks are square, although this is not necessary and our method extends to the non-square case in a straightforward way. Similarly, the chosen ordering of blocks should not be viewed as a limiting assumption; it merely simplifies notation.

The following definition is given in [14], where $\|\cdot\|_2$ denotes the matrix 2-norm and I the $n \times n$ identity matrix.

DEFINITION 2.1. Let $M \in \mathbb{C}^{n \times n}$ and consider a set \mathbb{B} of the form (2.1). Then the SSV (or μ -value) $\mu_{\mathbb{B}}(M)$ is defined as

$$\mu_{\mathbb{B}}(M) := \frac{1}{\min \{\|\Delta\|_2 : \Delta \in \mathbb{B}, \det(I - M\Delta) = 0\}}. \quad (2.2)$$

In Definition (2.1) and in the following, we use the convention that the minimum over an empty set is $+\infty$. In particular, $\mu_{\mathbb{B}}(M) = 0$ if $\det(I - M\Delta) \neq 0$ for all $\Delta \in \mathbb{B}$.

Note that μ_{Δ} is a positively homogeneous function, i.e.,

$$\mu_{\mathbb{B}}(\alpha M) = \alpha \mu_{\mathbb{B}}(M) \quad \text{for any } \alpha \geq 0.$$

For $\mathbb{B} = \mathbb{C}^{n \times n}$, it follows directly from Definition 2.1 that $\mu_{\mathbb{B}}(M) = \|M\|_2$. For general \mathbb{B} , the SSV can only become smaller and we thus have the upper bound $\mu_{\mathbb{B}}(M) \leq \|M\|_2$. This can be refined further by exploiting the properties of $\mu_{\mathbb{B}}$, see [20]. These relations between $\mu_{\mathbb{B}}(M)$ and $\|M\|_2$, the largest singular value of M , justifies the name *structured singular value* for $\mu_{\mathbb{B}}(M)$.

The important special case when \mathbb{B} only allows for complex perturbations, that is, $S = S'$ and $F = F'$, deserves particular attention. In this case we will write \mathbb{B}^* instead of \mathbb{B} . Note that $\Delta \in \mathbb{B}^*$ implies $e^{i\varphi}\Delta \in \mathbb{B}^*$ for any $\varphi \in \mathbb{R}$. In turn, there is $\Delta \in \mathbb{B}^*$ such that $\rho(M\Delta) = 1$ if and only if there is $\Delta' \in \mathbb{B}^*$, with the same norm, such that $M\Delta'$ has the eigenvalue 1, which implies $\det(I - M\Delta') = 0$. This gives the following alternative expression:

$$\mu_{\mathbb{B}^*}(M) = \frac{1}{\min \{\|\Delta\|_2 : \Delta \in \mathbb{B}^*, \rho(M\Delta) = 1\}}, \quad (2.3)$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix. For any nonzero eigenvalue λ of M , the matrix $\Delta = \lambda^{-1}I$ satisfies the constraints of the minimization problem in (2.3). This establishes the lower bound $\rho(M) \leq \mu_{\mathbb{B}^*}(M)$ for the case of purely complex perturbations. Note that $\mu_{\mathbb{B}^*}(M) = \rho(M)$ for $\mathbb{B}^* = \{\delta I : \delta \in \mathbb{C}\}$. Hence, both the spectral radius and the matrix 2-norm are included as (trivial) special cases of the SSV.

2.1. A motivating example. Consider the 3×3 matrix

$$M = \begin{pmatrix} -1 + \mathbf{i} & 1 - \mathbf{i} & -1 + \mathbf{i} \\ -1 + \mathbf{i} & -1 & \mathbf{i} \\ \mathbf{i} & -1 - \mathbf{i} & 1 - \mathbf{i} \end{pmatrix},$$

where \mathbf{i} denotes the imaginary unit, along with the perturbation set

$$\mathbb{B} = \{\text{diag}(\delta_1 I_2, \Delta_1) : \delta_1 \in \mathbb{R}, \Delta_1 \in \mathbb{C}^{1,1}\}.$$

Applying the MATLAB function `mussv*` yields the bounds

$$0.9807 \dots \leq \mu_{\mathbb{B}}(M) \leq 2.2477 \dots \quad (2.4)$$

The large difference between the lower and upper bounds is caused by the lower bound. The perturbation determining the lower bound is given by $\hat{\varepsilon}\hat{\Delta}$ with

$$\hat{\Delta} = \begin{pmatrix} -0.368473881 \dots & 0 & 0 \\ 0 & -0.368473881 \dots & 0 \\ 0 & 0 & -0.673755352 \dots - 0.738954481 \dots \mathbf{i} \end{pmatrix}$$

*In all experiments we have used `mussv` with its default parameters.

and $\widehat{\varepsilon} = 1.019727084 \dots$. The scaling has been chosen such that $\|\widehat{\Delta}\|_2 = 1$. However, not all blocks of $\widehat{\Delta}$ have unit norm; the 2×2 repeated scalar block of $\widehat{\Delta}$ has norm $0.368473881 \dots$. We will see in Theorem 4.2 below that this violates a necessary optimality condition for an extremizer $\Delta \in \mathbb{B}$, which states that the spectral norm of all blocks of a normalized extremizer, under suitable conditions which are fulfilled here, should be one.

Applying our new algorithm, Algorithm 1 below, we obtain the perturbation $\varepsilon^* \Delta^*$ with

$$\Delta^* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -0.989237164 - 0.146320991 \dots i \end{pmatrix}.$$

and $\varepsilon^* = 0.445238645 \dots$, determining the lower bound

$$\mu_{\mathbb{B}}(M) \geq \mu_{\text{New}}^{\ell} = 2.2459865301 \dots,$$

which makes the estimate (2.4) substantially sharper. Note that both blocks of Δ^* have unit norm.

2.2. A reformulation based on structured spectral value sets. The structured spectral value set of $M \in \mathbb{C}^{n \times n}$ with respect to a perturbation level ε is defined as

$$\Lambda_{\varepsilon}^{\mathbb{B}}(M) = \{ \lambda \in \Lambda(\varepsilon M \Delta) : \Delta \in \mathbb{B}, \|\Delta\|_2 \leq 1 \}, \quad (2.5)$$

where $\Lambda(\cdot)$ denotes the spectrum of a matrix. Note that for purely complex \mathbb{B}^* , the set (2.5) is simply a disk centered at 0. The set

$$\Sigma_{\varepsilon}^{\mathbb{B}}(M) = \{ \zeta = 1 - \lambda : \lambda \in \Lambda_{\varepsilon}^{\mathbb{B}}(M) \} \quad (2.6)$$

allows us to express the SSV defined in (2.2) as

$$\mu_{\mathbb{B}}(M) = \frac{1}{\arg \min_{\varepsilon > 0} \{0 \in \Sigma_{\varepsilon}^{\mathbb{B}}(M)\}},$$

that is, as a structured distance to singularity problem. We have that $0 \notin \Sigma_{\varepsilon}^{\mathbb{B}}(M)$ if and only if $\mu_{\mathbb{B}}(M) < 1/\varepsilon$.

For a purely complex perturbation set \mathbb{B}^* , we can use (2.3) to alternatively express the SSV as

$$\mu_{\mathbb{B}^*}(M) = \frac{1}{\arg \min_{\varepsilon > 0} \left\{ \max_{\lambda \in \Lambda_{\varepsilon}^{\mathbb{B}^*}(M)} |\lambda| = 1 \right\}}. \quad (2.7)$$

We have that $\Lambda_{\varepsilon}^{\mathbb{B}^*}(M) \subset D$, where D denotes the open complex unit disk, if and only if $\mu_{\mathbb{B}^*}(M) < 1/\varepsilon$.

2.3. Overview of the proposed methodology. Let us consider the minimization problem

$$\zeta(\varepsilon) = \arg \min_{\zeta \in \Sigma_{\varepsilon}^{\mathbb{B}}(M)} |\zeta| \quad (2.8)$$

for some fixed $\varepsilon > 0$. By the discussion above, the SSV $\mu_{\mathbb{B}}(M)$ is the reciprocal of the smallest value of ε for which $\zeta(\varepsilon) = 0$. This suggests a two-level algorithm: In the inner algorithm, we attempt to solve (2.8). In the outer algorithm, we vary ε by an iterative procedure which exploits the knowledge of the exact derivative of an extremizer – say $\Delta(\varepsilon)$ – with respect to ε . We address (2.8) by solving a system of ODEs. In general, this only yields a local minimum of (2.8) which, in turn, gives an upper bound for ε and hence a lower bound for $\mu_{\mathbb{B}}(M)$. Due to the lack of global optimality criteria for (2.8), the only way to increase the robustness of the method is to compute several local optima.

The case of a purely complex perturbation set \mathbb{B}^* can be addressed analogously by letting the inner algorithm determine local optima for

$$\lambda(\varepsilon) = \arg \max_{\lambda \in \Lambda_{\varepsilon}^{\mathbb{B}^*}(M)} |\lambda|, \quad (2.9)$$

which then yields a lower bound for $\mu_{\mathbb{B}^*}(M)$.

3. Purely complex perturbations. In this section, we consider the solution of the inner problem (2.9) in the estimation of $\mu_{\mathbb{B}^*}(M)$ for $M \in \mathbb{C}^{n \times n}$ and a purely complex perturbation set

$$\mathbb{B}^* = \{ \text{diag}(\delta_1 \mathbf{I}_{r_1}, \dots, \delta_S \mathbf{I}_{r_S}, \Delta_1, \dots, \Delta_F) : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j} \}.$$

3.1. Extremizers. We will make use of the following standard eigenvalue perturbation result, see, e.g., [11, Section II.1.1]. Here and in the following, we denote $\dot{\cdot} = d/dt$.

LEMMA 3.1. *Consider a smooth matrix family $C : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$ and let $\lambda(t)$ be an eigenvalue of $C(t)$ converging to a simple eigenvalue λ_0 of $C_0 = C(0)$ as $t \rightarrow 0$. Then $\lambda(t)$ is analytic near $t = 0$ with*

$$\dot{\lambda}(0) = \frac{y_0^* C_1 x_0}{y_0^* x_0},$$

where $C_1 = \dot{C}(0)$ and x_0, y_0 are right and left eigenvectors of C_0 associated to λ_0 , that is, $(C_0 - \lambda_0 \mathbf{I})x_0 = 0$ and $y_0^*(C_0 - \lambda_0 \mathbf{I}) = 0$.

Our goal is to solve the maximization problem (2.9), which requires finding a perturbation Δ_{opt} such that $\rho(\varepsilon M \Delta_{\text{opt}})$ is maximal among all $\Delta \in \mathbb{B}^*$ with $\|\Delta\|_2 \leq 1$. In the following, we call λ a largest eigenvalue if $|\lambda|$ equals the spectral radius.

DEFINITION 3.2. *A matrix $\Delta \in \mathbb{B}^*$ such that $\|\Delta\|_2 \leq 1$ and $\varepsilon M \Delta$ has a largest eigenvalue that locally maximizes the modulus of $\Lambda_{\varepsilon}^{\mathbb{B}^*}(M)$ is called a local extremizer.*

The following result provides an important characterization of local extremizers.

THEOREM 3.3. *Let*

$$\Delta_{\text{opt}} = \text{diag}(\delta_1 \mathbf{I}_{r_1}, \dots, \delta_S \mathbf{I}_{r_S}, \Delta_1, \dots, \Delta_F), \quad \|\Delta_{\text{opt}}\|_2 = 1,$$

be a local extremizer of $\Lambda_{\varepsilon}^{\mathbb{B}^}(M)$. We assume that $\varepsilon M \Delta_{\text{opt}}$ has a simple largest eigenvalue $\lambda = |\lambda|e^{i\theta}$, with the right and left eigenvectors x and y scaled such that $s = e^{i\theta} y^* x > 0$. Partitioning*

$$x = (x_1^T \dots x_S^T, x_{S+1}^T \dots x_{S+F}^T)^T, \quad z = M^* y = (z_1^T \dots z_S^T, z_{S+1}^T \dots z_{S+F}^T)^T, \quad (3.1)$$

such that the size of the components x_k, z_k equals the size of the k th block in Δ_{opt} , we additionally assume that

$$z_k^* x_k \neq 0 \quad \forall k = 1, \dots, S \quad (3.2)$$

$$\|z_{S+h}\|_2 \cdot \|x_{S+h}\|_2 \neq 0 \quad \forall h = 1, \dots, F. \quad (3.3)$$

Then

$$|\delta_k| = 1 \quad \forall k = 1, \dots, S \quad \text{and} \quad \|\Delta_h\|_2 = 1 \quad \forall h = 1, \dots, F,$$

that is, all blocks of Δ_{opt} have unit 2-norm.

Proof. The result is proved by contradiction. We first assume that $\|\Delta_h\|_2 < 1$ for some $1 \leq h \leq F$ and consider the matrix-valued function

$$\Delta(t) = \text{diag}(\delta_1 \mathbf{I}_{r_1}, \dots, \delta_S \mathbf{I}_{r_S}, \Delta_1, \dots, \Delta_h + t z_{S+h} x_{S+h}^*, \dots, \Delta_F), \quad (3.4)$$

which satisfies $\Delta(0) = \Delta_{\text{opt}}$ and $\|\Delta(t)\|_2 \leq 1$ for t sufficiently small. Since $\lambda(0) = \lambda$ is simple, we can apply Lemma 3.1 to $\varepsilon \Delta(t) M$ and obtain

$$\begin{aligned} \frac{d}{dt} |\lambda(t)|^2 \Big|_{t=0} &= 2 \text{Re}(\bar{\lambda} \dot{\lambda}) = 2 \text{Re} \left(\bar{\lambda} \varepsilon \frac{y^* M \dot{\Delta} x}{y^* x} \right) \\ &= 2\varepsilon |\lambda| \text{Re} \left(\frac{y^* M \dot{\Delta} x}{e^{i\theta} y^* x} \right) = 2\varepsilon \frac{|\lambda|}{s} \text{Re}(z^* \dot{\Delta} x). \end{aligned} \quad (3.5)$$

Inserting (3.4) and exploiting (3.3), we obtain

$$\frac{d}{dt} |\lambda(t)|^2 \Big|_{t=0} = 2\varepsilon \frac{|\lambda|}{s} \|z_{S+h}\|_2^2 \cdot \|x_{S+h}\|_2^2 > 0,$$

which contradicts the extremality of $|\lambda|$.

Let us now assume that $|\delta_k| < 1$ for some $1 \leq k \leq S$ and consider the matrix valued function

$$\Delta(t) = \text{diag}(\delta_1 \mathbf{I}_1, \dots, \delta_k \mathbf{I}_k + t x_k^* z_k \mathbf{I}_k, \dots, \delta_S \mathbf{I}_S, \Delta_1, \dots, \Delta_F)$$

which again satisfies $\Delta(0) = \Delta_{\text{opt}}$ and $\|\Delta(t)\|_2 \leq 1$ for t sufficiently small. In analogy to the first part, Assumption (3.2) implies

$$\frac{d}{dt} |\lambda(t)|^2 \Big|_{t=0} = 2\varepsilon \frac{|\lambda|}{s} |z_k^* x_k|^2 > 0.$$

This again gives a contradiction. \square

REMARK 3.1. Note that Assumptions (3.2) and (3.3) as well as the simplicity of λ are generic and commonly found in the literature on algorithms for the SSV, see, e.g., [14, Sec. 7.2].

The following theorem allows us to replace the full blocks in a local extremizer by rank-1 matrices.

THEOREM 3.4. Let $\Delta_{\text{opt}} = \text{diag}(\delta_1 \mathbf{I}_{r_1}, \dots, \delta_S \mathbf{I}_{r_S}, \Delta_1, \dots, \Delta_F)$ be a local extremizer and let λ, x, z be defined and partitioned as in Theorem 3.3. Assuming that (3.3) holds, every block Δ_h has a singular value 1 with associated singular vectors

$u_h = \gamma_h z_{S+h} / \|z_{S+h}\|_2$ and $v_h = \gamma_h x_{S+h} / \|x_{S+h}\|_2$ for some $|\gamma_h| = 1$. Moreover, the matrix

$$\Delta_* = \text{diag}(\delta_1 \mathbf{I}_{r_1}, \dots, \delta_S \mathbf{I}_{r_S}, u_1 v_1^*, \dots, u_F v_F^*)$$

is also a local extremizer, i.e., $\rho(\varepsilon M \Delta_{\text{opt}}) = \rho(\varepsilon M \Delta_*)$.

Proof. Let $\hat{z}_{S+h} = z_{S+h} / \|z_{S+h}\|_2$, $\hat{x}_{S+h} = x_{S+h} / \|x_{S+h}\|_2$ and consider the matrix valued function

$$\Delta(t) = \text{diag}(\delta_1 \mathbf{I}_1, \dots, \delta_S \mathbf{I}_S, \Delta_1, \dots, (1-t)\Delta_h + t \hat{z}_{S+h} \hat{x}_{S+h}^*, \dots, \Delta_F)$$

which has 2-norm bounded by 1 for $t \in [0, 1]$. By Theorem 3.3, $\|\Delta_h\|_2 = 1$, which implies $|z_{S+h}^* \Delta_h x_{S+h}| \leq \|z_{S+h}\| \|x_{S+h}\|$. Consequently,

$$\begin{aligned} \text{Re}(z^* \dot{\Delta} x) &= \text{Re}(z_{S+h}^* \Delta_h x_{S+h} + z_{S+h}^* \hat{z}_{S+h} \hat{x}_{S+h}^* x_{S+h}) \\ &= \text{Re}(z_{S+h}^* \Delta_h x_{S+h}) + \|z_{S+h}\| \|x_{S+h}\| \geq 0. \end{aligned}$$

Combined with the extremality assumption, we obtain $\text{Re}(z^* \dot{\Delta} x) = 0$. This implies that Δ_h has singular vectors u_h and v_h , which completes the proof. \square

REMARK 3.2. Theorem 3.4 allows us to restrict the perturbations in the structured spectral value set (2.5) to those with rank-1 blocks, which was also shown in [14]. Since the Frobenius and the matrix 2-norms of a rank-1 matrix are equal, we can equivalently search for extremizers within the submanifold

$$\begin{aligned} \mathbb{B}_1^* &= \{\text{diag}(\delta_1 \mathbf{I}_{r_1}, \dots, \delta_S \mathbf{I}_{r_S}, \Delta_1, \dots, \Delta_F) : \\ &\quad \delta_i \in \mathbb{C}, |\delta_i| = 1, \Delta_j \in \mathbb{C}^{m_j \times m_j}, \|\Delta_j\|_F = 1\}. \end{aligned} \quad (3.6)$$

3.2. A system of ODEs to compute extremal points of $\Lambda_\varepsilon^{\mathbb{B}^*}(M)$. In order to compute a local maximizer for $|\lambda|$, with $\lambda \in \Lambda_\varepsilon^{\mathbb{B}^*}(M)$, we will first construct a matrix valued function $\Delta(t)$, where $\Delta(t) \in \mathbb{B}_1^*$, such that a largest eigenvalue $\lambda(t)$ of $\varepsilon M \Delta(t)$ has maximal local increase. We then derive a system of ODEs satisfied by this choice of $\Delta(t)$.

Orthogonal projection onto \mathbb{B}^* . In the following, we make use of the Frobenius inner product $\langle A, B \rangle = \text{trace}(A^* B)$ for two $m \times n$ matrices A, B . We let

$$C_{\mathbb{B}^*} = P_{\mathbb{B}^*}(C). \quad (3.7)$$

denote the orthogonal projection, with respect to the Frobenius inner product, of a matrix $C \in \mathbb{C}^{n \times n}$ onto \mathbb{B}^* . To derive a compact formula for this projection, we use the pattern matrix

$$\mathbb{1}_{\mathbb{B}^*} = \text{diag}(\mathbb{1}_{r_1}, \dots, \mathbb{1}_{r_S}, \mathbb{1}_{m_1}, \dots, \mathbb{1}_{m_F}), \quad (3.8)$$

where $\mathbb{1}_d$ denotes the $d \times d$ -matrix of all ones.

LEMMA 3.5. For $C \in \mathbb{C}^{n \times n}$, let

$$C \odot \mathbb{1}_{\mathbb{B}^*} = \text{diag}(C_1, \dots, C_{S+F})$$

denote the block diagonal matrix obtained by entrywise multiplication of C with the matrix $\mathbb{1}_{\mathbb{B}^*}$ defined in (3.8). Then the orthogonal projection of C onto \mathbb{B}^* is given by

$$C_{\mathbb{B}^*} = P_{\mathbb{B}^*}(C) = \text{diag}(\gamma_1 \mathbf{I}_{r_1}, \dots, \gamma_S \mathbf{I}_{r_S}, \Gamma_1, \dots, \Gamma_F) \quad (3.9)$$

where $\gamma_i = \text{trace}(C_i)/r_i$, $i = 1, \dots, S$, and $\Gamma_1 = C_{S+1}, \dots, \Gamma_F = C_{S+F}$.

Proof. The result follows directly from the fact that

$$\gamma_* = \arg \min_{\gamma \in \mathbb{C}} \|E - \gamma \mathbf{I}_r\|_{\mathbf{F}} = \frac{1}{r} \text{trace}(E)$$

holds for every $E \in \mathbb{C}^{r \times r}$. \square

If $C = uv^*$ is a rank-1 matrix, with the partitioning

$$u = (u_1^T \dots u_S^T, u_{S+1}^T \dots u_{S+F}^T)^T, \quad v = (v_1^T \dots v_S^T, v_{S+1}^T \dots v_{S+F}^T)^T,$$

then the diagonal blocks $\Gamma_j = u_{S+j} v_{S+j}^*$ of the orthogonal projection are again rank-1 matrices and, moreover, $\gamma_i = v_i^* u_i / r_i$.

The local optimization problem. Let us recall the setting from Section 3.1: We assume that $\lambda = |\lambda|e^{i\theta}$ is a simple eigenvalue with eigenvectors x, y normalized such that

$$\|y\| = \|x\| = 1, \quad y^* x = |y^* x| e^{-i\theta}. \quad (3.10)$$

As a consequence of Lemma 3.1, see also (3.5), we have

$$\frac{d}{dt} |\lambda|^2 = 2|\lambda| \text{Re} \left(\frac{z^* \dot{\Delta} x}{e^{i\theta} y^* x} \right) = \frac{2|\lambda|}{|y^* x|} \text{Re}(z^* \dot{\Delta} x), \quad (3.11)$$

where $z = M^* y$ and the dependence on t is intentionally omitted.

Letting $\Delta \in \mathbb{B}_1^*$, with \mathbb{B}_1^* as in (3.6), we now aim at determining a direction $\dot{\Delta} = Z$ that locally maximizes the increase of the modulus of λ . This amounts to determining

$$Z = \text{diag}(\omega_1 \mathbf{I}_{r_1}, \dots, \omega_S \mathbf{I}_{r_S}, \Omega_1, \dots, \Omega_F) \quad (3.12)$$

as a solution of the optimization problem

$$\begin{aligned} Z_* &= \arg \max \{ \text{Re}(z^* Z x) : Z \text{ takes the form (3.12)} \} \\ \text{subject to} \quad & \text{Re}(\bar{\delta}_i \omega_i) = 0, \quad i = 1, \dots, S, \\ \text{and} \quad & \text{Re}\langle \Delta_j, \Omega_j \rangle = 0, \quad j = 1, \dots, F. \end{aligned} \quad (3.13)$$

The target function in (3.13) follows from (3.11), while the constraints in (3.12) and (3.13) ensure that Z is in the tangent space of \mathbb{B}_1^* at Δ . In particular, (3.13) implies that the norms of the blocks of Δ are conserved. Note that (3.13) only becomes well-posed after imposing an additional normalization on the norm of Z . The scaling chosen in the following lemma aims at $Z \in \mathbb{B}_1^*$.

LEMMA 3.6. *With the notation introduced above and x, z partitioned as in (3.1), a solution of the optimization problem (3.13) is given by*

$$Z_* = \text{diag}(\omega_1 \mathbf{I}_{r_1}, \dots, \omega_S \mathbf{I}_{r_S}, \Omega_1, \dots, \Omega_F),$$

with

$$\omega_i = \nu_i (x_i^* z_i - \text{Re}(x_i^* z_i \bar{\delta}_i) \delta_i), \quad i = 1, \dots, S \quad (3.14)$$

$$\Omega_j = \zeta_j (z_{S+j} x_{S+j}^* - \text{Re}\langle \Delta_j, z_{S+j} x_{S+j}^* \rangle \Delta_j), \quad j = 1, \dots, F. \quad (3.15)$$

Here, $\nu_i > 0$ is the reciprocal of the absolute value of the right-hand side in (3.14), if this is different from zero, and $\nu_i = 1$ otherwise. Similarly, $\zeta_j > 0$ is the reciprocal of the Frobenius norm of the matrix on the right hand side in (3.15), if this is different from zero, and $\zeta_j = 1$ otherwise. If all right-hand sides are different from zero then $Z_* \in \mathbb{B}_1^*$.

Proof. The equality

$$z^* Z x = \sum_{i=1}^S \omega_i z_i^* x_i + \sum_{j=1}^F z_{S+j}^* \Omega_j x_{S+j} = \sum_{i=1}^S \omega_i \langle z_i, x_i \rangle + \sum_{j=1}^F \langle z_{S+j} x_{S+j}^*, \Omega_j \rangle$$

implies that the maximization problem (3.13) decouples, which allows us to maximize for each block of Z individually.

For a full block Ω_j , the term $\langle z_{S+j} x_{S+j}^*, \Omega_j \rangle$ is maximized by the orthogonal projection of $z_{S+j} x_{S+j}^*$ onto the (real linear) subspace $\{\Omega \in \mathbb{C}^{m_j \times m_j} : \operatorname{Re} \langle \Delta_j, \Omega \rangle = 0\}$. This gives (3.15), with the scaling chosen such that $\|\Omega_j\|_{\mathbf{F}} = 1$ unless $\Omega_j = 0$.

For a block $\omega_i I_{r_i}$, the term $\omega_i z_i^* x_i$ is maximized by projecting $x_i^* z_i$ onto $\{\omega_i \in \mathbb{C} : \operatorname{Re}(\bar{\delta}_i \omega_i) = 0\}$. This gives (3.14), with the scaling chosen such that $|\delta_i| = 1$ unless $\delta_i = 0$. \square

COROLLARY 3.7. *The result of Lemma 3.6 can be expressed as*

$$Z_* = D_1 P_{\mathbb{B}^*}(zx^*) - D_2 \Delta \quad (3.16)$$

where $P_{\mathbb{B}^*}(\cdot)$ is the orthogonal projection from Definition 3.7, and $D_1, D_2 \in \mathbb{B}^*$ are diagonal matrices with D_1 positive.

Proof. The statement is an immediate consequence of Lemma 3.5. \square

The system of ODEs. Lemma 3.6 and Corollary 3.7 suggest to consider the following differential equation on the manifold \mathbb{B}_1^* :

$$\dot{\Delta} = D_1 P_{\mathbb{B}^*}(zx^*) - D_2 \Delta, \quad (3.17)$$

where $x(t)$ is an eigenvector, of unit norm, associated to a simple eigenvalue $\lambda(t)$ of $\varepsilon M \Delta(t)$ for some fixed $\varepsilon > 0$. Note that $z(t), D_1(t), D_2(t)$ depend on $\Delta(t)$ as well. The differential equation (3.17) is a gradient system because, by definition, the right-hand side is the projected gradient of $Z \mapsto \operatorname{Re}(z^* Z x)$.

The following result follows directly from Lemmas 3.1 and 3.6.

THEOREM 3.8. *Let $\Delta(t) \in \mathbb{B}_1^*$ satisfy the differential equation (3.17). If $\lambda(t)$ is a simple eigenvalue of $\varepsilon M \Delta(t)$, then $|\lambda(t)|$ increases monotonically.*

The following lemma establishes a useful property for the analysis of stationary points of (3.17).

LEMMA 3.9. *Let $\Delta(t) \in \mathbb{B}_1^*$ satisfy the differential equation (3.17). If $\lambda(t)$ is a nonzero simple eigenvalue of $\varepsilon M \Delta(t)$, with right and left eigenvectors $x(t)$ and $y(t)$ scaled according to (3.10), then*

$$P_{\mathbb{B}^*}(z(t)x(t)^*) \neq 0, \quad (3.18)$$

where $z(t) = M^* y(t)$.

Proof. For convenience, we again omit the dependence on t and let $\lambda = |\lambda|e^{i\theta}$. Assume – in contradiction to the statement – that $P_{\mathbb{B}^*}(zx^*) = 0$. Because of the block diagonal structure of Δ , this implies

$$\operatorname{Re} \langle zx^*, \varepsilon \Delta \rangle = \operatorname{Re} \langle P_{\mathbb{B}^*}(zx^*), \varepsilon \Delta \rangle = 0. \quad (3.19)$$

On the other hand,

$$\operatorname{Re}\langle zx^*, \varepsilon \Delta \rangle = \operatorname{Re}\langle yx^*, \varepsilon M \Delta \rangle = \operatorname{Re}(y^* \varepsilon M \Delta x) = \operatorname{Re}(|\lambda| e^{i\theta} y^* x).$$

Exploiting the normalization (3.10) and the simplicity of λ , we obtain $\operatorname{Re}\langle zx^*, \varepsilon \Delta \rangle = |\lambda| |y^* x| > 0$. This, however, contradicts (3.19). \square

The differential equation (3.17) can be expressed in terms of the blocks of Δ , that is, through $\{\delta_i\}_{i=1}^S$ and $\{\Delta_j\}_{j=1}^F$, as follows:

$$\begin{aligned} \dot{\delta}_i &= \nu_i \left(x_i^* z_i - \operatorname{Re}(x_i^* z_i \bar{\delta}_i) \delta_i \right), & i &= 1, \dots, S \\ \dot{\Delta}_j &= \eta_j \left(z_{S+j} x_{S+j}^* - \operatorname{Re}\langle \Delta_j, z_{S+j} x_{S+j}^* \rangle \Delta_j \right), & j &= 1, \dots, F \end{aligned} \quad (3.20)$$

with the scalars ν_i, η_j defined in Lemma 3.6.

Because of $|\delta_i| = 1$, we can reparametrize $\delta_i = e^{i\beta_i}$ and rewrite the first set of equations in (3.20) as a system of ODEs in $\beta_i \in \mathbb{R}$. Setting $\gamma_i = \arg(z_i^* x_i)$, we obtain

$$\dot{\beta}_i e^{i\beta_i} = \nu_i |z_i^* x_i| \left(e^{-i\gamma_i} - \operatorname{Re}(e^{-i(\gamma_i + \beta_i)}) e^{i\beta_i} \right),$$

which gives $\dot{\beta}_i = -\nu_i |z_i^* x_i| \sin(\gamma_i + \beta_i)$. With the normalization imposed by ν_i , this finally yields

$$\dot{\beta}_i = -\operatorname{sign}(\sin(\gamma_i + \beta_i)).$$

In particular, this means that $\dot{\beta}_i = 0$ if and only if $\gamma_i + \beta_i = 0, \pm\pi$; maximizers correspond to $\beta_i = -\gamma_i$.

REMARK 3.3. *The choice of ν_i, η_j originating from Lemma 3.6, to achieve unit norm of all blocks in (3.16), is completely arbitrary. Other choices would be also acceptable and investigating an optimal one in terms of speed of convergence to stationary points would be an interesting issue.*

The following result characterizes stationary points of (3.17).

THEOREM 3.10. *Assume that $\Delta(t)$ is a solution of (3.17) and $\lambda(t)$ is a largest simple nonzero eigenvalue of $\varepsilon M \Delta(t)$ with right/left eigenvectors $x(t), y(t)$. Moreover, suppose that Assumptions (3.2) and (3.3) hold for $x(t)$ and $z(t) = M^* y(t)$. Then*

$$\frac{d}{dt} |\lambda(t)|^2 = 0 \iff \dot{\Delta}(t) = 0 \iff \Delta(t) = DP_{\mathbb{B}^*}(z(t)x(t)^*), \quad (3.21)$$

for a specific real diagonal matrix $D \in \mathbb{B}^*$. Moreover if $\lambda(t)$ has (locally) maximal modulus over the set $\Lambda_{\varepsilon}^{\mathbb{B}^*}(M)$ then D is positive.

Proof. By (3.11), $\frac{d}{dt} |\lambda(t)|^2 = 0$ implies $\operatorname{Re}(z^* \dot{\Delta} x) = 0$. Inserting (3.20) shows that each block of $\dot{\Delta}$ is necessarily zero and hence $\dot{\Delta} = 0$. The other direction of the first equivalence in (3.21) is trivial. The second equivalence in (3.21) follows directly from (3.17). By Theorem 3.3, all blocks of $\Delta(t)$ have norm 1 and hence none of the scalars defining D can be zero. Thus, D is nonsingular.

Assuming that $\lambda(t)$ has (locally) maximal modulus, we now prove positivity of D by contradiction. Suppose that the $(S+j)$ th full block of D is equal to $-\gamma_j \mathbf{I}_{m_j}$ with $\gamma_j > 0$, implying $\Delta_j = -\gamma_j z_{S+j} x_{S+j}^*$. Consider an ODE with the $(S+j)$ th block $\dot{\Delta}_j = z_{S+j} x_{S+j}^*$ and initial datum $\Delta_j(0) = -\gamma_j z_{S+j} x_{S+j}^*$, while leaving all other blocks of Δ unaltered. Such an ODE clearly decreases the norm of Δ_j for

$t \leq \bar{t}$, for some $\bar{t} > 0$ (implying that $\|\Delta(t)\|_{\mathbf{F}}$ does not exceed 1). By the usual derivative formula from Lemma 3.1 the largest eigenvalue $\lambda(t)$ of $\varepsilon\Delta(t)M$ is such that $\frac{d}{dt}|\lambda(t)| > 0$, which contradicts local maximality.

Similarly consider a repeated scalar block and assume that the i th block of D is equal to $-\gamma_i \mathbf{I}_{r_i}$ with $\gamma_i > 0$, which means $\delta_i = \gamma_i z_i^* x_i$. Consider, similarly to previous case, an ODE with the i th block $\dot{\delta}_i = x_i^* z_i$ and initial datum $\delta_i = -\gamma_i x_i^* z_i$. Again, $|\delta_i(t)|$ decreases in a sufficiently small time-interval $[0, \bar{t}]$ and $|\lambda(t)|$ increases in the same interval, yielding again a contradiction. \square

3.3. Projection of full blocks on rank-1 manifolds. In order to exploit the rank-1 property of extremizers established in Theorem 3.4, we can proceed in complete analogy to [6] in order to obtain for each full block an ODE on the manifold \mathcal{M}_1 of (complex) rank-1 matrices. We express $\Delta_j \in \mathcal{M}_1 \subset \mathbb{C}^{m_j \times m_j}$ as

$$\Delta_j = \sigma_j p_j q_j^*, \quad \dot{\Delta}_j = \dot{\sigma}_j p_j q_j^* + \sigma_j \dot{p}_j q_j^* + \sigma_j p_j \dot{q}_j^*$$

where $\sigma_j \in \mathbb{C}$ and $p_j, q_j \in \mathbb{C}^{m_j}$ have unit norm. The parameters $\dot{\sigma}_j \in \mathbb{C}$, $\dot{p}_j, \dot{q}_j \in \mathbb{C}^{m_j}$ are uniquely determined by σ_j, p_j, q_j and $\dot{\Delta}_j$ when imposing the orthogonality conditions $p_j^* \dot{p}_j = 0$, $q_j^* \dot{q}_j = 0$.

In the differential equation (3.20) we replace the right-hand side by its orthogonal projection onto the tangent space $T_{\Delta_j} \mathcal{M}_1$ (and also remove the normalization constant) to obtain

$$\dot{\Delta}_j = P_{\Delta_j} (z_{S+j} x_{S+j}^* - \operatorname{Re} \langle \Delta_j, z_{S+j} x_{S+j}^* \rangle \Delta_j). \quad (3.22)$$

Note that the orthogonal projection of a matrix $Z \in \mathbb{C}^{m_j \times m_j}$ onto $T_{\Delta_j} \mathcal{M}_1$ at $\Delta_j = \sigma_j p_j q_j^* \in \mathcal{M}_1$ is given by

$$P_{\Delta_j}(Z) = Z - (\mathbf{I} - p_j p_j^*) Z (\mathbf{I} - q_j q_j^*).$$

Following the arguments of [6], the equation $\dot{\Delta}_j = P_{\Delta_j}(Z)$ is equivalent to

$$\begin{aligned} \dot{\sigma}_j &= p_j^* Z q_j \\ \dot{p}_j &= (\mathbf{I} - p_j p_j^*) Z q_j \sigma_j^{-1} \\ \dot{q}_j &= (\mathbf{I} - q_j q_j^*) Z^* p_j \bar{\sigma}_j^{-1}. \end{aligned}$$

Inserting $Z = z_{S+j} x_{S+j}^* - \operatorname{Re} \langle \Delta_j, z_{S+j} x_{S+j}^* \rangle \Delta_j$, we obtain that the differential equation (3.22) is equivalent to the following system of differential equations for σ_j, p_j and q_j , where we set $\alpha_j = p_j^* z_{S+j} \in \mathbb{C}$, $\beta_j = q_j^* x_{S+j} \in \mathbb{C}$:

$$\begin{aligned} \dot{\sigma}_j &= \alpha_j \bar{\beta}_j - \operatorname{Re}(\bar{\alpha}_j \beta_j \sigma_j) \sigma_j = \mathbf{i} \operatorname{Im}(\alpha_j \bar{\beta}_j \bar{\sigma}_j) \sigma_j \\ \dot{p}_j &= (z_{S+j} - \alpha_j p_j) \bar{\beta}_j \sigma_j^{-1} \\ \dot{q}_j &= (x_{S+j} - \beta_j q_j) \bar{\alpha}_j \bar{\sigma}_j^{-1}. \end{aligned} \quad (3.23)$$

The derivation of this system of ODEs is straightforward; we refer the reader to [7] for details.

The monotonicity and the characterization of stationary points follows analogously to those obtained for (3.20); we refer to [6] for the proofs. As a consequence we can use the ODE (3.23) instead of (3.20) and gain in terms of computational complexity.

3.4. An illustrative example. Consider the matrix

$$M = \begin{pmatrix} -0.10 - 0.55\mathbf{i} & -0.57 - 1.59\mathbf{i} & -1.34 - 1.70\mathbf{i} & 0.04 + 0.49\mathbf{i} & -0.18 + 0.19\mathbf{i} \\ -1.48 - 2.17\mathbf{i} & 0.58 + 1.17\mathbf{i} & 0.05 + 0.53\mathbf{i} & 0.11 - 0.42\mathbf{i} & 0.26 + 1.19\mathbf{i} \\ -0.53 + 0.59\mathbf{i} & 0.78 - 1.48\mathbf{i} & 0.15 & -0.25 + 1.46\mathbf{i} & 0.33 + 1.32\mathbf{i} \\ 0.24 + 0.79\mathbf{i} & -0.12 - 0.65\mathbf{i} & 1.79 - 0.09\mathbf{i} & -0.63 + 1.39\mathbf{i} & -0.88 + 0.10\mathbf{i} \\ -2.03 + 1.33\mathbf{i} & -1.22 - 0.22\mathbf{i} & 0.45 - 1.49\mathbf{i} & 0.94 - 0.13\mathbf{i} & -1.02 + 2.33\mathbf{i} \end{pmatrix}$$

and a perturbation set given by

$$\mathbb{B}^* = \{ \text{diag}(\delta_1 \mathbf{I}_1, \delta_2 \mathbf{I}_1, \Delta_1, \delta_3 \mathbf{I}_1), \delta_1, \delta_2, \delta_3 \in \mathbb{C}, \Delta_1 \in \mathbb{C}^{2,2} \}.$$

Applying MATLAB's `mussv`, we obtain the perturbation $\widehat{\varepsilon}\widehat{\Delta}$ (with $\|\widehat{\Delta}\|_2 = 1$)

$$\widehat{\Delta} = \begin{pmatrix} e^{-\mathbf{i}0.48650737} & 0 & 0 & 0 \\ 0 & e^{-\mathbf{i}1.49644308} & 0 & 0 \\ 0 & 0 & u_1 v_1^* & 0 \\ 0 & 0 & 0 & e^{-\mathbf{i}2.155849308} \end{pmatrix}$$

$$u_1 = \begin{pmatrix} 0.41899793 + 0.68039781\mathbf{i} \\ 0.06834008 - 0.59735180\mathbf{i} \end{pmatrix}, \quad v_1 = \begin{pmatrix} 0.52696073 \\ 0.70477030 + 0.47498548\mathbf{i} \end{pmatrix},$$

and $\widehat{\varepsilon} = 0.228726413$, which determines the following lower bound for the μ -value: $\mu_{\mathbb{B}^*}(M) \geq \mu_{\text{PD}}^\ell = 4.372035505$.

Applying Algorithm 1 below we find the locally extremal perturbation

$$\Delta^* = \begin{pmatrix} e^{-\mathbf{i}2.49033999} & 0 & 0 & 0 \\ 0 & e^{\mathbf{i}1.24640446} & 0 & 0 \\ 0 & 0 & u_1 v_1^* & 0 \\ 0 & 0 & 0 & e^{-\mathbf{i}1.72494213} \end{pmatrix}$$

$$u_1 = \begin{pmatrix} 0.15703326 + 0.85130227\mathbf{i} \\ 0.29626531 - 0.40354908\mathbf{i} \end{pmatrix}, \quad v_1 = \begin{pmatrix} 0.68793173 \\ 0.28357426 + 0.66808351\mathbf{i} \end{pmatrix}.$$

and $\varepsilon^* = 0.222994978$, which determines the following lower bound for the μ -value: $\mu_{\mathbb{B}^*}(M) \geq \mu_{\text{PD}}^\ell = 4.484405922$. Thus, the lower bound has improved, in particular when taking into account that the upper bound computed by `mussv` is 4.48638...

The behavior of the spectral radius of the matrix $\varepsilon^* M \Delta(t)$ along the solution of the ODE is illustrated in Figure 3.1, which shows the monotonically increasing behavior of $|\lambda(\varepsilon)|$.

4. General complex/real perturbations. We now consider the more general case addressed by [14], that is,

$$\mathbb{B} = \{ \text{diag}(\delta_1 \mathbf{I}_{r_1}, \dots, \delta_s \mathbf{I}_{r_s}, \Delta_1, \dots, \Delta_F) : \delta_i \in \mathbb{C}(\text{or } \mathbb{R}), \Delta_j \in \mathbb{C}^{m_j \times m_j} \}, \quad (4.1)$$

where δ_i is either a complex or a real scalar. Without loss of generality, we assume that the first s' repeated scalar blocks are complex while all other repeated scalar blocks are real. Moreover, we set

$$\mathbb{B}_1 = \{ \text{diag}(\delta_1 \mathbf{I}_{r_1}, \dots, \delta_s \mathbf{I}_{r_s}, \Delta_1, \dots, \Delta_F) \in \mathbb{B} : |\delta_i| = 1, \|\Delta_j\|_{\mathbf{F}} = 1 \}.$$

This case differs qualitatively from the purely complex case discussed in Section 3, since it has to be formulated as a structured distance to singularity of the matrix $\mathbf{I} - \varepsilon M \Delta$. Due to the realness constraint for some of the repeated scalar blocks, $\Delta \in \mathbb{B}$ does not imply that $e^{\mathbf{i}\theta} \Delta \in \mathbb{B}$ for all $\theta \in [0, 2\pi)$, which means that the spectral value set $\Lambda_\varepsilon(M)$ is generally not a disk. In turn, we need to address the minimization problem (2.8) instead of the maximization problem (2.9).

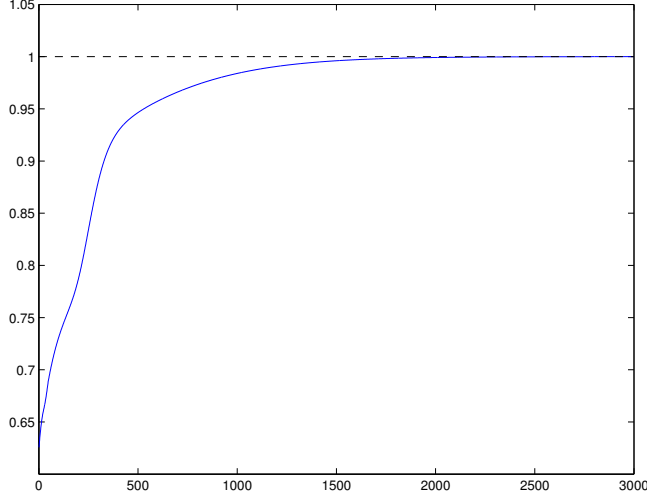


FIG. 3.1. Monotonic behavior of $\rho(\varepsilon^* M \Delta(t))$ along the solution of the ODE (3.17) for the example from Section 3.4.

4.1. Extremizers. DEFINITION 4.1. A matrix $\Delta \in \mathbb{B}$, such that $\|\Delta\|_2 \leq 1$ and $I - \varepsilon M \Delta$ has a smallest eigenvalue that locally minimizes the modulus of $\Sigma_\varepsilon^\mathbb{B}(M)$ is called a local extremizer.

We have the following result concerning local extremizers for the smallest (in modulus) complex number in $\Sigma_\varepsilon^\mathbb{B}(M)$.

THEOREM 4.2. Let

$$\Delta_{\text{opt}} = \text{diag}(\delta_1 I_{r_1}, \dots, \delta_{S'} I_{r_{S'}}, \delta_{S'+1} I_{r_{S'+1}}, \dots, \delta_S I_{r_S}, \Delta_1, \dots, \Delta_F), \quad \|\Delta_{\text{opt}}\|_2 = 1,$$

be a local extremizer of $\Sigma_\varepsilon^\mathbb{B}(M)$. Let $\zeta = |\zeta|e^{i\theta}$ be a simple smallest eigenvalue of the matrix $I - \varepsilon M \Delta_{\text{opt}}$, with the right and left eigenvectors x and y scaled such that $s = e^{i\theta} y^* x > 0$. Partitioning x and $z = M^* y$ as in (3.1), we assume that

$$z_k^* x_k \neq 0 \quad \forall k = 1, \dots, S' \quad (4.2)$$

$$\text{Re}(z_k^* x_k) \neq 0 \quad \forall k = S' + 1, \dots, S \quad (4.3)$$

$$\|z_{S+h}\|_2 \cdot \|x_{S+h}\|_2 \neq 0 \quad \forall h = 1, \dots, F \quad (4.4)$$

hold. Then

$$|\delta_k| = 1 \quad \forall k = 1, \dots, S \quad \text{and} \quad \|\Delta_h\|_2 = 1 \quad \forall h = 1, \dots, F,$$

that is, all blocks of Δ_{opt} have unit 2-norm.

Proof. The proof is analogous to the proof of Theorem 3.3. The only substantial difference is caused by repeated real scalar blocks. To address this case, suppose that $|\delta_k| < 1$ for some $S' + 1 \leq k \leq S$ with $\delta_k \in \mathbb{R}$. Let us consider the matrix valued function

$$\Delta(t) = \text{diag}(\delta_1 I_1, \dots, \delta_k I_k - t \text{Re}(x_k^* z_k) I_k, \dots, \delta_S I_S, \Delta_1, \dots, \Delta_F),$$

which satisfies $\Delta(0) = \Delta_{\text{opt}}$ and $\|\Delta(t)\|_2 \leq 1$ for t sufficiently small. Making use of Lemma 3.1 and (4.3), we obtain

$$\left. \frac{d}{dt} |\zeta(t)|^2 \right|_{t=0} = -2\varepsilon \frac{|\zeta|}{s} \text{Re}(z_k^* x_k)^2 < 0,$$

which contradicts the extremality of Δ_{opt} and thus completes the proof. \square

4.2. A system of ODEs to compute extremal points of $\Sigma_\varepsilon^{\mathbb{B}}(M)$. We can apply a procedure similar to the one developed in the Section 3.2 to develop a system of ODEs for solving (2.8), that is, $\arg \min_{\zeta \in \Sigma_\varepsilon^{\mathbb{B}}(M)} |\zeta|$.

The first step is to determine a steepest descent direction for $|\zeta|$, which then yields the right-hand side of a suitable gradient system for computing a local minimizer of the modulus of $\Sigma_\varepsilon(M)$. For this purpose, we use the same normalization (3.10) for the eigenvectors x and y associated to a simple smallest eigenvalue ζ of $I - \varepsilon M \Delta$, and recall from (3.11) that

$$\frac{d}{dt} |\zeta|^2 = - \frac{2|\zeta|}{|y^* x|} \operatorname{Re}(z^* \dot{\Delta} x), \quad \text{with } z = M^* y.$$

Rewriting the (constrained) minimization of this expression in terms of the blocks of

$$\dot{\Delta} = Z = \operatorname{diag}(\omega_1 I_{r_1}, \dots, \omega_s I_{r_s}, \Omega_1, \dots, \Omega_F) \in \mathbb{B} \quad (4.5)$$

yields the following optimization problem:

$$\begin{aligned} Z_* &= \arg \max \{ \operatorname{Re}(z^* Z x) : Z \text{ takes the form (4.5)} \} \\ \text{subject to } \quad & \operatorname{Re}(\delta_i^* \omega_i) = 0, \quad i = 1, \dots, S', \\ & \text{and } \operatorname{Re} \langle \Delta_j, \Omega_j \rangle = 0, \quad j = 1, \dots, F, \\ & \text{and } \delta_\ell \omega_\ell \leq 0 \text{ if } \delta_\ell = \pm 1, \quad \ell = S' + 1, \dots, S. \end{aligned} \quad (4.6)$$

As before, the first two constraints imply the conservation of the Frobenius norms for the full and repeated complex scalar blocks of Δ . The third constraint prevents $|\delta_\ell|$ from exceeding 1 for repeated real scalar blocks. To make (4.6) well-posed, we need to impose a normalization on Z and the following lemma aims at $Z \in \mathbb{B}_1$, whenever this is possible.

LEMMA 4.3. *With the notation introduced above and x, z partitioned as in (3.1), a solution of the optimization problem (4.6) is given by*

$$Z_* = \operatorname{diag}(\omega_1 I_{r_1}, \dots, \omega_s I_{r_s}, \Omega_1, \dots, \Omega_F)$$

with

$$\omega_i = \nu_i (x_i^* z_i - \operatorname{Re}(x_i^* z_i \bar{\delta}_i) \delta_i), \quad i = 1, \dots, S' \quad (4.7)$$

$$\omega_\ell = \begin{cases} 1 & \text{if } \operatorname{Re}(z_\ell^* x_\ell) > 0 \text{ and } \delta_\ell > -1 \\ -1 & \text{if } \operatorname{Re}(z_\ell^* x_\ell) < 0 \text{ and } \delta_\ell < 1, \\ 0 & \text{otherwise} \end{cases} \quad \ell = S' + 1, \dots, S \quad (4.8)$$

$$\Omega_j = \eta_j (z_{S+j} x_{S+j}^* - \operatorname{Re} \langle \Delta_j, z_{S+j} x_{S+j}^* \rangle \Delta_j), \quad j = 1, \dots, F. \quad (4.9)$$

Here, $\nu_i > 0$ is the reciprocal of the absolute value of the right-hand side in (4.7), if this is different from zero, and $\nu_i = 1$ otherwise; $\eta_j > 0$ is the reciprocal of the Frobenius norm of the matrix on the right hand side in (4.9), if this is different from zero, and $\eta_j = 1$ otherwise.

Proof. The equality

$$\begin{aligned} z^* Z x &= \sum_{i=1}^{S'} \omega_i z_i^* x_i + \sum_{\ell=S'+1}^S \omega_\ell z_\ell^* x_\ell + \sum_{j=1}^F z_{S+j}^* \Omega_j x_{S+j} \\ &= \sum_{i=1}^{S'} \omega_i \langle z_i, x_i \rangle + \sum_{\ell=S'+1}^S \omega_\ell \langle z_\ell, x_\ell \rangle + \sum_{j=1}^F \langle z_{S+j} x_{S+j}^*, \Omega_j \rangle. \end{aligned}$$

allows us to maximize for each block of Z individually. The expressions (4.7) and (4.9) follow from the proof of Lemma 3.6. The expression (4.8) trivially maximizes the real part of $\omega_\ell z_\ell^* x_\ell$ among $\omega_\ell = \pm 1$. \square

4.3. The system of ODEs. We use the result of Lemma 4.3 to build a gradient system for Δ to find a local minimizer of $|\zeta|$. In terms of the blocks of Δ , we obtain the following system of differential equations:

$$\begin{aligned} \dot{\delta}_i &= \nu_i \left(x_i^* z_i - \operatorname{Re} (x_i^* z_i \bar{\delta}_i) \delta_i \right), & i &= 1, \dots, S' \\ \dot{\delta}_\ell &= \operatorname{sign} \left(\operatorname{Re} (z_\ell^* x_\ell) \right) \mathbb{1}_{(-1,1)}(\delta_\ell), & \ell &= S' + 1, \dots, S \\ \dot{\Delta}_j &= \eta_j \left(z_{S+j} x_{S+j}^* - \operatorname{Re} \langle \Delta_j, z_{S+j} x_{S+j}^* \rangle \Delta_j \right), & j &= 1, \dots, F, \end{aligned} \quad (4.10)$$

where $\delta_i \in \mathbb{C}$ for $i = 1, \dots, S'$, $\delta_\ell \in \mathbb{R}$ for $\ell = S' + 1, \dots, S$, and $\mathbb{1}_E(\cdot)$ is the characteristic function for a set E .

Expressing $\delta_i = e^{i\beta_i}$ the first equation in (4.10) can again be rewritten as

$$\dot{\beta}_i = -\operatorname{sign}(\sin(\gamma_i + \beta_i)),$$

which means that $\dot{\beta}_i = 0$ if and only if $\gamma_i + \beta_i = 0, \pm\pi$; extremizers correspond to $\beta_i = -\gamma_i$.

A system of ODEs that exploits the rank-1 property of the full blocks in extremizers can be derived in a fashion completely analogous to Section 3.3.

4.4. An illustrative example. Consider the matrix

$$M = \begin{pmatrix} -1.54 - 1.28i & -0.56 + 0.57i & -0.03 - 0.63i & -0.64 - 0.55i & 0.46 - 0.22i \\ -1.08 + 1.91i & 1.16 - 0.08i & -0.41 - 0.13i & 0.04 - 0.06i & -0.01 - 0.71i \\ 0.11 - 2.16i & 0.53 + 0.79i & -0.33 + 0.26i & 0.44 + 0.02i & 0.20 + 0.96i \\ 0.52 + 0.29i & 2.38 + 0.09i & -0.03 + 0.06i & 0.01 + 1.12i & 0.51 - 0.77i \\ -1.30 + 0.34i & -1.72 + 0.14i & 1.02 + 1.34i & 0.35 - 0.75i & 0.48 + 0.04i \end{pmatrix}$$

and a perturbation set given by

$$\mathbb{B} = \left\{ \operatorname{diag}(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1, \delta_4 I_2), \delta_1, \delta_2 \in \mathbb{R}, \delta_3, \delta_4 \in \mathbb{C} \right\}.$$

Applying the MATLAB's `musv`, we obtain the perturbation $\widehat{\varepsilon}\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-i0.91357833} & 0 & 0 \\ 0 & 0 & 0 & e^{-i2.076961991} & 0 \\ 0 & 0 & 0 & 0 & e^{-i2.076961991} \end{pmatrix}.$$

and $\widehat{\varepsilon} = 0.30300829$, which yields the lower bound $\mu_{\mathbb{B}}(M) \geq \mu_{\text{PD}}^\ell = 3.300239739$. Applying the algorithms presented in this article we find the same solution $\mu_{\text{New}}^\ell = \mu_{\text{PD}}^\ell$.

Intensively sampling the set of all possible perturbations indicates that the computed value μ_{PD}^ℓ yields the exact value of $\mu_{\mathbb{B}}(M)$; see also Figure 4.1.

4.5. Choice of initial value matrix and ε_0 . In our two-level algorithm for determining ε , we use the perturbation Δ obtained for the previous value ε as the initial value matrix for the system of ODEs (4.10). However, it remains to discuss a suitable choice of the initial values $\Delta(0) = \Delta_0$ and ε_0 in the very beginning of the algorithm.

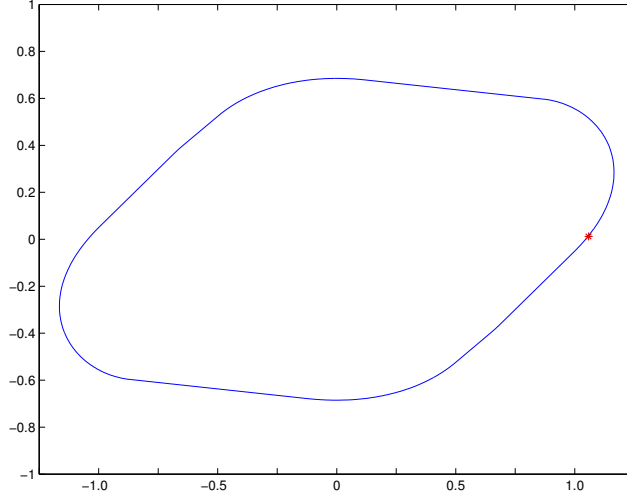


FIG. 4.1. Boundary of the set $\Lambda_\varepsilon^{\mathbb{B}}(M)$ for $\varepsilon = 0.30300829$. The point $\zeta = 1$ (the red asterisk in the picture) correctly lies on the boundary of $\Lambda_\varepsilon^{\mathbb{B}}(M)$.

For the moment, let us assume that M is invertible and write

$$I - \varepsilon_0 M \Delta_0 = M(M^{-1} - \varepsilon_0 \Delta_0),$$

which we aim to have as close as possible to singularity. To determine Δ_0 , we perform an asymptotic analysis around $\varepsilon_0 \approx 0$. For this purpose, let us consider the matrix valued function

$$G(\tau) = M^{-1} - \tau \Delta_0,$$

and let denote $\chi(\tau)$ denote an eigenvalue of $G(\tau)$ with smallest modulus. Letting x and y denote the right and left eigenvectors corresponding to $\chi(0) = \chi_0 = |\chi_0|e^{i\theta}$, scaled such that $e^{i\theta}y^*x > 0$, Lemma 3.1 implies

$$\begin{aligned} \frac{d}{d\tau} |\chi(\tau)|^2 \Big|_{\tau=0} &= 2 \operatorname{Re}(\bar{\chi} \dot{\chi}) = -2 \operatorname{Re} \left(\bar{\chi} \frac{y^* \Delta_0 x}{y^* x} \right) \\ &= -2 |\chi_0| \operatorname{Re} \left(\frac{y^* \Delta_0 x}{e^{i\theta} y^* x} \right) = -\frac{2 |\chi_0|}{|y^* x|} \operatorname{Re} \langle y x^*, \Delta_0 \rangle. \end{aligned}$$

In order to have the locally maximal decrease of $|\chi(\tau)|^2$ at $\tau = 0$ we choose

$$\Delta_0 = D P_{\mathbb{B}}(y x^*), \tag{4.11}$$

where the positive diagonal matrix D is chosen such that $\Delta_0 \in \mathbb{B}_1$. This is always possible under the genericity assumptions (4.2)–(4.4). The orthogonal projector $P_{\mathbb{B}}$ onto \mathbb{B} can be expressed in analogy to (3.9) for $P_{\mathbb{B}^*}$, with the notable difference that $\gamma_\ell = \operatorname{Re}(\operatorname{trace}(C_\ell))/r_\ell$ for $\ell = S' + 1, \dots, S$. Note that there is no need to form M^{-1} ; x and y can be obtained as the eigenvectors associated to a largest eigenvalue of M . However, attention needs to be paid to the scaling. Since the largest eigenvalue of M is $\frac{1}{|\chi_0|} e^{-i\theta}$, y and x have to be scaled accordingly.

A possible choice for ε_0 is obtained by solving the following simple linear equation, resulting from the first order expansion of the eigenvalue at $\tau = 0$:

$$|\chi(\varepsilon_0)|^2 + \frac{d}{d\tau} |\chi(\tau)|^2 \Big|_{\tau=0} \varepsilon_0 = 0.$$

This gives

$$\varepsilon_0 = \frac{|\chi_0| |y^* x|}{2\text{Re}\langle yx^*, \Delta_0 \rangle} = \frac{|\chi_0| |y^* x|}{2\|P_{\mathbb{B}}(yx^*)\|}. \quad (4.12)$$

This can be improved in a simple way by computing this expression for ε_0 for several eigenvalues of M (say, the m largest ones) and taking the smallest computed ε_0 . For a sparse matrix M , the MATLAB function `eigs` (an interface for ARPACK, which implements the implicitly restarted Arnoldi Method [12, 13]) allows to efficiently compute a predefined number m of Ritz values.

Another possible, very natural choice for ε_0 is given by

$$\varepsilon_0 = \frac{1}{\bar{\mu}_{\mathbb{B}}(M)} \quad (4.13)$$

where $\bar{\mu}_{\mathbb{B}}(M)$ is the upper bound for the SSV computed by the MATLAB function `muussv`.

4.6. The most general case. In the case where both repeated scalar blocks and full blocks can be either complex or real we have to add a dynamics for real full blocks. This can be done following an approach analogous to the one discussed in [7] and exploiting a rank-2 property of real blocks in the matrices. In order to derive a gradient system for this general case it is necessary to add systems of differential equations for $m \times m$ real blocks Δ , which are conveniently expressed in the form $\Delta = UQV^T$ with $U, V \in \mathbb{R}^{m \times 2}$ having orthonormal columns and a 2×2 orthogonal matrix Q .

A full discussion of this case is omitted for conciseness. However, our implemented algorithm includes this case.

5. Fast approximation of $\mu_{\mathbb{B}}(M)$. In this section, we discuss the outer algorithm for computing a lower bound of $\mu_{\mathbb{B}}(M)$. Since the principles are the same, we treat the case of purely complex perturbations in detail and provide a briefer discussion on the extension to the case of mixed complex/real perturbations.

5.1. Purely complex perturbations. In the following, we let $\lambda(\varepsilon)$ denote a continuous branch of (local) maximizers for

$$\max_{\lambda \in \Lambda_{\varepsilon}^{\mathbb{B}^*}(M)} |\lambda|,$$

computed by determining the stationary points $\Delta(\varepsilon)$ of the system of ODEs (3.17) (or, equivalently, (3.20)). The computation of the SSV is equivalent to the smallest solution ε of the equation $|\lambda(\varepsilon)| = 1$. In order to approximate this solution, we aim at computing ε^* such that the boundary of the ε^* -spectral value set is locally contained in the unit disk and its boundary $\partial \Lambda_{\varepsilon^*}^{\mathbb{B}^*}(M)$ is tangential to the unit circle. This provides a lower bound $1/\varepsilon^*$ for $\mu_{\mathbb{B}^*}(M)$.

In order to apply the Newton method for solving $|\lambda(\varepsilon)| = 1$ (see Figure 5.1 for an illustration of the function $|\lambda(\varepsilon)| - 1$), we need to compute the derivative of $|\lambda(\varepsilon)|$ with respect to ε . For this purpose, we make the following generic assumption.

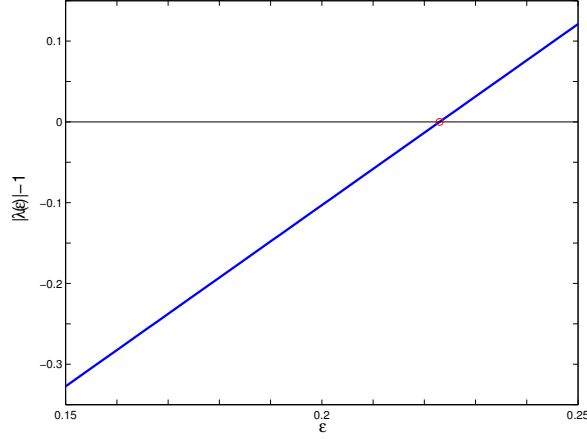


FIG. 5.1. The function $|\lambda(\varepsilon)| - 1$ for the example from Section 3.4.

ASSUMPTION 5.1. For a local extremizer $\Delta(\varepsilon)$ of $\Lambda_{\varepsilon}^{\mathbb{B}^*}(M)$, with corresponding largest eigenvalue $\lambda(\varepsilon)$, we assume that $\lambda(\varepsilon)$ is simple and that $\Delta(\cdot)$ and $\lambda(\cdot)$ are smooth in a neighborhood of ε .

The following theorem gives an explicit and easily computable expression for the derivative of $|\lambda(\varepsilon)|$.

THEOREM 5.1. Suppose that Assumption 5.1 holds for $\Delta(\varepsilon) \in \mathbb{B}_1^*$ and $\lambda(\varepsilon)$. Let $x(\varepsilon)$ and $y(\varepsilon)$ be the corresponding right and left eigenvectors of $\varepsilon M \Delta(\varepsilon)$, scaled according to (3.10). Consider the partitioning (3.1) of $x(\varepsilon)$, $z(\varepsilon) = M^* y(\varepsilon)$, and suppose that Assumptions (3.2) and (3.3) hold. Then

$$\frac{d|\lambda(\varepsilon)|}{d\varepsilon} = \frac{1}{|y(\varepsilon)^* x(\varepsilon)|} \left(\sum_{i=1}^S |z_i(\varepsilon)^* x_i(\varepsilon)| + \sum_{j=1}^F \|z_{S+j}(\varepsilon)\| \|y_{S+j}(\varepsilon)\| \right) > 0. \quad (5.1)$$

Proof. First, we observe that

$$\frac{d}{d\varepsilon} |\lambda(\varepsilon)| = \frac{1}{2|\lambda(\varepsilon)|} \frac{d}{d\varepsilon} |\lambda(\varepsilon)|^2 = \frac{1}{|\lambda(\varepsilon)|} \operatorname{Re} (\bar{\lambda}(\varepsilon) \lambda'(\varepsilon)), \quad (5.2)$$

where we let $' \equiv d/d\varepsilon$. Plugging

$$\lambda'(\varepsilon) = \frac{y(\varepsilon)^* (M \Delta(\varepsilon) + \varepsilon M \Delta'(\varepsilon)) x(\varepsilon)}{y(\varepsilon)^* x(\varepsilon)}$$

into (5.2) yields

$$\begin{aligned} \frac{d}{d\varepsilon} |\lambda(\varepsilon)| &= \frac{1}{|\lambda(\varepsilon)|} \operatorname{Re} \left(\bar{\lambda}(\varepsilon) \frac{y(\varepsilon)^* (M \Delta(\varepsilon) + \varepsilon M \Delta'(\varepsilon)) x(\varepsilon)}{y(\varepsilon)^* x(\varepsilon)} \right) \\ &= \frac{1}{|\lambda(\varepsilon)|} \operatorname{Re} \left(\frac{y(\varepsilon)^* (M \Delta(\varepsilon) + \varepsilon M \Delta'(\varepsilon)) x(\varepsilon)}{|y(\varepsilon)^* x(\varepsilon)| e^{-i\theta(\varepsilon)}} |\lambda(\varepsilon)| e^{-i\theta(\varepsilon)} \right) \\ &= \operatorname{Re} \left(\frac{\langle y(\varepsilon) x(\varepsilon)^*, M \Delta(\varepsilon) + \varepsilon M \Delta'(\varepsilon) \rangle}{|y(\varepsilon)^* x(\varepsilon)|} \right). \end{aligned} \quad (5.3)$$

We now aim to prove that the second term in the sum vanishes, that is,

$$\operatorname{Re} (y(\varepsilon)^* M \Delta'(\varepsilon) x(\varepsilon)) = 0. \quad (5.4)$$

The maximality property of the modulus of the eigenvalue $\lambda(\varepsilon)$ of $\varepsilon M \Delta(\varepsilon)$ yields $\operatorname{Re}(y(\varepsilon)^* M \Delta'(\varepsilon) x(\varepsilon)) \leq 0$. Now suppose that for some ε_0 , this inequality would actually be a strict inequality. Consider $\tilde{\Delta}(\varepsilon) \in \mathbb{B}_1^*$ such that $\tilde{\Delta}(\varepsilon_0) = \Delta(\varepsilon_0)$ and $\tilde{\Delta}'(\varepsilon_0) = -\Delta'(\varepsilon_0)$. Then, for all ε sufficiently close to ε_0 , (5.3) implies that the corresponding largest eigenvalue $\tilde{\lambda}(\varepsilon)$ of $\varepsilon M \tilde{\Delta}(\varepsilon)$ satisfies $|\tilde{\lambda}(\varepsilon)| > |\lambda(\varepsilon_0)|$. This, however, contradicts the extremality of $\Delta(\varepsilon)$ and hence (5.4) holds. In turn, (5.3) gives

$$\frac{d}{d\varepsilon} |\lambda(\varepsilon)| = \operatorname{Re} \left(\frac{\langle y(\varepsilon) x(\varepsilon)^*, M \Delta(\varepsilon) \rangle}{|y(\varepsilon)^* x(\varepsilon)|} \right) = \operatorname{Re} \left(\frac{\langle P_{\mathbb{B}^*}(z(\varepsilon) x(\varepsilon)^*), \Delta(\varepsilon) \rangle}{|y(\varepsilon)^* x(\varepsilon)|} \right).$$

The expression (5.1) now follows from the relation $\Delta(\varepsilon) = D(\varepsilon) P_{\mathbb{B}}(z(\varepsilon) x(\varepsilon)^*)$ established in Theorem 3.10, where the positive diagonal matrix $D(\varepsilon)$ is such that all blocks of $\Delta(\varepsilon)$ have unit Frobenius norm. The positivity of (5.1) is a consequence of Assumptions (3.2) and (3.3). \square

Theorem 5.1 allows us to easily realize the Newton method

$$\varepsilon^{(k+1)} = \varepsilon^{(k)} - \frac{|\lambda^{(k)}| - 1}{d|\lambda^{(k)}|}, \quad (5.5)$$

where $\lambda^{(k)} = \lambda(\varepsilon^{(k)})$ and $d|\lambda^{(k)}|$ is the derivative of $|\lambda(\varepsilon)|$ at $\varepsilon = \varepsilon^{(k)}$ given by (5.1). Note that Theorem 5.1 implies local quadratic convergence of (5.5) to ε^* , provided that the assumptions of the theorem hold for $\varepsilon = \varepsilon^*$. See Table 6.1 below for the numerical confirmation.

5.2. Mixed complex/real perturbations. Let $\zeta(\varepsilon)$ denote a continuous branch of (local) minimizers of the optimization problem

$$\min_{\zeta \in \Sigma_{\mathbb{B}}^{\mathbb{B}}(M)} |\zeta|.$$

We aim at computing the derivative of the function $\zeta(\varepsilon)$ with respect to ε ; see Figure 5.2 for an illustration. Since the function has a kink at the intersection with the

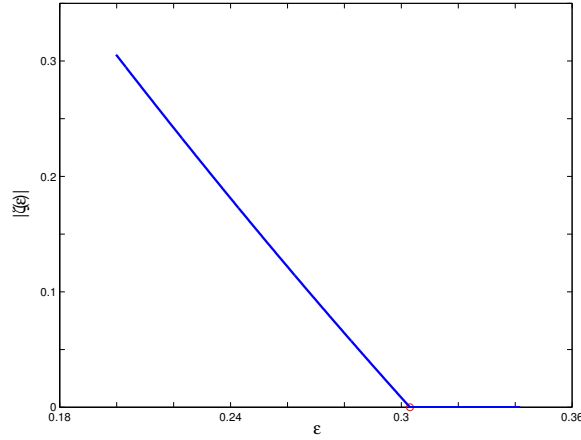


FIG. 5.2. The function $\zeta(\varepsilon)$ for the example from Section 4.4.

horizontal axis, the use of the derivative in a Newton method is meaningful only at values ε with $\zeta(\varepsilon) \neq 0$.

We make the following assumption analogous to Assumption 5.1.

ASSUMPTION 5.2. *For a local extremizer $\Delta(\varepsilon)$ of $\Sigma_{\varepsilon}^{\mathbb{B}}(M)$, with corresponding smallest eigenvalue $\zeta(\varepsilon)$ of $I - \varepsilon M \Delta(\varepsilon)$, we assume that $\zeta(\varepsilon) \neq 0$ is simple and that $\Delta(\cdot)$ and $\zeta(\cdot)$ are smooth in a neighborhood of ε .*

The following result is the analogue of Theorem 5.1; it gives an explicit and easily computable expression for the derivative of $|\zeta(\varepsilon)|$. Its proof is omitted for brevity, due to its similarity with the proof of Theorem 5.1.

THEOREM 5.2. *Suppose that Assumption 5.2 holds for $\Delta(\varepsilon)$ and $\zeta(\varepsilon)$. Let $x(\varepsilon)$ and $y(\varepsilon)$ be the corresponding right and left eigenvectors of $I - \varepsilon M \Delta(\varepsilon)$, scaled according to (3.10). Consider the partitioning (3.1) of $x(\varepsilon)$, $z(\varepsilon) = M^* y(\varepsilon)$, and suppose that Assumptions (4.2)–(4.4) hold. Then*

$$\begin{aligned} \frac{d|\zeta(\varepsilon)|}{d\varepsilon} = & - \frac{1}{|y(\varepsilon)^* x(\varepsilon)|} \left(\sum_{i=1}^{S'} |z_i(\varepsilon)^* x_i(\varepsilon)| + \sum_{i=1}^{S'} |\operatorname{Re}(z_i(\varepsilon)^* x_i(\varepsilon))| \right. \\ & \left. + \sum_{j=1}^F \|z_{S+j}(\varepsilon)\| \|y_{S+j}(\varepsilon)\| \right) < 0. \end{aligned} \quad (5.6)$$

We will make use of the following Newton method: For $\zeta^{(k)} = \zeta(\varepsilon^{(k)}) \neq 0$,

$$\varepsilon^{(k+1)} = \varepsilon^{(k)} + \frac{|\zeta^{(k)}|}{d|\zeta^{(k)}|} \quad (5.7)$$

where $d|\zeta^{(k)}|$ denotes the derivative of $|\zeta(\varepsilon)|$ at $\varepsilon = \varepsilon^{(k)}$, given by (5.6). Note that this formula cannot be used if $0 \in \Sigma_{\varepsilon^{(k)}}^{\mathbb{B}}(M)$.

Finally we remark that the discussed approach also allows to fix a threshold τ and approximate the problem

$$\min_{\varepsilon > 0} \mathcal{B}_{\tau} \cap \Sigma_{\varepsilon}^{\mathbb{B}}(M) = \emptyset,$$

where \mathcal{B}_{τ} is the sphere of radius τ in the complex plane.

5.3. Summary. Algorithm 1 describes the overall procedure for approximating the SSV of a matrix M with a prescribed block structure \mathbb{B} for admissible perturbations of complex/real form.

For the numerical integration of the ODEs we have made use of the forward Euler method with the step size controlled by the monotonicity of the extremal eigenvalue. The stopping rule is based on two criteria, the first is the condition for the stepsize to not decrease under a prescribed minimal value and the second relies on the difference of the extremal eigenvalues in two subsequent steps, which should not decrease under a given tolerance. More sophisticated numerical integrators might be the object of future research. As for the value of initially tested eigenvalues, in our implementation of Algorithm 1 we have made the choice $i_{\max} = \max(n/5, 5)$ for problems of dimension $n \geq 5$ and $i_{\max} = n$ otherwise.

6. Computational results. In this section we first provide some numerical tests for small matrices and then some statistics on the comparison between Algorithm 1 and the classical algorithm implemented in the MATLAB Control Toolbox `mussv` on a larger number of matrices having size between 5 to 100. For this purpose, we have developed a prototype MATLAB implementation of Algorithm 1. As this implementation is not particularly optimized, we do not provide timings but focus on the quality of the lower bounds. The fine tuning and efficient implementation of Algorithm 1 is beyond the scope of this paper and subject to future work.

Algorithm 1: Basic algorithm for computing ε^*

Data: $M, \mathbb{B}, \text{tol} > 0$ and $\varepsilon^{(0)}, \varepsilon_\ell$ (given lower bound), ε_u (given upper bound), i_{\max} (number of starting eigenvalues)

Result: ε_f (approximation of ε^*)

begin

```

1  for  $i \leftarrow 1$  to  $i_{\max}$  do
    Solve system of ODEs (4.10) starting from the initial matrix  $\Delta_i(0)$ 
    given by (4.11) associated to the  $i$ -th largest eigenvalue of  $M$ 
2  Let  $\Delta_i$  be the computed stationary solution and  $\zeta_i$  the smallest
    eigenvalue of  $\mathbf{I} - \varepsilon^{(0)} M \Delta_i$ 
3  Set  $i_* = \arg \min_{1 \leq i \leq i_{\max}} |\zeta_i|$ 
4  Set  $\Delta^{(0)} = \Delta_{i_*}$ ,  $\zeta^{(0)} = \zeta_{i_*}$ ,  $x^{(0)}, y^{(0)}$  the associated eigenvectors
5  Compute  $\varepsilon^{(1)}$  by one step of the Newton method (5.7)
6  Set  $k = 1$ 
7  while  $|\varepsilon^{(k)} - \varepsilon^{(k-1)}| \geq \text{tol}$  do
8  Solve ODEs (4.10) with  $\varepsilon = \varepsilon^{(k)}$ , starting from  $\Delta(0) = \Delta^{(k-1)}$ 
9  Let  $\Delta^{(k)}$  be the stationary solution of (4.10)
10 Let  $\zeta^{(k)}$  be the smallest eigenvalue of  $\mathbf{I} - \varepsilon^{(0)} M \Delta^{(k)}$ 
11 if  $|\zeta^{(k)}| > \text{tol}$  then
    Set  $\varepsilon_\ell = \varepsilon^{(k)}$ 
12 Compute  $\varepsilon^{(k+1)}$  by one step of the Newton method (5.7)
    else
        Set  $\varepsilon_u = \varepsilon^{(k)}$ 
        Set  $\varepsilon^{(k+1)} = (\varepsilon_\ell + \varepsilon_u)/2$ 
13 Set  $k = k + 1$ .
14 Set  $\varepsilon_f = \varepsilon^{(k)}$ .

```

6.1. Numerical tests. In the following examples we consider real / complex perturbations of the form (4.1) and do not impose a particular order of appearance of repeated scalar blocks and full blocks, which has been done for notational convenience only.

Example 1. Consider the following matrix from [16],

$$M = \begin{pmatrix} \mathbf{i} & \frac{1}{2} - \frac{1}{2}\mathbf{i} & 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \mathbf{i} & \mathbf{i} & \frac{1}{2} - \frac{1}{2}\mathbf{i} \\ \mathbf{i} & 1 - \frac{1}{2}\mathbf{i} & 1 & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} + \mathbf{i} & -\frac{1}{2} + \frac{1}{2}\mathbf{i} & 1 + \frac{1}{2}\mathbf{i} & \frac{1}{2} - \frac{1}{2}\mathbf{i} \\ \frac{1}{2} + \mathbf{i} & \frac{1}{2} + \frac{1}{2}\mathbf{i} & 0 & -\frac{1}{2} - \frac{1}{2}\mathbf{i} & \frac{1}{2} - \frac{1}{2}\mathbf{i} \end{pmatrix},$$

along with the perturbation set

$$\mathbb{B} = \{ \text{diag}(\delta_1 \mathbf{I}_3, \Delta_1), \delta_1 \in \mathbb{R}, \Delta_1 \in \mathbb{C}^{2,2} \}.$$

Applying MATLAB's `mu`, we obtain the perturbation $\widehat{\varepsilon}\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.661871043 - 0.048777846i & 0.114656146 - 0.401316361i & 0 \\ 0 & 0 & 0 & -0.325067916 + 0.037935543i & 0.018201081 - 0.205013390i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

and $\widehat{\varepsilon} = 0.546726635$, which gives the lower bound $\mu_{\mathbb{B}}(M) \geq \mu_{\text{PD}}^{\ell} = 1.829067647$. We note immediately that the 2×2 full complex block of $\widehat{\Delta}$ has norm $0.87\dots$, which violates the norm-1 condition for all blocks of an extremizer given by Theorem 3.3. Consequently we expect to be able to improve the lower bound.

TABLE 6.1
Values of $\varepsilon^{(k)}$ and $|\zeta^{(k)}|$ computed by Algorithm 1 applied to Example 1.

k	$\varepsilon^{(k)}$	$ \zeta^{(k)} $
0	0.321154624817	0.325206140643
1	0.475935094375	$6.509334991219 \cdot 10^{-6}$
2	0.475938192593	$2.803806976489 \cdot 10^{-12}$
3	0.475938192594	$1.037255102712 \cdot 10^{-16}$

Table 6.1 shows the result of Algorithm 1 with $\varepsilon^{(0)} = 1/\|M\|_2$. This gives

$$\varepsilon^* \approx \varepsilon_3 \implies \mu_{\text{New}}^{\ell} \approx 2.101113160408110,$$

which is very close to the upper bound $\mu_{\mathbb{B}}(M) \leq 2.110047520373674$ computed by `mu`, and hence provides a sharp estimate. The extremal perturbation $\varepsilon^*\Delta^*$ is given by

$$\Delta^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8414902738 - 0.0310321080i & -0.0774400898 + 0.4310267415i & 0 \\ 0 & 0 & 0 & 0.2196113059 + 0.1726644616i & -0.1120620799 + 0.0924665133i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The obtained result compares favorably with the approximate value 2.1007 computed in [16].

Example 2. Consider

$$M = \begin{pmatrix} -0.43 & 0.90 & -0.61 & 1.03 & 0.98 & 2.00 & 0.05 & 0.14 & 0.86 & 0.02 \\ -0.17 & -1.84 & -1.22 & -0.35 & -0.30 & 0.95 & 1.75 & -1.64 & 0.11 & -0.05 \\ -0.22 & 0.07 & 0.32 & 1.01 & 1.14 & -0.43 & 0.16 & -0.76 & 0.40 & 1.70 \\ 0.54 & 0.04 & -1.34 & 0.63 & -0.53 & 0.65 & -1.24 & -0.82 & 0.88 & -0.51 \\ 0.39 & 2.23 & -1.03 & -0.21 & 0.97 & -0.36 & -2.19 & 0.52 & 0.18 & 0.00 \\ 0.75 & -0.07 & 1.33 & -0.87 & -0.52 & 0.71 & -0.33 & -0.01 & 0.55 & 0.92 \\ 1.78 & -0.51 & -0.42 & -1.04 & 0.18 & 1.42 & 0.71 & -1.16 & 0.68 & 0.15 \\ 1.22 & 0.24 & -0.14 & -0.27 & 0.97 & -1.60 & 0.32 & -0.01 & 1.17 & 1.40 \\ -1.28 & 0.25 & 0.90 & -0.44 & -0.41 & 1.03 & 0.41 & -0.69 & 0.48 & 1.03 \\ -2.33 & 0.07 & -0.30 & -0.41 & -0.44 & 1.46 & -0.58 & -0.67 & 1.41 & 0.29 \end{pmatrix}$$

and

$$\mathbb{B} = \{ \text{diag}(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1, \delta_4 I_2, \Delta_1), \delta_1, \delta_2 \in \mathbb{R}, \delta_3, \delta_4 \in \mathbb{C}, \Delta_1 \in \mathbb{C}^{5,5} \}.$$

Applying MATLAB's `mu` gives the perturbation $\widehat{\varepsilon}\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \mathbf{0}^T \\ 0 & -1 & 0 & 0 & 0 & \mathbf{0}^T \\ 0 & 0 & -1 & 0 & 0 & \mathbf{0}^T \\ 0 & 0 & 0 & 1 & 0 & \mathbf{0}^T \\ 0 & 0 & 0 & 0 & 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{u}\widehat{v}^T \end{pmatrix}, \quad \widehat{u} = \begin{pmatrix} 0.93916167 \\ 0.06094908 \\ -0.22409849 \\ 0.25285464 \\ -0.01024501 \end{pmatrix}, \quad \widehat{v} = \begin{pmatrix} 0.21233474 \\ 0.27182946 \\ -0.57210258 \\ 0.41515717 \\ 0.61754828 \end{pmatrix}$$

and $\hat{\varepsilon} = 0.23674574$, yielding the lower bound $\mu_{\mathbb{B}}(M) \geq \mu_{\text{PD}}^{\ell} = 4.22394088$. The smallest eigenvalue of the matrix $\hat{\varepsilon}M\hat{\Delta}$ is correctly $\lambda_1 = 1$.

TABLE 6.2
Values of $\varepsilon^{(k)}$ and $|\zeta^{(k)}|$ computed by Algorithm 1 applied to Example 2.

k	$\varepsilon^{(k)}$	$ \zeta^{(k)} $
0	0.201123467713	0.117799670760
1	0.227979361395	$1.519798379258 \cdot 10^{-10}$
2	0.227979361429	$3.812788529246 \cdot 10^{-16}$

Algorithm 1 applied to this example results in Table 6.2, with $\varepsilon^{(0)} = 1/\|M\|_2$. The final (locally) extremal perturbation $\varepsilon^*\Delta^*$ is given by

$$\Delta^* = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \mathbf{0}^T \\ 0 & 1 & 0 & 0 & 0 & \mathbf{0}^T \\ 0 & 0 & -1 & 0 & 0 & \mathbf{0}^T \\ 0 & 0 & 0 & -1 & 0 & \mathbf{0}^T \\ 0 & 0 & 0 & 0 & -1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & uv^T \end{pmatrix}, \quad u = \begin{pmatrix} 0.85457765 \\ -0.04668806 \\ -0.28462457 \\ 0.41144779 \\ 0.13121292 \end{pmatrix}, \quad v = \begin{pmatrix} 0.15895464 \\ 0.22255005 \\ -0.28570067 \\ 0.49433879 \\ 0.77408603 \end{pmatrix}$$

and $\varepsilon^* \approx \varepsilon_2$. The corresponding lower bound for the μ -value is $\mu_{\mathbb{B}}(M) \geq \mu_{\text{New}}^{\ell} = 4.38636196596$, which improves the bound μ_{PD}^{ℓ} by about 3%. Note that the upper bound computed by `mussv` is $\mu_{\mathbb{B}}(M) \leq 4.45340809652$.

6.2. Numerical statistics. We now consider a test set of 100 matrices with random entries and perturbations with randomly chosen prescribed structure. Table 6.3 and Table 6.4 report the obtained results.

TABLE 6.3
Comparison between Algorithm 1 and MATLAB's `mussv`. The size of the randomly generated examples is given in the first column. The second column shows the number of cases (among a total number of 100) where the lower bound μ_{New}^{ℓ} computed with Algorithm 1 and the lower bound μ_{PD}^{ℓ} computed by `mussv` are equal, within a tolerance 10^{-3} . Third column shows the number of cases where Algorithm 1 is better than `mussv` and in the fourth column the number of cases where the opposite holds.

n	$\mu_{\text{New}}^{\ell} = \mu_{\text{PD}}^{\ell}$	$\mu_{\text{New}}^{\ell} > \mu_{\text{PD}}^{\ell}$	$\mu_{\text{New}}^{\ell} < \mu_{\text{PD}}^{\ell}$
5	63	26	11
10	66	24	10
25	39	50	11
50	37	57	6
100	34	63	3

For sizes $n = 25$, $n = 50$ and $n = 100$, our new method performs significantly better (that is, beyond the tolerance 10^{-3}) in more than the half of the cases compared to the MATLAB Control Toolbox.

Finally, let us mention the trivial consideration that one can always take the maximum of the lower bounds by Algorithm 1 and `mussv`. A little less trivial, one can take the output of `mussv` to initialize Algorithm 1, see Section 4.5. Any such hybrid algorithm will improve upon `mussv` and, as Tables 6.3 and 6.4 show, this improvement can often be quite significant. We therefore propose to complement `mussv` with such a hybrid strategy.

TABLE 6.4

Statistics on the difference $\delta = \mu_{\text{New}}^\ell - \mu_{\text{PD}}^\ell$, between the lower bound μ_{New}^ℓ computed by Algorithm 1 and the lower bound μ_{PD}^ℓ computed by **mussv**. The second column shows the maximal difference (i.e., in favor of μ_{New}^ℓ) and the third column shows the minimal difference (i.e., in favor of μ_{PD}^ℓ). The fourth and fifth column show the computed mean and variance, respectively.

n	δ_{\max}	δ_{\min}	$\langle \delta \rangle$	$\text{var}(\delta)$
5	0.8115	-1.1650	0.0277	0.0454
10	1.0082	-0.8674	0.0364	0.0382
25	1.6358	-0.5046	0.1506	0.1070
50	0.8504	-0.0016	0.1775	0.0586
100	6.1290	-0.0782	0.5793	1.2956

6.3. A possible combination with mussv. A possible combination of **mussv** with the method presented in this article could be as follows. Whenever there appear blocks of norm smaller than one in the normalized extremizer computed by **mussv**, one can reduce the value $\hat{\varepsilon}$ and apply the numerical integrator to the system of ODEs. The initial perturbation is chosen as the one computed by **mussv**. One diminishes ε until the smallest eigenvalue of the matrix $I - \varepsilon M \Delta$ is non zero. After following such a path (in ε) it would be natural to make use of a few steps of Algorithm 1.

Consider the following illustrative example:

$$M = \begin{pmatrix} -1+i & 0 & -1-2i & -1 & 1 & -2i & 1+i & 1 & 0 & 2-i \\ i & 1 & 1 & 1 & -1+i & 1+i & -1+i & 1 & -i & 2 \\ i & 0 & 0 & i & 0 & -2i & -1+i & -1+i & -1-i & -2-i \\ i & -4 & 0 & 1 & 1 & -1+i & -1+2i & -i & 2i & 3-i \\ 0 & -i & -1+i & 2i & -1+2i & -2+2i & 1+i & 2-i & 1+i & 1-i \\ -2 & -i & 1+i & -1-i & -i & -2i & -i & -1-i & -1-i & 0 \\ 1 & 1-i & 1-i & 0 & -1-i & -1 & 0 & 1 & -i & 2i \\ -1 & 2i & -2+i & 1 & 1-i & 1 & 0 & 1+i & -2i & 1-i \\ -1-2i & -i & -1+i & -1-2i & i & 0 & -1-i & 0 & 1 & i \\ -2i & 0 & 1+i & -1+i & -i & 0 & i & -2-i & 0 & i \end{pmatrix}$$

with

$$\mathbb{B} = \{ \text{diag}(\Delta_1, \delta_1 I_4, \delta_2 I_4), \Delta_1 \in \mathbb{C}^{2,2}, \delta_1, \delta_2 \in \mathbb{R} \}.$$

Applying **mussv** gives the following estimate:

$$1.87690862 \dots \leq \mu_{\mathbb{B}}(M) \leq 5.26766965 \dots$$

that is a significant gap. The perturbation associated to the lower bound is $\hat{\varepsilon} \hat{\Delta}$ with

$$\hat{\Delta} = \begin{pmatrix} 0.01622800 - 0.44875053i & 0.33074886 - 0.68259094i & \mathbf{0}^T & \mathbf{0}^T \\ -0.10388720 - 0.21700229i & -0.01277064 - 0.40618809i & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \hat{\delta}_1 I_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{\delta}_2 I_4 \end{pmatrix},$$

where $\hat{\Delta}$ has unit norm, but $\hat{\delta}_1 = 0.37144260 \dots$ and $\hat{\delta}_2 = -0.25823740 \dots$. This suggests that the necessary optimality conditions are not fulfilled. Indeed this can be checked by computing the left and right eigenvectors to the eigenvalue 1 of $\varepsilon M \Delta$.

Starting from the value $\varepsilon = 0.532790989$ which is the reciprocal of the lower bound computed by **mussv**, we proceed a few steps backward and diminish ε until reaching the value $\varepsilon_0 = 0.23$, for which we compute the point

$$z = -0.02016427 - 0.00149021i, \quad |z| = 0.0202192609,$$

which locally minimizes the modulus of $\Sigma_{\varepsilon_0}^{\mathbb{B}}(M)$.

Performing three iterations of Algorithm 1 determines the final (locally) extremal perturbation $\varepsilon^* \Delta^*$ with $\varepsilon^* = 0.23478601$ and

$$\Delta^* = \begin{pmatrix} 0.44211256 - 0.19582232i & 0.38904261 - 0.75366740i & \mathbf{0}^T & \mathbf{0}^T \\ 0.04777399 - 0.09593068i & -0.04015431 - 0.18364087i & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & -I_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -I_4 \end{pmatrix}.$$

This corresponds to the bound $4.259161456 \leq \mu_{\mathbb{B}}(M)$, which improves significantly the one computed by `mussv` with default parameters.

Conclusions. In this article we have considered the problem of approximating structured singular values, which play an important role in robust control. Our main results provide a characterization of extremizers and gradient systems, which can be integrated numerically in order to provide approximations from below to the structured singular value of a matrix subject to general complex/real block perturbations. The experimental results show the effectiveness of the proposed method when compared to some classical algorithms proposed in the literature and implemented in the MATLAB Robust Control Toolbox.

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REFERENCES

- [1] B. BERNHARDSSON, A. RANTZER, AND L. QIU, *Real perturbation values and real quadratic forms in a complex vector space*, Linear Algebra Appl., 270 (1998), pp. 131–154.
- [2] R.P. BRAATZ, P.M. YOUNG, J.C. DOYLE, AND M. MORARI, *Computational complexity of μ calculation*, IEEE Trans. Automat. Control, 39 (1994), pp. 1000–1002.
- [3] J. CHEN, M.K.H. FAN, AND C.N. NETT, *Structured singular values with nondiagonal structures. I. Characterizations*, IEEE Trans. Automat. Control, 41 (1996), pp. 1507–1511.
- [4] ———, *Structured singular values with nondiagonal structures. II. Computation*, IEEE Trans. Automat. Control, 41 (1996), pp. 1511–1516.
- [5] M.K.H. FAN, A.L. TITS, AND J.C. DOYLE, *Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics*, IEEE Trans. Automat. Control, 36 (1991), pp. 25–38.
- [6] N. GUGLIELMI AND C. LUBICH, *Differential equations for roaming pseudospectra: paths to extremal points and boundary tracking*, SIAM J. Numer. Anal., 49 (2011), pp. 1194–1209.
- [7] ———, *Low-rank dynamics for computing extremal points of real pseudospectra*, SIAM J. Matrix Anal. Appl., 34 (2013), pp. 40–66.
- [8] D. HINRICHSSEN AND A.J. PRITCHARD, *Mathematical systems theory I*, vol. 48 of Texts in Applied Mathematics, Springer, Heidelberg, 2010.
- [9] M. KAROW, *μ -values and spectral value sets for linear perturbation classes defined by a scalar product*, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 845–865.
- [10] M. KAROW, D. HINRICHSSEN, AND A.J. PRITCHARD, *Interconnected systems with uncertain couplings: explicit formulae for μ -values, spectral value sets, and stability radii*, SIAM J. Control Optim., 45 (2006), pp. 856–884.
- [11] T. KATO, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
- [12] R. B. LEHOUCQ AND D. C. SORESENSEN, *Deflation techniques for an implicitly restarted Arnoldi iteration*, SIAM J. Matrix Anal. Appl., 17 (1996), pp. 789–821.
- [13] R. B. LEHOUCQ, D. C. SORESENSEN, AND C. YANG, *ARPACK users' guide*, vol. 6 of Software, Environments, and Tools, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1998. Solution of large-scale eigenvalue problems with implicitly restarted Arnoldi methods.

- [14] A. PACKARD AND J.C. DOYLE, *The complex structured singular value*, Automatica J. IFAC, 29 (1993), pp. 71–109.
- [15] A. PACKARD, M.K.H. FAN, AND J.C. DOYLE, *A power method for the structured singular value*, in Proceedings of the 27th IEEE Conference on Decision and Control, vol. 3, 1988, pp. 2132–2137.
- [16] D. PIGA, *Computation of the structured singular value via moment lmi relaxations*, IEEE Trans. Automat. Control, in press (2016), p. 6.
- [17] L. QIU, B. BERNHARDSSON, A. RANTZER, E.J. DAVISON, P.M. YOUNG, AND J.C. DOYLE, *A formula for computation of the real stability radius*, Automatica J. IFAC, 31 (1995), pp. 879–890.
- [18] P.M. YOUNG, M.P. NEWLIN, J.C. DOYLE, AND A. PACKARD, *Theoretical and computational aspects of the structured singular value*, Systems Control Inform., 38 (1994), pp. 129–138.
- [19] P. M. YOUNG, M.P. NEWLIN, AND J.C. DOYLE, *Practical computation of the mixed problem*, in Proceedings of the American Control Conference, 1992, pp. 2190–2194.
- [20] K. ZHOU, J.C. DOYLE, AND K. GLOVER, *Robust and optimal control*, Prentice Hall, Upper Saddle River, NJ, 1996.