Periodic orbits of the ABC flow with
$A = B = C = 1$

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Abstract

In this paper, we prove that the ODE system

$$\dot{x} = \sin z + \cos y$$
$$\dot{y} = \sin x + \cos z$$
$$\dot{z} = \sin y + \cos x,$$

whose right-hand side is the Arnold-Beltrami-Childress (ABC) flow with parameters $A = B = C = 1$, has periodic orbits on $(2\pi\mathbb{T})^3$ with rotation vectors parallel to $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. An application of this result is that the well-known G-equation model for turbulent combustion with this ABC flow on $\mathbb{R}^3$ has a linear (i.e., maximal possible) flame speed enhancement rate as the amplitude of the flow grows.

1 Introduction

Arnold-Beltrami-Childress (ABC) flows are important examples of three dimensional periodic incompressible flows [1, 6, 7]. The general form is

$$u_{A,B,C}(x,y,z) = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x)$$

with $A, B, C \in \mathbb{R}$ some given parameters. These flows are steady solutions of the Euler equation, and the corresponding system of ODE

$$\dot{X}(t) = u_{A,B,C}(X(t))$$

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becomes integrable when one of the parameters is zero. We refer the reader to [6] and to Section 2.6 in [5] for more background information.

The most interesting case is when $A = B = C = 1$ (the 1-1-1 ABC flow), introduced by Childress [3, 4] in connection with dynamo studies. The 1-1-1 ABC flow system is known to be chaotic in the sense that a web of chaos occupies a (fairly complex) region of the phase space, as seen from the Poincaré section plots of Figure 10 in [6]. Though chaotic orbits for other ABC parameters have been identified analytically (see, e.g., [1, 13]), it is not known how to construct them rigorously for the 1-1-1 ABC flow.

In this paper, we prove that the 1-1-1 ABC flow system has unbounded orbits which are periodic up to shifts by $2\pi\mathbb{Z}$-multiples of one of the standard basis vectors. Our interest in this question is motivated by recent progress as well as open questions in the study of flame speed enhancement by turbulent fluid motion in combustion models (see, e.g., [12, 14]), especially in three spatial dimensions. Indeed, linear (and hence maximal possible) growth of flame front speeds as the amplitude of the flow grows is related to existence of unbounded (roughly) "periodic" orbits of the turbulent flow (see below), rather than to existence of chaotic orbits (which one might expect). The study of the former orbits thus becomes natural, and is a distinct departure from the long history of the study of chaos in ABC flows.

Note that since the 1-1-1 ABC flow is not a small perturbation of an integrable case, classical tools in dynamical systems (such as Melnikov analysis or Smale horseshoe) are difficult to apply. Nevertheless, using the analysis of the orbits of the flow inside a triangular prism region (see Figure 1) and certain symmetries of the flow, we are able to establish the following main result of this paper.

Theorem 1.1 *There exists $t_0 > 0$ and a solution $X(t) = (x(t), y(t), z(t))$ to the 1-1-1 ABC flow system such that for each $t \in \mathbb{R}$ we have*

$$X(t + t_0) = X(t) + (2\pi, 0, 0).$$

Then $Y(t) = (z(t), x(t), y(t))$ and $Z(t) = (y(t), z(t), x(t))$ are clearly also solutions and satisfy

$$Y(t + t_0) = Y(t) + (0, 2\pi, 0) \qquad \text{and} \qquad Z(t + t_0) = Z(t) + (0, 0, 2\pi).$$

Remarks. 1. Then $\tilde{X}(t) = X(-t) - (\pi, \pi, \pi)$, $\tilde{Y}(t) = Y(-t) - (\pi, \pi, \pi)$, and $\tilde{Z}(t) = Z(-t) - (\pi, \pi, \pi)$ are also solutions and they satisfy $\tilde{X}(t + t_0) = \tilde{X}(t) - (2\pi, 0, 0)$, $\tilde{Y}(t+t_0) = \tilde{Y}(t) - (0, 2\pi, 0)$, and $\tilde{Z}(t+t_0) = \tilde{Z}(t) - (0, 0, 2\pi)$.

2. Our method can be adjusted to obtain unbounded "periodic" orbits along the x, y and z directions for some other values of A, B, C. For instance, if A, B and C are all close to 1, then this follows from the proof of our result via a simple perturbation argument. Another example is the case $0 < A \ll 1$ and $B = C = 1$, which is a perturbation of the integrable case $A = 0$ and $B = C = 1$. We note that the corresponding system now possesses a large KAM region as well as a chaotic thin layer near the separatrix walls of the integrable flow (i.e., near $\{\sin y + \cos x = 0\}$), and we refer to [8] for both theoretical and numerical analysis of this case.

Applications to combustion models. Finding the turbulent flame speed (or effective burning velocity) is one of the most important unsolved problems in turbulent combustion. Roughly speaking, turbulent flame speed is the flame propagation speed in the presence of (and enhanced by) a strong flow of the ambient fluid medium. Two typical examples are the spread of wildfires fanned by winds and combustion of rotating air-gasoline mixtures inside internal combustion engines.

For simplicity, let us assume that the flow velocity profile $V : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, periodic, and incompressible (i.e., $\nabla \cdot V = 0$). A well-known approach to the study of flame propagation and turbulent flame speed is the G-equation model (see, e.g., [9, 10]), the level set Hamilton-Jacobi equation

$$G_t + KV(x) \cdot \nabla G + s_l |\nabla G| = 0,$$

with $K \geq 0$ the amplitude of the turbulent flow and $s_l > 0$ the laminar flame speed. A basic question is to understand how the turbulent flame speed depends on the flow amplitude as $K \to \infty$ (i.e., for strong flows). Let $s_T(p, K)$ be the turbulent flame speed given by the G-equation model along a fixed unit direction $p \in \mathbb{R}^n$ (see [12] for the precise definition and further references). Then it is proved in [12] that

$$\lim_{K \to \infty} \frac{s_T(p, K)}{K} = \max_{\{\xi | \ \dot{\xi} = V(\xi)\}} \limsup_{t \to \infty} \frac{p \cdot \xi(t)}{t}.$$

In particular, $s_T(p, K)$ grows linearly (which is the maximal possible growth rate) as $K \to \infty$ precisely when there exists an orbit of $\dot{\xi} = V(\xi)$ which travels roughly linearly in the direction p.

This yields the following corollary of Theorem 1.1 (and of Remark 1):

Corollary 1.1 *If V is the 1-1-1 ABC flow, then for any $p \in \mathbb{R}^3$ we have*

$$\lim_{K \to \infty} \frac{s_T(p, K)}{K} > 0.$$

3

Another well-known model used in the study of turbulent flame speeds involves traveling front solutions of the reaction-advection-diffusion equation

$$T_t + KV(x) \cdot \nabla T = d\Delta T + f(T).$$

Here T represents the temperature of the reactant, $d > 0$ is the molecular diffusivity, and f is a nonlinear reaction function (see, e.g., [2, 11]). Consider the case of a KPP reaction f (e.g., $f(T) = T(1 - T)$) and let $c^*(p, K)$ be the turbulent flame speed in the direction p given by this model (i.e., the minimal speed of a traveling front in direction p; see [2, 11] for details). It is established in [14] that

$$\lim_{K \to \infty} \frac{c^*(p, K)}{K} = \sup_{w \in \Gamma} \int_{\mathbb{T}^n} (V \cdot p) w^2 dx,$$

where

$$\Gamma = \left\{ w \in H^1(\mathbb{T}^n) \,\middle|\, V \cdot \nabla w = 0 \,\&\, \|w\|_{L^2(\mathbb{T}^n)} = 1 \,\&\, \|\nabla w\|^2_{L^2(\mathbb{T}^n)} \le f'(0) \right\}.$$

Hence, in contrast to the G-equation model, one now needs a positive measure of orbits of $\dot{\xi} = V(\xi)$ which travel roughly linearly in the direction p to obtain linear-in-K turbulent flame speed enhancement.

When $n = 2$, stability of periodic orbits was used in [12] to establish that $\lim_{K \to \infty} \frac{c^*(p,K)}{K} = 0$ if and only $\lim_{K \to \infty} \frac{s_T(p,K)}{K} = 0$. However, this is not true in general in three dimensions. An example from [12] is the so-called Robert cell flow, for which $\lim_{K \to \infty} \frac{s_T(p,K)}{K} > 0$ and $\lim_{K \to \infty} \frac{c^*(p,K)}{K} = 0$ when $p = (0, 0, 1)$.

The analysis becomes much more difficult for the much more interesting 1-1-1 ABC flow due to the presence of chaotic structures. Nevertheless, numerical simulations suggest that there are "vortex tubes" composed of orbits which travel roughly linearly in the $\pm x$, $\pm y$, and $\pm z$ directions [6], suggesting that $c^*(p, K)$ should also grow linearly in K for each p. One could expect to find such vortex tubes, if they indeed exist, in the vicinity of the "periodic" orbits constructed in this paper, but a rigorous proof of their existence is currently not known.

2 Proof of Theorem 1.1

Let R be the open triangle in the xy-plane with vertices $(0, -\frac{\pi}{2})$, $(0, \frac{3\pi}{2})$, $(-\pi, \frac{\pi}{2})$, and let $D = R \times (0, \frac{\pi}{2})$. Our proof is based on showing that there exists a solution $X_{\bar{a}}$ to the 1-1-1 ABC flow system which starts from $(-\frac{\pi}{2}, 0, \bar{a})$

for some $\bar{a} \in [0, \frac{\pi}{2})$ and passes through the segment $\{(0, y, \frac{\pi}{2}) \mid y \in [-\frac{\pi}{2}, \frac{3\pi}{2}]\}$ (see Figure 1). We do this in Step 1 below, and then use the symmetries of the flow to construct the desired solution X in Step 2.
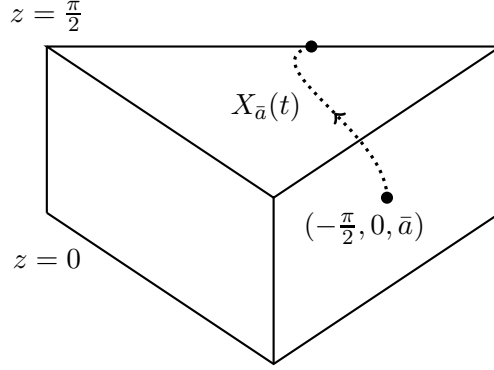

Figure 1: The region D, rotated counter-clockwise by 90 degrees, and $X_{\bar{a}}$.

Step 1: For any $a \in [0, \frac{\pi}{2})$, let $X_a(t) = (x(t), y(t), z(t))$ satisfy $X_a(0) = (-\frac{\pi}{2}, 0, a)$ and

$$\dot{x} = \sin z + \cos y$$
$$\dot{y} = \sin x + \cos z$$
$$\dot{z} = \sin y + \cos x.$$

Obviously, $(x(t), y(t)) \in R$ for all small $t > 0$ (and any $a \in [0, \frac{\pi}{2})$). Since $\cos x + \sin y > 0$ for $(x, y) \in R$, we have that $X_a(t) \in D$ for all small $t > 0$.

The question now is whether and where X_a will (first) exit D. Clearly $\dot{z}(t) > 0$ when $X_a(t) \in D$, and we also have

$$\dot{X}_a(t) \cdot (1, 1, 0) = \sin z(t) + \cos z(t) > 0$$

when $X_a(t) \in \{x + y = -\frac{\pi}{2}\} \cap \partial D$. Hence X_a cannot exit D through either the plane $\{z = 0\}$ or the plane $\{x + y = -\frac{\pi}{2}\}$.

Let us now consider any $T > 0$ such that $X_a(t) \in \bar{D} = \bar{R} \times [0, \frac{\pi}{2}]$ for all $t \in [0, T]$. The following result shows, in particular, that the first exit cannot happen through the plane $\{y - x = \frac{3\pi}{2}\}$ either.

Lemma 2.1 *We have $y(t) < \frac{\pi}{4}$ for $t \in [0, T]$.*

5

Proof. We argue by contradiction. If not, then by $y(0) = 0$, there are $0 \le t_1 < t_2 \le T$ such that $y(t) \in [0, \frac{\pi}{4}]$ for $t \in [t_1, t_2]$, $y(t_1) = 0$, and $y(t_2) = \frac{\pi}{4}$. Let us choose the smallest such t_1, t_2. Then $y(t) \in [-\frac{\pi}{2}, \frac{\pi}{4}]$ for all $t \in [0, t_2]$ and therefore $\dot{x}(t) \ge 0$ for $t \in [0, t_2]$. Therefore

$$w(t) = x(t) + \frac{\pi}{2}$$

satisfies $w(0) = 0$ and $w(t) \in [0, \frac{\pi}{2}]$ for $t \in [0, t_2]$.

In the following consider only $t \in [t_1, t_2]$. We have

$$\dot{z} - \dot{y} = \sin y + \cos x - \sin x - \cos z \ge \sin y + 1 - \cos z \ge \sin y \ge 0,$$

hence $z(t) \ge y(t)$ for $t \in [t_1, t_2]$ due to $z(t_1) \ge 0 = y(t_1)$. Then

$$\dot{w} = \sin z + \cos y \ge \sin y + \cos y \ge 1$$

$$\dot{y} = \cos z - \cos w \le 1 - \cos w \le \min\left\{\frac{w^2}{2}, 1\right\}$$

since $w \in [0, \frac{\pi}{2}]$. In particular, $\dot{y} \le 1$ shows that $t_2 - t_1 \ge \frac{\pi}{4}$. Let us now consider three cases.

Case 1: $w(t_1) \ge \frac{\pi}{3}$. Then $\dot{w} \ge 1$ shows $w(t_2) \ge \frac{7\pi}{12} > \frac{\pi}{2}$, a contradiction.

Case 2: $w(t_1) < \frac{\pi}{3}$ and $w(t_2) \le \frac{\pi}{3}$. We have $\dot{y} \le \dot{w}\frac{w^2}{2}$, which after integration over $[t_1, t_2]$ yields $y(t_2) \le \frac{1}{6}w^3(t_2) \le \frac{\pi^3}{162} < \frac{\pi}{4}$, a contradiction.

Case 3: $w(t_1) < \frac{\pi}{3}$ and $w(t_2) > \frac{\pi}{3}$. Then there exists $t^* \in [t_1, t_2]$ such that $w(t^*) = \frac{\pi}{3}$, and the computation in Case 2 shows $y(t^*) \le \frac{\pi^3}{162}$. Then

$$y(t_2) - y(t^*) \ge \frac{\pi}{4} - \frac{\pi^3}{162} > \frac{\pi}{6} = \frac{\pi}{2} - \frac{\pi}{3} \ge w(t_2) - w(t^*),$$

a contradiction with $\dot{w} \ge 1 \ge \dot{y}$ on $[t_1, t_2]$. The proof is finished. $\square$

Lemma 2.2 *If $a = 0$, then $x(t) + \frac{\pi}{2} > z(t)$ for all $t \in (0, T]$.*

Proof. Again let $w(t) = x(t) + \frac{\pi}{2}$. From $w(0) = y(0) = z(0) = 0$ and

$$\dot{w} = \sin z + \cos y$$
$$\dot{z} = \sin w + \sin y$$

we have $\dot{w}(0) > \dot{z}(0)$, hence $w(t) > z(t)$ for all small $t > 0$. Assume that there is $t^* \in (0, T]$ such that $w(t^*) = z(t^*)$ and $w > z$ on $(0, t^*)$. Lemma 2.1 shows that $\cos y > \sin y$ on $[0, T]$, so

$$0 \geq \dot{w}(t^*) - \dot{z}(t^*) > \sin z(t^*) - \sin w(t^*) = 0,$$

a contradiction. $\square$

Lemma 2.1 implies $\cos y \geq 0$ on $[0, T]$. Since $z(t) \in (0, \frac{\pi}{2}]$ and $\dot{z}(t) > 0$ for $t \in (0, T]$, it follows that $\dot{x}$ is bounded below by a positive constant on $[\delta, T]$ for each $\delta > 0$. Hence X_a will reach ∂D in finite time, and we denote by $t_a > 0$ the first such positive time. The discussion before Lemma 2.1 shows that

$$X_a(t_a) \notin \{z = 0\} \cup \{x + y = \frac{\pi}{2}\} \cup \{y - x = \frac{3\pi}{2}\},$$

and hence

$$X_a(t_a) \in \{x = 0\} \cup \{z = \frac{\pi}{2}\}.$$

Let S_0, S_1 be the sets of all $a \in [0, \frac{\pi}{2})$ such that $X_a(t_a) \in \{x = 0\} \setminus \{z = \frac{\pi}{2}\}$ and $X_a(t_a) \in \{z = \frac{\pi}{2}\} \setminus \{x = 0\}$, respectively. We have $S_0 \neq \emptyset \neq S_1$ since obviously $a \in S_1$ when a is close to $\frac{\pi}{2}$, while Lemma 2.2 implies $0 \in S_0$. Moreover, $\dot{X}_a(t_a)$ is transversal to ∂D for any $a \in S_0 \cup S_1$ (for $a \in S_0$ we have $\dot{x}(t_a) > 0$ by the argument in the last paragraph, while for $a \in S_1$ obviously $\dot{z}(t_a) > 0$). It follows that S_0, S_1 are both relatively open in $[0, \frac{\pi}{2})$ and t_a and $X_a(t_a)$ are continuous on them. Hence $[0, \frac{\pi}{2}) \setminus (S_0 \cup S_1) \neq \emptyset$, and then for any $\bar{a}$ from this set we must have

$$X_{\bar{a}}(t_{\bar{a}}) \in \partial D \cap \{x = 0\} \cap \{z = \frac{\pi}{2}\} = \left\{ (0, y, \frac{\pi}{2}) \,\middle|\, y \in [-\frac{\pi}{2}, \frac{\pi}{4}] \right\}.$$

Step 2: We now use the symmetry of the ABC flow to show that for any $\bar{a}$ as above, $X_{\bar{a}}(t) = (x(t), y(t), z(t))$ is the desired solution. For $t \in \mathbb{R}$ let

$$\tilde{X}(t) = (-\pi - x(-t), \; -y(-t), \; z(-t))$$

(this is the reflection across the line $(-\frac{\pi}{2}, 0) \times \mathbb{R}$) and

$$\hat{X}(t) = (-x(2t_{\bar{a}} - t), \; y(2t_{\bar{a}} - t), \; \pi - z(2t_{\bar{a}} - t))$$

(this is the reflection across the line $\{x = 0\} \cap \{z = \frac{\pi}{2}\}$). Clearly, both $\tilde{X}$ and $\hat{X}$ are solutions to the 1-1-1 ABC flow system. Since $\tilde{X}(0) = X_{\bar{a}}(0)$ and $\hat{X}(t_{\bar{a}}) = X_{\bar{a}}(t_{\bar{a}})$, we have $X_{\bar{a}} = \tilde{X} = \hat{X}$. Thus

$$(x(-t_{\bar{a}}), \; y(-t_{\bar{a}}), \; z(-t_{\bar{a}})) = X_{\bar{a}}(-t_{\bar{a}}) = \tilde{X}(-t_{\bar{a}}) = \left(-\pi, -y(t_{\bar{a}}), \frac{\pi}{2}\right),$$

7

and then

$$\hat{X}(3t_{\bar{a}}) = (-x(-t_{\bar{a}}), y(-t_{\bar{a}}), \pi - z(-t_{\bar{a}})) = \left(\pi, -y(t_{\bar{a}}), \frac{\pi}{2}\right).$$

So

$$X_{\bar{a}}(3t_{\bar{a}}) = \hat{X}(3t_{\bar{a}}) = X_{\bar{a}}(-t_{\bar{a}}) + (2\pi, 0, 0),$$

and it follows from 2π periodicity of the ABC flow that

$$X_{\bar{a}}(t + 4t_{\bar{a}}) = X_{\bar{a}}(t) + (2\pi, 0, 0).$$

for each $t \in \mathbb{R}$. $\square$

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