# Well-posed Bayesian Inverse Problems: Priors with Exponential Tails * 

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#### Abstract

We consider the well-posedness of Bayesian inverse problems when the prior measure has exponential tails. In particular, we consider the class of convex (log-concave) probability measures which include the Gaussian and Besov measures as well as certain classes of hierarchical priors. We identify appropriate conditions on the likelihood distribution and the prior measure which guarantee existence, uniqueness and stability of the posterior measure with respect to perturbations of the data. We also consider consistent approximations of the posterior such as discretization by projection. Finally, we present a general recipe for construction of convex priors on Banach spaces which will be of interest in practical applications where one often works with spaces such as $L^{2}$ or the continuous functions.


Key words. Inverse problems, Bayesian, non-Gaussian, Convex measure, Log-concave distribution
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1. Introduction. Readers are likely familiar with the generic inverse problem: locate a $u \in X$ from some data $y \in Y$ given the model

$$
\begin{equation*}
y=\tilde{\mathcal{G}}(u), \tag{1.1}
\end{equation*}
$$

where $\tilde{\mathcal{G}}$ is a generic stochastic mapping referred to as the parameter to observation map that models the relationship between the parameter and the observed data by taking the measurement noise into account (be it additive, multiplicative etc). Here $X$ and $Y$ are Banach spaces with norms $\|\cdot\|_{X},\|\cdot\|_{Y}$ respectively. As an example, if the measurement noise is additive then we can write

$$
\begin{equation*}
\tilde{\mathcal{G}}(u)=\mathcal{G}(u)+\eta, \tag{1.2}
\end{equation*}
$$

where $\mathcal{G}$ is referred to as the forward model which is a deterministic mapping that associates $u$ to $y$. Stated in this generality, of course, it is not at all obvious that we can find a solution to (1.1), nor how we should locate it.

The Bayesian approach to solution of inverse problems of the form (1.1) has attracted much attention in the past decade $[28,37]$. These methods are well established in the statistics literature $[11,4]$ where they are often applied to problems where $u$ belongs to a finite-dimensional space. However, the Bayesian approach in the setting of infinite-dimensional inverse problems, where the unknown $u$ belongs to an infinite dimensional function space, is less developed. The ultimate goal of the Bayesian approach is to identify a (posterior) probability measure on the unknown parameter $u$ using noisy measurements and our prior knowledge about $u$.

One of the first questions that one might ask is whether or not this posterior probability measure is well-defined. While the answer to this question is relatively straightforward in

[^0]finite dimensions, it is far from obvious in the infinite-dimensional setting and this is the main focus of this article.

In order to proceed, we introduce some terminology typically associated with (1.1). We seek the solution or parameter $u$ in (the parameter space) $X$. The Banach space $X$ may be infinite dimensional, such as the $L^{p}$ spaces for $p \geq 1$ or the space of continuous functions. We consider a Borel prior probability measure $\mu_{0}$ on $X$. This measure will reflect our prior knowledge of the parameter $u$. For example, if $u$ belongs to a function space then the prior measure can dictate whether it is smooth or merely continuous. The measurement noise $\eta$ is assumed $\eta \sim \varrho$ (distributed according to $\varrho$ ) where $\varrho$ is a Borel probability measure on $Y$.

Given $u \in X$ we let $\varrho^{u}$ denote the probability measure of $y$ conditioned on $u$. Assuming that $\varrho^{u} \ll \varrho$ (i.e. $\varrho^{u}$ is absolutely continuous with respect to $\varrho$ ) and has a density then we can define the likelihood potential $\Phi(u ; y): X \times Y \rightarrow \mathbb{R}$ so that

$$
\frac{\mathrm{d} \varrho^{u}}{\mathrm{~d} \varrho}(y)=\exp (-\Phi(u ; y)), \quad \int_{Y} \exp (-\Phi(u ; y)) \mathrm{d} \varrho(y)=1 .
$$

In finite dimensions $\Phi$ is simply the conditional distribution of $y$ given $u$ and encapsulates our assumptions regarding the distribution of the noise $\eta$ as well as those on the forward map $\mathcal{G}$. For example, let $Y=\mathbb{R}^{m}$ and consider the additive Gaussian measurement noise model

$$
y=\mathcal{G}(u)+\eta, \quad \eta \sim \mathcal{N}(0, \boldsymbol{\Gamma}) .
$$

Here, $\boldsymbol{\Gamma}$ is a $m \times m$ positive definite matrix. Then one can use the density of $\eta$ with respect to the Lebesgue measure in $\mathbb{R}^{m}$ to obtain

$$
\begin{equation*}
\Phi(u ; y)=\frac{1}{2}\left\|\Gamma^{-1 / 2}(\mathcal{G}(u)-y)\right\|_{2}^{2} . \tag{1.3}
\end{equation*}
$$

We can now define the posterior probability measure $\mu^{y}$ (on $X$ ) via Bayes' rule

$$
\begin{equation*}
\frac{\mathrm{d} \mu^{y}}{\mathrm{~d} \mu_{0}}(u)=\frac{1}{Z(y)} \exp (-\Phi(u ; y)) \quad \text { where } \quad Z(y)=\int_{X} \exp (-\Phi(u ; y)) \mathrm{d} \mu_{0}(u) . \tag{1.4}
\end{equation*}
$$

The Bayesian methodology for inverse problems consists of identifying the posterior measure $\mu^{y}$ which is interpreted as an updated version of the prior that is informed by the data. The constant $Z(y)$ is simply a normalizing constant that makes $\mu^{y}$ a probability measure on $X$. Equation (1.4) is a generalization of the well-known Bayes' rule to general state spaces (see [37, Sec. 6.6] for an in depth discussion of this generalization). Here, the relationship between the posterior and the prior is understood in the sense of the Radon-Nikodym theorem [6, Thm. 3.2.2] and so $\mu^{y} \ll \mu_{0}$.

Analogous to the situation for partial differential equations (PDEs), we may ask under what conditions (1.4) is uniquely solvable, and whether the posterior probability $\mu^{y}$ depends continuously on the data. This is loosely what we mean by well-posedness of this problem, though we shall make these definitions precise shortly. We note that our notion of wellposedness of the posterior is quite different from the consistency of the posterior introduced by Freedman and Diaconis [18, 21]. In their definition, the posterior measure $\mu^{y}$ is consistent
if it concentrates around the true value of the unknown parameter as more and more data is collected. In contrast, well-posedness is concerned with the behavior of the posterior $\mu^{y}$ when the data $y$ is perturbed and not augmented.

Our broad goal in this paper is to develop a well-posedness theory for Bayesian inverse problems in a large class of priors. Specifically, we study the well-posedness and consistent approximation of Bayesian inverse problems with priors that are convex. Convex measures are also known as "log-concave" distributions in the literature. We prefer the term "convex measure" due to the connection between convex priors and convex regularization techniques in variational inverse problems. This connection is investigated in detail in the recent articles $[10,24]$ where the authors study the maximum a posteriori points of Bayesian inverse problems with convex priors.

The central contribution of our article is the following. If $X$ is a Banach space and $\mu_{0}$ is a probability measure with certain properties (notably, convexity), then the Bayesian inverse problem of finding the measure $\mu^{y} \ll \mu_{0}$ given by (1.4) is well-posed under reasonable assumptions on the likelihood, which will be made precise shortly.

Our results will expand the class of prior measures that are available for modelling of prior information in inverse problems. As we will see later on the class of convex measures already includes the Gaussian and Besov priors and so our results will unify some of the existing results in the literature. Furthermore, the class of convex measures includes many of the priors that are commonly used in the statistics and inverse problems literature but no theory of well-posedness exists for them, such as the hierarchical priors of [34, 3].

The questions of well-posedness and consistent approximation of the Bayesian inverse problems have been studied in $[37,13]$ for Gaussian priors, in [15] for Besov priors and more recently in [27] for geometric priors and in [39] for heavy-tailed and stable priors. We now present three concrete motivating examples of inverse problems that use convex prior measures.

Example 1 ( $\ell^{1}$-regularization of inverse problems).
A popular form of regularization, particularly in the context of sparse recovery, is $\ell^{1}$ regularization. Let $X=\mathbb{R}^{n}$ and $Y=\mathbb{R}^{m}$ for fixed integers $m, n>0$. Suppose that $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a fixed matrix and that the data $y$ is obtained via

$$
y=\mathbf{A} u+\eta
$$

where $u \in \mathbb{R}^{n}$ and $\eta \sim \mathcal{N}\left(0, \sigma^{2} \mathbf{I}\right)$. Here, $\sigma>0$ is a fixed constant and $\mathbf{I} \in \mathbb{R}^{m \times m}$ is the identity matrix. Our goal in this problem is to estimate the parameter $u$ from the data $y$.

If we employ the Bayesian perspective then we need to identify the likelihood potential $\Phi$ and the prior measure $\mu_{0}$. A straightforward calculation yields

$$
\begin{equation*}
\Phi: \mathbb{R}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}, \quad \Phi(u ; y)=\frac{1}{2 \sigma^{2}}\|\mathbf{A} u-y\|_{2}^{2} \tag{1.5}
\end{equation*}
$$

As for the prior measure $\mu_{0}$, we postulate a model

$$
\begin{equation*}
\frac{d \mu_{0}}{d \Theta}(u)=\frac{1}{(2 \lambda)^{n}} \exp \left(-\frac{\|u\|_{1}}{\lambda}\right) \tag{1.6}
\end{equation*}
$$

which is a multivariate version of the Laplace distribution (see Table 1). Here, $\Theta$ denotes the Lebesgue measure on $\mathbb{R}^{n}$. We can now use Bayes' rule (1.4) to identify the posterior measure
$\mu^{y}$ as

$$
\frac{d \mu^{y}}{d \Theta}(u)=\frac{1}{Z(y)} \exp \left(-\frac{1}{2 \sigma^{2}}\|\mathbf{A} u-y\|_{2}^{2}-\frac{1}{\lambda}\|u\|_{1}\right)
$$

Finding the maximizer of this density (i.e. the maximum a posteriori (MAP) point) corresponds to solving the optimization problem

$$
u_{M A P}:=\operatorname{argmin}_{z \in \mathbb{R}^{n}} \frac{1}{2}\|\mathbf{A} z-y\|_{2}^{2}+\frac{\sigma^{2}}{2 \lambda}\|z\|_{1}
$$

This is an instance of the well-known $\ell^{1}$-regularization technique which is commonly used in recovery of sparse solutions [20].

The prior measure (1.6) is not Gaussian but, as we will see in Section 3, it is convex. This example demonstrates the potential benefits of using non-Gaussian prior measures in a finite dimensional setting. We now consider a second example that utilizes a non-Gaussian prior on a function space. This example can also be viewed as an infinite dimensional analog of Example 1.

Example 2 (Deconvolution). Let $X=L^{2}(\mathbb{T})$ where $\mathbb{T}$ is the circle of radius $(2 \pi)^{-1}$ and let $Y=\mathbb{R}^{m}$ for a fixed integer $m$. Let $S: C(\mathbb{T}) \rightarrow \mathbb{R}^{m}$ be a bounded linear operator that collects point values of a continuous function on a collection of $m$ points over $\mathbb{T}$. Finally, given a fixed kernel $g \in C^{\infty}(\mathbb{T})$, define the forward map $\mathcal{G}(\cdot): X \rightarrow Y$ as

$$
\begin{equation*}
\mathcal{G}(u)=S(g * u) \quad \text { where } \quad(g * u)(x):=\int_{\mathbb{T}} g(x-y) u(x) d y \tag{1.7}
\end{equation*}
$$

Now suppose that the data $y$ is obtained via

$$
y=\mathcal{G}(u)+\eta
$$

where $u \in L^{2}(\mathbb{T}), \eta \sim \mathcal{N}\left(0, \sigma^{2} \mathbf{I}\right), \sigma>0$ is a fixed constant and $\mathbf{I}$ is the $m \times m$ identity matrix. Our goal is to approximate $u \in L^{2}(\mathbb{T})$ from the data $y$. Our assumptions imply a likelihood potential of a similar form to (1.5),

$$
\begin{equation*}
\Phi: L^{2}(\mathbb{T}) \times \mathbb{R}^{m} \rightarrow \mathbb{R}, \quad \Phi(u ; y)=\frac{1}{2 \sigma^{2}}\|\mathcal{G}(u)-y\|_{2}^{2} \tag{1.8}
\end{equation*}
$$

We now identify the prior measure $\mu_{0}$ via spectral expansion of its samples. Let $\left\{\psi_{k}\right\}_{k \in \mathbb{Z}}$ denote the Fourier basis in $L^{2}(\mathbb{T})$ and consider a model of the form

$$
\begin{align*}
u:= & \sum_{k \in \mathbb{Z}} \gamma_{k} \xi_{k} \psi_{k}  \tag{1.9}\\
\text { where } \quad \xi_{k} & \sim \operatorname{Lap}(0,1), \quad \gamma_{k}:=\left(1+|k|^{2}\right)^{-5 / 4}, \quad \forall k \in \mathbb{Z}
\end{align*}
$$

The distribution function of the Laplace random variable $\xi$ is given in Table 1. The prior measure $\mu_{0}$ will be the probability measure induced by the random variable $u$. The posterior $\mu^{y}$ can be identified via (1.4). In order to demonstrate the potential benefits of using this prior measure we shall discretize the problem by truncating the spectral expansion in (1.9).

Consider the projection

$$
\Psi_{N}: L^{2}(\mathbb{T}) \rightarrow L^{2}(\mathbb{T}) \quad \Psi_{N}(u):=\sum_{k=-N}^{N-1}\left\langle u, \psi_{k}\right\rangle \psi_{k},
$$

where $\langle\cdot, \cdot\rangle$ denotes the usual $L^{2}$-inner product. We can approximate the likelihood potential as

$$
\Phi_{N}: L^{2}(\mathbb{T}) \times \mathbb{R}^{m} \rightarrow \mathbb{R}, \quad \Phi_{N}(u ; y)=\frac{1}{2 \sigma^{2}}\left\|\mathcal{G}\left(\Psi_{N}(u)\right)-y\right\|_{2}^{2}
$$

and apply Bayes' rule to obtain

$$
\frac{d \mu_{N}^{y}}{d \mu_{0}}(u) \propto \exp \left(-\frac{1}{2 \sigma^{2}}\left\|\mathcal{G}\left(\Psi_{N}(u)\right)-y\right\|_{2}^{2}\right) .
$$

Here we think of $\Phi_{N}$ and $\mu_{N}^{y}$ as approximations to the true likelihood $\Phi$ and posterior $\mu^{y}$ respectively. Now considering the MAP point of $\mu_{N}^{y}$ corresponds to an $\ell^{1}$-regularized optimization problem similar to Example 1 [30].

The above example is a linear inverse problem due to the fact that we assume that the convolution kernel $g$ is known. If the convolution kernel $g$ is unknown then this problem is known as the blind-deconvolution problem, which gives rise to a nonlinear inverse problem. The deconvolution problem is a classic ill-posed problem that arises widely in optics and imaging, especially in deblurring applications [41, 22].

The connection between $\ell^{1}$-regularization and the Laplace prior is well-known [28, 30, 10] and it serves as motivation for the study of non-Gaussian prior measures. One can prove the well-posedness of Example 2 using the already established theory of well-posedness for Besov priors [31, 15]. In this article, we present an alternative proof of well-posedness for this problem using a more general framework. More importantly, our well-posedness results will include more interesting choices of prior measures. We now present an example of such a prior measure in the context of deconvolution.

Example 3 (Deconvolution with a hierarchical prior). Consider the deconvolution problem of Example 2. Now assume the prior samples have the form

$$
u=\sum_{k \in \mathbb{Z}} \gamma_{k} \zeta_{k} \xi_{k} \psi_{k}
$$

where $\gamma_{k}=\left(1+|k|^{2}\right)^{-1}$ and the $\psi_{k}$ are the Fourier basis functions on $L^{2}(\mathbb{T})$. Let $\left\{\zeta_{k}\right\}$ and $\left\{\xi_{k}\right\}$ be two sequences of i.i.d. random variables so that $\zeta_{k} \sim \operatorname{Gamma}(2,1)$ and $\xi_{k} \sim \mathcal{N}(0,1)$ (refer to Table 1 for the density of the Gamma random variable). This construction of the prior can be thought of as a hierarchical prior model where the modes $\xi_{k}$ are assumed to be Gaussian random variables with unknown variances, since $\pi\left(\xi_{k} \zeta_{k} \mid \zeta_{k}\right)=\mathcal{N}\left(0, \zeta_{k}^{2}\right)$. We can think of the $\zeta_{k}$ as a model for the standard deviation of the $\xi_{k}$.

Hierarchical priors of this form are common in the literature [28, 3, 1, 34, 12] and have wide applications. The prior $\mu_{0}$ that is induced by the random variable $u$ above is not of the Besov or Gaussian form and so previous well-posedness results no longer apply.

As an immediate application of our theoretical results, we will be able to conclude that the inverse problem in Example 1 and 2 are well-posed. Furthermore, we can prove the wellposedness of Example 3 by using the fact that Gamma and Gaussian distributions are convex. We return to the proof of these results in Section 4.

The rest of this article is organized as follows. First we precisely define the notion of wellposedness and consistent approximation in the next subsection. In Section 2 we present a set of general conditions on $\Phi$ and $\mu_{0}$ which guarantee well-posedness and consistent approximation of Bayesian inverse problems. Throughout this section we will not assume that $\mu_{0}$ is convex. In Section 3 we collect some results about convex probability measures on Banach spaces and show that as priors, these measures will result in well-posed inverse problems. Section 4 is devoted to a general framework for the construction of convex priors on separable Banach spaces by means of countable products of one dimensional convex measures. In Section 4 we return to consistent approximation of the posterior measure and we present sharper results concerning the convergence of the approximate posterior. At the end of this section we present four example problems that use convex prior measures.
1.1. Key definitions. We gather here some key definitions and assumptions. In what follows, we shall consider the prior probability $\mu_{0}$ to be in the class of Radon probability measures on $X$. That is, $\mu_{0}$ will be an inner regular probability measure on the Borel sets of $X$, (meaning that the measure of every set can be approximated by the measure of compact sets from within). We also assume that $\mu_{0}$ is a complete measure i.e., subsets of sets of $\mu_{0}$-measure zero are measurable. Throughout this article the symbol $\nu$ denotes a generic probability measure that is used in the proofs and technical arguments and its definition is presented in each context. We use the shorthand notation $a \lesssim b$ for $a, b \in \mathbb{R}^{+}$when there exists a constant $C>0$ independent of $a, b$ such that $0<a \leq C b$.

To make precise the notion of distance between measures, we shall use the Hellinger metric on the space of probability measures on $X$. Assuming that $\mu_{1}$ and $\mu_{2}$ are both absolutely continuous with respect to a third measure $\Lambda$, then the Hellinger distance is defined as

$$
\begin{equation*}
d_{H}\left(\mu_{1}, \mu_{2}\right):=\left(\frac{1}{2} \int_{X}\left(\sqrt{\frac{\mathrm{~d} \mu_{1}}{\mathrm{~d} \Lambda}(u)}-\sqrt{\frac{\mathrm{d} \mu_{2}}{\mathrm{~d} \Lambda}(u)}\right)^{2} \mathrm{~d} \Lambda(u)\right)^{1 / 2} \tag{1.10}
\end{equation*}
$$

Alternatively, one can also work with the total variation metric

$$
\begin{equation*}
d_{T V}\left(\mu_{1}, \mu_{2}\right):=\frac{1}{2} \int_{X}\left|\frac{\mathrm{~d} \mu_{1}}{\mathrm{~d} \Lambda}(u)-\frac{\mathrm{d} \mu_{2}}{\mathrm{~d} \Lambda}(u)\right| \mathrm{d} \Lambda(u) . \tag{1.11}
\end{equation*}
$$

The Hellinger and total variation metrics are independent of the choice of the measure $\Lambda$ [6, Lem. 4.7.35]. Furthermore, they impose equivalent topologies due to the following set of inequalities [6, Thm. 4.7.35].

$$
2 d_{H}\left(\mu_{1}, \mu_{2}\right) \leq d_{T V}\left(\mu_{1}, \mu_{2}\right) \leq \sqrt{8} d_{H}\left(\mu_{1}, \mu_{2}\right)
$$

Thus, convergence in one metric implies convergence in the other. We note that the Hellinger metric bounds the difference in the expectations of certain functions in a particularly simple manner. Suppose that $h: X \rightarrow \mathbb{R}$ is a function so that $\int_{X} h^{2}(u) \mathrm{d} \mu_{1}(u)<\infty$ and
$\int_{X} h^{2}(u) \mathrm{d} \mu_{2}<\infty$. Then using the Radon-Nikodym theorem and Hölder's inequality we have

$$
\begin{aligned}
& \left|\int_{X} h(u) \mathrm{d} \mu_{1}(u)-\int_{X} h(u) \mathrm{d} \mu_{2}(u)\right| \\
& \leq \int_{X}\left|h(u)\left(\sqrt{\frac{\mathrm{d} \mu_{1}}{\mathrm{~d} \Lambda}(u)}+\sqrt{\frac{\mathrm{d} \mu_{2}}{\mathrm{~d} \Lambda}(u)}\right)\right|\left|\left(\sqrt{\frac{\mathrm{d} \mu_{1}}{\mathrm{~d} \Lambda}(u)}-\sqrt{\frac{\mathrm{d} \mu_{2}}{\mathrm{~d} \Lambda}(u)}\right)\right| \mathrm{d} \Lambda(u) \\
& \leq\left(\int_{X}\left|h(u)\left(\sqrt{\frac{\mathrm{d} \mu_{1}}{\mathrm{~d} \Lambda}(u)}+\sqrt{\frac{\mathrm{d} \mu_{2}}{\mathrm{~d} \Lambda}(u)}\right)\right|^{2} \mathrm{~d} \Lambda(u)\right)^{1 / 2}\left(\int_{X}\left|\left(\sqrt{\frac{\mathrm{~d} \mu_{1}}{\mathrm{~d} \Lambda}(u)}-\sqrt{\frac{\mathrm{d} \mu_{2}}{\mathrm{~d} \Lambda}(u)}\right)\right|^{2} \mathrm{~d} \Lambda(u)\right)^{1 / 2} \\
& \leq 2\left(\int_{X} h^{2}(u) \mathrm{d} \mu_{1}(u)+\int_{X} h^{2}(u) \mathrm{d} \mu_{2}(u)\right)^{1 / 2} d_{H}\left(\mu_{1}, \mu_{2}\right)
\end{aligned}
$$

Definition 1.1 (Hellinger Well-posedness). Suppose that $X$ is a Banach space and $d_{H}(\cdot, \cdot)$ is the Hellinger metric on the space of Borel probability measures on $X$. Then for a choice of the prior measure $\mu_{0}$ and the likelihood potential $\Phi$, the Bayesian inverse problem given by (1.4) is called well-posed if:

1. (Existence and uniqueness) There exists a unique posterior probability measure $\mu^{y} \ll$ $\mu_{0}$ given by (1.4).
2. (Stability) Given $\epsilon>0$, there is a constant $C>0$ such that if $\left\|y-y^{\prime}\right\|_{Y}<C, d_{H}\left(\mu^{y}, \mu^{y^{\prime}}\right)<$ $\epsilon$.
We will also define the notion of consistent approximation of a Bayesian inverse problem in the context of practical applications where one often discretizes the likelihood and approximates the posterior by sampling $[14,28]$. Let $\Phi_{N}: X \times Y \rightarrow \mathbb{R}$ denote an approximation to $\Phi$ that is parameterized by $N$, and define an approximation $\mu_{N}^{y}$ to $\mu^{y}$ via

$$
\begin{equation*}
\frac{\mathrm{d} \mu_{N}^{y}}{\mathrm{~d} \mu_{0}}=\frac{1}{Z_{N}(y)} \exp \left(-\Phi_{N}(u ; y)\right) \quad \text { where } \quad Z_{N}(y)=\int_{X} \exp \left(-\Phi_{N}(u ; y)\right) \mathrm{d} \mu_{0}(u) \tag{1.12}
\end{equation*}
$$

Definition 1.2 (Consistent approximation). The approximate Bayesian inverse problem (1.12) is a consistent approximation to (1.4) for a choice of $\mu_{0}, \Phi$ and $\Phi_{N}$ if $d_{H}\left(\mu^{y}, \mu_{N}^{y}\right) \rightarrow 0$ as $\left|\Phi(u ; y)-\Phi_{N}(u ; y)\right| \rightarrow 0$.
2. Well-posedness. In this section we collect certain conditions on the prior measure $\mu_{0}$ and the likelihood potential $\Phi$ that result in well-posed inverse problems. The results in this section are applicable to exponentially tailed prioirs. We emphasize that the assumption of convexity of the prior measure is not necessary and will only be considered in Section 3.

Following [15] we start with a collection of assumptions on the likelihood potential $\Phi$.
Assumption 1. Suppose that $X$ and $Y$ are Banach spaces. Then the function $\Phi: X \times Y \rightarrow \mathbb{R}$ has the following properties :
(i) (Lower bound in $u$ ): For an $\alpha_{1} \geq 0$ and every $r>0$, there is a constant $M\left(\alpha_{1}, r\right) \in \mathbb{R}$ such that $\forall u \in X$ and $\forall y \in Y$ with $\|y\|_{Y}<r$,

$$
\Phi(u ; y) \geq M-\alpha_{1}\|u\|_{X} .
$$

(ii) (Boundedness above) For every $r>0$ there is a constant $K(r)>0$ such that $\forall u \in X$ and $\forall y \in Y$ with $\max \left\{\|u\|_{X},\|y\|_{Y}\right\}<r$,

$$
\Phi(u ; y) \leq K .
$$

(iii) (Continuity in $u$ ) For every $r>0$ there exists a constant $L(r)>0$ such that $\forall u_{1}, u_{2} \in X$ and $y \in Y$ with $\max \left\{\left\|u_{1}\right\|_{X},\left\|u_{2}\right\|_{X},\|y\|_{Y}\right\}<r$,

$$
\left|\Phi\left(u_{1} ; y\right)-\Phi\left(u_{2}, y\right)\right| \leq L\left\|u_{1}-u_{2}\right\|_{X} .
$$

(iv) (Continuity in $y$ ) For an $\alpha_{2} \geq 0$ and for every $r>0$, there is a constant $C\left(\alpha_{2}, r\right) \in \mathbb{R}$ such that $\forall y_{1}, y_{2} \in Y$ with $\max \left\{\left\|y_{1}\right\|_{Y},\left\|y_{2}\right\|_{Y}\right\}<r$ and $\forall u \in X$,

$$
\left|\Phi\left(u ; y_{1}\right)-\Phi\left(u, y_{2}\right)\right| \leq \exp \left(\alpha_{2}\|u\|_{X}+C\right)\left\|y_{1}-y_{2}\right\|_{Y} .
$$

These four assumptions are required to ensure the well-posedness of the inverse problem according to Definition 1.1. For example, conditions (i) to (iii) are used to show the existence and uniqueness of the posterior in Theorem 2.1 while (i),(ii) and (iv) are used to show continuous dependence of the posterior on the data in Theorem 2.3. We note that these assumptions are mild and hold in many practical applications such as in the case of finite dimensional data and additive noise models (see the examples considered in Section 4 for details). However, one can still construct examples where these assumptions no longer hold. We now present a brief example where Assumption 1(i) is not satisfied.

Example 4. (Multiplicative noise model) Let $Y=\mathbb{R}$ and consider the model

$$
y=\eta \mathcal{G}(u), \quad \eta \sim \mathcal{U}(-1,1), \quad \mathcal{G}(u)=\|u\|_{X} .
$$

Thus, the measurement noise is multiplicative. As before let $\varrho^{u}$ denote the measure of $y$ conditioned on $u$ and let $\varrho^{u, \eta}$ denote the measure of $y$ conditioned on both $u$ and $\eta$. Letting $\delta$ denote the Dirac delta distribution we can write

$$
\varrho^{u, \eta}(y)=\delta(y-\eta \mathcal{G}(u)) .
$$

We obtain the distribution of $y$ conditioned on $u$ as

$$
\frac{d \varrho^{u}}{d \Theta}(y)=\frac{1}{2} \int_{\mathbb{R}} \delta(y-\eta \mathcal{G}(u)) \mathbf{1}_{(-1,1)}(\eta) d \Theta(\eta)=\frac{1}{2 \mathcal{G}(u)} \mathbf{1}_{(-1,1)}\left(\frac{y}{\mathcal{G}(u)}\right),
$$

where $\Theta$ is the Lebesgue measure as before and the above integral is understood in the sense of distributions. Therefore, the likelihood potential $\Phi$ takes the form

$$
\Phi(u ; y)= \begin{cases}\ln \left(\|u\|_{X}\right) & \|u\|_{X} \in[0, y) \\ \infty & \text { Otherwise }\end{cases}
$$

Clearly, this form of $\Phi$ does not satisfy Assumption 1(i) and (ii).
We also impose certain conditions on the prior measure $\mu_{0}$. Recall that $\mu_{0}$ by assumption is a Radon measure on $X$, and $\mu_{0}(X)=1$.

Assumption 2. The Radon probability measure $\mu_{0}$ on the Banach space $X$ has exponential tails i.e.

$$
\begin{equation*}
\exists \kappa>0 \quad \text { s.t. } \quad \int_{X} \exp \left(\kappa\|u\|_{X}\right) d \mu_{0}(u)<\infty \tag{2.1}
\end{equation*}
$$

The inner regularity assumption on the prior is not very restrictive in practice since one often works in separable Banach or Hilbert spaces where all Borel probability measures are automatically Radon [8, Thm. 1.2.5]. Assumption 2 constrains the well-posedness results of this paper to prior measures $\mu_{0}$ that have exponential tails. We borrow the term "exponential tails" from the theory of probability distributions in finite dimensions. For example if $\mu_{0}$ were a measure on $\mathbb{R}$ with a Lebesgue density $\pi_{0}(x)$ then the condition in (2.1) would reduce to the requirement that $\pi_{0}(x)$ should decay like an exponential as $|x| \rightarrow \infty$. Assumption 2 is central to many of the arguments leading to the proof of the well-posedness and consistent approximation.

We start with the question of existence and uniqueness of the posterior.
Theorem 2.1. Suppose that $X$ and $Y$ are Banach spaces and let the likelihood function $\Phi: X \times Y \rightarrow \mathbb{R}$ satisfy Assumptions 1 (i), (ii) and (iii) with some constant $\alpha_{1}$. Also let the prior Radon probability measure $\mu_{0}$ satisfy Assumption 2, with a constant $\kappa>0$. If $\kappa \geq \alpha_{1}$ then the posterior $\mu^{y}$ given by (1.4) is a well-defined Radon probability measure on $X$.

Proof. Our proof will closely follow the method of [37, Thm. 4.1]. Assumption 1(iii) ensures the continuity of $\Phi$ on $X$ and so $\Phi$ is $\mu_{0}$-measurable. Recall the normalization constant $Z(y):=\int_{X} \exp (-\Phi(u ; y)) \mathrm{d} \mu_{0}(u)$. It remains for us to show that $0<Z(y)<\infty$ in order to conclude that $\mu^{y}$ is well-defined. The fact that $\mu^{y}$ is Radon will then follow from the assumption that $\mu_{0}$ is Radon [7, Lem. 7.1.11]. To show the boundedness of the normalizing constant, we use Assumption 1(i) to get

$$
\begin{aligned}
Z(y) & =\int_{X} \exp (-\Phi(u ; y)) \mathrm{d} \mu_{0}(u) \\
& \leq \int_{X} \exp \left(\alpha_{1}\|u\|_{X}-M\right) \mathrm{d} \mu_{0}(u)=\exp (-M) \int_{X} \exp \left(\alpha_{1}\|u\|_{X}\right) \mathrm{d} \mu_{0}(u),
\end{aligned}
$$

which is bounded when $\alpha_{1} \leq \kappa$.
We now need to show that the normalizing constant $Z(y)$ does not vanish. It follows from Assumption 1(ii) that for each $R>0$

$$
\begin{aligned}
Z(y) & =\int_{X} \exp (-\Phi(u ; y)) d \mu_{0}(u) \\
& \geq \int_{\left\{\|u\|_{X}<R\right\}} \exp (-K) \mathrm{d} \mu_{0}(u)=\exp (-K) \mu_{0}\left(\left\{\|u\|_{X}<R\right\}\right) .
\end{aligned}
$$

In order to see that $\mu_{0}\left(\left\{\|u\|_{X}<R\right\}\right)>0$ for large enough $R$, consider the disjoint sets $A_{k}:=\left\{u \mid k-1 \leq\|u\|_{X}<k\right\}$ for $k \in \mathbb{N}$. The $A_{k}$ are open and hence measurable and $\sum_{k=1}^{\infty} \mu_{0}\left(A_{k}\right)=\mu_{0}\left(\bigcup_{k=1}^{\infty} A_{k}\right)=\mu(X)=1$. Then the measure of at least one of the $A_{k}$ has to be nonzero.

Observe that Assumptions 1(i) and (ii) are needed for the existence and uniqueness of the posterior measure. Therefore, in the case of Example 4 above where both of these assumptions
are violated, we are unable to prove the existence and uniqueness of the posterior measure. Furthermore, note that in the case where the likelihood potential $\Phi$ is bounded from below, i.e. Assumption 1(i) is satisfied with $\alpha_{1}=0$, we no longer require the prior $\mu_{0}$ to have exponential tails for the posterior $\mu^{y}$ to be well-defined. However, we still require the exponential tails assumption in Theorem 2.3 in order to show the stability of the posterior. Finally, We note that an alternative proof of the fact that $\mu_{0}\left(\left\{\|u\|_{X}<R\right\}\right)>0$ can be obtained by using the following theorem concerning the concentration of Radon measures on Banach spaces which is interesting in and of itself.

Theorem 2.2 ([7, Thm. 7.12.4]). Let $\mu$ be a Radon probability measure on a Banach space $X$. Then there exists a reflexive and separable Banach space $\left(E,\|\cdot\|_{E}\right)$ embedded in $X$ such that $\mu(X \backslash E)=0$ and the closed balls of $E$ are compact in $X$.

The measure $\mu_{0}$ is supported on the separable space $E$ and the closed balls of $E$ are compact in $X$. Given $R>0$ we can find $R^{\prime}>0$ so that $\left\{\|u\|_{E} \leq R^{\prime}\right\} \subseteq\left\{\|u\|_{X}<R\right\}$. If the measure of centred closed balls of $E$ are zero then the measure of all balls of $E$ would have to be zero. Since $E$ is separable, it can be covered by a countable union of balls which would imply $\mu_{0}(E)=0$. This contradicts the fact that $\mu_{0}$ is concentrated on $E$. Also, observe that since $\mu^{y} \ll \mu_{0}$ it follows from the definition of absolute continuity that $\mu^{y}(X \backslash E)=0$ as well. Then the posterior $\mu^{y}$ is also concentrated on the same separable subspace of $X$.

We now establish the stability of Bayesian inverse problems with respect to perturbations in the data.

Theorem 2.3. Suppose that $X$ is a Banach space, $\Phi$ satisfies Assumptions 1(i), (ii) and (iv) with constants $\alpha_{1}, \alpha_{2} \geq 0$ and let $\mu_{0}$ satisfy Assumption 2 with a constant $\kappa>0$. Let $\mu^{y}$ and $\mu^{y^{\prime}}$ be two measures defined via (1.4) for $y$ and $y^{\prime} \in Y$, both absolutely continuous with respect to $\mu_{0}$. If $\kappa \geq \alpha_{1}+2 \alpha_{2}$ then there exists a constant $C(r)>0$ such that whenever $\max \left\{\|y\|_{Y},\left\|y^{\prime}\right\|_{Y}\right\}<r$,

$$
d_{H}\left(\mu^{y}, \mu^{y^{\prime}}\right) \leq C\left\|y-y^{\prime}\right\|_{Y}
$$

The result of this theorem can be viewed as a guideline for choosing prior measures in practice. Given a likelihood potential $\Phi$ we need to choose a prior measure $\mu_{0}$ that has sufficiently heavy tails so that $\kappa \geq \alpha_{1}+2 \alpha_{2}$ in order to achieve stability. Furthermore, if we already have a prior measure $\mu_{0}$ that satisfies Assumption 2 but with a constant $\kappa<\alpha_{1}+2 \alpha_{2}$ then we can simply dilate this measure to construct a new measure $\overline{\mu_{0}}=\mu_{0} \circ c^{-1}$ where $c \in\left(0, \kappa /\left(\alpha_{1}+2 \alpha_{2}\right)\right)$. This new measure $\bar{\mu}_{0}$ will satisfy the conditions of Theorem 2.3 and results in a well-posed inverse problem.

A version of Theorem 2.3 along with Theorem 2.4 below is available for Gaussian priors in [37, Thm. 4.2] and for Besov priors in [15, Thm. 3.3]. Similar results on stability under perturbation of data and consistent approximation can also be found in the lecture notes [17, Sec. 4], where the results are established for separable Banach spaces and with slightly different assumptions on the prior $\mu_{0}$. Here we present an analog of those proofs under Assumption 2 (exponential integrability), and do not require X to be separable.

Proof. Consider the normalizing constants $Z(y)$ and $Z\left(y^{\prime}\right)$ associated with $y, y^{\prime} \in Y$ via (1.4). We have already established in the proof of Theorem 2.1 that neither of these constants will vanish. Now applying the mean value theorem to the exponential function and using

Assumptions 1(i), (iv) and Assumption 2 with $\kappa>\alpha_{1}+2 \alpha_{2}$ we obtain

$$
\begin{aligned}
\left|Z(y)-Z\left(y^{\prime}\right)\right| & \leq \int_{X} \exp (-\Phi(u ; y))\left|\Phi(u ; y)-\Phi\left(u ; y^{\prime}\right)\right| \mathrm{d} \mu_{0}(u) \\
& \leq\left(\int_{X} \exp \left(\alpha_{1}\|u\|_{X}-M\right) \exp \left(\alpha_{2}\|u\|_{X}+C\right) \mathrm{d} \mu_{0}(u)\right)\left\|y-y^{\prime}\right\|_{Y} \\
& \lesssim\left\|y-y^{\prime}\right\|_{Y}
\end{aligned}
$$

On the other hand, following the definition of the Hellinger metric and using the inequality $(a+b)^{2} \leq 2\left(a^{2}+b^{2}\right)$, we can write

$$
\begin{aligned}
2 d_{H}^{2}\left(\mu^{y}, \mu^{y^{\prime}}\right)= & \int_{X}\left(Z(y)^{-1 / 2} \exp \left(-\frac{1}{2} \Phi(u ; y)\right)-Z\left(y^{\prime}\right)^{-1 / 2} \exp \left(-\frac{1}{2} \Phi\left(u ; y^{\prime}\right)\right)\right)^{2} \mathrm{~d} \mu_{0}(u) \\
\leq & \frac{2}{Z(y)} \int_{X}\left(\exp \left(-\frac{1}{2} \Phi(u ; y)\right)-\exp \left(-\frac{1}{2} \Phi\left(u ; y^{\prime}\right)\right)\right)^{2} \mathrm{~d} \mu_{0}(u) \\
& +2\left|Z(y)^{-1 / 2}-Z\left(y^{\prime}\right)^{-1 / 2}\right|^{2} \int_{X} \exp \left(-\Phi\left(u ; y^{\prime}\right)\right) \mathrm{d} \mu_{0}(u) . \\
& =: I_{1}+I_{2} .
\end{aligned}
$$

Once again by the mean value theorem and Assumptions 1(iv), (i) and Assumption 2 with $\kappa>\alpha_{1}+2 \alpha_{2}$ we have

$$
\begin{aligned}
\frac{Z(y)}{2} I_{1} & \leq \int_{X} \frac{1}{4} \exp (-\Phi(u ; y))\left|\Phi\left(u ; y^{\prime}\right)-\Phi(u ; y)\right|^{2} \mathrm{~d} \mu_{0}(u) \\
& \leq \int_{X} \frac{1}{4} \exp (-\Phi(u ; y)) \exp \left(2 \alpha_{2}\|u\|_{X}+2 C\right)\left\|y-y^{\prime}\right\|_{Y}^{2} \mathrm{~d} \mu_{0}(u) \\
& \leq \int_{X} \frac{1}{4} \exp \left(\alpha_{1}\|u\|_{X}-M\right) \exp \left(2 \alpha_{2}\|u\|_{X}+2 C\right)\left\|y-y^{\prime}\right\|_{Y}^{2} \mathrm{~d} \mu_{0}(u) \\
& \lesssim\left\|y-y^{\prime}\right\|_{Y}^{2}
\end{aligned}
$$

Furthermore,

$$
I_{2}=2\left|Z(y)^{-1 / 2}-Z\left(y^{\prime}\right)^{-1 / 2}\right|^{2} Z\left(y^{\prime}\right) \lesssim\left|Z(y)-Z\left(y^{\prime}\right)\right|^{2} \lesssim\left\|y-y^{\prime}\right\|_{Y}^{2}
$$

This yields the desired result.
We now turn our attention to consistent approximation of the inverse problem as per Definition 1.2.

Theorem 2.4. Let $X$ and $Y$ be Banach spaces, and $\mu_{0}$ be a prior probability measure on $X$ satisfying Assumption 2 with a constant $\kappa>0$. Assume that the measures $\mu^{y}$ and $\mu_{N}^{y}$ are defined via (1.4) and (1.12), for a fixed $y \in Y$, and are absolutely continuous with respect to the prior $\mu_{0}$. Also assume that both likelihood potentials $\Phi$ and $\Phi^{N}$ satisfy Assumptions 1(i) and (ii) with a constant $\alpha_{1} \geq 0$, uniformly for all $N$ and that for an $\alpha_{3} \geq 0$ there exists a constant $C\left(\alpha_{3}\right) \in \mathbb{R}$ so that

$$
\begin{equation*}
\left|\Phi(u ; y)-\Phi_{N}(u ; y)\right| \leq \exp \left(\alpha_{3}\|u\|_{X}+C\right) \psi(N) \tag{2.2}
\end{equation*}
$$

where $\psi(N) \rightarrow 0$ as $N \rightarrow \infty$. If $\kappa \geq \alpha_{1}+2 \alpha_{3}$ then there exists a constant $D$ independent of $N$ so that

$$
d_{H}\left(\mu^{y}, \mu_{N}^{y}\right) \leq D \psi(N) .
$$

Proof. The proof of this theorem is very similar to that of Theorem 2.3. First note that

$$
\begin{aligned}
\left|Z(y)-Z_{N}(y)\right| & \leq \int_{X} \exp (-\Phi(u ; y))\left|\Phi(u ; y)-\Phi_{N}(u ; y)\right| \mathrm{d} \mu_{0}(u) \\
& \leq\left(\int_{X} \exp \left(\alpha_{1}\|u\|_{X}-M\right) \exp \left(\alpha_{3}\|u\|_{X}+C\right) \mathrm{d} \mu_{0}(u)\right) \psi(N) \\
& \lesssim \psi(N)
\end{aligned}
$$

which follows from applying the mean value theorem followed by Assumption 1(i) and (2.2) above as well as Assumption 2 with $\kappa \geq \alpha_{1}+2 \alpha_{3}$. Furthermore, we have

$$
\begin{aligned}
2 d_{H}^{2}\left(\mu^{y}, \mu_{N}^{y}\right)= & \int_{X}\left(Z(y)^{-1 / 2} \exp \left(-\frac{1}{2} \Phi(u ; y)\right)-Z_{N}(y)^{-1 / 2} \exp \left(-\frac{1}{2} \Phi_{N}(u ; y)\right)\right)^{2} \mathrm{~d} \mu_{0}(u) \\
\leq & \frac{2}{Z(y)} \int_{X}\left(\exp \left(-\frac{1}{2} \Phi(u ; y)\right)-\exp \left(-\frac{1}{2} \Phi_{N}(u ; y)\right)\right)^{2} \mathrm{~d} \mu_{0}(u) \\
& +2\left|Z(y)^{-1 / 2}-Z_{N}(y)^{-1 / 2}\right|^{2} \int_{X} \exp \left(-\Phi_{N}(u ; y)\right) \mathrm{d} \mu_{0}(u) . \\
& =: I_{1}+I_{2} .
\end{aligned}
$$

It then follows in a similar manner to proof of Theorem 2.3 that $I_{1} \lesssim \psi(N)$ and $I_{2} \lesssim \psi(N)$ if $\kappa>\alpha_{1}+2 \alpha_{3}$ which gives the desired result.

In order to provide more details about the rate of convergence of $\mu_{N}^{y}$ to $\mu^{y}$, we need to impose further assumptions on $\Phi$ such as stronger continuity assumptions or the behavior of the function $\psi(N)$. We will return to this question in Section 4 and provide finer results in the case where $\Phi_{N}$ is obtained from projections of $u$ onto finite dimensional subspaces of $X$. We emphasize that the result of Theorem 2.4 holds for a generic approximation $\Phi_{N}$ of the likelihood potential $\Phi$ and is not restricted to cases where $\Phi_{N}$ is obtained via discretization. For example [38] studies consistent approximations of the posterior measure by Gaussian process emulators.
3. Convex measures. Up to this point, we have shown that Assumptions 1 and 2 are sufficient for establishing the well-posedness of Bayesian inverse problems for a broad class of priors. Assumptions 1 are properties of the model for the measurements and do not depend on the prior. Since the focus of this article is on the prior measure we dedicate this section to showing that Assumption 2 holds for a large class of priors that go beyond the Gaussians [37] and the Besov priors [15]. We start by collecting some results on the class of convex probability measures on Banach spaces. Our main reference is [9] where the theory of convex probability measures on topological vector spaces is developed. In what follows $\mathcal{B}(X)$ denotes the Borel $\sigma$-algebra on a Banach space $X$.

Definition 3.1. Let $\mu$ be a Radon measure on a Banach space $X$. We say that $\mu$ belongs to the class of convex measures on $X$ if for all $0 \leq \lambda \leq 1$ and sets $A, B \in \mathcal{B}(X)$ we have

$$
\begin{equation*}
\mu(\lambda A+(1-\lambda) B) \geq \mu(A)^{\lambda} \mu(B)^{1-\lambda} \tag{3.1}
\end{equation*}
$$

Equivalently, convex measures on Banach spaces can be identified by their finite dimensional projections.

Theorem 3.2 ([9, Thm. 2.1]). A Radon probability measure $\mu$ on a Banach space $X$ is convex precisely when $\mu_{\ell_{1}, \cdots, \ell_{n}}$ is a convex measure on $\mathbb{R}^{n}$ for all integers $n$ and elements $\ell_{i} \in X^{\prime}$ the dual of $X$ where

$$
\mu_{\ell_{1}, \cdots, \ell_{n}}(A):=\mu\left(\left\{B \in \mathcal{B}(X):\left(\ell_{1}(B), \cdots, \ell_{n}(B)\right) \in A\right\}\right) \text { for all sets } A \in \mathcal{B}\left(\mathbb{R}^{n}\right)
$$

In the case of $\mathbb{R}^{n}$, convex measures are easily identified by their Lebesgue density.
Theorem 3.3.([9, Thm. 1.1]) A Radon probability measure $\mu$ on $\mathbb{R}^{n}$ is convex precisely when there exists an integer $0<k \leq n$, a Radon probability measure $\nu$ on $\mathbb{R}^{k}$ with Lebesgue density $\pi_{\nu}$ and an affine mapping $h$ such that $\mu=\nu \circ h^{-1}$ and $-\log \left(\pi_{\nu}\right)$ is convex.

The above theorem allows us to easily identify or construct convex measures in finite dimensions. We summarize a list of common convex measures on $\mathbb{R}$ in Table 1. Note that some of these measures such as the exponential, Gamma and uniform distribution are supported on subsets of $\mathbb{R}$. This fact does not affect the results of this section. The proof of convexity for these distributions follows from showing that their Lebesgue densities are log-concave (see [2] for details).

| Distribution | Symbol | Lebesgue density $\pi(x)$ | Parameter Range |
| :--- | :---: | :---: | :---: |
| Gaussian | $\mathcal{N}\left(m, \sigma^{2}\right)$ | $\frac{1}{\sqrt{2 \pi \sigma^{2}}} \exp \left(-\frac{(x-m)^{2}}{2 \sigma^{2}}\right)$ | $m \in \mathbb{R}$ and $\sigma>0$ |
| Exponential | $\operatorname{Exp}(\lambda)$ | $\mathbf{1}_{[0, \infty)}(x) \lambda \exp (-\lambda x)$ | $\lambda>0$ |
| Laplace | $\operatorname{Lap}(m, \sigma)$ | $\frac{1}{2 \sigma} \exp \left(-\frac{\|x-m\|}{\sigma}\right)$ | $m \in \mathbb{R}$ and $\sigma>0$ |
| Logistic | $\operatorname{Logistic}(m, s)$ | $\frac{\exp \left(-\frac{x-m}{s}\right)}{s\left(1+\exp \left(-\frac{x-m}{s}\right)\right)^{2}}$ | $m \in \mathbb{R}$ and $s>0$ |
| Gamma | $\operatorname{Gamma}(k, \lambda)$ | $\mathbf{1}_{[0, \infty)}(x) \frac{1}{\Gamma(k) \lambda^{k}} x^{k-1} \exp (-x / \lambda)$ | $k \geq 1$ and $\lambda>0$ |
| Uniform | $\mathcal{U}(a, b)$ | $\mathbf{1}_{[a, b]}(x)$ | $a, b \in \mathbb{R}$ and $b>a$ |

List of common distributions on $\mathbb{R}$ that are convex within the prescribed parameter range.

The following corollary to Theorem 3.2 is the cornerstone of our recipe for construction of convex prior measures in Section 3.1.

Corollary 3.4.
(i) The image of a convex measure under a continuous linear mapping is convex.
(ii) Let $X$ be a Banach space and let $\left\{\mu_{n}\right\}_{n=1}^{\infty}$ be a sequence of convex measures on $X$ converging weakly to a measure $\mu$. Then $\mu$ is also convex.
(iii) The product or convolution of two convex measures is also convex.

Proof. Parts (i) and (ii) are proven in [9, Lem. 2.1,Thm. 2.2] and so we only provide the details for (iii). Suppose that $\mu$ and $\rho$ are convex measures on Banach spaces $X$ and $Z$ and
let $\mu \otimes \rho$ denote the product measure on $X \times Z$. Also, let $\mathcal{B}(X) \otimes \mathcal{B}(Z)$ denote the product Borel $\sigma$-algebra on $X$ and $Z$. For every set $D \in \mathcal{B}(X) \otimes \mathcal{B}(Z)$ and given $z \in Z$, define the slice $D_{z}:=\{x \in X \mid(x, z) \in D\}$. For $A, B \in \mathcal{B}(X) \otimes \mathcal{B}(Z)$ and a constant $0 \leq \lambda \leq 1$ we have

$$
(\lambda A+(1-\lambda) B)_{z} \supseteq \lambda A_{z}+(1-\lambda) B_{z} .
$$

Then, by Fubini's theorem [7, Theorem 3.4.1] we have

$$
\begin{aligned}
\mu \otimes \rho(\lambda A+(1-\lambda) B) & =\int_{Z} \mu\left((\lambda A+(1-\lambda) B)_{z}\right) \rho(d z) \\
& \geq \int_{Z} \mu\left(\lambda A_{z}+(1-\lambda) B_{z}\right) \rho(d z) \\
& \geq \int_{Z} \mu\left(A_{z}\right)^{\lambda} \mu\left(B_{z}\right)^{(1-\lambda)} \rho(d z) \geq\left(\int_{Z} \mu\left(A_{z}\right) \rho(d z)\right)^{\lambda}\left(\int_{Z} \mu\left(B_{z}\right) \rho(d z)\right)^{(1-\lambda)}
\end{aligned}
$$

where the last two inequalities follow from the convexity of $\mu$ and Hölder's inequality. The assertion on the convexity of the convolution follows by considering the convolution $\mu * \rho$ as the image of $\mu \otimes \rho$ under the linear mapping $(x, z) \mapsto x+z$.

We also recall the following result which identifies the condition for convexity of measures that are absolutely continuous with respect to another convex measure.

Theorem 3.5 ([8, Prop. 4.3.8]). Let $\mu$ be a convex probability measure on a Banach space $X$ and let $V$ denote a continuous, measurable and convex function such that $\exp (-V)$ is $\mu$ measurable. If $0<Z=\int_{X} \exp (-V(u)) d \mu(u)<\infty$ then the measure $\nu$ defined via $\frac{d \nu}{d \mu}=$ $\frac{1}{Z} \exp (-V)$ is a convex probability measure.

The most attractive feature of convex measures in the context of inverse problems is that, as priors, they satisfy Assumption 2. First we need a technical result concerning convex measures on Banach spaces.

Theorem 3.6 ([8, Thm. 4.3.7]). Let $\mu$ be a convex measure on a Banach space $X$ and let $q$ be a $\mu$-measurable seminorm which is finite $\mu$-a.e. Then $\int_{X} \exp (c q(u)) d \mu(u)<\infty$ provided that $\mu\left(\left\{u \mid q(u) \leq c^{-1}\right\}\right)>\frac{e^{2}}{1+e^{2}}$.

Theorem 3.7. Let $X$ be a Banach space and let $\mu$ be a convex probability measure so that $\mu(X)=1$ and $\|u\|_{X}<\infty \mu$-a.s. Then there exists a $\epsilon_{0}>0$ such that $\mu\left(\left\{\epsilon_{0}\|u\|_{X} \leq 1\right\}\right)>\frac{e^{2}}{1+e^{2}}$. Moreover, $\int_{X} \exp \left(\epsilon\|u\|_{X}\right) d \mu(u)<\infty$ provided $0 \leq \epsilon \leq \epsilon_{0}$.

Proof. Since $\mu$ is a convex measure then it is Radon by definition and so there exists a compact set $K \subset X$ such that $\mu(K)>\frac{e^{2}}{1+e^{2}}$. Since $K$ is compact then it can be covered by a finite collection of open balls which in turn implies that it can be covered by a single open ball in $X$. This guarantees the existence of the constant $\epsilon_{0}>0$ so that $\mu\left(\left\{\epsilon_{0}\|u\|_{X} \leq 1\right\}\right)>\frac{e^{2}}{1+e^{2}}$. Then from the convexity of $\mu_{0}$ and Theorem 3.6 it follows that $\exp \left(\epsilon_{0}\|u\|_{X}\right)$ is integrable.

We emphasize that the key step in the proof of Theorem 3.6 is the assumption that the measure $\mu$ is convex. The proof of this theorem is rather technical [8, pp. 114]. To gain some intuition about this result consider a convex random variable $\xi$ in $\mathbb{R}$ with Lebesgue density $\pi_{\xi}(x)=\exp (-\mathcal{R}(x))$, where $\mathcal{R}$ is a convex function. The tail behavior of $\pi_{\xi}$ is tied to the rate of growth of $\mathcal{R}$. The slower the growth of $\mathcal{R}$ is the heavier the tails of $\pi_{\xi}$ will be. However, if $\mathcal{R}$ is convex then at the very least $\mathcal{R}(x)=\mathcal{O}(|x|)$ for large $|x|$. This means that the tails of $\pi_{\xi}$
will decay at least like that of the Laplace distribution (with Lebesgue density $\frac{1}{2 \lambda} \exp (-\lambda|x|)$ ), which also has exponential tails. Therefore, $\xi$ has exponential tails as well.

We are now in a position to connect the theory of convex measures to the question of well-posedness of inverse problems from the previous section.

Theorem 3.8. Let $X, Y$ be Banach spaces and let $\mu_{0}$ be a convex probability measure on $X$. Let $\kappa>0$ be the largest constant so that $\int_{X} \exp \left(\kappa\|u\|_{X}\right) d \mu_{0}(u)<\infty$, following Theorem 3.7. Let $\Phi: X \times Y \rightarrow \mathbb{R}$ be a likelihood potential satisfying Assumption 1 with constants $\alpha_{1}, \alpha_{2} \geq 0$. Given a fixed $y \in Y$, consider the Bayesian inverse problem of finding the measure $\mu^{y} \ll \mu_{0}$ given by (1.4). Then if $\kappa \geq \alpha_{1}+2 \alpha_{2}$
(i) The inverse problem (1.4) is well-posed.
(ii) If $\Phi_{N}$ satisfies the conditions of Theorem 2.4 with $\alpha_{3} \geq 0$ and $\kappa \geq \alpha_{1}+2 \alpha_{3}$ then the resulting approximation $\mu_{N}^{y}$ to the posterior $\mu^{y}$ is consistent.
(iii) If $\Phi(\cdot ; y): X \rightarrow \mathbb{R}$ is a convex function then $\mu^{y}$ is a convex measure.

Proof. The assertions follow from Theorems 2.1, 2.3, 2.4, 3.5.
A few remarks are in order regarding the condition on the constant $\kappa$. First, Theorem 3.7 guarantees that a constant $\kappa>0$ exists as long as $\mu_{0}$ is a convex measure. As before, given the constants $\alpha_{1}$ and $\alpha_{2}$, if we start with a convex measure $\mu_{0}$ for which $\kappa<\alpha_{1}+2 \alpha_{2}$ then we can simply dilate this measure to obtain a new measure $\bar{\mu}_{0}=\mu_{0} \circ c^{-1}$ where $c \in\left(0, \kappa /\left(\alpha_{1}+2 \alpha_{2}\right)\right)$. This new measure will satisfy the conditions of Theorem 3.8. Finally, we have the following corollary to this theorem which gives the well-posedness of the inverse problem without the need to specify the parameter $\kappa$ whenever the likelihood $\Phi$ satisfies a stronger version of Assumption 1(i) and (iv).

Corollary 3.9. Consider the setting of Theorem 3.8. If $\Phi$ satisfies Assumption 1(i) and (iv) for any constants $\alpha_{1}, \alpha_{2}>0$ then the result of Theorem 3.8 holds if $\mu_{0}$ is any convex measure such that $\mu_{0}(X)=1$.
3.1. Construction of convex priors. So far we have established that convex measures are a convenient choice as priors for inverse problems mainly due to the fact that they have exponentially decaying tails. However, we need to discussed a method for construction of convex prior measures in practical situations. It is often difficult to find a convex measure with a simple form, such as a Gaussian that can be identified by a mean function and a covariance operator. In this section we consider a general recipe for the construction of convex priors on Banach spaces that have an unconditional basis. This is relevant to practical applications since interesting Banach or Hilbert spaces are often separable and have a basis. Interesting spaces include the $L^{p}$ spaces, Sobolev and Besov spaces or the space of continuous functions on bounded intervals.

In the following we use the shorthand notation $\left\{c_{k}\right\}$ to denote a sequence of elements $\left\{c_{k}\right\}_{k=1}^{\infty}$. Also, $\mathbb{E} \xi$ and $\operatorname{Var} \xi$ denote the expectation and variance of the random variable $\xi$. We will need an assumption regarding the space $X$, which will greatly simplify the construction:

Assumption 3. $X$ is a Banach space with an unconditional and normalized Schauder basis $\left\{x_{k}\right\}$.

Our construction of the convex priors is inspired by the Karhunen-Loéve expansion of Gaussian measures [5, Thm. 3.5.1] and the construction of the Besov priors in [31, 15]. Consider a sequence of convex probability measures $\left\{\mu_{k}\right\}$ on $\mathbb{R}$ and corresponding random variables
$\xi_{k} \sim \mu_{k}$ as well as a fixed sequence $\left\{\gamma_{k}\right\}$. Next, define the random variable

$$
\begin{equation*}
u=\sum_{k=1}^{\infty} \gamma_{k} \xi_{k} x_{k} \tag{3.2}
\end{equation*}
$$

and take the prior measure $\mu_{0}$ to be the probability measure that is induced by this random variable. The prior measures in Examples 2 and 3 in Section 1 are concrete examples of this type of prior. Certain conditions on $\left\{\xi_{k}\right\}$ and $\left\{\gamma_{k}\right\}$ are needed to ensure that the induced measure $\mu_{0}$ is well-defined and is supported on $X$.

Theorem 3.10. Suppose that $X$ satisfies Assumption 3. Let u be defined as in (3.2) and let $\mu$ be the probability measure induced by $u$ on $X$. If $\left\{\gamma_{k}^{2}\right\} \in \ell^{p}(\mathbb{R})$ and $\left\{\operatorname{Var}\left|\xi_{k}\right|\right\} \in \ell^{q}(\mathbb{R})$ for $1 \leq p, q \leq \infty$ so that $1 / p+1 / q=1$ (with $p=1$ in the limiting case when $q=\infty$ ) then
(i) $\|u\|_{X}<\infty$ a.s.
(ii) $\mu$ is a convex measure on $X$.
(iii) $\int_{X} \exp \left(\epsilon\|u\|_{X}\right)<\infty$ for a small enough $\epsilon>0$.

Proof. (i) By Hölder's inequality we have

$$
\sum_{k=1}^{\infty} \operatorname{Var}\left|\gamma_{k} \xi_{k}\right|=\sum_{k=1}^{\infty}\left|\gamma_{k}\right|^{2} \operatorname{Var}\left|\xi_{k}\right| \leq\left\|\left\{\left|\gamma_{k}\right|^{2}\right\}\right\|_{\ell^{\rho}}\left\|\left\{\operatorname{Var}\left|\xi_{k}\right|\right\}\right\|_{\ell q}<\infty .
$$

Therefore, by Kolmogorov's two series theorem [29, Lemma 3.16] we have that $\sum_{k=1}^{\infty}\left|\gamma_{k} \xi_{k}\right|<$ $\infty$ a.s. Now consider positive integers $M>N>0$ and let $u_{N}=\sum_{k=1}^{N} \gamma_{k} \xi_{k} x_{k}$. Then, using the triangle inequality we have

$$
\left\|u_{M}-u_{N}\right\|_{X}=\left\|\sum_{k=N+1}^{M} \gamma_{k} \xi_{k} x_{k}\right\|_{X} \leq \sum_{k=N+1}^{\infty}\left|\gamma_{k} \xi_{k}\right| .
$$

Taking the limit as $M, N \rightarrow \infty$ the sum on the right hand side will vanish a.s. Then, by the dominated convergence theorem the sequence $\left\{u_{N}\right\}_{N=1}^{\infty}$ is Cauchy a.s. and so $\|u\|_{X}<\infty$ a.s.
(ii) Let $K_{i}$ denote the support of $\mu_{i}$ on $\mathbb{R}$ and let $\nu_{i}$ denote the sequence of probability measures that are obtained by restricting $\mu_{i}$ to $K_{i}$. Let $\tilde{\nu}:=\bigotimes_{i=1}^{\infty} \nu_{i}$ denote the quasi-measure that is generated by the countable product of the $\nu_{i}$ on $K:=\bigotimes_{i=1}^{\infty} K_{i}$. By [7, Thm. 7.6.2] $\tilde{\nu}$ has a unique extension to a Radon measure $\nu$ on $K$. Now consider the operator

$$
Q: K \rightarrow X \quad Q\left(\left\{c_{k}\right\}\right)=\sum_{k=1}^{\infty} \gamma_{k} c_{k} x_{k} .
$$

This operator is well defined following (i), and it is linear and continuous. Take $\mu$ to be the push-forward of the measure $\nu$ under the mapping $Q$. Then it follows from Corollary 3.4 that $\mu$ is a convex measure on $X$.
(iii) This result follows directly from the previous assertions and Theorem 3.7.

Corollary 3.11. Let $u$ be defined as in (3.2) and assume that the $\left\{\xi_{k}\right\}_{k=1}^{\infty}$ are i.i.d. convex random variables that are centered. If $\left\{\gamma_{k}^{2}\right\} \in \ell^{1}(\mathbb{R})$ then the result of Theorem 3.13 holds.

Proof. By Theorem 3.7 we have that $\operatorname{Var} \xi_{1}^{2}<\infty$ and so $\left\{\operatorname{Var} \xi_{k}\right\} \in \ell^{\infty}(\mathbb{R})$. The result now follows from Theorem 3.10.

It is possible to prove different versions of Theorem 3.13 with other conditions on the sequences $\left\{\gamma_{k}\right\}$ and $\left\{\xi_{k}\right\}$. We shall consider one more version of this theorem that may be of interest in practical applications. First, we recall the following classical result on the convergence of a sequence of positive random numbers.

Lemma 3.12 ([29, Proposition 3.14]). Let $\left\{\xi_{k}\right\}$ be a sequence of random variables on $\mathbb{R}^{+}$. Then $\sum_{k=1}^{\infty} \xi_{k}<\infty$ a.s. if and only if $\sum_{i=1}^{\infty} \mathbb{E} \min \left(\xi_{k}, 1\right)<\infty$.

Theorem 3.13. Let $u$ be defined as in (3.2) and let $\mu$ be the probability measure induced by $u$ on $X$. Then the result of Theorem 3.10 holds if the assumptions on $\left\{\gamma_{k}\right\}$ and $\left\{\xi_{k}\right\}$ are replaced by the assumption that $\sum_{i=1}^{\infty} \mathbb{E} \min \left(\left|\gamma_{k} \xi_{k}\right|, 1\right)<\infty$.

Proof. The proof is very similar to that of Theorem 3.10. The only difference is that the sum $\sum_{k=1}^{\infty}\left|\gamma_{k} \xi_{k}\right|<\infty$ a.s. following Lemma 3.12.

In general, the choice of the conditions on $\left\{\gamma_{k}\right\}$ and $\left\{\xi_{k}\right\}$ is an exercise in convergence of series of independent random variables. We refer the reader to the set of lecture notes $[17$, Section 2] for more examples of this approach.

We can further refine our result on the support of the measure $\mu$ that is induced by $u$. Assumption 3 implies that the space $X$ is isomorphic to the sequence space (see [23, Thm. 4.12])

$$
\begin{align*}
& W:=\left\{\left\{w_{k}\right\} \in \mathbb{R}: \sum_{k=1}^{\infty} w_{k} x_{k} \text { converges in } X\right\}, \\
& \|\{w\}\|_{W}=\sup _{n \in \mathbb{N}}\left\|\sum_{k=1}^{n} w_{k} x_{k}\right\|_{X}, \tag{3.3}
\end{align*}
$$

and the synthesis operator $T:\left\{w_{k}\right\} \mapsto \sum_{k=1}^{\infty} w_{k} x_{k}$ is an isomorphism between $X$ and $W$. To this end, we can show that the support of $\mu$ is a Hilbert space under mild conditions.

Theorem 3.14. Let $u$ be defined as in (3.2) and let $\mu$ be the probability measure that is induced by this random variable on $X$. Now suppose that the $\left\{\xi_{k}\right\}$ are i.i.d. convex random variables on $\mathbb{R}$. Also, let $\left\{\left|\gamma_{k}\right|^{2}\right\} \in \ell^{1}(\mathbb{R})$. Then $\mu$ is concentrated on a separable Hilbert space $H \subseteq X$ where

$$
H:=\left\{u \in X: \sum_{k=1}^{\infty} c_{k}^{2} \hat{u}_{k}^{2}<\infty\right\} \quad \text { with inner product }\langle u, v\rangle_{H}:=\sum_{k=1}^{\infty} c_{k}^{2} \hat{h}_{k} \hat{v}_{k} .
$$

Here, $\left\{\hat{u}_{k}\right\}:=T^{-1} u$ denotes the sequence of basis coefficients of an element $u \in X$ and $\left\{c_{k}\right\}$ is a fixed sequence that decays sufficiently fast so that $\sum_{i=1}^{\infty} c_{k}^{2} \gamma_{k}^{2} \xi_{k}^{2}<\infty$ a.s. In particular, it is sufficient if $\left\{c_{k}\right\} \in \ell^{2}(\mathbb{R})$.

Proof. We only consider the case when $\left\{c_{k}\right\} \in \ell^{2}(\mathbb{R})$. First, by Corollary 3.11 we know that $\|u\|_{X}<\infty$ a.s. and so we can work directly with the random sequence $\left\{\hat{u}_{k}\right\}$ and the measure that this sequence will induce on $W$. This measure will be equivalent to $\mu \circ T$ and its support is isomorphic to the support of $\mu$. Define the centered random variables $\zeta_{k}=\left|\xi_{k}\right|^{2}-\mathbb{E}\left|\xi_{k}\right|^{2}$ and write

$$
\sum_{k=1}^{\infty} c_{k}^{2} \gamma_{k}^{2} \xi_{k}^{2}=\sum_{k=1}^{\infty} c_{k}^{2} \gamma_{k}^{2} \zeta_{k}+\sum_{k=1}^{\infty} c_{k}^{2} \gamma_{k}^{2} \mathbb{E}\left|\xi_{k}\right|^{2}
$$

Since the $\mu_{k}$ are convex they have bounded moments of all orders and so we can define the constant $C=\operatorname{Var} \zeta_{1}$. Using Hölder's inequality and the fact that $\ell^{p}(\mathbb{R}) \subset \ell^{q}(\mathbb{R})$ for $1 \leq p<q<\infty$ we have

$$
\begin{aligned}
\sum_{k=1}^{\infty} \operatorname{Var}\left|c_{k}^{2} \gamma_{k}^{2}\right| \zeta_{k} & =C \sum_{k=1}^{\infty}\left|c_{k}\right|^{4}\left|\gamma_{k}\right|^{4} \leq C\left\|\left\{\left|c_{k}\right|^{4}\left|\gamma_{k}\right|^{4}\right\}\right\|_{\ell^{1}} \\
& \leq C\left\|\left\{\left|c_{k}\right|^{4}\right\}\right\|_{\ell^{2}}\left\|\left\{\left|\gamma_{k}\right|^{4}\right\}\right\|\left\|_{\ell^{2}}=C\right\|\left\{c_{k}\right\}\left\|_{\ell^{8}}^{4}\right\|\left\{\left|\gamma_{k}\right|^{2}\right\} \|_{\ell^{4}}^{2} \\
& \leq C\left\|\left\{c_{k}\right\}\right\|_{\ell^{8}}^{4}\left\|\left\{\left|\gamma_{k}\right|^{2}\right\}\right\|_{\ell^{1}}^{2} \\
& \leq C\left\|\left\{c_{k}\right\}\right\|_{\ell^{2}}^{4}\left\|\left\{\left|\gamma_{k}\right|^{2}\right\}\right\|_{\ell^{1}}^{2}<\infty .
\end{aligned}
$$

By Kolmogorov's two series theorem [29, Lemma 3.16] we have that $\sum_{k=1}^{\infty} c_{k}^{2} \gamma_{k}^{2} \zeta_{k}<\infty$ a.s. Using a similar argument as above we can show that $\sum_{k=1}^{\infty} c_{k}^{2} \gamma_{k}^{2} \mathbb{E}\left|\xi_{k}\right|^{2}<\infty$ and so $\sum_{k=1}^{\infty} c_{k}^{2} \gamma_{k}^{2} \xi_{k}^{2}<\infty$ a.s. Observe that our method of proof only requires $\left\{c_{k}\right\} \in \ell^{8}(\mathbb{R})$ and the assumptions on $\left\{c_{k}\right\}$ can be relaxed.

At the end of this section we note that the construction of the prior via (3.2) can be generalized to the setting where one starts from a Banach space $X$ and constructs a measure that is supported on a larger space that contains $X$. For example, we can start with $X=L^{2}(\mathbb{T})$ and construct a measure on a Sobolev space with a negative index.

To this end, consider a random variable of the form

$$
\begin{equation*}
u=\sum_{k=1}^{\infty} \xi_{k} x_{k} \tag{3.4}
\end{equation*}
$$

where $\left\{\xi_{k}\right\}$ are i.i.d. and centered convex random variables and $\left\{x_{k}\right\}$ is a normal basis in $X$. It is easy to see that the samples no longer belong to $X$ almost surely. However, they belong to a larger space $\tilde{X}$. Let $\tilde{W}$ be the space of real valued sequences $\left\{w_{k}\right\}$ so that

$$
\|\{w\}\|_{\tilde{W}}:=\left\|\sum_{k=1}^{\infty} c_{k} w_{k} x_{k}\right\|_{X}<\infty .
$$

Here we take $\left\{c_{k}\right\}$ to be a fixed sequence that decays sufficiently fast for $\sum_{k=1}^{\infty} c_{k} \xi_{k}$ to converge a.s. A convenient choice would be $\left\{c_{k}^{2}\right\} \in \ell^{1}(\mathbb{R})$. Now define $\tilde{X}$ to be the image of $\tilde{W}$ under the synthesis map

$$
T: \tilde{W} \rightarrow \tilde{X} \quad T(\{w\})=\sum_{i=1}^{\infty} w_{i} x_{i} .
$$

It is straightforward to check that the measure $\mu$ that is induced by (3.4) is a convex measure on $\tilde{X}$ using similar arguments to the proof of Theorem 3.10. It is also clear that $X \subset \tilde{X}$.
4. Practical considerations and examples. In this section we will discuss certain problems that are particularly interesting in applications. Throughout this section we consider the additive noise model of Section 1:

$$
y=\mathcal{G}(u)+\eta, \quad y \in Y, \quad u \in X, \quad \mathcal{G}: X \rightarrow Y
$$

where $Y=\mathbb{R}^{m}$ for some integer $m$ and $\eta \sim \mathcal{N}(0, \Gamma)$ where $\Gamma$ is a positive-definite matrix and $X$ is a Banach space that satisfies Assumption 3. We have already seen that under these assumptions the likelihood potential has the form

$$
\begin{equation*}
\Phi(u ; y)=\frac{1}{2}\left\|\Gamma^{-1 / 2}(\mathcal{G}(u)-y)\right\|_{2}^{2}=: \frac{1}{2}\|\mathcal{G}(u)-y\|_{\Gamma}^{2} \tag{4.1}
\end{equation*}
$$

We can now reduce Assumption 1 to a smaller set of assumptions on $\mathcal{G}$ (resp. $\Phi$ ). Note that, with minor modifications, the proof of the following theorem is applicable to other types of additve noise models such as Laplace noise.

Theorem 4.1. Suppose that $X$ is a Banach space and assume that the operator $\mathcal{G}: X \rightarrow \mathbb{R}^{m}$ satisfies the following conditions:
(i) For an $\epsilon>0$ there exists a constant $M=M(\epsilon) \in \mathbb{R}$ such that $\forall u \in X$

$$
\|\mathcal{G}(u)\|_{\Gamma} \leq \exp \left(\epsilon\|u\|_{X}+M\right)
$$

(ii) For every $r>0$ there is a constant $K=K(r)>0$ such that for all $u_{1}, u_{2} \in X$ with $\max \left\{\left\|u_{1}\right\|_{X},\left\|u_{2}\right\|_{X}\right\}<r$

$$
\left\|\mathcal{G}\left(u_{1}\right)-\mathcal{G}\left(u_{2}\right)\right\|_{\Gamma} \leq K\left\|u_{1}-u_{2}\right\|_{X}
$$

Then the likelihood potential $\Phi$ given by (4.1) satisfies the conditions of Assumption 1.
Proof. We will go through the conditions of Assumption 1 one by one. First, it is clear that by taking $\alpha_{1}=0$ and $M(r)=0$ Assumption 1 (i) is satisfied. Now fix an $r>0$ and consider $u \in X$ and $y \in Y$ so that $\max \left\{\|u\|_{X},\|y\|_{Y}\right\}<r$. It then follows from condition (i) on $\mathcal{G}$ that for $\epsilon>0$
$\Phi(u ; y) \lesssim\|y\|_{\Gamma}^{2}+\|\mathcal{G}(u)\|_{\Gamma}^{2}+\|y\|_{\Gamma}\|\mathcal{G}(u)\|_{\Gamma} \lesssim r^{2}+\exp (2 \epsilon r+M(\epsilon))+r \exp (\epsilon r+M(\epsilon))=K(r)$
which gives Assumption 1(ii). Once again fix $r>0$ and consider $u_{1}, u_{2} \in X$ and $y \in Y$ so that $\max \left\{\left\|u_{1}\right\|_{X},\left\|u_{2}\right\|_{X},\|y\|_{Y}\right\}<r$. Then, using conditions (i) and (ii) on $\mathcal{G}$, it is easy to verify that

$$
\begin{aligned}
\left|\Phi\left(u_{1}, y\right)-\Phi\left(u_{2}, y\right)\right| & \leq\left|\left\|\mathcal{G}\left(u_{1}\right)\right\|_{\Gamma}^{2}-\left\|\mathcal{G}\left(u_{2}\right)\right\|_{\Gamma}^{2}+2\|y\|_{\Gamma}^{2}\left\|\mathcal{G}\left(u_{1}\right)-\mathcal{G}\left(u_{2}\right)\right\|_{\Gamma}\right| \\
& \leq K(r)\left\|\mathcal{G}\left(u_{1}\right)-\mathcal{G}\left(u_{2}\right)\right\|_{\Gamma}
\end{aligned}
$$

Finally, Assumption 1(iv) follows from a similar argument using both conditions on $\mathcal{G}$.

Remark 1. If condition (i) of Theorem 4.1 is satisfied for any $\epsilon>0$ then $\Phi$ satisfies Assumption 1 (iv) for any constant $\alpha_{2}>0$. In particular this is true if $\|\mathcal{G}\|_{\Gamma}$ is polynomially bounded in $\|u\|_{X}$.

Remark 2. If $\mathcal{G}$ is a bounded linear operator from $X$ to $\mathbb{R}^{m}$ then it satisfies the conditions of Theorem 4.1 for any constant $\epsilon>0$.
4.1. Consistent discretization by projection. In Section 2 we mentioned that an important aspect of solving a Bayesian inverse problem is to have a consistent approximation. In practice we often use a numerical algorithm in order to extract interesting statistics from the posterior (be it sampling or optimization). Since we cannot compute the solution of the infinite dimensional problem we need to discretize it. In the case of the product priors of Section 3.1 the most convenient method of discretization is to truncate the spectral expansion of the samples.

Suppose that $X$ is a Banach space that satisfies Assumption 3, i.e. it has a unconditional normalized basis $\left\{x_{k}\right\}$. Furthermore, let $\left(X_{N},\|\cdot\|_{X}\right)$ for $N=1,2, \cdots$ be a sequence of finite dimensional linear subspaces of $X$ (not necessarily nested) where each $X_{N}$ is simply the span of a finite number of the $x_{k}$. Then for each $N$ the space can be factored as $X=X_{N} \oplus X_{N}^{\perp}$ i.e. every element $u \in X$ can be written as $u=u_{N}+u_{N}^{\perp}$ where $u_{N} \in X_{N}$ and $u_{N}^{\perp} \in X_{N}^{\perp}:=$ ( $X \backslash X_{N}$ ). Now suppose that there are projection operators $P_{N}: X \rightarrow X_{N}$ so that

$$
P_{N} u=u_{N} \quad \text { and } \quad\left\|u-P_{N} u\right\|_{X} \leq \psi(N) \rightarrow 0 \quad \text { as } \quad N \rightarrow \infty \quad \forall u \in X
$$

A simple example of this setting is to let $X_{N}=\operatorname{span}\left\{x_{k}\right\}_{k=1}^{N}$ and take $P_{N} u=\sum_{k=1}^{N} c_{k}(u) x_{k}$ where $\left\{c_{k}(u)\right\}$ are the basis coefficients of $u \in X$. If $X$ is a Hilbert space with an orthonormal basis then one can take $P_{N}$ to be the usual Galerkin projection.

Now consider the Bayesian inverse problem given by (1.4) with a likelihood potential $\Phi$ that satisfies the conditions of Assumption 1. Suppose that $\mu_{0}$ is a convex prior measure that satisfies Assumption 2 with a sufficiently large constant $\kappa$. In addition, suppose that for each $X_{N}$ the prior can be factored as

$$
\mu_{0}=\mu_{N} \otimes \mu_{N}^{\perp}
$$

where $\mu_{N}$ is a measure on $X_{N}$ and $\mu_{N}^{\perp}$ is a measure on $X_{N}^{\perp}$. An example of such a prior $\mu_{0}$ is our construction of the product convex measures in Section 3.1. Now consider the collection of approximate posterior measures $\mu_{N}^{y}$ given by

$$
\frac{\mathrm{d} \mu_{N}^{y}(u)}{\mathrm{d} \mu_{0}(u)}=\frac{1}{Z_{N}(y)} \exp \left(-\Phi\left(P_{N} u\right)\right) \quad \text { where } \quad Z_{N}(y)=\int_{X} \exp \left(-\Phi\left(P_{N} u\right)\right) \mathrm{d} \mu_{0}(u) .
$$

Observe that for every set $A \in \mathcal{B}(X)$, using Fubini's theorem [7, Theorem 3.4.1] we have

$$
\begin{aligned}
\mu_{N}^{y}(A) & =\int_{A} \frac{1}{Z_{N}(y)} \exp \left(-\Phi\left(P_{N} u ; y\right) \mathrm{d} \mu_{N}^{\perp} \otimes \mathrm{d} \mu_{N}(u)\right. \\
& =\int_{X_{N}} \int_{A_{v}} \frac{1}{Z_{N}(y)} \exp (-\Phi(v ; y)) \mathrm{d} \mu_{N}^{\perp}(w) \mathrm{d} \mu_{N}(v) \\
& =\int_{X_{N}} \mu_{N}^{\perp}\left(A_{v}\right)\left(\frac{1}{Z_{N}} \exp (-\Phi(v ; y))\right) \mathrm{d} \mu_{N}(v)
\end{aligned}
$$

Where $A_{v}=\left\{w \in X_{N}^{\perp}: v+w \in A\right\}$. This implies that the posterior measure $\mu_{N}^{y}$ can be factored as $\mu_{N}^{y}=\tilde{\mu}_{N}^{y} \otimes \mu_{N}^{\perp}$ where

$$
\frac{\mathrm{d} \tilde{\mu}_{N}^{y}}{\mathrm{~d} \mu_{N}}=\frac{1}{Z_{N}(y)} \exp (-\Phi(v ; y))
$$

Thus, the posterior measure inherits the product structure of the prior and it only differs from the prior on the subspace $X_{N}$.

It is straightforward to check that the measures $\mu_{N}^{y}$ are well-defined and absolutely continuous with respect to the prior following our assumptions on $\Phi$ and $\mu_{0}$. However, it remains for us to show that the measures will converge to the posterior $\mu^{y}$ in an appropriate sense.

Lemma 4.2. Consider the above setting where the posterior and the prior have the prescribed product structures and the $X_{N}$ are linear subpaces of $X$. Suppose that $\Phi$ is given by (1.3) and $\Phi_{N}(u ; y):=\Phi\left(P_{N} u ; y\right)$. Furthermore, suppose that for any $\epsilon>0$ there exist constants $K(\epsilon)>0$ and $M(\epsilon) \in \mathbb{R}$ so that
(i) $\forall u \in X,\|\mathcal{G}(u)\|_{2} \leq \exp \left(\epsilon\|u\|_{X}+M\right)$.
(ii) $\left\|\mathcal{G}\left(u_{1}\right)-\mathcal{G}\left(u_{2}\right)\right\|_{2} \leq K \exp \left(\epsilon \max \left\{\left\|u_{1}\right\|_{X},\left\|u_{2}\right\|_{X}\right\}\right)\left\|u_{1}-u_{2}\right\|_{X}$.

Then there exists a constant $C(\epsilon)>0$ such that

$$
\left|\Phi(u ; y)-\Phi_{N}(u ; y)\right| \leq C \exp \left(\epsilon\|u\|_{X}\right)\left\|u-P_{N} u\right\|_{X} .
$$

Proof. Let $\mathcal{G}_{N}(u):=\mathcal{G}\left(P_{N} u\right)$ and fix an $\epsilon>0$. Then using the elementary identity $\left(a^{2}-b^{2}\right)=(a+b)(a-b)$, the triangle inequality and the fact that the data $y$ is a fixed vector, we have:

$$
\begin{aligned}
\left|\Phi(u ; y)-\Phi_{N}(u ; y)\right| & \leq \frac{1}{2}\left\|2 y-\mathcal{G}(u)-\mathcal{G}_{N}(u)\right\|_{\Gamma}\left\|\mathcal{G}(u)-\mathcal{G}_{N}(u)\right\|_{\Gamma} \\
& \leq \frac{1}{2}\left(\|2 y\|_{\Gamma}+\|\mathcal{G}(u)\|_{\Gamma}+\left\|\mathcal{G}\left(P_{N} u\right)\right\|_{\Gamma}\right)\left\|\mathcal{G}(u)-\mathcal{G}_{N}(u)\right\|_{\Gamma} \\
& \leq \frac{1}{2}\left(\|2 y\|_{\Gamma}+C \exp \left(\epsilon\|u\|_{X}+M(\epsilon)\right)\left\|\mathcal{G}(u)-\mathcal{G}_{N}(u)\right\|_{\Gamma}\right. \\
& \leq C(\epsilon) \exp \left(\epsilon\|u\|_{X}\right)\left\|u-P_{N} u\right\|_{X}
\end{aligned}
$$

Corollary 4.3. Consider the setting of Lemma 4.2 above and let $\psi(N)$ be a function such that $\psi(N) \rightarrow 0$ as $N \rightarrow \infty$. If for every $\epsilon \geq 0$ there exists a constant $M(\epsilon) \in \mathbb{R}$ so that

$$
\left\|u-P_{N} u\right\|_{X} \leq \exp \left(\epsilon\|u\|_{X}+M\right) \psi(N)
$$

then there exists a constant $D$ independent of $N$ so that

$$
d_{H}\left(\mu^{y}, \mu_{N}^{y}\right) \leq D \psi(N)
$$

In other words, the error in the Hellinger metric decays at the same rate as the approximation error $\left\|u-P_{N} u\right\|_{X}$ as $N \rightarrow \infty$.

Proof. Note that our Assumption on continuity of $\mathcal{G}$ in Lemma 4.2 is stronger than the conditions of Theorem 4.1. Then it follows from Theorem 2.1 that the posterior $\mu^{y}$ and the approximations $\mu_{N}^{y}$ are well-defined. Finally, the desired result follows directly from Lemma 4.2 and Theorem 2.4.
4.2. Example 2: Deconvolution with a Laplace prior. We now return to Example 2 where we considered the deconvolution of an image on the circle. Recall that $X=L^{2}(\mathbb{T})$ and the data $y$ was generated by considering a fixed function $v \in L^{2}(\mathbb{T})$ and letting $y=\mathcal{G}(v)+\eta$ where $\eta \sim \mathcal{N}\left(0, \sigma^{2} \mathbf{I}\right)$ and the forward map is given by

$$
\mathcal{G}: L^{2}(\mathbb{T}) \rightarrow \mathbb{R}^{m}, \quad \mathcal{G}(u):=S(g * u) .
$$

Here $g$ is a $C^{\infty}(\mathbb{T})$ kernel and $S$ is a bounded linear operator that collects point values of the convolved image. Since the measurement noise is additive and Gaussian then the likelihood potential $\Phi$ is quadratic (1.8). The prior measure in (1.9) has the form

$$
u=\sum_{k \in \mathbb{Z}} \gamma_{k} \xi_{k} \psi_{k}
$$

where $\left\{\xi_{k}\right\}$ are i.i.d. $\operatorname{Lap}(0,1)$ random variables and $\gamma_{k}=\left(1+|k|^{2}\right)^{-5 / 4}$. Thus, $\left\{\gamma_{k}^{2}\right\} \in \ell^{1}(\mathbb{R})$. Furthermore, we took $\psi_{k}(x)=(2 \pi)^{-1 / 2} e^{-2 \pi i k x}$ which form an orthogonal basis in $L^{2}(\mathbb{T})$ and so by Corollary 3.11 the random variable $u \in L^{2}(\mathbb{T})$ a.s. and $\mu_{0}$ is a convex measure.

Next, it follows from Young's inequality [23, Thm. 13.8] that $(g * \cdot): L^{2}(\mathbb{T}) \rightarrow L^{2}(\mathbb{T})$ is a bounded linear operator and furthermore, $(g * u) \in C^{\infty}(\mathbb{T})$ for all $u \in L^{2}(\mathbb{T})$. Since pointwise evaluation is a bounded linear functional on $C^{\infty}(\mathbb{T})$ then the forward map $\mathcal{G}: L^{2}(\mathbb{T}) \rightarrow \mathbb{R}^{m}$ is a bounded linear operator. Therefore, by Remark 2 and Theorem 4.1 we know that the likelihood potential $\Phi$ satisfies Assumption 1. Putting this together with Theorem 3.8 implies that this deconvolution problem is well-posed.

Now we will consider a consistent approximation of this problem. Given $w \in L^{2}(\mathbb{T})$, we let $\left\{\hat{w}_{k}\right\}_{k \in \mathbb{Z}}$ denote the Fourier modes of $w$ i.e.

$$
\hat{w}_{k}=(2 \pi)^{-1 / 2} \int_{0}^{1} w(x) e^{-2 \pi i k x} d x, \quad \forall w \in L^{2}(\mathbb{T}) \quad \text { and } \quad k \in \mathbb{Z}
$$

We also define the Sobolev space $H^{2}(\mathbb{T})$ as

$$
H^{2}(\mathbb{T}):=\left\{w \in L^{2}(\mathbb{T}), \quad\|w\|_{H^{2}}:=\sum_{k \in \mathbb{Z}}\left(1+|k|^{2}\right)^{2}\left|\hat{w}_{k}\right|^{2}<\infty\right\}
$$

Since the prior samples $u \in L^{2}(\mathbb{T})$ a.s. then we can consider their Fourier modes $\hat{u}_{k}=\gamma_{k} \xi_{k}$. We can show that in fact such a sample $u \in H^{2}(\mathbb{T})$ a.s., and are therefore in $C^{1}(\mathbb{T})$. To this end, note that by substituting the values of the $\gamma_{k}$ we can write

$$
\|u\|_{H^{2}}:=\sum_{k \in \mathbb{Z}}\left(1+|k|^{2}\right)^{2} \frac{1}{\left(1+|k|^{2}\right)^{5 / 2}}\left|\xi_{k}\right|^{2}=\sum_{k \in \mathbb{Z}}\left(1+|k|^{2}\right)^{-1 / 2}\left|\xi_{k}\right|^{2} .
$$

This sum will converge almost surely following Kolmogorov's two series theorem [29, Lemma 3.16] and dominated convergence and so $u \in H^{2}(\mathbb{T})$ a.s. This means that the samples can be approximated by truncation of their Fourier expansions.

Suppose that we discretize the prior by truncating the series in (1.9) and define the projections

$$
P_{N} u=\sum_{k=-N}^{N-1} \gamma_{k} \xi_{k} \psi_{k} .
$$

Since $u \in H^{2}(\mathbb{T})$ a.s. then

$$
\begin{equation*}
\left\|u-P_{N} u\right\|_{L^{2}(\mathbb{T})} \lesssim\|u\|_{L^{2}(\mathbb{T})} \frac{1}{N^{2}} . \tag{4.2}
\end{equation*}
$$

Now let $\mu_{N}^{y}$ denote an approximation to the posterior that is obtained by defining $\mathcal{G}_{N}(u):=$ $\mathcal{G}\left(P_{N} u\right)$. Since $\mathcal{G}$ is a bounded linear operator then it satisfies the conditions of Lemma 4.2. Furthermore, the likelihood is quadratic and so by Corollary 4.3 and (4.2) we have that

$$
d_{H}\left(\mu^{y}, \mu_{N}^{y}\right) \lesssim \frac{1}{N^{2}} .
$$

4.3. Example 3: Deconvolution with a hierarchical prior. We now return to Example 3 that was first introduced in Section 1. Recall that the problem is in an identical setting as in Example 2 above with the exception that the prior measure is now induced by the random variable

$$
\begin{equation*}
u=\sum_{k \in \mathbb{Z}} \gamma_{k} \zeta_{k} \xi_{k} \psi_{k} \tag{4.3}
\end{equation*}
$$

where $\gamma_{k}=\left(1+|k|^{2}\right)^{-1}$ and the $\psi_{k}$ are the Fourier basis functions on $L^{2}(\mathbb{T})$. Here, $\left\{\zeta_{k}\right\}$ and $\left\{\xi_{k}\right\}$ are two sequences of i.i.d. random variables so that $\zeta_{1} \sim \operatorname{Gamma}(2,1)$ and $\xi_{1} \sim \mathcal{N}(0,1)$. The main difficulty in the study of this problem is the fact that even though $\xi_{k}$ and $\zeta_{k}$ have convex distributions their product may not be convex. To get around this issue and prove the well-posedness of the problem we shall cast it directly on the sequence space and work with a nonlinear forward map.

First, note that $\mathbb{E} \zeta_{k} \xi_{k}=\mathbb{E} \zeta_{k} \mathbb{E} \xi_{k}=0$ and $\operatorname{Var} \zeta_{k} \xi_{k}=\left(\boldsymbol{\operatorname { V a r }} \zeta_{k}\right)\left(\boldsymbol{\operatorname { V a r }} \xi_{k}\right)+\left(\boldsymbol{\operatorname { V a r }} \zeta_{k}\right) \mathbb{E} \xi_{k}+$ $\left(\operatorname{Var} \xi_{k}\right) \mathbb{E} \zeta_{k}<\infty$. We can use the same technique as in the proof of Theorem 3.10(i) to show that $u \in L^{2}(\mathbb{T})$ a.s. Furthermore, the random sequences $\left\{\sqrt{\gamma}_{k} \zeta_{k}\right\}$ and $\left\{\sqrt{\gamma}_{k} \xi_{k}\right\}$ belong to the sequence space $\ell^{1}$ a.s. and so they belong to $\ell^{2}$ as well. We now consider the product space $\ell^{2} \otimes \ell^{2}=\left\{\left\{c_{k}, b_{k}\right\}:\left\{c_{k}\right\} \in \ell^{2},\left\{b_{k}\right\} \in \ell^{2}\right\}$ equipped with the norm $\left\|\left\{c_{k}, b_{k}\right\}\right\|_{\ell^{2} \otimes \ell^{2}}:=\max \left\{\left\|\left\{c_{k}\right\}\right\|_{\ell^{2}},\left\|\left\{b_{k}\right\}\right\|_{\ell^{2}}\right\}$ along with the probability measure $\mu_{0}$ that is induced by $\left\{\sqrt{\gamma}_{k} \zeta_{k}, \sqrt{\gamma}_{k} \xi_{k}\right\}$ on the product space. We will take $\mu_{0}$ to be our prior. Note that $\mu_{0}$ can be obtained from the product of two convex measures on $\ell^{2}$ and so it is a convex measure by Corollary 3.4(ii).

Define the operator

$$
Q: \ell^{2} \otimes \ell^{2} \rightarrow L^{2}(\mathbb{T}) \quad Q\left(\left\{c_{k}, b_{k}\right\}\right)=\sum_{k \in \mathbb{Z}} c_{k} b_{k} \psi_{k} .
$$

Using the triangle and Hölder's inequalities we can easily check that

$$
\left\|Q\left(\left\{c_{k}, b_{k}\right\}\right)\right\|_{L^{2}(\mathbb{T})} \leq\left\|\left\{c_{k}, b_{k}\right\}\right\|_{\ell^{2} \otimes \ell^{2}}^{2}
$$

This bound together with the fact that $\mathcal{G}: L^{2}(\mathbb{T}) \rightarrow \mathbb{R}^{m}$ is a bounded linear operator implies that

$$
\left\|\mathcal{G} \circ Q\left(\left\{c_{k}, b_{k}\right\}\right)\right\|_{2} \leq C\left\|\left\{c_{k}, b_{k}\right\}\right\|_{\ell^{2} \otimes \ell^{2}}^{2} .
$$

Since $\|\mathcal{G} \circ Q\|_{2}$ is bounded by a quadratic function of $\left\|\left\{c_{k}, b_{k}\right\}\right\|_{\ell^{2} \otimes \ell^{2}}$ then for every $\epsilon>0$ we can always find $M(\epsilon)>0$ so that

$$
\begin{equation*}
\left\|\mathcal{G} \circ Q\left(\left\{c_{k}, b_{k}\right\}\right)\right\|_{2} \leq C \exp \left(\epsilon\left\|\left\{c_{k}, b_{k}\right\}\right\|_{\ell^{2} \otimes \ell^{2}}+M(\epsilon)\right) . \tag{4.4}
\end{equation*}
$$

Now consider $\left\{c_{k}, b_{k}\right\}$ and $\left\{g_{k}, f_{k}\right\}$ in $\ell^{2} \otimes \ell^{2}$. Using Parseval's identity we can write

$$
\begin{aligned}
\left\|Q\left(\left\{c_{k}, b_{k}\right\}\right)-Q\left(\left\{g_{k}, f_{k}\right\}\right)\right\|_{L^{2}(\mathbb{T})} & =\left(\sum_{k \in \mathbb{Z}}\left|c_{k} b_{k}-g_{k} f_{k}\right|^{2}\right)^{1 / 2} \\
& =\left(\sum_{k \in \mathbb{Z}}\left|c_{k}\left(b_{k}-f_{k}\right)+f_{k}\left(c_{k}-g_{k}\right)\right|^{2}\right)^{1 / 2} \\
& \leq C \max \left(\left\|\left\{c_{k}, b_{k}\right\}\right\|_{\ell^{2} \otimes \ell^{2}},\left\|\left\{g_{k}, f_{k}\right\}\right\|_{\ell^{2} \otimes \ell^{2}}\right)\left\|\left\{c_{k}-g_{k}, b_{k}-f_{k}\right\}\right\|_{\ell^{2} \otimes \ell^{2}} .
\end{aligned}
$$

Putting this result together with the fact that $\mathcal{G}: L^{2}(\mathbb{T}) \rightarrow \mathbb{R}^{m}$ is a bounded linear operator implies that for any $r>0$ such that $\max \left(\left\|\left\{c_{k}, b_{k}\right\}\right\|_{\ell^{2} \otimes \ell^{2}},\left\|\left\{g_{k}, f_{k}\right\}\right\|_{\ell^{2} \otimes \ell^{2}}\right)<r$ there is a constant $K(r)>0$ so that

$$
\begin{equation*}
\left\|\mathcal{G} \circ Q\left(\left\{c_{k}, b_{k}\right\}\right)-\mathcal{G} \circ Q\left(\left\{g_{k}, f_{k}\right\}\right)\right\|_{2} \leq K(r)\left\|\left\{c_{k}-g_{k}, b_{k}-f_{k}\right\}\right\|_{\ell^{2} \otimes \ell^{2}} . \tag{4.5}
\end{equation*}
$$

This bound along with (4.4) and Theorem 4.1 implies that the underlying likelihood potential $\Phi=\frac{1}{2 \sigma^{2}}\|\mathcal{G}(u)-y\|_{2}^{2}$ satisfies the conditions of Assumption 1 for any choice of $\alpha_{1}, \alpha_{2}>0$. Since the prior measure $\mu_{0}$ is convex this inverse problem is well posed by Corollary 3.9.
4.4. Example 5: Source inversion in atmospheric dispersion. In this example we consider the problem of estimating the source term in a parabolic PDE from linear measurements of the solution. This problem is closely related to the inverse problem of estimating the sources of emissions in an atmospheric dispersion model [26] and we shall present this example in that context. Let $D \subset \mathbb{R}^{3}$ be a smooth and connected domain and define $\Omega:=D \times(0, T]$ for some constant $T>0$. Now consider the PDE

$$
\left\{\begin{array}{lll}
\partial_{t} c-g(x, t) c-\sum_{i=1}^{3} a_{i}(x, t) \partial_{i} c-\sum_{i, j=1}^{3} b_{i j} \partial_{i j} c=u, & \text { in } & D \times(0, T)  \tag{4.6}\\
c(x, t)=0 & \text { on } & \partial D \times(0, T), \\
c(x, 0)=0 . & &
\end{array}\right.
$$

where $\partial_{i} c$ is the shorthand notation for $\frac{\partial c}{\partial x_{i}}$ and $\partial_{i j} c=\partial_{i} \partial_{j} c$. In the context of atmospheric dispersion modelling $c(x, t)$ is the pollutant concentration, $u(x, t)$ is the source term and $g, a_{i}, b_{i j}$ coefficients are used to model the wind velocity field and the eddy diffusivity coefficients $[25,36]$. To this end, we have the following result concerning the existence and uniqueness
of the solution to (4.6) (see [19, Section 7.1] or [35, Theorem 11.3 and Example 11.5] for a proof).

Theorem 4.4. Suppose that $\Omega=D \times(0, T)$ where $D \subset \mathbb{R}^{3}$ is defined as above and $T>0$. Also, assume that $u \in L^{2}(\Omega)$ and $g, a_{i}, b_{i j}$ are in $C(\Omega)$ for $i, j \in\{1,2,3\}$. Furthermore, assume that $b_{i j}=b_{j i}$ and there exists a uniform constant $K>0$ such that

$$
\sum_{i, j=1}^{3} b_{i j}(x, t) y_{i} y_{j}>K \sum_{i=1}^{3} y_{i}^{2} \quad \forall y_{i} \in \mathbb{R} \backslash\{0\} \quad \text { and } \quad \forall(x, t) \in \Omega
$$

Then there exists a unique solution $c(x, t)$ of (4.6) and a positive constant $C$ independent of $u$ so that

$$
\|c\|_{L^{2}(\Omega)} \leq C\|u\|_{L^{2}(\Omega)}
$$

This result along with the fact that $c$ depends linearly on $u$ allows us to define a bounded linear operator $S: L^{2}(\Omega) \rightarrow L^{2}(\Omega)$ so that $S(u)=c$ whenever the conditions of Theorem 4.4 are satisfied.

Now consider positive constants $r$ and $\tau$ and a sequence of points $x_{i} \in D$ and $t_{i} \in(0, T)$ for $i=1,2, \cdots, m$. Define the sets

$$
Q_{i}=B\left(x_{i}, r\right) \times\left[t_{i}, t_{i}+\tau\right]
$$

where $B\left(x_{i}, r\right) \subset D$ is the ball of radius $r$ centered at $x_{i}$. Suppose that $\Omega$ is large enough so that $Q_{i} \subset \Omega$ for all $i$ and also $Q_{i} \cap Q_{j}=\emptyset$ if $i \neq j$. To this end, given a solution $u$ of (4.6), we define the bounded linear measurement operators

$$
M_{i}: L^{2}(\Omega) \rightarrow \mathbb{R} \quad M_{i}(c):=\int_{Q_{i}} c d x d t
$$

The elaborate construction of the $M_{i}$ corresponds to a common method of measurement in the study of deposition of particulate matter where a number of plastic jars (also known as dust-fall jars) are left in the field for a given period of time [26, 32]. At the end of this period the jars are taken to the lab and the concentration of deposited material in each jar is measured.

Putting everything together we can define the forward map

$$
\begin{equation*}
\mathcal{G}: L^{2}(\Omega) \rightarrow \mathbb{R}^{m} \quad \mathcal{G}(u):=\left(M_{1}(S(u)), \cdots, M_{m}(S(u))\right)^{T} \tag{4.7}
\end{equation*}
$$

This operator is bounded and linear since $S$ and the $M_{i}$ are as well. Now suppose that $y \in \mathbb{R}^{m}$ is the data and consider the usual additive Gaussian noise model

$$
\begin{equation*}
y=\mathcal{G}(u)+\eta \quad \text { where } \quad \eta \sim \mathcal{N}\left(0, \sigma^{2} \mathbf{I}\right) \quad \text { and } \quad \sigma>0 \tag{4.8}
\end{equation*}
$$

Now we turn our attention to the construction of a prior measure for the source term.
Let $X=L^{2}(\Omega)$ and suppose that we have prior knowledge that the source is constant in time and it is supported within a smooth domain $\tilde{D} \subset D \subset \mathbb{R}^{3}$. For example, the domain $\tilde{D}$ can denote an industrial site which is known as the main polluter in an area. We let $\left\{\psi_{k}\right\}_{k=1}^{\infty}$
and $\left\{\lambda_{k}\right\}_{k=1}^{\infty}$ denote the eigenfunctions and eigenvalues of the Laplacian on $\tilde{D}$. The $\psi_{k}$ form an orthonormal basis for $L^{2}(\tilde{D})$ and so they are a good candidate for prior construction. Consider the random variable

$$
u=\sum_{k=1}^{\infty} \gamma_{k} \xi_{k} \psi_{k}
$$

where $\gamma_{k}=\lambda_{k}^{-1}$ and $\xi_{k} \sim \operatorname{Exp}(1)$. Let $\nu$ denote the probability measure that is induced by this random variable on $L^{2}(\tilde{D})$. By Corollary 3.11 this measure is convex. Now we uniquely extend the elements of $L^{2}(\tilde{D})$ by zero to $L^{2}(D)$ and then extend them as constant functions in direction of $t$ to elements of $L^{2}(\Omega)$. Let $E: L^{2}(\tilde{D}) \rightarrow L^{2}(\Omega)$ denote this extension operator which is both bounded and linear. We now define our prior measure $\mu_{0}$ to be the push-forward of the measure $\nu$ under $E$ which is a probability measure on $L^{2}(\Omega)$. This measure is convex following Corollary 3.4(i). Putting this fact together with the forward model given by (4.7) and (4.8) as well as Theorem 4.1 and Corollary 3.9 implies that the Bayesian inverse problem of finding $u$ from linear measurements of $c$ is well-posed.
4.5. Example 6: Estimating the permeability in porous medium flow. In this section we consider an example problem which involves estimating the coefficients of an elliptic PDE. We present this example in the context of flow in a porous medium, such as groundwater flow. The inverse problem involves estimating the permeability of the porous medium from measurements of pressure at multiple points. This example was considered in [15] with a Besov prior and in [16] with Gaussian priors. Here we will present the same example using a convex prior that is based on the Gamma distribution. This is an example of a nonlinear inverse problem with a convex prior measure where the forward map does not satisfy condition (i) of Theorem 4.1 for every $\epsilon>0$. Therefore, we have to take extra care to make sure that the prior results in a well-posed inverse problem.

Consider the elliptic PDE

$$
\begin{equation*}
-\nabla \cdot(\exp (u(x)) \nabla p(x))=f, \quad x \in \mathbb{T}^{2}, \quad f \in L^{2}\left(\mathbb{T}^{2}\right) \tag{4.9}
\end{equation*}
$$

where the boundary conditions are periodic and $\mathbb{T}^{2}=(0,1]^{2}$. Here $\exp (u(x))$ is the permeability of the medium. We choose to work with the exponential form to ensure that the permeability is positive. Assume that the data $y \in \mathbb{R}^{m}$ consists of noisy pointwise measurements of the pressure $p(x)$ (alternatively, one can consider local averages of the pressure field if $p$ is not defined pointwise) i.e.

$$
y=\mathcal{G}(u)+\eta \quad \text { where } \quad \eta \sim \mathcal{N}\left(0, \sigma^{2} \mathbf{I}\right) \quad \text { and } \quad \sigma>0
$$

and

$$
\begin{equation*}
\mathcal{G}(u):=\left(p\left(x_{1}\right), \cdots, p\left(x_{m}\right)\right)^{T} \quad \text { where } \quad x_{1}, x_{2}, \cdots, x_{m} \in \mathbb{T}^{2} . \tag{4.10}
\end{equation*}
$$

The noise variance $\sigma$ and the points $\left\{x_{k}\right\}_{k=1}^{m}$ are fixed.
We are interested in the setting where $u \in C^{1}\left(\mathbb{T}^{2}\right)$. Under this assumption we have the following theorem concerning the boundedness and continuity of the forward map.

Theorem 4.5 ([16, Corollary 3.5]). Suppose that $\mathcal{G}$ is given by (4.10) where $p(x)$ is given by (4.9) and $f \in L^{2}\left(\mathbb{T}^{2}\right)$. Then for any $u \in C^{1}\left(\mathbb{T}^{2}\right)$ there exists a constant $D_{1}=D_{1}\left(m,\|f\|_{L^{2}(\mathbb{T})}\right)$ such that

$$
\|\mathcal{G}(u)\|_{2} \leq D_{1} \exp \left(\|u\|_{C^{1}\left(\mathbb{T}^{2}\right)}\right) .
$$

If $u_{1}, u_{2} \in C^{1}\left(\mathbb{T}^{2}\right)$, then for any $\epsilon>0$

$$
\left\|\mathcal{G}\left(u_{1}\right)-\mathcal{G}\left(u_{2}\right)\right\|_{2} \leq D_{2} \exp \left(c \max \left\{\left\|u_{1}\right\|_{C^{1}\left(\mathbb{T}^{2}\right)},\left\|u_{2}\right\|_{C^{1}\left(\mathbb{T}^{2}\right)}\right)\left\|u_{1}-u_{2}\right\|_{C^{1}\left(\mathbb{T}^{2}\right)},\right.
$$

where $D_{2}=D_{2}\left(M, \epsilon,\|f\|_{L^{2}\left(\mathbb{T}^{2}\right)}\right)$ and $c=4+8+\epsilon$.
This theorem suggests that we need to construct a prior that is supported on $C^{1}\left(\mathbb{T}^{2}\right)$. Unfortunately, since $C^{1}\left(\mathbb{T}^{2}\right)$ does not have an unconditional basis our recipe for construction of the prior measures from Section 3.1 does not apply directly. Instead, we shall construct the prior to be supported within a Sobolev space that is sufficiently regular and we use the Sobolev embedding theorem to show that the prior will be supported on $C^{1}\left(\mathbb{T}^{2}\right)$ as desired.

Let $\left\{\psi_{j}\right\}_{j=1}^{\infty}$ be an $r$-regular wavelet basis for $L^{2}\left(\mathbb{T}^{2}\right)$ [33, Section 2.1] where $r>2$. Then for $f \in L^{2}\left(\mathbb{T}^{2}\right)$ we can write

$$
f(x)=\sum_{k=1}^{\infty} \hat{f}_{k} \psi_{k}(x)
$$

where $\left\{\hat{f}_{k}\right\}$ is the sequence of basis coefficients of $f$ (see [31, Appendix A] for how the wavelet basis indices are reordered to form a single sum). For $s<r$ we can identify the Sobolev space $H^{s}\left(\mathbb{T}^{2}\right)$ as

$$
H^{s}\left(\mathbb{T}^{2}\right):=\left\{f \in L^{2}\left(\mathbb{T}^{2}\right): \sum_{k=1}^{\infty} k^{s}\left|\hat{f}_{k}\right|^{2}<\infty\right\} .
$$

As usual, we construct the prior measure by randomizing the basis coefficients. Let $\left\{\xi_{k}\right\}$ and $\left\{\zeta_{k}\right\}$ be two sequences of i.i.d. random variables on $\mathbb{R}$ that are distributed according to $\operatorname{Gamma}(2,1)$ and define $\theta_{k}=\xi_{k}-\zeta_{k}$. This construction ensures that $\theta_{k}$ has a symmetric distribution. By Corollary 3.4(i) and (ii) the $\theta_{k}$ are convex. Now consider the random variable

$$
\begin{equation*}
v=\sum_{k=1}^{\infty} \gamma_{k} \theta_{k} \psi_{k} \tag{4.11}
\end{equation*}
$$

where $\gamma_{k}=k^{-2}$. By Corollary 3.11 we know that $\|v\|_{L^{2}\left(\mathbb{T}^{2}\right)}<\infty$ a.s. Furthermore,

$$
\sum_{k=1}^{\infty} k^{3}\left|\gamma_{k}\right|^{2}\left|\eta_{k}\right|^{2}=\sum_{k=1}^{\infty} k^{-1}\left|\eta_{k}\right|^{2} .
$$

But this sum converges almost surely by Kolmogorov's two series theorem and dominated convergence and so $\|v\|_{H^{3}\left(\mathbb{T}^{2}\right)}<\infty$ a.s. By the Sobolev embedding theorem [40, Proposition 3.3] $H^{3}\left(\mathbb{T}^{2}\right) \subset C^{1}\left(\mathbb{T}^{2}\right)$ and so the prior measure induced by the random variable $v$ in (4.11) is supported in $C^{1}\left(\mathbb{T}^{2}\right)$ as desired and it is a convex measure.

Before we proceed to proving the well-posedness of this problem we will need to reweight the samples $v$. The reason for this issue is that $\mathbb{E} \exp \left(\|v\|_{X}\right)$ may not be finite. However, by

Theorem 3.7 we know that there exists a constant $\kappa>0$ so that $\mathbb{E} \exp \left(\kappa\|v\|_{X}\right)<\infty$. Thus, we take the prior samples $u=\beta v$ for some $\beta \in(0, \kappa]$ which ensures that $\mathbb{E} \exp \left(\|u\|_{X}\right)<\infty$. We are now able to apply Theorems 4.5, 4.1 and 3.8 to the prior measure that is induced by $u$ in order to show that this inverse problem is well-posed.

Note that estimating the constant $\kappa$ for a given prior is in general a difficult task and to the best of our knowledge a general recipe for estimating this constant does not exist in the literature. Of course the actual value of this constant is only important when the forward map is exponentially bounded by the parameter norm. For example, Theorem 4.5 above dictates that $\exp \left(\kappa\|u\|_{C^{1}\left(\mathbb{T}^{2}\right)}\right)$ must be integrable under the prior for $\kappa>1$ in order for us to achieve well-posedness. Thus, the properties of the forward map can be used to identify the minimum "allowed" value of the constant $\kappa$. On the other hand, if the forward map is polynomially bounded then well-posedness can be achieved regardless of the actual value of $\kappa$ according to Corollary 3.9
5. Closing remarks. We started this article by defining the notions of well-posedness and consistency of Bayesian inverse problems in the general setting where the parameter of interest belongs to an infinite-dimensional Banach space. We presented a set of assumptions on the prior measure $\mu_{0}$ and the likelihood potential $\Phi$ under which the resulting inverse problem would be well-posed. Furthermore, we discussed consistent approximation of the posterior measure via an approximation $\Phi_{N}$ of the likelihood potential $\Phi$. We discussed mild conditions on the forward map and the likelihood potential that allowed us to obtain bounds on the rate of convergence of the approximate posterior in the Hellinger metric. We particularly focused on the setting where $\Phi_{N}$ is obtained by discretizing the forward problem using finite dimensional projections.

We mainly focused on our assumptions concerning the prior measure $\mu_{0}$ and showed that the class of convex measures is a good choice for modelling of prior knowledge as its elements result in well-posed inverse problems. This class already includes well known measures such as Gaussian and Besov measures and so our results can be viewed as a generalization of existing results regarding well-posedness of Bayesian inverse problems.

Afterwards, we presented a general framework for the construction of priors on separable Banach spaces that have an unconditional basis. Inspired by the Karhunen-Loéve expansion of Gaussian random variables, our framework uses a countable product of one dimensional convex measures on $\mathbb{R}$. Next, we considered some of the practical aspects of solving the Bayesian inverse problems that arise from choosing convex priors. Finally, we presented four concrete examples of well-posed Bayesian inverse problems that used convex prior measures.

An important consequence of this work is that now we have access to a much larger class of measures for modelling of prior knowledge in Bayesian inverse problems. For example, if one is interested in imposing a constraint such as positivity then one can use the Gamma distribution or the uniform distribution to do so. The resulting measure will still be convex. More interestingly, recent results in [34] and [3] hint at the use of specific convex measures in order to promote sparsity of the parameters. Even in the case of the Besov priors of [31], the resulting maximum a posteriory estimator is equivalent to solving a least squares problem with an $\ell^{1}$ regularization term which is a common technique for promoting sparsity of the solution. Then the well-posedness result for the class of convex measures is a first step towards the
study of sparse solutions within the Bayesian approach to inverse problems.
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