

Model-free portfolio theory and its functional master formula

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Abstract

We use pathwise Itô calculus to prove two strictly pathwise versions of the master formula in Fernholz' stochastic portfolio theory. Our first version is set within the framework of Föllmer's pathwise Itô calculus and works for portfolios generated from functions that may depend on the current states of the market portfolio and an additional path of finite variation. The second version is formulated within the functional pathwise Itô calculus of Dupire (2009) and Cont & Fournié (2010) and allows for portfolio-generating functionals that may depend additionally on the entire path of the market portfolio. Our results are illustrated by several examples and shown to work on empirical market data.

Keywords: Pathwise Itô calculus; Föllmer integral; functional Itô formula; portfolio analysis; market portfolio; portfolio generating functionals; functional master formula on path space; entropy weighting

1 Introduction

The purpose of this paper is twofold. On the one hand, it deals with an extension of the master formula in stochastic portfolio theory to path-dependent portfolio generating functions. On the other hand, it yields a new case study in which continuous-time trading strategies can be constructed in a probability-free manner by means of pathwise Itô calculus.

Stochastic portfolio theory (SPT) was introduced by Fernholz [8, 9, 10]; see also Karatzas and Fernholz [13] for an overview. Its goal is to construct investment strategies that outperform a certain reference portfolio such as the market portfolio $\mu(t)$; see, e.g., [12, 30, 18, 22]. Here, our focus is mainly on functionally generated portfolios, which in standard SPT are generated from functions $G(t, \mu(t))$ of the current state, $\mu(t)$, of the market portfolio. The performance of the functionally generated portfolio relative to the market portfolio can be described in a very convenient way by the so-called *master formula* of SPT. See Strong [29] for an extension of the master formula to the

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case in which G may additionally depend on the current state of a continuous trajectory of bounded variation.

The first contribution of this paper concerns the basis for the modeling framework of SPT. While price processes for SPT are usually modeled as Itô processes, it has often been remarked that both sides of the master formula can be understood in a strictly pathwise manner. So the question arises to what extent a stochastic model is actually needed in setting up SPT. Do price processes really need to be modeled as Itô processes driven by Brownian motion, or is it possible to relax this condition and consider more general processes, perhaps even beyond the class of semimartingales? Can one get rid of the nullsets that are inherent in stochastic models? That is, can one prove the master formula strictly path by path?

Our approach gives affirmative answers to the questions raised in the preceding paragraph. To this end, we show that SPT can be formulated within the pathwise Itô calculus introduced by Föllmer [14] and further developed to path-dependent functionals by Dupire [7] and Cont and Fournié [3, 4]. Thus, the only assumption on the trajectories of the price evolution is that they are continuous and admit quadratic variations and covariations in the sense of [14]. This assumption is satisfied by all typical sample paths of a continuous semimartingale but also by non-semimartingales, such as fractional Brownian motion with Hurst index $H \geq 1/2$ and many deterministic fractal curves [19, 25]. In this sense, our paper is also a contribution to *robust finance*, which aims to reduce the reliance on a probabilistic model and, thus, to model uncertainty; see, e.g., [2, 5, 6, 23, 24, 26] for similar analyses on other financial problems. Robustness results for discrete-time SPT were previously obtained also by Pal and Wong [20], where the relative performance of portfolios with respect to a certain benchmark is analyzed using the discrete-time energy-entropy framework [21, 32, 33].

To discuss the second contribution of this paper, note that in practice portfolios are often constructed not just from current market prices or capitalizations but also from past data, such as econometric estimates, moving or rolling averages, running maxima, realized covariance, Bollinger bands, etc. It is therefore a natural question whether it is possible to develop a master formula for portfolios that are generated by functionals of the entire past evolution, $\mu^t := (\mu(s \wedge t))_{s \geq 0}$, of the market portfolio and maybe also other factors. In this paper, we give an affirmative answer to this question. Our main result, Theorem 2.16, contains a master formula for portfolios that are generated by sufficiently smooth functionals of the form $G(t, \mu^t, A^t)$, where A is an additional m -dimensional continuous trajectory of bounded variation that may depend on μ in an adaptive manner. We then turn to analyzing several concrete examples for portfolios that are generated by functions of mixtures of current market portfolio weights and their moving averages. Our analysis is carried out both on a mathematical level and with empirical market data.

The paper is organized as follows. In Section 2.1, we first provide a master formula based on Föllmer's [14] pathwise Itô calculus. It works for portfolios generated by functions that, as in Strong [29], may depend on the current states of the market portfolio and an additional continuous trajectory A of bounded variation. In this case, both the formulations and the proofs of many results from SPT, including the master formula, can be extended relatively easily to the pathwise setting. In Section 2.2, we extend the results from Section 2.1 to portfolios that may depend on the entire past evolutions of the market portfolio and A . To this end, we use the pathwise functional Itô calculus developed by Dupire [7] and Cont and Fournié [3]. The main difficulty in achieving this extension is that now the Itô integral is based on "Riemann sums" involving approximations of the integrator path. Therefore, a functional dependence on the integrator paths must be retained in the integrands, and new arguments are needed so as to prove, e.g., the corresponding master formula. Our above-mentioned examples are discussed in Section 3. All proofs are given in Section 4, and key concepts from functional Itô calculus are recalled in the Appendix.

2 Statement of main results

Throughout this paper, we work in a strictly pathwise setting. Our goal is to derive, first, a pathwise master formula for Stochastic Portfolio Theory (SPT) [8, 9, 10, 13]. To this end, we will use the pathwise Itô calculus developed by Föllmer [14]. In a second step, we will use the functional extension of pathwise Itô calculus, as developed by Dupire [7] and Cont and Fournié [3], so as to extend the pathwise master formula also to path-dependent functionals.

In the sequel, we consider a financial market model consisting of d risky assets and a locally riskless bond. The price of the bond is given by

$$B(t) = \exp \left(\int_0^t r(s) \, ds \right),$$

where $r : [0, \infty) \rightarrow \mathbb{R}$ is a measurable short rate function satisfying $\int_0^T |r(s)| \, ds < \infty$ for all $T > 0$. The prices of the risky assets are described by a single d -dimensional continuous path $S : [0, \infty) \rightarrow \mathbb{R}^d$. We emphasize that no probabilistic assumptions are made on the dynamics of r and S . All that we require is that the components of S , besides being strictly positive, admit continuous covariations in the pathwise sense proposed by Föllmer [14]. To recall this notion, let $(\mathbb{T}_n)_{n \in \mathbb{N}}$ be a refining sequence of partitions of $[0, \infty)$, which will remain fixed for the remainder of this paper. That is, for fixed n , the partition $\mathbb{T}_n = \{t_0, t_1, \dots\}$ is such that $0 = t_0 < t_1 < \dots$ and $t_k \rightarrow \infty$ as $k \rightarrow \infty$. Moreover, we have $\mathbb{T}_1 \subset \mathbb{T}_2 \subset \dots$, and the mesh of \mathbb{T}_n tends to zero on each compact interval as $n \uparrow \infty$. For fixed n , it will be convenient to denote the successor of $t \in \mathbb{T}_n$ by t' , i.e.,

$$t' = \min\{u \in \mathbb{T}_n \mid u > t\}.$$

We then assume that for $1 \leq i, j \leq d$ and $t \geq 0$ the sequence

$$\sum_{\substack{s \in \mathbb{T}_n \\ s \leq t}} (S_i(s') - S_i(s)) (S_j(s') - S_j(s)) \quad (2.1)$$

converges to a finite limit, called the pathwise covariation of S_i and S_j and denoted by $[S_i, S_j](t)$, such that $t \rightarrow [S_i, S_j](t)$ is continuous. As usual, we write $[S_i] := [S_i, S_i]$ and call this the pathwise quadratic variation of the real-valued path S_i . The class of all continuous functions $S : [0, \infty) \rightarrow \mathbb{R}^d$ satisfying the preceding requirement will be denoted by QV^d . As mentioned above, we will require in addition that $S_i(t) > 0$ for all i and t . The corresponding subset of QV^d will be denoted by QV_+^d .

Note that QV^d depends strongly on the choice of the partition sequence $(\mathbb{T}_n)_{n \in \mathbb{N}}$ and is typically not a vector space [25]. Moreover, polarization of the sums in (2.1) implies that $[S_i, S_j]$ exists for $S_i, S_j \in QV^1$, if and only if the quadratic variation $[S_i + S_j]$ exists and that

$$[S_i, S_j](t) = \frac{1}{2} \left([S_i + S_j](t) - [S_i](t) - [S_j](t) \right). \quad (2.2)$$

Thus, for $d > 1$, the assumption that the covariations $[S_i, S_j]$ exist cannot be reduced to the existence of the quadratic variations $[S_i] = [S_i, S_i]$. If S is a typical sample path of a continuous semimartingale then $[S_i, S_j]$ clearly coincides with the quadratic variation taken in the usual sense—provided that this sample path does not belong to a certain nullset, which in turn depends strongly on the partition sequence $(\mathbb{T}_n)_{n \in \mathbb{N}}$. As a matter of fact, the union of all these nullsets is equal to the entire sample space Ω , because for every continuous function there exists some refining sequence of partitions along which this function has vanishing quadratic variation; see [16, p. 47]. Some authors, such as [1] or

[3], make the dependence of the pathwise quadratic variation (and the pathwise Itô integral) on the sequence of partitions $(\mathbb{T}_n)_{n \in \mathbb{N}}$ explicit by including a corresponding symbol in the notation. We refrain from this so as not to burden the notation.

According to [14], the requirement $S \in QV^d$ guarantees that Itô's formula holds in a pathwise sense. By taking those functions that appear naturally inside the Itô integral from Itô's formula, we arrive at a natural class of admissible integrands for Itô integrals with integrator S . In Section 2.1, we will use the corresponding integration theory to develop a strictly pathwise theory of SPT and state the corresponding master formula. In this context, it will be possible to extend the arguments used, e.g., in [13]. In Section 2.2, we will then consider a further extension to path-dependent portfolios that appear in the functional extension of Itô's formula given recently by Dupire [7] and Cont and Fournié [3]. Here, additional care will be needed in setting up the problem formulation, and a new proof strategy will be needed for the corresponding master formula.

2.1 The master formula within Föllmer's pathwise Itô calculus

We refer to [28] and [24, Section 3] for background on Föllmer's pathwise Itô calculus, including an English translation of [14] as provided in the Appendix of [28]. For open sets $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^n$, the class $C^{2,1}(U, V)$ will consist of all functions $f : U \times V \rightarrow \mathbb{R}$ that are continuously differentiable in $(x, a) \in U \times V$ and twice continuously differentiable in $x \in U$. We will write $f_{x_k} = \partial f / \partial x_k$ and $f_{a_k} = \partial f / \partial a_k$ for the partial derivatives of f with respect to the k^{th} components of $x = (x_1, \dots, x_d)$ and $a = (a_1, \dots, a_n)$, respectively. The gradient of f in direction $x = (x_1, \dots, x_d)$ will be denoted by $\nabla_x f = (f_{x_1}, \dots, f_{x_d})$, and we will write f_{x_k, x_m} for the second partial derivatives with respect to the components x_k and x_m of the vector x . The Euclidean inner product of two vectors v and w will be denoted by $v \cdot w$. We let \mathbb{R}_+^d be set of all those vectors in \mathbb{R}^d that have only strictly positive components. The space $CBV([0, T], V)$ will consist of all continuous functions $A : [0, T] \rightarrow V$ whose components are of bounded variation. The following definition is taken from [24, Definition 11].

Definition 2.1. A function $\xi : [0, \infty) \rightarrow \mathbb{R}^d$ is called an *admissible integrand* for S if for each $T > 0$ there exists $n \in \mathbb{N}$, open sets $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^n$, a function $f \in C^{2,1}(U, V)$, and $A \in CBV([0, T], V)$ such that $S(t) \in U$ and $\xi(t) = \nabla_x f(S(t), A(t))$ for $0 \leq t \leq T$.

If ξ is an admissible integrand for S , then Föllmer's pathwise Itô formula, e.g., in the form of [24, Theorem 9], implies that for every $T > 0$ the pathwise Itô integral exists as the following limit of Riemann sums:

$$\int_0^T \xi(t) dS(t) = \lim_{n \uparrow \infty} \sum_{t \in \mathbb{T}_n, t \leq T} \xi(t) \cdot (S(t') - S(t)). \quad (2.3)$$

This integral will be called the *Föllmer integral* in the sequel. Suppose that ξ is an admissible integrand for S and η is a real-valued measurable function on $[0, \infty)$ such that $\int_0^t |\eta(s)| d|B|(s) < \infty$ for all $t > 0$, where $d|B|(s)$ denotes Stieltjes integration with respect to the total variation of B . Then the pair (ξ, η) is called a *trading strategy*. As usual, the interpretation is that $\xi_i(t)$ corresponds to the number of shares held in the i^{th} stock at time t and $\eta(t)$ is the number of shares held in the riskless bond at time t .

Definition 2.2. Let ξ be an admissible integrand for S and η a real-valued measurable function on $[0, \infty)$ such that $\int_0^T |\eta(s)| d|B|(s) < \infty$ for all $T > 0$. The trading strategy (ξ, η) is called *self-financing* if the associated wealth $V(t) := \xi(t) \cdot S(t) + \eta(t)B(t)$ satisfies the identity

$$V(t) = V(0) + \int_0^t \xi(s) dS(s) + \int_0^t \eta(s) dB(s), \quad t \geq 0. \quad (2.4)$$

Note that the first integral in (2.4) is a Föllmer integral, whereas $\int_0^t \eta(s) dB(s)$ can be understood in the Riemann–Stieltjes sense, since both $\eta(t)$ and $B(t)$ are continuous functions of t . Indeed, $V(t)$ is continuous by the continuity of $t \mapsto \int_0^t \xi(s) dS(s)$ and (2.4), the continuity of $\xi(t)$ follows from the definition of an admissible integrand, and therefore $\eta(t) = (V(t) - \xi(t) \cdot S(t))/B(t)$ is continuous as well.

Lemma 2.3. *Suppose that X is a continuous function from $[0, \infty)$ to \mathbb{R}_+^d . Then $X \in QV_+^d$ if and only if $\log X := (\log X_1, \dots, \log X_d)^\top \in QV^d$. In this case,*

$$[\log X_i, \log X_j](t) = \int_0^t \frac{1}{X_i(s)X_j(s)} d[X_i, X_j](s). \quad (2.5)$$

The preceding lemma implies in particular that $\log S = (\log S_1, \dots, \log S_d)^\top \in QV^d$. The following lemmas are standard in stochastic calculus. In our pathwise setting, however, their proofs need some additional care.

Lemma 2.4. *A function $\pi : [0, \infty) \rightarrow \mathbb{R}^d$ is an admissible integrand for $\log S$ if and only if $\frac{\pi(t)}{S(t)} := \left(\frac{\pi_1(t)}{S_1(t)}, \dots, \frac{\pi_d(t)}{S_d(t)} \right)^\top$ is an admissible integrand for S . In this case,*

$$\int_0^t \pi(s) d \log S(s) = \int_0^t \frac{\pi(s)}{S(s)} dS(s) - \frac{1}{2} \sum_{i=1}^d \int_0^t \frac{\pi_i(s)}{S_i^2(s)} d[S_i](s). \quad (2.6)$$

The preceding lemma gives rise to a special class of self-financing trading strategies:

Lemma 2.5. *Suppose that π is an admissible integrand for $\log S$. If we let*

$$V^\pi(t) := \exp \left(\int_0^t \frac{\pi(s)}{S(s)} dS(s) - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\pi_i(s)\pi_j(s)}{S_i(s)S_j(s)} d[S_i, S_j](s) + \int_0^t \left(1 - \sum_{i=1}^d \pi_i(s) \right) r(s) ds \right), \quad (2.7)$$

then

$$\xi_i(t) := \frac{\pi_i(t)V^\pi(t)}{S_i(t)}, \quad i = 1, \dots, d, \quad \text{and} \quad \eta(t) := \frac{(1 - \sum_{i=1}^d \pi_i(t))V^\pi(t)}{B(t)} \quad (2.8)$$

defines a self-financing strategy with wealth V^π .

The preceding lemma justifies the following definition.

Definition 2.6. A function $\pi : [0, \infty) \rightarrow \mathbb{R}^d$ is called a *portfolio* for S if it is an admissible integrand for $\log S$ and if

$$\pi_1(t) + \dots + \pi_d(t) = 1, \quad t \geq 0. \quad (2.9)$$

A portfolio for S is called *long-only* if $\pi_i(t) \geq 0$ for all t . The function V^π from (2.7) will be called the *portfolio value* of π with unit initial wealth.

As in [13, Section 2], we normalize the market, i.e., we suppose that at any time t each stock has only one share outstanding. Then the stock prices $S_i(t)$ represent the capitalizations of the individual companies.

Lemma 2.7. *The quantities*

$$\mu_i(t) := \frac{S_i(t)}{S_1(t) + \dots + S_d(t)}, \quad i = 1, \dots, d,$$

form a portfolio for S , called the market portfolio. The corresponding portfolio value with unit initial wealth is given by

$$V^\mu(t) = \frac{S_1(t) + \dots + S_d(t)}{S_1(0) + \dots + S_d(0)}. \quad (2.10)$$

As in [13] and [29], we can now consider the pathwise dynamics of a portfolio that is generated by a *portfolio-generating function*. These are $C^{2,1}$ -functions that may depend on the current composition $\mu(t)$ of the market portfolio and on certain other paths $A_0(t), \dots, A_m(t)$ of (locally) finite variation. Examples for these additional paths include time, $A_0(t) = t$, running maximum, $A_j(t) = \max_{s \leq t} S_i(s)$, moving average, $A_k(t) = \frac{1}{\theta} \int_{t-\theta}^t S_i(0 \vee s) ds$, or realized quadratic variation, $A_\ell(t) = [S_i](t)$; see also Section 3. We denote by Δ^d the standard simplex in \mathbb{R}^d and let $\Delta_+^d := \Delta^d \cap \mathbb{R}_+^d$.

Lemma 2.8. *Let $V \supset \Delta_+^d$ and $W \subset \mathbb{R}^m$ be open sets, G a strictly positive function in $C^{2,1}(U, W)$, and $A : [0, \infty) \rightarrow W$ a continuous function whose restriction to compact intervals is of finite variation. Then*

$$\pi_i(t) = \mu_i(t) + \mu_i(t) \frac{\partial}{\partial x_i} \log G(\mu(t), A(t)) - \sum_{j=1}^d \mu_i(t) \mu_j(t) \frac{\partial}{\partial x_j} \log G(\mu(t), A(t)), \quad 1 \leq i \leq d, \quad (2.11)$$

is a portfolio for S and called the portfolio generated by G . Moreover, π is an admissible integrand for $\log \mu$, where $\log \mu(t) := (\log \mu_1(t), \dots, \log \mu_d(t))$ is defined in analogy to $\log S$.

The following theorem extends the “master equation” from Fernholz [9] and Strong [29] to our strictly pathwise setting.

Theorem 2.9 (Pathwise master equation). *For G as in Lemma 2.8, the relative wealth of the portfolio π generated by G with respect to the market portfolio is given by the following master equation*

$$\log \left(\frac{V^\pi(T)}{V^\mu(T)} \right) = \log \left(\frac{G(\mu(T), A(T))}{G(\mu(0), A(0))} \right) + \mathfrak{g}([0, T]) + \mathfrak{h}([0, T]), \quad 0 \leq T < \infty,$$

where the (possibly signed) Radon measures \mathfrak{g} and \mathfrak{h} are given by

$$\mathfrak{g}(dt) := -\frac{1}{2} \sum_{i,j=1}^d \frac{G_{x_i x_j}(\mu(t), A(t))}{G(\mu(t), A(t))} d[\mu_i, \mu_j](t) \quad \text{and} \quad \mathfrak{h}(dt) := -\sum_{k=1}^m \frac{G_{a_k}(\mu(t), A(t))}{G(\mu(t), A(t))} dA_k(t).$$

2.2 The master formula for path-dependent portfolios

Our goal in this section is to extend the pathwise portfolio theory from Section 2.1 to portfolios that are path-dependent and to prove a corresponding master formula. The main difficulty we are encountering here is that the pathwise Itô integral is then no longer the limit of ordinary Riemann sums as in (2.3). Instead, the integrands in the approximating “Riemann sums” (A.5) involve approximations of the integrator path. Therefore, one either needs to make strong regularity assumptions as in [1, Theorem 3.2] so as to be able to replace (A.5) by standard Riemann sums, or to retain the functional dependence

in the integrands. Here, we take the latter route. It will require a proof strategy that is different from the one we use to prove Theorem 2.9.

Let $S \in QV_+^d$ and recall from the Appendix the basic definitions and notations from pathwise functional Itô calculus. To keep notations simple and close to [7, 3, 27], we will work in the sequel with a fixed and finite time horizon $T > 0$. It is easy to extend our results to the case $T = \infty$ by using localization. By $D([0, T], U)$ we denote the usual Skorokhod space of U -valued càdlàg functions. For an open set $V \subset \mathbb{R}^m$ and a non-anticipative functional $F : [0, T] \times D([0, T], U) \times CBV([0, T], V) \rightarrow \mathbb{R}$, we denote by

$$\mathcal{D}_0 F, \dots, \mathcal{D}_m F$$

the corresponding horizontal derivatives defined in (A.1) and by

$$\nabla_X F = (\partial_i F)_{i=1, \dots, d}$$

the vertical derivative defined in (A.3), provided that all these derivatives exist. The second partial vertical derivatives of F will be denoted by $\partial_{ij} F$. We will also need the space $\mathbb{C}_c^{1,2}(U, W)$, whose definition is recalled in the Appendix. The following definition is based on a suggestion in [1].

Definition 2.10. Let $U \subset \mathbb{R}^d$ be an open set. A functional $\xi : [0, T] \times D([0, T], U) \rightarrow \mathbb{R}^d$ is called an *admissible functional integrand on U* if there exist $m \in \mathbb{N}$, an open set $W \subset \mathbb{R}^m$, $A \in CBV([0, T], W)$, and $F \in \mathbb{C}_c^{1,2}(U, W)$ such that $\xi(t, X) = \nabla_X F(t, X, A)$ for $t \in [0, T]$.

If ξ is an admissible functional integrand on $U \subset \mathbb{R}^d$ and $X \in QV^d$ is U -valued, then the Itô integral of $\xi(t, X)$ against X can be defined through (A.5). Moreover, it can be shown that $t \mapsto \int_0^t \xi(s, X) dX(s)$ is a continuous function [27, Lemma 3.1 (c)].

Definition 2.11. An admissible functional integrand π on \mathbb{R}^d is called a *functional portfolio* if $\pi_1(t, X) + \dots + \pi_d(t, X) = 1$ for all $t \in [0, T]$ and $X \in D([0, T], \mathbb{R}^d)$ and if $t \mapsto \int_0^t \pi(s, \log S) d \log S$ admits the quadratic variation

$$\left[\int_0^t \pi(s, \log S) d \log S \right] (t) = \sum_{i,j=1}^d \int_0^t \pi_i(s, \log S) \pi_j(s, \log S) d[\log S_i, \log S_j](s) \quad (2.12)$$

In contrast to the case without path dependence, the validity of (2.12) is no longer clear *a priori* in the path-dependent setting, which is why (2.12) is included as a requirement in the preceding definition. See [1, Theorem 2.1] for sufficient conditions on S and π under which (2.12) holds.

Remark 2.12. If π is a functional portfolio, then

$$\frac{\pi(t, \log Y)}{Y(t)} := \left(\frac{\pi_1(t, \log Y)}{Y_1(t)}, \dots, \frac{\pi_d(t, \log Y)}{Y_d(t)} \right)^\top, \quad Y \in D([0, T], \mathbb{R}_+^d),$$

is an admissible functional integrand on \mathbb{R}_+^d and we have the following change of variables formula:

$$\int_0^t \pi(s, \log S) d \log S(s) = \int_0^t \frac{\pi(s, \log S)}{S(s)} dS(s) - \frac{1}{2} \sum_{i=1}^d \int_0^t \frac{\pi_i(s, \log S)}{(S_i(s))^2} d[S_i, S_i](s).$$

This follows from [27, Corollary 2.1]. In addition, (2.5) and the associativity of the Riemann–Stieltjes integral (see, e.g., [31, Theorem I.6b]) yield the identity

$$\int_0^t \pi_i(s, \log S) \pi_j(s, \log S) d[\log S_i, \log S_j](s) = \int_0^t \frac{\pi_i(s, \log S) \pi_j(s, \log S)}{S_i(s) S_j(s)} d[S_i, S_j](s).$$

If π is a functional portfolio, we can in analogy to (2.6) and (2.7) define the corresponding *portfolio value with unit initial wealth* as

$$\begin{aligned} V^\pi(t) &:= \exp \left(\int_0^t \pi(s, \log S) d \log S(s) \right. \\ &\quad \left. + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \left(\delta_{ij} \pi_i(s, \log S) - \pi_i(s, \log S) \pi_j(s, \log S) \right) d[\log S_i \log S_j](t) \right) \\ &= \exp \left(\int_0^t \frac{\pi(s, \log S)}{S(s)} dS(s) - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\pi_i(s, \log S) \pi_j(s, \log S)}{S_i(s) S_j(s)} d[S_i, S_j](s) \right) \end{aligned} \quad (2.13)$$

where δ_{ij} is the Kronecker delta and where we have used Remark 2.12 for the second identity. However, in our path-dependent setting, it is not evident that $V^\pi(t)$ can be obtained, in analogy to Lemma 2.5, as the wealth of a self-financing trading strategy. For such a statement, a functional extension of V^π is needed.

Lemma 2.13. *For a functional portfolio π , let $W \subset \mathbb{R}^m$ be open, $F \in \mathbb{C}_c^{1,2}(\mathbb{R}^d, W)$, and $A \in CBV([0, T], W)$ be such that $\pi(t, X) = \nabla_X F(t, X, A)$. Define for $X \in D([0, T], \mathbb{R}_+^d)$*

$$\bar{V}^\pi(t, X, (A, B^\pi)) := \exp \left(F(t, \log X, A) - F(0, \log X, A) - B^\pi(t) \right),$$

where, for $A_0(t) = t$,

$$\begin{aligned} B^\pi(t) &= \sum_{i=0}^m \int_0^t \mathcal{D}_i F(s, \log S, A) dA_i(s) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \left((\partial_{ij} F)(s, \log S, A) + \pi_i(s, \log S) \pi_j(s, \log S) - \delta_{ij} \pi_i(s, \log S) \right) d[\log S_i, \log S_j](s). \end{aligned} \quad (2.14)$$

Then $\bar{V}^\pi \in \mathbb{C}_c^{1,2}(\mathbb{R}_+^d, W \times \mathbb{R})$, the functional $\xi : [0, T] \times D([0, T], \mathbb{R}_+^d) \rightarrow \mathbb{R}^d$ defined through

$$\xi_i(t, X) = \frac{\pi_i(t, \log X) V^\pi(t, X, (A, B))}{X_i(t)}, \quad i = 1, \dots, d,$$

is an admissible integrand on \mathbb{R}_+^d , and

$$\bar{V}^\pi(t, S, (A, B^\pi)) = 1 + \int_0^t \xi(s, S) dS(s) = V^\pi(t) \quad \text{for } 0 \leq t \leq T. \quad (2.15)$$

We will also need the following functional extension of the market portfolio. For $X \in D([0, T], \mathbb{R}^d)$, let

$$\bar{\mu}_i(t, X) := \frac{e^{X_i(t)}}{e^{X_1(t)} + \dots + e^{X_d(t)}}, \quad i = 1, \dots, d. \quad (2.16)$$

Then $\bar{\mu}(t, X) \in \Delta_+^d$, and $\bar{\mu}(t, \log S)$ is equal to the market portfolio $\mu(t)$ from Lemma 2.7.

Lemma 2.14. *The functional $\bar{\mu}$ is a functional portfolio, and*

$$\bar{V}^{\bar{\mu}}(t, X, B^{\bar{\mu}}) = \frac{e^{X_1(t)} + \dots + e^{X_d(t)}}{e^{X_1(0)} + \dots + e^{X_d(0)}}, \quad X \in D([0, T], \mathbb{R}^d).$$

The following lemma extends Lemma 2.8 to our present functional setting.

Lemma 2.15. *Let $U \supset \Delta_+^d$ and $W \in \mathbb{R}^m$ be open sets. For strictly positive $G \in \mathbb{C}_c^{1,2}(U, W)$ and $A \in \text{CBV}([0, T], W)$ we let*

$$g_i(t, X) := \partial_i \log G(t, X, A). \quad (2.17)$$

Then

$$\pi_i(t, X) := \bar{\mu}_i(t, X) \left(1 + g_i(t, \bar{\mu}(\cdot, X)) - \sum_{j=1}^d \bar{\mu}_j(t, X) g_j(t, \bar{\mu}(\cdot, X)) \right) \quad (2.18)$$

is an admissible integrand on \mathbb{R}^d satisfying $\pi_1 + \dots + \pi_d = 1$.

If π defined by (2.18) satisfies (2.12), then it is a functional portfolio and will be called the *functional portfolio generated by G* . Now we can state the path-dependent version of Theorem 2.9.

Theorem 2.16. (Path-dependent pathwise master formula) *Let G and π be as in Lemma 2.15 and assume that (2.12) holds. Let furthermore V^π be as in (2.13) and V^μ be the portfolio value (2.10) of the market portfolio μ . Then*

$$\log \left(\frac{V^\pi(T)}{V^\mu(T)} \right) = \log \left(\frac{G(T, \mu, A)}{G(0, \mu, A)} \right) + \mathfrak{g}([0, T]) + \mathfrak{h}([0, T]), \quad 0 \leq T < \infty,$$

where, for $A_0(t) := t$,

$$\mathfrak{g}(dt) := -\frac{1}{2} \sum_{i,j=1}^d \frac{\partial_{ij} G(t, \mu, A)}{G(t, \mu, A)} d[\mu_i, \mu_j](t) \quad \text{and} \quad \mathfrak{h}(dt) := -\sum_{k=0}^m \frac{\mathcal{D}_k G(t, \mu, A)}{G(t, \mu, A)} dA_k(t).$$

3 Examples

In this section, we will discuss a class of examples, along with an empirical analysis. The general idea is to use portfolio-generating functions of a convex combination of the market portfolio $\mu(t)$ and its moving average defined by

$$\alpha(t) = \frac{1}{\theta} \int_{t-\theta}^t \mu(0 \vee s) ds,$$

where $\theta > 0$ is given. The portfolio-generating function is then of the form $\varphi(\lambda\mu(t) + (1-\lambda)\alpha(t))$, where φ is a strictly positive and twice continuously differentiable function defined on an open and convex neighborhood U of Δ_+^d . It can be considered within the contexts of both Section 2.1 and Section 2.2. In the context of the first section, $\alpha(t)$ may be regarded as an additional component of bounded variation, and so we may work with the function

$$G(\mu(t), \alpha(t)) = \varphi(\lambda\mu(t) + (1-\lambda)\alpha(t)). \quad (3.1)$$

In the context of Section 2.2, however, we may consider a non-anticipative functional \widehat{G} that depends on the *path* of μ via α . To make this latter idea precise, we let for $X \in D([0, T], U)$,

$$\widehat{G}(t, X) := \varphi \left(\lambda X(t) + \frac{1-\lambda}{\theta} \int_{t-\theta}^t X(0 \vee s) ds \right).$$

Then $\widehat{G} \in \mathbb{C}_c^{1,2}(U, \emptyset)$ and

$$G(t, \mu(t)) = \widehat{G}(t, \mu). \quad (3.2)$$

The following proposition states in particular that the two descriptions (3.1) and (3.2) lead to the same results.

Proposition 3.1. *Let $\pi(t)$ be the portfolio generated by the function G in (3.1) and $\widehat{\pi}(t, X)$ as in Lemma 2.15 for \widehat{G} from (3.2). Then, for*

$$g_i(t) := \frac{\lambda \varphi_{x_i}(\lambda \mu(t) + (1 - \lambda)\alpha(t))}{\varphi(\lambda \mu(t) + (1 - \lambda)\alpha(t))}, \quad i = 1, \dots, d,$$

we have

$$\widehat{\pi}_i(t, \log S) = \pi_i(t) = \mu_i(t) \left(1 + g_i(t) - \sum_{j=1}^d \mu_j(t) g_j(t) \right). \quad (3.3)$$

Moreover, $\widehat{\pi}$ satisfies (2.12) and is hence a functional portfolio. Furthermore, we have $V^\pi(T) = V^{\widehat{\pi}}(T)$ and

$$\log \left(\frac{V^\pi(T)}{V^\mu(T)} \right) = \frac{\varphi(\lambda \mu(T) + (1 - \lambda)\alpha(T))}{\varphi(\lambda \mu(0) + (1 - \lambda)\alpha(0))} + \mathfrak{g}([0, T]) + \mathfrak{h}([0, T]), \quad (3.4)$$

where

$$\begin{aligned} \mathfrak{h}([0, T]) &= -\frac{1 - \lambda}{\lambda} \sum_{i=1}^d \int_0^T g_i(t) \alpha'_i(t) dt \\ \mathfrak{g}([0, T]) &= -\frac{\lambda^2}{2} \sum_{i,j=1}^d \int_0^T \frac{\varphi_{x_i, x_j}(\lambda \mu(t) + (1 - \lambda)\alpha(t))}{\varphi(\lambda \mu(t) + (1 - \lambda)\alpha(t))} d[\mu_i, \mu_j](t) \end{aligned} \quad (3.5)$$

In the following examples, we analyze empirically the situation for particular choices for φ . To this end, it will be convenient to use the shorthand notation

$$\widetilde{\mu}_i(t) = \lambda \mu_i(t) + (1 - \lambda)\alpha_i(t) \quad i = 1, \dots, d.$$

Example 3.2 (Geometric average). Consider $\varphi(x) = \prod_{i=1}^d x_i^{1/d}$, which generates the portfolio

$$\pi_i(t) = \left(1 + \frac{\lambda}{d \widetilde{\mu}_i(t)} - \sum_{j=1}^d \frac{\lambda \mu_j(t)}{d \widetilde{\mu}_j(t)} \right) \mu_i(t), \quad i = 1, \dots, d. \quad (3.6)$$

By Proposition 3.1, we have, with δ_{ij} denoting again the Kronecker delta,

$$\begin{aligned} \mathfrak{g}([0, T]) &= \frac{\lambda^2}{2d} \sum_{i,j=1}^d \int_0^T \left(\sum_{i=1}^d \frac{\delta_{ij}}{(\widetilde{\mu}_i(t))^2} - \frac{1}{d \widetilde{\mu}_i(t) \widetilde{\mu}_j(t)} \right) d[\mu_i, \mu_j](t) \\ \mathfrak{h}([0, T]) &= -\frac{1 - \lambda}{d} \sum_{i=1}^d \int_0^T \frac{\alpha'_i(t)}{\widetilde{\mu}_i(t)} dt. \end{aligned}$$

In Figures 1 and 2, we display the results of an empirical analysis of such a geometrically weighted portfolio with the parameters $\theta = 60$ days and $\lambda = 0, 7$. We used the stock data base from Reuters Datastream and considered those 30 stocks that were the constituents of the DAX in May 2015. Our 30 time series represent daily closing prices for those stocks during the period January 31, 2005 – May 15, 2015, and we work along the finite partition formed by the corresponding time points. In Figure 1 we see the relative performance of the portfolio (3.6) with respect to this stock index. In Figure 2 we see the decomposition of the curve(s) in the left-hand panel according to the master equation (3.4). The blue curve is the change in the generating functional, while the red and the green ones are the

respective drift terms. Each curve shows the cumulative value of the daily changes induced in the corresponding quantities by capital gains and losses. As can be seen, the cumulative second-order drift term was the dominant part over the period, with a total contribution of about 15 percentage points to the relative return. The second-order drift term was quite stable over the considered period, with an exception during the period around the financial crisis of 2008.

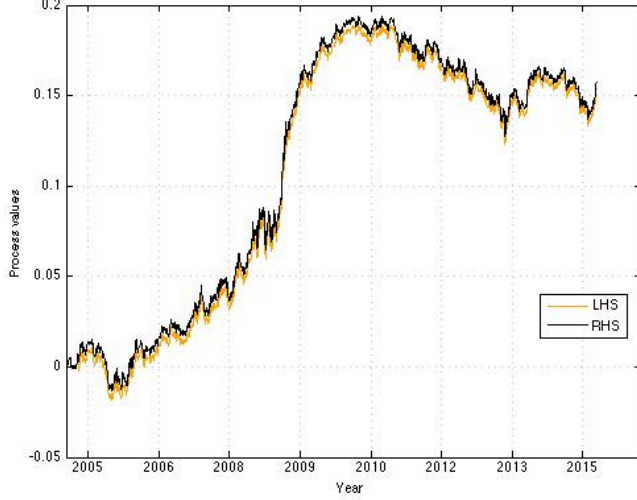


Figure 1: LHS vs. RHS of the master formula (3.4) for geometric average

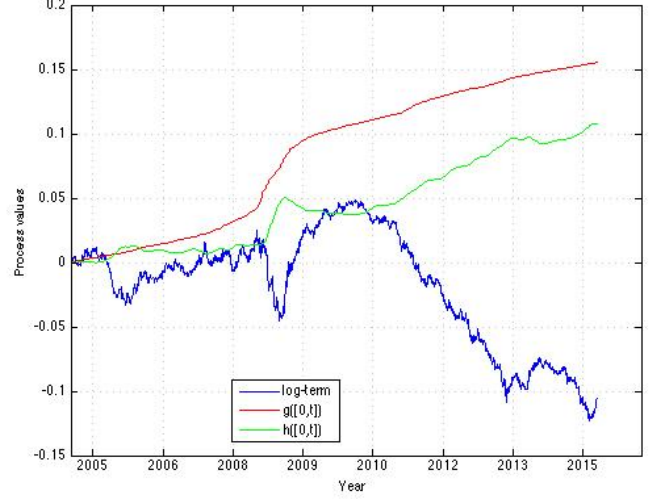


Figure 2: Componentwise representation of the RHS of (3.4) for geometric average

Example 3.3 (Functional diversity weighting). Here we take $p \in (0, 1)$ and $\varphi(x) = (\sum_{i=1}^d x_i^p)^{1/p}$, which generates the portfolio with weights

$$\pi_i(t) = \left(1 + \frac{\lambda (\tilde{\mu}_i(t))^{p-1}}{\sum_{k=1}^d (\tilde{\mu}_k(t))^p} - \sum_{j=1}^d \frac{\lambda \mu_j(t) (\tilde{\mu}_j(t))^{p-1}}{\sum_{k=1}^d (\tilde{\mu}_k(t))^p} \right) \mu_i(t), \quad i = 1, \dots, d. \quad (3.7)$$

By Proposition 3.1, we have

$$\begin{aligned} \mathfrak{g}([0, T]) &= \frac{\lambda^2(1-p)}{2} \sum_{i,j=1}^d \int_0^T \left(\frac{\delta_{ij} (\tilde{\mu}_i(t))^{p-2}}{\sum_{k=1}^d (\tilde{\mu}_k(t))^p} - \frac{(\tilde{\mu}_i(t))^{p-1} (\tilde{\mu}_j(t))^{p-1}}{(\sum_{k=1}^d (\tilde{\mu}_k(t))^p)^2} \right) d[\mu_i, \mu_j](t) \\ \mathfrak{h}([0, T]) &= - \int_0^T \frac{1-\lambda}{\sum_{k=1}^d (\tilde{\mu}_k(t))^p} \sum_{i=1}^d (\tilde{\mu}_i(t))^{p-1} \alpha'_i(t) dt. \end{aligned}$$

For our empirical analysis for the diversity-weighted portfolio, we used again Reuters Datastream to obtain our data base; we considered 207 fixed stocks that were among the constituents of the S&P 500 index in May 2015. Our 207 time series represent monthly average prices for the period February 2, 1973 – April 2, 2015. The results of a simulation of the portfolio (3.7) using the parameters $\theta = 12$ months, $\lambda = 0,6$, and $p = 0,1$ are presented below. Figure 3 shows the relative performance of this portfolio with respect to this filtered index, and Figure 4 shows its decomposition in the three components according to the master equation (3.4). Each curve represents the cumulative value of the monthly changes induced in the corresponding quantities by capital gains and losses, but in contrast to Example 3.2, it is now the cumulative change in the generating functional that yields the dominant

part over the period, with a total contribution of about 70 percentage points to the relative return. The second-order drift term was quite stable over the period with a total contribution of about 30 percentage points, whereas the horizontal drift term has a negative contribution.

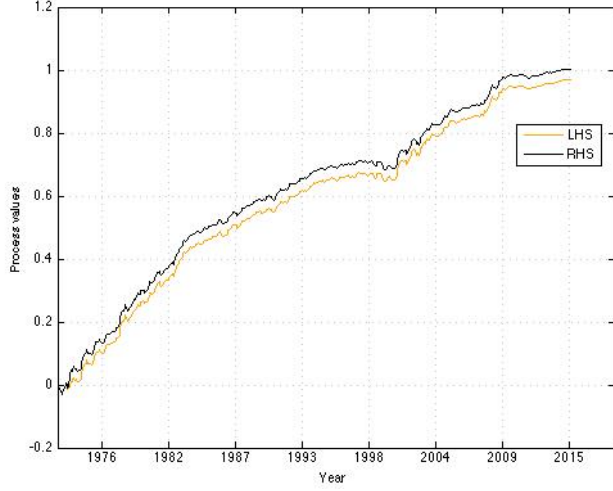


Figure 3: LHS vs. RHS of the master formula (3.4) for diversity weighting

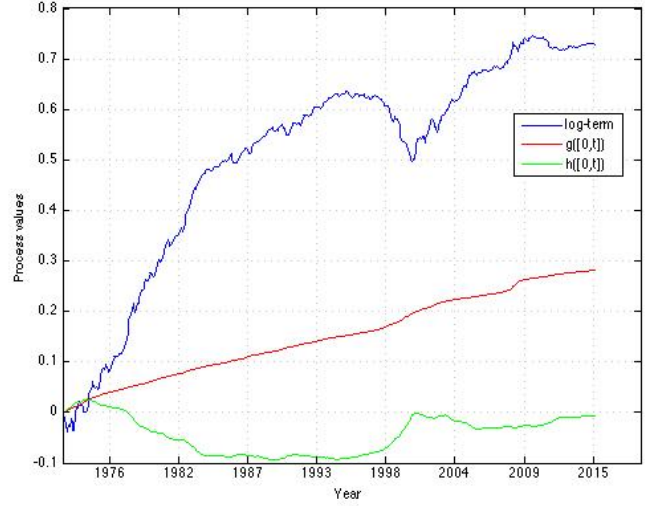


Figure 4: Componentwise representation of the RHS of (3.4) for diversity weighting

Example 3.4 (Functional entropy-weighting). Here we take $\varphi(x) = -\sum_{i=1}^d x_i \log x_i$, which generates the portfolio with weights

$$\pi_i(t) = \left(1 - \frac{\lambda \log(\tilde{\mu}_i(t))}{\varphi(\tilde{\mu}(t))} + \sum_{j=1}^d \frac{\lambda \mu_j(t) \log(\tilde{\mu}_j(t))}{\varphi(\tilde{\mu}(t))} \right) \mu_i(t), \quad (3.8)$$

and associated drift rates

$$\begin{aligned} \mathfrak{g}([0, T]) &= -\lambda^2 \sum_{i=1}^d \int_0^T \frac{1}{\varphi(\tilde{\mu}(t)) \tilde{\mu}_i(t)} d[\mu_i, \mu_i](t), \\ \mathfrak{h}([0, T]) &= (1 - \lambda) \sum_{i=1}^d \int_0^T \frac{1 + \log \tilde{\mu}_i(t)}{\varphi(\tilde{\mu}(t))} \alpha'_i(t) dt. \end{aligned}$$

Using the same data set as in Example 3.3, we conducted an empirical analysis of the portfolio (3.8) with respect to the filtered index taking the parameters $\theta = 6$ months and $\lambda = 0, 9$, which is presented in Figure 5 and Figure 6, respectively. Note that compared to the entropy-weighted portfolio in the classical Fernholz' setting (see [13, Example11.1]) the additional drift term $\mathfrak{h}([0, T])$ can be either positive or negative. In this sense, the horizontal drift term $\mathfrak{h}([0, T])$ may be regarded as quantifying the trade-off between outperforming the market and the greater flexibility within the more general framework. This is also supported by real market data, as can be seen in Figure 5 and Figure 6. Indeed, the cumulative second-order drift term is continually increasing. Moreover, the horizontal drift term does not seem to have a large influence on the relative performance of the entropy-weighted portfolio, with a total contribution of less than 1 percentage point. Thus, entropy-weighting should significantly outperform the market on the considered time interval, which is confirmed in the case study of Figure 5.

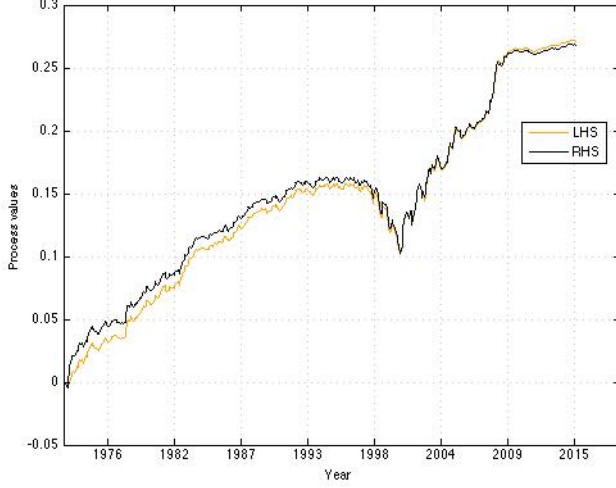


Figure 5: LHS vs. RHS of the master formula (3.4) for entropy weighting

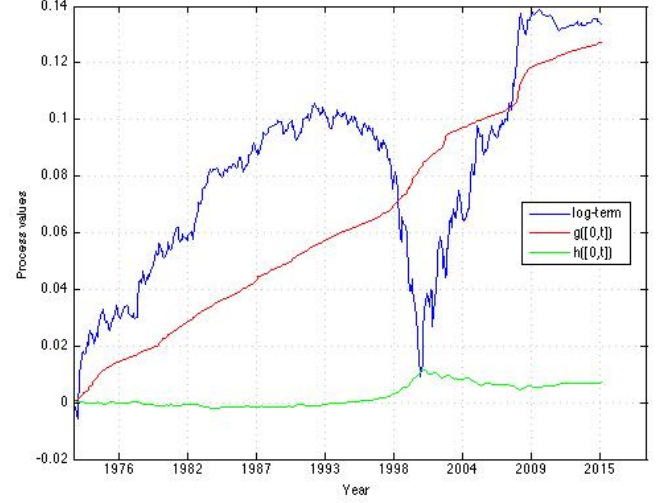


Figure 6: Componentwise representation of the RHS of (3.4) for entropy weighting

4 Proofs

4.1 Proofs of the lemmas from Section 2.1

Proof of Lemma 2.3. We first let $Z : [0, \infty) \rightarrow \mathbb{R}_+$ be any continuous path. It follows from [28, Proposition 2.2.10] that $Z \in QV_+^1$ if and only $\log Z \in QV^1$ and that, in this case,

$$[\log Z](t) = \int_0^t \frac{1}{Z(s)^2} d[Z](s). \quad (4.1)$$

Now suppose that $S \in QV_+^d$. By the polarization identity (2.2), we may conclude that $\log S \in QV^d$, if we can show that $f(S_i(t), S_j(t))$ belongs to QV^1 for all i, j , where $f(x_1, x_2) := \log x_1 + \log x_2$. The pathwise Itô formula from [14] applied to $f(S_i(t), S_j(t))$ yields that

$$\begin{aligned} f(S_i(t), S_j(t)) &= f(S_i(0), S_j(0)) + \int_0^t \nabla f(S_i(s), S_j(s)) d \begin{pmatrix} S_i(s) \\ S_j(s) \end{pmatrix} \\ &\quad + \frac{1}{2} \sum_{k, \ell=1}^2 \int_0^t f_{x_k, x_\ell}(S_i(s), S_j(s)) d[S_k, S_\ell](s). \end{aligned}$$

Remark 8 and Proposition 12 from [24] now imply that $f(S_i, S_j) \in QV^1$ with quadratic variation

$$\begin{aligned} [f(S_i, S_j)](t) &= \sum_{k, \ell=1}^2 \int_0^t f_{x_k}(S_i(s), S_j(s)) f_{x_\ell}(S_i(s), S_j(s)) d[S_k, S_\ell](s) \\ &= \int_0^t \frac{1}{S_i(s)^2} d[S_i](s) + \int_0^t \frac{1}{S_j(s)^2} d[S_j](s) + 2 \int_0^t \frac{1}{S_i(s) S_j(s)} d[S_i, S_j](s). \end{aligned}$$

Therefore, $\log S \in QV^d$ and (2.5) now follow from (2.2) and (4.1). That $\log S \in QV^d$ implies $S \in QV_+^d$ follows by an analogous argument in which the logarithm is replaced by the exponential function. \square

Proof of Lemma 2.4. Let $X := \log S$ and write $\exp(X)$ for the path with components e^{X_1}, \dots, e^{X_d} . That is, $\exp(X) = S$. Now suppose that π is an admissible integrand for X and $T > 0$ is given. Then there exist $n \in \mathbb{N}$, open sets $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^n$, a function $f \in C^{2,1}(U, V)$, and $A \in CBV([0, T], V)$ such that $X(t) \in U$ and $\xi(t) = \nabla_x f(X(t), A(t))$ for $0 \leq t \leq T$. We let $g(z, a) := f(\log z, a)$. Then $g \in C^{2,1}(\tilde{U}, V)$ for the open set $\tilde{U} = \{\exp(x) \mid x \in U\}$ and $g_{z_i}(z, a) = f_{x_i}(\log z, a)/z_i$. Thus, $g_{z_i}(S(t), A(t)) = \pi_i(t)/S_i(t)$, $i = 1, \dots, d$, are the components of an admissible integrand for S . The converse assertion follows analogously. To prove the formula (2.6), we use Föllmer's pathwise Itô formula, which yields

$$d \log S_i(t) = \frac{1}{S_i(t)} dS_i(t) - \frac{1}{2S_i(t)^2} d[S_i](t). \quad (4.2)$$

Thus, the associativity rule [24, Theorem 13] implies (2.6). \square

Proof of Lemma 2.5. It is clear from Lemma 2.4 that the Itô integral in (2.7) is well defined. Moreover, the two rightmost integrals in (2.7) exist as Riemann–Stieltjes integrals, because $\pi(t)/S(t)$ as admissible integrand for S is in particular a continuous function of t . Therefore, V^π is well defined and indeed the wealth of (ξ, η) , since $V^\pi(t) = \xi(t) \cdot S(t) + \eta(t)B(t)$ by (2.8). Letting $Z(t) := \int_0^t \frac{\pi(s)}{S(s)} dS(s)$ and $R(t) := \int_0^t (1 - \sum_{i=1}^d \pi(s))r(s) ds$, (2.7) can be re-written as $V^\pi(t) = \exp(Z(t) - \frac{1}{2}[Z](t) + R(t))$. Applying the pathwise Itô formula thus gives $dV^\pi(t) = V^\pi(t) dZ(t) + V^\pi(t) dR(t)$. The associativity rule [24, Theorem 13] thus implies that $V^\pi(t)\pi(t)/S(t) = \xi(t)$ is an admissible integrand for S and that $V^\pi(t) dZ(t) = \xi(t) dS(t)$. Moreover, the identity $V^\pi(t) dR(t) = \eta(t) dB(t)$ follows from the associativity of the Riemann–Stieltjes integral (see, e.g., [31, Theorem I.6b]). \square

Proof of Lemma 2.7. For $x = (x_1, \dots, x_d)$ let $h(x) := \log(e^{x_1} + \dots + e^{x_d})$. Then $h_{x_i}(x) = e^{x_i - h(x)}$ and so $\nabla h(\log S(t)) = \mu(t)$. Thus, μ is an admissible integrand for $\log S$. Moreover, the fact that $\mu_1(t) + \dots + \mu_d(t) = 1$ is obvious. Therefore, μ is a portfolio in the sense of Definition 2.6.

To prove the formula for V^μ , let $g(x_1, \dots, x_d) := \log(x_1 + \dots + x_d)$ for $x_i > 0$. The pathwise Itô formula yields

$$g(S(t)) - g(S(0)) = \int_0^t \frac{\mu(s)}{S(s)} dS(s) - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\mu_i(s)\mu_j(s)}{S_i(s)S_j(s)} d[S_i, S_j](s).$$

By Lemma 2.5, the right-hand side is equal to $\log V^\mu(t) - \log V^\mu(0)$, which proves the claim. \square

Proof of Lemma 2.8. We clearly have $\pi_1(t) + \dots + \pi_d(t) = 1$. We show next that π is an admissible integrand for $\log S$ and, hence, a portfolio for S . To this end, let h be as in the proof of Lemma 2.7, so that $\mu(t) = \nabla h(X(t))$, where again $X(t) = \log S(t)$. Observe that $\nabla_x h(x) \in \Delta_+^d$ for all $x \in \mathbb{R}^d$. Therefore, we may define

$$\gamma(x, a) := \log G(\nabla h(x), a), \quad x \in \mathbb{R}^d, a \in W.$$

Then

$$\gamma_{x_i}(x, a) = \sum_{j=1}^d \left(\frac{\partial}{\partial x_j} \log G \right) (\nabla h(x), a) h_{x_i, x_j}(x).$$

Since

$$h_{x_i, x_j}(X(t)) = \delta_{ij} e^{X_i(t) - h(X(t))} - e^{X_i(t) + X_j(t) - 2h(X(t))} = \delta_{ij} \mu_i(t) - \mu_i(t)\mu_j(t),$$

we get that

$$\tilde{\pi}(t) := \gamma_{x_i}(X(t), A(t)) = \mu_i(t) \frac{\partial}{\partial x_i} \log G(\mu(t), A(t)) - \sum_{j=1}^d \mu_i(t) \mu_j(t) \frac{\partial}{\partial x_j} \log G(\mu(t), A(t)) \quad (4.3)$$

is an admissible integrand for $\log S$. With Lemma 2.7 we can conclude now that $\pi = \mu + \tilde{\pi}$ is an admissible integrand for $\log S$.

Now we prove that π is an admissible integrand for $\log \mu$. To this end, observe first that $h(\log \mu(t)) = \log \sum_i \mu_i(t) = 0$. Hence, $h_{x_i}(\log \mu(t)) = e^{\log \mu_i(t) - h(\log \mu(t))} = \mu_i(t)$, and it follows that $\nabla h(\log S(t)) = \mu(t) = \nabla h(\log \mu(t))$. Therefore, the identity (4.3) also holds for $X := \log \mu$, and it follows that $\tilde{\pi}$ is an admissible integrand also for $\log \mu$. Hence, it only remains to prove that μ is an admissible integrand for $\log \mu$. To this end, let $f(x) := e^{h(x)}$ and note that $f_{x_i}(x) = e^{x_i}$. Thus, $\nabla f(\log \mu(t)) = \mu(t)$, and so μ is indeed an admissible integrand for $\log \mu$. \square

4.2 Pathwise portfolio dynamics without path dependence

In this section, we analyze the dynamics of portfolios and portfolio values. In standard stochastic portfolio theory, many of these results are well known (see, e.g., [13]), but occasionally additional care is needed in our pathwise setting. The results in this section will serve as preparation for the proof of the functional master formula, Theorem 2.9, but they are also of independent interest.

Throughout this section, π will be a portfolio for $S \in QV_+^d$ in the sense of Definition 2.6 and V^π the corresponding portfolio value with unit initial wealth as given in (2.7). Due to the condition that $\pi^1(t) + \dots + \pi^d(t) = 1$, formula (2.7) simplifies to

$$V^\pi(t) = \exp \left(\int_0^t \frac{\pi(s)}{S(s)} dS(s) - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\pi_i(s) \pi_j(s)}{S_i(s) S_j(s)} d[S_i, S_j](s) \right). \quad (4.4)$$

In our model-free version of portfolio theory, we do not wish to make assumptions on the structure of the covariations $[S_i, S_j]$ apart from their existence (2.1). In particular, we do not assume that $[S_i, S_j](t)$ is absolutely continuous in t . Growth rates and covariances, which in [13] can be taken as functions, therefore need to be modeled as measures.

Definition 4.1. The *covariance* of the stocks in the market is described by the positive semidefinite matrix-valued Radon measure $a = (a_{ij})_{1 \leq i, j \leq d}$ defined as

$$a_{ij}(dt) := d[\log S_i, \log S_j](t) = \frac{1}{S_i(t) S_j(t)} d[S_i, S_j](t), \quad i, j = 1, \dots, d.$$

The *excess growth rate* of a portfolio π is defined as the signed Radon measure

$$\gamma_\pi^*(dt) := \frac{1}{2} \left(\sum_{i=1}^d \pi_i(t) a_{ii}(dt) - \sum_{i,j=1}^d \pi_i(t) \pi_j(t) a_{ij}(dt) \right).$$

For any portfolio π , we define the *covariances* of the individual stocks *relative to the portfolio* π as follows for $i, j = 1, \dots, d$,

$$\tau_{ij}^\pi(dt) := (\pi(t) - e_i)^\top a(dt) (\pi(t) - e_j). \quad (4.5)$$

Lemma 4.2. *We have*

$$\log V^\pi(t) = \int_0^t \pi(s) d \log S(s) + \gamma_\pi^*([0, t]).$$

Proof. Using (4.2), Lemma 2.3, the fact that $V^\pi(0) = 1$, and the associativity of the Stieltjes and Föllmer integrals from [31, Theorem I.6 b] and [24, Theorem 13], respectively, we get

$$\log V^\pi(t) = \int_0^t \pi(s) d \log S(s) + \frac{1}{2} \sum_{i=1}^d \int_0^t \pi_i(s) d[\log S_i](s) - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \pi_i(s) \pi_j(s) d[\log S_i, \log S_j](s),$$

which implies the assertion via the definition of γ_π^* . \square

Applying the preceding lemma to the market portfolio yields the following results.

Lemma 4.3. *We have the following formulas for the dynamics of the market weights μ_i .*

- (a) $d \log \mu_i(t) = (e_i - \mu(t)) d \log S(t) - \gamma_\mu^*(dt).$
- (b) $\tau_{ij}^\mu([0, t]) = [\log \mu_i, \log \mu_j](t)$ and $d[\mu_i, \mu_j](t) = \mu_i(t) \mu_j(t) \tau_{ij}^\mu(dt).$
- (c) $d\mu_i(t) = \mu_i(t)(e_i - \mu(t)) d \log S(t) - \mu_i(t) \gamma_\mu^*(dt) + \frac{1}{2} \mu_i(t) \tau_{ii}^\mu(dt).$

Proof. (a) The definition of μ_i and the formula for V^μ from Lemma 2.7 yield that

$$\begin{aligned} \log \mu_i(t) &= \log S_i(t) - \log(S_1(t) + \cdots + S_d(t)) \\ &= \log S_i(t) - \log V^\mu(t) - \log(S_1(0) + \cdots + S_d(0)). \end{aligned} \tag{4.6}$$

Taking differentials and using Lemma 4.2 proves the claim.

(b) First, it follows, e.g., from (4.6) that $\log \mu \in QV^d$. Next, since $t \mapsto \gamma_\mu^*([0, t])$ is continuous and of bounded variation, it has vanishing quadratic variation. Hence, (a) and [24, Remark 8 and Proposition 12] imply that

$$\begin{aligned} [\log \mu_i](t) &= \left[\int_0^t (e_i - \mu(s)) d \log S(s) \right](t) \\ &= \sum_{k,l=1}^d \int_0^t ((e_i)_k - \mu_k(s)) ((e_i)_l - \mu_l(s)) d[\log S_k, \log S_l](t) \\ &= \int_0^t (\mu(t) - e_i)^\top a(dt) (\mu(t) - e_i) = \tau_{ii}^\mu([0, t]). \end{aligned} \tag{4.7}$$

The polarization identity (2.2) now yields the first claim in (b). The second one then follows with Lemma 2.3.

(c) Setting

$$I(t) := \int_0^t (e_i - \mu(s)) d \log S(s) = \log S_i(t) - \log S_i(0) - \int_0^t \mu(s) d \log S(s)$$

and integrating (a) gives $\mu_i(t) = \mu_i(0) \exp(I(t) - \gamma_\mu^*([0, t]))$. Using that $t \mapsto \gamma_\mu^*([0, t])$ is of bounded variation and hence has vanishing quadratic variation, the pathwise Itô formula gives

$$\mu_i(t) = \mu_i(0) + \int_0^t \mu_i(s) dI(s) + \frac{1}{2} \int_0^t \mu_i(s) d[I](s) - \int_0^t \mu_i(s) \gamma_\mu^*(ds).$$

To deal with the first integral on the right-hand side, the associativity rule of [24, Theorem 13] yields that $\int_0^t \mu_i(s) dI(s) = \int_0^t \mu_i(s)(e_i - \mu(s)) d \log S(s)$. For the second integral, (4.7) and the associativity of the Stieltjes integral [31, Theorem I.6 b] imply that $\int_0^t \mu_i(s) d[I](s) = \int_0^t \mu_i(s) \tau_{ii}^\mu(ds)$. This yields (c). \square

The proof of the following lemma is left to the reader, since it follows straightforwardly by adapting the proof from [13, Lemma 3.3].

Lemma 4.4. *For any pair of portfolios π and ρ we have the following numéraire invariance property*

$$\gamma_\pi^*(dt) = \frac{1}{2} \left(\sum_{i=1}^d \pi_i(t) \tau_{ii}^\rho(dt) - \sum_{i=1}^d \sum_{j=1}^d \pi_i(t) \pi_j(t) \tau_{ij}^\rho(dt) \right).$$

The following lemma is valid if π is a portfolio for S and in addition an admissible integrand for $\log \mu$. Lemma 2.4 (applied with $\log \mu$ in place of $\log S$) states that the latter requirement is equivalent to $\frac{\pi}{\mu}$ being an admissible integrand for μ . By Lemma 2.8, this requirement is satisfied for the functionally generated portfolio from (2.11).

Lemma 4.5. *Suppose that π is both a portfolio for S and an admissible integrand for $\log \mu$. Then*

$$\log \left(\frac{V^\pi(t)}{V^\mu(t)} \right) = \int_0^t \frac{\pi(s)}{\mu(s)} d\mu(s) - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \pi_i(s) \pi_j(s) \tau_{ij}^\mu(ds). \quad (4.8)$$

Proof. First, as noted above, $\frac{\pi}{\mu}$ is an admissible integrand for μ . Using Lemma 4.3 and the associativity of the Stieltjes integral from [31, Theorem I.6 b] together with the associativity of the Föllmer integral [24, Theorem 13] yields that

$$\frac{\pi(t)}{\mu(t)} d\mu(t) = (\pi(t) - \mu(t)) d \log S(t) - \gamma_\mu^*(dt) + \frac{1}{2} \sum_{i=1}^d \pi_i(t) \tau_{ii}^\mu(dt),$$

due to the fact that the portfolio weights sum up to one. Furthermore, applying the numéraire invariance property from Lemma 4.4 gives us

$$\frac{\pi(t)}{\mu(t)} d\mu(t) = (\pi(t) - \mu(t)) d \log S(t) - \gamma_\mu^*(dt) + \frac{1}{2} \left(\sum_{i,j=1}^d \pi_i(t) \pi_j(t) \tau_{ij}^\mu(dt) \right) + \gamma_\pi^*(dt).$$

On the other hand, Lemma 4.2 yields that

$$d \log \left(\frac{V^\pi(t)}{V^\mu(t)} \right) = (\pi(t) - \mu(t)) d \log S(t) + (\gamma_\pi^* - \gamma_\mu^*)(dt).$$

Subtracting these formulas from each other now yields the assertion. \square

As a preparation for the proof of Theorem 2.9, the following lemma further calculates the right-hand side of (4.8) in case π is the functionally generated portfolio from (2.11).

Lemma 4.6. *Let G be as in Lemma 2.8, π be the portfolio generated by G , and let us write*

$$g(t) = (g_1(t), \dots, g_d(t))^\top := \nabla_x \log G(\mu(t), A(t)). \quad (4.9)$$

Then

$$\log \left(\frac{V^\pi(t)}{V^\mu(t)} \right) = \int_0^t g(s) d\mu(s) - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \mu_i(s) \mu_j(s) g_i(s) g_j(s) \tau_{ij}^\mu(ds).$$

Proof. First, we deal with the Itô integral in (4.8). With the shorthand notation (4.9), the definition (2.11) of π becomes $\pi_i = \mu_i(t)(g_i(t) + 1 - \mu(t)^\top g(t))$, and so $\frac{\pi(t)}{\mu(t)} = g(t) + (1 - \mu(t)^\top g(t))\mathbf{1}$, where $\mathbf{1} := (1, \dots, 1)^\top \in \mathbb{R}^d$ denotes the vector whose entries are all 1. Since both $\frac{\pi}{\mu}$ and g are admissible integrands for μ , the function $\frac{\pi}{\mu} - g = (1 - \mu^\top g)\mathbf{1}$ must also be an admissible integrand for μ . But $\mathbf{1}^\top(\mu(t) - \mu(s)) = 0$ for all s and t , and therefore the Riemann sums in the approximation (2.3) of the Föllmer integral $\int_0^t (1 - \mu(s)^\top g(s))\mathbf{1} d\mu(s)$ must all vanish. It follows that this integral is zero, and hence that $\int_0^t \frac{\pi(s)}{\mu(s)} \mu(ds) = \int_0^t g(s) \mu(ds)$.

To deal with the rightmost integral in (4.8), we first note that

$$\tau_{ij}^\mu(dt) = a_{ij}(dt) - \sum_{\ell=1}^d \mu_\ell(t) a_{i\ell}(dt) - \sum_{k=1}^d \mu_k(t) a_{kj}(dt) + \sum_{k,\ell=1}^d \mu_k(t) \mu_\ell(t) a_{k\ell}(dt).$$

Thus, using the fact that the components of μ sum up to 1 we get

$$\sum_{j=1}^d \mu_j(t) \tau_{ij}^\mu(dt) = 0, \quad i = 1, \dots, d. \quad (4.10)$$

Using that $\pi_i = \mu_i(t)(g_i(t) + 1 - \mu(t)^\top g(t))$, the preceding identity yields that

$$\sum_{i,j=1}^d \pi_i(t) \pi_j(t) \tau_{ij}^\mu(dt) = \sum_{i,j=1}^d g_i(t) g_j(t) \mu_i(t) \mu_j(t) \tau_{ij}^\mu(dt).$$

This proves the lemma. \square

4.3 Proof of Theorem 2.9

Letting g be as in (4.9) and using the fact that

$$(\log G)_{x_i, x_j} = \frac{G_{x_i, x_j}}{G} - g_i g_j$$

the pathwise Itô formula and Lemma 4.3 (b) yield

$$\begin{aligned} \log \left(\frac{G(T, \mu(T), A(T))}{G(0, \mu(0), A(0))} \right) &= \int_0^T g(t) d\mu(t) + \sum_{k=0}^m \int_0^T \frac{\partial}{\partial a_k} \log G(t, \mu(t), A(t)) dA_k(t) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^T \left(\frac{\partial_{ij}^2 G(t, \mu(t), A(t))}{G(t, \mu(t), A(t))} - g_i(t) g_j(t) \right) \mu_i(t) \mu_j(t) \tau_{ij}^\mu(dt). \end{aligned}$$

Comparing this formula with Lemma 4.6 now gives the assertion. \square

4.4 Proofs of the results from Section 2.2

Proof of Lemma 2.13. The chain rule [27, Lemma 3.3] gives that $\bar{V}^\pi \in \mathbb{C}_c^{1,2}(\mathbb{R}_+^d, W \times \mathbb{R})$ and that $\nabla_X \bar{V}^\pi(t, X, (A, B^\pi)) = \xi(t, X)$. Therefore, ξ is indeed an admissible integrand on \mathbb{R}_+^d . Next, applying once again the chain rule [27, Lemma 3.3] gives

$$\begin{aligned} \partial_{ij} \bar{V}^\pi(t, X, (A, B^\pi)) &= \partial_i \xi_j(t, X) \\ &= \frac{\bar{V}^\pi(t, X, (A, B^\pi))}{X_i(t) X_j(t)} \left((\partial_{ij} F)(t, \log X, A) + \pi_i(t, \log X) \pi_j(t, \log X) - \delta_{ij} \pi_i(t, \log X) \right) \end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_i \bar{V}^\pi(t, X, (A, B^\pi)) &= \bar{V}^\pi(t, X, (A, B^\pi)) \mathcal{D}_i F(t, \log X, A), \quad i = 0, \dots, m, \\ \mathcal{D}_{m+1} \bar{V}^\pi(t, X, (A, B^\pi)) &= -\bar{V}^\pi(t, X, (A, B^\pi)).\end{aligned}$$

The functional Itô formula, the formula (2.5), and the associativity of the Stieltjes integral now give (2.15). \square

Proof of Lemma 2.14. The functional $H(t, X) := \log(e^{X_1(t)} + \dots + e^{X_d(t)})$ clearly belongs to $\mathbb{C}_c^{1,2}(\mathbb{R}^d, \emptyset)$, and we have $\bar{\mu}(t, X) = \nabla_X H(t, X)$. Therefore, $\bar{\mu}$ is indeed an admissible functional integrand. It is also clear that $\bar{\mu}_1 + \dots + \bar{\mu}_d = 1$. Next, we argue that

$$\int_0^t \bar{\mu}(s, \log S) d \log S(s) = \int_0^t \mu(s) d \log S(s). \quad (4.11)$$

That is, the functional Itô integral $\int_0^t \bar{\mu}(s, \log S) d \log S(s)$ coincides with the Föllmer integral of the market portfolio $\mu(t)$ with respect to $\log S$. (This fact is not entirely obvious, because both integrals are defined as the respective limits of different “Riemann sums”). To prove (4.11), note that $\mathcal{D}_0 H(t, X) = 0$ and

$$\partial_{ij} H(t, X) = \partial_i \mu_j(t, X) = \delta_{ij} \bar{\mu}_i(t, X) - \bar{\mu}_i(t, X) \bar{\mu}_j(t, X), \quad (4.12)$$

where δ_{ij} is again the Kronecker delta. Hence,

$$\begin{aligned}\int_0^t \bar{\mu}(s, \log S) d \log S(s) &= H(t, \log S) - H(0, \log S) - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \left(\delta_{ij} \mu_i(s) - \mu_i(s) \mu_j(s) \right) d[\log S_i, \log S_j](s) \\ &= \int_0^t \mu(s) d \log S(s),\end{aligned}$$

which gives (4.11). The identity (4.11) and [24, Proposition 12] imply in turn that

$$\begin{aligned}\left[\int_0^t \bar{\mu}(s, \log S) d \log S(s) \right](t) &= \sum_{i,j=1}^d \int_0^t \mu_i(s) \mu_j(s) d[\log S_i, \log S_j](s) \\ &= \sum_{i,j=1}^d \int_0^t \bar{\mu}_i(s, \log S) \bar{\mu}_j(s, \log S) d[\log S_i, \log S_j](s),\end{aligned}$$

and this completes the proof that $\bar{\mu}$ is a functional portfolio. Finally, it follows from $\mathcal{D}_0 H(t, X) = 0$ and (4.12) that $B^{\bar{\mu}}$ as defined in Lemma 2.13 with $F := H$ vanishes identically. Thus, that lemma yields the claimed formula for $\bar{V}^{\bar{\mu}}$. \square

Proof of Lemma 2.15. Let $\Gamma(t, X, A) := \log G(t, \bar{\mu}(\cdot, X), A)$. The chain rule [27, Lemma 3.3] implies that $\Gamma \in \mathbb{C}_c^{1,2}(\mathbb{R}^d, W)$ and that its i^{th} partial vertical derivative is given by

$$\begin{aligned}\partial_i \Gamma(t, X, A) &= \sum_{j=1}^d g_j(t, \bar{\mu}(\cdot, X)) \partial_i \bar{\mu}_j(t, X) \\ &= \bar{\mu}_i(t, X) \left(g_i(t, \bar{\mu}(\cdot, X)) - \sum_{j=1}^d \bar{\mu}_j(t, X) g_j(t, \bar{\mu}(\cdot, X)) \right) \\ &= \pi_i(t, X) - \bar{\mu}_i(t, X),\end{aligned} \quad (4.13)$$

where we have used (4.12) in the second step. Therefore, $\pi - \bar{\mu}$ is an admissible functional integrand on \mathbb{R}^d . Finally, by Lemma 2.14, $\bar{\mu}$, and in turn π , are admissible functional integrands on \mathbb{R}^d . \square

We will also need the second vertical derivatives of the functional Γ introduced in the proof of the preceding lemma.

Lemma 4.7. *For G as in Lemma 2.15 and $\Gamma(t, X, A) = \log G(t, \bar{\mu}(\cdot, X), A)$, we have*

$$\partial_{ij}\Gamma = \sum_{k,\ell=1}^d \frac{\partial_{\ell k}G}{G} \bar{\mu}_\ell \bar{\mu}_k (\bar{\mu}_j - \delta_{j\ell}) (\bar{\mu}_i - \delta_{ik}) - \pi_i \pi_j + \bar{\mu}_i \bar{\mu}_j + \delta_{ij} (\pi_i - \bar{\mu}_i). \quad (4.14)$$

Proof. Using (4.13) and the chain rule [27, Lemma 3.3], we find that

$$\begin{aligned} \partial_{ij}\Gamma(t, X, A) &= \partial_j \sum_{k=1}^d g_k(t, \bar{\mu}(\cdot, X)) \partial_i \bar{\mu}_k(t, X) \\ &= \sum_{k=1}^d \left(\sum_{\ell=1}^d \partial_\ell g_k(t, \bar{\mu}(\cdot, X)) \partial_j \bar{\mu}_\ell(t, X) \partial_i \bar{\mu}_k(t, X) + g_k(t, \bar{\mu}(\cdot, X)) \partial_{ij} \bar{\mu}_k(t, X) \right). \end{aligned}$$

By (4.12), we have $\partial_j \bar{\mu}_k = \delta_{jk} \bar{\mu}_k - \bar{\mu}_k \bar{\mu}_j$ and hence

$$\partial_{ij} \bar{\mu}_k = \mu_k \delta_{ik} \delta_{jk} - \mu_k \mu_i \delta_{kj} - \mu_j \mu_k \delta_{ki} - \mu_k \mu_i \delta_{ij} + 2\mu_i \mu_j \mu_k.$$

Therefore, when letting $\tilde{\pi} = \pi - \bar{\mu}$,

$$\sum_{k=1}^d g_k \partial_{ij} \bar{\mu}_k = \left(\delta_{ij} \mu_i - \bar{\mu}_i \bar{\mu}_j \right) \left(g_i - \sum_{k=1}^d g_k \bar{\mu}_k \right) - \bar{\mu}_i \bar{\mu}_j \left(g_j - \sum_{k=1}^d g_k \bar{\mu}_k \right) = \tilde{\pi}_i \delta_{ij} - \bar{\mu}_j \tilde{\pi}_i - \bar{\mu}_i \tilde{\pi}_j.$$

Moreover, $\partial_\ell g_k = \frac{\partial_{\ell k}G}{G} - g_\ell g_k$, and so

$$\sum_{k,\ell=1}^d \partial_\ell g_k \partial_j \bar{\mu}_\ell \partial_i \bar{\mu}_k = \sum_{k,\ell=1}^d \frac{\partial_{\ell k}G}{G} (\delta_{j\ell} \bar{\mu}_\ell - \bar{\mu}_\ell \bar{\mu}_j) (\delta_{ik} \bar{\mu}_k - \bar{\mu}_k \bar{\mu}_i) - \bar{\mu}_i \bar{\mu}_j \left(g_i - \sum_{k=1}^d g_k \bar{\mu}_k \right) \left(g_j - \sum_{k=1}^d g_k \bar{\mu}_k \right).$$

Putting everything together yields the assertion after a short computation. \square

Proof of Theorem 2.16. As in Lemma 2.13, we consider the functionals \bar{V}^π and \bar{V}^μ . It was shown in Lemma 2.15 that $B^\mu = 0$ and

$$\bar{V}^\mu(t, X) = \exp(H(t, \log X) - H(0, \log X)),$$

where $H(t, X) = \log(e^{X_1(t)} + \dots + e^{X_d(t)})$. Moreover, it was shown in the proof of Lemma 2.15 that $\pi = \nabla_X(\Gamma + H)$, where Γ is as in Lemma 4.7. Hence, by Lemma 2.13,

$$\log \left(\frac{V^\pi(T)}{V^\mu(T)} \right) = \Gamma(T, \log S, A) - \Gamma(0, \log S, A) - B^\pi(T) = \log \left(\frac{G(T, \mu, A)}{G(0, \mu, A)} \right) - B^\pi(T).$$

It thus remains to compute $B^\pi(T)$. By (2.14), (4.12), and (4.14), we have

$$\begin{aligned}
& dB^\pi(t) + \mathfrak{h}(dt) \\
&= \frac{1}{2} \sum_{i,j=1}^d \left((\partial_{ij}\Gamma)(t, \log S, A) + (\partial_{ij}H)(t, \log S) \right. \\
&\quad \left. + \pi_i(t, \log S)\pi_j(t, \log S) - \delta_{ij}\pi_i(t, \log S) \right) d[\log S_i, \log S_j](t) \\
&= \frac{1}{2} \sum_{i,j,k,\ell=1}^d \left(\frac{(\partial_{\ell k}G)(t, \log S, A)}{G(t, \log S, A)} (\mu_\ell(t)\mu_j(t) - \delta_{j\ell}\mu_\ell(t)) (\mu_k(t)\mu_i(t) - \delta_{ik}\mu_k(t)) \right) d[\log S_i, \log S_j](t).
\end{aligned}$$

Recalling the notation $a_{ij}(dt) = d[\log S_i, \log S_j](t)$ and $\tau_{\ell k}^\mu(dt) = \sum_{i,j=1}^d (\mu_j - \delta_{j\ell})a_{ij}(\mu_i - \delta_{ik})$ as well as the fact that $d[\mu_\ell, \mu_k](t) = \mu_\ell(t)\mu_k(t)\tau_{\ell k}(dt)$ by Lemma 4.3, we finally arrive at the desired identity $dB^\pi(t) + \mathfrak{h}(dt) = -\mathfrak{g}(dt)$. \square

4.5 Proof of Proposition 3.1

By the chain rule for functional derivatives, [27, Lemma 3.3], $\log \widehat{G} \in \mathbb{C}_c^{1,2}(U, \emptyset)$, and its vertical derivative is given by

$$\partial_i \log \widehat{G}(t, X) = \frac{\lambda}{\widehat{G}(t, X)} \varphi_{x_i} \left(\lambda X(t) + \frac{1-\lambda}{\theta} \int_{t-\theta}^t X(0 \vee s) ds \right). \quad (4.15)$$

When evaluating this expression at $X = \mu$, it becomes equal to $g_i(t)$. Since we know that $\mu(t) = \bar{\mu}(t, \log S)$, the identity (3.3) follows. Later on in this proof, we will also need the horizontal derivative $\mathcal{D}_0 \log \widehat{G}(t, X)$. By (A.2), it is given by

$$\mathcal{D}_0 \log \widehat{G}(t, X) = \frac{1-\lambda}{\theta \widehat{G}(t, X)} \sum_{i=1}^d \varphi_{x_i} \left(\lambda X(t) + \frac{1-\lambda}{\theta} \int_{t-\theta}^t X(0 \vee s) ds \right) (X_i(t-) - X_i(0 \vee (t-\theta)-)),$$

where we put $X(0-) = X(0)$. Therefore,

$$\mathcal{D}_0 \log \widehat{G}(t, \mu) = \frac{1-\lambda}{\lambda} \sum_{i=1}^d g_i(t) \alpha'_i(t), \quad (4.16)$$

In analogy to the proof of Lemma 2.14, it suffices to establish the identity

$$\int_0^t \widehat{\pi}(s, \log S) d \log S(s) = \int_0^t \pi(s) d \log S(s) \quad (4.17)$$

so as to conclude that $\widehat{\pi}$ satisfies (2.12) and is hence a functional portfolio. To this end, recall from the proof of Lemma 2.15 that $\widehat{\pi}(t, X) = \nabla_X \Gamma(t, X) + \nabla_X H(t, X)$, where $\Gamma(t, X) = \log \widehat{G}(t, \bar{\mu}(\cdot, X))$ and $H(t, X) = h(X(t))$ for $h(x) = \log(e^{x_1} + \dots + e^{x_d})$. Since we already know from the proof of Lemma 2.14 that $\int_0^t \nabla_X H(s, \log S) d \log S(s) = \int_0^t \mu(s) d \log S(s)$, it will be enough to show that

$$\int_0^t \nabla_X \Gamma(s, \log S) d \log S(s) = \int_0^t \nabla_x \gamma(\log S(t), \alpha(t)) d \log S(s), \quad (4.18)$$

where $\gamma(x, a) = \log G(\nabla_x h(x), a)$ is as in the proof of Lemma 2.8. The functional Itô formula implies that the left-hand side of (4.18) is given by

$$\Gamma(t, \log S) - \Gamma(0, \log S) - \int_0^t \mathcal{D}_0 \Gamma(s, \log S) ds - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij} \Gamma(s, \log S) d[\log S_i, \log S_j](s).$$

We clearly have $\Gamma(t, \log S) = \gamma(\log S(t), \alpha(t))$. Moreover, Lemma 4.7 implies that $\partial_{ij} \Gamma(t, \log S) = \gamma_{x_i, x_j}(\log S(t), \alpha(t))$. Next, the chain rule for functional derivatives [27, Lemma 3.3] yields that

$$\mathcal{D}_0 \Gamma(t, X) = (\mathcal{D}_0 \log \widehat{G})(t, \bar{\mu}(\cdot, X)) + \sum_{i=1}^d (\partial_i \log \widehat{G})(t, \bar{\mu}(\cdot, X)) \mathcal{D}_0 \bar{\mu}_i(t, X) = (\mathcal{D}_0 \log \widehat{G})(t, \bar{\mu}(\cdot, X)),$$

since $\mathcal{D}_0 \bar{\mu}_i = 0$. Together with (4.16) we thus obtain

$$\mathcal{D}_0 \Gamma(t, \log S) = \frac{1-\lambda}{\lambda} \sum_{i=1}^d g_i(t) \alpha'_i(t). \quad (4.19)$$

Re-writing the Föllmer integral on the right-hand side of (4.18) by means of the Itô formula for $\gamma(\log S(t), \alpha(t))$ and putting everything together thus yields (4.18).

Next, by the computations we have already completed in this proof, it is clear that

$$\begin{aligned} \mathfrak{h}([0, T]) &= - \int_0^T \frac{G_a(\mu(t), \alpha(t))}{G(\mu(t), \alpha(t))} d\alpha(t) = - \frac{1-\lambda}{\lambda} \sum_{i=1}^d \int_0^T g_i(t) \alpha'_i(t) dt = - \int_0^T \frac{\mathcal{D}_0 \widehat{G}(t, \mu)}{G(t, \mu)} dt, \\ \mathfrak{g}([0, T]) &= - \frac{1}{2} \sum_{i,j=1}^d \int_0^T \frac{G_{x_i, x_j}(\mu(t), \alpha(t))}{G(\mu(t), \alpha(t))} d[\mu_i, \mu_j](t) = - \frac{1}{2} \sum_{i,j=1}^d \int_0^T \frac{\partial_{ij} \widehat{G}(\mu(t), \alpha(t))}{\widehat{G}(\mu(t), \alpha(t))} d[\mu_i, \mu_j](t) \\ &= - \frac{\lambda^2}{2} \sum_{i,j=1}^d \int_0^T \frac{\varphi_{x_i, x_j}(\lambda \mu(t) + (1-\lambda)\alpha(t))}{\varphi(\lambda \mu(t) + (1-\lambda)\alpha(t))} d[\mu_i, \mu_j](t) \end{aligned}$$

Since, moreover, $G(\mu(t), \alpha(t)) = \widehat{G}(t, \mu)$, Theorems 2.9 and 2.16 imply that $V^\pi(T) = V^{\widehat{\pi}}(T)$. \square

Remark 4.8. As mentioned before, a main difficulty in dealing with *functional* pathwise Itô calculus is the fact that here the pathwise Itô integral is then no longer the limit of ordinary Riemann sums as in (2.3). Instead, the integrands in the approximating “Riemann sums” (A.5) involve approximations of the integrator path. Ananova and Cont [1, Theorem 3.2] provide strong regularity assumptions on both the integrand and the integrator to guarantee that (A.5) can be replaced by ordinary Riemann sums. In this context, it is interesting to note that in the special cases (4.11) and (4.17) we could obtain similar results without the regularity conditions on the integrators required in [1].

A Appendix on pathwise Itô calculus

For the convenience of the reader, we give here a short overview of the definitions and notations of pathwise functional Itô calculus as developed by Dupire [7] and Cont and Fournier [3, 4]. Our presentation and notation is close to [27]. In the sequel, we fix $T > 0$ and open sets $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^m$. The Skorokhod space $D([0, T], U)$ will be equipped with the supremum norm $\|X\|_\infty =$

$\sup_{u \in [0, T]} |X(u)|$. For $X \in D([0, T], U)$ and $t \in [0, T]$, we let $X^t = (X(t \wedge s))_{s \in [0, T]}$ denote the path stopped in t . A functional $F : [0, T] \times D([0, T], U) \times CBV([0, T], V) \mapsto \mathbb{R}$ is called *non-anticipative* if $F(t, X, A) = F(t, X^t, A^t)$ for all $(t, X, A) \in [0, T] \times D([0, T], U) \times CBV([0, T], V)$. We now recall several regularity properties for non-anticipative (and possibly vector-valued) functionals F .

- F is called *boundedness-preserving* if for every $A \in CBV([0, T], V)$ and any compact subset $K \subset U$ there exist a constant C such that $|F(t, X, A)| \leq C$ for all $t \in [0, T]$ and $X \in D([0, T], K)$.
- F is called *continuous at fixed times*, if for all $\varepsilon > 0$, $t \in [0, T]$, $X \in D([0, T], U)$, and $A \in CBV([0, T], V)$, there exists $\eta > 0$ such that $|F(t, X, A) - F(t, Y, A)| < \varepsilon$ for all $Y \in D([0, T], U)$ for which $\|X^t - Y^t\|_\infty < \eta$.
- F is called *left-continuous* if for all $t \in (0, T]$, $\varepsilon > 0$, $X \in D([0, T], U)$, and $A \in CBV([0, T], V)$, there exists $\eta > 0$ such that $|F(t, X, A) - F(t-h, Y, A)| < \varepsilon$ for all $h \in [0, \eta]$ and $Y \in D([0, T], U)$ for which $\|X^t - Y^{t-h}\|_\infty < \eta$.
- F is called *continuous in X locally uniformly in t* , if for all $\varepsilon > 0$ and $(t, X, A) \in [0, T] \times D([0, T], U) \times CBV([0, T], V)$ there is some $\eta > 0$ such that $|F(u, X, A) - F(u, Y, A)| < \varepsilon$ for all $(u, Y) \in [0, T] \times D([0, T], U)$ for which $\|X - Y\|_\infty < \eta$ and $|t - u| < \eta$.

Next, we recall the notions of horizontal and vertical derivatives, which are also called Dupire derivatives and which were proposed in [7, 3]. The following notion of a horizontal derivative extends the one from [7, 3] and was proposed in [27]. We say that F is *horizontally differentiable*, if there exist non-anticipative and boundedness preserving functionals $\mathcal{D}_0 F, \mathcal{D}_1 F, \dots, \mathcal{D}_m F$ on $[0, T] \times D([0, T], U) \times CBV([0, T], V)$ such that for $0 \leq s < t \leq T$ and $(X, A) \in D([0, T], U) \times CBV([0, T], V)$, the functions $[s, t] \ni r \mapsto \mathcal{D}_i F(r, X^s, A)$ are Borel measurable and

$$F(t, X^s, A) - F(s, X^s, A) = \sum_{i=0}^m \int_s^t \mathcal{D}_i F(r, X^s, A) A_i(dr), \quad (\text{A.1})$$

where we put $A_0(r) := r$. As discussed in [27, Remark 2.2], F will be horizontally differentiable with horizontal derivative $\mathcal{D}F = (\mathcal{D}_0 F, \mathcal{D}_1 F, \dots, \mathcal{D}_m F)$, if the following limits exist for all (t, X, A) and if they give rise to locally bounded and non-anticipative functionals on $[0, T] \times D([0, T], U) \times CBV([0, T], V)$ satisfying the above measurability requirement,

$$\begin{aligned} \mathcal{D}_0 F(t, X^t, A^t) &= \lim_{h \downarrow 0} \frac{F(t+h, X^t, A^t) - F(t, X^t, A^t)}{h} \\ \mathcal{D}_k F(t, X^t, A^t) &= \lim_{h \downarrow 0} \frac{F(t, X^t, A_1^t, \dots, A_k^{t+h}, \dots, A_m^t) - F(t, X^t, A^t)}{A_k(t+h) - A_k(t)} \mathbb{1}_{\{A_k(t+h) \neq A_k(t)\}}, \end{aligned} \quad (\text{A.2})$$

for $k = 1, \dots, m$.

A non-anticipative functional F is said to be *vertically differentiable* at (t, X, A) if the map $\mathbb{R}^d \ni v \rightarrow F(t, X + v \mathbb{1}_{[t, T]}, A^t)$ is differentiable at 0. The *vertical derivative* of F at (t, X, A) will then be the gradient of that map at $v = 0$. It will be denoted by

$$\nabla_X F(t, X, A) = (\partial_i F(t, X, A))_{i=1, \dots, d}, \quad (\text{A.3})$$

where $\partial_i F(t, X, A)$ is the i^{th} partial vertical derivative,

$$\partial_i F(t, X, A) = \lim_{h \rightarrow 0} \frac{F(t, X + h e_i \mathbb{1}_{[t, T]}, A) - F(t, X, A)}{h}.$$

If the functional F admits horizontal and vertical derivatives $\mathcal{D}F$ and $\nabla_X F$, we may iterate the corresponding operations so as to define higher order horizontal and vertical derivatives. We denote by $\mathbb{C}_b^{1,2}(U, V)$ the set of all non-anticipative functionals F on $[0, T] \times D([0, T], U) \times CBV([0, T], V)$ such that F is left-continuous, horizontally differentiable, and twice vertically differentiable; the horizontal derivative $\mathcal{D}F$ is continuous at fixed times; the vertical derivatives $\nabla_X F$ and $\nabla_X^2 F$ are left-continuous and boundedness-preserving. With $\mathbb{C}_c^{1,2}(U, V)$ we denote the class of all functionals $F \in \mathbb{C}_b^{1,2}(U, V)$ that are continuous in X locally uniformly in t and boundedness preserving. Functionals in $\mathbb{C}_b^{1,2}(U, V)$ satisfy the following pathwise Itô formula, which is taken from [27] and which slightly extends the ones from [7, 3].

Theorem A.1. *Let us fix a path $X \in C([0, T], U)$ with continuous quadratic variation, a path $A \in CBV([0, T], V)$, and a functional $F \in \mathbb{C}_b^{1,2}(U, V)$. For $n \in \mathbb{N}$, define the approximating path $X^n \in D([0, T], U)$ by*

$$X^n(t) := \sum_{s \in \mathbb{T}_n} X(s') \mathbb{1}_{[s, s')}(t) + X(T) \mathbb{1}_{\{T\}}(t), \quad 0 \leq t \leq T, \quad (\text{A.4})$$

and let $X^{n, s^-} := \lim_{r \uparrow s} X^{n, r}$. Then the pathwise Itô integral along (\mathbb{T}_n) ,

$$\int_0^T \nabla_X F(s, X, A) dX(s) := \lim_{n \uparrow \infty} \sum_{s \in \mathbb{T}_n} \nabla_X F(s, X^{n, s^-}, A) \cdot (X(s') - X(s)), \quad (\text{A.5})$$

exists and, with $A_0(t) = t$,

$$\begin{aligned} F(T, X, A) - F(0, X, A) &= \int_0^T \nabla_X F(s, X, A) dX(s) + \sum_{i=0}^m \int_0^T \mathcal{D}_i F(s, X, A) dA_i(s) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^T \partial_{ij} F(s, X, A) d[X_i, X_j](s). \end{aligned} \quad (\text{A.6})$$

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