# On well-posedness of semilinear stochastic evolution equations on $L_{p}$ spaces 

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#### Abstract

We establish well-posedness in the mild sense for a class of stochastic semilinear evolution equations on $L_{p}$ spaces, driven by multiplicative Wiener noise, with a drift term given by an evaluation operator that is assumed to be quasi-monotone and polynomially growing, but not necessarily continuous. In particular, we consider a notion of mild solution ensuring that the evaluation operator applied to the solution is still function-valued, but satisfies only minimal integrability conditions. The proofs rely on stochastic calculus in Banach spaces, monotonicity and convexity techniques, and weak compactness in $L_{1}$ spaces.


## 1 Introduction

The purpose of this work is to prove well-posedness (existence, uniqueness and continuous dependence of solutions on the initial datum) to stochastic evolution equations (SEEs) of the type

$$
\begin{equation*}
d u(t)+A u(t) d t+f(u(t)) d t=\eta u(t) d t+B(t, u(t)) d W(t), \quad u(0)=u_{0}, \tag{1}
\end{equation*}
$$

where $t \in[0, T], A$ is a linear $m$-accretive operator on $L_{q}(D)$, with $D$ a bounded domain in $\mathbb{R}^{n}$ and $q \geq 2, f: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function of polynomial growth (without any continuity assumption), $W$ is a cylindrical Wiener noise on a separable Hilbert space $H$, and $B(t, \cdot)$ is a (random) map from $L_{q}(D)$ to $\mathscr{L}\left(H, L_{q}(D)\right)$ satisfying suitable Lipschitz continuity conditions. Precise assumptions on the notion of solution and on the data of the problem are given in Section 3. In particular, we adopt three notions of solution, that depend on the integrability properties of $f(u)$ : strict mild and mild solution are defined to be such that $f(u) \in L_{1}\left(0, T ; L_{q}(D)\right)$ almost surely and that $f(u) \in L_{1}(\Omega \times[0, T] \times D)$, respectively (here $\Omega$ stands for the underlying probability space); on the other hand, generalized solutions are defined as limits of strict mild solutions, so that, in general, $f(u)$ may not have any integrability. The first notion of solution is the simplest but also the most restrictive in terms of assumptions on the data of the problem. The second notion is the most natural if one wants $f(u)$ to be function-valued, while satisfying minimal integrability conditions. The last notion, motivated by analogous constructions

[^0]in the deterministic setting, apart of being the least demanding, is useful in several contexts, for instance in the study of Kolmogorov operators and Markovian semigroups associated to SPDEs (cf. e.g. [12]).

Our approach to the well-posedness problem is based, on the probabilistic side, on stochastic calculus for processes with values in Banach spaces (of which we use only the "simpler" version on spaces with type 2), and, on the analytic side, on methods from the theory of (nonlinear) $m$-accretive operators and convex analysis. Some ideas developed here, concerning strict mild and generalized solutions, already appeared, in a more primitive form, in [21, 23] and, in a slightly different context, in [24].

There is a rather large literature on semilinear dissipative SEEs, up-to-date references to which can be found, e.g., in [13]. Here we shall only discuss how our results compare to other recent ones that are most closely related. A widely used technique to study the well-posedness of (11) consists in the reduction of the equation to a deterministic evolution equation with random coefficients, roughly speaking "by subtracting the stochastic convolution". To the best of our knowledge, the sharpest result obtained through this reduction is due to Barbu [2], who proved existence and uniqueness of mild solutions to (11) assuming that $q=2, A$ is the negative Laplacian, $B$ does not depend on $u$ (i.e. the noise is additive), and, most importantly, the stochastic convolution

$$
S \diamond B:=t \mapsto \int_{0}^{t} S(t-s) B(s) d W(s)
$$

where $S$ denotes the semigroup generated by $-A$, is continuous in time and space, and satisfies $F(S \diamond B) \in L_{1}(\Omega \times[0, T] \times D)$, where $F$ is a primitive of $f$. On the other hand, no polynomial bound on $f$ is assumed. Our setting allows much more flexibility and no assumption is made on the stochastic convolution, but we need an extra polynomial growth assumption on $f$. A (partial) extension of our results to the case of general $f$ (i.e. removing the growth assumption) and $q=2$ is provided in a forthcoming joint work with L. Scarpa [25], thus considerably improving on the result of [2]. In another vein, global well-posedness in the mild sense of (1) is obtained in [19] assuming that $S$ is an analytic semigroup and that $f$ is polynomially bounded and locally Lipschitz continuous on $L_{q}(D)$ (not as function of $\mathbb{R}!$ ). The approach is through approximation of the coefficients and extension of local solutions. Even though the condition on $f$ is very restrictive, adapting ideas from [10], and considerably improving results thereof, wellposedness in spaces of continuous functions is obtained, allowing $f$ to be monotone and locally Lipschitz, now only as a function of $\mathbb{R}$. Incidentally, in [10], hence also in [19], the above-mentioned reduction to a PDE with random coefficients is again used, although in a more sophisticated way. While the reasoning in [10] relies on stochastic calculus in Hilbert spaces and ad hoc arguments, the improvements in [19] depend in an essential way on stochastic calculus in Banach spaces. We also use techniques from this calculus (although in a less sophisticated way), but we do not need any local Lipschitzianity assumption, although we obviously cannot consider solvability in spaces of continuous functions.

Our proofs do not employ at any stage the reduction to a deterministic equation with random coefficients. In fact, following the classical approach of constructing solutions to regularized equations and then passing to the limit in an appropriate topology (cf. e.g. [1, 7] for the deterministic theory), all the necessary estimates are obtained by stochastic
calculus arguments, rather than by classical calculus. Namely, the essential tool is Itô's formula for $L_{q}$-valued processes (even though, as explained in Remark 4 below, the classical formula for real processes would suffice). Using techniques from convex analysis and the theory of nonlinear $m$-accretive operators, we then show that, thanks to the above-mentioned estimates, solutions to regularized equations converge to a process that solves the original equation.

The rest of the text is organized as follows: in Section 2 we collect several tools used in the proof of the main results. Everything except the content of the last subsection is known and is included here for the readers' convenience. Our main results are stated in Section 3. In Sections [4, 苛, and we prove well-posedness in the strict mild, generalized, and mild sense, respectively.
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## 2 Preliminaries

In this section we introduce notation and recall some facts that will be used in the rest of the text.

Let $\left(\Omega, \mathcal{F},\left(\mathcal{F}_{t}\right)_{0 \leq t \leq T}, \mathbb{P}\right)$, with $T>0$ fixed, be a filtered probability space satisfying the "usual" conditions (see e.g. [14]), and let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. All stochastic elements will be defined on this stochastic basis, and any expression involving random quantities will be meant to hold $\mathbb{P}$-almost surely, unless otherwise stated. Throughout the paper, $W$ stands for a cylindrical Wiener process on a (fixed) separable Hilbert space $H$.

Given $p>0$ and a Banach space $X$, we shall denote by $\mathbb{L}_{p}(X)$ the set of $X$-valued random variables $\zeta$ such that

$$
\|\zeta\|_{\mathbb{L}_{p}(X)}:=\left(\mathbb{E}\|\zeta\|_{X}^{p}\right)^{1 / p}<\infty
$$

and by $\mathbb{H}_{p}(X)$ the set of measurabl $\mathbb{1}$, adapted $X$-valued processes such that

$$
\|u\|_{\mathbb{H}_{p}(X)}:=\left(\mathbb{E} \sup _{t \leq T}\|u(t)\|_{X}^{p}\right)^{1 / p}<\infty
$$

Both spaces are Banach spaces for $p \geq 1$, and quasi-Banach spaces for $0<p<1$. The space $\mathbb{H}_{p}(X)$, when endowed with the equivalent (quasi-)norm

$$
\|u\|_{\mathbb{H}_{p, \alpha}(X)}:=\left(\mathbb{E}_{t \leq T} \sup _{t \leq}\left\|e^{-\alpha t} u(t)\right\|_{X}^{p}\right)^{1 / p}, \quad \alpha \in \mathbb{R}_{+}
$$

will be denoted by $\mathbb{H}_{p, \alpha}(X)$.
The domain and range of a map $T$ will be denoted by $\mathrm{D}(T)$ and $\mathrm{R}(T)$, respectively. The standard notation $\mathscr{L}(E, F)$ will be used for the space of linear bounded operators

[^1]between two Banach spaces $E$ and $F$. If $E$ and $F$ are metric spaces, $\dot{C}^{0,1}(E, F)$ stands for the set of Lipschitz maps $\phi: E \rightarrow F$ such that
$$
\|\phi\|_{\dot{C}^{0,1}(E, F)}:=\sup _{\substack{x, y \in E \\ x \neq y}} \frac{d(\phi(x), \phi(y))}{d(x, y)}<\infty
$$

We shall omit the indication of the spaces $E$ and $F$ when it is clear what they are.
Throughout this section we shall simply write $L_{q}, q \in[0, \infty]$, to mean the usual Lebesgue spaces over a generic $\sigma$-finite measure space $(Y, \mathcal{A}, \mu)$.

Finally, we shall use the notation $a \lesssim b$ to mean that $a$ is less than or equal to $b$ modulo a constant, with subscripts to emphasize its dependence on specific quantities. Completely analogous meaning have the symbols $\gtrsim$ and $\approx$.

### 2.1 Convex functions and subdifferentials

Let $F: \mathbb{R} \rightarrow \mathbb{R} \cup\{+\infty\}$ be a convex function. Then, for any $x, y \in \mathrm{D}(F)$,

$$
\begin{equation*}
F(y)-F(x) \geq z(y-x) \quad \forall z \in \partial F(x) \tag{2}
\end{equation*}
$$

where $\partial F(x)$ denotes the subdifferential of $F$ at $x$. The above inequality defines $\partial F(x)$, which is a subset of $\mathbb{R}$, in the sense that $z \in \partial F(x)$, by definition, if it satisfies (2) for all $y \in \mathrm{D}(F)$. If $F$ is differentiable at $x \in \mathbb{R}$, then $\partial F(x)$ reduces to a singleton and coincides with $F^{\prime}(x)$. The following mean-value theorem holds (cf. [16, Theorem 2.3.4, p. 179]): if $F$ is finite-valued, one has, for any $x, y \in \mathbb{R}$,

$$
F(y)-F(x)=\int_{x}^{y} s(r) d r
$$

where $s(r)$ is any selection of the subdifferential $\partial F(r)$.
Given a maximal monotone graph $f \subset \mathbb{R}^{2}$ (see 2.3 below), there exists a convex function $F$, called the potential of $f$, such that $f=\partial F$. The converse is also true, i.e. the map $x \mapsto \partial F(x)$ defines a maximal monotone graph of $\mathbb{R}^{2}$ for any convex function $F$.

The (Legendre-Fenchel) conjugate $F^{*}: \mathbb{R} \rightarrow \mathbb{R} \cup\{+\infty\}$ of the convex (proper, lower semicontinuous) function $F: \mathbb{R} \rightarrow \mathbb{R} \cup\{+\infty\}$ is defined as

$$
F^{*}(x):=\sup _{y \in \mathrm{D}(F)}(x y-F(y))
$$

$F^{*}$ is itself a convex (proper, lower semicontinuous) function. The definition obviously implies $x y \leq F(x)+F^{*}(y)$ for all $x, y \in \mathbb{R}$, with equality if and only if $y \in \partial F(x)$, which in turn is equivalent to $x \in \partial F^{*}(y)$. Moreover, if $F$ is everywhere finite on $\mathbb{R}$, then $F^{*}$ is superlinear at infinity, i.e.

$$
\lim _{|x| \rightarrow \infty} \frac{F^{*}(x)}{|x|}=+\infty
$$

In particular, if $\partial F(x) \neq \varnothing$ for all $x \in \mathbb{R}$, or, equivalently, the domain of $f:=\partial F$ is $\mathbb{R}$, then $F$ is finite-valued on $\mathbb{R}$ and $F^{*}$ is superlinear at infinity (see, e.g., [16, Chapter E] for all these facts).

### 2.2 Duality mapping and differentiability of the norm

Let $X$ be a Banach space with (topological) dual $X^{*}$. The duality mapping of $X$ is the map

$$
\begin{aligned}
J: X & \rightarrow 2^{X^{*}} \\
x & \mapsto\left\{x^{*} \in X^{*}:\left\langle x^{*}, x\right\rangle=\|x\|_{X}^{2}=\left\|x^{*}\right\|_{X^{*}}^{2}\right\} .
\end{aligned}
$$

If $X^{*}$ is strictly convex, then $J$ is single-valued and continuous from $X$, endowed with the strong topology, to $X^{*}$, endowed with the weak topology (i.e. $J$ is demicontinuous). Moreover, if $X^{*}$ is uniformly convex, then $J$ is uniformly continuous on bounded subsets of $X$. For instance, all Hilbert spaces and all $L_{q}$ spaces with $1<q<\infty$ are uniformly convex (hence also strictly convex), and their duality mappings are single-valued and demicontinuous. In particular, if $X=L_{q}, 1<q<\infty$, one has

$$
J: u \mapsto\|u\|_{L_{q}}^{2-q}|u|^{q-2} u
$$

On the other hand, the duality mapping of $L_{1}$ is multivalued: in fact, if $X=L_{1}$, one has

$$
J: u \mapsto\left\{v \in L_{\infty}: v \in\|u\|_{L_{1}} \operatorname{sgn} u \text { a.e. }\right\} .
$$

Moreover, one has $J=\partial \phi$, where $\phi=\frac{1}{2}\|\cdot\|_{X}^{2}$ and $\partial$ stands for the subdifferential in the sense of convex analysis. The ( $q$-th power of the) norm of $L_{q}$ spaces with $q \geq 2$ is in fact very regular: setting $\Phi_{q}:=\|\cdot\|_{L_{q}}^{q}$, one has $\Phi_{q} \in C^{2}\left(L_{q}\right)$, with

$$
\begin{aligned}
& \Phi_{q}^{\prime}: L_{q} \rightarrow \mathscr{L}\left(L_{q}, \mathbb{R}\right) \simeq L_{q^{\prime}}, \quad \Phi_{q}^{\prime \prime}: L_{q} \rightarrow \mathscr{L}\left(L_{q}, \mathscr{L}\left(L_{q}, \mathbb{R}\right)\right) \simeq \mathscr{L}_{2}\left(L_{q}\right), \\
& \Phi_{q}^{\prime}(u): v\left.\left.\longmapsto q\langle | u\right|^{q-2} u, v\right\rangle \equiv q \int_{Y}|u|^{q-2} u v d \mu, \\
& \Phi_{q}^{\prime \prime}(u):(v, w)\left.\left.\longmapsto q(q-1)\langle | u\right|^{q-2} v, w\right\rangle \equiv q(q-1) \int_{Y}|u|^{q-2} v w d \mu,
\end{aligned}
$$

where $\mathscr{L}_{2}\left(L_{q}\right)$ stands for the space of bilinear forms on $L_{q}$. In particular, for any $u \in L_{q}$,

$$
\begin{equation*}
\Phi_{q}^{\prime}(u)=q\|u\|_{L_{q}}^{q-2} J(u), \quad\left\|\Phi^{\prime}(x)\right\|_{L_{q^{\prime}}}=q\|x\|_{L_{q}}^{q-1} \tag{3}
\end{equation*}
$$

and, by Hölder's inequality,

$$
\begin{equation*}
\left\|\Phi_{q}^{\prime \prime}(u)\right\|_{\mathscr{L}_{2}\left(L_{q}\right)} \leq q(q-1)\|u\|_{L_{q}}^{q-2} . \tag{4}
\end{equation*}
$$

A detailed treatment of duality mappings and related geometric properties of $L_{q}$ spaces can be found, for instance, in [11], while most results needed here are also recalled in, e.g., [1, Chapter 1].

## 2.3 m-accretive operators

A subset $A$ of $X \times X$ is called accretive if, for every $\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right) \in A$, there exists $z \in J\left(x_{1}-x_{2}\right)$ such that $\left\langle y_{1}-y_{2}, z\right\rangle \geq 0$. An accretive set $A$ is called $m$-accretive if $\mathrm{R}(I+A)=X$. One often says that $A$ is a multivalued (nonlinear) mapping on $X$, rather
than a subset of $X \times X$. Through the rest of this subsection, we shall assume that $A$ is an $m$-accretive subset of $X \times X$.

The Yosida approximation (or regularization) of $A$ is the family $\left\{A_{\lambda}\right\}_{\lambda>0}$ of (singlevalued) operators on $X$ defined by

$$
A_{\lambda}:=\frac{1}{\lambda}\left(I-(I+\lambda A)^{-1}\right), \quad \lambda>0
$$

The following properties will be extensively used:
(a) $A_{\lambda}$ is $m$-accretive;
(b) $A_{\lambda} \in \dot{C}^{0,1}(X, X)$ with Lipschitz constant bounded by $2 / \lambda$;
(c) $\left\|A_{\lambda} x\right\| \leq \inf _{y \in A x}\|y\|$ for all $x \in X$;
(d) $A_{\lambda} x \in A(I+\lambda A)^{-1} x$ for all $x \in X$;
(d') if $A$ is single-valued and $X, X^{*}$ are uniformly convex, then $A_{\lambda} x \rightarrow A x$ as $\lambda \rightarrow 0$ for all $x \in \mathrm{D}(A)$.
(e) $(I+\lambda A)^{-1} \in \dot{C}^{0,1}(X, X)$ with Lipschitz constant bounded by 1 ;
(f) $(I+\lambda A)^{-1} x \rightarrow x$ as $\lambda \rightarrow 0$ for all $x \in \overline{\mathrm{D}(A)}$.

If $X^{*}$ is uniformly convex, then the $m$-accretive set $A$ is demiclosed, i.e. it is closed in $X \times X_{w}$, where $X_{w}$ stands for $X$ endowed with its weak topology. More precisely, if $x_{n} \rightarrow x$ strongly in $X$ and $A_{\lambda_{n}} x_{n} \rightarrow y$ weakly in $X$ as $n \rightarrow \infty$, then $(x, y) \in A$.

Let $X=L_{q}, 1 \leq q<\infty$. If $g$ is a maximal monotone graph in $\mathbb{R}^{2}$, then the (multivalued) evaluation operator $\bar{g}$ associated to $g$ is an $m$-accretive subset of $X \times X$. The operator $\bar{g}$ is defined on $X$ as

$$
\bar{g}: u \mapsto\{v \in X: v \in g(u) \quad \mu \text {-a.e. }\} .
$$

Note that the graph of a (discontinuous) increasing function $g_{0}: \mathbb{R} \rightarrow \mathbb{R}$ is a monotone subset of $\mathbb{R}^{2}$, but it is not maximal monotone. However, the graph $g \subset \mathbb{R}^{2}$ defined by

$$
g(x)= \begin{cases}g_{0}(x), & x \in \mathbb{R} \backslash I, \\ {\left[g_{0}(x-), g_{0}(x+)\right],} & x \in I\end{cases}
$$

where $I$ is the jump set of $g_{0}$, is maximal monotone and (clearly) extends $g_{0}$. We shall not explicitly distinguish below among an increasing function $g_{0}$, its maximal monotone extension $g$, and the associated evaluation operator $\bar{g}$.

The proofs of the above facts (and much more) can be found, for instance, in [1, $\S 2.3] 2$

[^2]
## $2.4 \quad \gamma$-Radonifying operators

We shall use only basic facts from the rich and powerful theory of $\gamma$-Radonifying operators. For more information we refer to, e.g., the survey [27].

Let $H, H^{\prime}$ be real separable Hilbert spaces and $X, X^{\prime}$ Banach spaces. An operator $T \in \mathscr{L}(H, X)$ is said to be $\gamma$-Radonifying if there exists an orthonormal basis $\left(h_{n}\right)_{n \in \mathbb{N}}$ of $H$ such that

$$
\|T\|_{\gamma(H, X)}:=\left(\mathbb{E}^{\prime}\left\|\sum_{n \in \mathbb{N}} \gamma_{n} T h_{n}\right\|\right)^{1 / 2}<\infty,
$$

where $\left(\gamma_{n}\right)$ is a sequence of independent identically distributed standard Gaussian random variable on a probability space $\left(\Omega^{\prime}, \mathcal{F}^{\prime}, \mathbb{P}^{\prime}\right)$. One can shows that $\|T\|_{\gamma(H, X)}$ does not depend on the choice of the orthonormal basis $\left(h_{n}\right)$. The set of all $T \in \mathscr{L}(H, X)$ such that $\|T\|_{\gamma(H, X)}$ is finite is itself a Banach space with norm $\|\cdot\|_{\gamma(H, X)}$, and $\gamma(H, X)$ is a two-sided ideal of $\mathscr{L}(H, X)$, i.e. $L \in \mathscr{L}\left(X, X^{\prime}\right)$ and $R \in \mathscr{L}\left(H^{\prime}, H\right)$ imply

$$
\|L T R\|_{\gamma\left(H^{\prime}, X^{\prime}\right)} \leq\|L\|_{\mathscr{L}\left(X, X^{\prime}\right)}\|T\|_{\gamma(H, X)}\|R\|_{\mathscr{L}\left(H^{\prime}, H\right)} .
$$

The following convergence result is a simple corollary of the ideal property: if $L_{n} \rightarrow L$ strongly in $\mathscr{L}\left(X, X^{\prime}\right)$ as $n \rightarrow \infty$, i.e. $L_{n} x \rightarrow L x$ in $X^{\prime}$ for all $x \in X$, then

$$
\left\|L_{n} T-L T\right\|_{\gamma\left(H, X^{\prime}\right)} \xrightarrow{n \rightarrow \infty} 0 .
$$

If $X=L_{q}, q \geq 1$, by a simple application of the Khinchin-Kahane inequalities it follows that $T \in \gamma\left(H, L_{q}\right)$ if and only if

$$
\left\|\left(T h_{n}\right)\right\|_{L_{q}\left(\ell_{2}\right)}=\left\|\left(\sum_{n \in \mathbb{N}}\left|T h_{n}\right|^{2}\right)^{1 / 2}\right\|_{L_{q}}<\infty
$$

for all orthonormal bases $\left(h_{n}\right)$ of $H$, and $\|T\|_{\gamma\left(H, L_{q}\right)} \bar{\sim}\left\|\left(T h_{n}\right)\right\|_{L_{q}\left(\ell_{2}\right)}$. Moreover, the mapping

$$
\begin{aligned}
L_{q}(H) & \longrightarrow \gamma\left(H, L_{q}\right) \\
f & \longmapsto T_{f}: g \mapsto\langle f(\cdot), g\rangle_{H}
\end{aligned}
$$

is an isomorphism of Banach spaces, where one can take

$$
f=\left\|\left(T h_{n}\right)\right\|_{\ell_{2}}=\left(\sum_{n \in \mathbb{N}}\left|T h_{n}\right|^{2}\right)^{1 / 2} .
$$

### 2.5 Stochastic calculus in Banach spaces

Let $X$ be a UMD Banach space. For any $1<p<\infty$ and progressively measurable process $G \in \mathbb{L}_{p}\left(\gamma\left(L_{2}((0, T) ; H), X\right)\right)$, the stochastic integral of $G$ with respect to $W$ is a well-defined $X$-valued local martingale that satisfies Burkholder inequality

$$
\mathbb{E} \sup _{t \leq T}\left\|\int_{0}^{t} G(s) d W(s)\right\|_{X}^{p} \bar{\sim}_{p, X} \mathbb{E}\|G\|_{\gamma\left(L_{2}((0, T) ; H), X\right)}^{p} .
$$

If $X$ has type 2 (this is the case if $X=L_{q}, q \geq 2$ ), one has the continuous embedding ${ }^{3}$

$$
L_{2}(0, T ; \gamma(H, X)) \hookrightarrow \gamma\left(L_{2}((0, T) ; H), X\right),
$$

hence

$$
\begin{equation*}
\mathbb{E} \sup _{t \leq T}\left\|\int_{0}^{t} G(s) d W(s)\right\|_{X}^{p} \lesssim_{p, X} \mathbb{E}\left(\int_{0}^{T}\|G(s)\|_{\gamma(H, X)}^{2} d s\right)^{p / 2} \tag{5}
\end{equation*}
$$

for all $p>0$ (the case $0<p \leq 1$ follows by Lenglart's domination inequality, see [20]). Note that, if $X=L_{q}$, in view of the isomorphism mentioned at the end of last subsection, the above inequalities can be equivalently written only in terms of $L_{q}$ norms. In other words, for our purposes the use of $\gamma$-Radonifying norms amounts only to adopting a convenient language. For further details we refer to [28] and references therein.

We shall also need Itô's formula for $L_{q}$-valued processes, and we use the version of [9], which is valid for UMD-valued processes. For our purposes, however, previous less general versions (cited in 9$]$ ) would also do, as well as the very specific one of [18], where only the $q$-th power of the norm is considered. Let us first introduce some notation: if $\Phi \in \mathscr{L}_{2}(X)$ is a bilinear form on $X$ and $T \in \gamma(H, X)$, we set

$$
\operatorname{Tr}_{T} \Phi:=\sum_{n \in \mathbb{N}} \Phi\left(T h_{n}, T h_{n}\right),
$$

for which it is easily seen that

$$
\begin{equation*}
\left|\operatorname{Tr}_{T} \Phi\right| \leq\|\Phi\|_{\mathscr{L}_{2}(X)}\|T\|_{\gamma(H, X)}^{2} . \tag{6}
\end{equation*}
$$

Theorem 1. Let $X$ be a UMD Banach space, and consider the $X$-valued process

$$
u(t)=u_{0}+\int_{0}^{t} b(s) d s+\int_{0}^{t} G(s) d W(s)
$$

where
(a) $u_{0}: \Omega \rightarrow X$ is $\mathcal{F}_{0}$-measurable;
(b) $b: \Omega \times[0, T] \rightarrow X$ is measurable, adapted and such that $b \in L_{1}(0, T ; X)$;
(c) $G: \Omega \times[0, T] \rightarrow \mathscr{L}(H, X)$ is $H$-measurable, adapted, stochastically integrable with respect to $W$, and such that $G \in L_{2}(0, T ; \gamma(H, X))$.

For any $\varphi \in C^{2}(X)$, one has

$$
\begin{aligned}
\varphi(u(t))= & \varphi\left(u_{0}\right)+\int_{0}^{t} \varphi^{\prime}(u(s)) b(s) d s+\int_{0}^{t} \varphi^{\prime}(u(s)) G(s) d W(s) \\
& +\frac{1}{2} \int_{0}^{t} \operatorname{Tr}_{G(s)} \varphi^{\prime \prime}(u(s)) d s
\end{aligned}
$$

[^3]
### 2.6 Estimates for linear equations

Given a Banach space $X$ and a linear $m$-accretive operator $A$ on $X$, for any $X$-valued or $\mathscr{L}(H, X)$-mapping $h$, we shall write, for any $\varepsilon>0, h^{\varepsilon}:=(I+\varepsilon A)^{-1} h$.

We first prove an estimate that will be used repeatedly in the following.
Proposition 2. Let $A$ be a linear $m$-accretive operator on $L_{q}$, and consider the unique mild solution $u$ to the equation

$$
d u(t)+A u(t)=b(t) d t+G(t) d W(t), \quad u(0)=u_{0}
$$

where $u_{0}, b$, and $G$ satisfy the assumptions of Theorem $\mathbb{1}$ (with $X=L_{q}$ ). If $u \in L_{\infty}\left(L_{q}\right)$, then

$$
\begin{aligned}
\|u(t)\|_{L_{q}}^{q} \leq & \left\|u_{0}\right\|_{L_{q}}^{q}+\int_{0}^{t} \Phi_{q}^{\prime}(u(s)) b(s) d s+\int_{0}^{t} \Phi_{q}^{\prime}(u(s)) G(s) d W(s) \\
& +\frac{1}{2} q(q-1) \int_{0}^{t}\|G(s)\|_{\gamma\left(H, L_{q}\right)}^{2}\|u(s)\|_{L_{q}}^{q-2} d s
\end{aligned}
$$

Proof. It is not difficult to verify that $u^{\varepsilon}$ is the unique strong solution to

$$
d u^{\varepsilon}+A u^{\varepsilon}=b^{\varepsilon} d t+G^{\varepsilon} d W, \quad u^{\varepsilon}(0)=u_{0}^{\varepsilon}
$$

(cf. e.g. [22, Lemma 6]). Itô's formula then yield\{4]

$$
\begin{aligned}
\left\|u^{\varepsilon}(t)\right\|_{L_{q}}^{q}+\int_{0}^{t} \Phi_{q}^{\prime}\left(u^{\varepsilon}\right) A u^{\varepsilon} d s= & \left\|u_{0}^{\varepsilon}\right\|_{L_{q}}^{q}+\int_{0}^{t} \Phi_{q}^{\prime}\left(u^{\varepsilon}\right) b^{\varepsilon} d s+\int_{0}^{t} \Phi_{q}^{\prime}\left(u^{\varepsilon}\right) G^{\varepsilon} d W \\
& +\frac{1}{2} \int_{0}^{t} \operatorname{Tr}_{G^{\varepsilon}} \Phi_{q}^{\prime \prime}\left(u^{\varepsilon}\right) d s
\end{aligned}
$$

where $\Phi_{q}^{\prime}\left(u^{\varepsilon}\right) A u^{\varepsilon}=q\left\|u^{\varepsilon}\right\|_{L_{q}}^{q-2}\left\langle A u^{\varepsilon}, J\left(u^{\varepsilon}\right)\right\rangle \geq 0$ by accretivity of $A$ on $L_{q}$, and $\left\|u_{0}^{\varepsilon}\right\|_{L_{q}} \leq$ $\left\|u_{0}\right\|_{L_{q}}$ by contractivity of $(I+\varepsilon A)^{-1}$ on $L_{q}$. We are thus left with

$$
\begin{aligned}
\left\|u^{\varepsilon}(t)\right\|_{L_{q}}^{q} \leq & \left\|u_{0}\right\|_{L_{q}}^{q}+\int_{0}^{t} \Phi_{q}^{\prime}\left(u^{\varepsilon}\right) b^{\varepsilon} d s+\int_{0}^{t} \Phi_{q}^{\prime}\left(u^{\varepsilon}\right) G^{\varepsilon} d W \\
& +\frac{1}{2} \int_{0}^{t} \operatorname{Tr}_{G^{\varepsilon}} \Phi_{q}^{\prime \prime}\left(u^{\varepsilon}\right) d s .
\end{aligned}
$$

We are now going to pass to the limit as $\varepsilon \rightarrow 0$ in this inequality. One clearly has $\left\|u^{\varepsilon}(t)\right\|_{L_{q}} \rightarrow\|u(t)\|_{L_{q}}$ as $\varepsilon \rightarrow 0$ because $(I+\varepsilon A)^{-1}$ converges strongly to the identity in $\mathscr{L}\left(L_{q}\right)$ as $\varepsilon \rightarrow 0$. By the triangle inequality,

$$
\begin{aligned}
\sup _{t \leq T}\left|\int_{0}^{t} \Phi_{q}^{\prime}\left(u^{\varepsilon}\right) b^{\varepsilon} d s-\int_{0}^{t} \Phi_{q}^{\prime}(u) b d s\right| \leq & \int_{0}^{T}\left|\left(\Phi_{q}^{\prime}\left(u^{\varepsilon}\right)-\Phi_{q}^{\prime}(u)\right) b^{\varepsilon}\right| d s \\
& +\int_{0}^{T}\left|\Phi_{q}^{\prime}(u)\left(b^{\varepsilon}-b\right)\right| d s .
\end{aligned}
$$

[^4]The following reasoning is to be understood to hold for each fixed $\omega$ in a subset of $\Omega$ of full $\mathbb{P}$-measure. Since $u^{\varepsilon}(s) \rightarrow u(s)$ and $b^{\varepsilon}(s) \rightarrow b(s)$ in $L_{q}$, hence also in measure, for all $s \in[0, T]$, and $\Phi_{q}^{\prime}$ is continuous, it follows that

$$
\left|\left(\Phi_{q}^{\prime}\left(u^{\varepsilon}(s)\right)-\Phi_{q}^{\prime}(u(s))\right) b^{\varepsilon}(s)\right| \xrightarrow{\varepsilon \rightarrow 0} 0
$$

in measure for all $s$. Moreover,

$$
\begin{aligned}
\left|\left(\Phi_{q}^{\prime}\left(u^{\varepsilon}(s)\right)-\Phi_{q}^{\prime}(u(s))\right) b^{\varepsilon}(s)\right| & \leq\left\|\Phi_{q}^{\prime}\left(u^{\varepsilon}(s)\right)-\Phi_{q}^{\prime}(u(s))\right\|_{L_{q^{\prime}}}\|b(s)\|_{L_{q}} \\
& \lesssim\|u(s)\|_{L_{q}}^{q-1}\|b(s)\|_{L_{q}} \leq\|u\|_{L_{\infty}\left(L_{q}\right)}^{q-1}\|b(s)\|_{L_{q}}
\end{aligned}
$$

and $\|u\|_{L_{\infty}\left(L_{q}\right)}^{q-1}\|b\|_{L_{q}} \in L_{1}(0, T)$, which imply, by the dominated convergence theorem,

$$
\int_{0}^{T}\left|\left(\Phi_{q}^{\prime}\left(u^{\varepsilon}\right)-\Phi_{q}^{\prime}(u)\right) b^{\varepsilon}\right| d s \xrightarrow{\varepsilon \rightarrow 0} 0 .
$$

By a completely analogous argument one shows that $\int_{0}^{T}\left|\Phi_{q}^{\prime}(u)\left(b^{\varepsilon}-b\right)\right| d s \xrightarrow{\varepsilon \rightarrow 0} 0$, hence also that

$$
\int_{0}^{t} \Phi_{q}^{\prime}\left(u^{\varepsilon}\right) b^{\varepsilon} d s \xrightarrow{\varepsilon \rightarrow 0} \int_{0}^{t} \Phi_{q}^{\prime}(u) b d s
$$

Let us now show that

$$
M_{\varepsilon}(t):=\int_{0}^{t} \Phi_{q}^{\prime}\left(u^{\varepsilon}\right) G^{\varepsilon} d W \xrightarrow{\varepsilon \rightarrow 0} M(t):=\int_{0}^{t} \Phi_{q}^{\prime}(u) G d W
$$

in probability. Recall that, for the sequence of continuous local martingales $\left(M_{\varepsilon}-M\right)$, one has $\sup _{t \leq T}\left|M_{\varepsilon}(t)-M(t)\right| \rightarrow 0$ in probability if and only if $\left[M_{\varepsilon}-M, M_{\varepsilon}-M\right](T) \rightarrow 0$ in probability (see e.g. [17, Proposition 17.6]). We have

$$
\left[M_{\varepsilon}-M, M_{\varepsilon}-M\right](T)=\int_{0}^{T}\left\|\Phi_{q}^{\prime}\left(u^{\varepsilon}\right) G^{\varepsilon}-\Phi_{q}^{\prime}(u) G\right\|_{\gamma(H, \mathbb{R})}^{2} d s
$$

and, by the triangle inequality,

$$
\begin{aligned}
& \left\|\Phi_{q}^{\prime}\left(u^{\varepsilon}\right) G^{\varepsilon}-\Phi_{q}^{\prime}(u) G\right\|_{\gamma(H, \mathbb{R})} \\
& \left.\qquad\left\|\left\|\Phi_{q}^{\prime}\left(u^{\varepsilon}\right)-\Phi_{q}^{\prime}(u)\right\|_{L_{q^{\prime}}}\right\| G^{\varepsilon}\left\|_{\gamma\left(H, L_{q}\right)}+\right\| \Phi_{q}^{\prime}(u)\left\|_{L_{q^{\prime}}}\right\| G^{\varepsilon}-G \|_{\gamma\left(H, L_{q}\right)}\right)
\end{aligned}
$$

where

$$
\left\|\Phi_{q}^{\prime}\left(u^{\varepsilon}\right)-\Phi_{q}^{\prime}(u)\right\|_{L_{q^{\prime}}}\left\|G^{\varepsilon}\right\|_{\gamma\left(H, L_{q}\right)} \leq\left\|\Phi_{q}^{\prime}\left(u^{\varepsilon}\right)-\Phi_{q}^{\prime}(u)\right\|_{L_{q^{\prime}}}\|G\|_{\gamma\left(H, L_{q}\right)} \xrightarrow{\varepsilon \rightarrow 0} 0
$$

pointwise in the time variable, and $\left\|G^{\varepsilon}-G\right\|_{\gamma\left(H, L_{q}\right)} \rightarrow 0$ because $(I+\varepsilon A)^{-1}$ converges strongly to the identity in $\mathscr{L}\left(L_{q}\right)$. The above also yields

$$
\left\|\Phi_{q}^{\prime}\left(u^{\varepsilon}\right) G^{\varepsilon}-\Phi_{q}^{\prime}(u) G\right\|_{\gamma(H, \mathbb{R})} \lesssim\|u\|_{L_{q}}^{q-1}\|G\|_{\gamma\left(H, L_{q}\right)},
$$

where, since $G \in L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)$ and $u \in L_{\infty}\left(L_{q}\right)$,

$$
\int_{0}^{T}\|u(s)\|_{L_{q}}^{2(q-1)}\|G(s)\|_{\gamma\left(H, L_{q}\right)}^{2} d s \leq\|u\|_{L_{\infty}\left(L_{q}\right)}^{2(q-1)} \int_{0}^{T}\|G(s)\|_{\gamma\left(H, L_{q}\right)}^{2} d s<\infty
$$

Therefore, by the dominated convergence theorem,

$$
\left[M_{\varepsilon}-M, M_{\varepsilon}-M\right](T) \xrightarrow{\varepsilon \rightarrow 0} 0
$$

in probability. Finally, by (4), (6), and the ideal property of $\gamma\left(H, L_{q}\right)$,

$$
\begin{aligned}
\int_{0}^{t} \operatorname{Tr}_{G^{\varepsilon}(s)} \Phi_{q}^{\prime \prime}\left(u^{\varepsilon}(s)\right) d s & \leq q(q-1) \int_{0}^{t}\left\|G^{\varepsilon}(s)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|u^{\varepsilon}(s)\right\|_{L_{q}}^{q-2} d s \\
& \leq q(q-1) \int_{0}^{t}\|G(s)\|_{\gamma\left(H, L_{q}\right)}^{2}\|u(s)\|_{L_{q}}^{q-2} d s
\end{aligned}
$$

We now establish a maximal inequality for stochastic convolutions that might be interesting in its own right (see Remark 4 below). We shall use the following notation, already used in the Introduction:

$$
S \diamond G(t):=\int_{0}^{t} S(t-s) G(s) d W(s)
$$

Theorem 3. Let $p>0$ and $q \geq 2$. If $G$ satisfies the hypothesis of Theorem 11, then the stochastic convolution $S \diamond G$ has (a modification with) continuous paths and

$$
\mathbb{E} \sup _{t \leq T}\|S \diamond G(t)\|_{L_{q}}^{p} \lesssim \mathbb{E}\left(\int_{0}^{T}\|G(s)\|_{\gamma\left(H, L_{q}\right)}^{2} d s\right)^{p / 2}
$$

Proof. We proceed in two steps, first assuming that $G$ takes values in $\mathrm{D}(A)$, then removing this assumption.
Step 1. Let us assume for the moment that $G \in L_{2}(0, T ; \gamma(H, \mathrm{D}(A)))$. As in the proof of Proposition 2, it is easy to see that $S \diamond G$ is the unique strong solution to

$$
d u(t)+A u(t) d t=G(t) d W(t), \quad u(0)=0
$$

Then Itô's formula yields

$$
\|u(t)\|_{L_{q}}^{q}+\int_{0}^{t} \Phi_{q}^{\prime}(u) A u d s=\int_{0}^{t} \Phi_{q}^{\prime}(u) G d W+\frac{1}{2} \int_{0}^{t} \operatorname{Tr}_{G} \Phi_{q}^{\prime \prime}(u) d s
$$

Setting

$$
v:=\|u\|_{L_{q}}^{q}, \quad b:=\frac{1}{2} \operatorname{Tr}_{G} \Phi_{q}^{\prime \prime}(u)-\Phi_{q}^{\prime}(u) A u, \quad g:=\Phi_{q}^{\prime}(u) G
$$

we can write

$$
v(t)=\int_{0}^{t} b(s) d s+\int_{0}^{t} g(s) d W(s)
$$

Let $\alpha \geq 1$ be arbitrary but fixed. Then $\varphi: x \mapsto x^{2 \alpha} \in C^{2}$ with

$$
\varphi^{\prime}(x)=2 \alpha x^{2 \alpha-1}, \quad \varphi^{\prime \prime}(x)=2 \alpha(2 \alpha-1) x^{2(\alpha-1)}
$$

Therefore, by Itô's formula for real processes,

$$
\begin{aligned}
\|u(t)\|_{L_{q}}^{2 \alpha q}=\varphi(v(t))= & \int_{0}^{t}\left(\varphi^{\prime}(v(s)) b(s)+\frac{1}{2} \varphi^{\prime \prime}(v(s))\|g(s)\|_{\gamma(H, \mathbb{R})}^{2}\right) d s \\
& +\int_{0}^{t} \varphi^{\prime}(v(s)) g(s) d W(s)
\end{aligned}
$$

where, by the accretivity of $A$, (4) and (6),

$$
\begin{aligned}
\int_{0}^{t} \varphi^{\prime}(v(s)) b(s) d s & =\int_{0}^{t}\|u(s)\|_{L_{q}}^{(2 \alpha-1) q}\left(\frac{1}{2} \operatorname{Tr}_{G} \Phi_{q}^{\prime \prime}(u(s))-\Phi_{q}^{\prime}(u(s)) A u(s)\right) d s \\
& \lesssim \int_{0}^{T}\|u(s)\|_{L_{q}}^{2(\alpha q-1)}\|G(s)\|_{\gamma\left(H, L_{q}\right)}^{2} d s \\
& \leq\|u\|_{L_{\infty}\left(L_{q}\right)}^{2(\alpha q-1)} \int_{0}^{T}\|G(s)\|_{\gamma\left(H, L_{q}\right)}^{2} d s \\
& \leq \varepsilon\|u\|_{L_{\infty}\left(L_{q}\right)}^{2 \alpha q}+N(\varepsilon)\left(\int_{0}^{T}\|G(s)\|_{\gamma\left(H, L_{q}\right)}^{2} d s\right)^{\alpha q}
\end{aligned}
$$

where we have applied Young's inequality ${ }^{5}$ in the form

$$
x y \leq \varepsilon x^{\frac{\alpha q}{\alpha q-1}}+N(\varepsilon) y^{\alpha q} \quad \forall x, y \geq 0, \varepsilon>0
$$

Similarly,

$$
\int_{0}^{t} \varphi^{\prime \prime}(v(s))\|g(s)\|_{\gamma(H, \mathbb{R})}^{2} d s \lesssim \int_{0}^{t}\|u(s)\|_{L_{q}}^{2(\alpha-1) q}\|g(s)\|_{\gamma(H, \mathbb{R})}^{2} d s
$$

where, by the ideal property of $\gamma$-Radonifying operators and (3),

$$
\|g\|_{\gamma(H, \mathbb{R})}=\left\|\Phi_{q}^{\prime}(u) G\right\|_{\gamma(H, \mathbb{R})} \leq\left\|\Phi_{q}^{\prime}(u)\right\|_{\mathscr{L}\left(L_{q}\right)}\|G\|_{\gamma\left(H, L_{q}\right)} \lesssim\|u\|_{L_{q}}^{q-1}\|G\|_{\gamma\left(H, L_{q}\right)}
$$

hence, proceeding exactly as before,

$$
\begin{aligned}
\int_{0}^{t} \varphi^{\prime \prime}(v(s))\|g(s)\|_{\gamma(H, \mathbb{R})}^{2} d s & \lesssim \int_{0}^{t}\|u(s)\|_{L_{q}}^{2(\alpha q-1)}\|G(s)\|_{\gamma\left(H, L_{q}\right)}^{2} d s \\
& \leq \varepsilon\|u\|_{L_{\infty}\left(L_{q}\right)}^{2 \alpha q}+N(\varepsilon)\left(\int_{0}^{T}\|G(s)\|_{\gamma\left(H, L_{q}\right)}^{2} d s\right)^{\alpha q}
\end{aligned}
$$

Finally, Davis’ inequality yields

$$
\mathbb{E} \sup _{t \leq T}\left|\int_{0}^{t} \varphi^{\prime}(v(s)) g(s) d W(s)\right| \lesssim \mathbb{E}\left(\int_{0}^{T}\left\|\varphi^{\prime}(v(s)) g(s)\right\|_{\gamma(H, \mathbb{R})}^{2} d s\right)^{1 / 2}
$$

where

$$
\left\|\varphi^{\prime}(v) g\right\|_{\gamma(H, \mathbb{R})} \lesssim\|u\|_{L_{q}}^{2 \alpha q-1}\|G\|_{\gamma\left(H, L_{q}\right)}
$$

which implies, by Young's inequality with exponents $2 \alpha q /(2 \alpha q-1)$ and $2 \alpha q$,

$$
\begin{aligned}
\left(\int_{0}^{T}\left\|\varphi^{\prime}(v(s)) g(s)\right\|_{\gamma(H, \mathbb{R})}^{2} d s\right)^{1 / 2} & \lesssim\|u\|_{L_{\infty}\left(L_{q}\right)}^{2 \alpha q-1}\left(\int_{0}^{T}\|G(s)\|_{\gamma\left(H, L_{q}\right)}^{2} d s\right)^{1 / 2} \\
& \leq \varepsilon\|u\|_{L_{\infty}\left(L_{q}\right)}^{2 \alpha q}+N(\varepsilon)\left(\int_{0}^{T}\|G(s)\|_{\gamma\left(H, L_{q}\right)}^{2} d s\right)^{\alpha q}
\end{aligned}
$$

Taking $\varepsilon$ small enough, the claim is proved in the case $p \geq 2 q$. The case $0<p<2 q$ follows by Lenglart's domination inequality (see [20]).

[^5]Step 2. Recall that $G^{\varepsilon}:=(I+\varepsilon A)^{-1} G \rightarrow G$ in $L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)$ as $\varepsilon \rightarrow 0$ and $G^{\varepsilon} \in L_{2}(0, T ; \gamma(H, \mathrm{D}(A)))$, and $u^{\varepsilon}:=S \diamond G^{\varepsilon}=(S \diamond G)^{\varepsilon}$ is the unique strong solution to

$$
d u^{\varepsilon}(t)+A u^{\varepsilon}(t) d t=G^{\varepsilon}(t) d W(t), \quad u^{\varepsilon}(0)=0 .
$$

It is elementary to show, by the previous step, that $\left(u^{\varepsilon}\right)$ is a Cauchy sequence in $\mathbb{H}_{p}\left(L_{q}\right)$, and that its limit is a modification of $S \diamond G$. Since $u^{\varepsilon}$ has continuous paths and the convergence in $\mathbb{H}_{p}\left(L_{q}\right)$ implies almost sure uniform convergence of paths, we conclude that $u$ has a modification with continuous paths.

Remark 4. The previous Theorem is actually a special case of [30], who considered the case of $X$-valued stochastic convolutions, with $X$ a 2-smooth Banach space. Our proof, although similar in spirit (the idea, as a matter of fact, goes back at least to [26]), is interesting in the sense that it does not use any infinite-dimensional calculus. To wit, Itô's formula for the $q$-th power of the $L_{q}$-norm reduces to nothing else than the one-dimensional Itô formula and Fubini's theorem (cf. the proof in [18]).

## 3 Main results

Let $D$ be an open bounded subset of $\mathbb{R}^{n}$ with smooth boundary. All Lebesgue spaces on $D$ will be denoted without explicit mention of the domain, e.g. $L_{q}:=L_{q}(D)$. The mixed-norm spaces $L_{p}\left(0, T ; L_{q}(D)\right)$ will simply be denoted by $L_{p}\left(L_{q}\right)$. Denoting the Lebesgue measure on $[0, T]$ and on $D$ by $d t$ and $d x$, respectively, we define the measure $m:=\mathbb{P} \otimes d t \otimes d x$ on $\Omega \times[0, T] \times D$.

To look for $L_{q}$-valued (mild) solutions to (1), it is clear that the linear operator $A$ should be taken as the generator of a $C_{0}$-semigroup on $L_{q}$, and that the map $B: L_{q} \rightarrow$ $\mathscr{L}\left(H, L_{q}\right)$ should satisfy suitable Lipschitz continuity assumptions. For later use, we introduce the following conditions, where $r>0, s \geq 2$ :
( $\mathrm{A}_{s}$ ) $A$ is a linear $m$-accretive operator on $L_{s}$.
$\left(\mathrm{B}_{r, s}\right)$ The map $B: \Omega \times[0, T] \times L_{s} \rightarrow \gamma\left(H, L_{s}\right)$ is such that $B(\cdot, \cdot, x)$ is $H$-measurable and adapted for all $x \in L_{s}$, there is a constant $\|B\|_{\dot{C}^{0,1}}$ such that

$$
\|B(\omega, t, u)-B(\omega, t, v)\|_{\gamma\left(H, L_{s}\right)} \leq\|B\|_{\dot{C}^{0,1}}\|u-v\|_{L_{s}} \quad \forall(\omega, t) \in \Omega \times[0, T]
$$

and $B(\cdot, \cdot, 0) \in \mathbb{L}_{r}\left(L_{2}\left(0, T ; \gamma\left(H, L_{s}\right)\right)\right)$.
If $A$ satisfies $\left(\mathrm{A}_{s}\right)$, the $C_{0}$-semigroup of contractions generated by $-A$ on $L_{s}$ will be denoted by $S$. Should $A$ satisfy $\left(\mathrm{A}_{s}\right)$ for different values of $s$, we shall not notationally distinguish among different (but consistent) realizations of $A$ and $S$ on different $L_{s}$ spaces.

We assume that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ is increasing and that there exists $d \geq 1$ such that $|f(x)| \lesssim 1+|x|^{d}$ for all $x \in \mathbb{R}$. In particular, $f$ is not assumed to be continuous. We shall denote the maximal monotone graph associated to $f$ (see $\sqrt[42.3]{ }$ ) by the same symbol.
Remark 5. Thanks to the linear term in $u$ on the right-hand side of (1), nothing changes assuming that $f$ (or $A$, or both) is quasi-monotone, i.e. that $f+\delta I$ is monotone for some $\delta>0$.

We shall establish well-posedness of (1) in several classes of processes. The most natural, and most restrictive, notion of solution is the following.

Definition 6. Let $u_{0}$ be an $L_{q}$-valued $\mathcal{F}_{0}$-measurable random variable. A measurable adapted $L_{q}$-valued processes $u$ is a strict mild solution to (11) if $u \in L_{\infty}\left(L_{q}\right)$, there exists an adapted $L_{q}$-valued process $g \in L_{1}\left(L_{q}\right)$, with $g \in f(u) m$-a.e., and, for all $t \in[0, T]$, $S(t-\cdot) B(\cdot, u)$ is stochastically integrable and

$$
\begin{equation*}
u(t)+\int_{0}^{t} S(t-s)(g(s)-\eta u(s)) d s=S(t) u_{0}+\int_{0}^{t} S(t-s) B(s, u(s)) d W(s) \tag{7}
\end{equation*}
$$

Our first main result provides sufficient conditions for the well-posedness of (1) in $\mathbb{H}_{p}\left(L_{q}\right)$. The proof is given in Section 4 below.

Theorem 7. Let $p>0$ and $q \geq 2$ be such that

$$
p^{*}:=\frac{p}{q}(2 d+q-2)>d .
$$

Assume that
(a) $u_{0} \in \mathbb{L}_{p^{*}}\left(L_{q d}\right)$;
(b) hypothesis $\left(\mathrm{A}_{s}\right)$ is satisfied for $s=q$ and $s=q d$;
(c) hypothesis $\left(\mathrm{B}_{r, s}\right)$ is satisfied for $r=p, s=q$ and $r=p^{*}, s=q d$.

Then there exists a unique strict mild solution $u \in \mathbb{H}_{p}\left(L_{q}\right)$ to (11). Moreover, $u$ has continuous paths and the solution map $u_{0} \mapsto u$ is Lipschitz continuous from $\mathbb{L}_{p}\left(L_{q}\right)$ to $\mathbb{H}_{p}\left(L_{q}\right)$.

Relaxing the definition of solution, well-posedness for (11) can be proved for any $p>0$ and $q \geq 2$. The following notion of solution derives from the definition of solution faible by Benilan and Brézis 4]. We first deal with with equations with additive noise, i.e. of the type

$$
\begin{equation*}
d u(t)+A u(t) d t+f(u(t)) d t=\eta u(t) d t+B(t) d W(t), \quad u(0)=u_{0} \tag{8}
\end{equation*}
$$

where $B \in \mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)$.
Definition 8. Let $p>0$ and $q \geq 2$. A process $u \in \mathbb{H}_{p}\left(L_{q}\right)$ is a generalized solution to (8) if there exist sequences $\left(u_{0 n}\right)_{n} \subset \mathbb{L}_{p}\left(L_{q}\right)$, $\left(B_{n}\right)_{n} \subset \mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)$, and $\left(u_{n}\right)_{n} \subset \mathbb{H}_{p}\left(L_{q}\right)$ such that $u_{0 n} \rightarrow u_{0}$ in $\mathbb{L}_{p}\left(L_{q}\right), B_{n} \rightarrow B$ in $\mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)$, and $u_{n} \rightarrow u$ in $\mathbb{H}_{p}\left(L_{q}\right)$ as $n \rightarrow \infty$, where $u_{n}$ is the (unique) strict mild solution to

$$
d u_{n}(t)+A u_{n}(t) d t+f\left(u_{n}(t)\right) d t=\eta u_{n}(t) d t+B_{n}(t) d W(t), \quad u_{n}(0)=u_{0 n}
$$

Theorem 9. Let $p>0$ and $q \geq 2$. Assume that
(a) $u_{0} \in \mathbb{L}_{p}\left(L_{q}\right) ;$
(b) hypothesis $\left(\mathrm{A}_{s}\right)$ is satisfied for $s=q$ and $s=q d$;
(c) $B \in \mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)$.

Then (8) admits a unique generalized solution $u \in \mathbb{H}_{p}\left(L_{q}\right)$. Moreover, $u$ has continuous paths and the solution map $u_{0} \mapsto u$ is Lipschitz continuous from $\mathbb{L}_{p}\left(L_{q}\right)$ to $\mathbb{H}_{p}\left(L_{q}\right)$.

In order to define generalized solutions to (1) we need some preparations. In particular, we (formally) introduce the map $\Gamma$ on $\mathbb{H}_{p}\left(L_{q}\right)$ defined by $\Gamma: v \mapsto w$, where $w$ is the unique generalized solution to

$$
d w(t)+A w(t) d t+f(w(t)) d t=\eta w(t) d t+B(v(t)) d W(t), \quad u(0)=u_{0}
$$

if it exists. Otherwise, if no generalized solution exists, then we set $\Gamma(v)=\varnothing$.
Definition 10. A process $u \in \mathbb{H}_{p}\left(L_{q}\right)$ is a generalized solution to equation (1) if it is a fixed point of $\Gamma$ in $\mathbb{H}_{p}\left(L_{q}\right)$.

Theorem 11. Let $p>0$ and $q \geq 2$. Assume that
(a) $u_{0} \in \mathbb{L}_{p}\left(L_{q}\right)$;
(b) hypothesis $\left(\mathrm{A}_{s}\right)$ is satisfied for $s=q$ and $s=q d$;
(c) hypothesis $\left(\mathrm{B}_{p, q}\right)$ is satisfied.

Then (1) admits a unique generalized solution $u \in \mathbb{H}_{p}\left(L_{q}\right)$. Moreover, $u$ has continuous paths and the solution map $u_{0} \mapsto u$ is Lipschitz continuous from $\mathbb{L}_{p}\left(L_{q}\right)$ to $\mathbb{H}_{p}\left(L_{q}\right)$.

The proofs of Theorems 9 and 11 are given in Section 5 below. Note that, if $u \in$ $\mathbb{H}_{p}\left(L_{q}\right)$ is a generalized solution to (1), we cannot claim that $f(u)$ admits a selection $g$ such that $\int_{0}^{t} S(t-s) g(s) d s$ in (7) is well defined, essentially because we do not have enough integrability for $g$.

Under additional assumptions we obtain existence of a (unique) solution $u$ for which $f(u)$ admits a selection $g$ satisfying the "minimal" integrability condition $g \in \mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$.

Definition 12. Let $u_{0}$ be an $L_{q}$-valued $\mathcal{F}_{0}$-measurable random variable. A measurable adapted $L_{q}$-valued process $u \in L_{\infty}\left(L_{q}\right)$ is a mild solution to equation (11) if there exists $g \in \mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$, with $g \in f(u) m$-a.e., and, for all $t \in[0, T]$, $S(t-\cdot) B(\cdot, u)$ is stochastically integrable and (17) is satisfied for all $t \in[0, T]$.

The corresponding well-posedness result holds in a subset of $\mathbb{H}_{p}\left(L_{q}\right)$ defined in terms of the potential of $f$, for which we assume that $0 \in f(0)$. We need some definitions first: for $q \geq 2$, let $\phi_{q}$ be the homeomorphism of $\mathbb{R}$ defined by $\phi_{q}: x \mapsto x|x|^{q-2}$, and $F: \mathbb{R} \rightarrow \mathbb{R}$ be the potential of $f$, i.e. a convex function such that $\partial F=f$, which we "normalize" so that $F(0)=0$ (in particular $F \geq 0$ ). Similarly, setting $\tilde{f}:=f \circ \phi_{q}^{-1}, \tilde{F}$ stands for the potential of $\tilde{f}$, subject to the same normalization, and $\tilde{F}^{*}$ for its Legendre-Fenchel conjugate. Finally, we set $\hat{F}:=\tilde{F} \circ \phi_{q}$. A simple computation shows that the convex function $\hat{F}$ is the potential of the maximal monotone graph $x \mapsto f(x) \phi_{q}^{\prime}(x)$.

Theorem 13. Let $p \geq q \geq 2$. Assume that
(a) $u_{0} \in \mathbb{L}_{p}\left(L_{q}\right)$;
(b) hypothesis $\left(\mathrm{A}_{s}\right)$ is satisfied for $s \in\{1, q, q d\}$;
(b') the resolvent $R_{\lambda}:=(I+\lambda A)^{-1}, \lambda>0$, is positivity preserving and such that $R_{\lambda}^{\sigma}\left(L_{1}\right) \subset L_{q}$ for some $\sigma \in \mathbb{N}$;
(c) hypothesis $\left(\mathrm{B}_{p, q}\right)$ is satisfied;
(d) $0 \in f(0)$ and $F$ is even.

Then (11) admits a unique mild solution $u \in \mathbb{H}_{p}\left(L_{q}\right)$ such that $\hat{F}(u), \tilde{F}^{*}(g) \in \mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$. Moreover, $u$ has continuous paths and the solution map $u_{0} \mapsto u$ is Lipschitz continuous from $\mathbb{L}_{p}\left(L_{q}\right)$ to $\mathbb{H}_{p}\left(L_{q}\right)$.

The proof is given in Section 6] below. It should be remarked that unconditional well-posedness in $\mathbb{H}_{p}\left(L_{q}\right)$, i.e. without any further conditions on $u$, remains an open problem.

Hypothesis (b) is satisfied by large classes of operators, for instance all generators of sub-Markovian semigroups on $L_{1}(D)$ and of symmetric semigroups on $L_{2}(D)$. Their resolvent are also positivity preserving. The "hypercontractivity" of the resolvent in hypothesis (b') is satisfied, for example, by non-degenerate second order elliptic operators, under very mild regularity assumptions on the coefficients, thanks to elliptic regularity results and Sobolev embedding theorems.

## 4 Strict mild solutions

Consider the regularized equation

$$
\begin{equation*}
d u_{\lambda}(t)+A u_{\lambda}(t) d t+f_{\lambda}\left(u_{\lambda}(t)\right) d t=\eta u_{\lambda}(t) d t+B\left(u_{\lambda}(t)\right) d W(t), \quad u_{\lambda}(0)=u_{0} \tag{9}
\end{equation*}
$$

where $f_{\lambda}: \mathbb{R} \rightarrow \mathbb{R}, \lambda>0$, is the Yosida approximation of $f$, so that $f_{\lambda}$ is Lipschitz continuous on $\mathbb{R}$, as well as on $L_{q}$ (when viewed as evaluation operator). As is natural to expect, well-posedness of (9) in the strict mild sense holds in $\mathbb{H}_{p}\left(L_{q}\right)$ for all $p>0$ and $q \geq 2$.

Proposition 14. Let $p>0$ and $q \geq 2$. Assume that hypotheses $\left(\mathrm{A}_{q}\right)$ and $\left(\mathrm{B}_{p, q}\right)$ are verified. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous and $u_{0} \in \mathbb{L}_{p}\left(L_{q}\right)$, then the equation

$$
d u(t)+A u(t) d t+f(u(t)) d t=B(u(t)) d W(t), \quad u(0)=u_{0}
$$

admits a unique strict mild solution $u \in \mathbb{H}_{p}\left(L_{q}\right)$ with continuous paths, and the solution map $u_{0} \mapsto u$ is Lipschitz continuous from $\mathbb{L}_{p}\left(L_{q}\right)$ to $\mathbb{H}_{p}\left(L_{q}\right)$.

Proof. Since the proof proceeds by the classical fixed point argument, we omit some simple details. Consider the map, formally defined for the moment,

$$
\Gamma:\left(u_{0}, u\right) \mapsto S(t) u_{0}-\int_{0}^{t} S(t-s) f(u(s)) d s+\int_{0}^{t} S(t-s) B(u(s)) d W(s)
$$

To prove existence and uniqueness, it suffices to show that $\Gamma\left(u_{0}, \cdot\right)$ is an everywhere defined contraction on $\mathbb{H}_{p}\left(L_{q}\right)$ for any $u_{0} \in \mathbb{L}_{p}\left(L_{q}\right)$. One has

$$
\begin{aligned}
\sup _{t \leq T} \| \int_{0}^{t} S(t & -s)(f(u(s))-f(v(s))) d s \|_{L_{q}} \\
& \leq \int_{0}^{T}\|f(u(s))-f(v(s))\|_{L_{q}} d s \leq T\|f\|_{\dot{C}^{0,1}} \sup _{t \leq T}\|u(t)-v(t)\|_{L_{q}},
\end{aligned}
$$

hence, writing $S * f$ to denote the second term in the above definition of $\Gamma$,

$$
\|S *(f(u)-f(v))\|_{\mathbb{H}_{p}\left(L_{q}\right)} \leq T\|f\|_{\dot{C}^{0,1}}\|u-v\|_{\mathbb{H}_{p}\left(L_{q}\right)} .
$$

Similarly, it follows by Theorem 3 that

$$
\begin{aligned}
\mathbb{E} \sup _{t \leq T} \| \int_{0}^{t} S & (t-s)(B(u(s))-B(v(s))) d W(s) \|_{L_{q}}^{p} \\
& \lesssim \mathbb{E}\left(\int_{0}^{T} \| B\left(u(s)-B(v(s)) \|_{\gamma\left(H, L_{q}\right)}^{2} d s\right)^{p / 2}\right. \\
& \leq T^{p / 2}\|B\|_{\dot{C}^{0,1}}^{p}\|u-v\|_{\mathbb{H}_{p}\left(L_{q}\right)}^{p}
\end{aligned}
$$

i.e., for a constant $N$ independent of $T$,

$$
\|S \diamond(B(u)-B(v))\|_{\mathbb{H}_{p}\left(L_{q}\right)} \leq N T^{1 / 2}\|B\|_{\dot{C}^{0,1}}\|u-v\|_{\mathbb{H}_{p}\left(L_{q}\right)} .
$$

Therefore, choosing $T$ small enough, one finds a constant $c \in] 0,1[$ such that

$$
\left\|\Gamma\left(u_{0}, u\right)-\Gamma\left(u_{0}, v\right)\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} \leq c\|u-v\|_{\mathbb{H}_{p}\left(L_{q}\right)}
$$

It is clear that $\Gamma\left(\mathbb{H}_{p}\left(L_{q}\right)\right) \subset \mathbb{H}_{p}\left(L_{q}\right)$. Recalling that the function

$$
\begin{aligned}
d: \mathbb{H}_{p}\left(L_{q}\right) \times \mathbb{H}_{p}\left(L_{q}\right) & \rightarrow \mathbb{R}_{+} \\
(x, y) & \mapsto\|x-y\|_{\mathbb{H}_{p}\left(L_{q}\right)}^{1 \wedge p}
\end{aligned}
$$

is a metric on $\mathbb{H}_{p}\left(L_{q}\right)$, Banach's contraction principle yields the existence of a unique fixed point of $\Gamma$ on the complete metric space $\left(\mathbb{H}_{p}\left(L_{q}\right), d\right)$, which is the unique strict mild solution we are looking for on the interval $[0, T]$. Writing $u=\Gamma\left(u_{0}, u\right), v=\Gamma\left(v_{0}, v\right)$, the Lipschitz continuity of the solution map follows by $c<1$ and

$$
\begin{aligned}
\|u-v\|_{\mathbb{H}_{p}\left(L_{q}\right)} & =\left\|\Gamma\left(u_{0}, u\right)-\Gamma\left(v_{0}, v\right)\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} \\
& \leq\left\|\Gamma\left(u_{0}, u\right)-\Gamma\left(u_{0}, v\right)\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}+\left\|\Gamma\left(u_{0}, v\right)-\Gamma\left(v_{0}, v\right)\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} \\
& \leq c\|u-v\|_{\mathbb{H}_{p}\left(L_{q}\right)}+\left\|u_{0}-v_{0}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}
\end{aligned}
$$

By a classical patching argument, the smallness restriction on $T$ can be removed. Continuity of paths follows by Theorem 3,

Remark 15. Even though quite sophisticated well-posedness results exist for SEEs on $L_{q}$ spaces with Lipschitz continuous coefficients (cf. e.g. [8, [29]), the previous simple Proposition does not seem to follow from the existing literature. For instance, in op. cit. the semigroup $S$ is assumed to be analytic (but not necessarily accretive), and (in [29]) solutions are sought in spaces strictly contained in $\mathbb{H}_{p}\left(L_{q}\right)$, and $p>2$. It may indeed be possible to deduce the above well-posedness result from op. cit., but it seems much easier to give a direct proof.

We now proceed to considering equation (11). In this section we first show that a priori estimates on $u_{\lambda}$ imply well-posedness of equation (1), then obtain such estimates (under additional assumptions on $A$ and $B$ ), thus proving Theorem 7 Our argument depends on passing to the limit as $\lambda \rightarrow 0$ in the mild form of the regularized equation (91).

### 4.1 A priori estimates imply well-posedness

We begin establishing sufficient conditions for $\left(u_{\lambda}\right)_{\lambda}$ to be a Cauchy sequence in $\mathbb{H}_{p}\left(L_{q}\right)$, whose limit is then a natural candidate as solution to (11).

Lemma 16. Let $p>0, q \geq 2, p^{*}:=p(2 d+q-2) / q$, and assume that hypotheses $\left(\mathrm{A}_{q}\right)$ and $\left(\mathrm{B}_{p, q}\right)$ are satisfied. If the sequence $\left(u_{\lambda}\right)$ is bounded in $\mathbb{H}_{p^{*}}\left(L_{2 d+q-2}\right)$, then $\left(u_{\lambda}\right)$ is a Cauchy sequence in $\mathbb{H}_{p}\left(L_{q}\right)$.

Proof. Let us define, for a constant parameter $\alpha>\eta$ to be chosen later, $v_{\lambda}(t):=$ $e^{-\alpha t} u_{\lambda}(t)$ for all $t \geq 0$, so that

$$
d v_{\lambda}(t)=-\alpha v_{\lambda}(t)+e^{-\alpha t} u_{\lambda}(t),
$$

hence also, for $\mu>0$,

$$
\begin{align*}
d\left(v_{\lambda}-v_{\mu}\right)+\left((\alpha-\eta)\left(v_{\lambda}-v_{\mu}\right)\right. & \left.+A\left(v_{\lambda}-v_{\mu}\right)+e^{-\alpha t}\left(f_{\lambda}\left(u_{\lambda}\right)-f_{\mu}\left(u_{\mu}\right)\right)\right) d t  \tag{10}\\
& =e^{-\alpha t}\left(B\left(u_{\lambda}\right)-B\left(u_{\mu}\right)\right) d W
\end{align*}
$$

in the (strict) mild sense, with initial condition $v_{\lambda}(0)-v_{\mu}(0)=0$. Proposition 2 yields

$$
\begin{align*}
\left\|v_{\lambda}(t)-v_{\mu}(t)\right\|_{L_{q}}^{q} & +q(\alpha-\eta) \int_{0}^{t}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{q} d s \\
& \quad+\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}-v_{\mu}\right)\left(f_{\lambda}\left(u_{\lambda}\right)-f_{\mu}\left(u_{\mu}\right)\right) d s \\
\leq & \int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}-v_{\mu}\right)\left(B\left(s, u_{\lambda}\right)-B\left(s, u_{\mu}\right)\right) d W  \tag{11}\\
& \quad+\frac{1}{2} q(q-1) \int_{0}^{t}\left\|e^{-\alpha s}\left(B\left(u_{\lambda}\right)-B\left(u_{\mu}\right)\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{q-2} d s .
\end{align*}
$$

We are going to estimate each term appearing in this inequality. Note that $\Phi_{q}^{\prime}(c x)=$ $c^{q-1} \Phi_{q}^{\prime}(x)$ for all $c \in \mathbb{R}_{+}$and $x \in L_{q}$, hence

$$
\Phi_{q}^{\prime}\left(v_{\lambda}-v_{\mu}\right)\left(f_{\lambda}\left(u_{\lambda}\right)-f_{\mu}\left(u_{\mu}\right)\right)=e^{-\alpha(q-1) s} \Phi_{q}^{\prime}\left(u_{\lambda}-u_{\mu}\right)\left(f_{\lambda}\left(u_{\lambda}\right)-f_{\mu}\left(u_{\mu}\right)\right)
$$

and

$$
\Phi_{q}^{\prime}\left(u_{\lambda}-u_{\mu}\right)\left(f_{\lambda}\left(u_{\lambda}\right)-f_{\mu}\left(u_{\mu}\right)\right)=q \int_{D}\left|u_{\lambda}-u_{\mu}\right|^{q-2}\left(u_{\lambda}-u_{\mu}\right)\left(f_{\lambda}\left(u_{\lambda}\right)-f_{\mu}\left(u_{\mu}\right)\right) d x
$$

where, setting $J_{\lambda}:=(I+\lambda f)^{-1}, \lambda>0$, and writing

$$
\begin{aligned}
u_{\lambda}-u_{\mu} & =u_{\lambda}-J_{\lambda} u_{\lambda}+J_{\lambda} u_{\lambda}-J_{\mu} u_{\mu}+J_{\mu} u_{\mu}-u_{\mu} \\
& =\lambda f_{\lambda}\left(u_{\lambda}\right)+J_{\lambda} u_{\lambda}-J_{\mu} u_{\mu}-\mu f_{\mu}\left(u_{\mu}\right),
\end{aligned}
$$

one has, by monotonicity of $f$ and recalling that $f_{\lambda}=f \circ J_{\lambda}$,

$$
\begin{aligned}
\left(f_{\lambda}\left(u_{\lambda}\right)-f_{\mu}\left(u_{\mu}\right)\right)\left(u_{\lambda}-u_{\mu}\right) & \geq\left(f_{\lambda}\left(u_{\lambda}\right)-f_{\mu}\left(u_{\mu}\right)\right)\left(\lambda f_{\lambda}\left(u_{\lambda}\right)-\mu f_{\mu}\left(u_{\mu}\right)\right) \\
& \geq \lambda\left|f_{\lambda}\left(u_{\lambda}\right)\right|^{2}+\mu\left|f_{\mu}\left(u_{\mu}\right)\right|^{2}-(\lambda+\mu)\left|f_{\lambda}\left(u_{\lambda}\right)\right|\left|f_{\mu}\left(u_{\mu}\right)\right| \\
& \geq-\frac{\mu}{2}\left|f_{\lambda}\left(u_{\lambda}\right)\right|^{2}-\frac{\lambda}{2}\left|f_{\mu}\left(u_{\mu}\right)\right|^{2} \\
& \geq-\frac{1}{2}(\lambda+\mu)\left(\left|f_{\lambda}\left(u_{\lambda}\right)\right|^{2}+\left|f_{\mu}\left(u_{\mu}\right)\right|^{2}\right) .
\end{aligned}
$$

Moreover, since $\left|f_{\lambda}(x)\right| \leq|f(x)| \lesssim 1+|x|^{d}$ for all $x \in \mathbb{R}$ and $|x-y|^{q-2} \lesssim(|x|+|y|)^{q-2}$ for all $x, y \in \mathbb{R}$ (the latter inequality holds because $q \geq 2$ ), one infers

$$
\begin{aligned}
\left(f_{\lambda}\left(u_{\lambda}\right)-f_{\mu}\left(u_{\mu}\right)\right) & \left(u_{\lambda}-u_{\mu}\right)\left|u_{\lambda}-u_{\mu}\right|^{q-2} \\
& \gtrsim-(\lambda+\mu)\left(1+\left|u_{\lambda}\right|^{2 d}+\left|u_{\mu}\right|^{2 d}\right)\left|u_{\lambda}-u_{\mu}\right|^{q-2} \\
& \gtrsim-(\lambda+\mu)\left(1+\left(\left|u_{\lambda}\right|+\left|u_{\mu}\right|\right)^{2 d}\right)\left(\left|u_{\lambda}\right|+\left|u_{\mu}\right|\right)^{q-2} \\
& \gtrsim-(\lambda+\mu)\left(1+\left|u_{\lambda}\right|^{2 d+q-2}+\left|u_{\mu}\right|^{2 d+q-2}\right),
\end{aligned}
$$

thus also

$$
\begin{aligned}
\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}-\right. & \left.v_{\mu}\right)\left(f_{\lambda}\left(u_{\lambda}\right)-f_{\mu}\left(u_{\mu}\right)\right) d s \\
& \gtrsim-(\lambda+\mu) \int_{0}^{t} e^{-q \alpha s}\left(1+\left\|u_{\lambda}\right\|_{L_{2 d+q-2}}^{2 d+q-2}+\left\|u_{\mu}\right\|_{L_{2 d+q-2}}^{2 d+q-2}\right) d s \\
& \gtrsim-(\lambda+\mu) \frac{1-e^{-q \alpha t}}{q \alpha}\left(1+\sup _{s \leq t}\left\|u_{\lambda}(s)\right\|_{L_{2 d+q-2}}^{2 d+q-2}+\sup _{s \leq t}\left\|u_{\mu}(s)\right\|_{L_{2 d+q-2}}^{2 d+q-2}\right)
\end{aligned}
$$

which estimates the third term on the left-hand side of (11).
Since $\Phi_{q}^{\prime \prime}(c x)=c^{q-2} \Phi_{q}^{\prime \prime}(x)$ for all $c \in \mathbb{R}_{+}$and $x \in L_{q}$, recalling (4) and (6), the Lipschitz continuity of $B$ implies that the integrand in the last term on the right-hand side of (11) is estimated by

$$
\begin{aligned}
& e^{-q \alpha s}\left\|u_{\lambda}-u_{\mu}\right\|_{L_{q}}^{q-2}\left\|B\left(u_{\lambda}\right)-B\left(u_{\mu}\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2} \\
& \leq\|B\|_{\dot{C}^{0,1}}^{2} e^{-q \alpha s}\left\|u_{\lambda}-u_{\mu}\right\|_{L_{q}}^{q}=\|B\|_{\dot{C}^{0,1}}^{2}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{q} .
\end{aligned}
$$

In particular, collecting the second term on the right-hand side and the second term on the left-hand side of (11), we obtain

$$
\begin{aligned}
\left\|v_{\lambda}(t)-v_{\mu}(t)\right\|_{L_{q}}^{q} & +q\left(\alpha-\eta-\|B\|_{\dot{C}^{0,1}}^{2}(q-1) / 2\right) \int_{0}^{t}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{q} d s \\
\lesssim & (\lambda+\mu) \frac{1-e^{-q \alpha t}}{q \alpha}\left(1+\sup _{s \leq t}\left\|u_{\lambda}(s)\right\|_{L_{2 d+q-2}}^{2 d+q-2}+\sup _{s \leq t}\left\|u_{\mu}(s)\right\|_{L_{2 d+q-2}}^{2 d+q-2}\right) \\
& +\left|\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}-v_{\mu}\right)\left(B\left(u_{\lambda}\right)-B\left(u_{\mu}\right)\right) d W\right|
\end{aligned}
$$

Raising both sides to the power $p / q$, taking supremum in time 6 then expectation, one gets, setting $p^{*}:=(2 d+q-2) p / q$,

$$
\begin{align*}
\mathbb{E} \sup _{t \leq T} \| v_{\lambda}(t)- & v_{\mu}(t) \|_{L_{q}}^{p} \\
& +q^{p / q}\left(\alpha-\eta-\frac{1}{2}\|B\|_{\dot{C}^{0,1}}^{2}(q-1)\right)^{p / q} \mathbb{E}\left(\int_{0}^{T}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{q} d s\right)^{p / q} \\
& \lesssim(\lambda+\mu)^{p / q}\left(1+\mathbb{E} \sup _{t \leq T}\left\|u_{\lambda}(t)\right\|_{L_{2 d+q-2}}^{p^{*}}+\mathbb{E} \sup _{t \leq T}\left\|u_{\mu}(t)\right\|_{L_{2 d+q-2}}^{p^{*}}\right)  \tag{12}\\
& \quad+\mathbb{E} \sup _{t \leq T}\left|\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}-v_{\mu}\right)\left(B\left(u_{\lambda}\right)-B\left(u_{\mu}\right)\right) d W\right|^{p / q}
\end{align*}
$$

where, by the Burkholder-Davis-Gundy inequality,

$$
\begin{aligned}
& \mathbb{E} \sup _{t \leq T}\left|\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}-v_{\mu}\right)\left(B\left(u_{\lambda}\right)-B\left(u_{\mu}\right)\right) d W\right|^{p / q} \\
& \lesssim \mathbb{E}\left(\int_{0}^{T}\left\|e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}-v_{\mu}\right)\left(B\left(u_{\lambda}\right)-B\left(u_{\mu}\right)\right)\right\|_{\gamma(H, \mathbb{R})}^{2} d s\right)^{\frac{p}{2 q}}
\end{aligned}
$$

Thanks to the ideal property of $\gamma$-Radonifying operators, identity (3), and the Lipschitz continuity of $B$, one has

$$
\begin{aligned}
& \int_{0}^{T}\left\|e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}-v_{\mu}\right)\left(B\left(u_{\lambda}\right)-B\left(u_{\mu}\right)\right)\right\|_{\gamma(H, \mathbb{R})}^{2} d s \\
& \leq \int_{0}^{T}\left\|\Phi_{q}^{\prime}\left(v_{\lambda}-v_{\mu}\right)\right\|_{L_{q^{\prime}}}^{2}\left\|e^{-\alpha s}\left(B\left(u_{\lambda}\right)-B\left(u_{\mu}\right)\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2} d s \\
& \lesssim \int_{0}^{T}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{2(q-1)}\left\|e^{-\alpha s}\left(B\left(u_{\lambda}\right)-B\left(u_{\mu}\right)\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2} d s \\
& \leq\|B\|_{\dot{C}^{0,1}}^{2} \sup _{t \leq T}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{2(q-1)} \int_{0}^{T}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{2} d s
\end{aligned}
$$

[^6]This implies

$$
\begin{aligned}
\mathbb{E} \sup _{t \leq T} \mid & \left.\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}-v_{\mu}\right)\left(B\left(u_{\lambda}\right)-B\left(u_{\mu}\right)\right) d W\right|^{p / q} \\
& \left.\lesssim\|B\|_{\dot{C}^{0}, 1}^{p / q} \mathbb{E} \sup _{t \leq T}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{p(q-1) / q}\left(\int_{0}^{T} \| v_{\lambda}-v_{\mu}\right) \|_{L_{q}}^{2} d s\right)^{\frac{p}{2 q}} \\
& \left.\leq \varepsilon\|B\|_{\dot{C}^{0,1}}^{p / q} \mathbb{E} \sup _{t \leq T}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{p}+N_{1}(\varepsilon)\|B\|_{\dot{C}^{0}, 1}^{p / q} \mathbb{E}\left(\int_{0}^{T} \| v_{\lambda}-v_{\mu}\right) \|_{L_{q}}^{2} d s\right)^{\frac{p}{2}} \\
& \left.\leq \varepsilon\|B\|_{\dot{C}^{0}, 1}^{p / q} \mathbb{E} \sup _{t \leq T}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{p}+N_{1}(\varepsilon)\|B\|_{\dot{C}^{0,1}}^{p / q} T^{1-\frac{2}{q}} \mathbb{E}\left(\int_{0}^{T} \| v_{\lambda}-v_{\mu}\right) \|_{L_{q}}^{q} d s\right)^{\frac{p}{q}},
\end{aligned}
$$

where we have used Young's inequality with exponents $q /(q-1)$ and $q$ in the second-last step, and Hölder's inequality with exponents $q / 2$ and $q /(q-2)$ in the last step (recall that $q \geq 2$ ). By (12), we conclude that there exist constants $N_{2}, N_{3}$, independent of $\lambda$, $\mu$ and $\alpha$, with $N_{2}$ also independent of $\varepsilon$, such that

$$
\begin{aligned}
\mathbb{E} \sup _{t \leq T} \| v_{\lambda}(t)- & v_{\mu}(t) \|_{L_{q}}^{p}+q^{p / q}\left(\alpha-\eta-\frac{1}{2}(q-1)\|B\|_{\dot{C}^{0,1}}^{2}\right)^{p / q} \mathbb{E}\left(\int_{0}^{T}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{q} d s\right)^{p / q} \\
\leq & \varepsilon N_{2} \mathbb{E} \sup _{t \leq T}\left\|v_{\lambda}(t)-v_{\mu}(t)\right\|_{L_{q}}^{p}+N_{3} \mathbb{E}\left(\int_{0}^{T}\left\|v_{\lambda}-v_{\mu}\right\|_{L_{q}}^{q} d s\right)^{p / q} \\
& +(\lambda+\mu)^{p / q}\left(1+\mathbb{E} \sup _{t \leq T}\left\|u_{\lambda}(t)\right\|_{L_{2 d+q-2}}^{p^{*}}+\mathbb{E} \sup _{t \leq T}\left\|u_{\mu}(t)\right\|_{L_{2 d+q-2}}^{p^{*}}\right)
\end{aligned}
$$

It is immediately seen that, choosing $\varepsilon$ small enough and $\alpha$ large enough, we are left with

$$
\mathbb{E} \sup _{t \leq T}\left\|v_{\lambda}(t)-v_{\mu}(t)\right\|_{L_{q}}^{p} \lesssim(\lambda+\mu)^{p / q}\left(1+\mathbb{E} \sup _{t \leq T}\left\|u_{\lambda}(t)\right\|_{L_{2 d+q-2}}^{p^{*}}+\mathbb{E} \sup _{t \leq T}\left\|u_{\mu}(t)\right\|_{L_{2 d+q-2}}^{p^{*}}\right)
$$

which implies, by the boundedness of $\left(u_{\lambda}\right)$ in $\mathbb{H}_{p^{*}}\left(L_{2 d+q-2}\right)$, that $\left(u_{\lambda}\right)$ is a Cauchy sequence in $\mathbb{H}_{p, \alpha}\left(L_{q}\right)$, hence also in $\mathbb{H}_{p}\left(L_{q}\right)$ by equivalence of (quasi-)norms.

The strong convergence of $u_{\lambda}$ to a process $u \in \mathbb{H}_{p}\left(L_{q}\right)$ just established does not seem sufficient, unfortunately, to prove that $u$ is a strict mild solution to (1). In fact, writing the regularized equation (9) in its integral form

$$
\begin{align*}
& u_{\lambda}(t)+\int_{0}^{t} S(t-s) f_{\lambda}\left(u_{\lambda}(s)\right) d s \\
&=S(t) u_{0}+\eta \int_{0}^{t} S(t-s) u_{\lambda}(s) d s+\int_{0}^{t} S(t-s) B\left(s, u_{\lambda}(s)\right) d W(s) \tag{13}
\end{align*}
$$

difficulties appear, as is natural to expect, when trying to pass to the limit in the integral on the left-hand side. We are going to show that boundedness assumptions on $\left(u_{\lambda}\right)$ in a smaller space imply convergence of the term containing $f_{\lambda}\left(u_{\lambda}\right)$ in a suitable norm, which is turn yields well-posedness in the strict mild sense. First we state and prove a Lipschitz continuity result for the solution map $u_{0} \mapsto u$ of strict mild solution, which immediately implies uniqueness.

Lemma 17. Let $u_{1}, u_{2}$ be strict mild solutions in $\mathbb{H}_{p}\left(L_{q}\right)$ to (11) with initial conditions $u_{01}$ and $u_{02}$, respectively. Then

$$
\left\|u_{1}-u_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} \lesssim\left\|u_{01}-u_{02}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}
$$

In particular, if (11) admits a strict mild solution $u \in \mathbb{H}_{p}\left(L_{q}\right)$, then it is unique and the solution map is Lipschitz continuous from $\mathbb{L}_{p}\left(L_{q}\right)$ to $\mathbb{H}_{p}\left(L_{q}\right)$.

Proof. We use again an argument based on Itô's formula and elementary inequalities. By definition of strict mild solution, we have $f\left(u_{1}\right), f\left(u_{2}\right) \in L_{1}\left(L_{q}\right)$. Therefore, from

$$
\begin{aligned}
d\left(u_{1}-u_{2}\right)+ & A\left(u_{1}-u_{2}\right) d t+\left(f\left(u_{1}\right)-f\left(u_{2}\right)\right) d t \\
& =\eta\left(u_{1}-u_{2}\right) d t+\left(B\left(u_{1}\right)-B\left(u_{2}\right)\right) d W, \quad u_{1}(0)-u_{2}(0)=u_{01}-u_{02}
\end{aligned}
$$

and Proposition 2, it follows

$$
\begin{aligned}
\| v_{1}(t)-v_{2}(t) & \left\|_{L_{q}}^{q}+q(\alpha-\eta) \int_{0}^{t}\right\| v_{1}-v_{2} \|_{L_{q}}^{q} d s \\
\leq & \left\|u_{01}-u_{02}\right\|_{L_{q}}^{q}+M(t) \\
& \quad+\frac{1}{2} q(q-1) \int_{0}^{t}\left\|e^{-\alpha s}\left(B\left(u_{1}\right)-B\left(u_{2}\right)\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|v_{1}-v_{2}\right\|_{L_{q}}^{q-2} d s,
\end{aligned}
$$

where $v_{i}:=e^{-\alpha \cdot} u_{i}, i=1,2$, and

$$
M(t):=\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{1}(s)-v_{2}(s)\right)\left(B\left(u_{1}(s)\right)-B\left(u_{2}(s)\right)\right) d W(s)
$$

By the Lipschitz continuity of $B$,

$$
\left\|e^{-\alpha s}\left(B\left(u_{1}\right)-B\left(u_{2}\right)\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|v_{1}-v_{2}\right\|_{L_{q}}^{q-2} \leq\|B\|_{\dot{C}^{0,1}}^{2}\left\|v_{1}-v_{2}\right\|_{L_{q}}^{q}
$$

hence

$$
\begin{aligned}
& \left\|v_{1}(t)-v_{2}(t)\right\|_{L_{q}}^{q}+q\left(\alpha-\eta-\frac{1}{2}(q-1)\|B\|_{\dot{C}^{0,1}}^{2}\right) \int_{0}^{t}\left\|v_{1}-v_{2}\right\|_{L_{q}}^{q} d s \\
& \leq\left\|u_{01}-u_{02}\right\|_{L_{q}}^{q}+M(t)
\end{aligned}
$$

and we choose $\alpha$ so that $\left(\alpha-\eta-(q-1)\|B\|_{\dot{C}^{0,1}}^{2} / 2\right)>0$. Taking suprema in time, raising to the power $p / q$, taking expectation, and raising to the power $1 / p$, we get

$$
\begin{gathered}
\left\|v_{1}-v_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}+q^{1 / q}\left(\alpha-\eta-\frac{1}{2}(q-1)\|B\|_{\dot{C}^{0,1}}^{2}\right)^{1 / q}\left\|v_{1}-v_{2}\right\|_{\left.\mathbb{L}_{p}\left(L_{q}\left(L_{q}\right)\right)\right)} \\
\lesssim\left\|u_{01}-u_{02}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}+\left\|M_{T}^{*}\right\|_{\mathbb{L}_{p / q}}^{1 / q}
\end{gathered}
$$

where the implicit constant depends only on $p$ and $q$, and $M_{T}^{*}:=\sup _{t \leq T}\left|M_{t}\right|$. The Burkholder-Davis-Gundy inequality yields

$$
\begin{aligned}
\left\|M_{T}^{*}\right\|_{\mathbb{L}_{p / q}}^{1 / q} & \lesssim\left\|[M, M]_{T}^{1 / 2}\right\|_{\mathbb{L}_{p / q}}^{1 / q}=\left\|[M, M]_{T}^{1 / 2 q}\right\|_{\mathbb{L}_{p}} \\
& =\left\|\left(\int_{0}^{T}\left\|v_{1}-v_{2}\right\|_{L_{q}}^{2(q-1)}\left\|e^{-\alpha s}\left(B\left(u_{1}\right)-B\left(u_{2}\right)\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\right)^{\frac{1}{2 q}}\right\|_{\mathbb{L}_{p}} \\
& \leq\|B\|_{\dot{C}^{0}, 1}^{2 / q}\left\|v_{1}-v_{2}\right\|_{\mathbb{L}_{p}\left(L_{2 q}\left(L_{q}\right)\right)} \\
& \leq \varepsilon\|B\|_{\dot{C}^{0}, 1}^{2 / q}\left\|v_{1}-v_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}+N(\varepsilon)\left\|v_{1}-v_{2}\right\|_{\mathbb{L}_{p}\left(L_{q}\left(L_{q}\right)\right)},
\end{aligned}
$$

where we have used the Lipschitz continuity of $B$ and the inequality

$$
\|\phi\|_{L_{2 q}} \leq\|\phi\|_{L_{q}}^{1 / 2}\|\phi\|_{L_{\infty}}^{1 / 2} \leq \varepsilon\|\phi\|_{L_{\infty}}+N(\varepsilon)\|\phi\|_{L_{q}} \quad \forall \phi \in L_{q} \cap L_{\infty} .
$$

We are thus left with

$$
\begin{aligned}
& \left\|v_{1}-v_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}+q^{1 / q}\left(\alpha-\eta-\frac{1}{2}(q-1)\|B\|_{\dot{C}^{0,1}}^{2}\right)^{1 / q}\left\|v_{1}-v_{2}\right\|_{\left.\mathbb{L}_{p}\left(L_{q}\left(L_{q}\right)\right)\right)} \\
& \quad \lesssim\left\|u_{01}-u_{02}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}+\varepsilon\|B\|_{\dot{C}^{0,1}}^{2 / q}\left\|v_{1}-v_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}+N(\varepsilon)\left\|v_{1}-v_{2}\right\|_{\mathbb{L}_{p}\left(L_{q}\left(L_{q}\right)\right)} .
\end{aligned}
$$

Since the implicit constant is independent of $\alpha$ and $\varepsilon$, this implies, upon choosing $\alpha$ large enough and $\varepsilon$ small enough, and recalling that the (quasi-)norms $\|\cdot\|_{\mathbb{H}_{p, \alpha}\left(L_{q}\right)}$ and $\|\cdot\|_{\mathbb{H}_{p}\left(L_{q}\right)}$ are equivalent,

$$
\left\|u_{1}-u_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} \bar{\sim}\left\|u_{1}-u_{2}\right\|_{\mathbb{H}_{p, \alpha}\left(L_{q}\right)}=\left\|v_{1}-v_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} \lesssim\left\|u_{01}-u_{02}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)} .
$$

To prove uniqueness of solutions in $\mathbb{H}_{p}\left(L_{q}\right)$ we have used in a crucial way the condition $f(u) \in L_{1}\left(L_{q}\right)$, which allows one to apply Proposition 2 (i.e. to use Itô's formula). It is thus natural to look for conditions ensuring weak compactness of $f_{\lambda}\left(u_{\lambda}\right)$ in a functional space contained in $\mathbb{L}_{0}\left(L_{1}\left(L_{q}\right)\right)$. This is the motivation for the following well-posedness result, conditional on boundedness of $\left(u_{\lambda}\right)$ in a suitable norm.

Proposition 18. Let $p>0, q \geq 2$ and $p^{*}:=p(2 d+q-2) / q>d$. If the sequence $\left(u_{\lambda}\right)$ is bounded in $\mathbb{H}_{p^{*}}\left(L_{q d}\right)$, then (1) admits a unique strict mild solution $u \in \mathbb{H}_{p}\left(L_{q}\right)$ with continuous paths, and $u_{0} \mapsto u$ is Lipschitz continuous from $\mathbb{L}_{p}\left(L_{q}\right)$ to $\mathbb{H}_{p}\left(L_{q}\right)$.

Proof. Since $d \geq 1$ implies $q d \geq 2 d+q-2$ and $L_{q d} \hookrightarrow L_{2 d+q-2}$, it follows by Lemma 16 that $u_{\lambda}$ converges strongly to $u \in \mathbb{H}_{p}\left(L_{q}\right)$ as $\lambda \rightarrow 0$. We are going to pass to the limit as $\lambda \rightarrow 0$ in mild form of equation (9), i.e. in (13) above. Let us show that

$$
\int_{0}^{t} S(t-s) B\left(s, u_{\lambda}(s)\right) d W(s) \xrightarrow{\lambda \rightarrow 0} \int_{0}^{t} S(t-s) B(s, u(s)) d W(s)
$$

in probability for all $t \leq T$. In fact $\sqrt[7]{7}$

$$
\begin{aligned}
\mathbb{E} \| \int_{0}^{t} S(t-r) & \left(B\left(u_{\lambda}(r)\right)-B(u(r)) d W(r)\right) \|_{L_{q}}^{p} \\
& \lesssim \mathbb{E}\left(\int_{0}^{t} \| S(t-r)\left(B\left(u_{\lambda}(r)\right)-B(u(r)) \|_{\gamma\left(H, L_{q}\right)}^{2} d r\right)^{p / 2}\right. \\
& \lesssim T \mathbb{E}\left(\int_{0}^{t}\left\|u_{\lambda}(r)-u(r)\right\|_{L_{q}}^{2} d r\right)^{p / 2},
\end{aligned}
$$

thanks to the ideal property of $\gamma\left(H, L_{q}\right)$, the contractivity of $S$, and the Lipschitz continuity of $B$. The last term tends to zero as $\lambda \rightarrow 0$ because $u_{\lambda} \rightarrow u$ in $\mathbb{H}_{p}\left(L_{q}\right)$ and $\mathbb{H}_{p}\left(L_{q}\right) \hookrightarrow \mathbb{L}_{p}\left(L_{2}\left(L_{q}\right)\right)$.

[^7]We can now consider the term in (13) involving $f_{\lambda}\left(u_{\lambda}\right)$. It follows from $\left|f_{\lambda}\right| \leq|f|$ and $|f(x)| \lesssim 1+|x|^{d}$ for all $x \in \mathbb{R}$ that, for any $s>1$,

$$
\left\|f_{\lambda}\left(u_{\lambda}\right)\right\|_{\mathbb{I}_{p^{*} / d}\left(L_{s}\left(L_{q}\right)\right)} \lesssim 1+\left\|u_{\lambda}\right\|_{\mathbb{H}_{p^{*}}\left(L_{q d}\right)}^{d},
$$

so that $f_{\lambda}\left(u_{\lambda}\right)=f\left(J_{\lambda} u_{\lambda}\right)$ is bounded, hence weakly compact, in the reflexive Banach space $E:=\mathbb{L}_{p^{*} / d}\left(L_{s}\left(L_{q}\right)\right)$ (recall that $p^{*} / d>1$ by assumption). In particular, there exists $g \in E$ and a subsequence of $\lambda$, denoted by the same symbol, such that $f\left(J_{\lambda} u_{\lambda}\right) \rightarrow g$ weakly in $E$ as $\lambda \rightarrow 0$. Since $J_{\lambda} u_{\lambda} \rightarrow u$ strongly in $E$ as $\lambda \rightarrow 0$ and $f$, as an $m$-accretive operator on $E$, is also strongly-weakly closed thereon, we infer that $g \in f(u) m$-a.e.. Since the linear operator

$$
\phi \mapsto \int_{0} S(\cdot-s) \phi(s) d s
$$

is strongly (hence also weakly) continuous on $E$, we infer that

$$
\int_{0} S(\cdot-s) f_{\lambda}\left(u_{\lambda}(s)\right) d s \xrightarrow{\lambda \rightarrow 0} \int_{0} S(\cdot-s) g(s) d s
$$

weakly in $E$, hence that

$$
u(t)=S(t) u_{0}-\int_{0}^{t} S(t-s)(g(s)-\eta u(s)) d s+\int_{0}^{t} S(t-s) B(s, u(s)) d W(s)
$$

for almost all $t \in[0, T]$. However, since $u$ admits a continuous $L_{q}$-valued modification, the identity must be satisfied for all $t \in[0, T]$. Existence is thus proved, and uniqueness as well as continuous dependence on the initial datum follow by the previous Lemma. The mild solution $u$, being a strong limit in $\mathbb{H}_{p}\left(L_{q}\right)$ of $\left(u_{\lambda}\right)$, inherits the path continuity of the latter.

### 4.2 A priori estimates

As we have just seen, well-posedness in the strict mild sense in $\mathbb{H}_{p}\left(L_{q}\right)$ for (1) can be reduced to obtaining a priori estimates for $\left(u_{\lambda}\right)$ in $\mathbb{H}_{p_{1}}\left(L_{q_{1}}\right)$, with $p_{1}>p$ and $q_{1}>q$ suitably chosen.

Proposition 19. Let $p>0$ and $q \geq 2$. If $u_{0} \in \mathbb{L}_{p}\left(\mathcal{F}_{0} ; L_{q}\right)$ and hypotheses $\left(\mathrm{A}_{q}\right)$, $\left(\mathrm{B}_{p, q}\right)$ are satisfied, then there exists a constant $N$, independent of $\lambda$, such that

$$
\mathbb{E} \sup _{t \leq T}\left\|u_{\lambda}(t)\right\|_{L_{q}}^{p} \leq N\left(1+\mathbb{E}\left\|u_{0}\right\|_{L_{q}}^{p}\right) .
$$

Proof. The proof uses arguments analogous to ones already seen, hence we omit some detail. As in previous proofs, we begin observing that the regularized equation (9) admits a unique $L_{q}$-valued solution $u_{\lambda}$, and, setting $v_{\lambda}(t):=e^{-\alpha t} u_{\lambda}(t)$ for all $t \geq 0$, with $\alpha>\eta$ a constant to be fixed later, Proposition 2 implies

$$
\begin{aligned}
\left\|v_{\lambda}(t)\right\|_{L_{q}}^{q}+ & q(\alpha-\eta) \int_{0}^{t}\left\|v_{\lambda}\right\|_{L_{q}}^{q} d s+\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}\right) f_{\lambda}\left(u_{\lambda}\right) d s \\
\leq & \int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}\right) B\left(u_{\lambda}\right) d W \\
& +\frac{1}{2} q(q-1) \int_{0}^{t}\left\|e^{-\alpha s} B\left(u_{\lambda}\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|v_{\lambda}\right\|_{L_{q}}^{q-2} d s .
\end{aligned}
$$

We shall denote the stochastic integral in the previous inequality by $M$. By the homogeneity of order $q-1$ of $\Phi_{q}^{\prime}$, the monotonicity of $f_{\lambda}$, the inequality $\left|f_{\lambda}\right| \leq|f|$, the identity $\left\|\Phi_{q}^{\prime}(x)\right\|_{L_{q^{\prime}}}=q\|x\|_{L_{q}}^{q-1}$, and the elementary inequality $a^{q-1} \leq 1+a^{q}$ for all $a \geq 0$, we have

$$
\begin{aligned}
e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}\right) f_{\lambda}\left(u_{\lambda}\right) & =e^{-q \alpha s} \Phi_{q}^{\prime}\left(u_{\lambda}\right)\left(f_{\lambda}\left(u_{\lambda}\right)-f_{\lambda}(0)+f_{\lambda}(0)\right) \\
& \geq e^{-q \alpha s} \Phi_{q}^{\prime}\left(u_{\lambda}\right) f_{\lambda}(0) \geq-e^{-q \alpha s} \Phi_{q}^{\prime}\left(u_{\lambda}\right)|f(0)| \\
& \geq-q e^{-q \alpha s}|f(0)|\left\|u_{\lambda}\right\|_{L_{q}}^{q-1} \\
& \geq-q e^{-q \alpha s}|f(0)|-q e^{-q \alpha s}|f(0)|\left\|u_{\lambda}\right\|_{L_{q}}^{q}
\end{aligned}
$$

hence

$$
\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{\lambda}\right) f_{\lambda}\left(u_{\lambda}\right) d s \geq-\frac{|f(0)|}{\alpha}\left(1-e^{-q \alpha t}\right)-q|f(0)| \int_{0}^{t}\left\|v_{\lambda}\right\|_{L_{q}}^{q} d s
$$

Similarly, by the triangle inequality and Lipschitz continuity of $B$,

$$
\left\|e^{-\alpha s} B\left(u_{\lambda}\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2} \leq 2\|B\|_{\dot{C}^{0,1}}^{2}\left\|v_{\lambda}\right\|_{L_{q}}^{2}+2 e^{-2 \alpha s}\|B(0)\|_{\gamma\left(H, L_{q}\right)}^{2}
$$

hence, thanks to the elementary inequality $a^{q-2} \leq 1+a^{q}, a \geq 0$,

$$
\begin{aligned}
& \frac{1}{2} q(q-1) \int_{0}^{t}\left\|e^{-\alpha s} B\left(u_{\lambda}\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|v_{\lambda}\right\|_{L_{q}}^{q-2} d s \\
& \leq \\
& \quad q(q-1)\|B\|_{\dot{C}^{0,1}}^{2} \int_{0}^{t}\left(1+e^{-2 \alpha s}\|B(0)\|_{\gamma\left(H, L_{q}\right)}^{2}\right)\left\|v_{\lambda}\right\|_{L_{q}}^{q} d s \\
& \quad+\frac{q(q-1)}{2 \alpha}\|B(0)\|_{\gamma\left(H, L_{q}\right)}^{2}\left(1-e^{-2 \alpha t}\right)
\end{aligned}
$$

We can thus write

$$
\begin{aligned}
\left\|v_{\lambda}(t)\right\|_{L_{q}}^{q}+ & q\left(\alpha-\eta-|f(0)|-(q-1)\|B\|_{\dot{C}^{0}, 1}^{2}\right) \int_{0}^{t}\left\|v_{\lambda}\right\|_{L_{q}}^{q} d s \\
& \leq \frac{|f(0)|}{\alpha}+\frac{q(q-1)}{2 \alpha}\|B(0)\|_{\gamma\left(H, L_{q}\right)}^{2}+M_{T}^{*}
\end{aligned}
$$

and we choose the constant $\alpha$ larger than $\eta+|f(0)|+(q-1)\|B\|_{\dot{C}^{0,1}}^{2}$. Raising to the power $p / q$, taking suprema, then expectation, and taking the power $1 / p$, we are left with

$$
\left\|v_{\lambda}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}+N_{1}\left\|v_{\lambda}\right\|_{\mathbb{L}_{p}\left(L_{q}\left(L_{q}\right)\right)} \lesssim N_{2}+\left\|M_{T}^{*}\right\|_{\mathbb{L}_{p / q}}^{1 / q}
$$

where

$$
\begin{aligned}
& N_{1}:=q^{1 / q}\left(\alpha-\eta-|f(0)|-(q-1)\|B\|_{\dot{C}^{0,1}}^{2}\right)^{1 / q} \\
& N_{2}:=\left(\frac{|f(0)|}{\alpha}+\frac{q(q-1)}{2 \alpha}\|B(0)\|_{\gamma\left(H, L_{q}\right)}^{2}\right)^{1 / q}
\end{aligned}
$$

and, by an argument based on the Burkholder-Davis-Gundy inequality and norm interpolation, as in the proof of Lemma 17 ,

$$
\begin{aligned}
\left\|M_{T}^{*}\right\|_{\mathbb{L}_{p / q}}^{1 / q} & \lesssim\left\|[M, M]_{T}^{1 / 2}\right\|_{\mathbb{L}_{p / q}}^{1 / q}=\left\|[M, M]_{T}^{\frac{1}{2 q}}\right\|_{\mathbb{I}_{p}} \\
& \leq \varepsilon\|B\|^{2 / q}\left\|v_{\lambda}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}+N_{3}(\varepsilon)\left\|v_{\lambda}\right\|_{\mathbb{L}_{p}\left(L_{q}\left(L_{q}\right)\right)}
\end{aligned}
$$

Collecting estimates yields

$$
\begin{aligned}
\left\|v_{\lambda}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} & +N_{1}(\alpha)\left\|v_{\lambda}\right\|_{\mathbb{L}_{p}\left(L_{q}\left(L_{q}\right)\right)} \\
& \leq N_{4}\left(N_{2}+\varepsilon\|B\|_{\dot{C}^{0,1}}^{2 / q}\left\|v_{\lambda}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}+N_{3}(\varepsilon)\left\|v_{\lambda}\right\|_{\mathbb{L}_{p}\left(L_{q}\left(L_{q}\right)\right)}\right)
\end{aligned}
$$

for a constant $N_{4}$ independent of $\alpha$ and $\varepsilon$. The proof is completed choosing first $\varepsilon$ small enough, and then $\alpha$ large enough.

### 4.3 Proof of Theorem 7

Since $u_{0} \in \mathbb{L}_{p^{*}}\left(L_{q d}\right)$ and hypotheses $\left(\mathrm{A}_{q d}\right),\left(\mathrm{B}_{p^{*}, q d}\right)$ are satisfied, the regularized equation (9) admits a unique $L_{q d}$-valued (strict) mild solution $u_{\lambda}$ for all $\lambda>0$. By Proposition 19 the sequence $\left(u_{\lambda}\right)$ is bounded in $\mathbb{H}_{p^{*}}\left(L_{q d}\right)$, hence Proposition 18 allows us to conclude that (11) admits a unique strict mild solution $u \in \mathbb{H}_{p}\left(L_{q}\right)$ and that the solution map $u_{0} \mapsto u$ is Lipschitz continuous from $\mathbb{L}_{p}\left(L_{q}\right)$ to $\mathbb{H}_{p}\left(L_{q}\right)$.

## 5 Generalized solutions

In this section we prove Theorems 9 and 11. The main tool is the Lipschitz continuity of the map $\left(u_{0}, B\right) \mapsto u$ established in the next Lemma. For reasons of notational compactness, we set $L_{r, \alpha}(0, T ; X):=L_{r}([0, T], \mu ; X)$, where $\mu$ is the measure on $[0, T]$ with density $t \mapsto e^{-r \alpha t}$.

Lemma 20. Assume that $p>0$ and $q \geq 2$. Let $u_{1}, u_{2} \in \mathbb{H}_{p}\left(L_{q}\right)$ be strict mild solutions to

$$
d u_{1}+A u_{1} d t+f\left(u_{1}\right) d t=\eta u_{1} d t+B_{1} d W, \quad u_{1}(0)=u_{01}
$$

and

$$
d u_{2}+A u_{2} d t+f\left(u_{2}\right) d t=\eta u_{2} d t+B_{2} d W, \quad u_{2}(0)=u_{02}
$$

respectively, where $B_{1}, B_{2} \in \mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)$ and $u_{01}$, $u_{02} \in \mathbb{L}_{p}\left(L_{q}\right)$. Then, for any $\alpha>\eta$,

$$
\begin{equation*}
\left\|u_{1}-u_{2}\right\|_{\mathbb{H}_{p, \alpha}\left(L_{q}\right)} \lesssim\left\|u_{01}-u_{02}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}+\left\|B_{1}-B_{2}\right\|_{\mathbb{L}_{p}\left(L_{2, \alpha}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)} \tag{14}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\left\|u_{1}-u_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} \lesssim\left\|u_{01}-u_{02}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}+\left\|B_{1}-B_{2}\right\|_{\mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)} \tag{15}
\end{equation*}
$$

Moreover, the same estimates hold for generalized solutions.

Proof. The proof uses again arguments analogous to those used in the proof of Lemma 17, therefore some detail will be omitted.

Setting $v_{i}(t):=e^{-\alpha t} u_{i}(t), i=1,2$, for all $t \geq 0$, with $\alpha>\eta$, it follows by Proposition 2 and monotonicity of $f$,

$$
\begin{aligned}
\| v_{1}(t)-v_{2}(t) & \left\|_{L_{q}}^{q}+q(\alpha-\eta) \int_{0}^{t}\right\| v_{1}-v_{2} \|_{L_{q}}^{q} d s \\
\leq & \left\|u_{01}-u_{02}\right\|_{L_{q}}^{q}+\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{1}-v_{2}\right)\left(B_{1}-B_{2}\right) d W \\
& +\frac{1}{2} q(q-1) \int_{0}^{t}\left\|e^{-\alpha s}\left(B_{1}-B_{2}\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|v_{1}-v_{2}\right\|_{L_{q}}^{q-2} d s
\end{aligned}
$$

where, by Young's inequality with exponents $q /(q-2)$ and $q / 2$, the last term is estimated by

$$
\begin{aligned}
\sup _{s \leq t} \| v_{1}(s)- & v_{2}(s)\left\|_{L_{q}}^{q-2} \int_{0}^{t} e^{-2 \alpha s}\right\| B_{1}-B_{2} \|_{\gamma\left(H, L_{q}\right)}^{2} d s \\
& \leq \varepsilon \sup _{s \leq t}\left\|v_{1}(s)-v_{2}(s)\right\|_{L_{q}}^{q}+N(\varepsilon)\left(\int_{0}^{t} e^{-2 \alpha s}\left\|B_{1}-B_{2}\right\|_{\gamma\left(H, L_{q}\right)}^{2} d s\right)^{q / 2}
\end{aligned}
$$

Choosing $\varepsilon$ smaller than one, hence, as before, raising to the power $p / q$, taking suprema in time, then expectation, and finally power $1 / p$, we get

$$
\begin{aligned}
&\left\|v_{1}-v_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}+(\alpha-\eta)^{1 / q}\left\|v_{1}-v_{2}\right\|_{\mathbb{L}_{p}\left(L_{q}\left(L_{q}\right)\right)} \\
& \lesssim\left\|u_{01}-u_{02}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}+\left\|B_{1}-B_{2}\right\|_{\mathbb{L}_{p}\left(L_{2, \alpha}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)}+\left\|M_{T}^{*}\right\|_{\mathbb{L}_{p / q}}^{1 / q}
\end{aligned}
$$

where $M$ denotes the stochastic integral with respect to $W$ in the first inequality above. Applying the Burkholder-Davis-Gundy inequality, elementary estimates, and Young's inequality with exponents $q /(q-1)$ and $q$, we have

$$
\begin{aligned}
\left\|M_{T}^{*}\right\|_{\mathbb{L}_{p / q}}^{1 / q} & \lesssim\left\|[M, M]_{T}^{\frac{1}{2 q}}\right\|_{\mathbb{L}_{p}} \\
& =\left\|\left(\int_{0}^{T}\left\|v_{1}-v_{2}\right\|_{L_{q}}^{2(q-1)}\left\|B_{1}-B_{2}\right\|_{\gamma\left(H, L_{q}\right)}^{2} e^{-2 \alpha s} d s\right)^{\frac{1}{2 q}}\right\|_{\mathbb{L}_{p}} \\
& \lesssim \varepsilon\left\|v_{1}-v_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}+N(\varepsilon)\left\|B_{1}-B_{2}\right\|_{\mathbb{L}_{p}\left(L_{2, \alpha}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)}
\end{aligned}
$$

Choosing $\varepsilon$ suitably small, the last two inequalities yield

$$
\left\|v_{1}-v_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} \lesssim\left\|u_{01}-u_{02}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}+\left\|B_{1}-B_{2}\right\|_{\mathbb{L}_{p}\left(L_{2, \alpha}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)}
$$

which establishes the claim because

$$
\begin{aligned}
\left\|u_{1}-u_{2}\right\|_{\mathbb{H}_{p, \alpha}\left(L_{q}\right)} & =\left\|v_{1}-v_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} \\
\left\|u_{1}-u_{2}\right\|_{\mathbb{L}_{p}\left(L_{q, \alpha}\left(L_{q}\right)\right)} & =\left\|v_{1}-v_{2}\right\|_{\mathbb{L}_{p}\left(L_{q}\left(L_{q}\right)\right)}
\end{aligned}
$$

and equivalence of the norms in $\mathbb{H}_{p, \alpha}\left(L_{q}\right), \alpha \geq 0$. It is easily seen that estimates (14) and (15) are stable with respect to passage to the limit, hence the remain true for generalized solutions.

Proof of Theorem 9. Let $p_{1} \geq p$ be such that

$$
p_{1}^{*}:=\frac{p_{1}}{q}(2 d+q-2)>d
$$

Note that $d \geq 1$ implies $p_{1}^{*} \geq p_{1} \geq p$, hence $\mathbb{L}_{p_{1}^{*}}\left(L_{q d}\right)$ is dense in $\mathbb{L}_{p}\left(L_{q}\right)$, so that there exists a sequence

$$
\left(u_{0 n}\right)_{n \in \mathbb{N}} \subset \mathbb{L}_{p_{1}^{*}}\left(L_{q d}\right)
$$

such that $u_{0 n} \rightarrow u_{0}$ in $\mathbb{L}_{p}\left(L_{q}\right)$ as $n \rightarrow \infty$. Similarly, recalling the isomorphism $\gamma\left(H, L_{q}\right) \simeq$ $L_{q}(H)$, there also exists a sequence

$$
\left(B_{n}\right)_{n \in \mathbb{N}} \subset \mathbb{L}_{p_{1}^{*}}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q d}\right)\right)\right)
$$

such that $B_{n} \rightarrow B$ in $\mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)$ as $n \rightarrow \infty$. Then, by Theorem 7, for each $n \in \mathbb{N}$ the equation

$$
d u_{n}(t)+A u_{n}(t) d t+f\left(u_{n}(t)\right) d t=\eta u(t) d t+B_{n}(t) d W(t), \quad u_{n}(0)=u_{0 n}
$$

admits a unique strict mild solution $u_{n} \in \mathbb{H}_{p_{1}}\left(L_{q}\right) \hookrightarrow \mathbb{H}_{p}\left(L_{q}\right)$, and the previous lemma yields

$$
\left\|u_{n}-u_{m}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} \lesssim\left\|u_{0 n}-u_{0 m}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}+\left\|B_{n}-B_{m}\right\|_{\mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)}
$$

hence $\left(u_{n}\right)$ is a Cauchy sequence in $\mathbb{H}_{p}\left(L_{q}\right)$. This implies that its strong limit $u \in \mathbb{H}_{p}\left(L_{q}\right)$ is a generalized solution to (8). Uniqueness and Lipschitz dependence on the initial datum follow immediately by (14).

Proof of Theorem 11. Let $w_{1}, w_{2} \in \mathbb{H}_{p}\left(L_{q}\right)$ and consider the equation, for $i=1,2$,

$$
d u_{i}(t)+A u_{i}(t) d t+f\left(u_{i}(t)\right) d t=\eta u_{i}(t) d t+B\left(w_{i}(t)\right) d W(t), \quad u_{i}(0)=u_{0 i}
$$

The assumptions on $B$ immediately imply that $B(w) \in \mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)$ for all $w \in \mathbb{H}_{p}\left(L_{q}\right)$, hence the previous equation admits a unique generalized solution $u_{i} \in$ $\mathbb{H}_{p}\left(L_{q}\right)$ by Theorem 9, In particular, the domain of the map $\Gamma$ is the whole $\mathbb{H}_{p}\left(L_{q}\right)$ and its image is contained in $\mathbb{H}_{p}\left(L_{q}\right)$. We are now going to show that $\Gamma$ is a contraction in $\mathbb{H}_{p, \alpha}\left(L_{q}\right)$ for small $T$. In fact, inequality (14) yields

$$
\left\|u_{1}-u_{2}\right\|_{\mathbb{H}_{p, \alpha}\left(L_{q}\right)} \lesssim\left\|u_{01}-u_{02}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}+\left\|B\left(w_{1}\right)-B\left(w_{2}\right)\right\|_{\mathbb{L}_{p}\left(L_{2, \alpha}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)},
$$

where, by the Lipschitz continuity of $B$,

$$
\begin{aligned}
\| B\left(w_{1}\right)-B & \left(w_{2}\right) \|_{\mathbb{L}_{p}\left(L_{2, \alpha}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)}^{p} \\
& =\mathbb{E}\left(\int_{0}^{T} e^{-2 \alpha s}\left\|B\left(w_{1}\right)-B\left(w_{2}\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2} d s\right)^{p / 2} \\
& \leq\|B\|_{\dot{C}^{0,1}}^{p} \mathbb{E}\left(\int_{0}^{T} e^{-2 \alpha s}\left\|w_{1}-w_{2}\right\|_{L_{q}}^{2} d s\right)^{p / 2} \\
& \leq\|B\|_{\dot{C}^{0,1}}^{p} T^{p / 2}\left\|w_{1}-w_{2}\right\|_{\mathbb{H}_{p, \alpha}\left(L_{q}\right)}^{p}
\end{aligned}
$$

This implies that $\Gamma$ is a contraction on $\mathbb{H}_{p, \alpha}\left(L_{q}\right)$ for $T$ small enough, hence that a unique generalized solution exists that depends Lipschitz continuously on the initial datum. By a classical patching procedure, the result can be extended to arbitrary finite $T$.

## 6 Mild solutions

In this section we prove Theorem 13. The proof is split into two parts: first we prove existence, showing that one can pass to the limit in the term of (13) containing $f_{\lambda}\left(u_{\lambda}\right)$ in the weak topology of $\mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$. Then we prove uniqueness, as a consequence of continuous dependence on the initial datum, via an extension of Proposition 2, We proceed this way because, as will be apparent soon, the symmetry condition on $F$ and the "regularizing" assumptions on $A$ are needed only to prove uniqueness.

### 6.1 Existence

We shall use the following weak convergence criterion (see [7, Theorem 18]).
Lemma 21. Let $(Y, \mathcal{A}, \mu)$ be a finite measure space. Assume that $\gamma$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $\mathrm{D}(\gamma)=\mathbb{R}$ and $0 \in \gamma(0)$. If the sequences of functions $\left(z_{n}\right)$, $\left(y_{n}\right) \subset L_{0}(Y, \mathcal{A}, \mu)$ indexed by $n \in \mathbb{N}$ are such that $\lim _{n \rightarrow \infty} y_{n}=y \mu$-a.e., $z_{n} \in \gamma\left(y_{n}\right)$ $\mu$-a.e. for all $n$, and there exists a constant $N$ such that

$$
\int_{Y} z_{n} y_{n} d \mu<N \quad \forall n \in \mathbb{N}
$$

then there exist $z \in L_{1}(Y, \mu)$ and a subsequence $\left(n_{k}\right)_{k}$ such that $z_{n_{k}} \rightarrow z$ weakly in $L_{1}(Y, \mu)$ as $k \rightarrow \infty$ and $z \in \gamma(y) \mu$-a.e.

Sketch of proof. The weak compactness in $L_{1}(Y, \mu)$ of $\left(z_{n}\right)$ is a consequence of $z_{n} y_{n}=$ $G\left(y_{n}\right)+G^{*}\left(z_{n}\right)$, where $G$ is a convex function with $G(0)=0$ such that $\gamma=\partial G$. In fact, $\mathrm{D}(\gamma)=\mathbb{R}$ implies that $G^{*}$ is superlinear at infinity (see 2.1 ), which in turn implies, by the criterion of de la Vallée-Poussin (see e.g. [5, Theorem 4.5.9]), that $\left(z_{n}\right)$ is uniformly integrable in $L_{1}(Y, \mu)$, hence, by the Dunford-Pettis theorem, it is relatively weakly compact thereon (see e.g. [5] Corollary 4.7.19]). The fact that $z \in \gamma(y) \mu$-a.e. requires a further (short) argument based on monotonicity (see [7] for details).

Let us also recall some further notation. For $q \geq 2$, define the homeomorphism $\phi_{q}$ of $\mathbb{R}$ and the maximal monotone graph $\tilde{f}$ in $\mathbb{R} \times \mathbb{R}$ as

$$
\phi_{q}: x \mapsto x|x|^{q-2} \equiv|x|^{q-1} \operatorname{sgn} x, \quad \tilde{f}:=f \circ \phi_{q}^{-1}
$$

Since $0 \in \tilde{f}(0)$, there exists a convex function $\tilde{F}: \mathbb{R} \rightarrow \mathbb{R}$ with $\tilde{F}(0)=0$ such that $\partial \tilde{F}=\tilde{f}$. As usual, we shall denote by $\tilde{F}^{*}$ the convex conjugate of $\tilde{F}$. We recall that $\tilde{F}^{*}$ is convex and superlinear at infinity, because $\tilde{f}$ is finite on the whole real line (see 82.1 ).

In the next statement $g$ stands for the process defined in Definition 12 ,
Proposition 22. Let $p \geq q \geq 2$ and $0 \in f(0)$. Assume that
(a) $u_{0} \in \mathbb{L}_{p}\left(L_{q}\right)$;
(b) hypothesis $\left(\mathrm{A}_{s}\right)$ is satisfied for $s \in\{1, q, q d\}$;
(c) hypothesis $\left(\mathrm{B}_{p, q}\right)$ is satisfied.

Then there exists a mild solution $u \in \mathbb{H}_{p}\left(L_{q}\right)$ to (11). Moreover, $u$ has continuous paths and satisfies $\hat{F}(u), \tilde{F}^{*}(g) \in \mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$.

Proof. We proceed in several steps.
Step 1. We begin showing that the generalized solution to (11), which exists and is unique thanks to Theorem 11, can be approximated by strict mild solutions to suitable equations. Let $u$ be the generalized solution to (1) and $\delta>0$. Then there exists $n_{0} \in \mathbb{N}$ such that $\left\|u-u_{n}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}<\delta$ for all $n>n_{0}$, where $u_{n}$ is the unique generalized solution to

$$
d u_{n}+A u_{n} d t+f\left(u_{n}\right) d t=\eta u_{n} d t+B\left(u_{n-1}\right) d W, \quad u_{n}(0)=u_{0}
$$

In turn, for any $n>n_{0}$, there exists $\nu=\nu(n)$ and $u_{0 \nu},\left[B\left(u_{n-1}\right)\right]_{\nu}$ such that

$$
\left\|u_{n}-u_{n}^{\nu}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} \lesssim\left\|u_{0}-u_{0 \nu}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}+\left\|\left[B\left(u_{n-1}\right)\right]_{\nu}-B\left(u_{n-1}\right)\right\|_{\mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)}<\delta
$$

where $u_{n}^{\nu}$ is the unique strict mild solution to

$$
d u_{n}^{\nu}+A u_{n}^{\nu} d t+f\left(u_{n}^{\nu}\right) d t=\eta u_{n}^{\nu} d t+\left[B\left(u_{n-1}\right)\right]_{\nu} d W, \quad u_{n}^{\nu}(0)=u_{0 \nu}
$$

In particular, by the triangle inequality, one can construct a sequence of strict mild solutions $\bar{u}_{n}:=u_{n}^{\nu(n)}$ such that $\left\|u-\bar{u}_{n}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}$ is less than (a constant times) $\delta$, i.e., being $\delta$ arbitrary, the sequence $\bar{u}_{n}:=u_{n}^{\nu(n)}$ converges to $u$ in $\mathbb{H}_{p}\left(L_{q}\right)$.
Step 2. We are now going to show that there exists a constant $N$, independent of $n$, such that

$$
\mathbb{E} \int_{0}^{T} \Phi_{q}^{\prime}\left(\bar{u}_{n}(s)\right) \bar{g}_{n}(s) d s<N
$$

where $\bar{g}_{n} \in f\left(\bar{u}_{n}\right) m$-a.e. is the selection of $f\left(\bar{u}_{n}\right)$ appearing in the definition of strict mild solution. Let $\bar{B}_{n}:=\left[B\left(u_{n-1}\right)\right]_{\nu(n)}$ and $v_{n}(t):=e^{-\alpha t} \bar{u}_{n}(t)$ for all $t \in[0, T]$, with $\alpha>\eta$ a constant. Proposition 2 and obvious estimates yield

$$
\begin{aligned}
\left\|v_{n}(t)\right\|_{L_{\infty}\left(0, T ; L_{q}\right)}^{q}+ & q(\alpha-\eta) \int_{0}^{T}\left\|v_{n}(s)\right\|_{L_{q}}^{q} d s+\int_{0}^{T} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{n}(s)\right) \bar{g}_{n}(s) d s \\
\lesssim & \left\|u_{0 n}\right\|_{L_{q}}^{q}+\sup _{t \leq T}\left|\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{n}(s)\right) \bar{B}_{n}(s) d W\right| \\
& +\frac{1}{2} q(q-1) \int_{0}^{T}\left\|e^{-\alpha s} \bar{B}_{n}(s)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|v_{n}(s)\right\|_{L_{q}}^{q-2} d s
\end{aligned}
$$

where $M$, as before, stands for the stochastic integral above. By Young's inequality with exponents $q /(q-2)$ and $q / 2$, the last term can be estimated by

$$
\varepsilon\left\|v_{n}\right\|_{L_{\infty}\left(0, T ; L_{q}\right)}^{q}+N(\varepsilon)\left\|\bar{B}_{n}\right\|_{L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)}^{q}
$$

so that, choosing $\varepsilon$ small enough, raising to the power $p / q$, and taking expectation, we obtain

$$
\begin{aligned}
\left\|v_{n}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}^{p}+\mathbb{E}\left(\int_{0}^{T} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{n}(s)\right)\right. & \left.\bar{g}_{n}(s) d s\right)^{p / q} \\
& \therefore\left\|u_{0 n}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}^{p}+\mathbb{E}\left(M_{T}^{*}\right)^{p / q}+\left\|\bar{B}_{n}\right\|_{\mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)}^{p}
\end{aligned}
$$

By an argument already used before, based on the Burkholder-Davis-Gundy inequality and Young's inequality with exponents $q /(q-1)$ and $q$, we have

$$
\mathbb{E}\left(M_{T}^{*}\right)^{p / q} \lesssim \varepsilon\left\|v_{n}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}^{p}+N(\varepsilon)\left\|\bar{B}_{n}\right\|_{\mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)}^{p}
$$

thus also, choosing $\varepsilon$ small enough,

$$
\mathbb{E}\left(\int_{0}^{T} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{n}(s)\right) \bar{g}_{n}(s) d s\right)^{p / q} \lesssim\left\|u_{0 n}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}^{p}+\left\|\bar{B}_{n}\right\|_{\mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)}^{p}
$$

where the first term on the right-hand side is bounded because, by definition of generalized solution, $u_{0 n}$ converges to $u_{0}$ in $\mathbb{L}_{p}\left(L_{q}\right)$. Moreover, denoting the norm of $\mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)$ by $\|\cdot\|$ for simplicity,

$$
\left\|\bar{B}_{n}-B\left(u_{n-1}\right)\right\|=\left\|\left[B\left(u_{n-1}\right)\right]_{\nu(n)}-B\left(u_{n-1}\right)\right\|<\delta / 2
$$

hence $\left\|\bar{B}_{n}\right\|<\left\|B\left(u_{n-1}\right)\right\|+\delta / 2$, where, by the Lipschitz continuity of $B$,

$$
\begin{aligned}
\left\|B\left(u_{n-1}\right)\right\| & \leq\left\|B\left(u_{n-1}\right)-B(0)\right\|+\|B(0)\| \\
& \lesssim\left\|u_{n-1}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}+\|B(0)\|
\end{aligned}
$$

Since $u_{n} \rightarrow u$ in $\mathbb{H}_{p}\left(L_{q}\right)$ as $n \rightarrow \infty$, it follows that $\left\|u_{n}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)}$ is bounded, which in turn implies that $\left\|\bar{B}_{n}\right\|$ is also bounded. We conclude that there exists a constant $N$, independent of $n$, such that

$$
\mathbb{E}\left(\int_{0}^{T} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{n}(s)\right) \bar{g}_{n}(s) d s\right)^{p / q}<N
$$

Since $p / q \geq 1$, it follows by Jensen's inequality that

$$
\mathbb{E} \int_{0}^{T} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{n}(s)\right) \bar{g}_{n}(s) d s<N
$$

(where $N$ might differ from the previous one). The proof is finished observing that $e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{n}(s)\right)=e^{-q \alpha s} \Phi_{q}^{\prime}\left(\bar{u}_{n}(s)\right)$ and $e^{-q \alpha s} \geq e^{-q \alpha T}$ for all $s \in[0, T]$.
STEP 3. We are going to show that the uniform estimate of the previous step implies that the generalized solution $u$ is also a mild solution to (11).

The conclusion of the previous step can equivalently be written as

$$
\int_{\Xi} \phi_{q}\left(\bar{u}_{n}\right) \bar{g}_{n} d m<N
$$

where $\Xi:=\Omega \times[0, T] \times D$, and $N$ is a constant independent of $n$. Since $\phi_{q}$ is a homeomorphism of $\mathbb{R}$, setting $v_{n}:=\phi_{q}\left(\bar{u}_{n}\right)$ and recalling the definition $\tilde{f}:=f \circ \phi_{q}^{-1}$, we have (see e.g. [6, p. II.12] about the associativity of composition of graphs)

$$
\bar{g}_{n} \in f\left(\bar{u}_{n}\right)=f \circ \phi_{q}^{-1}\left(\phi_{q}\left(\bar{u}_{n}\right)\right)=\tilde{f}\left(v_{n}\right)
$$

hence the previous estimate can be written as

$$
\int_{\Xi} v_{n} \bar{g}_{n} d m<N
$$

where $\bar{g}_{n} \in \tilde{f}\left(v_{n}\right) m$-a.e. Since, by continuity of $\phi_{q}, v_{n}=\phi_{q}\left(u_{n}\right) \rightarrow \phi_{q}(u)=: v m$-a.e. as $n \rightarrow \infty$ and $\mathrm{D}\left(f \circ \phi_{q}^{-1}\right)=\mathbb{R}$, Lemma 21 implies that there exists $g \in L_{1}(m)$ and a subsequence $\left(n^{\prime}\right)$ of $(n)$ such that $\bar{g}_{n^{\prime}} \rightarrow g$ weakly in $L_{1}(m)$ as $n^{\prime} \rightarrow \infty$, and

$$
g \in f \circ \phi_{q}^{-1}(v)=f \circ \phi_{q}^{-1}\left(\phi_{q}(u)\right)=f(u) \quad m \text {-a.e. }
$$

Since $-A$ generates a $C_{0}$-semigroup of contractions on $L_{1}(D)$ by assumption, one obtains

$$
\int_{0}^{t} S(t-s) \tilde{g}_{n}(s) d s \rightarrow \int_{0}^{t} S(t-s) g(s) d s
$$

weakly in $\mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$ as $\lambda \rightarrow 0$, by a reasoning completely analogous to that used in the last part of the proof of Proposition 18, Similarly, convergence of the stochastic convolutions follows as in the proof just mentioned because

$$
\left\|\bar{B}_{n}-B(u)\right\|_{\mathbb{L}_{p}\left(L_{2}\left(0, T ; \gamma\left(H, L_{q}\right)\right)\right)} \xrightarrow{n \rightarrow \infty} 0 .
$$

STEP 4. It remains to prove that $\hat{F}(u), \tilde{F}^{*}(g) \in \mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$. In fact, we have $\bar{g}_{n} v_{n}=$ $\tilde{F}\left(v_{n}\right)+\tilde{F}^{*}\left(\tilde{g}_{n}\right)$ because $g_{n} \in \tilde{f}\left(v_{n}\right)=\partial \tilde{F}\left(v_{n}\right)$, hence, by the previous Step, there exists a constant $N$, independent of $n$, such that

$$
\int_{\Xi} \tilde{F}\left(v_{n}\right) d m<N, \quad \int_{\Xi} \tilde{F}^{*}\left(g_{n}\right) d m<N
$$

The convexity of $\tilde{F}$ and $\tilde{F}^{*}$ implies the weak lower semicontinuity in $L_{1}(m)$ of

$$
\phi \mapsto \int_{\Xi} \tilde{F}(\phi) d m, \quad \phi \mapsto \int_{\Xi} \tilde{F}^{*}(\phi) d m
$$

hence $\tilde{F}^{*}(g) \in L_{1}(m)$ because $\bar{g}_{n} \rightarrow g$ weakly in $L_{1}(m)$. Moreover, $v_{n} \rightarrow v$ in $m$-measure and

$$
\left\|v_{n}\right\|_{L_{1}(m)}=\left\|\bar{u}_{n}\right\|_{L_{q-1}(m)} \rightarrow\left\|u_{n}\right\|_{L_{q-1}(m)}=\|v\|_{L_{1}(m)}
$$

by virtue of the strong convergence $\bar{u}_{n} \rightarrow u$ in $\mathbb{H}_{p}\left(L_{q}\right)$ and the embedding $\mathbb{H}_{p}\left(L_{q}\right) \hookrightarrow$ ${\underset{\sim}{q}}_{q-1}(m)$. We thus have $v_{n} \rightarrow v$ strongly in $L_{1}(m)$, hence, similarly as above, $\hat{F}(u)=$ $\tilde{F}(v) \in L_{1}(m)$.

Remark 23. If $p<q$, the above proof does not work because Jensen's inequality reverses. However, a weaker integrability result can still be obtained. Namely, again by Jensen's inequality, we have

$$
\int_{\Xi}\left|f_{n} \phi_{q}\left(u_{n}\right)\right|^{p / q} d \mu<N
$$

uniformly over $n$, where the constant $N$ depends also on the Lebesgue measure of $D$. Setting $x^{\langle a\rangle}:=|x|^{a} \operatorname{sgn} x$ for all $x \in \mathbb{R}$ and $a>0$, taking into account that $0 \in f(0)$, the previous estimate can equivalently be written as

$$
\int_{\Xi} f_{n}^{\langle p / q\rangle} \phi_{q}^{\langle p / q\rangle}\left(u_{n}\right) d \mu<N
$$

For any $a>0$ the function $x \mapsto x^{\langle a\rangle}$ is a homeomorphism of $\mathbb{R}$, hence the function $\psi_{p, q}: x \mapsto \phi_{q}^{\langle p / q\rangle}(x)$ is also a homeomorphism of $\mathbb{R}$. We clearly have

$$
f_{n}^{\langle p / q\rangle} \in f^{\langle p / q\rangle} \circ \psi_{p, q}^{-1}\left(\psi_{p, q}\left(u_{n}\right)\right)
$$

$\mu$-a.e., hence there exists $z \in L_{1}(\mu)$ such that $f_{n_{k}}^{\langle p / q\rangle} \rightarrow z$ weakly in $L_{1}(\mu)$ along a subsequence $\left(n_{k}\right)$, with $z \in f^{\langle p / q\rangle} \circ \psi_{p, q}^{-1}\left(\psi_{p, q}(u)\right)=f^{\langle p / q\rangle}(u)$. We thus have, for a generalized solution, that $\zeta \in f(u)$ is only in $L_{p / q}(\mu)$, rather than in $L_{1}(\mu)$. This in particular implies that it does not seem possible any longer to claim that $u$ is a mild solution, even in a very weak sense, as the semigroup $S$ is not defined in $L_{q}(D)$ spaces with $0<q<1$.

### 6.2 Uniqueness

The aim of this subsection is to prove continuous dependence on the initial datum (from which uniqueness follows immediately) for mild solutions to (11), without assuming that $g \in L_{1}\left(L_{q}\right)$. We need to assume, however, the same integrability conditions on the solution that are established in the proof of Proposition 22, as well as positivity and regularizing properties for the resolvent of $A$ and a symmetry condition on $F$.

The key is the following estimate for the difference of two mild solutions to (1), whose proof is inspired by an analogous result, in a different setting, in 3.

Lemma 24. Under the hypotheses of Theorem 13, assume that $u_{i}, i=1,2$, satisfies

$$
u_{i}(t)+\int_{0}^{t} S(t-s)\left(g_{i}(s)-\eta u_{i}(s)\right) d s=S(t) u_{0 i}+\int_{0}^{t} S(t-s) B\left(u_{i}(s)\right) d W(s)
$$

for all $t \in[0, T]$, where $g_{i} \in \mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$, $g_{i} \in f\left(u_{i}\right) m$-a.e., and $\hat{F}\left(u_{i}\right)$, $\tilde{F}^{*}\left(g_{i}\right) \in$ $\mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$. Then, setting $v_{i}(t):=e^{-\alpha t} u_{i}(t), t \in[0, T]$, for $\alpha \geq 0$ constant, one has

$$
\begin{aligned}
\| v_{1}(t)- & v_{2}(t)\left\|_{L_{q}}^{q}+q(\alpha-\eta) \int_{0}^{t}\right\| v_{1}(s)-v_{2}(s) \|_{L_{q}}^{q} d s \\
& +\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{1}(s)-v_{2}(s)\right)\left(g_{1}(s)-g_{2}(s)\right) d s \\
\leq & \left\|u_{01}-u_{02}\right\|_{L_{q}}^{q} \\
& +\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{1}(s)-v_{2}(s)\right)\left(B\left(u_{1}(s)\right)-B\left(u_{2}(s)\right)\right) d W(s) \\
& +\frac{1}{2} q(q-1) \int_{0}^{t}\left\|e^{-\alpha s}\left(B\left(u_{1}(s)\right)-B\left(u_{2}(s)\right)\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|v_{1}(s)-v_{2}(s)\right\|_{L_{q}}^{q-2} d s
\end{aligned}
$$

for all $t \in[0, T]$.
Proof. Given $\sigma \in \mathbb{N}$ such that $(I+\varepsilon A)^{-\sigma}$ maps $L_{1}(D)$ to $L_{q}(D)$, set $h_{i}^{\varepsilon}:=(I+\varepsilon A)^{-\sigma} h$ for all $h \in\left\{u_{i}, u_{0 i}, g_{i}, v_{i}\right\}$, and $B_{i}^{\varepsilon}:=(I+\varepsilon)^{-\sigma} B\left(u_{i}\right)$. Then $g_{i} \in L_{1}\left(L_{q}\right)$ and $v_{i}^{\varepsilon}$ is the unique $L_{q}$-valued strong solution to

$$
d v_{i}^{\varepsilon}+A v_{i}^{\varepsilon} d t+(\alpha-\eta) v_{i}^{\varepsilon} d t+e^{-\alpha t} g_{i}^{\varepsilon} d t=e^{-\alpha t} B_{i}^{\varepsilon} d W, \quad u_{i}^{\varepsilon}(0)=u_{0 i}^{\varepsilon}
$$

Itô's formula and (6) then yield

$$
\begin{align*}
\| v_{1}^{\varepsilon}(t)- & v_{2}^{\varepsilon}(t)\left\|_{L_{q}}^{q}+q(\alpha-\eta) \int_{0}^{t}\right\| v_{1}^{\varepsilon}(s)-v_{2}^{\varepsilon}(s) \|_{L_{q}}^{q} d s \\
& +\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{1}^{\varepsilon}(s)-v_{2}^{\varepsilon}(s)\right)\left(g_{1}^{\varepsilon}(s)-g_{2}^{\varepsilon}(s)\right) d s \\
\leq & \left\|u_{01}^{\varepsilon}-u_{02}^{\varepsilon}\right\|_{L_{q}}^{q}  \tag{16}\\
& +\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{1}^{\varepsilon}(s)-v_{2}^{\varepsilon}(s)\right)\left(B_{1}^{\varepsilon}(s)-B_{2}^{\varepsilon}(s)\right) d W(s) \\
& +\frac{1}{2} q(q-1) \int_{0}^{t}\left\|e^{-\alpha s}\left(B_{1}^{\varepsilon}(s)-B_{2}^{\varepsilon}(s)\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|v_{1}^{\varepsilon}(s)-v_{2}^{\varepsilon}(s)\right\|_{L_{q}}^{q-2} d s
\end{align*}
$$

We are now going to pass to the limit as $\varepsilon \rightarrow 0$ in the above inequality. Since $(I+\varepsilon A)^{-\sigma}$ converges to the identity in $\mathscr{L}\left(L_{q}\right)$ in the strong operator topology, it immediately follows that

$$
\begin{aligned}
& \left\|v_{1}^{\varepsilon}(t)-v_{2}^{\varepsilon}(t)\right\|_{L_{q}} \xrightarrow{\varepsilon \rightarrow 0}\left\|v_{1}(t)-v_{2}(t)\right\|_{L_{q}} \quad \forall t \in[0, T] \\
& \quad\left\|u_{01}^{\varepsilon}-u_{02}^{\varepsilon}\right\|_{L_{q}} \xrightarrow{\varepsilon \rightarrow 0}\left\|u_{01}-u_{02}\right\|_{L_{q}}
\end{aligned}
$$

Since $(I+\varepsilon A)^{-\sigma}$ is contracting in $L_{q},\left\|v_{1}^{\varepsilon}-v_{2}^{\varepsilon}\right\|_{L_{q}} \leq\left\|v_{1}\right\|_{L_{q}}+\left\|v_{2}\right\|_{L_{q}}$ pointwise, hence, by Fubini's theorem, $v \in \mathbb{H}_{p}\left(L_{q}\right)$, and the dominated convergence theorem,

$$
\int_{0}^{t}\left\|v^{\varepsilon}(s)\right\|_{L_{q}}^{q} d s \xrightarrow{\varepsilon \rightarrow 0} \int_{0}^{t}\|v(s)\|_{L_{q}}^{q} d s \quad \forall t \leq T
$$

The dominated convergence theorem also immediately shows that

$$
\begin{aligned}
& \int_{0}^{t}\left\|e^{-\alpha s}\left(B_{1}^{\varepsilon}(s)-B_{2}^{\varepsilon}(s)\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|v_{1}^{\varepsilon}(s)-v_{2}^{\varepsilon}(s)\right\|_{L_{q}}^{q-2} d s \\
& \stackrel{\varepsilon \rightarrow 0}{\longrightarrow} \int_{0}^{t}\left\|e^{-\alpha s}\left(B\left(u_{1}(s)\right)-B\left(u_{2}(s)\right)\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|v_{1}(s)-v_{2}(s)\right\|_{L_{q}}^{q-2} d s
\end{aligned}
$$

Let us now consider the last term on the left-hand side of (16). Recalling the definition of the homeomorphism $\phi_{q}: x \mapsto x|x|^{q-2}$, we have

$$
\begin{aligned}
\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}( & \left.v_{1}^{\varepsilon}(s)-v_{2}^{\varepsilon}(s)\right)\left(g_{1}^{\varepsilon}(s)-g_{2}^{\varepsilon}(s)\right) d s \\
& =q \int_{0}^{t} e^{-q \alpha s} \Phi_{q}^{\prime}\left(u_{1}^{\varepsilon}(s)-u_{2}^{\varepsilon}(s)\right)\left(g_{1}^{\varepsilon}(s)-g_{2}^{\varepsilon}(s)\right) d s \\
& \approx \int_{0}^{t} \int_{D}\left(g_{1}^{\varepsilon}(s)-g_{2}^{\varepsilon}(s)\right) \phi_{q}\left(u_{1}^{\varepsilon}(s)-u_{2}^{\varepsilon}(s)\right) d x d s
\end{aligned}
$$

The properties of $(I+\varepsilon A)^{-\sigma}$ imply easily that $g_{i}^{\varepsilon} \rightarrow g_{i}$ in $\mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$, hence in $m$ measure, as $\varepsilon \rightarrow 0$. Similarly, since $u_{i}^{\varepsilon} \rightarrow u_{i}$ in $m$-measure and $\phi_{q}$ is continuous, $\phi_{q}\left(u_{1}^{\varepsilon}-\right.$ $\left.u_{2}^{\varepsilon}\right) \rightarrow \phi_{q}\left(u_{1}-u_{2}\right)$ in $m$-measure. In particular,

$$
\left(g_{1}^{\varepsilon}-g_{2}^{\varepsilon}\right) \phi_{q}\left(u_{1}^{\varepsilon}-u_{2}^{\varepsilon}\right) \xrightarrow{\varepsilon \rightarrow 0}\left(g_{1}-g_{2}\right) \phi_{q}\left(u_{1}-u_{2}\right)
$$

in $m$-measure. We are going to show that this convergence takes place in $\mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$. To this purpose, it suffices to show, by Vitali's theorem, that the sequence on the right-hand side is uniformly integrable (UI). Let $\delta \in] 0,1 / 2]$ be arbitrary but fixed. By Young's inequality with conjugate functions $\tilde{F}$ and $\tilde{F}^{*}$ and the definition $\hat{F}:=\tilde{F} \circ \phi_{q}$,

$$
\begin{aligned}
\left|\left(g_{1}^{\varepsilon}-g_{2}^{\varepsilon}\right) \phi_{q}\left(u_{1}^{\varepsilon}-u_{2}^{\varepsilon}\right)\right| & \approx\left|\delta\left(g_{1}^{\varepsilon}-g_{2}^{\varepsilon}\right)\right|\left|\phi_{q}\left(\delta\left(u_{1}^{\varepsilon}-u_{2}^{\varepsilon}\right)\right)\right| \\
& \leq \hat{F}\left(\left|\delta\left(u_{1}^{\varepsilon}-u_{2}^{\varepsilon}\right)\right|\right)+\tilde{F}^{*}\left(\left|\delta\left(g_{1}^{\varepsilon}-g_{2}^{\varepsilon}\right)\right|\right) \\
& =\hat{F}\left(\delta\left(u_{1}^{\varepsilon}-u_{2}^{\varepsilon}\right)\right)+\tilde{F}^{*}\left(\delta\left(g_{1}^{\varepsilon}-g_{2}^{\varepsilon}\right)\right)
\end{aligned}
$$

In the last step we have used that $\hat{F}$ and $\tilde{F}^{*}$ are even: in fact, since $F$ is even and $F(0)=0$, we infer that $f$ is odd, $\tilde{f}$ is odd, hence $\tilde{F}, \tilde{F}^{*}$ and $\hat{F}$ are even with $\tilde{F}(0)=$ $\tilde{F}^{*}(0)=\hat{F}(0)=0$. Then it follows that $\tilde{F}^{*}$ and $\hat{F}$ are increasing on $\mathbb{R}_{+}$(this can also be seen by $\partial \tilde{F}^{*}=\tilde{f}^{-1}=\phi_{q} \circ f^{-1} \geq 0$ and $\partial \hat{F}=f \phi_{q}^{\prime} \geq 0$ on $\mathbb{R}_{+}$). Therefore $\hat{F}(c x)=\hat{F}(c|x|) \leq \hat{F}(|x|)=\hat{F}(x)$ for all $x \in \mathbb{R}$ and $c \in[0,1]$, and the same holds for $\tilde{F}^{*}$. In particular,

$$
\begin{aligned}
\hat{F}\left(\delta\left(u_{1}^{\varepsilon}-u_{2}^{\varepsilon}\right)\right) & =\hat{F}\left(\frac{1}{2}\left(2 \delta u_{1}^{\varepsilon}\right)+\frac{1}{2}\left(-2 \delta u_{2}^{\varepsilon}\right)\right) \\
& \leq \frac{1}{2} \hat{F}\left(2 \delta u_{1}^{\varepsilon}\right)+\frac{1}{2} \hat{F}\left(2 \delta u_{2}^{\varepsilon}\right) \\
& \leq \frac{1}{2}\left(\hat{F}\left(u_{1}^{\varepsilon}\right)+\hat{F}\left(u_{2}^{\varepsilon}\right)\right)
\end{aligned}
$$

and, completely analogously,

$$
\tilde{F}^{*}\left(\delta\left(g_{1}^{\varepsilon}-g_{2}^{\varepsilon}\right)\right) \leq \frac{1}{2}\left(\tilde{F}^{*}\left(g_{1}^{\varepsilon}\right)+\tilde{F}^{*}\left(g_{2}^{\varepsilon}\right)\right)
$$

thus also

$$
\left|\left(g_{1}^{\varepsilon}-g_{2}^{\varepsilon}\right) \phi_{q}\left(u_{1}^{\varepsilon}-u_{2}^{\varepsilon}\right)\right| \lesssim \hat{F}\left(u_{1}^{\varepsilon}\right)+\hat{F}\left(u_{2}^{\varepsilon}\right)+\tilde{F}^{*}\left(g_{1}^{\varepsilon}\right)+\tilde{F}^{*}\left(g_{2}^{\varepsilon}\right)
$$

Let us now observe that, by Jensen's inequality for positive operators (see e.g. [15]),

$$
\begin{aligned}
& \tilde{F}^{*}\left(g_{i}^{\varepsilon}\right)=\tilde{F}^{*}\left((I+\varepsilon A)^{-\sigma} g_{i}\right) \leq(I+\varepsilon A)^{-\sigma} \tilde{F}^{*}\left(g_{i}\right) \\
& \hat{F}^{*}\left(u_{i}^{\varepsilon}\right)=\hat{F}\left((I+\varepsilon A)^{-\sigma} u_{i}\right) \leq(I+\varepsilon A)^{-\sigma} \hat{F}\left(u_{i}\right)
\end{aligned}
$$

But since $\hat{F}\left(u_{i}\right), \tilde{F}^{*}\left(g_{i}\right) \in \mathbb{L}_{\tilde{1}}\left(L_{1}\left(L_{1}\right)\right)$ by assumption, hence $(I+\varepsilon A)^{-\sigma} \hat{F}\left(u_{i}\right) \rightarrow \hat{F}\left(u_{i}\right)$ and $(I+\varepsilon A)^{-\sigma} \tilde{F}^{*}\left(g_{i}\right) \rightarrow \tilde{F}^{*}\left(g_{i}\right)$ as $\varepsilon \rightarrow 0$ in $\mathbb{L}_{1} L_{1} L_{1}$, it follows that the sequence $\left|\left(g_{1}^{\varepsilon}-g_{2}^{\varepsilon}\right) \phi_{q}\left(u_{1}^{\varepsilon}-u_{2}^{\varepsilon}\right)\right|$ is dominated by a convergent sequence of $\mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$, which is a fortiori UI. Then $\left(g_{1}^{\varepsilon}-g_{2}^{\varepsilon}\right) \phi_{q}\left(u_{1}^{\varepsilon}-u_{2}^{\varepsilon}\right)$ is also UI, because a (positive) sequence dominated by a UI sequence is itself UI. We have thus proved that the last term on the left-hand side of (16) converges in probability for all $t \in[0, T]$ to

$$
\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{1}(s)-v_{2}(s)\right)\left(g_{1}(s)-g_{2}(s)\right) d s
$$

It remains only to consider the stochastic integral on the right-hand side of (16), which converges to

$$
\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{1}(s)-v_{2}(s)\right)\left(B\left(u_{1}(s)\right)-B\left(u_{2}(s)\right)\right) d W(s)
$$

in probability for all $t \in[0, T]$ as $\varepsilon \rightarrow 0$. The proof is based on an argument entirely analogous to the one already used in the proof of Proposition 2, and is hence omitted.

We can now prove uniqueness of mild solution to (1) and their continuous dependence on the initial datum.

Proposition 25. Under the hypotheses of Theorem 13, assume that $u \in \mathbb{H}_{p}\left(L_{q}\right)$ is a mild solution to (11). Then $u$ is the unique mild solution such that $\hat{F}(u)+\tilde{F}^{*} g \in \mathbb{L}_{1}\left(L_{1}\left(L_{1}\right)\right)$. Moreover, the solution map $u_{0} \mapsto u$ is Lipschitz continuous from $\mathbb{L}_{p}\left(L_{q}\right)$ to $\mathbb{H}_{p}\left(L_{q}\right)$.

Proof. Let $u_{1}, u_{2}$ be as in the previous Lemma, with $u_{01}, u_{0,2} \in \mathbb{L}_{p}\left(L_{q}\right)$. Then

$$
\begin{aligned}
\| v_{1}(t)- & v_{2}(t)\left\|_{L_{q}}^{q}+q(\alpha-\eta) \int_{0}^{t}\right\| v_{1}-v_{2} \|_{L_{q}}^{q} d s+\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{1}-v_{2}\right)\left(g_{1}-g_{2}\right) d s \\
\leq & \left\|u_{01}-u_{02}\right\|_{L_{q}}^{q}+\int_{0}^{t} e^{-\alpha s} \Phi_{q}^{\prime}\left(v_{1}-v_{2}\right)\left(B\left(u_{1}\right)-B\left(u_{2}\right)\right) d W \\
& +\frac{1}{2} q(q-1) \int_{0}^{t}\left\|e^{-\alpha s}\left(B\left(u_{1}\right)-B\left(u_{2}\right)\right)\right\|_{\gamma\left(H, L_{q}\right)}^{2}\left\|v_{1}-v_{2}\right\|_{L_{q}}^{q-2} d s
\end{aligned}
$$

where

$$
\Phi_{q}^{\prime}\left(v_{1}-v_{2}\right)\left(g_{1}-g_{2}\right)=q e^{-(q-1) \alpha \cdot}\left\langle g_{1}-g_{2}, \phi_{q}\left(u_{1}-u_{2}\right)\right\rangle \geq 0
$$

We are now in the condition to use exactly the same proof of Lemma 17, arriving at

$$
\left\|u_{1}-u_{2}\right\|_{\mathbb{H}_{p}\left(L_{q}\right)} \lesssim\left\|u_{01}-u_{02}\right\|_{\mathbb{L}_{p}\left(L_{q}\right)}
$$

which proves that $u_{0} \mapsto u \in \dot{C}^{0,1}\left(\mathbb{H}_{p}\left(L_{q}\right), \mathbb{L}_{p}\left(L_{q}\right)\right)$ and, as an immediate consequence, uniqueness of the solution.

Remark 26. It is clear by the previous proof that we do not have well-posedness in the space $\mathbb{H}_{p}\left(L_{q}\right)$, as our uniqueness result holds only under additional assumptions on the solution itself. The problem of unconditional uniqueness in $\mathbb{H}_{p}\left(L_{q}\right)$ remains therefore open.

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[^1]:    ${ }^{1}$ Since we never need weak measurability, measurable will always mean strongly measurable.

[^2]:    ${ }^{2}$ Formula (3.12) in op. cit. contains a misprint: $A_{\lambda_{n}} x$ should be replaced by $A_{\lambda_{n}} x_{n}$.

[^3]:    ${ }^{3}$ If $X=L_{q}, q \geq 2$, the only case of interest for us, the embedding is just an obvious consequence of Minkowski's inequality: $L_{2}\left(0, t ; \gamma\left(H, L_{q}\right)\right) \simeq L_{2}\left(0, t ; L_{q}(H)\right) \hookrightarrow L_{q}\left(L_{2}(0, t ; H)\right) \simeq \gamma\left(L_{2}((0, t) ; H), L_{q}\right)$.

[^4]:    ${ }^{4}$ From now on we shall occasionally omit the indication of the time parameter, if no confusion may arise, for notational compactness.

[^5]:    ${ }^{5}$ From now on, whenever we apply Young's inequality, we shall mostly state only the exponents used.

[^6]:    ${ }^{6}$ Note that $A(t)+B(t) \leq C(t)$ for all $t$, with $A, B, C$ positive functions of $t$, $\operatorname{implies}^{\sup }{ }_{t} A(t) \leq$ $\sup _{t} C(t)$ and $\sup _{t} B(t) \leq \sup _{t} C(t)$, hence $\sup _{t} A(t)+\sup _{t} B(t) \leq 2 \sup _{t} C(t)$.

[^7]:    ${ }^{7}$ Note that here we are just using Itô's isomorphism for the stochastic integral, not Burkholder's inequality, which does not hold as the stochastic convolution is not, in general, a local martingale.

