# GLOBAL AND INTERIOR POINTWISE BEST APPROXIMATION RESULTS FOR THE GRADIENT OF GALERKIN SOLUTIONS FOR PARABOLIC PROBLEMS 

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#### Abstract

In this paper we establish best approximation property of fully discrete Galerkin solutions of second order parabolic problems on convex polygonal and polyhedral domains in the $L^{\infty}\left(I ; W^{1, \infty}(\Omega)\right)$ norm. The discretization method consists of continuous Lagrange finite elements in space and discontinuous Galerkin methods of arbitrary order in time. The method of the proof differs from the established fully discrete error estimate techniques and uses only elliptic results and discrete maximal parabolic regularity for discontinuous Galerkin methods established by the authors in [15]. In addition, the proof does not require any relationship between spatial mesh sizes and time steps. We also establish interior best approximation property that shows more local dependence of the error at a point.


Key words. optimal control, pointwise control, parabolic problems, finite elements, discontinuous Galerkin, error estimates, pointwise error estimates

## AMS subject classifications.

1. Introduction. Let $\Omega$ be a convex polygonal/polyhedral domain in $\mathbb{R}^{N}, N=2,3$ and $I=(0, T)$ with some $T>0$. We consider a second order parabolic problem

$$
\begin{align*}
& u_{t}(t, x)-\Delta u(t, x)=f(t, x), \quad(t, x) \in I \times \Omega, \\
& u(t, x)=0, \quad(t, x) \in I \times \partial \Omega,  \tag{1.1}\\
& u(0, x)=u_{0}(x), \quad x \in \Omega .
\end{align*}
$$

To discretize the problem we use continuous Lagrange finite elements in space and discontinuous Galerkin methods in time. The precise description of the method is given in section 2 Our main goal in this paper is to establish global and interior (local) space-time pointwise best approximation type results for the fully discrete error. The global estimate has the following structure:

$$
\begin{equation*}
\left\|\nabla\left(u-u_{k h}\right)\right\|_{L^{\infty}(I \times \Omega)} \leq C \ell_{k} \ell_{h}\|\nabla(u-\chi)\|_{L^{\infty}(I \times \Omega)}, \tag{1.2}
\end{equation*}
$$

where $u_{k h}$ denotes the fully discrete solution and $\chi$ is an arbitrary element of the finite dimensional space, $h$ stands for spatial mesh size and $k$ for the maximal time step, and $\ell_{k}, \ell_{h}$ stand for some logarithmic terms. Such results are sometimes called symmetric estimates, cf. [4, 8]. The interior (local) result provides an estimate of the error $\left|\nabla\left(u-u_{u_{k h}}\right)\left(\tilde{t}, x_{0}\right)\right|$ for given $\tilde{t} \in(0, T]$ and $x_{0} \in \Omega$ in terms of best approximation on a ball $B_{d}\left(x_{0}\right)$ and some global terms in weaker norms. Precise results are stated in section 2, see Theorem 2.1 and Theorem 2.2 For the global estimate (1.2) we assume that $f$ and $u_{0}$ are such that $\nabla u \in C(\bar{I} \times \bar{\Omega})$. For the interior result we essentially need only $\nabla u \in C\left(\bar{I} \times \bar{B}_{d}\left(x_{0}\right)\right) \cap L^{2}(I \times \Omega)$. Such best approximation type results have only natural assumptions on the problem data and are desirable in many applications, for example optimal control problems governed by parabolic equations with gradient constraints, cf. [18]. We refer to a recent paper [30] for a further discussion on the importance of best approximation results and difficulties associated with obtaining such estimates for parabolic problems.

For elliptic problems the best approximation property as (1.2), which is equivalent to the stability of the Ritz projection in $W^{1, \infty}(\Omega)$ norm, is well known. The first log-free result was established in [23] on convex polygonal domains. Later the result was extended to convex polyhedral domains with some restriction on angles in [2]. This restriction was removed in [12] and even extended to certain graded meshes in [6]. For parabolic problems similar results are rather scarce. The main body of the work on pointwise error estimates for parabolic problems are devoted to $L^{\infty}(I \times \Omega)$ error estimates, see [14] for review of the corresponding results. We are aware of only three publications dealing with pointwise error estimates for the gradient of the error.

In two space dimensions, semidiscrete error estimates were studied in [3] and the fully discrete CrankNicolson method was studied in [33]. Since the main motivation of both investigations was the question of superconvergence of the gradient of the error, it was assumed that the solution is sufficiently smooth. More general fully discrete error estimates using Padé time schemes were obtained in [16] for smooth domains in $\mathbb{R}^{N}$.

[^0]In both publications dealing with fully discrete error estimates, [16] and [33], the proofs are based on the splitting $u-u_{k h}=\left(u-R_{h} u\right)+\left(R_{h} u-u_{k h}\right)$, where $R_{h}$ is the Ritz projection. This idea was first introduced by M. Wheeler [34] in order to obtain optimal order error estimates in $L^{2}$ norm in space. The main idea of this approach is the following: the first part of the error is treated by elliptic results and the second part satisfies a certain parabolic equation with the right-hand side involving $\left(u-R_{h} u\right)$, which can be treated by results from rational approximation of analytic semigroups in Banach spaces (see also [31, Thm .8.6]). However, this approach requires additional smoothness of the solution, well beyond the natural regularity $\nabla u \in C(\bar{I} \times \bar{\Omega})$ of the exact solution. Our approach is completely different. It uses newly established discrete maximal parabolic regularity results [15] for discontinuous Galerkin time schemes, see section 5below, and the discrete resolvent estimate of the form:

$$
\begin{equation*}
\left\|\left\|\left(z+\Delta_{h}\right)^{-1} \chi \mid\right\| \leq \frac{M_{h}}{|z|}\right\| \chi \|, \quad \text { for } z \in \mathbb{C} \backslash \Sigma_{\gamma}, \quad \text { for all } \chi \in \mathbb{V}_{h}=V_{h}+i V_{h} \tag{1.3}
\end{equation*}
$$

where $M_{h}$ may depend on $|\ln h|$ but is independent of $h$ otherwise, $V_{h}$ is the space of continuous Lagrange finite elements of degree $r, \Delta_{h}$ is the discrete Laplace operator, see (3.9) below, and

$$
\Sigma_{\gamma}=\{z \in \mathbb{C}| | \arg (z) \mid \leq \gamma\},
$$

for some $\gamma \in\left(0, \frac{\pi}{2}\right)$. In [14] we showed this estimate for the triple norm $\left\|v_{h}\right\|\|=\| \sigma^{\frac{N}{2}} v_{h} \|_{L^{2}(\Omega)}$ with the weight function $\sigma(x)=\sqrt{\left|x-x_{0}\right|^{2}+K^{2} h^{2}}$. This norm behaves similar to the $L^{1}(\Omega)$ norm, and we used the corresponding discrete maximal parabolic result to prove (global and interior) pointwise best approximation for function values of the solution $u$. Here, we will in addition require the estimate (1.3) with respect to the norm

$$
\left\|v_{h}\right\|=\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} v_{h}\right\|_{L^{2}(\Omega)}
$$

which behaves similar to the the $W^{-1,1}(\Omega)$ norm, see Theorem 4.1 below. This allows us to prove our main results of (global and interior) pointwise best approximation for the gradient of the solution.

The rest of the paper is organized as follows. In the next section we describe the discretization method and state our main results. In section 3, we review some essential elliptic results in weighted norms. Section 4 is devoted to establishing the resolvent estimate in weighted norms. In section [5, we review our discrete maximal parabolic regularity result. Finally, in sections 6 and 7 , we provide proofs of the global and interior best approximation properties of the fully discrete solution.
2. Discretization and statement of main results. To introduce the time discontinuous Galerkin discretization for the problem, we partition $(0, T]$ into subintervals $I_{m}=\left(t_{m-1}, t_{m}\right]$ of length $k_{m}=t_{m}-t_{m-1}$, where $0=t_{0}<t_{1}<\cdots<t_{M-1}<t_{M}=T$. The maximal and minimal time steps are denoted by $k=\max _{m} k_{m}$ and $k_{\text {min }}=\min _{m} k_{m}$, respectively. We impose the following conditions on the time mesh (as in [15] or [19]):
(i) There are constants $c, \beta>0$ independent of $k$ such that

$$
k_{\min } \geq c k^{\beta}
$$

(ii) There is a constant $\kappa>0$ independent of $k$ such that for all $m=1,2, \ldots, M-1$

$$
\kappa^{-1} \leq \frac{k_{m}}{k_{m+1}} \leq \kappa
$$

(iii) It holds $k \leq \frac{1}{4} T$.

The semidiscrete space $X_{k}^{q}$ of piecewise polynomial functions in time is defined by

$$
X_{k}^{q}=\left\{v_{k} \in L^{2}\left(I ; H_{0}^{1}(\Omega)\right)\left|v_{k}\right|_{I_{m}} \in \mathcal{P}_{q}\left(I_{m} ; H_{0}^{1}(\Omega)\right), m=1,2, \ldots, M\right\}
$$

where $\mathcal{P}_{q}\left(I_{m} ; V\right)$ is the space of polynomial functions of degree $q$ in time with values in a Banach space $V$. We will employ the following notation for time dependent functions

$$
\begin{equation*}
v_{m}^{+}=\lim _{\varepsilon \rightarrow 0^{+}} v\left(t_{m}+\varepsilon\right), \quad v_{m}^{-}=\lim _{\varepsilon \rightarrow 0^{+}} v\left(t_{m}-\varepsilon\right), \quad[v]_{m}=v_{m}^{+}-v_{m}^{-} \tag{2.1}
\end{equation*}
$$

if these limits exist. Next we define the following bilinear form

$$
\begin{equation*}
B(v, \varphi)=\sum_{m=1}^{M}\left\langle v_{t}, \varphi\right\rangle_{I_{m} \times \Omega}+(\nabla v, \nabla \varphi)_{I \times \Omega}+\sum_{m=2}^{M}\left([v]_{m-1}, \varphi_{m-1}^{+}\right)_{\Omega}+\left(v_{0}^{+}, \varphi_{0}^{+}\right)_{\Omega}, \tag{2.2}
\end{equation*}
$$

where $(\cdot, \cdot)_{\Omega}$ and $(\cdot, \cdot)_{I_{m} \times \Omega}$ are the usual $L^{2}$ space and space-time inner-products, $\langle\cdot, \cdot\rangle_{I_{m} \times \Omega}$ is the duality product between $L^{2}\left(I_{m} ; H^{-1}(\Omega)\right)$ and $L^{2}\left(I_{m} ; H_{0}^{1}(\Omega)\right)$. We note, that the first sum vanishes for $v \in X_{k}^{0}$. Rearranging the terms in (2.2), we obtain an equivalent (dual) expression of $B$ :

$$
\begin{equation*}
B(v, \varphi)=-\sum_{m=1}^{M}\left\langle v, \varphi_{t}\right\rangle_{I_{m} \times \Omega}+(\nabla v, \nabla \varphi)_{I \times \Omega}-\sum_{m=1}^{M-1}\left(v_{m}^{-},[\varphi]_{m}\right)_{\Omega}+\left(v_{M}^{-}, \varphi_{M}^{-}\right)_{\Omega} \tag{2.3}
\end{equation*}
$$

To introduce the fully discrete approximation, let $\mathcal{T}_{h}$ for $h>0$ denote a quasi-uniform triangulation of $\Omega$ with mesh size $h$, i.e., $\mathcal{T}_{h}=\{\tau\}$ is a partition of $\Omega$ into cells (triangles or tetrahedrons) $\tau$ of diameter $h_{\tau}$ such that for $h=\max _{\tau} h_{\tau}$,

$$
\operatorname{diam}(\tau) \leq h \leq C|\tau|^{\frac{1}{N}}, \quad \text { for all } \tau \in \mathcal{T}_{h}
$$

hold. Let $V_{h}$ be the set of all functions in $H_{0}^{1}(\Omega)$ that are polynomials of degree $r$ on each $\tau$, i.e. $V_{h}$ is the usual space of conforming finite elements. To obtain the fully discrete approximation we consider the space-time finite element space

$$
\begin{equation*}
X_{k, h}^{q, r}=\left\{v_{k h} \in L^{2}\left(I ; H_{0}^{1}(\Omega)\right)\left|v_{k h}\right|_{I_{m}} \in \mathcal{P}_{q}\left(I_{m} ; V_{h}\right), m=1,2, \ldots, M\right\}, \quad q \geq 0, \quad r \geq 1 \tag{2.4}
\end{equation*}
$$

We define a fully discrete $\mathrm{dG}(q) \mathrm{cG}(r)$ solution $u_{k h} \in X_{k, h}^{q, r}$ by

$$
\begin{equation*}
B\left(u_{k h}, \varphi_{k h}\right)=\left(f, \varphi_{k h}\right)_{I \times \Omega}+\left(u_{0}, \varphi_{k h, 0}^{+}\right)_{\Omega} \quad \text { for all } \varphi_{k h} \in X_{k, h}^{q, r} . \tag{2.5}
\end{equation*}
$$

2.1. Main results. Now we state our main results. The first result establishes the global best approximation property of the fully discrete Galerkin solution in the $L^{\infty}\left(I ; W^{1, \infty}(\Omega)\right)$ norm.

THEOREM 2.1 (Global best approximation). Let $u$ and $u_{k h}$ satisfy (1.1) and (2.5) respectively. Then, there exists a constant $C$ independent of $k$ and $h$ such that

$$
\left\|\nabla\left(u-u_{k h}\right)\right\|_{L^{\infty}(I \times \Omega)} \leq C \ell_{k} \ell_{h} \inf _{\chi \in X_{k, h}^{q, r}}\|\nabla(u-\chi)\|_{L^{\infty}(I \times \Omega)}
$$

where $\ell_{k}=\ln \frac{T}{k}$ and $\ell_{h}=|\ln h|^{\frac{2 N-1}{N}}$.
The proof of this theorem is given in Section 6
For the error at a given point $x_{0} \in \Omega$ we obtain a sharper results. For elliptic problems similar results were obtained in [25, 27]. We denote by $B_{d}=B_{d}\left(x_{0}\right)$ the ball of radius $d$ centered at $x_{0}$.

THEOREM 2.2 (Interior best approximation). Let $u$ and $u_{k h}$ satisfy (1.1) and (2.5), respectively and let $d>4 h$. Assume $x_{0} \in \Omega$ and $\tilde{t} \in I_{m}$ for some $m=1,2, \ldots, M$ and $B_{d} \subset \subset \Omega$. Then there exists a constant $C$ independent of $h, k$, and $d$ such that

$$
\begin{aligned}
\left|\nabla\left(u-u_{k h}\right)\left(\tilde{t}, x_{0}\right)\right| & \leq C \ell_{k} \ell_{h} \inf _{\chi \in X_{k, h}^{q, r}}\left\{\|\nabla(u-\chi)\|_{L^{\infty}\left(\left(0, t_{m}\right) \times B_{d}\left(x_{0}\right)\right)}+d^{-1}\|u-\chi\|_{L^{\infty}\left(\left(0, t_{m}\right) \times B_{d}\left(x_{0}\right)\right)}\right. \\
& \left.+d^{-\frac{N}{2}}\left(\|\nabla(u-\chi)\|_{L^{\infty}\left(\left(0, t_{m}\right) ; L^{2}(\Omega)\right)}+d^{-1}\|u-\chi\|_{L^{\infty}\left(\left(0, t_{m}\right) ; L^{2}(\Omega)\right)}\right)\right\}
\end{aligned}
$$

with $\ell_{k}$ and $\ell_{h}$ defined as in Theorem 2.1]
The proof of this theorem is given in Section 7
3. Elliptic estimates in weighted norms. In this section we collect some estimates for the finite element discretization of elliptic problems in weighted norms on convex polygonal/polyhedral domains mainly taken from [13]. These results will be used in the following sections within the proofs of Theorem 4.1, Theorem 2.1 and Theorem 2.2

In this section we consider a fixed (but arbitrary) point $x_{0} \in \Omega$. Associated to this point we introduce a smoothed delta function [27, Appendix], which we will denote by $\tilde{\delta}$. This function is supported in one cell, which is denoted by $\tau_{0}$ with $x_{0} \in \bar{\tau}_{0}$, and satisfies

$$
\begin{equation*}
(\chi, \tilde{\delta})_{\tau_{0}}=\chi\left(x_{0}\right), \quad \text { for all } \chi \in \mathcal{P}_{r}\left(\tau_{0}\right) \tag{3.1}
\end{equation*}
$$

In addition we also have, see, e.g., [32, Lemma 2.2],

$$
\begin{equation*}
\|\tilde{\delta}\|_{W^{s, p}(\Omega)} \leq C h^{-s-N\left(1-\frac{1}{p}\right)}, \quad 1 \leq p \leq \infty, \quad s=0,1,2 . \tag{3.2}
\end{equation*}
$$

Thus in particular $\|\tilde{\delta}\|_{L^{1}(\Omega)} \leq C,\|\tilde{\delta}\|_{L^{2}(\Omega)} \leq C h^{-\frac{N}{2}}$, and $\|\tilde{\delta}\|_{L^{\infty}(\Omega)} \leq C h^{-N}$. Next we introduce a weight function

$$
\begin{equation*}
\sigma(x)=\sqrt{\left|x-x_{0}\right|^{2}+K^{2} h^{2}} \tag{3.3}
\end{equation*}
$$

where $K>0$ is a sufficiently large constant. This weight function was first introduced in [20, 21] to analyze pointwise finite element error estimates. One can easily check that $\sigma$ satisfies the following properties:

$$
\begin{align*}
\left\|\sigma^{-\frac{N}{2}}\right\|_{L^{2}(\Omega)} & \leq C|\ln h|^{\frac{1}{2}}  \tag{3.4a}\\
|\nabla \sigma| & \leq C  \tag{3.4b}\\
\left|\nabla^{2} \sigma\right| & \leq C \sigma^{-1},  \tag{3.4c}\\
\max _{\tau} \sigma & \leq C \min _{\tau} \sigma \quad \text { for all } \tau \in \mathcal{T}_{h} . \tag{3.4d}
\end{align*}
$$

For the finite element space $V_{h}$ we will utilize the $L^{2}$ projection $P_{h}: L^{2}(\Omega) \rightarrow V_{h}$ defined by

$$
\begin{equation*}
\left(P_{h} v, \chi\right)_{\Omega}=(v, \chi)_{\Omega} \quad \text { for all } \chi \in V_{h} \tag{3.5}
\end{equation*}
$$

the Ritz projection $R_{h}: H_{0}^{1}(\Omega) \rightarrow V_{h}$ defined by

$$
\begin{equation*}
\left(\nabla R_{h} v, \nabla \chi\right)_{\Omega}=(\nabla v, \nabla \chi)_{\Omega} \quad \text { for all } \chi \in V_{h} \tag{3.6}
\end{equation*}
$$

and the usual nodal interpolation operator $i_{h}: C_{0}(\Omega) \rightarrow V_{h}$ with usual approximation properties (cf., e. g., [5], Theorem 3.1.5])

$$
\begin{equation*}
\left\|u-i_{h} u\right\|_{L^{q}(\Omega)} \leq C h^{2+N\left(\frac{1}{q}-\frac{1}{p}\right)}\|u\|_{W^{2, p}(\Omega)}, \quad \text { for } \quad q \geq p>\frac{N}{2} \tag{3.7}
\end{equation*}
$$

as well as the Scott-Zhang interpolation operator $i_{h}^{S Z}: W_{0}^{1,1}(\Omega) \rightarrow V_{h}$ with the approximation properties (cf., e. g., [28]) for $N=3$ :

$$
\begin{equation*}
h\left\|\nabla\left(u-i_{h}^{S Z} u\right)\right\|_{L^{2}(\Omega)}+\left\|u-i_{h}^{S Z} u\right\|_{L^{2}(\Omega)} \leq C h^{\frac{3}{2}}\|u\|_{W^{2, \frac{3}{2}}(\Omega)} \quad \text { for all } u \in W^{2, \frac{3}{2}}(\Omega) \cap W_{0}^{1,1}(\Omega) \tag{3.8}
\end{equation*}
$$

Moreover we introduce the discrete Laplace operator $\Delta_{h}: V_{h} \rightarrow V_{h}$ defined by

$$
\begin{equation*}
\left(-\Delta_{h} v_{h}, \chi\right)_{\Omega}=\left(\nabla v_{h}, \nabla \chi\right)_{\Omega}, \quad \text { for all } \chi \in V_{h} \tag{3.9}
\end{equation*}
$$

The next lemma states an approximation result for the Ritz projection in the $L^{\infty}(\Omega)$ norm.
Lemma 3.1. There exists a constant $C>0$ independent of $h$, such that

$$
\left\|v-R_{h} v\right\|_{L^{\infty}(\Omega)} \leq C h|\ln h|\|\nabla v\|_{L^{\infty}(\Omega)} .
$$

For smooth domains such a result was established in [22, 25, 26] (logfree for higher order elements), for polygonal domains in [24] and [9, Theorem 3.2] (for mildly graded meshes), and for convex polyhedral domains it follows from stability of the Ritz projection in the $L^{\infty}(\Omega)$ norm in [13, Theorem 12].

The following lemma is a superapproximation result in weighted norms.
Lemma 3.2 (Lemma 3 in [13]). Let $v_{h} \in V_{h}$. Then the following estimates hold for any $\alpha, \beta \in \mathbb{R}$ and $K$ (in the definition (3.3) of the weight $\sigma$ ) large enough:

$$
\begin{align*}
& \left\|\sigma^{\alpha}\left(\operatorname{Id}-i_{h}\right)\left(\sigma^{\beta} v_{h}\right)\right\|_{L^{2}(\Omega)}+h\left\|\sigma^{\alpha} \nabla\left(\operatorname{Id}-i_{h}\right)\left(\sigma^{\beta} v_{h}\right)\right\|_{L^{2}(\Omega)} \leq c h\left\|\sigma^{\alpha+\beta-1} v_{h}\right\|_{L^{2}(\Omega)}  \tag{3.10a}\\
& \left\|\sigma^{\alpha}\left(\operatorname{Id}-P_{h}\right)\left(\sigma^{\beta} v_{h}\right)\right\|_{L^{2}(\Omega)}+h\left\|\sigma^{\alpha} \nabla\left(\operatorname{Id}-P_{h}\right)\left(\sigma^{\beta} v_{h}\right)\right\|_{L^{2}(\Omega)} \leq c h\left\|\sigma^{\alpha+\beta-1} v_{h}\right\|_{L^{2}(\Omega)} \tag{3.10b}
\end{align*}
$$

The next lemma describes a connection between the regularized delta function $\tilde{\delta}$ and the weight $\sigma$. Lemma 3.3. There hold

$$
\begin{equation*}
\left\|\sigma^{\frac{N}{2}} \tilde{\delta}\right\|_{L^{2}(\Omega)}+\left\|\sigma^{\frac{N+2}{2}} \nabla \tilde{\delta}\right\|_{L^{2}(\Omega)}+h\left\|\sigma^{\frac{N}{2}} \nabla \tilde{\delta}\right\|_{L^{2}(\Omega)} \leq C \tag{3.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\sigma^{\frac{N}{2}} P_{h} \tilde{\delta}\right\|_{L^{2}(\Omega)}+\left\|\sigma^{\frac{N+2}{2}} P_{h} \nabla \tilde{\delta}\right\|_{L^{2}(\Omega)}+h\left\|\sigma^{\frac{N}{2}} P_{h} \nabla \tilde{\delta}\right\|_{L^{2}(\Omega)} \leq C . \tag{3.12}
\end{equation*}
$$

The proof of the first two terms in (3.11) and (3.12) respectively can be found in [10] for $N=2$ and in [13], Lemma 4] for $N=3$. Using similar arguments it is straightforward to show the result for the other terms.

The following two lemmas provide the flexibility in manipulating weighted norms.
LEmma 3.4. For each $\alpha \in \mathbb{R}$, there is a constant $C>0$ such that for any $v \in H_{0}^{1}(\Omega) \cap H^{2}(\Omega)$ there holds

$$
\left\|\sigma^{\alpha} \nabla v\right\|_{L^{2}(\Omega)} \leq C\left(\left\|\sigma^{\alpha+1} \Delta v\right\|_{L^{2}(\Omega)}+\left\|\sigma^{\alpha-1} v\right\|_{L^{2}(\Omega)}\right)
$$

Proof. There holds

$$
\begin{aligned}
\left\|\sigma^{\alpha} \nabla v\right\|_{L^{2}(\Omega)}^{2} & =\left(\sigma^{2 \alpha} \nabla v, \nabla v\right)=\left(\nabla\left(\sigma^{2 \alpha} v\right), \nabla v\right)-2 \alpha\left(v \sigma^{2 \alpha-1} \nabla \sigma, \nabla v\right) \\
& =-\left(\sigma^{\alpha-1} v, \sigma^{\alpha+1} \Delta v\right)-2 \alpha\left(v \sigma^{\alpha-1} \nabla \sigma, \sigma^{\alpha} \nabla v\right) .
\end{aligned}
$$

Using $|\nabla \sigma| \leq C$ we obtain

$$
\left\|\sigma^{\alpha} \nabla v\right\|_{L^{2}(\Omega)}^{2} \leq\left\|\sigma^{\alpha-1} v\right\|_{L^{2}(\Omega)}\left\|\sigma^{\alpha+1} \Delta v\right\|_{L^{2}(\Omega)}+C\left\|\sigma^{\alpha-1} v\right\|_{L^{2}(\Omega)}\left\|\sigma^{\alpha} \nabla v\right\|_{L^{2}(\Omega)}
$$

Absorbing $\left\|\sigma^{\alpha} \nabla v\right\|_{L^{2}(\Omega)}$ we obtain the desired estimate. $\square$
Lemma 3.5. For each $\alpha \in \mathbb{R}$, there is a constant $C>0$ such that for any $v_{h} \in V_{h}$ there holds

$$
\left\|\sigma^{\alpha} \nabla v_{h}\right\|_{L^{2}(\Omega)} \leq C\left(\left\|\sigma^{\alpha+1} \Delta_{h} v_{h}\right\|_{L^{2}(\Omega)}+\left\|\sigma^{\alpha-1} v_{h}\right\|_{L^{2}(\Omega)}\right)
$$

Proof. Similar to the proof of the previous lemma we have

$$
\begin{aligned}
\left\|\sigma^{\alpha} \nabla v_{h}\right\|_{L^{2}(\Omega)}^{2} & =\left(\sigma^{2 \alpha} \nabla v_{h}, \nabla v_{h}\right)=\left(\nabla\left(\sigma^{2 \alpha} v_{h}\right), \nabla v_{h}\right)-2 \alpha\left(v_{h} \sigma^{2 \alpha-1} \nabla \sigma, \nabla v_{h}\right) \\
& =\left(\nabla P_{h}\left(\sigma^{2 \alpha} v_{h}\right), \nabla v_{h}\right)+\left(\nabla\left(I d-P_{h}\right)\left(\sigma^{2 \alpha} v_{h}\right), \nabla v_{h}\right)-2 \alpha\left(v_{h} \sigma^{2 \alpha-1} \nabla \sigma, \nabla v_{h}\right) \\
& =-\left(\sigma^{\alpha-1} v_{h}, \sigma^{\alpha+1} \Delta_{h} v_{h}\right)+\left(\sigma^{-\alpha} \nabla\left(I d-P_{h}\right)\left(\sigma^{2 \alpha} v_{h}\right), \sigma^{\alpha} \nabla v_{h}\right)-2 \alpha\left(v_{h} \sigma^{\alpha-1} \nabla \sigma, \sigma^{\alpha} \nabla v_{h}\right) .
\end{aligned}
$$

Applying Lemma 3.2 for the second term and using $|\nabla \sigma| \leq C$ we obtain

$$
\left\|\sigma^{\alpha} \nabla v_{h}\right\|_{L^{2}(\Omega)}^{2} \leq\left\|\sigma^{\alpha-1} v_{h}\right\|_{L^{2}(\Omega)}\left\|\sigma^{\alpha+1} \Delta_{h} v_{h}\right\|_{L^{2}(\Omega)}+C\left\|\sigma^{\alpha-1} v_{h}\right\|_{L^{2}(\Omega)}\left\|\sigma^{\alpha} \nabla v_{h}\right\|_{L^{2}(\Omega)}
$$

Absorbing $\left\|\sigma^{\alpha} \nabla v_{h}\right\|_{L^{2}(\Omega)}$ we obtain the desired estimate.
In the following proofs we will make a heavy use of pointwise estimates for the Green's function.
Lemma 3.6. Let $G(x, y)$ denotes the elliptic Green's function of the Laplace operator on the domain $\Omega$. Then for $N=2,3$ the following estimates hold,

$$
\begin{align*}
\left|\nabla_{x} G(x, y)\right| \leq C|x-y|^{1-N}, \quad \text { for all } x, y \in \Omega, \quad x \neq y  \tag{3.13a}\\
\left|\nabla_{y} G(x, y)\right| \leq C|x-y|^{1-N}, \quad \text { for all } x, y \in \Omega, \quad x \neq y  \tag{3.13b}\\
\left|\nabla_{y} \nabla_{x} G(x, y)\right| \leq C|x-y|^{-N}, \quad \text { for all } x, y \in \Omega, \quad x \neq y \tag{3.13c}
\end{align*}
$$

The proof of the first estimate can be found in [11, Prop 1] and the second one follows from the symmetry of the Green's function and the first estimate, i.e. $\left|\nabla_{y} G(x, y)\right|=\left|\nabla_{x} G(y, x)\right| \leq C|x-y|^{1-N}$. The third estimate is also proven in [11, Prop 1].

The next lemma can be thought of as weighted Gagliardo-Nirenberg interpolation inequality.
Lemma 3.7 (Lemma 5 in [13]). Let $N=3$. There exists a constant $C$ independent of $K$ and $h$ such that for any $f \in H_{0}^{1}(\Omega)$, any $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq-\frac{1}{2}$ and any $1 \leq p \leq \infty, \frac{1}{p}+\frac{1}{p^{\prime}}=1$ there holds:

$$
\left\|\sigma^{\alpha} f\right\|_{L^{2}(\Omega)}^{2} \leq C\left\|\sigma^{\alpha-\beta} f\right\|_{L^{p}(\Omega)}\left\|\sigma^{\alpha+1+\beta} \nabla f\right\|_{L^{p^{\prime}}(\Omega)}
$$

provided $\left\|\sigma^{\alpha-\beta} f\right\|_{L^{p}(\Omega)}$ and $\left\|\sigma^{\alpha+1+\beta} \nabla f\right\|_{L^{p^{\prime}}(\Omega)}$ are bounded.
Lemma 3.8. Let $D=\partial_{x_{i}}, i=1, \ldots, N$ denote any partial derivative. Then for $N=2,3$ there holds

$$
\begin{equation*}
\left\|\sigma^{\frac{N-2}{2}} \Delta^{-1} D \tilde{\delta}\right\|_{L^{2}(\Omega)}+\left\|\sigma^{\frac{N}{2}} \nabla \Delta^{-1} D \tilde{\delta}\right\|_{L^{2}(\Omega)} \leq C|\ln h|^{\frac{1}{2}} \tag{3.14}
\end{equation*}
$$

and for $N=3$ there holds

$$
\begin{equation*}
\left\|\Delta^{-1} D \tilde{\delta}\right\|_{L^{3}(\Omega)}+\left\|\nabla \Delta^{-1} D \tilde{\delta}\right\|_{L^{\frac{3}{2}}(\Omega)} \leq C h^{-1} \tag{3.15}
\end{equation*}
$$

Proof. Consider the following elliptic problem

$$
\begin{align*}
-\Delta g(x) & =D \tilde{\delta}(x), & & x \in \Omega  \tag{3.16}\\
g(x) & =0, & & x \in \partial \Omega
\end{align*}
$$

Thus, in order to obtain the estimate (3.14) we need to establish

$$
\left\|\sigma^{\frac{N-2}{2}} g\right\|_{L^{2}(\Omega)}+\left\|\sigma^{\frac{N}{2}} \nabla g\right\|_{L^{2}(\Omega)} \leq C|\ln h|^{\frac{1}{2}}
$$

To estimate the first term, we will be using the following Green's function representation

$$
\begin{equation*}
g(x)=\int_{\tau_{0}} G(x, y) \partial_{y_{i}} \tilde{\delta}(y) d y=-\int_{\tau_{0}} \partial_{y_{i}} G(x, y) \tilde{\delta}(y) d y \tag{3.17}
\end{equation*}
$$

Define $B_{h}=B_{3 h}\left(x_{0}\right) \cap \Omega$ and $B_{h}^{c}=\Omega \backslash B_{h}$ and consider two cases: $x \in B_{h}$ and $x \in B_{h}^{c}$. In the case $x \in B_{h}$, we obtain using polar coordinates centered at $x$ and using (3.2), 3.17), and Lemma 3.6,

$$
|g(x)| \leq\|\tilde{\delta}\|_{L^{\infty}\left(\tau_{0}\right)} \int_{\tau_{0}}\left|\nabla_{y} G(x, y)\right| d y \leq C h^{-N} \int_{\tau_{0}}|x-y|^{1-N} d y \leq C h^{-N} \int_{0}^{c h} d \rho \leq C h^{1-N}
$$

Hence by the Hölder inequality and using that $\sigma \leq C h$ on $B_{h}$, we have

$$
\left\|\sigma^{\frac{N-2}{2}} g\right\|_{L^{2}\left(B_{h}\right)} \leq C h^{\frac{N}{2}} h^{\frac{N-2}{2}}\|g\|_{L^{\infty}\left(B_{h}\right)} \leq C
$$

In the case $x \in B_{h}^{c}$, we have for any $y \in \tau_{0}$ by the triangle inequality

$$
|x-y| \geq\left|x-x_{0}\right|-\left|y-x_{0}\right| \geq\left|x-x_{0}\right|-h
$$

and therefore again by (3.17) and Lemma 3.6

$$
|g(x)| \leq\|\tilde{\delta}\|_{L^{1}\left(\tau_{0}\right)} \frac{C}{\left(\left|x-x_{0}\right|-h\right)^{N-1}} \leq \frac{C}{\left(\left|x-x_{0}\right|-h\right)^{N-1}}
$$

Hence, using polar coordinates with $\rho=\left|x-x_{0}\right|$, we obtain

$$
\left\|\sigma^{\frac{N-2}{2}} g\right\|_{L^{2}\left(B_{h}^{c}\right)}^{2} \leq C \int_{B_{h}^{c}} \frac{\left(\left|x-x_{0}\right|+K h\right)^{N-2}}{\left(\left|x-x_{0}\right|-h\right)^{2 N-2}} d x \leq C \int_{3 h}^{\operatorname{diam}(\Omega)} \frac{(\rho+K h)^{N-2}}{(\rho-h)^{2 N-2}} \rho^{N-1} d \rho \leq C|\ln h| .
$$

Thus, we established

$$
\begin{equation*}
\left\|\sigma^{\frac{N-2}{2}} g\right\|_{L^{2}(\Omega)} \leq C|\ln h|^{\frac{1}{2}} \tag{3.18}
\end{equation*}
$$

To estimate the second term in (3.14) we apply Lemma 3.4 and obtain

$$
\left\|\sigma^{\frac{N}{2}} \nabla g\right\|_{L^{2}(\Omega)} \leq C\left(\left\|\sigma^{\frac{N+2}{2}} D \tilde{\delta}\right\|+\left\|\sigma^{\frac{N-2}{2}} g\right\|\right) \leq C+C\left\|\sigma^{\frac{N-2}{2}} g\right\| \leq C|\ln h|^{\frac{1}{2}}
$$

where we have used Lemma 3.3 and (3.18).
The first term in (3.15) is estimated as follows. There holds

$$
\|g\|_{L^{3}(\Omega)}^{3}=\|g\|_{L^{3}\left(B_{h}\right)}^{3}+\|g\|_{L^{3}\left(B_{h}^{c}\right)}^{3}
$$

For the term on $B_{h}$ we obtain as above

$$
\|g\|_{L^{3}\left(B_{h}\right)}^{3} \leq C h^{3}\|g\|_{L^{\infty}\left(B_{h}\right)}^{3} \leq C h^{-3}
$$

For the second term we have

$$
\|g\|_{L^{3}\left(B_{h}^{c}\right)}^{3} \leq C \int_{B_{h}^{c}} \frac{1}{\left(\left|x-x_{0}\right|-h\right)^{6}} d x \leq C \int_{3 h}^{\operatorname{diam}(\Omega)} \frac{1}{(\rho-h)^{6}} \rho^{2} d \rho \leq C h^{-3}
$$

In order to estimate $\|\nabla g\|_{L^{\frac{3}{2}}(\Omega)}$ we use the pointwise representation

$$
\begin{equation*}
\nabla g(x)=\int_{\tau_{0}} \nabla_{x} G(x, y) \partial_{y_{i}} \tilde{\delta}(y) d y \tag{3.19}
\end{equation*}
$$

apply Lemma 3.6, and obtain for $x \in B_{h}$

$$
|\nabla g(x)| \leq\|\tilde{\delta}\|_{W^{1, \infty}\left(\tau_{0}\right)} \int_{\tau_{0}}\left|\nabla_{x} G(x, y)\right| d y \leq C h^{-N-1} \int_{\tau_{0}}|x-y|^{1-N} d y \leq C h^{-N-1} \int_{0}^{c h} d \rho \leq C h^{-N}
$$

Hence, for $N=3$, we have

$$
\|\nabla g\|_{L^{\frac{3}{2}}\left(B_{h}\right)} \leq C h^{-3}\left(h^{3}\right)^{\frac{2}{3}}=C h^{-1}
$$

For $x \in B_{h}^{c}$ we integrate by parts in (3.19),

$$
\nabla g(x)=-\int_{\tau_{0}} \partial_{y_{i}} \nabla_{x} G(x, y) \tilde{\delta}(y) d y
$$

and obtain using estimate (3.13c) from Lemma 3.6

$$
|\nabla g(x)| \leq\|\tilde{\delta}\|_{L^{1}\left(\tau_{0}\right)} \frac{C}{\left(\left|x-x_{0}\right|-h\right)^{3}} \leq \frac{C}{\left(\left|x-x_{0}\right|-h\right)^{3}} .
$$

Thus,

$$
\|\nabla g\|_{L^{\frac{3}{2}}\left(B_{h}^{c}\right)}^{\frac{3}{2}} \leq C \int_{B_{h}^{c}} \frac{1}{\left(\left|x-x_{0}\right|-h\right)^{\frac{9}{2}}} d x \leq C \int_{3 h}^{\operatorname{diam}(\Omega)} \frac{\rho^{2}}{(\rho-h)^{\frac{9}{2}}} d \rho \leq C h^{-\frac{3}{2}}
$$

This completes the proof. $\square$
We will also require a discrete version of the Lemma3.8
Lemma 3.9. For $N=2,3$, we have

$$
\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} P_{h} D \tilde{\delta}\right\|_{L^{2}(\Omega)} \leq C|\ln h|^{\frac{1}{2}} .
$$

Proof. Let $g$ be solution of (3.16) and let $g_{h} \in V_{h}$ satisfy

$$
\begin{equation*}
-\Delta_{h} g_{h}=P_{h} D \tilde{\delta} \tag{3.20}
\end{equation*}
$$

Notice that $g_{h}=R_{h} g$. Thus in order to establish the lemma, we need to show

$$
\left\|\sigma^{\frac{N}{2}} \nabla g_{h}\right\|_{L^{2}(\Omega)} \leq C|\ln h|^{\frac{1}{2}}
$$

For $N=2$ we apply Lemma 3.5 and obtain

$$
\left\|\sigma \nabla g_{h}\right\|_{L^{2}(\Omega)} \leq C\left(\left\|\sigma^{2} P_{h} D \tilde{\delta}\right\|_{L^{2}(\Omega)}+\left\|g_{h}\right\|_{L^{2}(\Omega)}\right) \leq C+C\left\|g_{h}\right\|_{L^{2}(\Omega)}
$$

where we have used Lemma3.3. Thus, for $N=2$ it remains to prove

$$
\left\|g_{h}\right\|_{L^{2}(\Omega)} \leq C|\ln h|^{\frac{1}{2}}
$$

To prove this estimate, we use Lemma 3.8, global error estimates in the $L^{2}(\Omega)$, the $H^{2}$ regularity, and the property (3.2) of $\tilde{\delta}$. Thus, we obtain

$$
\begin{aligned}
\left\|g_{h}\right\|_{L^{2}(\Omega)} \leq\|g\|_{L^{2}(\Omega)}+\left\|g-g_{h}\right\|_{L^{2}(\Omega)} & \leq C|\ln h|^{\frac{1}{2}}+C h^{2}\|g\|_{H^{2}(\Omega)} \\
& \leq C|\ln h|^{\frac{1}{2}}+C h^{2}\|D \tilde{\delta}\|_{L^{2}(\Omega)} \leq C|\ln h|^{\frac{1}{2}}
\end{aligned}
$$

The case $N=3$ is more challenging. By the triangle inequality we get

$$
\begin{equation*}
\left\|\sigma^{\frac{3}{2}} \nabla g_{h}\right\|_{L^{2}(\Omega)} \leq\left\|\sigma^{\frac{3}{2}} \nabla g\right\|_{L^{2}(\Omega)}+\left\|\sigma^{\frac{3}{2}} \nabla\left(g-g_{h}\right)\right\|_{L^{2}(\Omega)} \tag{3.21}
\end{equation*}
$$

For the first term we have by Lemma 3.8

$$
\left\|\sigma^{\frac{3}{2}} \nabla g\right\|_{L^{2}(\Omega)} \leq C|\ln h|^{\frac{1}{2}} .
$$

For the second term we apply [13, Lemma 10], which gives

$$
\left\|\sigma^{\frac{3}{2}} \nabla\left(g-g_{h}\right)\right\|_{L^{2}(\Omega)} \leq C h\left(\left\|\sigma^{\frac{3}{2}} \Delta_{h} g_{h}\right\|_{L^{2}(\Omega)}+\left\|\sigma^{\frac{1}{2}} \nabla g_{h}\right\|_{L^{2}(\Omega)}\right)
$$

For the term $\left\|\sigma^{\frac{3}{2}} \Delta_{h} g_{h}\right\|_{L^{2}(\Omega)}$ we get by Lemma3.3

$$
\left\|\sigma^{\frac{3}{2}} \Delta_{h} g_{h}\right\|_{L^{2}(\Omega)}=\left\|\sigma^{\frac{3}{2}} P_{h} D \tilde{\delta}\right\|_{L^{2}(\Omega)} \leq C h^{-1}
$$

Inserting this estimate into (3.21) we obtain

$$
\begin{equation*}
\left\|\sigma^{\frac{3}{2}} \nabla g_{h}\right\|_{L^{2}(\Omega)} \leq C|\ln h|^{\frac{1}{2}}+C h\left\|\sigma^{\frac{1}{2}} \nabla g_{h}\right\|_{L^{2}(\Omega)} \tag{3.22}
\end{equation*}
$$

Thus, it remains to estimate $\left\|\sigma^{\frac{1}{2}} \nabla g_{h}\right\|_{L^{2}(\Omega)}$. To this end we apply Lemma 3.5 and obtain

$$
\left\|\sigma^{\frac{1}{2}} \nabla g_{h}\right\|_{L^{2}(\Omega)} \leq C\left(\left\|\sigma^{\frac{3}{2}} P_{h} D \tilde{\delta}\right\|_{L^{2}(\Omega)}+\left\|\sigma^{-\frac{1}{2}} g_{h}\right\|_{L^{2}(\Omega)}\right)
$$

Using Lemma 3.3we obtain

$$
\begin{equation*}
\left\|\sigma^{\frac{1}{2}} \nabla g_{h}\right\|_{L^{2}(\Omega)} \leq C h^{-1}+C\left\|\sigma^{-\frac{1}{2}} g_{h}\right\|_{L^{2}(\Omega)} \tag{3.23}
\end{equation*}
$$

To estimate $\left\|\sigma^{-\frac{1}{2}} g_{h}\right\|_{L^{2}(\Omega)}$ we use Lemma3.7, with $\alpha=\beta=-\frac{1}{2}$ and $p=3$, to obtain

$$
\begin{equation*}
\left\|\sigma^{-\frac{1}{2}} g_{h}\right\|_{L^{2}(\Omega)} \leq C\left\|g_{h}\right\|_{L^{3}(\Omega)}^{\frac{1}{2}}\left\|\nabla g_{h}\right\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{1}{2}} \leq C\left\|g_{h}\right\|_{L^{3}(\Omega)}^{\frac{1}{2}}\|\nabla g\|_{L^{\frac{3}{2}}(\Omega)}^{\frac{1}{2}}, \tag{3.24}
\end{equation*}
$$

where in the last step we used stability of the Ritz projection in the $W^{1, \frac{3}{2}}(\Omega)$ seminorm, see [12]. Using the inverse and the triangle inequalities,

$$
\begin{aligned}
\left\|g_{h}\right\|_{L^{3}(\Omega)} & \leq\|g\|_{L^{3}(\Omega)}+\left\|g-g_{h}\right\|_{L^{3}(\Omega)} \leq\|g\|_{L^{3}(\Omega)}+\left\|i_{h} g-g_{h}\right\|_{L^{3}(\Omega)}+\left\|g-i_{h} g\right\|_{L^{3}(\Omega)} \\
& \leq\|g\|_{L^{3}(\Omega)}+C h^{-\frac{1}{2}}\left\|i_{h} g-g_{h}\right\|_{L^{2}(\Omega)}+\left\|g-i_{h} g\right\|_{L^{3}(\Omega)} \\
& \leq\|g\|_{L^{3}(\Omega)}+C h^{-\frac{1}{2}}\left\|g-g_{h}\right\|_{L^{2}(\Omega)}+C h^{-\frac{1}{2}}\left\|g-i_{h} g\right\|_{L^{2}(\Omega)}+\left\|g-i_{h} g\right\|_{L^{3}(\Omega)} .
\end{aligned}
$$

Using the approximation theory (3.7), the standard $L^{2}$ estimate, and the properties of $\tilde{\delta}$ function, we have

$$
\begin{equation*}
h^{-\frac{1}{2}}\left\|g-g_{h}\right\|_{L^{2}(\Omega)}+h^{-\frac{1}{2}}\left\|g-i_{h} g\right\|_{L^{2}(\Omega)}+\left\|g-i_{h} g\right\|_{L^{3}(\Omega)} \leq C h^{\frac{3}{2}}\|g\|_{H^{2}(\Omega)} \leq C h^{\frac{3}{2}}\|D \tilde{\delta}\|_{L^{2}(\Omega)} \leq C h^{-1} \tag{3.25}
\end{equation*}
$$

By Lemma 3.8 we have

$$
\|g\|_{L^{3}(\Omega)}+\|\nabla g\|_{L^{\frac{3}{2}}(\Omega)} \leq C h^{-1}
$$

Inserting this in (3.24) and (3.23) we obtain

$$
\left\|\sigma^{\frac{1}{2}} \nabla g_{h}\right\| \leq C h^{-1} .
$$

Using (3.22) we establish the lemma for $N=3 . \square$
4. Weighted resolvent estimates. In this section we will prove the weighted resolvent estimates in two and three dimensions. Since in this section (only) we will be dealing with complex valued function spaces, we need to modify the definition of the $L^{2}$-inner product as

$$
(u, v)_{\Omega}=\int_{\Omega} u(x) \bar{v}(x) d x
$$

where $\bar{v}$ is the complex conjugate of $v$. Moreover we introduce the spaces $\mathbb{V}=H_{0}^{1}(\Omega)+i H_{0}^{1}(\Omega)$ and $\mathbb{V}_{h}=$ $V_{h}+i V_{h}$.

In the continuous case for Lipschitz domains the following result was shown in [29]: For any $\gamma \in\left(0, \frac{\pi}{2}\right)$ there exists a constant $C=C_{\gamma}$ such that

$$
\begin{equation*}
\left\|(z+\Delta)^{-1} v\right\|_{L^{p}(\Omega)} \leq \frac{C}{|z|}\|v\|_{L^{p}(\Omega)}, \quad z \in \mathbb{C} \backslash \Sigma_{\gamma}, \quad 1 \leq p \leq \infty, \quad v \in L^{p}(\Omega) \tag{4.1}
\end{equation*}
$$

where $\Sigma_{\gamma}$ is defined by

$$
\begin{equation*}
\Sigma_{\gamma}=\{z \in \mathbb{C}| | \arg z \mid \leq \gamma\} \tag{4.2}
\end{equation*}
$$

In the finite element setting, it is also known that

$$
\begin{equation*}
\left\|\left(z+\Delta_{h}\right)^{-1} \chi\right\|_{L^{p}(\Omega)} \leq \frac{C}{|z|}\|\chi\|_{L^{p}(\Omega)}, \quad \text { for all } z \in \mathbb{C} \backslash \Sigma_{\gamma}, \quad \chi \in \mathbb{V}_{h} \tag{4.3}
\end{equation*}
$$

for $1 \leq p \leq \infty$. For smooth domains such result is established in [1] and for convex polyhedral domains in [13, 17]. In [14, Theorem 7] we also established the following weighted resolvent estimate:

$$
\begin{equation*}
\left\|\sigma^{\frac{N}{2}}\left(z+\Delta_{h}\right)^{-1} \chi\right\|_{L^{2}(\Omega)} \leq \frac{C|\ln h|}{|z|}\left\|\sigma^{\frac{N}{2}} \chi\right\|_{L^{2}(\Omega)}, \quad \text { for all } z \in \mathbb{C} \backslash \Sigma_{\gamma}, \chi \in \mathbb{V}_{h} \tag{4.4}
\end{equation*}
$$

Our goal in this section is to establish another resolvent estimate in the weighted norm, which will be required later.

Theorem 4.1. Let $N=2,3$. For any $\gamma \in\left(0, \frac{\pi}{2}\right)$, there exists a constant $C$ independent of $h$ and $z$ such that

$$
\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1}\left(z+\Delta_{h}\right)^{-1} \chi\right\|_{L^{2}(\Omega)} \leq \frac{C|\ln h|^{\frac{N-1}{N}}}{|z|}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} \chi\right\|_{L^{2}(\Omega)}, \quad \text { for all } z \in \mathbb{C} \backslash \Sigma_{\gamma}, \chi \in \mathbb{V}_{h}
$$

where $\Sigma_{\gamma}$ is defined in (4.2).
Before we provide a proof of the above theorem we collect some preliminary results.
4.1. Preliminary resolvent results. The following lemma will be often used if dealing resolvent estimates.

Lemma 4.2. Let for each $z \in \mathbb{C} \backslash \Sigma_{\gamma}$ the numbers $\alpha_{z}, \beta_{z} \in \mathbb{R}_{+}$be given and let $F_{z}=-z \alpha_{z}^{2}+\beta_{z}^{2}$. Then there exists a constant $C_{\gamma}$ such that

$$
|z| \alpha_{z}^{2}+\beta_{z}^{2} \leq C_{\gamma}\left|F_{z}\right| \quad \text { for all } z \in \mathbb{C} \backslash \Sigma_{\gamma} .
$$

Proof. We consider the polar representation $-z \alpha_{z}^{2}=|z| \alpha_{z}^{2} e^{i \phi_{z}}$ with $\left|\phi_{z}\right| \leq \pi-\gamma$, since $\gamma \leq|\arg z| \leq \pi$. This results in

$$
|z| \alpha_{z}^{2} e^{i \phi_{z}}+\beta_{z}^{2}=F_{z} .
$$

Multiplying it by $e^{-i \phi_{z} / 2}$ and taking real parts, we have

$$
|z| \alpha_{z}^{2}+\beta_{z}^{2} \leq\left(\cos \left(\phi_{z} / 2\right)\right)^{-1}\left|F_{z}\right| \leq(\sin (\gamma / 2))^{-1}\left|F_{z}\right|=C_{\gamma}\left|F_{z}\right|
$$

■
The following result is a best approximation type estimate in $H^{1}$ norm.
Lemma 4.3. Let $w \in \mathbb{V}$ and let $w_{h} \in \mathbb{V}_{h}$ with $e=w-w_{h}$ be defined by the orthogonality relation

$$
\begin{equation*}
z(e, \chi)-(\nabla e, \nabla \chi)=0, \quad \text { for all } \chi \in \mathbb{V}_{h} \tag{4.5}
\end{equation*}
$$

Then there exists a constant $C>0$ such that for any $\chi \in \mathbb{V}_{h}$

$$
\left\|\nabla\left(w-w_{h}\right)\right\|_{L^{2}(\Omega)} \leq C \inf _{\chi \in \mathbb{V}_{h}}\left(h^{-1}\|w-\chi\|_{L^{2}(\Omega)}+\|\nabla(w-\chi)\|_{L^{2}(\Omega)}\right)
$$

Proof. Although the proof is straightforward, we will provide it for a completeness. Using (4.5), for any $\chi \in \mathbb{V}_{h}$ we have

$$
-z\|e\|^{2}+\|\nabla e\|^{2}=-z(e, e)+(\nabla e, \nabla e)=-z(e, w-\chi)+(\nabla e, \nabla(w-\chi)):=F .
$$

Using the Cauchy-Schwarz inequality, we obtain

$$
|F| \leq|z|\|e\|\|w-\chi\|+\|\nabla e\|\|\nabla(w-\chi)\|
$$

Hence, by Lemma 4.2 and the Young's inequality, we have

$$
\begin{aligned}
|z|\|e\|^{2}+\|\nabla e\|^{2} & \leq C_{\gamma}(|z|\|e\|\|w-\chi\|+\|\nabla e\|\|\nabla(w-\chi)\|) \\
& \leq \frac{|z|}{2}\|e\|^{2}+\frac{C_{\gamma}^{2}}{2}|z|\|w-\chi\|^{2}+\frac{1}{2}\|\nabla e\|^{2}+\frac{C_{\gamma}^{2}}{2}\|\nabla(w-\chi)\|^{2} .
\end{aligned}
$$

Canceling, we obtain for all $z \in \mathbb{C} \backslash \Sigma_{\gamma}$

$$
\begin{equation*}
|z|\|e\|^{2}+\|\nabla e\|^{2} \leq C_{\gamma}^{2}\left(|z|\|w-\chi\|^{2}+\|\nabla(w-\chi)\|^{2}\right) . \tag{4.6}
\end{equation*}
$$

Now we consider two cases: $|z| \leq h^{-2}$ and $|z|>h^{-2}$.
Case 1: $|z| \leq h^{-2}$.
Using that $|z| \leq h^{-2}$ from (4.6) we immediately obtain

$$
\|\nabla e\| \leq C_{\gamma}\left(h^{-1}\|w-\chi\|+\|\nabla(w-\chi)\|\right)
$$

Case 2: $|z|>h^{-2}$.
In this case from (4.6), we conclude

$$
\|e\|^{2} \leq C_{\gamma}^{2}\left(\|w-\chi\|^{2}+\frac{1}{|z|}\|\nabla(w-\chi)\|^{2}\right) \leq C_{\gamma}^{2}\left(\|w-\chi\|^{2}+h^{2}\|\nabla(w-\chi)\|^{2}\right)
$$

To estimate $\|\nabla e\|$ we use the triangle and the inverse estimate to obtain

$$
\begin{aligned}
\|\nabla e\| & \leq\|\nabla(w-\chi)\|+\left\|\nabla\left(\chi-w_{h}\right)\right\| \\
& \leq\|\nabla(w-\chi)\|+C_{\text {inv }} h^{-1}\left\|\chi-w_{h}\right\| \\
& \leq\|\nabla(w-\chi)\|+C_{\text {inv }} h^{-1}(\|\chi-w\|+\|e\|) \\
& \leq C_{\text {inv }}\left(1+C_{\gamma}\right) h^{-1}\|w-\chi\|+\left(C_{\text {inv }} C_{\gamma}+1\right)\|\nabla(w-\chi)\| .
\end{aligned}
$$

Combining both cases, we complete the proof. $\mathrm{\square}$
We will also need the following lemma.
Lemma 4.4. Let $w_{h} \in \mathbb{V}_{h}$ be the solution of

$$
z\left(w_{h}, \varphi\right)_{\Omega}-\left(\nabla w_{h}, \nabla \varphi\right)_{\Omega}=(f, \varphi)_{\Omega}, \quad \text { for all } \varphi \in \mathbb{V}_{h}
$$

for some $f \in L^{\frac{3}{2}}(\Omega)+i L^{\frac{3}{2}}(\Omega)$. There exists a constant $c>0$ such that

$$
\left\|\nabla w_{h}\right\|_{L^{3}(\Omega)} \leq C\|f\|_{L^{\frac{3}{2}}(\Omega)} .
$$

Proof. Let $w=(z+\Delta)^{-1} f$. From the resolvent estimates [29] we have

$$
\left\|(z+\Delta)^{-1} f\right\|_{L^{\frac{3}{2}}(\Omega)} \leq \frac{C}{|z|}\|f\|_{L^{\frac{3}{2}}(\Omega)} \quad \text { and } \quad\left\|\Delta(z+\Delta)^{-1} f\right\|_{L^{\frac{3}{2}}(\Omega)} \leq C\|f\|_{L^{\frac{3}{2}}(\Omega)} .
$$

Therefore $\Delta w \in L^{\frac{3}{2}}(\Omega)$ and using the elliptic regularity, see [11, Corollary 1], we can conclude that $w \in$ $W^{2, \frac{3}{2}}(\Omega)$ with

$$
\begin{equation*}
\|w\|_{W^{2, \frac{3}{2}}(\Omega)} \leq C\|f\|_{L^{\frac{3}{2}}(\Omega)} \tag{4.7}
\end{equation*}
$$

Since $W^{2, \frac{3}{2}}(\Omega)$ is not embedded into $C(\Omega)$, we use the Scott-Zhang interpolant $i_{h}^{S Z}$. Thus, by the triangle inequality we have

$$
\left\|\nabla w_{h}\right\|_{L^{3}(\Omega)} \leq\|\nabla w\|_{L^{3}(\Omega)}+\left\|\nabla\left(w-i_{h}^{S Z} w\right)\right\|_{L^{3}(\Omega)}+\left\|\nabla\left(w_{h}-i_{h}^{S Z} w\right)\right\|_{L^{3}(\Omega)}:=J_{1}+J_{2}+J_{3}
$$

Using the Sobolev embedding $W^{2, \frac{3}{2}}(\Omega) \hookrightarrow W^{1,3}(\Omega)$ and (4.7) we have

$$
J_{1} \leq\|w\|_{W^{2, \frac{3}{2}}(\Omega)} \leq C\|f\|_{L^{\frac{3}{2}}(\Omega)}
$$

Similarly, using stability of $i_{h}^{S Z}$ we have

$$
J_{2} \leq\|w\|_{W^{2, \frac{3}{2}}(\Omega)} \leq C\|f\|_{L^{\frac{3}{2}}(\Omega)}
$$

To estimate $J_{3}$, we first use the inverse inequality, Lemma4.3 and (3.8), we have

$$
\begin{aligned}
J_{3} & \leq C h^{-\frac{1}{2}}\left\|\nabla\left(w_{h}-i_{h}^{S Z} w\right)\right\|_{L^{2}(\Omega)} \leq C h^{-\frac{1}{2}}\left(\left\|\nabla\left(w_{h}-w\right)\right\|_{L^{2}(\Omega)}+\left\|\nabla\left(w-i_{h}^{S Z} w\right)\right\|_{L^{2}(\Omega)}\right) \\
& \leq C h^{-\frac{1}{2}}\left(h^{-1}\left\|w-i_{h}^{S Z} w\right\|_{L^{2}(\Omega)}+\left\|\nabla\left(w-i_{h}^{S Z} w\right)\right\|_{L^{2}(\Omega)}\right) \leq C\|w\|_{W^{2, \frac{3}{2}}(\Omega)} \leq C\|f\|_{L^{\frac{3}{2}}(\Omega)}
\end{aligned}
$$

Combining estimates for $J_{1}, J_{2}$, and $J_{3}$ we obtain the lemma. $\square$
The following lemma is needed for the proof of our main resolvent estimate Theorem4.1
Lemma 4.5. Let $N=2$, 3. For a given $\chi \in \mathbb{V}_{h}$, let $u_{h}=\left(z+\Delta_{h}\right)^{-1} \chi$, or equivalently

$$
\begin{equation*}
z\left(u_{h}, \varphi\right)_{\Omega}+\left(\Delta_{h} u_{h}, \varphi\right)_{\Omega}=(\chi, \varphi)_{\Omega}, \quad \text { for all } \varphi \in \mathbb{V}_{h} \tag{4.8}
\end{equation*}
$$

Then for any $\gamma \in\left(0, \frac{\pi}{2}\right)$, there exists a constant $C$ independent of $h$ and $z$ such that

$$
\begin{equation*}
\left\|\sigma^{\frac{N-2}{2}} \Delta_{h}^{-1} u_{h}\right\|_{L^{2}(\Omega)} \leq \frac{C|\ln h|^{\frac{N-1}{N}}}{|z|}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} \chi\right\|_{L^{2}(\Omega)} \quad \text { for all } z \in \mathbb{C} \backslash \Sigma_{\gamma} \tag{4.9}
\end{equation*}
$$

Proof. Most arguments will be using $L^{2}$ inner-products and $L^{2}$ norms over the whole domain $\Omega$. To simplify the notation in this proof we will denote $\|\cdot\|_{L^{2}(\Omega)}$ by $\|\cdot\|$ and $(\cdot, \cdot)_{\Omega}$ by $(\cdot, \cdot)$.

We will consider the cases $N=2$ and $N=3$ separately. Thus, for $N=2$, we need to show

$$
\begin{equation*}
\left\|\Delta_{h}^{-1} u_{h}\right\| \leq \frac{C|\ln h|^{\frac{1}{2}}}{|z|}\left\|\sigma \nabla \Delta_{h}^{-1} \chi\right\| \tag{4.10}
\end{equation*}
$$

To accomplish that, we test (4.8) with $\varphi=-\Delta_{h}^{-2} u_{h}$. We obtain

$$
-z\left(u_{h}, \Delta_{h}^{-2} u_{h}\right)-\left(\Delta_{h} u_{h}, \Delta_{h}^{-2} u_{h}\right)=-\left(\chi, \Delta_{h}^{-2} u_{h}\right)
$$

Using that $\left(u_{h}, \Delta_{h}^{-2} u_{h}\right)=\left\|\Delta_{h}^{-1} u_{h}\right\|^{2}$ and $\left(\Delta_{h} u_{h}, \Delta_{h}^{-2} u_{h}\right)=-\left\|\nabla \Delta_{h}^{-1} u_{h}\right\|^{2}$ we obtain

$$
\begin{equation*}
-z\left\|\Delta_{h}^{-1} u_{h}\right\|^{2}+\left\|\nabla \Delta_{h}^{-1} u_{h}\right\|^{2}=-\left(\chi, \Delta_{h}^{-2} u_{h}\right)=-\left(\Delta_{h}^{-1} \chi, \Delta_{h}^{-1} u_{h}\right) \tag{4.11}
\end{equation*}
$$

Using Lemma 4.2 we obtain

$$
|z|\left\|\Delta_{h}^{-1} u_{h}\right\|^{2}+\left\|\nabla \Delta_{h}^{-1} u_{h}\right\|^{2} \leq C_{\gamma}\left|\left(\Delta_{h}^{-1} \chi, \Delta_{h}^{-1} u_{h}\right)\right|, \quad \text { for } \quad z \in \mathbb{C} \backslash \Sigma_{\gamma}
$$

For the right-hand side we have by the Cauchy-Schwarz and the Young's inequalities,

$$
\left|\left(\Delta_{h}^{-1} \chi, \Delta_{h}^{-1} u_{h}\right)\right| \leq\left\|\Delta_{h}^{-1} \chi\right\|\left\|\Delta_{h}^{-1} u_{h}\right\| \leq \frac{|z|}{2 C_{\gamma}}\left\|\Delta_{h}^{-1} u_{h}\right\|^{2}+\frac{C}{|z|}\left\|\Delta_{h}^{-1} \chi\right\|^{2}
$$

With the Sobolev $W^{1,1}(\Omega) \hookrightarrow L^{2}(\Omega)$ in two space dimensions, the Poincare inequality, and using the property of $\sigma$ (3.4a), we obtain

$$
\left\|\Delta_{h}^{-1} \chi\right\| \leq C\left\|\Delta_{h}^{-1} \chi\right\|_{W^{1,1}(\Omega)} \leq C\left\|\nabla \Delta_{h}^{-1} \chi\right\|_{L^{1}(\Omega)} \leq C|\ln h|^{\frac{1}{2}}\left\|\sigma \nabla \Delta_{h}^{-1} \chi\right\| .
$$

Thus, we have

$$
|z|\left\|\Delta_{h}^{-1} u_{h}\right\|^{2}+\left\|\nabla \Delta_{h}^{-1} u_{h}\right\|^{2} \leq \frac{C|\ln h|}{|z|}\left\|\sigma \nabla \Delta_{h}^{-1} \chi\right\|^{2}+\frac{|z|}{2}\left\|\Delta_{h}^{-1} u_{h}\right\|^{2} .
$$

Kicking back $\frac{|z|}{2}\left\|\Delta_{h}^{-1} u_{h}\right\|^{2}$, we establish (4.10) and hence the lemma for $N=2$.
For $N=3$, we need to show

$$
\begin{equation*}
\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\| \leq \frac{C|\ln h|^{\frac{2}{3}}}{|z|}\left\|\sigma^{\frac{3}{2}} \nabla \Delta_{h}^{-1} \chi\right\| \tag{4.12}
\end{equation*}
$$

To accomplish that, we test (4.8) with $\varphi=-\Delta_{h}^{-1} P_{h}\left(\sigma \Delta_{h}^{-1} u_{h}\right)$. We obtain

$$
-z\left(u_{h}, \Delta_{h}^{-1} P_{h}\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right)-\left(\Delta_{h} u_{h}, \Delta_{h}^{-1} P_{h}\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right)=-\left(\chi, \Delta_{h}^{-1} P_{h}\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right) .
$$

Using that

$$
\left(u_{h}, \Delta_{h}^{-1} P_{h}\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right)=\left(\Delta_{h}^{-1} u_{h}, P_{h}\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right)=\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2}
$$

and

$$
\begin{aligned}
\left(\Delta_{h} u_{h}, \Delta_{h}^{-1} P_{h}\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right)= & \left(\Delta_{h} \Delta_{h}^{-1} u_{h}, P_{h}\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right) \\
= & -\left(\nabla \Delta_{h}^{-1} u_{h}, \nabla P_{h}\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right) \\
= & -\left(\nabla \Delta_{h}^{-1} u_{h}, \nabla\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right)-\left(\nabla \Delta_{h}^{-1} u_{h}, \nabla\left(P_{h}-\mathrm{Id}\right)\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right) \\
= & -\left\|\sigma^{\frac{1}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2}-\left(\nabla \Delta_{h}^{-1} u_{h}, \nabla \sigma \Delta_{h}^{-1} u_{h}\right) \\
& \quad-\left(\sigma^{\frac{1}{2}} \nabla \Delta_{h}^{-1} u_{h}, \sigma^{-\frac{1}{2}} \nabla\left(P_{h}-\mathrm{Id}\right)\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right),
\end{aligned}
$$

we obtain

$$
\begin{equation*}
-z\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2}+\left\|\sigma^{\frac{1}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2}=F \tag{4.13}
\end{equation*}
$$

where

$$
\begin{aligned}
F=F_{1}+F_{2}+F_{3} & :=-\left(\chi, \Delta_{h}^{-1} P_{h}\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right)-\left(\nabla \Delta_{h}^{-1} u_{h}, \nabla \sigma \Delta_{h}^{-1} u_{h}\right) \\
& -\left(\sigma^{\frac{1}{2}} \nabla \Delta_{h}^{-1} u_{h}, \sigma^{-\frac{1}{2}} \nabla\left(P_{h}-\mathrm{Id}\right)\left(\sigma \Delta_{h}^{-1} u_{h}\right)\right) .
\end{aligned}
$$

Using Lemma 4.2 we conclude

$$
\begin{equation*}
|z|\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2}+\left\|\sigma^{\frac{1}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2} \leq C_{\gamma}|F|, \quad \text { for } \quad z \in \mathbb{C} \backslash \Sigma_{\gamma} \tag{4.14}
\end{equation*}
$$

By the Cauchy-Schwarz and the Young's inequalities,

$$
\begin{aligned}
\left|F_{1}\right| \leq\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} \chi\right\|\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\| & \leq \frac{C C_{\gamma}}{|z|}\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} \chi\right\|^{2}+\frac{|z|}{4 C_{\gamma}}\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2} \\
& \leq \frac{C C_{\gamma}}{|z|}\left\|\sigma^{\frac{3}{2}} \nabla \Delta_{h}^{-1} \chi\right\|^{2}+\frac{|z|}{4 C_{\gamma}}\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2},
\end{aligned}
$$

where in the last step we again use Lemma 3.7 with $\alpha=\frac{1}{2}, \beta=0$, and $p=2$. To estimate $F_{2}$ we use the Cauchy-Schwarz and the Young's inequalities, as well as the fact that $|\nabla \sigma| \leq C$.

$$
\left|F_{2}\right| \leq C\left\|\sigma^{\frac{1}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|\left\|\sigma^{-\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\| \leq \frac{1}{4 C_{\gamma}}\left\|\sigma^{\frac{1}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2}+C C_{\gamma}\left\|\sigma^{-\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2}
$$

Using Lemma 3.7 with $\alpha=\beta=-\frac{1}{2}, p=\frac{3}{2}$ and $p^{\prime}=3$, we obtain

$$
\left\|\sigma^{-\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2} \leq C\left\|\Delta_{h}^{-1} u_{h}\right\|_{L^{\frac{3}{2}}(\Omega)}\left\|\nabla \Delta_{h}^{-1} u_{h}\right\|_{L^{3}(\Omega)} .
$$

Using the properties of $\sigma$ and the Hölder inequality, we have

$$
\left\|\Delta_{h}^{-1} u_{h}\right\|_{L^{\frac{3}{2}}(\Omega)} \leq C|\ln h|^{\frac{1}{6}}\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\|,
$$

and as a result

$$
\begin{equation*}
\left|F_{2}\right| \leq \frac{1}{4 C_{\gamma}}\left\|\sigma^{\frac{1}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2}+\frac{|z|}{4 C_{\gamma}}\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2}+\frac{C}{|z|}|\ln h|^{\frac{1}{3}}\left\|\nabla \Delta_{h}^{-1} u_{h}\right\|_{L^{3}(\Omega)}^{2} \tag{4.15}
\end{equation*}
$$

Finally, using the Cauchy-Schwarz inequality, Lemma 3.2 and the Young's inequality, we obtain

$$
\left|F_{3}\right| \leq C\left\|\sigma^{\frac{1}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|\left\|\sigma^{-\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\| \leq \frac{1}{4 C_{\gamma}}\left\|\sigma^{\frac{1}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2}+C_{\gamma}\left\|\sigma^{-\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2}
$$

Similarly to the estimate of $F_{2}$ above we obtain,

$$
\begin{equation*}
\left|F_{3}\right| \leq \frac{1}{4 C_{\gamma}}\left\|\sigma^{\frac{1}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2}+\frac{|z|}{4 C_{\gamma}}\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2}+\frac{C}{|z|}|\ln h|^{\frac{1}{3}}\left\|\nabla \Delta_{h}^{-1} u_{h}\right\|_{L^{3}(\Omega)}^{2} \tag{4.16}
\end{equation*}
$$

Combining estimates for $F_{1}, F_{2}$, and $F_{3}$, inserting them into (4.14) and kicking back, we obtain

$$
\begin{equation*}
|z|\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2}+\left\|\sigma^{\frac{1}{2}} \nabla \Delta^{-1} u_{h}\right\|^{2} \leq \frac{C}{|z|}\left\|\sigma^{\frac{3}{2}} \nabla \Delta_{h}^{-1} \chi\right\|^{2}+\frac{C}{|z|}|\ln h|^{\frac{1}{3}}\left\|\nabla \Delta_{h}^{-1} u_{h}\right\|_{L^{3}(\Omega)}^{2} \tag{4.17}
\end{equation*}
$$

Thus, in order to establish the lemma for $N=3$, we need to show

$$
\begin{equation*}
\left\|\nabla \Delta_{h}^{-1} u_{h}\right\|_{L^{3}(\Omega)} \leq C|\ln h|^{\frac{1}{2}}\left\|\sigma^{\frac{3}{2}} \nabla \Delta_{h}^{-1} \chi\right\| . \tag{4.18}
\end{equation*}
$$

This estimates follows by Lemma 4.4, Sobolev embedding theorem $W^{1,1}(\Omega) \hookrightarrow L^{\frac{3}{2}}(\Omega)$ combined with the Poincare inequality, and the properties of $\sigma$. Indeed,

$$
\left\|\nabla \Delta_{h}^{-1} u_{h}\right\|_{L^{3}(\Omega)} \leq C\left\|\Delta_{h}^{-1} \chi\right\|_{L^{\frac{3}{2}}(\Omega)} \leq C\left\|\nabla \Delta_{h}^{-1} \chi\right\|_{L^{1}(\Omega)} \leq C|\ln h|^{\frac{1}{2}}\left\|\sigma^{\frac{3}{2}} \nabla \Delta_{h}^{-1} \chi\right\| .
$$

This concludes the proof of the lemma.
4.2. Proof of Theorem4.1. For an arbitrary $\chi \in \mathbb{V}_{h}$, the solution to resolvent equation $u_{h}$ satisfies

$$
\begin{equation*}
z\left(u_{h}, \varphi\right)+\left(\Delta_{h} u_{h}, \varphi\right)=(\chi, \varphi), \quad \text { for all } \varphi \in \mathbb{V}_{h} \tag{4.19}
\end{equation*}
$$

First we test (4.19) with $\varphi=\Delta_{h}^{-1} P_{h}\left(\sigma^{N} u_{h}\right)$ to obtain

$$
z\left(u_{h}, \Delta_{h}^{-1} P_{h}\left(\sigma^{N} u_{h}\right)\right)+\left(\Delta_{h} u_{h}, \Delta_{h}^{-1} P_{h}\left(\sigma^{N} u_{h}\right)\right)=\left(\chi, \Delta_{h}^{-1} P_{h}\left(\sigma^{N} u_{h}\right)\right)
$$

Using that

$$
\left(\Delta_{h} u_{h}, \Delta_{h}^{-1} P_{h}\left(\sigma^{N} u_{h}\right)\right)=\left(u_{h}, P_{h}\left(\sigma^{N} u_{h}\right)\right)=\left(u_{h}, \sigma^{N} u_{h}\right)=\left\|\sigma^{\frac{N}{2}} u_{h}\right\|^{2}
$$

and

$$
\begin{aligned}
& \left(u_{h}, \Delta_{h}^{-1} P_{h}\left(\sigma^{N} u_{h}\right)\right)=\left(\Delta_{h}^{-1} u_{h}, P_{h}\left(\sigma^{N} u_{h}\right)\right)=\left(\Delta_{h}^{-1} u_{h}, \sigma^{N} u_{h}\right)=\left(\sigma^{N} \Delta_{h}^{-1} u_{h}, \Delta_{h} \Delta_{h}^{-1} u_{h}\right) \\
& \quad=-\left(\nabla\left(P_{h} \sigma^{N} \Delta_{h}^{-1} u_{h}\right), \nabla \Delta_{h}^{-1} u_{h}\right) \\
& \quad=-\left(\nabla\left(\sigma^{N} \Delta_{h}^{-1} u_{h}\right), \nabla \Delta_{h}^{-1} u_{h}\right)-\left(\nabla\left(P_{h}-\operatorname{Id}\right)\left(\sigma^{N} \Delta_{h}^{-1} u_{h}\right), \nabla \Delta_{h}^{-1} u_{h}\right) \\
& \quad=-\left\|\sigma^{N} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2}-N\left(\sigma^{N-1} \nabla \sigma \Delta_{h}^{-1} u_{h}, \nabla \Delta_{h}^{-1} u_{h}\right)-\left(\nabla\left(P_{h}-\operatorname{Id}\right)\left(\sigma^{N} \Delta_{h}^{-1} u_{h}\right), \nabla \Delta_{h}^{-1} u_{h}\right),
\end{aligned}
$$

we obtain

$$
\begin{equation*}
-z\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2}+\left\|\sigma^{\frac{N}{2}} u_{h}\right\|^{2}=F \tag{4.20}
\end{equation*}
$$

where

$$
\begin{aligned}
F & =F_{1}+F_{2}+F_{3} \\
& :=\left(\chi, \Delta_{h}^{-1} P_{h}\left(\sigma^{N} u_{h}\right)\right)+N z\left(\sigma^{N-1} \nabla \sigma \Delta_{h}^{-1} u_{h}, \nabla \Delta_{h}^{-1} u_{h}\right)+z\left(\sigma^{-\frac{N}{2}} \nabla\left(P_{h}-\mathrm{Id}\right)\left(\sigma^{N} \Delta_{h}^{-1} u_{h}\right), \sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right) .
\end{aligned}
$$

By Lemma 4.2 we conclude

$$
|z|\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2}+\left\|\sigma^{\frac{N}{2}} u_{h}\right\|^{2} \leq C_{\gamma}|F|, \quad \text { for } \quad z \in \mathbb{C} \backslash \Sigma_{\gamma}
$$

To estimate $F_{1}$ we notice that

$$
\begin{aligned}
\left(\chi, \Delta_{h}^{-1} P_{h}\left(\sigma^{N} u_{h}\right)\right) & =\left(\Delta_{h}^{-1} \chi, P_{h}\left(\sigma^{N} u_{h}\right)\right)=\left(\sigma^{N} \Delta_{h}^{-1} \chi, u_{h}\right) \\
= & \left(P_{h}\left(\sigma^{N} \Delta_{h}^{-1} \chi\right), \Delta_{h} \Delta_{h}^{-1} u_{h}\right) \\
= & -\left(\nabla P_{h}\left(\sigma^{N} \Delta_{h}^{-1} \chi\right), \nabla \Delta_{h}^{-1} u_{h}\right) \\
= & -\left(\nabla\left(\sigma^{N} \Delta_{h}^{-1} \chi\right), \nabla \Delta_{h}^{-1} u_{h}\right)-\left(\nabla\left(P_{h}-\mathrm{Id}\right)\left(\sigma^{N} \Delta_{h}^{-1} \chi\right), \nabla \Delta_{h}^{-1} u_{h}\right) \\
= & -\left(\sigma^{N} \nabla \Delta_{h}^{-1} \chi, \nabla \Delta_{h}^{-1} u_{h}\right)-N\left(\sigma^{N-1} \nabla \sigma \Delta_{h}^{-1} \chi, \nabla \Delta_{h}^{-1} u_{h}\right) \\
& \quad-\left(\sigma^{-\frac{N}{2}} \nabla\left(P_{h}-\mathrm{Id}\right)\left(\sigma^{N} \Delta_{h}^{-1} \chi\right), \sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right) .
\end{aligned}
$$

Using $|\nabla \sigma| \leq C$, the Cauchy-Schwarz inequality, and the Young's inequality, we obtain,

$$
\begin{aligned}
\left|F_{1}\right| \leq & \left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} \chi\right\|+C\left\|\sigma^{\frac{N-2}{2}} \Delta_{h}^{-1} \chi\right\|\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\| \\
& +\left\|\sigma^{-\frac{N}{2}} \nabla\left(P_{h}-\mathrm{Id}\right)\left(\sigma^{N} \Delta_{h}^{-1} \chi\right)\right\|\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\| \\
\leq & \frac{C C_{\gamma}}{|z|}\left(\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} \chi\right\|^{2}+\left\|\sigma^{\frac{N-2}{2}} \Delta_{h}^{-1} \chi\right\|^{2}\right)+\frac{|z|}{4 C_{\gamma}}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2}
\end{aligned}
$$

where in the last step we used Lemma 3.2 to obtain

$$
\left\|\sigma^{-\frac{N}{2}} \nabla\left(P_{h}-\mathrm{Id}\right)\left(\sigma^{N} \Delta_{h}^{-1} \chi\right)\right\| \leq C\left\|\sigma^{\frac{N-2}{2}} \Delta_{h}^{-1} \chi\right\|
$$

For $N=2$, using the Sobolev embedding $W^{1,1}(\Omega) \hookrightarrow L^{2}(\Omega)$ and the Poincare inequality, we obtain

$$
\left\|v_{h}\right\| \leq C\left\|v_{h}\right\|_{W^{1,1}(\Omega)} \leq C\left\|\nabla v_{h}\right\|_{L^{1}(\Omega)}, \quad \text { for all } v_{h} \in V_{h}
$$

Using in addition the property of $\sigma$ (3.4a), we obtain

$$
\left\|\Delta_{h}^{-1} \chi\right\| \leq C\left\|\nabla \Delta_{h}^{-1} \chi\right\|_{L^{1}(\Omega)} \leq C|\ln h|^{\frac{1}{2}}\left\|\sigma \nabla \Delta_{h}^{-1} \chi\right\| .
$$

For $N=3$, we use Lemma 3.7 with $\alpha=\frac{1}{2}, \beta=0$, and $p=2$, to obtain

$$
\left\|\sigma^{\frac{1}{2}} \Delta_{h}^{-1} \chi\right\| \leq C\left\|\sigma^{\frac{3}{2}} \nabla \Delta_{h}^{-1} \chi\right\| .
$$

Thus,

$$
\left|F_{1}\right| \leq \frac{C C_{\gamma}|\ln h|^{3-N}}{|z|}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} \chi\right\|^{2}+\frac{|z|}{4 C_{\gamma}}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2} .
$$

To estimate $F_{2}$ we use the Cauchy-Schwarz and the Young's inequalities,

$$
\left|F_{2}\right| \leq C|z|\left\|\sigma^{\frac{N-2}{2}} \Delta_{h}^{-1} u_{h}\right\|\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\| \leq \frac{|z|}{4 C_{\gamma}}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2}+C C_{\gamma}|z|\left\|\sigma^{\frac{N-2}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2}
$$

To estimate $F_{3}$ we use Lemma 3.2, the Cauchy-Schwarz and the Young's inequalities,

$$
\left|F_{3}\right| \leq C|z|\left\|\sigma^{-\frac{N}{2}} \nabla\left(P_{h}-\mathrm{Id}\right)\left(\sigma^{N} \Delta_{h}^{-1} u_{h}\right)\right\|\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\| \leq C_{\gamma}|z|\left\|\sigma^{\frac{N-2}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2}+\frac{|z|}{4 C_{\gamma}}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2} .
$$

Combining estimates for $F_{1}, F_{2}, F_{3}$ and kicking back, we obtain

$$
\begin{equation*}
|z|\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} u_{h}\right\|^{2}+\left\|\sigma^{\frac{N}{2}} u_{h}\right\|^{2} \leq \frac{C|\ln h|^{3-N}}{|z|}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} \chi\right\|^{2}+C|z|\left\|\sigma^{\frac{N-2}{2}} \Delta_{h}^{-1} u_{h}\right\|^{2} . \tag{4.21}
\end{equation*}
$$

Now applying Lemma 4.5 to the last term concludes the proof of the theorem.
5. Discrete maximal parabolic estimates. In this section we state stability results for inhomogeneous problems that are central in establishing our main results. Since we apply the following results for different norms on $V_{h}$, namely, for $L^{p}(\Omega)$, weighted $L^{2}(\Omega)$, and weighted $H^{-1}(\Omega)$ norms, we state them for a general Banach norm $|||\cdot||$.

Let $|\|\cdot\||$ be a norm on $V_{h}$ (naturally extended to a norm on $\mathbb{V}_{h}$ ) such that for some $\gamma \in\left(0, \frac{\pi}{2}\right)$ the following resolvent estimate holds,

$$
\begin{equation*}
\left\lvert\,\left\|\left(z+\Delta_{h}\right)^{-1} \chi\right\|\left\|\leq \frac{M_{h}}{|z|}\right\| \chi\| \|\right., \quad \text { for all } z \in \mathbb{C} \backslash \Sigma_{\gamma}, \quad \chi \in \mathbb{V}_{h} \tag{5.1}
\end{equation*}
$$

where $\Sigma_{\gamma}$ is defined in (4.2) and the constant $M_{h}$ is independent of $z$.
This assumption is fulfilled for $\|\cdot\|\|=\| \cdot \|_{L^{p}(\Omega)}, 1 \leq p \leq \infty$, with a constant $M_{h} \leq C$ independent of $h$, see [17], for $\|\|\cdot\|\|=\left\|\sigma^{\frac{N}{2}}(\cdot)\right\|_{L^{2}(\Omega)}$ with $M_{h} \leq C|\ln h|$, see [14, Theorem 7], and for $\|\|\cdot\|\|=\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1}(\cdot)\right\|_{L^{2}(\Omega)}$ with $M_{h} \leq C|\ln h|^{\frac{N-1}{N}}$, see Theorem4.1

We consider the inhomogeneous heat equation (1.1), with $u_{0}=0$ and its discrete approximation $u_{k h} \in X_{k, h}^{q, r}$ defined by

$$
\begin{equation*}
B\left(u_{k h}, \varphi_{k h}\right)=\left(f, \varphi_{k h}\right), \quad \text { for all } \varphi_{k h} \in X_{k, h}^{q, r} . \tag{5.2}
\end{equation*}
$$

The next result is a discrete maximal parabolic regularity result [15, Theorem 14].
LEMMA 5.1 (Discrete maximal parabolic regularity). Let $\||\cdot|\|$ be a norm on $V_{h}$ fulfilling the resolvent estimate (5.1) and let $1 \leq s \leq \infty$. Let $u_{k h}$ be a solution of (5.2). Then, there exists a constant $C$ independent of $k$ and $h$ such that

$$
\begin{aligned}
\left(\sum_{m=1}^{M} \int_{I_{m}}\left\|\partial_{t} u_{k h}(t)\right\|^{s} d t\right)^{\frac{1}{s}}+\left(\sum_{m=1}^{M} \int_{I_{m}}\left\|\Delta_{h} u_{k h}(t)\right\|^{s} d t\right)^{\frac{1}{s}} & +\left(\sum_{m=1}^{M} k_{m}\| \| k_{m}^{-1}\left[u_{k h}\right]_{m-1} \|^{s}\right)^{\frac{1}{s}} \\
& \leq C M_{h} \ln \frac{T}{k}\left(\int_{I}\left\|P_{h} f(t)\right\|^{s} d t\right)^{\frac{1}{s}}
\end{aligned}
$$

with obvious change of notation in the case $s=\infty$. For $m=1$ the jump is understood as $\left[u_{k h}\right]_{0}=u_{k h, 0}^{+}$.
6. Proofs of pointwise global best approximation results. We are now ready to establish our main results.
6.1. Proof of Theorem 2.1. Proof. Let $\tilde{t} \in(0, T]$ and let $x_{0} \in \Omega$ be an arbitrary but fixed point. Without loss of generality we assume $\tilde{t} \in I_{M}=\left(t_{M-1}, T\right]$. Note, that the case $\tilde{t}=0$ is trivial, since $u_{k h}(0)=P_{h} u_{0}$ and the statement of the theorem follows by the stability of the $L^{2}$ projection in the $W^{1, \infty}(\Omega)$ norm. This stability result is a consequence of the stability in the $L^{\infty}(\Omega)$ norm, see [7] and the standard inverse inequality.

We consider the following regularized Green's function

$$
\begin{align*}
-\tilde{g}_{t}(t, x)-\Delta \tilde{g}(t, x) & =D \tilde{\delta}_{x_{0}}(x) \tilde{\theta}(t) & (t, x) \in I \times \Omega, \\
\tilde{g}(t, x) & =0, & (t, x) \in I \times \partial \Omega,  \tag{6.1}\\
\tilde{g}(T, x) & =0, & x \in \Omega,
\end{align*}
$$

where $\tilde{\delta}_{x_{0}}$ is the smoothed Dirac introduced in (3.1), $D$ denotes an arbitrary partial derivative in space, and $\tilde{\theta} \in C^{\infty}(0, T)$ is the regularized Delta function in time with properties $\operatorname{supp}(\tilde{\theta}) \subset I_{M},\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)} \leq C$ and

$$
\left(\tilde{\theta}, \varphi_{k}\right)_{I_{M}}=\varphi_{k}(\tilde{t}), \quad \text { for all } \varphi_{k} \in X_{k}^{q} .
$$

Let $\tilde{g}_{k h}$ be $\mathrm{dG}(q) \mathrm{cG}(r)$ approximation of $\tilde{g}$, i.e. $B\left(\varphi_{k h}, \tilde{g}-\tilde{g}_{k h}\right)=0$. Then we have

$$
\begin{aligned}
-D u_{k h}\left(\tilde{t}, x_{0}\right) & =\left(u_{k h}, D \tilde{\delta}_{x_{0}} \tilde{\theta}\right)=B\left(u_{k h}, \tilde{g}\right)=B\left(u_{k h}, \tilde{g}_{k h}\right)=B\left(u, \tilde{g}_{k h}\right) \\
& =-\sum_{m=1}^{M}\left(u, \partial_{t} \tilde{g}_{k h}\right)_{I_{m} \times \Omega}+\left(\nabla u, \nabla \tilde{g}_{k h}\right)_{I \times \Omega}-\sum_{m=1}^{M}\left(u_{m},\left[\tilde{g}_{k h}\right]_{m}\right)_{\Omega}=J_{1}+J_{2}+J_{3},
\end{aligned}
$$

where in the sum with jumps we included the last term by setting $\tilde{g}_{k h, M+1}=0$ and defining consequently $\left[\tilde{g}_{k h}\right]_{M}=-\tilde{g}_{k h, M}$. Using the Hölder inequality, stability of the Ritz projection in $W^{1, \infty}(\Omega)$ from [12] and the $L^{\infty}$ error estimate from Lemma 3.1 we have

$$
\begin{aligned}
J_{1} & =-\sum_{m=1}^{M}\left(\left(R_{h} u, \Delta_{h} \Delta_{h}^{-1} \partial_{t} \tilde{g}_{k h}\right)_{I_{m} \times \Omega}+\left(\left(I-R_{h}\right) u, \partial_{t} \tilde{g}_{k h}\right)_{I_{m} \times \Omega}\right) \\
& =\sum_{m=1}^{M}\left(\left(\nabla R_{h} u, \nabla \Delta_{h}^{-1} \partial_{t} \tilde{g}_{k h}\right)_{I_{m} \times \Omega}-\left(\left(I-R_{h}\right) u, \partial_{t} \tilde{g}_{k h}\right)_{I_{m} \times \Omega}\right) \\
& \leq \sum_{m=1}^{M}\left(\|\nabla u\|_{L^{\infty}\left(I_{m} \times \Omega\right)}\left\|\nabla \Delta_{h}^{-1} \partial_{t} \tilde{g}_{k h}\right\|_{L^{1}\left(I_{m} ; L^{1}(\Omega)\right)}+\left\|\left(I-R_{h}\right) u\right\|_{L^{\infty}\left(I_{m} \times \Omega\right)}\left\|\partial_{t} \tilde{g}_{k h}\right\|_{L^{1}\left(I_{m} ; L^{1}(\Omega)\right)}\right) \\
& \leq C|\ln h|^{\frac{1}{2}}\|\nabla u\|_{L^{\infty}(I \times \Omega)} \sum_{m=1}^{M}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} \partial_{t} \tilde{g}_{k h}\right\|_{L^{1}\left(I_{m} ; L^{2}(\Omega)\right)} \\
& +C h|\ln h|\|\nabla u\|_{L^{\infty}(I \times \Omega)} \sum_{m=1}^{M}\left\|\partial_{t} \tilde{g}_{k h}\right\|_{L^{1}\left(I_{m} ; L^{1}(\Omega)\right)} .
\end{aligned}
$$

Applying the discrete maximal parabolic regularity result from Lemma 5.1 with respect to $\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1}(\cdot)\right\|_{L^{2}(\Omega)}$ and with respect to the $L^{1}(\Omega)$ norm we get

$$
\begin{align*}
J_{1} & \leq C \ln \frac{T}{k}\|\nabla u\|_{L^{\infty}(I \times \Omega)}\left(|\ln h|^{\frac{1}{2}+\frac{N-1}{N}}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} P_{h} D \tilde{\delta}\right\|_{L^{2}(\Omega)}\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)}+h|\ln h|\left\|P_{h} D \tilde{\delta}\right\|_{L^{1}(\Omega)}\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)}\right) \\
& \leq C|\ln h|^{\frac{2 N-1}{N}} \ln \frac{T}{k}\|\nabla u\|_{L^{\infty}(I \times \Omega)}, \tag{6.2}
\end{align*}
$$

where in the last step we used Lemma 3.9, Lemma 3.3 and the fact that $\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)} \leq C$. Similarly, using the Hölder inequality, properties of $\sigma$, Lemma5.1 and Lemma 3.9, we have

$$
\begin{align*}
J_{2}=\left(\nabla u, \nabla g_{k h}\right)_{I \times \Omega} & \leq\|\nabla u\|_{\left.L^{\infty}(I \times \Omega)\right)}\left\|\nabla \tilde{g}_{k h}\right\|_{L^{1}\left(I ; L^{1}(\Omega)\right)} \\
& \leq C|\ln h|^{\frac{1}{2}}\|\nabla u\|_{L^{\infty}(I \times \Omega)}\left\|\sigma^{\frac{N}{2}} \nabla \tilde{g}_{k h}\right\|_{L^{1}\left(I ; L^{2}(\Omega)\right)} \\
& \leq C|\ln h|^{\frac{1}{2}+\frac{N-1}{N}} \ln \frac{T}{k}\|\nabla u\|_{L^{\infty}(I \times \Omega)}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} P_{h} D \tilde{\delta}\right\|_{L^{2}(\Omega)}\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)}  \tag{6.3}\\
& \leq C|\ln h|^{\frac{2 N-1}{N}} \ln \frac{T}{k}\|\nabla u\|_{L^{\infty}(I \times \Omega)} .
\end{align*}
$$

Similarly to the estimate of $J_{1}$, using the Hölder inequality, properties of $\sigma$, and Lemma 3.1 we have

$$
\begin{aligned}
J_{3} & =-\sum_{m=1}^{M}\left(\left(R_{h} u_{m},\left[\tilde{g}_{k h}\right]_{m}\right)_{\Omega}+\left(\left(I-R_{h}\right) u_{m},\left[\tilde{g}_{k h}\right]_{m}\right)_{\Omega}\right) \\
& =\sum_{m=1}^{M}\left(\left(\nabla u_{m},\left[\nabla \Delta_{h}^{-1} \tilde{g}_{k h}\right]_{m}\right)_{\Omega}-\left(\left(I-R_{h}\right) u_{m},\left[\tilde{g}_{k h}\right]_{m}\right)_{\Omega}\right) \\
& \leq \sum_{m=1}^{M}\left\|\nabla u_{m}\right\|_{L^{\infty}(\Omega)}\left\|\left[\nabla \Delta_{h}^{-1} \tilde{g}_{k h}\right]_{m}\right\|_{L^{1}(\Omega)}+\sum_{m=1}^{M}\left\|\left(I-R_{h}\right) u_{m}\right\|_{L^{\infty}(\Omega)}\left\|\left[\tilde{g}_{k h}\right]_{m}\right\|_{L^{1}(\Omega)} \\
& \leq C|\ln h|^{\frac{1}{2}}\|\nabla u\|_{L^{\infty}(I \times \Omega)} \sum_{m=1}^{M}\left\|\sigma^{\frac{N}{2}}\left[\nabla \Delta_{h}^{-1} \tilde{g}_{k h}\right]_{m}\right\|_{L^{2}(\Omega)}+C h|\ln h|\|\nabla u\|_{L^{\infty}(I \times \Omega)} \sum_{m=1}^{M}\left\|\left[\tilde{g}_{k h}\right]_{m}\right\|_{L^{1}(\Omega)}
\end{aligned}
$$

Applying the discrete maximal parabolic regularity result from Lemma5.1 with respect to $\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1}(\cdot)\right\|_{L^{2}(\Omega)}$ and with respect to the $L^{1}(\Omega)$ norm we get

$$
\begin{align*}
J_{3} & \leq C \ln \frac{T}{k}\|\nabla u\|_{L^{\infty}(I \times \Omega)}\left(|\ln h|^{\frac{1}{2}+\frac{N-1}{N}}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} P_{h} D \tilde{\delta}\right\|_{L^{2}(\Omega)}\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)}+h|\ln h|\left\|P_{h} D \tilde{\delta}\right\|_{L^{1}(\Omega)}\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)}\right) \\
& \leq C|\ln h|^{\frac{2 N-1}{N}} \ln \frac{T}{k}\|\nabla u\|_{L^{\infty}(I \times \Omega)}, \tag{6.4}
\end{align*}
$$

where in the last step we again used Lemma 3.9, Lemma 3.3, and the fact that $\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)} \leq C$. Combining the estimates for $J_{1}, J_{2}$, and $J_{3}$, and taking supremum over all partial derivatives, we conclude that

$$
\left|\nabla u_{k h}\left(\tilde{t}, x_{0}\right)\right| \leq C \ell_{h} \ell_{k}\|\nabla u\|_{L^{\infty}(I \times \Omega)} .
$$

Using that the $\mathrm{dG}(q) \mathrm{cG}(r)$ method is invariant on $X_{k, h}^{q, r}$, by replacing $u$ and $u_{k h}$ with $u-\chi$ and $u_{k h}-\chi$ for any $\chi \in X_{k, h}^{q, r}$, and using the triangle inequality we obtain Theorem 2.1 प

## 7. Proof of pointwise interior best approximation results.

7.1. Proof of Theorem 2.2, To obtain the interior estimate we introduce a smooth cut-off function $\omega$ with the properties that

$$
\begin{align*}
& \omega(x) \equiv 1, \quad x \in B_{d}  \tag{7.1a}\\
& \omega(x) \equiv 0, \quad x \in \Omega \backslash B_{2 d}  \tag{7.1b}\\
& |\nabla \omega| \leq C d^{-1}, \quad\left|\nabla^{2} \omega\right| \leq C d^{-2} \tag{7.1c}
\end{align*}
$$

where $B_{d}=B_{d}\left(x_{0}\right)$ is a ball of radius $d$ centered at $x_{0}$.
As in the proof of Theorem 2.1, we obtain

$$
\begin{equation*}
-D u_{k h}\left(\tilde{t}, x_{0}\right)=B\left(u_{k h}, \tilde{g}_{k h}\right)=B\left(u, \tilde{g}_{k h}\right)=B\left(\omega u, \tilde{g}_{k h}\right)+B\left((1-\omega) u, \tilde{g}_{k h}\right) \tag{7.2}
\end{equation*}
$$

where $\tilde{g}_{k h}$ is the solution of (6.1). The first term can be estimated using the global result from Theorem 2.1. To this end we introduce $\tilde{u}=\omega u$ and the corresponding $\mathrm{dG}(q) \mathrm{cG}(r)$ solution $\tilde{u}_{k h} \in X_{k, h}^{q, r}$ defined by

$$
B\left(\tilde{u}_{k h}-\tilde{u}, \varphi_{k h}\right)=0 \quad \text { for all } \varphi_{k h} \in X_{k, h}^{q, r}
$$

There holds

$$
\begin{aligned}
B\left(\tilde{u}, \tilde{g}_{k h}\right)=B\left(\tilde{u}_{k h}, g_{k h}\right)=-D \tilde{u}_{k h}\left(\tilde{t}, x_{0}\right) & \leq C \ell_{k} \ell_{h}\|\nabla \tilde{u}\|_{L^{\infty}(I \times \Omega)} \\
& \leq C \ell_{k} \ell_{h}\left(d^{-1}\|u\|_{L^{\infty}\left(I \times B_{2 d}\right)}+\|\nabla u\|_{L^{\infty}\left(I \times B_{2 d}\right)}\right) .
\end{aligned}
$$

This results in

$$
\begin{equation*}
\left|\nabla u_{k h}\left(\tilde{t}, x_{0}\right)\right| \leq C \ell_{k} \ell_{h}\left(d^{-1}\|u\|_{L^{\infty}\left(I \times B_{2 d}\right)}+\|\nabla u\|_{L^{\infty}\left(I \times B_{2 d}\right)}\right)+B\left((1-\omega) u, \tilde{g}_{k h}\right) . \tag{7.3}
\end{equation*}
$$

It remains to estimate the term $B\left((1-\omega) u, \tilde{g}_{k h}\right)$. Using the dual expression (2.3) of the bilinear form $B$ we obtain

$$
\begin{align*}
B\left((1-\omega) u, \tilde{g}_{k h}\right) & =-\sum_{m=1}^{M}\left((1-\omega) u, \partial_{t} \tilde{g}_{k h}\right)_{I_{m} \times \Omega}+\left(\nabla((1-\omega) u), \nabla \tilde{g}_{k h}\right)_{I \times \Omega} \\
& -\sum_{m=1}^{M}\left((1-\omega) u_{m},\left[\tilde{g}_{k h}\right]_{m}\right)_{\Omega}=J_{1}+J_{2}+J_{3} \tag{7.4}
\end{align*}
$$

where again in the sum with jumps we included the last term by setting $\tilde{g}_{k h, M+1}=0$ and defining consequently $\left[\tilde{g}_{k h}\right]_{M}=-\tilde{g}_{k h, M}$. For $J_{1}$, adding and subtracting $\left(R_{h}(1-\omega) u, \partial_{t} \tilde{g}_{k h}\right)_{I \times \Omega}$, we obtain

$$
J_{1}=-\sum_{m=1}^{M}\left(R_{h}(1-\omega) u, \partial_{t} \tilde{g}_{k h}\right)_{I_{m} \times \Omega}+\sum_{m=1}^{M}\left(\left(I-R_{h}\right)(1-\omega) u, \partial_{t} \tilde{g}_{k h}\right)_{I_{m} \times \Omega}=J_{11}+J_{12}
$$

Using that $\sigma^{-\frac{N}{2}} \leq C d^{-\frac{N}{2}}$ on $\Omega \backslash B_{d}$ and $(1-\omega) \leq 1$, we obtain

$$
\begin{aligned}
J_{11} & =\sum_{m=1}^{M}\left(\nabla((1-\omega) u), \nabla \Delta_{h}^{-1} \partial_{t} \tilde{g}_{k h}\right)_{I_{m} \times \Omega} \\
& =\sum_{m=1}^{M}\left(\sigma^{-\frac{N}{2}} \nabla((1-\omega) u), \sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} \partial_{t} \tilde{g}_{k h}\right)_{I_{m} \times \Omega} \\
& \leq C d^{-\frac{N}{2}} \sum_{m=1}^{M}\|\nabla((1-\omega) u)\|_{L^{\infty}\left(I_{m} ; L^{2}(\Omega)\right)}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} \partial_{t} \tilde{g}_{k h}\right\|_{L^{1}\left(I_{m} ; L^{2}(\Omega)\right)} \\
& \leq C d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right) \sum_{m=1}^{M}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} \partial_{t} \tilde{g}_{k h}\right\|_{L^{1}\left(I_{m} ; L^{2}(\Omega)\right)}
\end{aligned}
$$

Applying the discrete maximal parabolic regularity result from Lemma 5.1] with respect to $\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1}(\cdot)\right\|_{L^{2}(\Omega)}$ we get

$$
\begin{align*}
J_{11} & \leq C \ln \frac{T}{k}|\ln h|^{\frac{N-1}{N}} d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right)\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} P_{h} D \tilde{\delta}\right\|_{L^{2}(\Omega)}\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)} \\
& \leq C \ln \frac{T}{k}|\ln h|^{\frac{N-1}{N}+\frac{1}{2}} d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right) \tag{7.5}
\end{align*}
$$

where in the last step we used Lemma3.9, Lemma 3.3 and the fact that $\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)} \leq C$.
The estimate for $J_{12}$ is slightly more involved since $R_{h}$ is a global operator. Put $\psi=(1-\omega) u$, then pointwise in time we obtain

$$
\left(\left(I-R_{h}\right) \psi, \partial_{t} \tilde{g}_{k h}\right)_{\Omega}=\left(\left(I-R_{h}\right) \psi, \partial_{t} \tilde{g}_{k h}\right)_{B_{d / 2}}+\left(\left(I-R_{h}\right) \psi, \partial_{t} \tilde{g}_{k h}\right)_{\Omega \backslash B_{d / 2}}=I_{1}+I_{2}
$$

Using local pointwise error estimates [25], the fact that $\psi$ is supported on $\Omega \backslash B_{d}$, and the standard error estimate for $R_{h}$ we have

$$
\begin{aligned}
I_{1} & \leq\left\|\left(I-R_{h}\right) \psi\right\|_{L^{\infty}\left(B_{d / 2}\right)}\left\|\partial_{t} \tilde{g}_{k h}\right\|_{L^{1}\left(B_{d / 2}\right)} \\
& \leq C\left(|\ln h|\|\psi\|_{L^{\infty}\left(B_{d}\right)}+d^{-\frac{N}{2}}\left\|\left(I-R_{h}\right) \psi\right\|_{L^{2}(\Omega)}\right)\left\|\partial_{t} \tilde{g}_{k h}\right\|_{L^{1}(\Omega)} \\
& \leq C h d^{-\frac{N}{2}}\|\nabla \psi\|_{L^{2}(\Omega)}\left\|\partial_{t} \tilde{g}_{k h}\right\|_{L^{1}(\Omega)} \leq C h d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{2}(\Omega)}+\|\nabla u\|_{L^{2}(\Omega)}\right)\left\|\partial_{t} \tilde{g}_{k h}\right\|_{L^{1}(\Omega)}
\end{aligned}
$$

Using that $\sigma \geq C d$ on $\Omega \backslash B_{d / 2}$ we have for $I_{2}$ :

$$
\begin{aligned}
I_{2} & =\left(\sigma^{-\frac{N}{2}}\left(I-R_{h}\right) \psi, \sigma^{\frac{N}{2}} \partial_{t} \tilde{g}_{k h}\right)_{\Omega \backslash B_{d / 2}} \leq C d^{-\frac{N}{2}}\left\|\left(I-R_{h}\right) \psi\right\|_{L^{2}(\Omega)}\left\|\sigma^{\frac{N}{2}} \partial_{t} \tilde{g}_{k h}\right\|_{L^{2}(\Omega)} \\
& \leq C h d^{-\frac{N}{2}}\|\nabla \psi\|_{L^{2}(\Omega)}\left\|\sigma^{\frac{N}{2}} \partial_{t} \tilde{g}_{k h}\right\|_{L^{2}(\Omega)} \leq C h d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{2}(\Omega)}+\|\nabla u\|_{L^{2}(\Omega)}\right)\left\|\sigma^{\frac{N}{2}} \partial_{t} \tilde{g}_{k h}\right\|_{L^{2}(\Omega)}
\end{aligned}
$$

Combining estimates for $I_{1}$ and $I_{2}$ and using discrete maximal parabolic regularity from Lemma 5.1 with respect to the $L^{1}(\Omega)$ norm and $\left\|\sigma^{\frac{N}{2}}(\cdot)\right\|_{L^{2}(\Omega)}$, we obtain

$$
\begin{align*}
J_{12} & \leq C h d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right) \sum_{m=1}^{M}\left(\left\|\partial_{t} \tilde{g}_{k h}\right\|_{L^{1}\left(I_{m} \times \Omega\right)}+\left\|\sigma^{\frac{N}{2}} \partial_{t} \tilde{g}_{k h}\right\|_{L^{1}\left(I_{m} ; L^{2}(\Omega)\right)}\right) \\
& \leq C \ln \frac{T}{k} h d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right)\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)} \times \\
& \left(\left\|P_{h} D \tilde{\delta}\right\|_{L^{1}(\Omega)}+|\ln h|\left\|\sigma^{\frac{N}{2}} P_{h} D \tilde{\delta}\right\|_{L^{2}(\Omega)}\right) \leq C \ln \frac{T}{k}|\ln h| d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right), \tag{7.6}
\end{align*}
$$

where in the last step we used Lemma 3.3. Thus, combining estimates for $J_{11}$ and $J_{12}$ we obtain

$$
J_{1} \leq C \ln \frac{T}{k}|\ln h|^{\frac{N-1}{N}+\frac{1}{2}} d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right)
$$

To estimate $J_{2}$, we use the Hölder inequality, Lemma5.1 and Lemma 3.9 to obtain

$$
\begin{align*}
J_{2} & =\left(\sigma^{-\frac{N}{2}} \nabla((1-\omega) u), \sigma^{\frac{N}{2}} \nabla \tilde{g}_{k h}\right)_{I \times \Omega} \\
& \leq C d^{-\frac{N}{2}}\|\nabla((1-\omega) u)\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\left\|\sigma^{\frac{N}{2}} \nabla \tilde{g}_{k h}\right\|_{L^{1}\left(I ; L^{2}(\Omega)\right)} \\
& \leq C \ln \frac{T}{k}|\ln h|^{\frac{N-1}{N}} d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right)\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} P_{h} D \tilde{\delta}\right\|_{L^{2}(\Omega)}\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)} \\
& \leq C \ln \frac{T}{k}|\ln h|^{\frac{N-1}{N}+\frac{1}{2}} d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{\left.L^{\infty}\left(I ; L^{2}(\Omega)\right)\right)}\right) . \tag{7.7}
\end{align*}
$$

Similarly to $J_{1}$, to estimate $J_{3}$, we, add and subtract $\left(R_{h}(1-\omega) u,\left[\tilde{g}_{k h}\right]_{m}\right)_{\Omega}$, to obtain

$$
J_{3}=-\sum_{m=1}^{M}\left(R_{h}\left((1-\omega) u_{m}\right),\left[\tilde{g}_{k h}\right]_{m}\right)_{\Omega}+\sum_{m=1}^{M}\left(\left(I-R_{h}\right)\left((1-\omega) u_{m}\right),\left[\tilde{g}_{k h}\right]_{m}\right)_{\Omega}=J_{31}+J_{32}
$$

Similarly to $J_{11}$, using that $\sigma^{-\frac{N}{2}} \leq C d^{-\frac{N}{2}}$ on $\Omega \backslash B_{d}$ and $(1-\omega) \leq 1$, we obtain

$$
\begin{align*}
J_{31} & =\sum_{m=1}^{M}\left(\nabla((1-\omega) u), \nabla \Delta_{h}^{-1}\left[\tilde{g}_{k h}\right]_{m}\right)_{\Omega} \\
& =\sum_{m=1}^{M}\left(\sigma^{-\frac{N}{2}} \nabla((1-\omega) u), \sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1}\left[\tilde{g}_{k h}\right]_{m}\right)_{\Omega} \\
& \leq C d^{-\frac{N}{2}} \sum_{m=1}^{M}\|\nabla((1-\omega) u)\|_{L^{\infty}\left(I_{m} ; L^{2}(\Omega)\right)}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1}\left[\tilde{g}_{k h}\right]_{m}\right\|_{L^{2}(\Omega)} \\
& \leq C d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right) \sum_{m=1}^{M}\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1}\left[\tilde{g}_{k h}\right]_{m}\right\|_{L^{2}(\Omega)} \\
& \leq C \ln \frac{T}{k}|\ln h|^{\frac{N-1}{N}} d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right)\left\|\sigma^{\frac{N}{2}} \nabla \Delta_{h}^{-1} P_{h} D \tilde{\delta}\right\|_{L^{2}(\Omega)}\|\tilde{\theta}\|_{L^{1}\left(I_{M}\right)} \\
& \leq C \ln \frac{T}{k}|\ln h|^{\frac{N-1}{N}+\frac{1}{2}} d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right) \tag{7.8}
\end{align*}
$$

Similarly to $J_{12}$ we also obtain

$$
J_{32} \leq C \ln \frac{T}{k}|\ln h| d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right) .
$$

Combining the estimates for $J_{1}, J_{2}$, and $J_{3}$, and taking supremum over all partial derivatives, we conclude that

$$
\begin{aligned}
&\left|\nabla u_{k h}\left(\tilde{t}, x_{0}\right)\right| \leq C \ell_{k} \ell_{h}\left(d^{-1}\|u\|_{L^{\infty}\left(I \times B_{2 d}\right)}+\|\nabla u\|_{L^{\infty}\left(I \times B_{2 d}\right)}\right. \\
&\left.+d^{-\frac{N}{2}}\left(d^{-1}\|u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}+\|\nabla u\|_{L^{\infty}\left(I ; L^{2}(\Omega)\right)}\right)\right)
\end{aligned}
$$

Using that the $\mathrm{dG}(q) \mathrm{cG}(r)$ method is invariant on $X_{k, h}^{q, r}$, by replacing $u$ and $u_{k h}$ with $u-\chi$ and $u_{k h}-\chi$ for any $\chi \in X_{k, h}^{q, r}$, we obtain Theorem 2.2.

Acknowledgments. The authors would like to thank Dominik Meidner for the careful reading of the manuscript and providing valuable suggestions that help to improve the presentation of the paper.

## REFERENCES

[1] N. Y. BaKaev, V. Thomée, and L. B. Wahlbin, Maximum-norm estimates for resolvents of elliptic finite element operators, Math. Comp., 72 (2003), pp. 1597-1610 (electronic).
[2] S. C. Brenner and L. R. Scott, The mathematical theory of finite element methods, vol. 15 of Texts in Applied Mathematics, Springer-Verlag, New York, 1994.
[3] H. ChEN, An $L^{2}$-and $L^{\infty}$ error analysis for parabolic finite element equations with applications to superconvergence and error expansions, PhD thesis, Ruprecht-Karls-Universität Heidelberg, Germany, 1993.
[4] K. ChRysafinos and N. J. WALKIngton, Error estimates for the discontinuous Galerkin methods for parabolic equations, SIAM J. Numer. Anal., 44 (2006), pp. 349-366 (electronic).
[5] P. G. Ciarlet, The finite element method for elliptic problems, North-Holland, Amsterdam, New York, 1978.
[6] A. Demlow, D. Leykekhman, A. H. Schatz, and L. B. Wahlbin, Best approximation property in the $W_{\infty}^{1}$ norm for finite element methods on graded meshes, Math. Comp., 81 (2012), pp. 743-764.
[7] J. DOUGLAS, JR., T. DUPONT, AND L. WAhLbin, The stability in $L^{q}$ of the $L^{2}$-projection into finite element function spaces, Numer. Math., 23 (1974/75), pp. 193-197.
[8] T. F. DUPONT AND Y. LIU, Symmetric error estimates for moving mesh Galerkin methods for advection-diffusion equations, SIAM J. Numer. Anal., 40 (2002), pp. 914-927 (electronic).
[9] K. Eriksson, An adaptive finite element method with efficient maximum norm error control for elliptic problems, Math. Models Methods Appl. Sci., 4 (1994), pp. 313-329.
[10] K. ERIKSSON AND C. JOhnson, Adaptive finite element methods for parabolic problems. II. Optimal error estimates in $L_{\infty} L_{2}$ and $L_{\infty} L_{\infty}$, SIAM J. Numer. Anal., 32 (1995), pp. 706-740.
[11] S. J. Fromm, Potential space estimates for Green potentials in convex domains, Proc. Amer. Math. Soc., 119 (1993), pp. 225-233.
[12] J. Guzmán, D. Leykekhman, J. Rossmann, and A. H. Schatz, Hölder estimates for Green's functions on convex polyhedral domains and their applications to finite element methods, Numer. Math., 112 (2009), pp. 221-243.
[13] D. LEYKEKHMAN AND B. VEXLER, Finite element pointwise results on convex polyhedral domains, SIAM J. Numer. Anal., 54 (2016), pp. 561-587.
[14] ——, Pointwise best approximation results for Galerkin finite element solutions of parabolic problems, SIAM J. Numer. Anal., 54 (2016), pp. 1365-1384.
[15] ——, Discrete maximal parabolic regularity for Galerkin finite element methods, Numer. Math., 135 (2017), pp. 923-952.
[16] D. LEYKEKHMAN AND L. B. WAHLBIN, A posteriori error estimates by recovered gradients in parabolic finite element equations, BIT, 48 (2008), pp. 585-605.
[17] B. Li and W. Sun, Maximal $L^{p}$ analysis of finite element solutions for parabolic equations with nonsmooth coefficients in convex polyhedra, Math. Comp., 86 (2017), pp. 1071-1102.
[18] F. LUDOVICI AND W. WOLLNER, A priori error estimates for a finite element discretization of parabolic optimization problems with pointwise constraints in time on mean values of the gradient of the state, SIAM J. Control Optim., 53 (2015), pp. 745-770.
[19] D. MEIDNER, R. RANNACHER, AND B. VEXLER, A priori error estimates for finite element discretizations of parabolic optimization problems with pointwise state constraints in time, SIAM J. Control Optim., 49 (2011), pp. 1961-1997.
[20] F. Natterer, Über die punktweise Konvergenz finiter Elemente, Numer. Math., 25 (1975/76), pp. 67-77.
[21] J. A. Nitsche, Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind, Abh. Math. Sem. Univ. Hamburg, 36 (1971), pp. 9-15.
[22] J. A. NITSCHE, $L_{\infty}$-convergence of finite element approximation, in Journées "Éléments Finis" (Rennes, 1975), Univ. Rennes, Rennes, 1975, p. 18.
[23] R. Rannacher and R. Scott, Some optimal error estimates for piecewise linear finite element approximations, Math. Comp., 38 (1982), pp. 437-445.
[24] A. H. Schatz, A weak discrete maximum principle and stability of the finite element method in $L_{\infty}$ on plane polygonal domains. I, Math. Comp., 34 (1980), pp. 77-91.
[25] A. H. Schatz and L. B. WAhLbin, Interior maximum norm estimates for finite element methods, Math. Comp., 31 (1977), pp. 414442.
[26] - On the quasi-optimality in $L_{\infty}$ of the $\dot{H}^{1}$-projection into finite element spaces, Math. Comp., 38 (1982), pp. 1-22.
[27] _-, Interior maximum-norm estimates for finite element methods. II, Math. Comp., 64 (1995), pp. 907-928.
[28] L. R. Scott and S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions, Math. Comp., 54 (1990), pp. 483-493.
[29] Z. W. Shen, Resolvent estimates in $L^{p}$ for elliptic systems in Lipschitz domains, J. Funct. Anal., 133 (1995), pp. 224-251.
[30] F. Tantardini and A. Veeser, The $L^{2}$-projection and quasi-optimality of Galerkin methods for parabolic equations, SIAM J. Numer. Anal., 54 (2016), pp. 317-340.
[31] V. Thоме́e, Galerkin finite element methods for parabolic problems, vol. 25 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, second ed., 2006.
[32] V. Thomée and L. B. Wahlbin, Stability and analyticity in maximum-norm for simplicial Lagrange finite element semidiscretizations of parabolic equations with Dirichlet boundary conditions, Numer. Math., 87 (2000), pp. 373-389.
[33] V. Thomée, J. Xu, and N. Y. Zhang, Superconvergence of the gradient in piecewise linear finite-element approximation to a parabolic problem, SIAM J. Numer. Anal., 26 (1989), pp. 553-573.
[34] M. F. WHEELER, A priori $L_{2}$ error estimates for Galerkin approximations to parabolic partial differential equations, SIAM J. Numer. Anal., 10 (1973), pp. 723-759.


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