# Plane graphs without 4- and 5-cycles and without ext-triangular 7-cycles are 3-colorable 

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#### Abstract

Listed as No. 53 among the one hundred famous unsolved problems in [J. A. Bondy, U. S. R. Murty, Graph Theory, Springer, Berlin, 2008] is Steinberg's conjecture, which states that every planar graph without 4 - and 5 -cycles is 3 -colorable. In this paper, we show that plane graphs without 4 - and 5 -cycles are 3 -colorable if they have no ext-triangular 7 -cycles. This implies that (1) planar graphs without 4-, 5-, 7 -cycles are 3 -colorable, and (2) planar graphs without 4-, 5-, 8-cycles are 3-colorable, which cover a number of known results in the literature motivated by Steinberg's conjecture.


## 1 Introduction

In the field of 3 -colorings of planar graphs, one of the most active topics is about a conjecture proposed by Steinberg in 1976: every planar graph without cycles of length 4 and 5 is 3colorable. There had been no progress on this conjecture for a long time, until Erdös [14] suggested a relaxation of it: does there exist a constant $k$ such that every planar graph without cycles of length from 4 to $k$ is 3 -colorable? Abbott and Zhou [1] confirmed that such $k$ exists and $k \leq 11$. This result was later on improved to $k \leq 9$ by Borodin [2] and, independently, by Sanders and Zhao [13], and to $k \leq 7$ by Borodin, Glebov, Raspaud and Salavatipour (3].

Theorem 1.1 ([3]). Planar graphs without cycles of length from 4 to 7 are 3-colorable.

[^0]We remark that Steinberg's conjecture was recently shown to be false in [6], by constructing a counterexample to the conjecture. The question whether every planar graph without cycles of length from 3 to 5 is 3 -colorable is still open.

A more general problem than Steinberg's Conjecture was formulated in [11, 9]:
Problem 1.2. What is $\mathcal{A}$, a set of integers between 5 and 9, such that for $i \in \mathcal{A}$, every planar graph with cycles of length neither 4 nor $i$ is 3-colorable?

Thus, Steinberg's Conjecture states that $5 \in \mathcal{A}$. Since so far no element of $\mathcal{A}$ has been confirmed, it seems reasonable to consider a relaxation of Problem 1.2 where more integers are forbidden to be the length of a cycle in planar graphs. Due to a famous theorem of Grötzsch that planar graphs without triangles are 3-colorable, triangles are always allowed in further sufficient conditions. Several papers together contribute to the result below:

Theorem 1.3. For any three integers $i, j, k$ with $5 \leq i<j<k \leq 9$, it holds true that planar graphs having no cycles of length $4, i, j, k$ are 3 -colorable.

Later on, the sufficient conditions, concerning three integers forbidden to be the length of a cycle, were considered. The corresponding problem can be formulated as follows:

Problem 1.4. What is $\mathcal{B}$, a set of pairs of integers $(i, j)$ with $5 \leq i<j \leq 9$, such that planar graphs without cycles of length $4, i, j$ are 3 -colorable?

It has been proved by Borodin et al. [4] and independently by Xu [17] that every planar graph having neither 5 - and 7 -cycles nor adjacent 3 -cycles is 3 -colorable. Hence, $(5,7) \in \mathcal{B}$, which improves on Theorem 1.1. More elements of $B$ have been confirmed: $(6,8) \in \mathcal{B}$ by Wang and Chen [15], $(7,9) \in \mathcal{B}$ by Lu et al. [11], and $(6,9) \in \mathcal{B}$ by Jin et al. [9]. The result $(6,7) \in \mathcal{B}$ is implied in the following theorem, which reconfirms the results $(5,7) \in \mathcal{B}$ and $(6,8) \in \mathcal{B}$.

Theorem 1.5 ([5]). Planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable.

In this paper, we show that $(5,8) \in \mathcal{B}$, which leaves four pairs of integers $(5,6),(5,9),(7,8)$, $(8,9)$ unconfirmed as elements of $\mathcal{B}$.

Recently, Mondal gave a proof of the result $(5,8) \in \mathcal{B}$ in [12]. Here we exhibit two couterexamples to the theorem proved in that paper which yields the result $(5,8) \in \mathcal{B}$. We restated this theorem as follows. Let $C$ be a cycle of length at most 12 in a plane graph without 4 -, 5 - and 8 -cycles. $C$ is bad if it is of length 9 or 12 and the subgraph inside $C$ has a partition into 3 - and 6-cycles; otherwise, $C$ is good.

Theorem 1.6 (Theorem 2 in [12]). Let $G$ be a graph without 4-, 5-, and 8-cycles. If $D$ is a good cycle of $G$, then every proper 3-coloring of $D$ can be extended to a proper 3-coloring of the whole graph $G$.

Counterexamples to Theorem 1.6. A plane graph $G_{1}$ consisting of a cycle $C$ of length 12 , say $C:=\left[v_{1} \ldots v_{12}\right]$, and a vertex $u$ inside $C$ connected to all of $v_{1}, v_{2}, v_{6}$. The graph $G_{1}$ contradicts Theorem 1.6, since any proper 3 -coloring of $C$ where $v_{1}, v_{2}, v_{6}$ receive pairwise distinct colors can not be extended to $G_{1}$. Also, a plane graph $G_{2}$ consisting of a cycle $C$ of length 12 and a triangle $T$ inside $C$, say $C:=\left[v_{1} \ldots v_{12}\right]$ and $T:=\left[u_{1} u_{2} u_{3}\right]$, and three more edges $u_{1} v_{1}, u_{2} v_{4}, u_{3} v_{7}$. The graph $G_{2}$ contradicts Theorem 1.6 , since any proper 3-coloring of $C$ where $v_{1}, v_{4}, v_{7}$ receive the same color can not be extended to $G_{2}$ (see Figure 11).


Figure 1: two graphs as counterexamples to Theorem 1.6.

### 1.1 Notations and formulation of the main theorem

The graphs considered in this paper are finite and simple. A graph is planar if it is embeddable into the Euclidean plane. A plane graph $(G, \Sigma)$ is a planar graph $G$ together with an embedding $\Sigma$ of $G$ into the Euclidean plane, that is, $(G, \Sigma)$ is a particular drawing of $G$ in the Euclidean plane. In what follows, we will always say a plane graph $G$ instead of $(G, \Sigma)$, which causes no confusion since no two embeddings of the same graph $G$ will be involved in.

Let $G$ be a plane graph and $C$ be a cycle of $G$. By $\operatorname{Int}(C)$ (or $\operatorname{Ext}(C)$ ) we denote the subgraph of $G$ induced by the vertices lying inside (or outside) $C$. The cycle $C$ is separating if neither $\operatorname{Int}(C)$ nor $\operatorname{Ext}(C)$ is empty. By $\overline{\operatorname{Int}}(C)$ (or $\overline{\operatorname{Ext}}(C)$ ) we denote the subgraph of $G$ consisting of $C$ and its interior (or exterior). The cycle $C$ is triangular if it is adjacent to a triangle, and $C$ is ext-triangular if it is adjacent to a triangle of $\overline{E x t}(C)$.

The following theorem is the main result of this paper.
Theorem 1.7. Plane graphs with neither 4- and 5-cycles nor ext-triangular 7-cycles are 3colorable.

As a consequence of Theorem 1.7, the following corollary holds true.

Corollary 1.8. Planar graphs without cycles of length 4, 5, 8 are 3-colorable, that is, $(5,8) \in \mathcal{B}$.
We remark that Theorem 1.7 implies the known result that $(5,7) \in \mathcal{B}$ as well.
Denote by $d(v)$ the degree of a vertex $v$, by $|P|$ the number of edges of a path $P$, by $|C|$ the length of a cycle $C$ and by $d(f)$ the size of a face $f$. A $k$-vertex (or $k^{+}$-vertex, or $k^{-}$-vertex) is a vertex $v$ with $d(v)=k$ (or $d(v) \geq k$, or $d(v) \leq k$ ). Similar notations are used for paths, cycles, faces with $|P|,|C|, d(f)$ instead of $d(v)$, respectively.

Let $G[S]$ denote the subgraph of $G$ induced by $S$ with either $S \subseteq V(G)$ or $S \subseteq E(G)$. A chord of $C$ is an edge of $\overline{\operatorname{Int}}(C)$ that connects two nonconsecutive vertices on $C$. If $\operatorname{Int}(C)$ has a vertex $v$ with three neighbors $v_{1}, v_{2}, v_{3}$ on $C$, then $G\left[\left\{v v_{1}, v v_{2}, v v_{3}\right\}\right]$ is called a claw of $C$. If $\operatorname{Int}(C)$ has two adjacent vertices $u$ and $v$ such that $u$ has two neighbors $u_{1}, u_{2}$ on $C$ and $v$ has two neighbors $v_{1}, v_{2}$ on $C$, then $G\left[\left\{u v, u u_{1}, u u_{2}, v v_{1}, v v_{2}\right\}\right]$ is called a biclaw of $C$. If $\operatorname{Int}(C)$ has three pairwise adjacent vertices $u, v, w$ which has a neighbor $u^{\prime}, v^{\prime}, w^{\prime}$ on $C$ respectively, then $G\left[\left\{u v, v w, u w, u u^{\prime}, v v^{\prime}, w w^{\prime}\right\}\right]$ is called a triclaw of $C$. If $G$ has four vertices $x, u, v, w$ inside $C$ and four vertices $x_{1}, x_{2}, v_{1}, w_{1}$ on $C$ such that $S=\left\{u v, v w, w u, u x, x x_{1}, x x_{2}, v v_{1}, w w_{1}\right\} \subseteq E(G)$, then $G[S]$ is called a combclaw of $C$ (see Figure 2).


Figure 2: chord, claw, biclaw, triclaw and combclaw of a cycle

A good cycle is an $11^{-}$-cycle that has none of claws, biclaws, triclaws and combclaws. A bad cycle is an $11^{-}$-cycle that is not good.

Instead of Theorem 1.7, it is easier for us to prove the following stronger one:
Theorem 1.9. Let $G$ be a connected plane graph with neither 4- and 5-cycles nor ext-triangular 7 -cycles. If $D$, the boundary of the exterior face of $G$, is a good cycle, then every proper 3coloring of $G[V(D)]$ can be extended to a proper 3-coloring of $G$.

The proof of Theorem 1.9 will be proceeded by using discharging method and is given in the next section. For more information on the discharging method, we refer readers to [7]. The rest of this section contributes to other needed notations.

Let $C$ be a cycle and $T$ be one of chords, claws, biclaws, triclaws and combclaws of $C$. We call the graph $H$ consisting of $C$ and $T$ a bad partition of $C$. The boundary of any one of the parts, into which $C$ is divided by $H$, is called a cell of $H$. Clearly, every cell is a cycle. In case of confusion, let us always order the cells $c_{1}, \cdots, c_{t}$ of $H$ in the way as shown in Figure 2. Let $k_{i}$ be the length of $c_{i}$. Then $T$ is further called a $\left(k_{1}, k_{2}\right)$-chord, a $\left(k_{1}, k_{2}, k_{3}\right)$-claw, a $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$-biclaw, a $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$-triclaw and a ( $k_{1}, k_{2}, k_{3}, k_{4}, k_{5}$ )-combclaw, respectively.

A vertex is external if it lies on the exterior face; internal otherwise. A vertex (or an edge) is triangular if it is incident with a triangle. We say a vertex is bad if it is an internal triangular 3-vertex; good otherwise. A path is a splitting path of a cycle $C$ if it has the two end-vertices on $C$ and all other vertices inside $C$. A $k$-cycle with vertices $v_{1}, \ldots, v_{k}$ in cyclic order is denoted by $\left[v_{1} \ldots v_{k}\right]$.

Let $u v w$ be a path on the boundary of a face $f$ of $G$ with $v$ internal. The vertex $v$ is $f$-heavy if both $u v$ and $v w$ are triangular and $d(v) \geq 5$, and is $f$-Mlight if both $u v$ and $v w$ are triangular and $d(v)=4$, and $f$-Vlight if neither $u v$ nor $v w$ is triangular and $v$ is triangular and of degree 4. A vertex is $f$-light if it is either $f$-Mlight or $f$-Vlight.

Denote by $\mathcal{G}$ the class of connected plane graphs with neither 4 - and 5-cycles nor exttriangular 7-cycles.

## 2 The proof of Theorem 1.9

Suppose to the contrary that Theorem 1.9 is false. From now on, let $G$ be a counterexample to Theorem 1.9 with fewest vertices. Thus, we may assume that the boundary $D$ of the exterior face of $G$ is a good cycle, and there exists a proper 3-coloring $\phi$ of $G[V(D)]$ which cannot be extended to a proper 3 -coloring of $G$. By the minimality of $G$, we deduce that $D$ has no chord.

### 2.1 Structural properties of the minimal counterexample $G$

Lemma 2.1. Every internal vertex of $G$ has degree at least 3.
Proof. Suppose to the contrary that $G$ has an internal vertex $v$ with $d(v) \leq 2$. We can extend $\phi$ to $G-v$ by the minimality of $G$, and then to $G$ by coloring $v$ different from its neighbors.

Lemma 2.2. $G$ is 2-connected and therefore, the boundary of each face of $G$ is a cycle.
Proof. Otherwise, we can assume that $G$ has a pendant block $B$ with cut vertex $v$ such that $B-v$ does not intersect with $D$. We first extend $\phi$ to $G-(B-v)$, and then 3 -color $B$ so that the color assigned to $v$ is unchanged.

Lemma 2.3. G has no separating good cycle.
Proof. Suppose to the contrary that $G$ has a separating good cycle $C$. We extend $\phi$ to $G-$ $\operatorname{Int}(C)$. Furthermore, since $C$ is a good cycle, the color of $C$ can be extended to its interior.

By the definition of a bad cycle, one can easily conclude the lemma as follows.
Lemma 2.4. If $C$ is a bad cycle of a graph in $\mathcal{G}$, then $C$ has length either 9 or 11. Furthermore, if $|C|=9$, then $C$ has a (3,6,6)-claw or a (3,6,6,6)-triclaw; if $|C|=11$, then $C$ has a (3,6,8)claw, or a (3, 6, 6, 6)- or (6,3,6,6)-biclaw, or a (3,6,6,8)-triclaw, or a (3, 6, 6, 6, 6)-combclaw.

Notice that all 3- and 6- and 8-cycles of $G$ are facial, thus the following statement is a consequence of the previous lemma together with the fact that $G \in \mathcal{G}$.

Lemma 2.5. G has neither bad cycle with a chord nor ext-triangular bad 9-cycle.
Lemma 2.6. Let $P$ be a splitting path of $D$ which divides $D$ into two cycles $D^{\prime}$ and $D^{\prime \prime}$. The following four statements hold true.
(1) If $|P|=2$, then there is a triangle between $D^{\prime}$ and $D^{\prime \prime}$.
(2) $|P| \neq 3$.
(3) If $|P|=4$, then there is a 6 - or 7 -cycle between $D^{\prime}$ and $D^{\prime \prime}$.
(4) If $|P|=5$, then there is a $9^{-}$-cycle between $D^{\prime}$ and $D^{\prime \prime}$.

Proof. Since $D$ has length at most 11, we have $\left|D^{\prime}\right|+\left|D^{\prime \prime}\right|=|D|+2|P| \leq 11+2|P|$.
(1) Let $P=x y z$. Suppose to the contrary that $\left|D^{\prime}\right|,\left|D^{\prime \prime}\right| \geq 6$. By Lemma 2.1, $y$ has a neighbor other than $x$ and $z$, say $y^{\prime}$. It follows that $y^{\prime}$ is internal since otherwise $D$ is a bad cycle with a claw. Without loss of generality, let $y^{\prime}$ lie inside $D^{\prime}$. Now $D^{\prime}$ is a separating cycle. By Lemma 2.3, $D^{\prime}$ is not good, i.e., either $D^{\prime}$ is bad or $\left|D^{\prime}\right| \geq 12$. Since every bad cycle has length either 9 or 11 by Lemma 2.4, we have $\left|D^{\prime}\right| \geq 9$. Recall that $\left|D^{\prime}\right|+\left|D^{\prime \prime}\right| \leq 15$, thus $\left|D^{\prime}\right|=9$ and $\left|D^{\prime \prime}\right|=6$. Now $D^{\prime}$ has either a $(3,6,6)$-claw or a $(3,6,6,6)$-triclaw by Lemma 2.4 , which implies that $D$ has a biclaw or a combclaw respectively, a contradiction.
(2) Suppose to the contrary that $|P|=3$. Let $P=w x y z$. Clearly $\left|D^{\prime}\right|,\left|D^{\prime \prime}\right| \geq 6$. Let $x^{\prime}$ and $y^{\prime}$ be a neighbor of $x$ and $y$ not on $P$, respectively. If both $x^{\prime}$ and $y^{\prime}$ are external, then $D$ has a biclaw. Hence, we may assume $x^{\prime}$ lies inside $D^{\prime}$. By Lemmas 2.3 and 2.4 and the inequality $\left|D^{\prime}\right|+\left|D^{\prime \prime}\right| \leq 17$, we deduce that $D^{\prime}$ is a bad cycle and $D^{\prime \prime}$ is a good $8^{-}$-cycle. If $y^{\prime}$ is internal, then $y^{\prime}$ lies inside $D^{\prime}$. It follows with the specific interior of a bad cycle that $x^{\prime}=y^{\prime}$
and $D^{\prime}$ has either a claw or a biclaw, which implies that $D$ has either a triclaw or a combclaw respectively, a contradiction. Hence, $y^{\prime}$ is external. Since every bad cycle as well as every $6^{-}$- or 8 -cycle contains no chord by Lemma 2.5 , we deduce that $y y^{\prime}$ is a $(3,6)$-chord of $D^{\prime \prime}$. It follows that $D^{\prime}$ is a bad and ext-triangular 9 -cycle, contradicting Lemma 2.5 .
(3) Let $P=$ vwxyz. Suppose to the contrary that $\left|D^{\prime}\right|,\left|D^{\prime \prime}\right| \geq 8$. Since $\left|D^{\prime}\right|+\left|D^{\prime \prime}\right| \leq 19$, we have $\left|D^{\prime}\right|,\left|D^{\prime \prime}\right| \leq 11$. Since $G$ has no 4 - and 5 -cycles, if $G$ has an edge $e$ connecting two nonconsecutive vertices on $P$, then the cycle formed by $e$ and $P$ has to be a triangle, yielding a splitting 3-path of $D$, contradicting the statement (2). Therefore, no pair of nonconsecutive vertices on $P$ are adjacent.

Let $w^{\prime}, x^{\prime}, y^{\prime}$ be a neighbor of $w, x, y$ not on $P$, respectively. The statement (2)implies that $x^{\prime}$ is internal. Let $x^{\prime}$ lie inside of $D^{\prime}$. Thus $D^{\prime}$ is a bad 9 - or 11-cycle. If $D^{\prime}$ is a bad 11-cycle, then $D^{\prime \prime}$ is a facial 8-cycle, and thus both $w^{\prime}$ and $y^{\prime}$ lie in $\overline{\operatorname{Int}}\left(D^{\prime}\right)$, which is impossible by the interior of a bad cycle. Hence, $D^{\prime}$ is a bad 9-cycle. By the statement (1), if $w^{\prime} \in V\left(D^{\prime \prime}\right)$, then $G$ has the triangle $\left[v w w^{\prime}\right]$, which makes $D^{\prime}$ ext-triangular, a contradiction. Hence, $w^{\prime} \notin V\left(D^{\prime \prime}\right)$. Furthermore, as a bad cycle, $D^{\prime}$ has no chord by Lemma 2.5, thus $w^{\prime}$ is internal. If $w^{\prime}$ lies inside $D^{\prime}$, then it gives the interior of $D^{\prime}$ no other choices but $w^{\prime}=x^{\prime}$ and $D^{\prime}$ has a $(3,6,6)$-claw, in which case this claw contains a splitting 3 -path of $D$, a contradiction. Hence, $w^{\prime}$ lies inside $D^{\prime \prime}$. Similarly, we can deduce that $y^{\prime}$ lies inside $D^{\prime \prime}$ as well. Note that $\left|D^{\prime \prime}\right| \in\{8,9,10\}$, thus $D^{\prime \prime}$ is a bad 9-cycle but has to contain both $w^{\prime}$ and $y^{\prime}$ inside, which is impossible.
(4) Let $P=$ uvwxyz. Suppose to the contrary that $\left|D^{\prime}\right|,\left|D^{\prime \prime}\right| \geq 10$. Since $\left|D^{\prime}\right|+\left|D^{\prime \prime}\right| \leq 21$, we have $\left|D^{\prime}\right|,\left|D^{\prime \prime}\right| \leq 11$. By similar argument as in the proof of the statement (3), one can conclude that $G$ has no edge connecting two nonconsecutive vertices on $P$. Let $v^{\prime}, w^{\prime}, x^{\prime}, y^{\prime}$ be a neighbor of $v, w, x, y$ not on $P$, respectively.

The statement (2) implies that both $w^{\prime}$ and $x^{\prime}$ are internal. Let $w^{\prime}$ lie inside $D^{\prime}$. It follows that $D^{\prime}$ is a bad 11 -cycle and $D^{\prime \prime}$ is a 10 -cycle. Thus $x^{\prime}$ also lies inside $D^{\prime}$ and furthermore, $x^{\prime}=w^{\prime}$ and $D^{\prime}$ is a bad cycle with either a $(3,6,8)$-claw or a $(3,6,6,6)$-biclaw. It follows that $v^{\prime}, y^{\prime} \in V\left(D^{\prime \prime}\right)$. By the statement (1), $G$ has two triangles $\left[u v v^{\prime}\right]$ and $\left[y y^{\prime} z\right]$, at least one of them is adjacent to a 7 -cycle of $\overline{\operatorname{Int}}\left(D^{\prime}\right)$, a contradiction.

Lemma 2.7. Let $G^{\prime}$ be a connected plane graph obtained from $G$ by deleting a set of internal vertices and identifying two other vertices so that at most one pair of edges are merged. If we
(a) identify no two vertices of $D$, and create no edge connecting two vertices of $D$, and
(b) create no $6^{-}$-cycle and ext-triangular 7-cycle,
then $\phi$ can be extended to $G^{\prime}$.

Proof. The item (a) guarantees that $D$ is unchanged and bounds $G^{\prime}$, and $\phi$ is a proper 3coloring of $G^{\prime}[V(D)]$. By item $(b)$, the graph $G^{\prime}$ is simple and $G^{\prime} \in \mathcal{G}$. Hence, to extend $\phi$ to $G^{\prime}$ by the minimality of $G$, it remains to show that $D$ is a good cycle of $G^{\prime}$.

Suppose to the contrary that $D$ has a bad partition $H$ in $G^{\prime}$. Clearly, $H$ has a 6 -cell $C^{\prime}$ such that the intersection between $D$ and $C^{\prime}$ is a path $v_{1} \ldots v_{k}$ of length $k-1$ with $k \in\{4,5\}$. Since we create no 6 -cycles, $C^{\prime}$ corresponds to a 6 -cycle $C$ of the original graph $G$. Recall that at most one pair of edges are merged during the process from $G$ to $G^{\prime}$, we deduce that the intersection between $D$ and $C$ is a path $P$ of one of the forms $v_{1} \ldots v_{k}, v_{1} \ldots v_{k-1}, v_{2} \ldots v_{k}$. Thus, $|P| \in\{3,4,5\}$. If $|P| \in\{4,5\}$, then $C$ contains a splitting 3 - or 2-path of $D$ in $G$, yielding a contradiction by Lemma 2.6. Hence, $|P|=3$ and so $k=4$. By the choice of the 6 -cell $C^{\prime}$, we may assume that the bad partition $H$ has either a $(3,6,6,6)$ - or $(3,6,6,8)$-triclaw. Now $H$ contains three splitting 3 -paths of $D$, at least one of them does not contain the identified vertex of $G^{\prime}$ no matter where it is, yielding the existence of a splitting 3-path of $D$ in $G$, contradicting Lemma 2.6.

Lemma 2.8. $G$ has no edge uv incident with a 6 -face and a 3-face such that both $u$ and $v$ are internal 3-vertices and therefore, every bad cycle of $G$ has either a (3,6,6)- or (3,6,8)-claw or a (3,6,6,6)-biclaw.

Proof. Suppose to the contrary that such an edge $u v$ exists. Denote by $[u v w x y z]$ and $[u v t]$ the 6 -face and 3 -face, respectively. Lemma 2.6 implies that not both of $w$ and $z$ are external vertices. Without loss of generality, we may assume that $w$ is internal. Let $G^{\prime}$ be the graph obtained from $G$ by deleting $u$ and $v$, and identifying $w$ with $y$ so that $w x$ and $y x$ are merged. Clearly, $G^{\prime}$ is a plane graph on fewer vertices than $G$. We will show that both the items in Lemma 2.7 are satisfied.

Since $w$ is internal, we identify no two vertices on $D$. If we create an edge connecting two vertices on $D$, then $w$ has a neighbor $w_{1}$ not adjacent to $y$ and both $y$ and $w_{1}$ are external. But now, Lemma 2.6 implies that $x$ is external and thus, $\left[w w_{1} x\right]$ is a triangle which makes the 7-cycle [utvwxyz] ext-triangular. Hence, the item (a) holds.

Suppose we create a $6^{-}$-cycle or an ext-triangular 7 -cycle $C^{\prime}$. Thus $G$ has a $7^{-}$-path $P$ between $w$ and $y$ corresponding to $C^{\prime}$. If $x \in V(P)$, then neither $w x$ nor $x y$ are on $P$ since otherwise, $C^{\prime}$ already exists in $G$. Hence, the paths $w x y$ and $P$ form two cycles, both of them has length at least 6. It follows that $|P| \geq 10$, a contradiction. Hence, we may assume that $x \notin V(P)$. The paths $P$ and $w x y$ form a $9^{-}$-cycle, say $C$. By Lemma 2.1, we may let $x_{1}$ be a neighbor of $x$ other than $y$ and $w$. We have $x_{1} \notin V(P)$, since otherwise $P$ has length at least 8. Now $C$ has to contain either $u$ and $v$ or $x_{1}$ inside, which implies that $C$ is a bad

9-cycle. By Lemma 2.5, $C$ is not ext-triangular. Thus $C^{\prime}$ is a 7 -cycle that is not ext-triangular, contradicting the supposition. Hence, the item (b) holds.

By Lemma 2.7, the pre-coloring $\phi$ can be extended to $G^{\prime}$. Since $z$ and $w$ receive different colors, we can properly color $v$ and $u$, extending $\phi$ further to $G$.

We follow the notations of $M$-face and $M M$-face in [3], and define weak tetrads. An $M$-face is an 8 -face $f$ containing no external vertices with boundary $\left[v_{1} \ldots v_{8}\right]$ such that the vertices $v_{1}, v_{2}, v_{3}, v_{5}, v_{6}, v_{7}$ are of degree 3 and the edges $v_{1} v_{2}, v_{3} v_{4}, v_{4} v_{5}, v_{6} v_{7}$ are triangular. An $M M$-face is an 8 -face $f$ containing no external vertices with boundary $\left[v_{1} \ldots v_{8}\right]$ such that $v_{2}$ and $v_{7}$ are of degree 4 and other six vertices on $f$ are of degree 3 , and the edges $v_{1} v_{2}, v_{2} v_{3}, v_{4} v_{5}, v_{6} v_{7}, v_{7} v_{8}$ are triangular. A weak tetrad is a path $v_{1} \ldots v_{5}$ on the boundary of a face $f$ such that both the edges $v_{1} v_{2}$ and $v_{3} v_{4}$ are triangular, all of $v_{1}, v_{2}, v_{3}, v_{4}$ are internal 3 -vertices, and $v_{5}$ is either of degree 3 or $f$-light.

Lemma 2.9. $G$ has no weak tetrad and therefore, every face of $G$ contains no five consecutive bad vertices.

Proof. Suppose to the contrary that $G$ has a weak tetrad $T$ following the notation used in the definition. Denote by $v_{0}$ the neighbor of $v_{1}$ on $f$ with $v_{0} \neq v_{2}$. Denote by $x$ the common neighbor of $v_{1}$ and $v_{2}$, and $y$ the common neighbor of $v_{3}$ and $v_{4}$. If $x=v_{0}$, then $v_{1}$ is an internal 2-vertex, contradicting Lemma 2.1. Hence, $x \neq v_{0}$ and similarly, $x \neq v_{3}$. Since $G$ has no 4 - or 5-cycles, $x \notin\left\{v_{4}, v_{5}\right\}$. Concluding above, $x \notin v_{0} \cup V(T)$. Similarly, $y \notin v_{0} \cup V(T)$. Moreover, $x \neq y$ since otherwise $\left[v_{1} v_{2} v_{3} x\right]$ is a 4 -cycle. We delete $v_{1}, \ldots, v_{4}$ and identity $v_{0}$ with $y$, obtaining a plane graph $G^{\prime}$ on fewer vertices than $G$. We will show that both the items in Lemma 2.7 are satisfied.

Suppose that we create a $6^{-}$-cycle or an ext-triangular 7 -cycle $C^{\prime}$. Thus $G$ has a $7^{-}$-path $P$ between $v_{0}$ and $y$ corresponding to $C^{\prime}$. If $x \in V(P)$, then the cycle formed by $P$ and $v_{0} v_{1} x$ has length at least 6 and the one formed by $P$ and $x v_{2} v_{3} y$ has length at least 8 , which gives $|P| \geq 9$, a contradiction. Hence, $x \notin V(P)$. The paths $P$ and $v_{0} v_{1} v_{2} v_{3} y$ form a $11^{-}$-cycle, say $C$. Now $C$ contains either $x$ or $v_{4}$ inside. Thus, $C$ is a bad cycle. By Lemma 2.8, $C$ has either a $(3,6,6)$ - or $(3,6,8)$-claw or a $(3,6,6,6)$-biclaw. Note that both the two faces incident with $v_{2} v_{3}$ has length at least 8 , thus $C$ has a bad partition owning an 8 -cell no matter which one of $x$ and $v_{4}$ lies inside $C$. It follows that $C$ has a $(3,6,8)$-claw. If $x$ lies inside $C$, then the 6 -cell is adjacent to the triangle $\left[x v_{1} v_{2}\right]$ with $d\left(v_{1}\right)=d(x)=3$, contradicting Lemma 2.8. Hence, $v_{4}$ lies inside $C$. Note that $v_{4} v_{5}$ is incident with the 6 -cell and the 8 -cell, we deduce that $v_{5}$ is not $f$-light. By the assumption of $T$ as a weak tetrad, we may assume that $d\left(v_{5}\right)=3$. We delete
$v_{5}$ together with other vertices of $T$ and repeat the argument above, yielding a contradiction. Therefore, the item (b) holds.

Suppose we identify two vertices on $D$ or create an edge connecting two vertices on $D$. Thus there is a splitting 4 - or 5 -path $Q$ of $D$ containing the path $v_{0} v_{1} v_{2} v_{3} y$. By Lemma 2.6, $Q$ together with $D$ forms a $9^{-}$-cycle which corresponds to a $5^{-}$-cycle in $G^{\prime}$. Since we create no $6^{-}$-cycle, a contradiction follows. Hence, the item (a) holds.

By Lemma 2.7, the pre-coloring $\phi$ can be extended to $G^{\prime}$. We first properly color $v_{5}$ (if needed), $v_{4}, v_{3}$ in turn. Since $v_{0}$ and $v_{3}$ receive different colors, we can properly color $v_{1}$ and $v_{2}$, extending $\phi$ further to $G$.

Lemma 2.10. G has no M-face.
Proof. Suppose to the contrary that $G$ has an $M$-face $f$ following the notation used in the definition. For $(i, j) \in\{(1,2),(3,4),(4,5),(6,7)\}$, denote by $t_{i j}$ the common neighbor of $v_{i}$ and $v_{j}$. By similar argument as in the proof of previous lemma, we deduce that the vertices $t_{12}, t_{34}, t_{45}, t_{67}$ are pairwise distinct and not incident with $f$. We delete $v_{1}, v_{2}, v_{3}, v_{5}, v_{6}, v_{7}$ and identity $v_{4}$ with $v_{8}$, obtaining a plane graph $G^{\prime}$ on fewer vertices than $G$. We will show that both the items in Lemma 2.7 are satisfied.

Suppose that we create a $6^{-}$-cycle or an ext-triangular 7 -cycle $C^{\prime}$. Thus $G$ has a $7^{-}$-path $P$ between $v_{4}$ and $v_{8}$ corresponding to $C^{\prime}$. By the symmetry of an $M$-face, we may assume that $P$ together with the path $v_{4} \ldots v_{8}$ forms a $11^{-}$-cycle $C$ containing $v_{1}, v_{2}, v_{3}$ inside. It follows with Lemma 2.8 that $C$ is a bad cycle with a (3, $6,6,6$ )-claw. But now $\overline{\operatorname{Int}}(C)$ contains $f$ that is an 8-face, a contradiction. Therefore, the item (b) holds.

The satisfaction of the item $(a)$ can be proved in a similar way as in the proof of previous lemma.

By Lemma 2.7, the pre-coloring $\phi$ can be extended to $G^{\prime}$. Since we first color $v_{3}$ different from $v_{8}$, both $v_{1}$ and $v_{2}$ can be properly colored. Finally, color $v_{5}, v_{6}, v_{7}$ in the same way, extending $\phi$ further to $G$.

Lemma 2.11. G has no MM-face.
Proof. Suppose to the contrary that $G$ has an $M M$-face $f$ following the notation used in the definition. For $(i, j) \in\{(1,2),(2,3),(4,5),(6,7),(7,8)\}$, denote by $t_{i j}$ the common neighbor of $v_{i}$ and $v_{j}$. Similarly, we deduce that the vertices $t_{12}, t_{23}, t_{45}, t_{67}, t_{78}$ are pairwise distinct and not incident with $f$. We delete all the vertices of $f$ and identity $t_{12}$ with $t_{67}$, obtaining a plane graph $G^{\prime}$ on fewer vertices than $G$. To extend $\phi$ to $G^{\prime}$, it suffices to fulfill the item (a) of Lemma 2.7, as what we did in previous lemma.

Suppose that we create a $6^{-}$-cycle or an ext-triangular 7-cycle $C^{\prime}$. Thus $G$ has a $7^{-}$-path $P$ between $t_{12}$ and $t_{67}$ corresponding to $C^{\prime}$. If $t_{78} \in V(P)$, then both the cycles formed by $P$ and $t_{12} v_{1} v_{8} t_{78}$ and by $P$ and $t_{78} v_{7} t_{67}$ have length at least 8 , which gives $|P| \geq 11$, a contradiction. Hence, $t_{78} \notin V(P)$. The paths $P$ and $t_{12} v_{1} v_{8} v_{7} t_{67}$ form a $11^{-}$-cycle, say $C$. It follows that $C$ is a bad cycle containing either $t_{78}$ or $v_{2}, \ldots, v_{6}$ inside, that is, either $C$ has a bad partition owning two $8^{+}$-cell or $C$ contains five vertices inside, a contradiction in any case.

We further extend $\phi$ from $G^{\prime}$ to $G$ as follows. Let $\alpha, \beta$ and $\gamma$ be the three colors used in $\phi$. First regardless the edge $v_{1} v_{8}$, we can properly color $v_{2}, v_{1}, v_{3}$ and $v_{7}, v_{8}, v_{6}$. If $v_{1}$ and $v_{8}$ receive different colors and so do $v_{3}$ and $v_{6}$, then $v_{4}$ and $v_{5}$ can be properly colored, we are done. Hence, we may assume without loss of generality that $v_{1}$ and $v_{8}$ receive the same color, say $\beta$. Let $\alpha$ be the color assigned to $t_{12}$ and $t_{67}$. Thus $v_{2}$ and $v_{7}$ are colored with $\gamma$ and $t_{78}$ is colored with $\alpha$. We recolor $v_{8}, v_{7}, v_{6}$ with $\gamma, \beta, \gamma$ respectively. Now $v_{1}$ and $v_{8}$ receive different colors and so do $v_{3}$ and $v_{6}$. Again $v_{4}$ and $v_{5}$ can be properly colored, we are also done.

### 2.2 Discharging in $G$

Let $V=V(G), E=E(G)$, and $F$ be the set of faces of $G$. Denote by $f_{0}$ the exterior face of $G$. Give initial charge $\operatorname{ch}(x)$ to each element $x$ of $V \cup F$, where $\operatorname{ch}\left(f_{0}\right)=d\left(f_{0}\right)+4, \operatorname{ch}(v)=d(v)-4$ for $v \in V$, and $\operatorname{ch}(f)=d(f)-4$ for $f \in F \backslash\left\{f_{0}\right\}$. Discharge the elements of $V \cup F$ according to the following rules:
$R 1$. Every internal 3 -face receives $\frac{1}{3}$ from each incident vertex.
$R 2$. Every internal $6^{+}$-face sends $\frac{2}{3}$ to each incident 2 -vertex.
R3. Every internal $6^{+}$-face sends each incident 3 -vertex $v$ charge $\frac{2}{3}$ if $v$ is triangular, and charge $\frac{1}{3}$ otherwise.
$R 4$. Every internal $6^{+}$-face $f$ sends $\frac{1}{3}$ to each $f$-light vertex, and receives $\frac{1}{3}$ from each $f$-heavy vertex.
$R 5$. Every internal $6^{+}$-face receives $\frac{1}{3}$ from each incident external $4^{+}$-vertex.
$R 6$. The exterior face $f_{0}$ sends $\frac{4}{3}$ to each incident vertex.
Let $c h^{*}(x)$ denote the final charge of each element $x$ of $V \cup F$ after discharging. On one hand, by Euler's formula we deduce $\sum_{x \in V \cup F} c h(x)=0$. Since the sum of charges over all elements of $V \cup F$ is unchanged, it follows that $\sum_{x \in V \cup F} c h^{*}(x)=0$. On the other hand, we show that
$c h^{*}(x) \geq 0$ for $x \in V \cup F \backslash\left\{f_{0}\right\}$ and $c h^{*}\left(f_{0}\right)>0$. Hence, this obvious contradiction completes the proof of Theorem 1.9. It remains to show that $c h^{*}(x) \geq 0$ for $x \in V \cup F \backslash\left\{f_{0}\right\}$ and $c h^{*}\left(f_{0}\right)>0$.

We remark that the discharging rules can be tracked back to the one used in [3].
Lemma 2.12. $c h^{*}(v) \geq 0$ for $v \in V$.
Proof. First suppose that $v$ is external. Since $D$ is a cycle, $d(v) \geq 2$. If $d(v)=2$, then since $D$ has no chord, the internal face incident with $v$ is not a triangle and sends $\frac{2}{3}$ to $v$ by $R 2$. Moreover, $v$ receives $\frac{4}{3}$ from $f_{0}$ by $R 6$, which gives $c h^{*}(v)=d(v)-4+\frac{2}{3}+\frac{4}{3}=0$. If $d(v)=3$, then $v$ sends charge to at most one 3 -face by $R 1$ and thus $c h^{*}(v) \geq d(v)-4-\frac{1}{3}+\frac{4}{3}=0$. If $d(v) \geq 4$, then $v$ sends at most $\frac{1}{3}$ to each incident internal face by $R 1$ and $R 5$, yielding $c h^{*}(v) \geq d(v)-4-\frac{1}{3}(d(v)-1)+\frac{4}{3}>0$. Hence, we are done in any case.

It remains to suppose that $v$ is internal. By Lemma 2.1, $d(v) \geq 3$. If $d(v)=3$, then we have $c h^{*}(v)=d(v)-4-\frac{1}{3}+\frac{2}{3} \times 2=0$ by $R 1$ and $R 3$ when $v$ is triangular, and $c h^{*}(v)=$ $d(v)-4+\frac{1}{3} \times 3=0$ by $R 3$ when $v$ not. If $d(v)=4$, then $v$ is incident with $k 3$-faces with $k \leq 2$. By $R 1$ and $R 4$, we have $c h^{*}(v)=d(v)-4-\frac{1}{3} \times 2+\frac{1}{3} \times 2=0$ when $k=2$, $c h^{*}(v)=d(v)-4-\frac{1}{3}+\frac{1}{3}=0$ when $k=1$, and $c h^{*}(v)=d(v)-4=0$ when $k=0$. If $d(v)=5$, then $v$ sends charge to at most two 3 -faces by $R 1$ and to at most one $6^{+}$-face by $R 4$, which gives $c h^{*}(v) \geq d(v)-4-\frac{1}{3} \times 2-\frac{1}{3}=0$. Hence, we may next assume that $d(v) \geq 6$. Since $v$ sends at most $\frac{1}{3}$ to each incident face by our rules, we get $c h^{*}(v) \geq d(v)-4-\frac{1}{3} d(v) \geq 0$.

Lemma 2.13. $c h^{*}\left(f_{0}\right)>0$.
Proof. Recall that $\operatorname{ch}\left(f_{0}\right)=d\left(f_{0}\right)+4$ and $d\left(f_{0}\right) \leq 11$. We have $c h^{*}\left(f_{0}\right) \geq d\left(f_{0}\right)+4-\frac{4}{3} d\left(f_{0}\right)>0$ by $R 6$.

Lemma 2.14. ch ${ }^{*}(f) \geq 0$ for $f \in F \backslash\left\{f_{0}\right\}$.
Proof. We distinguish cases according to the size of $f$. Since $G$ has no 4- and 5-cycle, $d(f) \notin$ $\{4,5\}$.

If $d(f)=3$, then $f$ receives $\frac{1}{3}$ from each incident vertices by $R 1$, which gives $c h^{*}(f)=$ $d(f)-4+\frac{1}{3} \times 3=0$.

Let $d(f)=6$. For any incident vertex $v$, by the rules, $f$ sends to $v$ charge $\frac{2}{3}$ if $v$ is either of degree 2 or bad, and charge at most $\frac{1}{3}$ otherwise. Since $G$ has no ext-triangular 7 -cycles, $f$ is adjacent to at most one 3 -face. Furthermore, by Lemma 2.8, $f$ contains at most one bad vertex. If $f$ contains a 2 -vertex, say $u$, we can deduce with Lemma 2.6 that $u$ is the unique 2-vertex of $f$ and the two neighbors of $u$ on $f$ are external $3^{+}$-vertices which receive nothing
from $f$. It follows that $c h^{*}(f) \geq d(f)-4-\frac{2}{3}-\frac{2}{3}-\frac{1}{3} \times 2=0$. Hence, we may assume that $f$ contains no 2 -vertices. If $f$ has no bad vertices, then $f$ sends each incident vertex at most $\frac{1}{3}$, which gives $c h^{*}(f) \geq d(f)-4-\frac{1}{3} d(f)=0$. Hence, we may let $x$ be a bad vertex of $f$. Denote by $y$ the other common vertex between $f$ and the triangle adjacent to $f$. By Lemma 2.8 again, $y$ is not a bad vertex, i.e., $y$ is either an internal $4^{+}$-vertex or an external $3^{+}$-vertex. By our rules, $f$ sends nothing to $y$, yielding $c h^{*}(f) \geq d(f)-4-\frac{2}{3}-\frac{1}{3} \times 4=0$.

Let $d(f)=7$. Since $G$ has no ext-triangular 7-cycles, $f$ contains no bad vertices. Moreover, by Lemma 2.6 , we deduce that $f$ has at most two 2 -vertices. Thus, $c h^{*}(f) \geq d(f)-4-\frac{2}{3} \times$ $2-\frac{1}{3} \times 5=0$.

Let $d(f) \geq 8$. On the hand, if $f$ contains precisely one external vertex, say $w$, then $d(w) \geq 4$ and so $f$ receives $\frac{1}{3}$ from $w$ by $R 5$. Furthermore, since $f$ contains no weak tetrad by Lemma 2.9, $f$ has a good vertex other than $w$ and sends at most $\frac{1}{3}$ to it. Hence, $c h^{*}(f) \geq$ $d(f)-4+\frac{1}{3}-\frac{1}{3}-\frac{2}{3}(d(f)-2) \geq 0$. On the other hand, if $f$ contains at least two external vertices, then at least two of them are of degree more than 2 . Since $f$ sends nothing to external $3^{+}$-vertices, we have $c h^{*}(f) \geq d(f)-4-\frac{2}{3}(d(f)-2) \geq 0$. By the two hands above, we may assume that all the vertices of $f$ are internal. We distinguish two cases.

Case 1: assume that $d(f)=8$. Denote by $r$ the number of bad vertices of $f$. We have $c h^{*}(f) \geq d(f)-4-\frac{2}{3} r-\frac{1}{3}(d(f)-r)=\frac{4-r}{3} \geq 0$, provided by $r \leq 4$. Since $f$ contains no weak tetrad, $r \leq 6$. Hence, we may assume that $r \in\{5,6\}$. For $r=5$, we claim that $f$ has a vertex failing to take charge from $f$, which gives $c h^{*}(f) \geq d(f)-4-\frac{2}{3} \times 5-\frac{1}{3} \times 2=0$. Suppose to the contrary that no such vertex exists. Thus, the bad vertices of $f$ can be paired so that any good vertex of the path of $f$ between each pair is $f$-Mlight, contradicting the parity of $r$. For $r=6$, since again $f$ contains no five consecutive bad vertices, these six bad vertices of $f$ are divided by the two good ones into cyclically either $3+3$ or $2+4$. We may assume that $f$ has a good vertex that is either $f$-light or of degree 3 , since otherwise we are done with $c h^{*}(f) \geq d(f)-4-\frac{2}{3} \times 6=0$. Denote by $u$ such a good vertex and by $v$ the other one. By the drawing of $u$ and of the 3 -faces adjacent to $f$, we deduce that, for the case $3+3, f$ is an $M$-face, contradicting Lemma 2.10, and for the case $2+4$, if $u$ is $f$-Mlight then either $f$ is an $M M$-face or $v$ is $f$-heavy; otherwise $f$ contains a weak tetrad. It follows with Lemmas 2.11 and 2.9 that $v$ is $f$-heavy, which is the only possible case. Hence, $f$ receives $\frac{1}{3}$ from $v$ by $R 4$, yielding $c h^{*}(f) \geq \operatorname{ch}(f)-4-\frac{2}{3} \times 6+\frac{1}{3}-\frac{1}{3}=0$.

Case 2: assume that $d(f) \geq 9$. By Lemma 2.9, we deduce that $f$ contains at least two good vertices, each of them receives at most $\frac{1}{3}$ from $f$. Thus, $c h^{*}(f) \geq d(f)-4-\frac{2}{3}(d(f)-$ 2) $-\frac{1}{3} \times 2=\frac{d(f)-10}{3} \geq 0$, provided by $d(f) \geq 10$. It remains to suppose $d(f)=9$. If $f$
has at most six bad vertices, then $c h^{*}(f) \geq d(f)-4-\frac{2}{3} \times 6-\frac{1}{3} \times 3=0$. Hence, we may assume that $f$ has precisely seven bad vertices. By the same argument as for the case $d(f)=8$ and $f$ has five bad vertices above, $f$ has a vertex failing to take charge from $f$, which gives $c h^{*}(f) \geq d(f)-4-\frac{2}{3} \times 7-\frac{1}{3}=0$.

By the previous three lemmas, the proof of Theorem 1.9 is completed.

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