# Plane graphs without 4- and 5-cycles and without ext-triangular 7-cycles are 3-colorable

Ligang Jin, Yingli Kang<sup>\*</sup>, Michael Schubert<sup>\*</sup>, Yingqian Wang<sup>‡</sup>

#### Abstract

Listed as No. 53 among the one hundred famous unsolved problems in [J. A. Bondy, U. S. R. Murty, Graph Theory, Springer, Berlin, 2008] is Steinberg's conjecture, which states that every planar graph without 4- and 5-cycles is 3-colorable. In this paper, we show that plane graphs without 4- and 5-cycles are 3-colorable if they have no ext-triangular 7-cycles. This implies that (1) planar graphs without 4-, 5-, 7-cycles are 3-colorable, and (2) planar graphs without 4-, 5-, 8-cycles are 3-colorable, which cover a number of known results in the literature motivated by Steinberg's conjecture.

# 1 Introduction

In the field of 3-colorings of planar graphs, one of the most active topics is about a conjecture proposed by Steinberg in 1976: every planar graph without cycles of length 4 and 5 is 3colorable. There had been no progress on this conjecture for a long time, until Erdös [14] suggested a relaxation of it: does there exist a constant k such that every planar graph without cycles of length from 4 to k is 3-colorable? Abbott and Zhou [1] confirmed that such k exists and  $k \leq 11$ . This result was later on improved to  $k \leq 9$  by Borodin [2] and, independently, by Sanders and Zhao [13], and to  $k \leq 7$  by Borodin, Glebov, Raspaud and Salavatipour [3].

**Theorem 1.1** ([3]). Planar graphs without cycles of length from 4 to 7 are 3-colorable.

<sup>\*</sup>Institute of Mathematics and Paderborn Center for Advanced Studies, Paderborn University, 33102 Paderborn, Germany; ligang@mail.upb.de (Ligang Jin), Yingli@mail.upb.de (Yingli Kang), mischub@upb.de (Michael Schubert)

<sup>&</sup>lt;sup>†</sup>Fellow of the International Graduate School "Dynamic Intelligent Systems"

<sup>&</sup>lt;sup>‡</sup>Department of Mathematics, Zhejiang Normal University, 321004 Jinhua, China; yqwang@zjnu.cn (Yingqian Wang)

We remark that Steinberg's conjecture was recently shown to be false in [6], by constructing a counterexample to the conjecture. The question whether every planar graph without cycles of length from 3 to 5 is 3-colorable is still open.

A more general problem than Steinberg's Conjecture was formulated in [11, 9]:

**Problem 1.2.** What is A, a set of integers between 5 and 9, such that for  $i \in A$ , every planar graph with cycles of length neither 4 nor i is 3-colorable?

Thus, Steinberg's Conjecture states that  $5 \in \mathcal{A}$ . Since so far no element of  $\mathcal{A}$  has been confirmed, it seems reasonable to consider a relaxation of Problem 1.2 where more integers are forbidden to be the length of a cycle in planar graphs. Due to a famous theorem of Grötzsch that planar graphs without triangles are 3-colorable, triangles are always allowed in further sufficient conditions. Several papers together contribute to the result below:

**Theorem 1.3.** For any three integers i, j, k with  $5 \le i < j < k \le 9$ , it holds true that planar graphs having no cycles of length 4, i, j, k are 3-colorable.

Later on, the sufficient conditions, concerning three integers forbidden to be the length of a cycle, were considered. The corresponding problem can be formulated as follows:

**Problem 1.4.** What is  $\mathcal{B}$ , a set of pairs of integers (i, j) with  $5 \le i < j \le 9$ , such that planar graphs without cycles of length 4, i, j are 3-colorable?

It has been proved by Borodin et al. [4] and independently by Xu [17] that every planar graph having neither 5- and 7-cycles nor adjacent 3-cycles is 3-colorable. Hence,  $(5,7) \in \mathcal{B}$ , which improves on Theorem 1.1. More elements of B have been confirmed:  $(6,8) \in \mathcal{B}$  by Wang and Chen [15],  $(7,9) \in \mathcal{B}$  by Lu et al. [11], and  $(6,9) \in \mathcal{B}$  by Jin et al. [9]. The result  $(6,7) \in \mathcal{B}$ is implied in the following theorem, which reconfirms the results  $(5,7) \in \mathcal{B}$  and  $(6,8) \in \mathcal{B}$ .

**Theorem 1.5** ([5]). Planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable.

In this paper, we show that  $(5,8) \in \mathcal{B}$ , which leaves four pairs of integers (5,6), (5,9), (7,8),(8,9) unconfirmed as elements of  $\mathcal{B}$ .

Recently, Mondal gave a proof of the result  $(5,8) \in \mathcal{B}$  in [12]. Here we exhibit two couterexamples to the theorem proved in that paper which yields the result  $(5,8) \in \mathcal{B}$ . We restated this theorem as follows. Let C be a cycle of length at most 12 in a plane graph without 4-, 5- and 8-cycles. C is bad if it is of length 9 or 12 and the subgraph inside C has a partition into 3- and 6-cycles; otherwise, C is good. **Theorem 1.6** (Theorem 2 in [12]). Let G be a graph without 4-, 5-, and 8-cycles. If D is a good cycle of G, then every proper 3-coloring of D can be extended to a proper 3-coloring of the whole graph G.

**Counterexamples to Theorem 1.6.** A plane graph  $G_1$  consisting of a cycle C of length 12, say  $C := [v_1 \ldots v_{12}]$ , and a vertex u inside C connected to all of  $v_1, v_2, v_6$ . The graph  $G_1$  contradicts Theorem 1.6, since any proper 3-coloring of C where  $v_1, v_2, v_6$  receive pairwise distinct colors can not be extended to  $G_1$ . Also, a plane graph  $G_2$  consisting of a cycle C of length 12 and a triangle T inside C, say  $C := [v_1 \ldots v_{12}]$  and  $T := [u_1 u_2 u_3]$ , and three more edges  $u_1 v_1, u_2 v_4, u_3 v_7$ . The graph  $G_2$  contradicts Theorem 1.6, since any proper 3-coloring of C where  $v_1, v_4, v_7$  receive the same color can not be extended to  $G_2$  (see Figure 1).



Figure 1: two graphs as counterexamples to Theorem 1.6.

## **1.1** Notations and formulation of the main theorem

The graphs considered in this paper are finite and simple. A graph is planar if it is embeddable into the Euclidean plane. A plane graph  $(G, \Sigma)$  is a planar graph G together with an embedding  $\Sigma$  of G into the Euclidean plane, that is,  $(G, \Sigma)$  is a particular drawing of G in the Euclidean plane. In what follows, we will always say a plane graph G instead of  $(G, \Sigma)$ , which causes no confusion since no two embeddings of the same graph G will be involved in.

Let G be a plane graph and C be a cycle of G. By Int(C) (or Ext(C)) we denote the subgraph of G induced by the vertices lying inside (or outside) C. The cycle C is separating if neither Int(C) nor Ext(C) is empty. By  $\overline{Int}(C)$  (or  $\overline{Ext}(C)$ ) we denote the subgraph of G consisting of C and its interior (or exterior). The cycle C is triangular if it is adjacent to a triangle, and C is ext-triangular if it is adjacent to a triangle of  $\overline{Ext}(C)$ .

The following theorem is the main result of this paper.

**Theorem 1.7.** Plane graphs with neither 4- and 5-cycles nor ext-triangular 7-cycles are 3-colorable.

As a consequence of Theorem 1.7, the following corollary holds true.

#### **Corollary 1.8.** Planar graphs without cycles of length 4, 5, 8 are 3-colorable, that is, $(5,8) \in \mathcal{B}$ .

We remark that Theorem 1.7 implies the known result that  $(5,7) \in \mathcal{B}$  as well.

Denote by d(v) the degree of a vertex v, by |P| the number of edges of a path P, by |C|the length of a cycle C and by d(f) the size of a face f. A *k*-vertex (or  $k^+$ -vertex, or  $k^-$ -vertex) is a vertex v with d(v) = k (or  $d(v) \ge k$ , or  $d(v) \le k$ ). Similar notations are used for paths, cycles, faces with |P|, |C|, d(f) instead of d(v), respectively.

Let G[S] denote the subgraph of G induced by S with either  $S \subseteq V(G)$  or  $S \subseteq E(G)$ . A chord of C is an edge of  $\overline{Int}(C)$  that connects two nonconsecutive vertices on C. If Int(C) has a vertex v with three neighbors  $v_1, v_2, v_3$  on C, then  $G[\{vv_1, vv_2, vv_3\}]$  is called a *claw* of C. If Int(C) has two adjacent vertices u and v such that u has two neighbors  $u_1, u_2$  on C and v has two neighbors  $v_1, v_2$  on C, then  $G[\{uv, uu_1, uu_2, vv_1, vv_2\}]$  is called a *biclaw* of C. If Int(C) has three pairwise adjacent vertices u, v, w which has a neighbor u', v', w' on C respectively, then  $G[\{uv, vw, uw, uu', vv', ww'\}]$  is called a *triclaw* of C. If G has four vertices x, u, v, w inside Cand four vertices  $x_1, x_2, v_1, w_1$  on C such that  $S = \{uv, vw, wu, ux, xx_1, xx_2, vv_1, ww_1\} \subseteq E(G)$ , then G[S] is called a *combclaw* of C (see Figure 2).



Figure 2: chord, claw, biclaw, triclaw and combclaw of a cycle

A good cycle is an  $11^-$ -cycle that has none of claws, biclaws, triclaws and combclaws. A bad cycle is an  $11^-$ -cycle that is not good.

Instead of Theorem 1.7, it is easier for us to prove the following stronger one:

**Theorem 1.9.** Let G be a connected plane graph with neither 4- and 5-cycles nor ext-triangular 7-cycles. If D, the boundary of the exterior face of G, is a good cycle, then every proper 3-coloring of G[V(D)] can be extended to a proper 3-coloring of G.

The proof of Theorem 1.9 will be proceeded by using discharging method and is given in the next section. For more information on the discharging method, we refer readers to [7]. The rest of this section contributes to other needed notations. Let C be a cycle and T be one of chords, claws, biclaws, triclaws and combclaws of C. We call the graph H consisting of C and T a bad partition of C. The boundary of any one of the parts, into which C is divided by H, is called a *cell* of H. Clearly, every cell is a cycle. In case of confusion, let us always order the cells  $c_1, \dots, c_t$  of H in the way as shown in Figure 2. Let  $k_i$  be the length of  $c_i$ . Then T is further called a  $(k_1, k_2)$ -chord, a  $(k_1, k_2, k_3)$ -claw, a  $(k_1, k_2, k_3, k_4)$ -biclaw, a  $(k_1, k_2, k_3, k_4)$ -triclaw and a  $(k_1, k_2, k_3, k_4, k_5)$ -combclaw, respectively.

A vertex is *external* if it lies on the exterior face; *internal* otherwise. A vertex (or an edge) is triangular if it is incident with a triangle. We say a vertex is *bad* if it is an internal triangular 3-vertex; *good* otherwise. A path is a *splitting path* of a cycle C if it has the two end-vertices on C and all other vertices inside C. A k-cycle with vertices  $v_1, \ldots, v_k$  in cyclic order is denoted by  $[v_1 \ldots v_k]$ .

Let uvw be a path on the boundary of a face f of G with v internal. The vertex v is f-heavy if both uv and vw are triangular and  $d(v) \ge 5$ , and is f-Mlight if both uv and vw are triangular and d(v) = 4, and f-Vlight if neither uv nor vw is triangular and v is triangular and of degree 4. A vertex is f-light if it is either f-Mlight or f-Vlight.

Denote by  $\mathcal{G}$  the class of connected plane graphs with neither 4- and 5-cycles nor exttriangular 7-cycles.

# 2 The proof of Theorem 1.9

Suppose to the contrary that Theorem 1.9 is false. From now on, let G be a counterexample to Theorem 1.9 with fewest vertices. Thus, we may assume that the boundary D of the exterior face of G is a good cycle, and there exists a proper 3-coloring  $\phi$  of G[V(D)] which cannot be extended to a proper 3-coloring of G. By the minimality of G, we deduce that D has no chord.

## **2.1** Structural properties of the minimal counterexample G

**Lemma 2.1.** Every internal vertex of G has degree at least 3.

*Proof.* Suppose to the contrary that G has an internal vertex v with  $d(v) \leq 2$ . We can extend  $\phi$  to G - v by the minimality of G, and then to G by coloring v different from its neighbors.  $\Box$ 

**Lemma 2.2.** G is 2-connected and therefore, the boundary of each face of G is a cycle.

*Proof.* Otherwise, we can assume that G has a pendant block B with cut vertex v such that B - v does not intersect with D. We first extend  $\phi$  to G - (B - v), and then 3-color B so that the color assigned to v is unchanged.

Lemma 2.3. G has no separating good cycle.

*Proof.* Suppose to the contrary that G has a separating good cycle C. We extend  $\phi$  to G - Int(C). Furthermore, since C is a good cycle, the color of C can be extended to its interior.  $\Box$ 

By the definition of a bad cycle, one can easily conclude the lemma as follows.

**Lemma 2.4.** If C is a bad cycle of a graph in  $\mathcal{G}$ , then C has length either 9 or 11. Furthermore, if |C| = 9, then C has a (3,6,6)-claw or a (3,6,6,6)-triclaw; if |C| = 11, then C has a (3,6,8)-claw, or a (3,6,6,6)- or (6,3,6,6)-biclaw, or a (3,6,6,8)-triclaw, or a (3,6,6,6,6)-combclaw.

Notice that all 3- and 6- and 8-cycles of G are facial, thus the following statement is a consequence of the previous lemma together with the fact that  $G \in \mathcal{G}$ .

Lemma 2.5. G has neither bad cycle with a chord nor ext-triangular bad 9-cycle.

**Lemma 2.6.** Let P be a splitting path of D which divides D into two cycles D' and D". The following four statements hold true.

- (1) If |P| = 2, then there is a triangle between D' and D".
- (2)  $|P| \neq 3$ .

(3) If |P| = 4, then there is a 6- or 7-cycle between D' and D".

(4) If |P| = 5, then there is a 9<sup>-</sup>-cycle between D' and D".

*Proof.* Since D has length at most 11, we have  $|D'| + |D''| = |D| + 2|P| \le 11 + 2|P|$ .

(1) Let P = xyz. Suppose to the contrary that  $|D'|, |D''| \ge 6$ . By Lemma 2.1, y has a neighbor other than x and z, say y'. It follows that y' is internal since otherwise D is a bad cycle with a claw. Without loss of generality, let y' lie inside D'. Now D' is a separating cycle. By Lemma 2.3, D' is not good, i.e., either D' is bad or  $|D'| \ge 12$ . Since every bad cycle has length either 9 or 11 by Lemma 2.4, we have  $|D'| \ge 9$ . Recall that  $|D'| + |D''| \le 15$ , thus |D'| = 9 and |D''| = 6. Now D' has either a (3,6,6)-claw or a (3,6,6,6)-triclaw by Lemma 2.4, which implies that D has a biclaw or a combclaw respectively, a contradiction.

(2) Suppose to the contrary that |P| = 3. Let P = wxyz. Clearly  $|D'|, |D''| \ge 6$ . Let x' and y' be a neighbor of x and y not on P, respectively. If both x' and y' are external, then D has a biclaw. Hence, we may assume x' lies inside D'. By Lemmas 2.3 and 2.4 and the inequality  $|D'| + |D''| \le 17$ , we deduce that D' is a bad cycle and D'' is a good 8<sup>-</sup>-cycle. If y' is internal, then y' lies inside D'. It follows with the specific interior of a bad cycle that x' = y'

and D' has either a claw or a biclaw, which implies that D has either a triclaw or a combclaw respectively, a contradiction. Hence, y' is external. Since every bad cycle as well as every 6<sup>-</sup>- or 8-cycle contains no chord by Lemma 2.5, we deduce that yy' is a (3,6)-chord of D''. It follows that D' is a bad and ext-triangular 9-cycle, contradicting Lemma 2.5.

(3) Let P = vwxyz. Suppose to the contrary that  $|D'|, |D''| \ge 8$ . Since  $|D'| + |D''| \le 19$ , we have  $|D'|, |D''| \le 11$ . Since G has no 4- and 5-cycles, if G has an edge e connecting two nonconsecutive vertices on P, then the cycle formed by e and P has to be a triangle, yielding a splitting 3-path of D, contradicting the statement (2). Therefore, no pair of nonconsecutive vertices on P are adjacent.

Let w', x', y' be a neighbor of w, x, y not on P, respectively. The statement (2) implies that x' is internal. Let x' lie inside of D'. Thus D' is a bad 9- or 11-cycle. If D' is a bad 11-cycle, then D'' is a facial 8-cycle, and thus both w' and y' lie in  $\overline{Int}(D')$ , which is impossible by the interior of a bad cycle. Hence, D' is a bad 9-cycle. By the statement (1), if  $w' \in V(D'')$ , then G has the triangle [vww'], which makes D' ext-triangular, a contradiction. Hence,  $w' \notin V(D'')$ . Furthermore, as a bad cycle, D' has no chord by Lemma 2.5, thus w' is internal. If w' lies inside D', then it gives the interior of D' no other choices but w' = x' and D' has a (3, 6, 6)-claw, in which case this claw contains a splitting 3-path of D, a contradiction. Hence, w' lies inside D''. Similarly, we can deduce that y' lies inside D'' as well. Note that  $|D''| \in \{8, 9, 10\}$ , thus D'' is a bad 9-cycle but has to contain both w' and y' inside, which is impossible.

(4) Let P = uvwxyz. Suppose to the contrary that  $|D'|, |D''| \ge 10$ . Since  $|D'| + |D''| \le 21$ , we have  $|D'|, |D''| \le 11$ . By similar argument as in the proof of the statement (3), one can conclude that G has no edge connecting two nonconsecutive vertices on P. Let v', w', x', y' be a neighbor of v, w, x, y not on P, respectively.

The statement (2) implies that both w' and x' are internal. Let w' lie inside D'. It follows that D' is a bad 11-cycle and D'' is a 10-cycle. Thus x' also lies inside D' and furthermore, x' = w' and D' is a bad cycle with either a (3,6,8)-claw or a (3,6,6,6)-biclaw. It follows that  $v', y' \in V(D'')$ . By the statement (1), G has two triangles [uvv'] and [yy'z], at least one of them is adjacent to a 7-cycle of  $\overline{Int}(D')$ , a contradiction.

**Lemma 2.7.** Let G' be a connected plane graph obtained from G by deleting a set of internal vertices and identifying two other vertices so that at most one pair of edges are merged. If we

(a) identify no two vertices of D, and create no edge connecting two vertices of D, and

(b) create no 6<sup>-</sup>-cycle and ext-triangular 7-cycle,

then  $\phi$  can be extended to G'.

*Proof.* The item (a) guarantees that D is unchanged and bounds G', and  $\phi$  is a proper 3coloring of G'[V(D)]. By item (b), the graph G' is simple and  $G' \in \mathcal{G}$ . Hence, to extend  $\phi$  to G' by the minimality of G, it remains to show that D is a good cycle of G'.

Suppose to the contrary that D has a bad partition H in G'. Clearly, H has a 6-cell C' such that the intersection between D and C' is a path  $v_1 \ldots v_k$  of length k - 1 with  $k \in \{4, 5\}$ . Since we create no 6-cycles, C' corresponds to a 6-cycle C of the original graph G. Recall that at most one pair of edges are merged during the process from G to G', we deduce that the intersection between D and C is a path P of one of the forms  $v_1 \ldots v_k, v_1 \ldots v_{k-1}, v_2 \ldots v_k$ . Thus,  $|P| \in \{3, 4, 5\}$ . If  $|P| \in \{4, 5\}$ , then C contains a splitting 3- or 2-path of D in G, yielding a contradiction by Lemma 2.6. Hence, |P| = 3 and so k = 4. By the choice of the 6-cell C', we may assume that the bad partition H has either a (3, 6, 6, 6)- or (3, 6, 6, 8)-triclaw. Now H contains three splitting 3-paths of D, at least one of them does not contain the identified vertex of G' no matter where it is, yielding the existence of a splitting 3-path of D in G, contradicting Lemma 2.6.

**Lemma 2.8.** G has no edge uv incident with a 6-face and a 3-face such that both u and v are internal 3-vertices and therefore, every bad cycle of G has either a (3,6,6)- or (3,6,8)-claw or a (3,6,6,6)-biclaw.

*Proof.* Suppose to the contrary that such an edge uv exists. Denote by [uvwxyz] and [uvt] the 6-face and 3-face, respectively. Lemma 2.6 implies that not both of w and z are external vertices. Without loss of generality, we may assume that w is internal. Let G' be the graph obtained from G by deleting u and v, and identifying w with y so that wx and yx are merged. Clearly, G' is a plane graph on fewer vertices than G. We will show that both the items in Lemma 2.7 are satisfied.

Since w is internal, we identify no two vertices on D. If we create an edge connecting two vertices on D, then w has a neighbor  $w_1$  not adjacent to y and both y and  $w_1$  are external. But now, Lemma 2.6 implies that x is external and thus,  $[ww_1x]$  is a triangle which makes the 7-cycle [utvwxyz] ext-triangular. Hence, the item (a) holds.

Suppose we create a 6<sup>-</sup>-cycle or an ext-triangular 7-cycle C'. Thus G has a 7<sup>-</sup>-path P between w and y corresponding to C'. If  $x \in V(P)$ , then neither wx nor xy are on P since otherwise, C' already exists in G. Hence, the paths wxy and P form two cycles, both of them has length at least 6. It follows that  $|P| \ge 10$ , a contradiction. Hence, we may assume that  $x \notin V(P)$ . The paths P and wxy form a 9<sup>-</sup>-cycle, say C. By Lemma 2.1, we may let  $x_1$  be a neighbor of x other than y and w. We have  $x_1 \notin V(P)$ , since otherwise P has length at least 8. Now C has to contain either u and v or  $x_1$  inside, which implies that C is a bad

9-cycle. By Lemma 2.5, C is not ext-triangular. Thus C' is a 7-cycle that is not ext-triangular, contradicting the supposition. Hence, the item (b) holds.

By Lemma 2.7, the pre-coloring  $\phi$  can be extended to G'. Since z and w receive different colors, we can properly color v and u, extending  $\phi$  further to G.

We follow the notations of M-face and MM-face in [3], and define weak tetrads. An M-face is an 8-face f containing no external vertices with boundary  $[v_1 \dots v_8]$  such that the vertices  $v_1, v_2, v_3, v_5, v_6, v_7$  are of degree 3 and the edges  $v_1v_2, v_3v_4, v_4v_5, v_6v_7$  are triangular. An MM-face is an 8-face f containing no external vertices with boundary  $[v_1 \dots v_8]$  such that  $v_2$  and  $v_7$  are of degree 4 and other six vertices on f are of degree 3, and the edges  $v_1v_2, v_2v_3, v_4v_5, v_6v_7, v_7v_8$ are triangular. A weak tetrad is a path  $v_1 \dots v_5$  on the boundary of a face f such that both the edges  $v_1v_2$  and  $v_3v_4$  are triangular, all of  $v_1, v_2, v_3, v_4$  are internal 3-vertices, and  $v_5$  is either of degree 3 or f-light.

**Lemma 2.9.** G has no weak tetrad and therefore, every face of G contains no five consecutive bad vertices.

Proof. Suppose to the contrary that G has a weak tetrad T following the notation used in the definition. Denote by  $v_0$  the neighbor of  $v_1$  on f with  $v_0 \neq v_2$ . Denote by x the common neighbor of  $v_1$  and  $v_2$ , and y the common neighbor of  $v_3$  and  $v_4$ . If  $x = v_0$ , then  $v_1$  is an internal 2-vertex, contradicting Lemma 2.1. Hence,  $x \neq v_0$  and similarly,  $x \neq v_3$ . Since G has no 4- or 5-cycles,  $x \notin \{v_4, v_5\}$ . Concluding above,  $x \notin v_0 \cup V(T)$ . Similarly,  $y \notin v_0 \cup V(T)$ . Moreover,  $x \neq y$  since otherwise  $[v_1v_2v_3x]$  is a 4-cycle. We delete  $v_1, \ldots, v_4$  and identity  $v_0$  with y, obtaining a plane graph G' on fewer vertices than G. We will show that both the items in Lemma 2.7 are satisfied.

Suppose that we create a 6<sup>-</sup>-cycle or an ext-triangular 7-cycle C'. Thus G has a 7<sup>-</sup>-path P between  $v_0$  and y corresponding to C'. If  $x \in V(P)$ , then the cycle formed by P and  $v_0v_1x$  has length at least 6 and the one formed by P and  $xv_2v_3y$  has length at least 8, which gives  $|P| \ge 9$ , a contradiction. Hence,  $x \notin V(P)$ . The paths P and  $v_0v_1v_2v_3y$  form a 11<sup>-</sup>-cycle, say C. Now C contains either x or  $v_4$  inside. Thus, C is a bad cycle. By Lemma 2.8, C has either a (3,6,6)- or (3,6,8)-claw or a (3,6,6,6)-biclaw. Note that both the two faces incident with  $v_2v_3$  has length at least 8, thus C has a bad partition owning an 8-cell no matter which one of x and  $v_4$  lies inside C. It follows that C has a (3,6,8)-claw. If x lies inside C, then the 6-cell is adjacent to the triangle  $[xv_1v_2]$  with  $d(v_1) = d(x) = 3$ , contradicting Lemma 2.8. Hence,  $v_4$  lies inside C. Note that  $v_4v_5$  is incident with the 6-cell and the 8-cell, we deduce that  $v_5$  is not f-light. By the assumption of T as a weak tetrad, we may assume that  $d(v_5) = 3$ . We delete

 $v_5$  together with other vertices of T and repeat the argument above, yielding a contradiction. Therefore, the item (b) holds.

Suppose we identify two vertices on D or create an edge connecting two vertices on D. Thus there is a splitting 4- or 5-path Q of D containing the path  $v_0v_1v_2v_3y$ . By Lemma 2.6, Q together with D forms a 9<sup>-</sup>-cycle which corresponds to a 5<sup>-</sup>-cycle in G'. Since we create no 6<sup>-</sup>-cycle, a contradiction follows. Hence, the item (a) holds.

By Lemma 2.7, the pre-coloring  $\phi$  can be extended to G'. We first properly color  $v_5$  (if needed),  $v_4, v_3$  in turn. Since  $v_0$  and  $v_3$  receive different colors, we can properly color  $v_1$  and  $v_2$ , extending  $\phi$  further to G.

#### Lemma 2.10. G has no M-face.

*Proof.* Suppose to the contrary that G has an M-face f following the notation used in the definition. For  $(i, j) \in \{(1, 2), (3, 4), (4, 5), (6, 7)\}$ , denote by  $t_{ij}$  the common neighbor of  $v_i$  and  $v_j$ . By similar argument as in the proof of previous lemma, we deduce that the vertices  $t_{12}, t_{34}, t_{45}, t_{67}$  are pairwise distinct and not incident with f. We delete  $v_1, v_2, v_3, v_5, v_6, v_7$  and identity  $v_4$  with  $v_8$ , obtaining a plane graph G' on fewer vertices than G. We will show that both the items in Lemma 2.7 are satisfied.

Suppose that we create a 6<sup>-</sup>-cycle or an ext-triangular 7-cycle C'. Thus G has a 7<sup>-</sup>-path P between  $v_4$  and  $v_8$  corresponding to C'. By the symmetry of an M-face, we may assume that P together with the path  $v_4 \ldots v_8$  forms a 11<sup>-</sup>-cycle C containing  $v_1, v_2, v_3$  inside. It follows with Lemma 2.8 that C is a bad cycle with a (3, 6, 6, 6)-claw. But now  $\overline{Int}(C)$  contains f that is an 8-face, a contradiction. Therefore, the item (b) holds.

The satisfaction of the item (a) can be proved in a similar way as in the proof of previous lemma.

By Lemma 2.7, the pre-coloring  $\phi$  can be extended to G'. Since we first color  $v_3$  different from  $v_8$ , both  $v_1$  and  $v_2$  can be properly colored. Finally, color  $v_5, v_6, v_7$  in the same way, extending  $\phi$  further to G.

#### Lemma 2.11. G has no MM-face.

Proof. Suppose to the contrary that G has an MM-face f following the notation used in the definition. For  $(i, j) \in \{(1, 2), (2, 3), (4, 5), (6, 7), (7, 8)\}$ , denote by  $t_{ij}$  the common neighbor of  $v_i$  and  $v_j$ . Similarly, we deduce that the vertices  $t_{12}, t_{23}, t_{45}, t_{67}, t_{78}$  are pairwise distinct and not incident with f. We delete all the vertices of f and identity  $t_{12}$  with  $t_{67}$ , obtaining a plane graph G' on fewer vertices than G. To extend  $\phi$  to G', it suffices to fulfill the item (a) of Lemma 2.7, as what we did in previous lemma.

Suppose that we create a 6<sup>-</sup>-cycle or an ext-triangular 7-cycle C'. Thus G has a 7<sup>-</sup>-path P between  $t_{12}$  and  $t_{67}$  corresponding to C'. If  $t_{78} \in V(P)$ , then both the cycles formed by P and  $t_{12}v_1v_8t_{78}$  and by P and  $t_{78}v_7t_{67}$  have length at least 8, which gives  $|P| \ge 11$ , a contradiction. Hence,  $t_{78} \notin V(P)$ . The paths P and  $t_{12}v_1v_8v_7t_{67}$  form a 11<sup>-</sup>-cycle, say C. It follows that C is a bad cycle containing either  $t_{78}$  or  $v_2, \ldots, v_6$  inside, that is, either C has a bad partition owning two 8<sup>+</sup>-cell or C contains five vertices inside, a contradiction in any case.

We further extend  $\phi$  from G' to G as follows. Let  $\alpha, \beta$  and  $\gamma$  be the three colors used in  $\phi$ . First regardless the edge  $v_1v_8$ , we can properly color  $v_2, v_1, v_3$  and  $v_7, v_8, v_6$ . If  $v_1$  and  $v_8$  receive different colors and so do  $v_3$  and  $v_6$ , then  $v_4$  and  $v_5$  can be properly colored, we are done. Hence, we may assume without loss of generality that  $v_1$  and  $v_8$  receive the same color, say  $\beta$ . Let  $\alpha$  be the color assigned to  $t_{12}$  and  $t_{67}$ . Thus  $v_2$  and  $v_7$  are colored with  $\gamma$  and  $t_{78}$  is colored with  $\alpha$ . We recolor  $v_8, v_7, v_6$  with  $\gamma, \beta, \gamma$  respectively. Now  $v_1$  and  $v_8$  receive different colors and so do  $v_3$  and  $v_6$ . Again  $v_4$  and  $v_5$  can be properly colored, we are also done.  $\Box$ 

## **2.2** Discharging in G

Let V = V(G), E = E(G), and F be the set of faces of G. Denote by  $f_0$  the exterior face of G. Give initial charge ch(x) to each element x of  $V \cup F$ , where  $ch(f_0) = d(f_0) + 4$ , ch(v) = d(v) - 4for  $v \in V$ , and ch(f) = d(f) - 4 for  $f \in F \setminus \{f_0\}$ . Discharge the elements of  $V \cup F$  according to the following rules:

- R1. Every internal 3-face receives  $\frac{1}{3}$  from each incident vertex.
- R2. Every internal 6<sup>+</sup>-face sends  $\frac{2}{3}$  to each incident 2-vertex.
- R3. Every internal 6<sup>+</sup>-face sends each incident 3-vertex v charge  $\frac{2}{3}$  if v is triangular, and charge  $\frac{1}{3}$  otherwise.
- R4. Every internal 6<sup>+</sup>-face f sends  $\frac{1}{3}$  to each f-light vertex, and receives  $\frac{1}{3}$  from each f-heavy vertex.
- R5. Every internal 6<sup>+</sup>-face receives  $\frac{1}{3}$  from each incident external 4<sup>+</sup>-vertex.
- R6. The exterior face  $f_0$  sends  $\frac{4}{3}$  to each incident vertex.

Let  $ch^*(x)$  denote the final charge of each element x of  $V \cup F$  after discharging. On one hand, by Euler's formula we deduce  $\sum_{x \in V \cup F} ch(x) = 0$ . Since the sum of charges over all elements of  $V \cup F$  is unchanged, it follows that  $\sum_{x \in V \cup F} ch^*(x) = 0$ . On the other hand, we show that  $ch^*(x) \ge 0$  for  $x \in V \cup F \setminus \{f_0\}$  and  $ch^*(f_0) > 0$ . Hence, this obvious contradiction completes the proof of Theorem 1.9. It remains to show that  $ch^*(x) \ge 0$  for  $x \in V \cup F \setminus \{f_0\}$  and  $ch^*(f_0) > 0$ .

We remark that the discharging rules can be tracked back to the one used in [3].

## **Lemma 2.12.** $ch^*(v) \ge 0$ for $v \in V$ .

Proof. First suppose that v is external. Since D is a cycle,  $d(v) \ge 2$ . If d(v) = 2, then since D has no chord, the internal face incident with v is not a triangle and sends  $\frac{2}{3}$  to v by R2. Moreover, v receives  $\frac{4}{3}$  from  $f_0$  by R6, which gives  $ch^*(v) = d(v) - 4 + \frac{2}{3} + \frac{4}{3} = 0$ . If d(v) = 3, then v sends charge to at most one 3-face by R1 and thus  $ch^*(v) \ge d(v) - 4 - \frac{1}{3} + \frac{4}{3} = 0$ . If  $d(v) \ge 4$ , then v sends at most  $\frac{1}{3}$  to each incident internal face by R1 and R5, yielding  $ch^*(v) \ge d(v) - 4 - \frac{1}{3}(d(v) - 1) + \frac{4}{3} > 0$ . Hence, we are done in any case.

It remains to suppose that v is internal. By Lemma 2.1,  $d(v) \ge 3$ . If d(v) = 3, then we have  $ch^*(v) = d(v) - 4 - \frac{1}{3} + \frac{2}{3} \times 2 = 0$  by R1 and R3 when v is triangular, and  $ch^*(v) = d(v) - 4 + \frac{1}{3} \times 3 = 0$  by R3 when v not. If d(v) = 4, then v is incident with k 3-faces with  $k \le 2$ . By R1 and R4, we have  $ch^*(v) = d(v) - 4 - \frac{1}{3} \times 2 + \frac{1}{3} \times 2 = 0$  when k = 2,  $ch^*(v) = d(v) - 4 - \frac{1}{3} + \frac{1}{3} = 0$  when k = 1, and  $ch^*(v) = d(v) - 4 = 0$  when k = 0. If d(v) = 5, then v sends charge to at most two 3-faces by R1 and to at most one  $6^+$ -face by R4, which gives  $ch^*(v) \ge d(v) - 4 - \frac{1}{3} \times 2 - \frac{1}{3} = 0$ . Hence, we may next assume that  $d(v) \ge 6$ . Since v sends at most  $\frac{1}{3}$  to each incident face by our rules, we get  $ch^*(v) \ge d(v) - 4 - \frac{1}{3}d(v) \ge 0$ .

Lemma 2.13.  $ch^*(f_0) > 0$ .

*Proof.* Recall that  $ch(f_0) = d(f_0) + 4$  and  $d(f_0) \le 11$ . We have  $ch^*(f_0) \ge d(f_0) + 4 - \frac{4}{3}d(f_0) > 0$  by R6.

**Lemma 2.14.**  $ch^*(f) \ge 0$  for  $f \in F \setminus \{f_0\}$ .

*Proof.* We distinguish cases according to the size of f. Since G has no 4- and 5-cycle,  $d(f) \notin \{4, 5\}$ .

If d(f) = 3, then f receives  $\frac{1}{3}$  from each incident vertices by R1, which gives  $ch^*(f) = d(f) - 4 + \frac{1}{3} \times 3 = 0$ .

Let d(f) = 6. For any incident vertex v, by the rules, f sends to v charge  $\frac{2}{3}$  if v is either of degree 2 or bad, and charge at most  $\frac{1}{3}$  otherwise. Since G has no ext-triangular 7-cycles, fis adjacent to at most one 3-face. Furthermore, by Lemma 2.8, f contains at most one bad vertex. If f contains a 2-vertex, say u, we can deduce with Lemma 2.6 that u is the unique 2-vertex of f and the two neighbors of u on f are external 3<sup>+</sup>-vertices which receive nothing from f. It follows that  $ch^*(f) \ge d(f) - 4 - \frac{2}{3} - \frac{2}{3} - \frac{1}{3} \times 2 = 0$ . Hence, we may assume that f contains no 2-vertices. If f has no bad vertices, then f sends each incident vertex at most  $\frac{1}{3}$ , which gives  $ch^*(f) \ge d(f) - 4 - \frac{1}{3}d(f) = 0$ . Hence, we may let x be a bad vertex of f. Denote by y the other common vertex between f and the triangle adjacent to f. By Lemma 2.8 again, y is not a bad vertex, i.e., y is either an internal 4<sup>+</sup>-vertex or an external 3<sup>+</sup>-vertex. By our rules, f sends nothing to y, yielding  $ch^*(f) \ge d(f) - 4 - \frac{2}{3} - \frac{1}{3} \times 4 = 0$ .

Let d(f) = 7. Since G has no ext-triangular 7-cycles, f contains no bad vertices. Moreover, by Lemma 2.6, we deduce that f has at most two 2-vertices. Thus,  $ch^*(f) \ge d(f) - 4 - \frac{2}{3} \times 2 - \frac{1}{3} \times 5 = 0$ .

Let  $d(f) \ge 8$ . On the hand, if f contains precisely one external vertex, say w, then  $d(w) \ge 4$  and so f receives  $\frac{1}{3}$  from w by R5. Furthermore, since f contains no weak tetrad by Lemma 2.9, f has a good vertex other than w and sends at most  $\frac{1}{3}$  to it. Hence,  $ch^*(f) \ge d(f) - 4 + \frac{1}{3} - \frac{1}{3} - \frac{2}{3}(d(f) - 2) \ge 0$ . On the other hand, if f contains at least two external vertices, then at least two of them are of degree more than 2. Since f sends nothing to external  $3^+$ -vertices, we have  $ch^*(f) \ge d(f) - 4 - \frac{2}{3}(d(f) - 2) \ge 0$ . By the two hands above, we may assume that all the vertices of f are internal. We distinguish two cases.

Case 1: assume that d(f) = 8. Denote by r the number of bad vertices of f. We have  $ch^*(f) \ge d(f) - 4 - \frac{2}{3}r - \frac{1}{3}(d(f) - r) = \frac{4-r}{3} \ge 0$ , provided by  $r \le 4$ . Since f contains no weak tetrad,  $r \le 6$ . Hence, we may assume that  $r \in \{5, 6\}$ . For r = 5, we claim that f has a vertex failing to take charge from f, which gives  $ch^*(f) \ge d(f) - 4 - \frac{2}{3} \times 5 - \frac{1}{3} \times 2 = 0$ . Suppose to the contrary that no such vertex exists. Thus, the bad vertices of f can be paired so that any good vertex of the path of f between each pair is f-Mlight, contradicting the parity of r. For r = 6, since again f contains no five consecutive bad vertices, these six bad vertices of f are divided by the two good ones into cyclically either 3+3 or 2+4. We may assume that  $ch^*(f) \ge d(f) - 4 - \frac{2}{3} \times 6 = 0$ . Denote by u such a good vertex and by v the other one. By the drawing of u and of the 3-faces adjacent to f, we deduce that, for the case 3+3, f is an M-face or v is f-heavy; otherwise f contains a weak tetrad. It follows with Lemmas 2.11 and 2.9 that v is f-heavy, which is the only possible case. Hence, f receives  $\frac{1}{3}$  from v by R4, yielding  $ch^*(f) \ge ch(f) - 4 - \frac{2}{3} \times 6 + \frac{1}{3} - \frac{1}{3} = 0$ .

Case 2: assume that  $d(f) \ge 9$ . By Lemma 2.9, we deduce that f contains at least two good vertices, each of them receives at most  $\frac{1}{3}$  from f. Thus,  $ch^*(f) \ge d(f) - 4 - \frac{2}{3}(d(f) - 2) - \frac{1}{3} \times 2 = \frac{d(f)-10}{3} \ge 0$ , provided by  $d(f) \ge 10$ . It remains to suppose d(f) = 9. If f

has at most six bad vertices, then  $ch^*(f) \ge d(f) - 4 - \frac{2}{3} \times 6 - \frac{1}{3} \times 3 = 0$ . Hence, we may assume that f has precisely seven bad vertices. By the same argument as for the case d(f) = 8and f has five bad vertices above, f has a vertex failing to take charge from f, which gives  $ch^*(f) \ge d(f) - 4 - \frac{2}{3} \times 7 - \frac{1}{3} = 0$ .

By the previous three lemmas, the proof of Theorem 1.9 is completed.

# **3** Acknowledgement

The first author is supported by Deutsche Forschungsgemeinschaft (DFG) grant STE 792/2-1. The fourth author is supported by National Natural Science Foundation of China (NSFC) 11271335.

# References

- H. L. ABBOTT AND B. ZHOU, On small faces in 4-critical graphs, Ars Combin. 32 (1991) 203-207.
- [2] O. V. BORODIN, Structural properties of plane graphs without adjacent triangles and an application to 3-colorings, J. Graph Theory 21 (1996) 183-186.
- [3] O. V. BORODIN, A. N. GLEBOV, A. RASPAUD AND M. R. SALAVATIPOUR, Planar graphs without cycles of length from 4 to 7 are 3-colorable, J. Combin. Theory Ser. B 93 (2005) 303-311.
- [4] O. V. BORODIN, A. N. GLEBOV, M. MONTASSIER AND A. RASPAUD, Planar graphs without 5- and 7-cycles and without adjacent triangles are 3-colorable, J. Combin. Theory Ser. B 99 (2009) 668-673.
- [5] O. V. BORODIN, A. N. GLEBOV AND A. RASPAUD, Planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable, Discrete Math. **310** (2010) 2584-2594.
- [6] V. COHEN-ADDAD, M. HEBDIGE, D. KRÁL', Z. LI AND E. SALGADO, Steinberg's Conjecture is false, (2016) arXiv: 1604.05108v1.
- [7] D. W. CRANSTON AND D. B. WEST, A guide to the discharging method, (2013) arXiv: 1306.4434.

- [8] H. GRÖTZSCH, Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel, Wiss. Z. Martin Luther Univ. Halle Wittenberg, Math.-Nat. Reihe 8 (1959) 109-120.
- [9] L. JIN, Y. KANG AND Y. WANG, The 3-colorability of planar graphs without cycles of length 4, 6 and 9, Discrete Mathematics 339 (2016) 299-307.
- [10] Y. KANG AND Y. WANG, Distance constraints on short cycles for 3-colorability of planar graphs, Graphs and Combinatorics 31 (2015) 1497-1505.
- [11] H. LU, Y. WANG, W. WANG, Y. BU, M. MONTASSIER AND A. RASPAUD, On the 3-colorability of planar graphs without 4-, 7- and 9-cycles, Discrete Math. 309 (2009) 4596-4607.
- [12] S.A. MONDAL, Planar graphs without 4-, 5- and 8-cycles are 3-colorable, Discuss. Math. Graph Theory **31** (2011) 775-789.
- [13] D. P. SANDERS AND Y. ZHAO, A note on the three color problem, Graphs Combin. 11 (1995) 91-94.
- [14] R. STEINBERG, The state of the three color problem, in: J. Gimbel, J. W. Kennedy & L.
  V. Quintas (eds.), Quo Vadis, Graph Theory? Ann Discrete Math 55 (1993) 211-248.
- [15] W. WANG AND M. CHEN, Planar graphs without 4, 6, 8-cycles are 3-colorable, Science in China Ser. A: Mathematics 50 (2007) 1552-1562.
- [16] Y. WANG, L. JIN AND Y. KANG, Planar graphs without cycles of length from 4 to 6 are (1,0,0)-colorable (in Chinese), Sci. Sin. Math. 43 (2013) 1145-1164.
- [17] B. XU, On 3-colorable plane graphs without 5- and 7-cycles, Discrete Mathematics, Algorithms and applications 1 (2009) 347-353.