

AN ALGORITHMIC REGULARITY LEMMA FOR L_p REGULAR SPARSE MATRICES

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ABSTRACT. We prove an algorithmic regularity lemma for L_p regular matrices ($1 < p \leq \infty$), a class of sparse $\{0, 1\}$ matrices which obey a natural pseudorandomness condition. This extends a result of Coja-Oghlan, Cooper and Frieze who treated the case of L_∞ regular matrices. We also present applications of this result for tensors and MAX-CSP instances.

1. INTRODUCTION

1.1. Overview. It is well known that it is NP-hard not only to compute the optimal solution for the MAX-CSP problem, but also to find “good” approximations of this optimal solution (see, e.g., [10, 11, 17]).

In a seminal paper [9], Frieze and Kannan proved several results concerning *dense* instances of the previous problems. Later on, Coja-Oghlan, Cooper and Frieze [4] showed that such results may be extended to the *sparse* setting if we assume a pseudorandomness condition known as (C, η) -*boundedness* (see [12, 13]). Specifically, in [4] the authors found an algorithm for approximating a sparse $\{0, 1\}$ matrix f by a sum of *cut matrices* under the assumption that f is (C, η) -bounded. The crucial fact is that the number of summands is independent of the size of the matrix and its density. Then, using this result, they proved a similar theorem for tensors which in turn yields approximations for sparse MAX-CSP instances.

The purpose of this paper is to extend these results to a larger class of sparse $\{0, 1\}$ matrices, namely, the L_p regular matrices introduced recently by Borgs, Chayes, Cohn and Zhao [3].

1.1.1. To proceed with our discussion it is useful at this point to introduce some pieces of notation and some terminology. Unless otherwise stated, in the rest of this paper by n_1 and n_2 we denote two positive integers. As usual, for every positive integer n we set $[n] := \{1, \dots, n\}$. The cardinality of a finite set S is denoted by $|S|$.

If X is a nonempty finite set, then by μ_X we denote the uniform probability measure on X , that is, $\mu_X(A) := |A|/|X|$ for every $A \subseteq X$. For notational simplicity, the probability measures $\mu_{[n_1]}$, $\mu_{[n_2]}$ and $\mu_{[n_1] \times [n_2]}$ will be denoted by μ_1, μ_2

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and μ respectively. If \mathcal{P} is a partition of $[n_1] \times [n_2]$, then by $\mathcal{A}_{\mathcal{P}}$ we denote the (finite) σ -algebra on $[n_1] \times [n_2]$ generated by \mathcal{P} .

Next, let X_1, X_2 be nonempty finite sets and set

$$\mathcal{S}_{X_1 \times X_2} := \{A_1 \times A_2 : A_1 \subseteq X_1 \text{ and } A_2 \subseteq X_2\}.$$

If X_1 and X_2 are understood from the context (in particular, if $X_1 = [n_1]$ and $X_2 = [n_2]$), then we shall denote $\mathcal{S}_{X_1 \times X_2}$ simply by \mathcal{S} . Moreover, for every partition \mathcal{P} of $X_1 \times X_2$ with $\mathcal{P} \subseteq \mathcal{S}_{X_1 \times X_2}$ we set

$$\iota(\mathcal{P}) := \min \left\{ \min \{ \mu_{X_1}(P_1), \mu_{X_2}(P_2) \} : P = P_1 \times P_2 \in \mathcal{P} \right\}.$$

Namely, the quantity $\iota(\mathcal{P})$ is the minimal density of each side of each rectangle $P_1 \times P_2$ belonging to the partition \mathcal{P} .

Now recall that a *cut matrix* is a matrix $g : [n_1] \times [n_2] \rightarrow \mathbb{R}$ for which there exist two sets $S \subseteq [n_1]$ and $T \subseteq [n_2]$, and a real number c such that $g = c \cdot \mathbf{1}_{S \times T}$; the set $S \times T$ is called the *support* of the matrix g . Also recall that for every matrix $f : [n_1] \times [n_2] \rightarrow \mathbb{R}$ the *cut norm* of f is the quantity

$$\|f\|_{\square} = \max_{\substack{S \subseteq [n_1] \\ T \subseteq [n_2]}} \left| \sum_{(x_1, x_2) \in S \times T} f(x_1, x_2) \right| = (n_1 \cdot n_2) \cdot \max_{\substack{S \subseteq [n_1] \\ T \subseteq [n_2]}} \left| \int_{S \times T} f d\mu \right|.$$

Finally, let $f : [n_1] \times [n_2] \rightarrow \{0, 1\}$ be a matrix and let \mathcal{P} be a partition of $[n_1] \times [n_2]$ with $\mathcal{P} \subseteq \mathcal{S}$. Recall that the *conditional expectation* of f with respect to $\mathcal{A}_{\mathcal{P}}$ is defined by

$$\mathbb{E}(f | \mathcal{A}_{\mathcal{P}}) = \sum_{P \in \mathcal{P}} \frac{\int_P f d\mu}{\mu(P)} \mathbf{1}_P.$$

Notice, in particular, that $\mathbb{E}(f | \mathcal{A}_{\mathcal{P}})$ is a sum of cut matrices with disjoint supports; this observation will be useful later on. Also note that if $1 \leq p < \infty$, then we have

$$\|\mathbb{E}(f | \mathcal{A}_{\mathcal{P}})\|_{L_p} = \left(\sum_{P \in \mathcal{P}} \left| \frac{\int_P f d\mu}{\mu(P)} \right|^p \mu(P) \right)^{1/p}$$

while if $p = \infty$, then

$$\|\mathbb{E}(f | \mathcal{A}_{\mathcal{P}})\|_{L_{\infty}} = \max \left\{ \left| \frac{\int_P f d\mu}{\mu(P)} \right| : P \in \mathcal{P} \right\}.$$

In particular, observe that $\|f\|_{L_1}$ is equal to the *density* of f , that is, the number of ones in the matrix divided by $n_1 \cdot n_2$. Also notice that $\|f\|_{\square} = \|f\|_{L_p}^p \cdot (n_1 \cdot n_2)$ for every $1 \leq p < \infty$.

1.1.2. We are now in a position to introduce the class of $\{0, 1\}$ matrices which we consider in this paper.

Definition 1.1 (L_p regular matrices [3]). *Let $0 < \eta \leq 1$, $C \geq 1$ and $1 \leq p \leq \infty$. A matrix $f : [n_1] \times [n_2] \rightarrow \{0, 1\}$ is called (C, η, p) -regular (or simply L_p regular*

if C and η are understood) if for every partition \mathcal{P} of $[n_1] \times [n_2]$ with $\mathcal{P} \subseteq \mathcal{S}$ and $\iota(\mathcal{P}) \geq \eta$ we have

$$(1.1) \quad \|\mathbb{E}(f \mid \mathcal{A}_{\mathcal{P}})\|_{L_p} \leq C \|f\|_{L_1}.$$

Notice that, by the monotonicity of the L_p norms, if $1 \leq p_1 \leq p_2 \leq \infty$ and f is L_{p_2} regular, then f is L_{p_1} regular. Thus, L_p regularity is less restrictive when p gets smaller. Also observe that for $p = 1$ the previous definition is essentially of no interest since every $\{0, 1\}$ matrix is L_1 regular. On the other hand, the case $p = \infty$ in Definition 1.1 is equivalent to the aforementioned (C, η) -boundedness condition. Indeed, recall that a matrix $f: [n_1] \times [n_2] \rightarrow \{0, 1\}$ is said to be (C, η) -bounded if for every $S \subseteq [n_1]$ and every $T \subseteq [n_2]$ with $\mu_1(S) \geq \eta$ and $\mu_2(T) \geq \eta$ we have

$$\frac{\int_{S \times T} f d\mu}{\mu(S \times T)} \leq C \|f\|_{L_1}.$$

We have the following simple fact. (See also Lemma 3.1 below.)

Fact 1.2. *Let $0 < \eta \leq 1$ and $C \geq 1$, and let $f: [n_1] \times [n_2] \rightarrow \{0, 1\}$ be a matrix. If f is (C, η) -bounded, then f is (C, η, ∞) -regular. Conversely, if f is (C, η, ∞) -regular, then f is $(4C, \eta)$ -bounded.*

Between the extreme cases “ $p = 1$ ” and “ $p = \infty$ ”, there is a large class of sparse matrices which are very well behaved. The examples which are easiest to grasp are random. Specifically, by [3, Theorem 2.14], for every symmetric measurable function $W: [0, 1] \times [0, 1] \rightarrow \mathbb{R}^+$ with $W \in L_p$ ($1 < p \leq \infty$) and every positive integer n there exists a natural model¹ of sparse random n -by- n $\{0, 1\}$ matrices which are L_p regular asymptotically almost surely. (On the other hand, if $W \notin L_p$, then a typical matrix in this model is not L_p regular.) Further (deterministic) examples, which are relevant from a number theoretic perspective, are given in [7].

1.2. The main result. The following theorem is the main result of this paper.

Theorem 1.3. *There exist absolute constants $a_1, a_2 > 0$, an algorithm and a polynomial² Π_0 such that the following holds. Let $0 < \varepsilon < 1/2$ and $C \geq 1$. Also let $1 < p \leq \infty$, set $p^\dagger = \min\{2, p\}$ and let q denote the conjugate exponent of p^\dagger (that is, $1/p^\dagger + 1/q = 1$). We set*

$$(1.2) \quad \tau = \left\lceil \frac{a_1 \cdot C^2}{(p^\dagger - 1) \varepsilon^2} \right\rceil \quad \text{and} \quad \eta = \left(\frac{a_2 \cdot \varepsilon}{C} \right)^{\sum_{i=1}^{\tau+1} (\frac{2}{p^\dagger} + 1)^{i-1} q^i}.$$

If we input

INP: *a (C, η, p) -regular matrix $f: [n_1] \times [n_2] \rightarrow \{0, 1\}$,*

¹This model encompasses the classical Erdős–Rényi model—see, e.g., [2].

²Here, and in the rest of this paper, by the term *polynomial* we mean a real polynomial Π with non-negative coefficients, that is, $\Pi(x) = a_d x^d + \dots + a_1 x + a_0$ where $d \in \mathbb{N}$ and $a_0, \dots, a_d \in \mathbb{R}^+$. Moreover, unless otherwise stated, we will assume that the degree d and the coefficients a_0, \dots, a_d are absolute and independent of the rest of the parameters.

then the algorithm outputs

OUT: a partition \mathcal{P} of $[n_1] \times [n_2]$ with $\mathcal{P} \subseteq \mathcal{S}$, $|\mathcal{P}| \leq 4^\tau$ and $\iota(\mathcal{P}) \geq \eta$, such that

$$(1.3) \quad \|f - \mathbb{E}(f \mid \mathcal{A}_{\mathcal{P}})\|_{\square} \leq \varepsilon \|f\|_{\square}.$$

This algorithm has running time $(\tau 4^\tau) \cdot \Pi_0(n_1 \cdot n_2)$.

Theorem 1.3 extends [4, Theorem 1] which corresponds to the case $p = \infty$ ³. Note that, by (1.2) and (1.3), the matrix f is well approximated by a sum of at most 4^τ cut matrices with disjoint supports and, moreover, the positive integer τ is independent of the size of f and its density. Also observe that, as expected, the running time of the algorithm in Theorem 1.3 increases as p decreases to 1.

1.3. Organization of the paper. The paper is organized as follows. In Section 2 we recall some results which are needed for the proof of Theorem 1.3, and in Section 3 we present some preparatory lemmas. The proof of Theorem 1.3 is completed in Section 4. Finally, in Section 5 we present applications for tensors and sparse MAX-CSP instances.

2. BACKGROUND MATERIAL

2.1. Martingale difference sequences. Recall that a finite sequence $(d_i)_{i=0}^n$ of integrable real-valued random variables on a probability space (X, Σ, μ) is said to be a *martingale difference sequence* if there exists a martingale $(f_i)_{i=0}^n$ such that $d_0 = f_0$ and $d_i = f_i - f_{i-1}$ if $i \geq 1$ and $i \in [n]$. We will need the following result due to Ricard and Xu [15] which can be seen as an extension of the basic fact that martingale difference sequences are orthogonal in L_2 . (See also [5, Appendix A] for a discussion on this result and its proof.)

Proposition 2.1. *Let (X, Σ, μ) be a probability space and $1 < p \leq 2$. Then for every martingale difference sequence $(d_i)_{i=0}^n$ in $L_p(X, \Sigma, \mu)$ we have*

$$(2.1) \quad \left(\sum_{i=0}^n \|d_i\|_{L_p}^2 \right)^{1/2} \leq \left(\frac{1}{p-1} \right)^{1/2} \left\| \sum_{i=0}^n d_i \right\|_{L_p}.$$

We point out that the constant $(p-1)^{-1/2}$ appearing in the right-hand side of (2.1) is best possible.

2.2. The algorithmic version of Grothendieck's inequality. We will need the following result due to Alon and Naor [1].

Proposition 2.2. *There exist a constant $a_0 > 0$, an algorithm and a polynomial Π_{AN} such that the following holds. If we input*

INP: a matrix $f: [n_1] \times [n_2] \rightarrow \mathbb{R}$,

³Actually, the argument in [4] works for the more general case $p \geq 2$. We also remark that the cut matrices obtained by [4, Theorem 1] do not necessarily have disjoint supports, but this can be easily arranged—see [4, Corollary 1] for more details.

then the algorithm outputs

OUT: a set $A \in \mathcal{S}$ such that $(n_1 \cdot n_2) \left| \int_A f d\mu \right| \geq a_0 \|f\|_{\square}$.

This algorithm has running time $\Pi_{\text{AN}}(n_1 \cdot n_2)$.

The constant a_0 in Proposition 2.2 is closely related to Grothendieck's constant K_G (see, e.g., [14]).

3. PREPARATORY LEMMAS

In this section we prove some preparatory results concerning L_p regular matrices. We begin with the following lemma.

Lemma 3.1. *There exist an algorithm and a polynomial Π_1 such that the following holds. Let X_1, X_2 be nonempty finite sets, and let $0 < \vartheta < 1/2$. If we input*

INP: two sets $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$ with $\mu_{X_1}(A_1) \geq \vartheta$ and $\mu_{X_2}(A_2) \geq \vartheta$,

then the algorithm outputs

OUT1: a partition $\mathcal{Q} \subseteq \mathcal{S}$ with $|\mathcal{Q}| \leq 4$ and $\iota(\mathcal{Q}) \geq \vartheta$, and

OUT2: a set $B \in \mathcal{Q}$ such that $A_1 \times A_2 \subseteq B$ and $\mu_{X_1 \times X_2}(B \setminus (A_1 \times A_2)) \leq 2\vartheta$.

This algorithm has running time $\Pi_1(|X_1| \cdot |X_2|)$.

Proof. We distinguish the following four (mutually exclusive) cases.

CASE 1: We have $\mu_{X_1}(A_1) < 1 - \vartheta$ and $\mu_{X_2}(A_2) < 1 - \vartheta$. In this case the algorithm outputs $\mathcal{Q} = \{A_1 \times A_2, (X_1 \setminus A_1) \times A_2, A_1 \times (X_2 \setminus A_2), (X_1 \setminus A_1) \times (X_2 \setminus A_2)\}$ and $B = A_1 \times A_2$. Notice that \mathcal{Q} and B satisfy the requirements of the lemma.

CASE 2: We have $\mu_{X_1}(A_1) < 1 - \vartheta$ and $\mu_{X_2}(A_2) \geq 1 - \vartheta$. In this case the algorithm outputs $\mathcal{Q} = \{A_1 \times X_2, (X_1 \setminus A_1) \times X_2\}$ and $B = A_1 \times X_2$. Again, it is easy to see that \mathcal{Q} and B satisfy the requirements of the lemma.

CASE 3: We have $\mu_{X_1}(A_1) \geq 1 - \vartheta$ and $\mu_{X_2}(A_2) < 1 - \vartheta$. This case is similar to Case 2. In particular, we set $\mathcal{Q} = \{X_1 \times A_2, X_1 \times (X_2 \setminus A_2)\}$ and $B = X_1 \times A_2$.

CASE 4: We have $\mu_{X_1}(A_1) \geq 1 - \vartheta$ and $\mu_{X_2}(A_2) \geq 1 - \vartheta$. In this case the algorithm outputs $\mathcal{Q} = \{X_1 \times X_2\}$ and $B = X_1 \times X_2$. As before, it is easy to see that \mathcal{Q} and B are as desired.

Finally, notice that the most costly part of this algorithm is to estimate the quantities $\mu_{X_1}(A_1)$ and $\mu_{X_2}(A_2)$, but of course this can be done in polynomial time of $|X_1| \cdot |X_2|$. Thus, this algorithm will stop in polynomial time of $|X_1| \cdot |X_2|$. \square

The next result is a Hölder-type inequality for L_p regular matrices. To motivate this inequality, let $f: [n_1] \times [n_2] \rightarrow \{0, 1\}$ be a matrix, let $1 < p < \infty$, let q denote its conjugate exponent and observe that, by Hölder's inequality, for every $A \subseteq [n_1] \times [n_2]$ we have

$$(3.1) \quad \int_A f d\mu \leq \|f\|_{L_1}^{1/p} \cdot \mu(A)^{1/q}.$$

Unfortunately, this estimate is not particularly useful if f is sparse—that is, in the regime $\|f\|_{L_1} = o(1)$ —since in this case the quantity $\|f\|_{L_1}^{1/p}$ is *not* comparable to the density $\|f\|_{L_1}$ of f . Nevertheless, we can improve upon (3.1) provided that the matrix f is L_p regular and $A \in \mathcal{S}$. Specifically, we have the following lemma (see also [6, Proposition 4.1]).

Lemma 3.2. *Let $0 < \eta < 1/2$ and $C \geq 1$. Also let $1 < p \leq 2$ and let q denote its conjugate exponent. Finally, let $f: [n_1] \times [n_2] \rightarrow \{0, 1\}$ be (C, η, p) -regular. Then for every $A \subseteq [n_1] \times [n_2]$ with $A \in \mathcal{S}$ we have*

$$(3.2) \quad \int_A f d\mu \leq C \|f\|_{L_1} (\mu(A) + 6\eta)^{1/q}.$$

Proof. Fix a nonempty subset A of $[n_1] \times [n_2]$ with $A \in \mathcal{S}$, and let $A_1 \subseteq [n_1]$ and $A_2 \subseteq [n_2]$ such that $A = A_1 \times A_2$. If $\mu_1(A_1) \geq \eta$ and $\mu_2(A_2) \geq \eta$, then we claim that

$$(3.3) \quad \int_A f d\mu \leq C \|f\|_{L_1} (\mu(A) + 2\eta)^{1/q}.$$

Indeed, by Lemma 3.1 applied for $X_1 = [n_1]$ and $X_2 = [n_2]$, we obtain a partition \mathcal{Q} of $[n_1] \times [n_2]$ with $\mathcal{Q} \in \mathcal{S}$ and $\iota(\mathcal{Q}) \geq \eta$, and a set $B \in \mathcal{Q}$ such that $A \subseteq B$ and $\mu(B \setminus A) \leq 2\eta$. By the L_p regularity of f , we have

$$\frac{\int_B f d\mu}{\mu(B)} \mu(B)^{1/p} \leq \|\mathbb{E}(f | \mathcal{A}_{\mathcal{Q}})\|_{L_p} \leq C \|f\|_{L_1}$$

and so

$$\int_A f d\mu \leq \int_B f d\mu \leq C \|f\|_{L_1} \mu(B)^{1/q} \leq C \|f\|_{L_1} (\mu(A) + 2\eta)^{1/q}.$$

Next, we assume that $\mu_1(A_1) \geq \eta$ and $\mu_2(A_2) < \eta$ and observe that we may select a set $B \subseteq [n_2]$ with $\eta < \mu_2(B) \leq 2\eta$. Then, we have

$$\begin{aligned} \int_A f d\mu &\leq \int_{A_1 \times (A_2 \cup B)} f d\mu \stackrel{(3.3)}{\leq} C \|f\|_{L_1} (\mu(A_1 \times (A_2 \cup B)) + 2\eta)^{1/q} \\ &\leq C \|f\|_{L_1} (\mu(A) + 2\eta \mu_1(A_1) + 2\eta)^{1/q} \leq C \|f\|_{L_1} (\mu(A) + 4\eta)^{1/q}. \end{aligned}$$

The case $\mu_1(A_1) < \eta$ and $\mu_2(A_2) \geq \eta$ is identical.

Finally, assume that $\mu_1(A_1) < \eta$ and $\mu_2(A_2) < \eta$, and observe that there exist $B_1 \subseteq [n_1]$ and $B_2 \subseteq [n_2]$ such that $\eta < \mu_1(B_1) \leq 2\eta$ and $\eta < \mu_2(B_2) \leq 2\eta$. Then,

$$\begin{aligned} \int_A f d\mu &\leq \int_{(A_1 \cup B_1) \times (A_2 \cup B_2)} f d\mu \\ &\stackrel{(3.3)}{\leq} C \|f\|_{L_1} (\mu((A_1 \cup B_1) \times (A_2 \cup B_2)) + 2\eta)^{1/q} \\ &\leq C \|f\|_{L_1} (\mu(A) + 8\eta^2 + 2\eta)^{1/q} \leq C \|f\|_{L_1} (\mu(A) + 6\eta)^{1/q} \end{aligned}$$

and the proof of the lemma is completed. \square

Lemmas 3.1 and 3.2 will be used in the proof of the following result.

Lemma 3.3. *There exist an algorithm and a polynomial Π_2 such that the following holds. Let $0 < \varepsilon < 1/2$ and $C \geq 1$. Let $1 < p \leq \infty$, set $p^\dagger = \min\{2, p\}$ and let q denote the conjugate exponent of p^\dagger . Also let a_0 be as in Proposition 2.2, and set*

$$\vartheta = \frac{a_0 \varepsilon}{16C} \quad \text{and} \quad \eta \leq \left(\vartheta \cdot \iota(\mathcal{P})^{\frac{2}{p^\dagger}+1} \right)^q.$$

If we input

INP1: a partition \mathcal{P} of $[n_1] \times [n_2]$ with $\mathcal{P} \subseteq \mathcal{S}$,

INP2: a subset A of $[n_1] \times [n_2]$ with $A \in \mathcal{S}$, and

INP3: a (C, η, p) -regular matrix $f: [n_1] \times [n_2] \rightarrow \{0, 1\}$,

then the algorithm outputs

OUT1: a refinement \mathcal{Q} of \mathcal{P} with $\mathcal{Q} \subseteq \mathcal{S}$, $|\mathcal{Q}| \leq 4|\mathcal{P}|$ and $\iota(\mathcal{Q}) \geq (\vartheta \cdot \iota(\mathcal{P})^{\frac{2}{p^\dagger}+1})^q$, and

OUT2: a set $B \in \mathcal{A}_{\mathcal{Q}}$ such that

$$(3.4) \quad \int_{A \Delta B} \mathbb{E}(f | \mathcal{A}_{\mathcal{P}}) d\mu \leq 2C \|f\|_{L_1} \vartheta \quad \text{and} \quad \int_{A \Delta B} f d\mu \leq 6C \|f\|_{L_1} \vartheta.$$

If we additionally assume that the matrix f in INP3 satisfies

$$(3.5) \quad \left| \int_A (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}})) d\mu \right| \geq a_0 \varepsilon \|f\|_{L_1},$$

then the partition \mathcal{Q} in OUT2 satisfies

$$(3.6) \quad \|\mathbb{E}(f | \mathcal{A}_{\mathcal{Q}}) - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}})\|_{L_{p^\dagger}} \geq \frac{a_0 \varepsilon \|f\|_{L_1}}{2}.$$

Finally, this algorithm has running time $|\mathcal{P}| \cdot \Pi_2(n_1 \cdot n_2)$.

Lemma 3.3 is an algorithmic version of [6, Lemmas 5.1 and 5.2]. We notice that if the matrix f satisfies the estimate in (3.5), then inequality (3.6) implies that the partition \mathcal{Q} is a genuine refinement of \mathcal{P} . We also point out that the polynomial Π_2 obtained by Lemma 3.3 is absolute and independent of the parameters ε, C and p . We proceed to the proof.

Proof of Lemma 3.3. We may (and we will) assume that A is nonempty. We select $A_1 \subseteq [n_1]$ and $A_2 \subseteq [n_2]$ such that $A = A_1 \times A_2$, and we set

$$\theta = \vartheta^q \cdot \iota(\mathcal{P})^{\frac{2q}{p^\dagger}}.$$

Also let

$$\mathcal{P}^1 = \{P = P_1 \times P_2 \in \mathcal{P} : \mu_1(A_1 \cap P_1) < \theta \mu_1(P_1) \text{ and } \mu_2(A_2 \cap P_2) < \theta \mu_2(P_2)\},$$

$$\mathcal{P}^2 = \{P = P_1 \times P_2 \in \mathcal{P} : \mu_1(A_1 \cap P_1) < \theta \mu_1(P_1) \text{ and } \mu_2(A_2 \cap P_2) \geq \theta \mu_2(P_2)\},$$

$$\mathcal{P}^3 = \{P = P_1 \times P_2 \in \mathcal{P} : \mu_1(A_1 \cap P_1) \geq \theta \mu_1(P_1) \text{ and } \mu_2(A_2 \cap P_2) < \theta \mu_2(P_2)\},$$

$$\mathcal{P}^4 = \{P = P_1 \times P_2 \in \mathcal{P} : \mu_1(A_1 \cap P_1) \geq \theta \mu_1(P_1) \text{ and } \mu_2(A_2 \cap P_2) \geq \theta \mu_2(P_2)\}.$$

Clearly, the family $\{\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3, \mathcal{P}^4\}$ is a partition of \mathcal{P} .

Now for every $P \in \mathcal{P}$ we perform the following subroutine. First, assume that $P \in \mathcal{P}^1 \cup \mathcal{P}^2 \cup \mathcal{P}^3$ and notice that in this case we have $\mu(A \cap P) \leq \theta \mu(P)$. Then we set $B_P = \emptyset$ and $\mathcal{Q}_P = \{P\}$. On the other hand, if $P = P_1 \times P_2 \in \mathcal{P}^4$, then

we apply Lemma 3.1 for $X_1 = P_1$ and $X_2 = P_2$, and we obtain⁴ a partition \mathcal{Q}_P of P with $|\mathcal{Q}_P| \leq 4$ and $\iota(\mathcal{Q}_P) \geq \theta \cdot \iota(\mathcal{P})$, and a set $B_P \in \mathcal{Q}_P$ such that $A \cap P \subseteq B_P$ and $\mu(B_P \setminus (A \cap P)) \leq 2\theta\mu(P)$.

Once this is done, the algorithm outputs

$$\mathcal{Q} = \bigcup_{P \in \mathcal{P}} \mathcal{Q}_P \quad \text{and} \quad B = \bigcup_{P \in \mathcal{P}} B_P.$$

Notice that there exists a polynomial Π_2 such that this algorithm has running time $|\mathcal{P}| \cdot \Pi_2(n_1 \cdot n_2)$. Indeed, recall that the algorithm in Lemma 3.1 runs in polynomial time and observe that we have applied Lemma 3.1 at most $|\mathcal{P}|$ times.

We proceed to show that the partition \mathcal{Q} and the set B satisfy the requirements of the lemma. To this end, we first observe that \mathcal{Q} satisfies the requirements in OUT1. Moreover, we have $B \in \mathcal{A}_{\mathcal{Q}}$ and

$$(3.7) \quad A \triangle B = \left(\bigcup_{i=1}^3 \bigcup_{P \in \mathcal{P}^i} (A \cap P) \right) \cup \left(\bigcup_{P \in \mathcal{P}^4} (B_P \setminus (A \cap P)) \right).$$

Therefore,

$$(3.8) \quad \mu(A \triangle B) \leq 2\theta$$

and so, by the L_p regularity of f , Hölder's inequality, the monotonicity of the L_p norms and the fact that $p^\dagger \leq p$, we obtain that

$$\begin{aligned} \int_{A \triangle B} \mathbb{E}(f | \mathcal{A}_{\mathcal{P}}) d\mu &\leq \|\mathbb{E}(f | \mathcal{A}_{\mathcal{P}})\|_{L_{p^\dagger}} \cdot \mu(A \triangle B)^{1/q} \leq \|\mathbb{E}(f | \mathcal{A}_{\mathcal{P}})\|_{L_p} \cdot \mu(A \triangle B)^{1/q} \\ &\leq C \|f\|_{L_1} (2\theta)^{1/q} \leq 2C \|f\|_{L_1} \vartheta \end{aligned}$$

which proves the first inequality in (3.4). For the second inequality, by (3.7), we have

$$(3.9) \quad \int_{A \triangle B} f d\mu = \sum_{P \in \mathcal{P}^1 \cup \mathcal{P}^2 \cup \mathcal{P}^3} \int_{A \cap P} f d\mu + \sum_{P \in \mathcal{P}^4} \int_{B_P \setminus (A \cap P)} f d\mu$$

and, by the definition of θ and the fact that $\eta \leq (\vartheta \cdot \iota(\mathcal{P})^{\frac{2}{p^\dagger} + 1})^q$, we have $\eta \leq \theta\mu(P)$ for every $P \in \mathcal{P}$. Thus, if $P \in \mathcal{P}^1 \cup \mathcal{P}^2 \cup \mathcal{P}^3$, then, by Lemma 3.2 and our assumption that f is (C, η, p) -regular (and, consequently, (C, η, p^\dagger) -regular), we have

$$\int_{A \cap P} f d\mu \leq C \|f\|_{L_1} (\mu(A \cap P) + 6\eta)^{1/q} \leq 3C \|f\|_{L_1} (\theta\mu(P))^{1/q}$$

which yields that

$$(3.10) \quad \sum_{P \in \mathcal{P}^1 \cup \mathcal{P}^2 \cup \mathcal{P}^3} \int_{A \cap P} f d\mu \leq 3C \|f\|_{L_1} \theta^{1/q} \sum_{P \in \mathcal{P}^1 \cup \mathcal{P}^2 \cup \mathcal{P}^3} \mu(P)^{1/q}.$$

On the other hand, by the choice of the family $\{B_P : P \in \mathcal{P}^4\}$ and Lemma 3.2,

$$(3.11) \quad \sum_{P \in \mathcal{P}^4} \int_{B_P \setminus (A \cap P)} f d\mu \leq 6C \|f\|_{L_1} \theta^{1/q} \sum_{P \in \mathcal{P}^4} \mu(P)^{1/q}.$$

⁴Notice that for every $A \subseteq X_1$ we have $\mu_{X_1}(A) = \mu_1(A)/\mu_1(X_1)$, and similarly for X_2 .

Moreover, since $q \geq 2$ we have that $x^{1/q}$ is concave on \mathbb{R}^+ , and so

$$(3.12) \quad \sum_{P \in \mathcal{P}} \mu(P)^{1/q} \leq |\mathcal{P}|^{\frac{1}{p^\dagger}} \leq \iota(\mathcal{P})^{-\frac{2}{p^\dagger}}.$$

Combining (3.10)–(3.12), we see that the second inequality in (3.4) is satisfied.

Finally, assume that the matrix f satisfies (3.5). By (3.4) and the choice of ϑ ,

$$\begin{aligned} & \left| \int_A (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}})) d\mu - \int_B (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}})) d\mu \right| \\ & \leq \int_{A \Delta B} \mathbb{E}(f | \mathcal{A}_{\mathcal{P}}) d\mu + \int_{A \Delta B} f d\mu \leq \frac{a_0 \varepsilon \|f\|_{L_1}}{2} \end{aligned}$$

and so, by (3.5), we have

$$(3.13) \quad \left| \int_B (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}})) d\mu \right| \geq \frac{a_0 \varepsilon \|f\|_{L_1}}{2}.$$

Moreover, the fact that $B \in \mathcal{A}_{\mathcal{Q}}$ yields that

$$(3.14) \quad \int_B (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}})) d\mu = \int_B (\mathbb{E}(f | \mathcal{A}_{\mathcal{Q}}) - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}})) d\mu.$$

Thus, by the monotonicity of the L_p norms, we conclude that

$$\begin{aligned} & \|\mathbb{E}(f | \mathcal{A}_{\mathcal{Q}}) - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}})\|_{L_{p^\dagger}} \geq \|\mathbb{E}(f | \mathcal{A}_{\mathcal{Q}}) - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}})\|_{L_1} \\ & \geq \left| \int_B (\mathbb{E}(f | \mathcal{A}_{\mathcal{Q}}) - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}})) d\mu \right| \stackrel{(3.14)}{=} \left| \int_B (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}})) d\mu \right| \stackrel{(3.13)}{\geq} \frac{a_0 \varepsilon \|f\|_{L_1}}{2} \end{aligned}$$

and the proof of Lemma 3.3 is completed. \square

4. PROOF OF THEOREM 1.3

We will describe a recursive algorithm that performs the following steps. Starting from the trivial partition of $[n_1] \times [n_2]$ and using Lemma 3.3 as a subroutine, the algorithm will produce an increasing family of partitions of $[n_1] \times [n_2]$. Simultaneously, using Proposition 2.2 as a subroutine, the algorithm will be checking if the partition that is produced at each step satisfies the requirements in OUT of Theorem 1.3. The fact that this algorithm will eventually terminate is based on Proposition 2.1.

Proof of Theorem 1.3. Let a_0 be as in Proposition 2.2, and set

$$(4.1) \quad \vartheta = \frac{a_0 \varepsilon}{16C}, \quad \tau = \left\lceil \frac{4C^2}{(p^\dagger - 1) \varepsilon^2 a_0^2} \right\rceil \quad \text{and} \quad \eta = \vartheta^{\sum_{i=1}^{\tau+1} (\frac{2}{p^\dagger} + 1)^{i-1} q^i}.$$

Also fix a (C, η, p) -regular matrix $f: [n_1] \times [n_2] \rightarrow \{0, 1\}$. The algorithm performs the following steps.

InitialStep: We set $\mathcal{P}_0 := \{[n_1] \times [n_2]\}$ and we apply the algorithm in Proposition 2.2 for the matrix $f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_0})$. Thus, we obtain a set $A_0 \subseteq [n_1] \times [n_2]$ with $A_0 \in \mathcal{S}$ and such that $(n_1 \cdot n_2) \left| \int_{A_0} (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_0})) d\mu \right| \geq a_0 \|f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_0})\|_{\square}$. If $\left| \int_{A_0} (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_0})) d\mu \right| \leq a_0 \varepsilon \|f\|_{L_1}$, then the algorithm outputs the partition

\mathcal{P}_0 and **Halts**. Otherwise, the algorithm sets $m = 1$ and enters into the following loop.

GeneralStep: The algorithm will have as an input a positive integer $m \in [\tau - 1]$, a partition⁵ $\mathcal{P}_{m-1} \subseteq \mathcal{S}$ and a set $A_{m-1} \subseteq [n_1] \times [n_2]$ with $A_{m-1} \in \mathcal{S}$, such that

- (a) $|\mathcal{P}_{m-1}| \leq 4^m$,
- (b) $(\vartheta \cdot \iota(\mathcal{P}_{m-1})^{\frac{2}{p^\dagger}+1})^q \geq \vartheta^{\sum_{i=1}^m (\frac{2}{p^\dagger}+1)^{i-1} q^i}$, and
- (c) $|\int_{A_{m-1}} (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_{m-1}})) d\mu| > a_0 \varepsilon \|f\|_{L_1}$.

By (b) and the choice of η in (4.1), we have $\eta \leq (\vartheta \cdot \iota(\mathcal{P}_{m-1})^{\frac{2}{p^\dagger}+1})^q$. This fact together with the choice of ϑ in (4.1) allows us to perform the algorithm in Lemma 3.3 for the matrix f , the partition \mathcal{P}_{m-1} and the set A_{m-1} . Thus, we obtain a refinement \mathcal{P}_m of \mathcal{P}_{m-1} with $\mathcal{P}_m \subseteq \mathcal{S}$, $|\mathcal{P}_m| \leq 4|\mathcal{P}_{m-1}|$, $\iota(\mathcal{P}_m) \geq (\vartheta \cdot \iota(\mathcal{P}_{m-1})^{\frac{2}{p^\dagger}+1})^q$, such that

$$\|\mathbb{E}(f | \mathcal{A}_{\mathcal{P}_m}) - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_{m-1}})\|_{L_{p^\dagger}} \geq \frac{a_0 \varepsilon \|f\|_{L_1}}{2}.$$

Next, we apply the algorithm in Proposition 2.2 for the matrix $f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_m})$, and we obtain a set $A_m \subseteq [n_1] \times [n_2]$ with $A_m \in \mathcal{S}$ and such that

$$(n_1 \cdot n_2) \left| \int_{A_m} (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_m})) d\mu \right| \geq a_0 \|f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_m})\|_{\square}.$$

If $|\int_{A_m} (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_m})) d\mu| \leq a_0 \varepsilon \|f\|_{L_1}$, then the algorithm outputs the partition \mathcal{P}_m and **Halts**. Otherwise, if $m < \tau - 1$, then the algorithm reruns the loop we described above for the positive integer $m + 1$, the partition \mathcal{P}_m and the set A_m , while if $m = \tau - 1$, then the algorithm proceeds to the following step.

FinalStep: The algorithm will have as an input a partition $\mathcal{P}_{\tau-1} \subseteq \mathcal{S}$ and a set $A_{\tau-1} \subseteq [n_1] \times [n_2]$ with $A_{\tau-1} \in \mathcal{S}$, such that

- (d) $|\mathcal{P}_{\tau-1}| \leq 4^{\tau-1}$,
- (e) $(\vartheta \cdot \iota(\mathcal{P}_{\tau-1})^{\frac{2}{p^\dagger}+1})^q \geq \vartheta^{\sum_{i=1}^{\tau} (\frac{2}{p^\dagger}+1)^{i-1} q^i}$, and
- (f) $|\int_{A_{\tau-1}} (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_{\tau-1}})) d\mu| > a_0 \varepsilon \|f\|_{L_1}$.

Again observe that, by (e) and the choice of η in (4.1), we have $\eta \leq (\vartheta \cdot \iota(\mathcal{P}_{\tau-1})^{\frac{2}{p^\dagger}+1})^q$. Using this fact and the choice of ϑ in (4.1), we may apply the algorithm in Lemma 3.3 for the matrix f , the partition $\mathcal{P}_{\tau-1}$ and the set $A_{\tau-1}$. Therefore, we obtain a refinement \mathcal{P}_τ of $\mathcal{P}_{\tau-1}$ with $\mathcal{P}_\tau \subseteq \mathcal{S}$, $|\mathcal{P}_\tau| \leq 4|\mathcal{P}_{\tau-1}|$, $\iota(\mathcal{P}_\tau) \geq (\vartheta \cdot \iota(\mathcal{P}_{\tau-1})^{\frac{2}{p^\dagger}+1})^q$, and such that

$$\|\mathbb{E}(f | \mathcal{A}_{\mathcal{P}_\tau}) - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_{\tau-1}})\|_{L_{p^\dagger}} \geq \frac{a_0 \varepsilon \|f\|_{L_1}}{2}.$$

The algorithm outputs the partition \mathcal{P}_τ and **Halts**.

Notice that there exists a polynomial Π_0 such that the previous algorithm has running time $(\tau 4^\tau) \cdot \Pi_0(n_1 \cdot n_2)$. Indeed, by Proposition 2.2, there exists a polynomial Π'_0 such that the **InitialStep** runs in time $\Pi'_0(n_1 \cdot n_2)$. Moreover, by the running

⁵Notice that $\mathcal{P}_0 \subseteq \mathcal{S}$ and $\iota(\mathcal{P}_0) = 1$.

times of the algorithms in Lemma 3.3 and Proposition 2.2, there exists a polynomial Π_0'' such that each of the **GeneralStep** runs in time $4^\tau \cdot \Pi_0''(n_1 \cdot n_2)$. Finally, invoking again Lemma 3.3, we see that there exists a polynomial Π_0''' such that the **FinalStep** runs in time $\Pi_0'''(n_1 \cdot n_2)$. Therefore, the algorithm we described above runs in time

$$\Pi_0'(n_1 \cdot n_2) + (\tau - 1) 4^\tau \Pi_0''(n_1 \cdot n_2) + \Pi_0'''(n_1 \cdot n_2)$$

which in turn yields that there exists a polynomial Π_0 such that the algorithm has running time $(\tau 4^\tau) \cdot \Pi_0(n_1 \cdot n_2)$.

It remains to verify that the previous algorithm will produce a partition that satisfies the requirements in **OUT** of Theorem 1.3. As we have noted, the argument is based on Proposition 2.1 and can be seen as the L_p version of the, so called, *energy increment method* (see, e.g., [16, Lemmas 10.40 and 11.31]). For more information and further applications of this method we refer to [5, 6, 8].

We proceed to the details. First assume that the algorithm has stopped before the **FinalStep**. Then the output of the algorithm is one of the partitions we described in **InitialStep** and in **GeneralStep**, say \mathcal{P}_m for some $m \in \{0, \dots, \tau - 1\}$. Observe that \mathcal{P}_m satisfies $\mathcal{P}_m \subseteq \mathcal{S}$, $|\mathcal{P}_m| \leq 4^m$, and $\iota(\mathcal{P}_m) \geq \eta$; in other words, \mathcal{P}_m satisfies the first three requirements in **OUT** of Theorem 1.3. Moreover, recall that there exists a set $A_m \subseteq [n_1] \times [n_2]$ with $A_m \in \mathcal{S}$, and such that

$$(n_1 \cdot n_2) \left| \int_{A_m} (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_m})) d\mu \right| \geq a_0 \|f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_m})\|_{\square}.$$

On the other hand, since the output of the algorithm is the partition \mathcal{P}_m , we have $|\int_{A_m} (f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_m})) d\mu| \leq a_0 \varepsilon \|f\|_{L_1}$. Combining these estimates, we conclude that $\|f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_m})\|_{\square} \leq \varepsilon \|f\|_{\square}$.

Next, assume that the algorithm reaches the **FinalStep**. Recall that $\mathcal{P}_\tau \subseteq \mathcal{S}$ and observe that, by (d) above and the fact that $|\mathcal{P}_\tau| \leq 4|\mathcal{P}_{\tau-1}|$, we have $|\mathcal{P}_\tau| \leq 4^\tau$. Moreover, by (e) and the choice of η in (4.1),

$$(4.2) \quad \iota(\mathcal{P}_\tau) \geq (\vartheta \cdot \iota(\mathcal{P}_{\tau-1})^{\frac{2}{p^\dagger}+1})^q \geq \vartheta^{\sum_{i=1}^{\tau} (\frac{2}{p^\dagger}+1)^{i-1} q^i} \geq \eta.$$

Thus, we only need to show that $\|f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_\tau})\|_{\square} \leq \varepsilon \|f\|_{\square}$. To this end assume, towards a contradiction, that $\|f - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_\tau})\|_{\square} > \varepsilon \|f\|_{\square}$. Notice that, by the choice of η in (4.1) and (4.2), we have $(\vartheta \cdot \iota(\mathcal{P}_\tau)^{\frac{2}{p^\dagger}+1})^q \geq \eta$. Using the previous two estimates, Proposition 2.2, Lemma 3.3 and arguing precisely as in the **GeneralStep**, we may select a refinement $\mathcal{P}_{\tau+1}$ of \mathcal{P}_τ with $\mathcal{P}_{\tau+1} \subseteq \mathcal{S}$ and $\iota(\mathcal{P}_{\tau+1}) \geq \eta$, and such that $\|\mathbb{E}(f | \mathcal{A}_{\mathcal{P}_{\tau+1}}) - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_\tau})\|_{L_{p^\dagger}} \geq (a_0 \varepsilon \|f\|_{L_1})/2$. It follows that there exists an increasing finite sequence $(\mathcal{P}_i)_{i=0}^{\tau+1}$ of partitions with $\mathcal{P}_0 = \{[n_1] \times [n_2]\}$ and such that for every $i \in [\tau + 1]$ we have $\mathcal{P}_i \subseteq \mathcal{S}$, $\iota(\mathcal{P}_i) \geq \eta$, and

$$(4.3) \quad \|\mathbb{E}(f | \mathcal{A}_{\mathcal{P}_i}) - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_{i-1}})\|_{L_{p^\dagger}} \geq \frac{a_0 \varepsilon \|f\|_{L_1}}{2}.$$

Now set $d_0 = \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_0})$ and $d_i = \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_i}) - \mathbb{E}(f | \mathcal{A}_{\mathcal{P}_{i-1}})$ for every $i \in [\tau + 1]$, and observe that the sequence $(d_i)_{i=0}^{\tau+1}$ is a martingale difference sequence. Therefore,

by Proposition 2.1 and the fact that the matrix f is (C, η, p) -regular, we have

$$\begin{aligned}
\frac{a_0 \varepsilon \|f\|_{L_1}}{2} \cdot \sqrt{\tau+1} &\stackrel{(4.3)}{\leq} \left(\sum_{i=1}^{\tau+1} \|d_i\|_{L_{p^\dagger}}^2 \right)^{1/2} \leq \left(\sum_{i=0}^{\tau+1} \|d_i\|_{L_{p^\dagger}}^2 \right)^{1/2} \\
&\stackrel{(2.1)}{\leq} \frac{1}{\sqrt{p^\dagger-1}} \left\| \sum_{i=0}^{\tau+1} d_i \right\|_{L_{p^\dagger}} = \frac{1}{\sqrt{p^\dagger-1}} \|\mathbb{E}(f \mid \mathcal{A}_{\mathcal{P}_{\tau+1}})\|_{L_{p^\dagger}} \\
&\leq \frac{C}{\sqrt{p^\dagger-1}} \|f\|_{L_1}
\end{aligned}$$

which clearly contradicts the choice of τ in (4.1). The proof of Theorem 1.3 is thus completed. \square

5. APPLICATIONS

5.1. Tensor approximation algorithms. Throughout this subsection let $k \geq 2$ be an integer. Also let n_1, \dots, n_k be positive integers, and let μ_k denote the uniform probability measure on $[n_1] \times \dots \times [n_k]$.

Recall that a k -dimensional tensor is a function $F: [n_1] \times \dots \times [n_k] \rightarrow \mathbb{R}$. (Notice, in particular, that a 2-dimensional tensor is just a matrix.) Also recall, that a tensor $G: [n_1] \times \dots \times [n_k] \rightarrow \mathbb{R}$ is called a *cut tensor* if there exist a real number c and for every $i \in [k]$ a subset S_i of $[n_i]$ such that $G = c \cdot \mathbf{1}_{S_1 \times \dots \times S_k}$. Finally, recall that for every tensor $F: [n_1] \times \dots \times [n_k] \rightarrow \mathbb{R}$ its *cut norm* is defined as

$$\|F\|_{\square} = \left(\prod_{i=1}^k n_i \right) \cdot \max \left\{ \left| \int_{S_1 \times \dots \times S_k} F d\mu_k \right| : S_i \subseteq [n_i] \text{ for every } i \in [k] \right\}.$$

Next, set

$$(5.1) \quad k_1 := \lfloor k/2 \rfloor, \quad A_k := [n_1] \times \dots \times [n_{k_1}] \quad \text{and} \quad B_k := [n_{k_1+1}] \times \dots \times [n_k],$$

and for every tensor $F: [n_1] \times \dots \times [n_k] \rightarrow \{0, 1\}$ let the *respective matrix* f_F of F be the matrix $f_F: A_k \times B_k \rightarrow \{0, 1\}$ defined by the rule

$$(5.2) \quad f_F((i_1, \dots, i_{k_1}), (i_{k_1+1}, \dots, i_k)) = F(i_1, \dots, i_k)$$

for every $((i_1, \dots, i_{k_1}), (i_{k_1+1}, \dots, i_k)) \in A_k \times B_k = [n_1] \times \dots \times [n_k]$.

As in [4], we extend the notion of L_p regularity from matrices to tensors as follows.

Definition 5.1 (L_p regular tensors). *Let $0 < \eta \leq 1, C \geq 1$ and $1 \leq p \leq \infty$. A tensor $F: [n_1] \times \dots \times [n_k]$ is called (C, η, p) -regular if its respective matrix f_F is (C, η, p) -regular, that is, if for every partition \mathcal{P} of $A_k \times B_k$ with $\mathcal{P} \subseteq \mathcal{S}_{A_k \times B_k}$ and $\iota(\mathcal{P}) \geq \eta$ we have $\|\mathbb{E}(f_F \mid \mathcal{A}_{\mathcal{P}})\|_{L_p} \leq C$.*

To state our main result about L_p regular tensors we need to introduce some numerical invariants. Specifically, let $\varepsilon > 0$ and $C \geq 1$. Also let $1 < p \leq \infty$, set

$p^\dagger = \min\{2, p\}$ and let q denote the conjugate exponent of p^\dagger . Finally, let a_1, a_2 be as in Theorem 1.3, and define

$$(5.3) \quad \tau(\varepsilon, C, p) = \left\lceil \frac{a_1 C^2}{(p^\dagger - 1) \varepsilon^2} \right\rceil \quad \text{and} \quad \eta(\varepsilon, C, p) = \left(\frac{a_2 \varepsilon}{C} \right)^{\sum_{i=1}^{\tau(\varepsilon, C, p)+1} (\frac{2}{p^\dagger} + 1)^{i-1} q^i}.$$

We have the following theorem.

Theorem 5.2. *There exist a constant b , an algorithm and a polynomial Π_3 such that the following holds. Let $0 < \varepsilon < 1/2$ and $C \geq 1$. Also let $1 < p \leq \infty$, and let $\tau = \tau(\varepsilon/2, C, p)$ and $\eta = \eta(\varepsilon/2, C, p)$ be as in (5.3). If we input*

INP: a (C, η, p) -regular tensor $F: [n_1] \times \cdots \times [n_k] \rightarrow \{0, 1\}$,

then the algorithm outputs

OUT: cut tensors G_1, \dots, G_s with $s \leq \left(\frac{2bC}{\varepsilon \eta^2} \right)^{2(k-1)}$ and such that

$$(5.4) \quad \left\| F - \sum_{i=1}^s G_i \right\|_{\square} \leq \varepsilon \|F\|_{\square} \quad \text{and} \quad \sum_{i=1}^s \|G_i\|_{L_\infty}^2 \leq \left(\frac{C \|F\|_{L_1}}{\eta^2} \right)^2 b^{2k}.$$

This algorithm has running time $(\tau 4^\tau + (\frac{2C}{\varepsilon \eta^2})^{3k}) \cdot \Pi_3(\prod_{i=1}^k n_i)$.

Theorem 5.2 can be proved arguing precisely as in the proof of [4, Theorem 2] and using Theorem 1.3 instead of [4, Corollary 1]. We leave the details to the interested reader.

5.2. MAX-CSP instances approximation. In what follows let n, k denote two positive integers with $k \leq n$.

Let $V = \{x_1, \dots, x_n\}$ be a set of Boolean variables, and recall that an *assignment* σ on V is a map $\sigma: V \rightarrow \{0, 1\}$. Notice that if σ is an assignment on V and $W \subseteq V$, then $\sigma|_W: W \rightarrow \{0, 1\}$ is an assignment on W . Also recall that a *k -constraint* is a pair (ϕ, V_ϕ) where $V_\phi \subseteq V$ with $|V_\phi| = k$ and $\phi: \{0, 1\}^{V_\phi} \rightarrow \{0, 1\}$ is a not identically zero map. Finally, recall that a *k -CSP instance* over V is a family \mathcal{F} of k -constraints over V .

For every k -CSP instance \mathcal{F} we define

$$(5.5) \quad \text{OPT}(\mathcal{F}) = \max_{\sigma \in \{0, 1\}^V} \sum_{(\phi, V_\phi) \in \mathcal{F}} \phi(\sigma|_{V_\phi}).$$

Moreover, let Ψ_k be the set of all non-zero maps from $\{0, 1\}^k$ into $\{0, 1\}$. We have the following definition.

Definition 5.3. *Let $\psi \in \Psi_k$. Also let (ϕ, V_ϕ) be a k -constraint over V where $V_\phi = \{x_{i_1}, \dots, x_{i_k}\}$ for some $1 \leq i_1 < \dots < i_k \leq n$. We say that (ϕ, V_ϕ) is of type ψ if for every assignment $\sigma: V \rightarrow \{0, 1\}$ we have*

$$\psi(\sigma(x_{i_1}), \dots, \sigma(x_{i_k})) = \phi(\sigma|_{V_\phi}).$$

Observe that every k -CSP instance \mathcal{F} can be represented by a family $(F_{\mathcal{F}}^{\psi})_{\psi \in \Psi_k}$ of $2^{2^k} - 1$ tensors where for every $\psi \in \Psi_k$ the tensor $F_{\mathcal{F}}^{\psi}: [n]^k \rightarrow \{0, 1\}$ is defined by the rule

$$(5.6) \quad F_{\mathcal{F}}^{\psi}(i_1, \dots, i_k) = \begin{cases} 1 & \text{if there is } (\phi, V_{\phi}) \in \mathcal{F} \text{ of type } \psi \\ & \text{with } V_{\phi} = \{x_{i_1}, \dots, x_{i_k}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Having this representation in mind, we say that a k -constraint \mathcal{F} is (C, η, p) -regular for some $0 < \eta \leq 1$, $C \geq 1$ and $1 \leq p \leq \infty$, provided that for every $\psi \in \Psi_k$ the tensor $F_{\mathcal{F}}^{\psi}$ defined above is (C, η, p) -regular.

We have the following theorem which extends [4, Theorem 3]. It follows from Theorem 5.2 using the arguments in the proof of [4, Theorem 3]; as such, its proof is left to the reader.

Theorem 5.4. *There exist an algorithm, a constant $\gamma > 0$ and a polynomial Π_4 such that the following holds. Let k be a positive integer, and let $0 < \varepsilon < 1/2$, $C \geq 1$ and $1 < p \leq \infty$. Set $a = \varepsilon 2^{-(2^k + 2k + 2)}$, and let $\tau = \tau(a, C, p)$ and $\eta = \eta(a, C, p)$ be as in (5.3). If we input*

INP: *a (C, η, p) -regular k -CSP instance \mathcal{F} over a set $V = \{x_1, \dots, x_n\}$ of Boolean variables,*

then the algorithm outputs

OUT: *an assignment $\sigma: V \rightarrow \{0, 1\}$ such that*

$$\sum_{(\phi, V_{\phi}) \in \mathcal{F}} \phi(\sigma|_{V_{\phi}}) \geq (1 - \varepsilon) \cdot \text{OPT}(\mathcal{F}).$$

This algorithm has running time

$$\Pi_4 \left(n^k \cdot \exp \left(k 2^k 2^{2^k} \left(\frac{2C}{\varepsilon \eta^2} \right)^{2k} \ln \left(\frac{2C}{\varepsilon \eta^2} \right) \right) \right).$$

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