

CLOSED-FORM EXPRESSIONS FOR PROJECTORS ONTO POLYHEDRAL SETS IN HILBERT SPACES

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ABSTRACT. We provide formulas for projectors onto a polyhedral set, i.e. the intersection of a finite number of halfspaces. To this aim we formulate the problem of finding the projection as a convex optimization problem and we solve explicitly sufficient and necessary optimality conditions. This approach has already been successfully applied in deriving formulas for projection onto the intersection of two halfspaces. We also discuss possible generalizations to Banach spaces.

1. INTRODUCTION

Projections onto convex closed sets play an important role in constructions of algorithms for solving optimization problems (see [22] for nonlinear complementarity problems and variational inequalities). In general, projections onto intersections of convex sets are obtained as limits of iterative processes, see e.g. [2, 5, 6, 9, 11, 13, 16].

The idea of finding a closed-form expression for projector onto a linear subspace by solving explicitly the corresponding optimization problem goes back to Pshenichnyj [25, Theorem 1.19]. This idea has also been used by Bauschke and Combettes in [4, Proposition 28.19, Proposition 28.20] to provide explicit formulas for the projection onto the intersection of two halfspaces. The obtained formulas were at the core of the algorithm approximating the Kuhn-Tucker set for the pair of dual monotone inclusions as proposed in [1]. Recently, a finite algorithm for projection onto an isotone projection cone was given in [21]. This algorithm allows one to improve considerably the performance of a class of algorithms for solving complementarity problems [23].

In this paper we provide a closed-form expression for the projector onto polyhedral sets in Hilbert spaces. The results of the present paper are applied in constructing inertial algorithms for approximation of the Kuhn-Tucker set for monotone inclusions [7].

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The starting point of our considerations is Theorem 6.41 of [10] which provides the Kuhn-Tucker conditions for the convex optimization problem related to projections. Analogous approach was presented in [12] to solve explicitly an optimization problems with second order cone constraints. It is essential in our approach that: (1) in Hilbert space we have a simple formula for the derivative of the norm, (2) the number of halfspaces is finite. An analogous approach in Banach spaces depends strongly on differentiability properties of the norm.

Let H be a real Hilbert space equipped with a scalar product $\langle \cdot | \cdot \rangle : H \times H \rightarrow \mathbb{R}$ and the associated norm $\|\cdot\|$. For any closed convex subset $K \subset H$, let $P_K(x)$ denote the projection of $x \in H$ onto K . Let $C_i := \{h \in H \mid \langle h | u_i \rangle \leq \eta_i\}$, $u_i \in H \setminus \{0\}$, $\eta_i \in \mathbb{R}$, for $i = 1, \dots, n$ be a finite family of halfspaces. Halfspaces are clearly convex sets, and, by the Riesz representation theorem, halfspaces are also closed subsets of the Hilbert space H .

Let $C \subset H$ be defined as

$$C := \bigcap_{i=1}^n C_i.$$

We derive closed-form expressions for the projector $P_C(x)$ of an element $x \in H$ onto C when C is nonempty. Our framework takes into account all possible relationships between vectors u_i for $i = 1, \dots, n$, e.g. we do not assume that u_i , $i = 1, \dots, n$ are linearly independent. In Banach spaces this approach does not provide, in general, explicit formulas for projections. In some particular cases we give verifiable criteria to check whether a given \bar{x} is a projection.

The organization of the paper is as follows. In section 2 we provide a refinement of the existing theorem on formulas for the projection onto polyhedral sets (Proposition 2). It is achieved by the analysis of optimality conditions for the optimization problem related to finding the projection $P_C(x)$. In section 3 we use Proposition 2 to provide explicit formulas for projection $P_C(x)$. This is the content of Theorem 2 which is our main result. In section 4 we compare our approach with the already existing approaches to provide explicit formulas for projections, in particular we compare Theorem 2 of section 3 with Theorem 2 of [21]. In section 5 we investigate projections onto C in Banach spaces and we provide some criteria to verify whether \bar{x} is a projection of x .

Notation. Let n be a strictly positive integer number. We reserve the symbol N to the set defined as $N := \{1, 2, \dots, n\}$. Symbols H, B denote Hilbert and Banach space, respectively. A function $g : H \rightarrow (-\infty, \infty]$ is proper if it is not equal to $+\infty$ on the whole space. When G is a matrix of dimensions $m \times k$ and $I \subset \{1, \dots, m\}$, $J \subset \{1, \dots, k\}$, $I, J \neq \emptyset$ the symbol $G_{I,J}$ denotes the submatrix of G composed by rows indexed by I and columns indexed by J only. For any Gateaux differentiable function $f : B \rightarrow \mathbb{R}$, $f'(x)$ denote the Gateaux derivative of f at x and $f'(x, d)$ denotes the directional derivative of f at x in direction d . For any $i, j \in \mathbb{N}$ the symbol $\delta_{i,j}$ denotes the Kronecker delta.

2. PROJECTIONS

Let $x \in H$, $C = \bigcap_{i \in N} C_i$, where $C_i = \{h \in H \mid \langle h | u_i \rangle \leq \eta_i\}$, $i \in N$. The optimization problem

$$\min_{h \in C} \frac{1}{2} \|h - x\|^2 \tag{2.1}$$

is equivalent to finding the projection of x onto C . This is a quadratic programming problem with linear inequality constraints on H (see [8, 17] and [18] for applications).

To this problem we can apply the optimality conditions for general convex optimization problem of the form

$$\begin{aligned} \min_{x \in H} F_0(x) \\ F_i(x) \leq 0, i \in N, \end{aligned} \quad (2.2)$$

where $F_0, F_i : H \rightarrow \mathbb{R}, i \in N$ are functionals on H . In the sequel we use the following form of the Kuhn-Tucker conditions for the problem (2.2)¹.

Proposition 1. [19, Theorem 11.3] (see also [8, Proposition 3.118]) *Let $F_0, F_i : H \rightarrow \mathbb{R}, i \in N$ be convex and continuously differentiable on H and*

$$F_i(x) := \langle x | u_i \rangle - \eta_i, \quad 0 \neq u_i \in H, \quad \eta_i \in \mathbb{R}, \quad i \in N.$$

Sufficient and necessary conditions for minimum at $\bar{x} \in H$ are:

$$\begin{aligned} \nu_i \geq 0, \quad F_i(\bar{x}) \leq 0, \quad \nu_i F_i(\bar{x}) = 0, \quad i \in N, \\ F'_0(\bar{x}) + \nu_1 F'_1(\bar{x}) + \dots + \nu_n F'_n(\bar{x}) = 0. \end{aligned} \quad (2.3)$$

Applying Proposition 1 to functions $F_0(\cdot) = \frac{1}{2} \|\cdot - x\|^2$ and $F_i(\cdot) = \langle \cdot | u_i \rangle - \eta_i, i \in N$, we obtain the following theorem due to Deutsch [10].

Theorem 1. [10, Theorem 6.41] *Let $u_i \in H, \eta_i \in \mathbb{R}, i \in 1, \dots, n$ and $C = \bigcap_{i \in N} C_i \neq \emptyset$. If $x \in H$ then*

$$P_C(x) = x - \sum_{i=1}^n \nu_i u_i,$$

for any set of scalars ν_i that satisfy the following three conditions:

$$\nu_i \geq 0, \quad \text{for } i = 1, \dots, n, \quad (2.4)$$

$$\langle x, u_i \rangle - \eta_i - \sum_{j=1}^n \nu_j \langle u_j | u_i \rangle \leq 0 \quad \text{for } i = 1, \dots, n, \quad (2.5)$$

$$\nu_i (\langle x, u_i \rangle - \eta_i - \sum_{j=1}^n \nu_j \langle u_j | u_i \rangle) = 0 \quad \text{for } i = 1, \dots, n. \quad (2.6)$$

Consequently, if $x \in H$ and $\bar{x} \in C$, then $\bar{x} = P_C(x)$ if and only if

$$\bar{x} = x - \sum_{i \in I(\bar{x})} \nu_i u_i, \quad \text{for some } \nu_i \geq 0,$$

where $I(\bar{x}) := \{i \in N \mid \langle \bar{x} | u_i \rangle = \eta_i\}$.

This fact is proved in [10] as an immediate consequence of the projection theorem in Hilbert spaces and representations of dual cones. Note that conditions (2.4), (2.5), (2.6) of Theorem 1 are conditions (2.3) of Proposition 1 when applied to problem (2.2).

Let $\{u_i\} \in H, \eta_i \in \mathbb{R}, i \in N$ and let

$$G := \begin{bmatrix} \|u_1\|^2 & \langle u_1 | u_2 \rangle & \cdots & \langle u_1 | u_n \rangle \\ \langle u_2 | u_1 \rangle & \|u_2\|^2 & & \langle u_2 | u_n \rangle \\ \vdots & & \ddots & \vdots \\ \langle u_n | u_1 \rangle & \langle u_n | u_2 \rangle & \cdots & \|u_n\|^2 \end{bmatrix}.$$

¹Originally generalizations of the Kuhn-Tucker necessary optimality conditions to infinite dimensional spaces were given in [3], [24], [26].

The matrix G is called the *Gram matrix* and has the following well-known property: for any $I \subset N$ $\det G_{I,I} \geq 0$ and $\det G_{I,I} = 0$ if and only if vectors u_i , $i \in I$ are linearly dependent.

In Proposition 2 we derive equivalent conditions on scalars ν_i . Due to the form of (2.4), (2.5), (2.6) these conditions can be expressed in terms of the existence of positive solutions of systems of linear equations.

Proposition 2. *Let $C = \bigcap_{i \in N} C_i \neq \emptyset$, where $C_i = \{h \in H \mid \langle h \mid u_i \rangle \leq \eta_i\}$, $u_i \in H \setminus \{0\}$, $\eta_i \in \mathbb{R}$ for $i \in N$ and let $x \in H \setminus C$. The point \bar{x} is a projection of x onto C if and only if there exists $I \subset N$, $I \neq \emptyset$ such that (feasibility conditions)*

$$\bar{x} = x - \sum_{i \in I} \nu_i u_i \in C, \quad (2.7)$$

where ν_i , $i \in I$, solve the following system of linear equations (complementarity slackness conditions)

$$\forall i \in I \quad \begin{cases} \langle x \mid u_i \rangle - \eta_i = \sum_{k \in I} \nu_k \langle u_k \mid u_i \rangle, \\ \nu_i > 0. \end{cases} \quad (2.8)$$

Moreover, there always exists at least one I for which: (1) $\det G_{I,I} > 0$, (2) system (2.8) is solvable, (3) formula (2.7) holds.

The main contribution of this proposition is condition (2.8) which replaces conditions (2.4), (2.5), (2.6) of Theorem 1 and reduces the question of finding the projection onto C to solving a consistent system of linear equations.

The proof of Proposition 2 is based on the following technical lemma.

Lemma 1. (see, e.g. [10, Lemma 6.33]) *Let $u_i \in H$, $u_i \neq 0$, and $\tilde{\nu}_i \geq 0$ for $i \in N$, not all equal zero, and $w := \sum_{i \in N} \tilde{\nu}_i u_i \neq 0$. There exist $I \subset N$ and $\nu_i > 0$, $i \in I$ such that*

$$w = \sum_{i \in I} \nu_i u_i \quad \text{and} \quad \det G_{I,I} > 0.$$

Proof of Proposition 2. By Theorem 1, there exists $\tilde{\nu}_i$, $i \in N$, such that

$$\forall i \in N \quad \begin{cases} \langle x - \sum_{k \in N} \tilde{\nu}_k u_k \mid u_i \rangle - \eta_i \leq 0, \\ \tilde{\nu}_i (\langle x - \sum_{k \in N} \tilde{\nu}_k u_k \mid u_i \rangle - \eta_i) = 0, \\ \tilde{\nu}_i \geq 0 \end{cases} \quad (2.9)$$

and \bar{x} defined as

$$\bar{x} := x - \sum_{i \in N} \tilde{\nu}_i u_i \quad (2.10)$$

is the projection of x onto C . Let $J := \{i \in N \mid \tilde{\nu}_i > 0\}$. Since $x \notin C$, from (2.10) we deduce that $J \neq \emptyset$. We rewrite formula (2.10) in the form

$$\bar{x} = x - \sum_{i \in J} \tilde{\nu}_i u_i \quad (2.11)$$

and by (2.9) we have

$$\forall i \in J \quad \begin{cases} \langle x - \sum_{k \in J} \tilde{\nu}_k u_k \mid u_i \rangle - \eta_i = 0, \\ \tilde{\nu}_i > 0. \end{cases} \quad (2.12)$$

The system (2.12) is of the form

$$G_{J,J}[\tilde{\nu}_i]_{i \in J} = [\langle x \mid u_i \rangle - \eta_i]_{i \in J} \quad (2.13)$$

and, by (2.11), we know that (2.13) has a strictly positive solution $\tilde{\nu}_i > 0$, $i \in J$. If $\det G_{J,J} = 0$, then, by Lemma 1, there exists $I \subset J$ and $\nu_i > 0$, $i \in I$, such that $\det G_{I,I} \neq 0$ and $\sum_{j \in J} \tilde{\nu}_j u_j = \sum_{i \in I} \nu_i u_i$. The index set I satisfies the requirements given in the assertion of the proposition. \square

Let us note that the index set I might be a one element set.

3. MAIN RESULTS

In this section we provide explicit formulas for solutions to optimization problem (2.1). This is the content of Theorem 2 which is our main result.

Let $I \subset N$ and $s_I(a) := \{b \in I \mid b \leq a\}$. We define

$$B_I^a := \begin{cases} (-1)^{|s_I(a)|} & \text{if } a \in I, \\ (-1)^{|I|+1} & \text{if } a \notin I. \end{cases}$$

Let $w_i := \langle x \mid u_i \rangle - \eta_i$, $i \in N$.

Theorem 2. Let $C = \bigcap_{i=1}^n C_i \neq \emptyset$, where $C_i = \{h \in H \mid \langle h \mid u_i \rangle \leq \eta_i\}$, $u_i \neq 0$, $\eta_i \in \mathbb{R}$, $i \in N$, $x \notin C$. Let $\text{rank } G = k$. Let $\emptyset \neq I \subset N$, $|I| \leq k$ be such that $\det G_{I,I} \neq 0$. Let

$$\nu_i := \begin{cases} \sum_{j \in I} w_j B_I^j B_I^i \det G_{I \setminus j, I \setminus i} & \text{if } |I| > 1, \\ w_i & \text{if } |I| = 1 \end{cases} \quad \text{for all } i \in I \quad (3.1)$$

and, whenever $I' := N \setminus I$ is nonempty, let

$$\nu_{i'} := \sum_{j \in I \cup \{i'\}} w_j B_I^j B_I^{i'} \det G_{I, (I \cup i') \setminus j} \quad \text{for all } i' \in I'. \quad (3.2)$$

If $\nu_i > 0$ for $i \in I$ and $\nu_{i'} \leq 0$ for all $i' \in I'$, then

$$P_C(x) = x - \sum_{i \in I} \frac{\nu_i}{\det G_{I,I}} u_i.$$

Moreover, among all the elements of the set Δ of all subsets $I \subset N$ there exists at least one $I \in \Delta$ for which: (1) $\det G_{I,I} \neq 0$, (2) the coefficients ν_i , $i \in I$ given by (3.1) are positive, (3) the coefficients $\nu_{i'}$, $i' \in I'$ given by (3.2) are nonpositive.

Proof. Let $I \subset N$, $I \neq \emptyset$, $\det G_{I,I} \neq 0$. Let ν_i be given by (3.1) for $i \in I$ and let $\nu_{i'}$ be given by (3.2) for $i' \in I'$. Assume that $\nu_i > 0$ for $i \in I$ and $\nu_{i'} \leq 0$ for $i' \in I'$. The elements $\tilde{\nu}_i := \frac{\nu_i}{\det G_{I,I}}$ for $i \in I$ solve the system

$$\forall i \in I \quad \begin{cases} \langle x \mid u_i \rangle - \eta_i = \sum_{k \in I} \tilde{\nu}_k \langle u_k \mid u_i \rangle, \\ \tilde{\nu}_i > 0 \end{cases} \quad (3.3)$$

or, in the matrix form,

$$G_{I,I} [\tilde{\nu}_i]_{i \in I} = [\langle x \mid u_i \rangle - \eta_i]_{i \in I},$$

where the solution of this system satisfies $\tilde{\nu}_i > 0$, $i \in I$. Thus, for all $i \in I$ we have $\tilde{\nu}_i (\langle x - \sum_{k \in I} \tilde{\nu}_k u_k \mid u_i \rangle - \eta_i) = 0$ and consequently $\langle x - \sum_{k \in I} \tilde{\nu}_k u_k \mid u_i \rangle - \eta_i = 0$ for $i \in I$.

To prove that $\bar{x} = x - \sum_{i \in I} \nu_i u_i$ is the projection of x onto C it is enough to show that $\langle \bar{x} \mid u_{i'} \rangle - \eta_{i'} \leq 0$ for all $i' \in I'$. It is obvious in case $I' = \emptyset$, so suppose I' is nonempty.

(1) Suppose $I = \{m\}$, where $m \in N$. Let $i' \in I'$. Then

$$\begin{aligned} \|u_m\|^2(\langle x - \tilde{\nu}_m u_m \mid u_{i'} \rangle - \eta_{i'}) &= \|u_m\|^2 \langle x - \frac{\langle x \mid u_m \rangle - \eta_m}{\|u_m\|^2} u_m \mid u_{i'} \rangle - \|u_m\|^2 \eta_{i'} \\ &= (\langle x \mid u_{i'} \rangle - \eta_{i'}) \|u_m\|^2 - (\langle x \mid u_m \rangle - \eta_m) \langle u_m \mid u_{i'} \rangle \\ &= \sum_{j \in \{m, i'\}} (\langle x \mid u_j \rangle - \eta_j) B_I^j B_I^{i'} \det G_{I, (I \cup \{i'\}) \setminus \{j\}} \leq 0. \end{aligned}$$

Since $\|u_m\|^2 > 0$, $\langle \bar{x} \mid u_{i'} \rangle - \eta_{i'} \leq 0$ for all $i' \in I'$.

(2) Suppose $|I| \geq 2$. Let $i' \in I'$. Then

$$\begin{aligned} \det G_{I, I}(\langle x - \sum_{k \in I} \tilde{\nu}_k u_k \mid u_{i'} \rangle - \eta_{i'}) &= (\langle x \mid u_{i'} \rangle - \eta_{i'}) \det G_{I, I} - \sum_{k \in I} \nu_k \langle u_k \mid u_{i'} \rangle \\ &= (\langle x \mid u_{i'} \rangle - \eta_{i'}) \det G_{I, I} - \sum_{k \in I} \sum_{j \in I} (\langle x \mid u_j \rangle - \eta_j) B_I^j B_I^k \det G_{I \setminus \{j\}, I \setminus \{k\}} \langle u_k \mid u_{i'} \rangle \\ &= (\langle x \mid u_{i'} \rangle - \eta_{i'}) \det G_{I, I} - \sum_{j \in I} ((\langle x \mid u_j \rangle - \eta_j) \sum_{k \in I} B_I^j B_I^k \det G_{I \setminus \{j\}, I \setminus \{k\}} \langle u_k \mid u_{i'} \rangle) \\ &= (\langle x \mid u_{i'} \rangle - \eta_{i'}) \det G_{I, I} - \sum_{j \in I} (\langle x \mid u_j \rangle - \eta_j) B_I^j B_I^{i'} \det G_{(I \cup \{i'\}) \setminus \{j\}, I} \\ &= (\langle x \mid u_{i'} \rangle - \eta_{i'}) \det G_{I, I} + \sum_{j \in I} (\langle x \mid u_j \rangle - \eta_j) B_I^j B_I^{i'} \det G_{I, (I \cup \{i'\}) \setminus \{j\}} \\ &= \sum_{j \in I \cup \{i'\}} (\langle x \mid u_j \rangle - \eta_j) B_I^j B_I^{i'} \det G_{I, (I \cup \{i'\}) \setminus \{j\}} \leq 0. \end{aligned}$$

Since $\det G_{I, I} > 0$, $\langle \bar{x} \mid u_{i'} \rangle - \eta_{i'} \leq 0$ for all $i' \in I'$.

The existence of $I \subset N$ such that $\nu_i > 0$, $i \in I$ and $\nu_{i'} \leq 0$, $i' \in I'$ is guaranteed by Proposition 2. \square

It is easy to see from the proof of Theorem 2 that $\tilde{\nu}_i := \frac{\nu_i}{\det G_{I, I}}$, $i \in I$, with ν_i defined by (3.1), solve system (3.3) (complementarity slackness conditions) which can be rewritten as

$$G_{I, I}[\tilde{\nu}_i]_{i \in I} = [\langle x \mid u_i \rangle - \eta_i]_{i \in I},$$

whereas $(\nu_{i'})_{i' \in I'}$ allows us to check whether the resulting \bar{x} belongs to the set C (feasibility conditions).

Theorem 2 suggests the following finite algorithm for finding the projection $P_C(x)$ for $x \notin C$: Let $\Delta := \{I_1, I_2, \dots, I_{2^n-1}\}$ be the collection of all nonempty subsets of N and let $m = 1$.

Step 1. Check, if $\det G_{I_m, I_m} \neq 0$. If not, let $m := m + 1$ and repeat Step 1.

Step 2. Solve the linear system

$$\langle x \mid u_i \rangle - \eta_i = \sum_{k \in I_m} \nu_k \langle u_k \mid u_i \rangle, \quad i \in I_m \quad (3.4)$$

with respect to ν_k , $k \in I_m$. If there exists $k \in I_m$ such that $\nu_k \leq 0$ let $m := m + 1$ and go to Step 1.

Step 3. Check if the following formula is satisfied

$$\forall i' \in I'_m \quad \langle x - \sum_{k \in I_m} \nu_k u_k \mid u_{i'} \rangle \leq 0. \quad (3.5)$$

If not let $m := m + 1$ and go to Step 1.

Step 4. The projection of x onto C is given by formula

$$P_C(x) = x - \sum_{i \in I_m} \nu_i u_i. \quad (3.6)$$

By Theorem 2, among all the subsets $I \subset \Delta$ for which $\nu_k > 0$, $k \in I$ given by (3.4) there exists at least one for which (3.5) holds.

The proposed algorithm is suitable for parallelization. The parallelized version of the algorithm can be organized as follows. In Step 2 of the algorithm at most $2^n - 1$ systems are solved of at most n equations. In Step 3 for each solution of system from Step 2 we need to calculate at most $n - 1$ scalar products. Let us observe that according to Theorem 2 the representation (3.6) may not be unique.

4. ON LATTICIAL CONE

In [21] the authors proposed a finite algorithm for finding the projection onto a class of cones, called *latticeal cones*. In this section we compare our approach developed in section 3 with the approach proposed in [21], where the main tool was the Moreau decomposition theorem.

Let $K \subset H$ be a cone. The polar of K is the set

$$K^\circ := \{x \in H \mid \langle x \mid y \rangle \leq 0 \quad \forall y \in K\}.$$

Definition 1. Let $H = \mathbb{R}^n$. K is called *latticeal* if $K = \text{cone}\{b_1, b_2, \dots, b_n\}$, where $b_1, b_2, \dots, b_n \in \mathbb{R}^n$ are linearly independent and

$$\text{cone}\{b_1, b_2, \dots, b_n\} := \{x \in H \mid x = \sum_{i \in N} \alpha_i b_i, \alpha_i \geq 0 \text{ for } i \in N\}.$$

When $K = \text{cone}\{b_1, b_2, \dots, b_n\}$ we say that K is generated by b_1, b_2, \dots, b_n .

Any *latticeal cone* is closed and convex.

Lemma 2. [21] Let $K \subset \mathbb{R}^n$ be a *latticeal cone* generated by vectors b_1, b_2, \dots, b_n . The polar cone to K can be represented as

$$K^\circ = \{\mu_1 u_1 + \mu_2 u_2 + \dots + \mu_n u_n \mid \mu_i \geq 0, i \in N\},$$

where $u_j, j \in N$, solves the system

$$\langle u_j \mid b_i \rangle = -\delta_{i,j}, \quad i \in N,$$

Since K is closed and convex, $K = \{x \in \mathbb{R}^n \mid \langle y \mid x \rangle \leq 0 \quad \forall y \in K^\circ\}$. By Lemma 2,

$$x \in K \Leftrightarrow \langle x \mid u_i \rangle \leq 0 \quad \forall i \in N. \quad (4.1)$$

Corollary 1. [21] For each subset I of indices $I \subset N$, the vectors $b_i, i \in I, u_j, j \in N \setminus I$ are linearly independent.

Theorem 3. [20] (Moreau decomposition theorem) Let $K \subset \mathbb{R}^n$ be a closed convex cone and $x \in \mathbb{R}^n$. The following statements are equivalent.

- (1) $x = y + z, y \in K, z \in K^\circ$ and $\langle y \mid z \rangle = 0$,
- (2) $y = P_K x$ and $z = P_{K^\circ} x$.

The following fact has been proved in [21, Theorem 2]. Here we provide an alternative proof based on the tools developed in Section 2.

Theorem 4. Let $H = \mathbb{R}^n$ and let K be a laticial cone generated by vectors b_1, b_2, \dots, b_n and $x \notin K$. For each subset of indices $I \subset N$, x can be represented in the form

$$x = \sum_{i \in I'} \alpha_i b_i + \sum_{j \in I} \beta_j u_j \quad (4.2)$$

with $I' := N \setminus I$. Moreover, among the subsets $I \subset N$ of indices there exists exactly one (the case $I = \emptyset$ is not excluded, but we exclude the case $I = N$ since $x \notin K$) with the property that in (4.2) one has $\beta_j > 0$ for $j \in I$ and $\alpha_i \geq 0$ for $i \in I'$ and

$$P_K(x) = \sum_{i \in I'} \alpha_i b_i.$$

Proof. The representation 4.2 follows from Corollary 1. To see the second assertion note that, by Theorem 3, $x = a + b$, where $a \in K$, $b \in K^\circ$. Let u_j , $j \in N$ solve the system

$$\langle u_j | b_i \rangle = -\delta_{i,j}, \quad i \in N.$$

By formula (4.1), cone K can be represented as $K = \bigcap_{i \in N} C_i$, where $C_i := \{h \in \mathbb{R}^n \mid \langle h | u_i \rangle \leq 0\}$ for $i \in N$. By Theorem 2, there exists a set $I \subset N$ such that ν_i , $i \in I$, given by formula (3.1) are positive and for $i' \in I' := N \setminus I$, the coefficients $\nu_{i'}$ given by formula (3.2) are nonpositive and $P_K(x) = x - \sum_{i \in I} \tilde{\nu}_i u_i$, where $\tilde{\nu}_i = \nu_i / \det G_{I,I}$ for $i \in I$. Since $\bar{x} = P_K(x) \in K$,

$$\bar{x} = \sum_{k \in N} \alpha_k b_k, \quad \alpha_k \geq 0 \text{ for } k \in N. \quad (4.3)$$

Due to the linear independence of b_k , $k \in N$ and uniqueness of the projection, there exists exactly one system of coefficients α_k , $k \in N$ such that \bar{x} given by (4.3) is the projection of x onto K . Let $i \in I$. Taking the scalar product with vector u_i at both sides of (4.3) we obtain

$$\langle \bar{x} | u_i \rangle = \sum_{k \in N} \alpha_k \langle b_k | u_i \rangle \Leftrightarrow \langle \bar{x} | u_i \rangle = -\alpha_i.$$

From (2.7) and (2.8) we have $\langle \bar{x} | u_i \rangle = 0$ for $i \in I$. Thus $\alpha_i = 0$ for any $i \in I$. Hence, (4.3) reduces to

$$\bar{x} = \sum_{k \in I'} \alpha_k b_k, \quad \alpha_k \geq 0 \text{ for } k \in I'.$$

Thus, by Theorem 3 and Lemma 2, x can be represented as

$$x = \sum_{i \in I'} \alpha_i b_i + \sum_{j \in N} \beta_j u_j \quad \alpha_i \geq 0 \text{ for } i \in I' \text{ and } \beta_j \geq 0 \text{ for } j \in N, \quad (4.4)$$

where $P_K(x) = \sum_{i \in I'} \alpha_i b_i$, $P_{K^\circ}(x) = \sum_{j \in N} \beta_j u_j$ and K° is generated by vectors $\{u_1, u_2, \dots, u_n\}$. Due to the linear independence of vectors u_1, u_2, \dots, u_n , the representation $P_{K^\circ}(x) = \sum_{j \in N} \beta_j u_j$ is unique.

Formula (4.4) can be rewritten as

$$P_K(x) = x - \sum_{j \in N} \beta_j u_j. \quad (4.5)$$

Since the representation (4.5) is unique and vectors u_i , $i \in N$ are orthogonal, by Theorem, 2 we obtain

$$\beta_k = \begin{cases} \nu_k / \det G_{I,I} & \text{if } k \in I, \\ 0 & \text{if } k \notin I. \end{cases} \quad (4.6)$$

Thus, (4.4) can be written as

$$x = \sum_{i \in I'} \alpha_i b_i + \sum_{j \in I} \beta_j u_j \quad \alpha_i \geq 0 \text{ for } i \in I' \text{ and } \beta_j > 0 \text{ for } j \in I.$$

Since the representation is unique the proof is completed. \square

In case of projections onto latticial cones in \mathbb{R}^n , the algorithm proposed in section 3 differs from algorithm proposed in Section 3 of [21]. The differences follows from the fact that algorithm proposed in [21] is based on the Moreau decomposition theorem, whereas our algorithm is based on the Kuhn-Tucker conditions for the corresponding convex optimization problem. Formula (4.6) shows the relationship between the two algorithms. Namely, $\beta_k = \tilde{\nu}_k$ for $k \in I$, where $\tilde{\nu}_k$ is given as in the proof of Theorem 2, but the formula (4.4) is operational only in the finite-dimensional case for vectors $\{u_1, \dots, u_n\}$ which are linearly independent. Moreover, the computational cost of algorithm proposed in [21] depends strongly on the dimensionality of x .

5. THE CASE OF BANACH SPACES

Let $(B, \|\cdot\|)$ be a Banach space. Let $C = \bigcap_{i \in N} C_i$, where $C_i = \{h \in B \mid \langle f_i \mid h \rangle \leq \eta_i\}$, $f_i \in B^* \setminus \{0\}$, $\eta_i \in \mathbb{R}$ and $\langle \cdot \mid \cdot \rangle$ denotes the duality mapping.

Finding the projection of x onto C is equivalent to solving the optimization problem

$$\min_{h \in C} \frac{1}{r} \|h - x\|^r, \quad r \geq 1. \quad (5.1)$$

For this problem the following Pshenichnyi-Rockafellar optimality conditions hold (see e.g. Theorem 2.9.1 of [27]).

Theorem 5. *A point $\bar{x} \in B$ solves (5.1) if and only if $\partial_{\frac{1}{r}\|\cdot-x\|^r}(\bar{x}) \cap (-N(C, \bar{x})) \neq \emptyset$, where $N(C, \bar{x})$ is the normal cone to C at \bar{x} and $N(C, \bar{x}) := \{h^* \in B^* \mid \forall h \in C \quad \langle h^* \mid h - \bar{x} \rangle \leq 0\}$.*

In the case of strictly convex reflexive Banach space X every closed convex set D is Chebyshev, i.e. for each $x \in X$ there exists a *unique* point $P_D(x) \in D$ such that $\|x - P_D(x)\| = \inf\{\|x - d\|, d \in D\}$.

Let $I(x) := \{i \in N \mid \langle f_i \mid x \rangle = \eta_i\}$. We have $N(C, x) = \text{cone}(\{f_i, i \in I(x)\})$. When $k(\cdot) := \frac{1}{r}\|\cdot - x\|^r$ is Gateaux differentiable on B we have $\bar{x} = P_C(x)$ if, and only if,

$$\begin{aligned} 0 &\in k'(\bar{x}) + N(C, \bar{x}) \\ \iff \exists \nu_i \geq 0, i \in I(\bar{x}) \quad k'(\bar{x}) &= - \sum_{i \in I(\bar{x})} \nu_i f_i \\ \iff \exists \nu_i \geq 0, i \in I(\bar{x}) \quad \forall y \in B \quad \langle k'(\bar{x}) \mid y \rangle &= - \sum_{i \in I(\bar{x})} \nu_i \langle f_i \mid y \rangle \\ \iff \exists \nu_i \geq 0, i \in I(\bar{x}) \quad \forall y \in B \quad k'(\bar{x}, y) &= - \sum_{i \in I(\bar{x})} \nu_i \langle f_i \mid y \rangle. \end{aligned} \quad (5.2)$$

In Banach spaces ℓ_p , $p > 1$ of all sequences $f = \{f_1, f_2, \dots\}$ such that

$$\|f\|_{\ell_p} := \left(\sum_{i=1}^{+\infty} \|f_i\|^p \right)^{\frac{1}{p}} < +\infty$$

it was shown in [15, Example 8.1] that the directional derivative of $k(\cdot) = \frac{1}{p} \|\cdot - x\|_{\ell_p}^p$ at u in direction v is given by formula

$$k'(u, v) = \sum_{i=1}^{+\infty} |u_i - x_i|^{p-2} (u_i - x_i) v_i. \quad (5.3)$$

In Banach spaces $L_p(\Omega)$, $p \geq 1$ of all functions $f: L_p(\Omega) \rightarrow \mathbb{R}$ such that

$$\|f\|_{L^p} := \left(\int_{t \in \Omega} |f(t)|^p \mu(dt) \right)^{\frac{1}{p}} < +\infty$$

it was shown in [15, Example 8.2], [14, Example 13.12] that the directional derivative of $k(\cdot) = \frac{1}{p} \|\cdot - x\|_{\ell_p}^p$ at u in direction v is given by formula

$$k'(u, v) = \int_{\Omega} |u(t) - x(t)|^{p-2} (u(t) - x(t)) v(t) \mu(dt). \quad (5.4)$$

We start by discussing formulas for projections in spaces ℓ_p , $p > 1$. For any matrix $L = (\lambda_i^j)$, $\lambda_i^j \in \mathbb{R}$, $i, j \in N$ and any finite subsets $A_1, A_2 \subset N$ let L_{A_1, A_2} be the matrix with entries λ_j^i , where $i \in A_1$, $j \in A_2$. Let $e_i = (0, \dots, 0, 1, 0, \dots)$, with 1 at the position i denote the standard basis of ℓ_p .

Proposition 3. *In the space ℓ_p , $p > 1$ consider $C_i = \{h \in \ell_p \mid \langle f_i \mid h \rangle \leq \eta_i\}$, where $\eta_i \in \mathbb{R}$, $f_i = \sum_{j \in N} \lambda_j^i e_j \in \ell_q \setminus \{0\}$, $i \in N$. Let $W := \{k \in N \mid \forall i \in N \lambda_k^i = 0\}$. Then \bar{x} is the projection of $x \in \ell_p \setminus C$ onto C if and only if there exists $I \subset I(\bar{x})$ such that $\det L_{I, I} \neq 0$, and for $i \in I$*

$$\begin{aligned} \bar{x}_i &= \frac{1}{\det L_{I, I}} \sum_{k \in I} \tilde{\eta}_k B_I^k B_I^i \det L_{I \setminus \{k\}, I \setminus \{i\}}, \\ \frac{1}{\det L_{I, I}} \sum_{k \in I} \xi_k B_I^k B_I^i \det L_{I \setminus \{k\}, I \setminus \{i\}} &> 0, \end{aligned}$$

for $j \in N \setminus (I \cup W)$

$$|\bar{x}_j - x_j|^{p-2} (\bar{x}_j - x_j) = - \sum_{i \in I} \lambda_j^i \frac{1}{\det L_{I, I}} \left(\sum_{k \in I} \xi_k B_I^k B_I^i \det L_{I \setminus \{k\}, I \setminus \{i\}} \right) \quad (5.5)$$

where $\tilde{\eta}_k = \eta_k - \sum_{j \in N \setminus I} \lambda_j^k \bar{x}_j$, $\xi_k = -|\bar{x}_k - x_k|^{p-2} (\bar{x}_k - x_k)$, $k \in I$, and

$$\bar{x}_k = x_k \quad \text{for } k \in W.$$

Proof. By (5.2) and (5.3), $\bar{x} \in C$ is the projection of $x \in \ell_p \setminus C$ onto $C = \bigcap_{i \in N} C_i$ if and only if

$$\exists \{\nu_j\}_{j \in I(\bar{x})} \geq 0 \quad \forall y \in \ell_p \quad \sum_{k=1}^{+\infty} |\bar{x}_k - x_k|^{p-2} (\bar{x}_k - x_k) y_k = - \sum_{i \in I(\bar{x})} \nu_i \langle f_i \mid y \rangle, \quad (5.6)$$

where $I(\bar{x}) = \{i \in N \mid \langle f_i \mid \bar{x} \rangle = \eta_i\}$. Formula (5.6) is equivalent to the following two conditions

$$\exists \{\nu_i\}_{i \in I(\bar{x})} \geq 0 \quad \forall_{k \in N \setminus W} \quad |\bar{x}_k - x_k|^{p-2} (\bar{x}_k - x_k) = - \sum_{i \in I(\bar{x})} \lambda_k^i \nu_i, \quad (5.7)$$

$$\forall_{k \in W} \quad |\bar{x}_k - x_k|^{p-2} (\bar{x}_k - x_k) = 0. \quad (5.8)$$

These conditions are obtained by taking $y = e_k = (0, \dots, 0, 1, 0, \dots)$, $k \in \mathbb{N}$, in (5.6). Hence for all $k \in W$, $\bar{x}_k = x_k$. For any $i \in I(\bar{x})$ we have

$$\langle f_i \mid \bar{x} \rangle = \eta_i \iff \sum_{j \in \mathbb{N}} \lambda_j^i \bar{x}_j = \eta_i. \quad (5.9)$$

If $\det L_{I(\bar{x}), I(\bar{x})} = 0$, by applying Lemma 1 to vectors $u_i := [\lambda_k^i]_{k \in I(\bar{x})}$, $i \in I(\bar{x})$, we obtain the existence of an index set $I \subset I(\bar{x})$, $I \neq \emptyset$ such that $\det L_{I, I} \neq 0$ and

$$\forall i \in I(\bar{x}) \exists \tilde{\nu}_k > 0, k \in I \quad \sum_{k \in I(\bar{x})} \lambda_k^i \nu_k = \sum_{k \in I} \lambda_k^i \tilde{\nu}_k. \quad (5.10)$$

If $\det L_{I(\bar{x}), I(\bar{x})} \neq 0$ put $I := I(\bar{x})$. Let $\tilde{\eta}_i := \eta_i - \sum_{j \in \mathbb{N} \setminus I} \lambda_j^i \bar{x}_j$, $i \in I$. By (5.9), for any $i \in I$

$$\sum_{k \in \mathbb{N}} \lambda_k^i \bar{x}_k = \eta_i \iff \sum_{k \in I} \lambda_k^i \bar{x}_k = \eta_i - \sum_{j \in \mathbb{N} \setminus I} \lambda_j^i \bar{x}_j,$$

hence

$$L_{I, I}[\bar{x}_i]_{i \in I} = [\tilde{\eta}_i]_{i \in I},$$

and consequently

$$\bar{x}_i = \frac{1}{\det L_{I, I}} \sum_{k \in I(\bar{x})} \tilde{\eta}_k B_I^k B_I^i \det L_{I \setminus \{k\}, I \setminus \{i\}}, \quad i \in I.$$

Let $\xi_k := -|\bar{x}_k - x_k|^{p-2}(\bar{x}_k - x_k)$, $k \in I$. By (5.7) and (5.10),

$$\exists \tilde{\nu}_i > 0, i \in I \quad \forall k \in I \quad \sum_{i \in I} \lambda_k^i \tilde{\nu}_i = \xi_k.$$

Hence

$$\exists \tilde{\nu}_i > 0, i \in I \quad L_{I, I}[\tilde{\nu}_i]_{i \in I} = [\xi_i]_{i \in I},$$

and consequently

$$\tilde{\nu}_i = \frac{1}{\det L_{I, I}} \sum_{k \in I} \xi_k B_I^k B_I^i \det L_{I \setminus \{k\}, I \setminus \{i\}}, \quad i \in I.$$

Thus, for all $j \in \mathbb{N} \setminus (I \cup W)$

$$|\bar{x}_j - x_j|^{p-2}(\bar{x}_j - x_j) = - \sum_{i \in I} \lambda_j^i \frac{1}{\det L_{I, I}} \left(\sum_{k \in I} \xi_k B_I^k B_I^i \det L_{I \setminus \{k\}, I \setminus \{i\}} \right),$$

which completes the proof. \square

Remark 1. Let us note that if $\{m, m+1, \dots\} \subset W$, where $m \in \mathbb{N}$, then condition (5.5) is actually required for $i \in \{1, \dots, m\} \setminus (I \cup W)$ only.

Example 1. Recall that $N = \{1, \dots, n\}$. Let $J \subset N$, $J \neq \emptyset$. Let $|\delta_k| = 1$ if $k \in J$ and $\delta_k = 0$ otherwise. Consider $C_i = \{h \in \ell_p \mid \langle f_i \mid h \rangle \leq \eta_i\}$, where $f_i = \{0, \dots, 0, \delta_i, 0, \dots\} \in \ell_q$, $\eta_i \in \mathbb{R}$, $i \in N$, i.e. n is the highest index i such that $\delta_i \neq 0$ appearing in the sets C_i and $C_k = B$ for $k \in N \setminus J$.

Let \bar{x} be the projection of $x \in \ell_p \setminus C$ onto $C = \bigcap_{i \in N} C_i$, $C \neq \emptyset$. By (5.7)-(5.8) we obtain

$$\begin{aligned} \exists \{\nu_i\}_{i \in I(\bar{x})} \geq 0 \quad \forall k \in I(\bar{x}) \quad |\bar{x}_k - x_k|^{p-2}(\bar{x}_k - x_k) &= -\delta_k \nu_k \\ \forall i \in J \setminus I(\bar{x}) \quad \bar{x}_i &= x_i. \end{aligned}$$

We will show that \bar{z} given by

$$\bar{z} := x - \sum_{k \in J} \xi_k e_k, \text{ where } \xi_k := \begin{cases} x_k - \delta_k \eta_k & \text{when } \delta_k x_k \geq \eta_k, \\ 0 & \text{otherwise.} \end{cases} \quad (5.11)$$

is a projection of x onto C . For any $i \in I(\bar{x})$ we have

$$\langle f_i | \bar{x} \rangle = \eta_i \iff \delta_i \bar{x}_i = \eta_i \iff \bar{x}_i = \delta_i \eta_i.$$

Consider the case $j \notin I(\bar{x})$. Then $\bar{x}_j = x_j$, and moreover, if $j \in J$, then $x_j = \bar{x}_j \leq \eta_j$ since $\bar{x} \in C_j$.

Consider the case $i \in I(\bar{x})$ and $x_i > \delta_i \eta_i$. Then $\bar{x}_i = \delta_i \eta_i$ and by (5.7)

$$0 > |\delta_i \eta_i - x_i|^{p-2} (\delta_i \eta_i - x_i) = -\delta_i \nu_i.$$

Since $\nu_i \geq 0$, $\delta_i = 1$ and $\delta_i x_i > \eta_i$.

Consider the case $i \in I(\bar{x})$ and $x_i \leq \delta_i \eta_i$. Then $\bar{x}_i = \delta_i \eta_i$ and by (5.7)

$$0 \leq |\delta_i \eta_i - x_i|^{p-2} (\delta_i \eta_i - x_i) = -\delta_i \nu_i.$$

Since $\nu_i \geq 0$ one of the following appears:

- $\delta_i = 1$ and $x_i = \delta_i \eta_i = \bar{x}_i$,
- $\delta_i = -1$, $\delta_i x_i \geq \eta_i$.

Thus $\bar{x}_i = \delta_i \eta_i$ when $\delta_i x_i \geq \eta_i$, $i \in J$ and $\bar{x}_i = x_i$ otherwise, which proves (5.11).

Remark 2. Let $B = L^p(\Omega)$, $p \geq 1$. Let $f_i \in L^q(\Omega)$ and $\eta_i \in \mathbb{R}$, $i \in N$. Consider $C_i = \{g \in L^p \mid \langle f_i | g \rangle \leq \eta_i\}$, $i \in \{1, \dots, n\}$. Let $W := \{t \in \Omega \mid \forall i \in N, f_i(t) = 0\}$. Then by (5.2) and (5.4), \bar{x} is a projection of $x \in L^p \setminus C$ onto $C = \bigcap_{i \in N} C_i$ if and only if there exists $\nu_i \geq 0$, $i \in I(\bar{x})$ such that for all $y \in B$

$$\int_{\Omega} |\bar{x}(t) - x(t)|^{p-2} (\bar{x}(t) - x(t)) y(t) \mu(dt) = - \sum_{i \in I(\bar{x})} \nu_i \int_{\Omega} f_i(t) y(t) \mu(dt), \quad (5.12)$$

where $I(\bar{x}) = \{i \in N \mid \int_{\Omega} f_i(t) \bar{x}(t) \mu(dt) = \eta_i\}$.

Taking into account $y(t) = 0$ for $t \in \Omega \setminus W$ and $y(t) = \bar{x}(t) - x(t)$ for $t \in W$ in (5.12) we obtain

$$\int_W |\bar{x}(t) - x(t)|^p \mu(dt) = 0.$$

Hence $\bar{x}(t) = x(t)$ for almost all $t \in W$.

Remark 3. Let us note, that in nonreflexive spaces, the projection may not exist. In consequence, the conditions given in Remark 2, when applied to the space $L_1(\Omega)$ do not assure the existence of $\nu_i \geq 0$, $i \in I(\bar{x})$ such that for all $y \in B$ formula (5.12) holds. Proposition 3 concerns the spaces ℓ_p , $p > 1$ because in the space ℓ_1 we do not have the formula for the directional derivative of the norm.

6. CONCLUSIONS

The main advantage of our approach is that no requirement is needed for any mutual relationships between vectors u_i , $i \in N$, which generate the halfspaces. In our approach the problem of finding projection reduces to the problem of finding Kuhn-Tucker multipliers ν_1, \dots, ν_n , the number of which coincides with the number of halfspaces and is independent of the dimensionality of x . The crucial point of the result is that we work in Hilbert spaces. We showed through examples that, in general, in Banach space, even if the norm is Gateaux-differentiable and C is of particular form one cannot expect explicit formulas for projections onto C .

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