# Boundary feedback stabilization of the isothermal Euler-equations with uncertain boundary data 

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#### Abstract

In a gas transport system, the customer behavior is uncertain. Motivated by this situation, we consider a boundary stabilization problem for the flow through a gas pipeline, where the outflow at one end of the pipe is uncertain. The control action is located at the other end of the pipe. The feedback law is a classical Neumann velocity feedback with a feedback parameter $k>0$.

We show that as long as the $H^{1}$-norm of the function that describes the noise in the customer's behavior decays exponentially with a rate that is sufficiently large, the velocity of the gas can be stabilized exponentially fast in the sense that a suitably chosen Lyapunov function decays exponentially. For the exponential stability it is sufficient that the feedback parameter $k$ is sufficiently large and the stationary state to which the system is stabilized is sufficiently small. The stability result is local, that is it holds for initial states that are sufficiently close to the stationary state.

This result is an example for the exponential boundary feedback stabilization of a quasilinear hyperbolic system with uncertain boundary data. The analysis is based upon the choice of a suitably Lyapunov function. The decay of this Lyapunov function implies that also the $L^{2}$-norm of the difference of the system state and the stationary state decays exponentially.


Keywords: Boundary feedback stabilization, exponential decay, uncertainty, Lyapunov function, quasilinear hypberbolic system, wave equation, isothermal Euler-equations.

AMS Subject Classification: 76N25, 35L50, 93C20

[^0]
## 1 Introduction

There are many studies on boundary feedback stabilization where exponential stability is shown for a family initial states under the assumption that the action at the boundary is certain. In this paper we consider a different situation, where not only the initial state, but also the behavior at a part of the boundary of the system is uncertain. We show that if the uncertain boundary action converges sufficiently fast to a stationary state, during this process a suitably chosen Lyapunov function decays exponentially. As stated in [20], where semilinear parabolic systems are considered, Lyapunov function based techniques are central in the study of partial differential equations (pdes). In particular, in 20] input-to-state stable (ISS) Lyapunov functions for pdes with disturbances are considered. In this paper we consider a system with a quasilinear hyperbolic pde with uncertainties in the boundary data. The concept of input-to-state stability is discussed in detail in [21].

The transient behavior of pipeline gas flow can be modeled very precisely by a quasilinear hyperbolic partial differential equation. Often the consumer behavior in gas transport networks is uncertain. Therefore we consider a problem of boundary stabilization with uncertain boundary data

$$
b(t, \omega)
$$

that depend on an uncertain decision $\omega \in \Omega$ that models the uncertainty in the customer's behavior. This customer behavior can be considered as a noise in the boundary data. We assume that the consumer behavior approaches some desired state in $H^{1}$, that is that the noise has the following structure:

For all $\omega \in \Omega$ the function $b(\cdot, \omega)$ is twice continuously differentiable and there exists numbers $T_{\text {period }}>0$ and $T>T_{\text {period }}$, such that for all $t \in$ $\left(T_{\text {period }}, T\right)$

$$
\begin{equation*}
\int_{t-T_{\text {period }}}^{t}|b(\tau, \omega)|^{2}+\left|b_{t}(\tau, \omega)\right|^{2} d \tau \leq C_{\nu} \exp (-\nu t) \tag{1}
\end{equation*}
$$

for some $\nu>0$ and $C_{\nu}>0$. This means that after some variations the noise in the customer behavior decays exponentially fast to zero.

We consider the questions: How does the uncertainty in the demand influence the stabilization of the system? Is there a boundary feedback law that leads to exponential decay of the difference of the system state and a desired stationary state in spite of the uncertain boundary data?

For this purpose we consider a boundary feedback law that for the case without noise, that is for $b(t, \omega)=0$ for all $t \geq 0$, stabilizes the system exponentially fast.

Output-feedback stabilization of stochastic nonlinear systems driven by noise of unknown covariance has been studied in 7] where a controller is presented that guarantees regulation to the desired state with probability one. Our result is of a different type. We present a Lyapunov function that decays exponentially
under suitable smallness assumptions for the variations in the consumer behavior. Our Lyapunov function for the case with uncertainty is a time average of a strict Lyapunov function for the case without uncertainty.

In [8] a strict $H^{1}$-Lyapunov function and feedback stabilization for the isothermal Euler equations with friction have been studied without uncertainty. The stabilization of the gas flow in pipeline networks has been considered in [11]. The novelty in the present contribution is that we include uncertainty in the boundary data in our analysis. The isothermal Euler equations are equivalent to a quasilinear wave equation for the gas velocity. Therefore for the stabilization of our system we consider the same feedback law that is used for the stabilization for the linear wave equation that has been studied for example in [15]. For the linear wave equation, this feedback law stabilizes the system as long as the feedback parameter has the right sign.

In our system, due to its nonlinearity, the stationary states are not constant and blow up after a finite critical length. Due to the nonlinearity of the system, in our analysis we have to assume that the stationary states are sufficiently small and we can only stabilize the system locally around the stationary state. To guarantee the exponential decay of the Lyapunov function, we have to assume that the feedback parameter is sufficiently large. Apart from the obvious restriction that the length of the pipe is less than the critical length, in this paper we do not impose any additional restrictions on the length of the pipe. This is in contrast to the earlier contributions 9 and 13 that are only applicable if the lengths of the pipes are sufficiently small.

## 2 The model for the pipeline flow

Let a finite time $T>0$ be given. The system dynamics for the flow of ideal gas in a single pipe can be modeled by the isothermal Euler equations (see [2], [3], [8]):

$$
\begin{align*}
& \rho_{t}+q_{x}=0  \tag{2}\\
& q_{t}+\left(\frac{q^{2}}{\rho}+a^{2} \rho\right)_{x}=-\frac{1}{2} \theta \frac{q|q|}{\rho} \tag{3}
\end{align*}
$$

where $\rho=\rho(t, x)>0$ is the density of the gas, $q=q(t, x)$ is the mass flux, $\theta=\frac{f_{g}}{\delta}$ where the constant $f_{g} \geq 0$ is a friction factor and $\delta>0$ is the diameter of the pipe. The constant $a>0$ is the speed of sound in the gas. We consider the equations on the domain $\Omega:=[0, T] \times[0, L]$ where $L>0$ and $T>0$ are given. Equation (2) states the conservation of mass and equation (3) is the balance of momentum. Define the velocity $\tilde{u}$ of the gas flow as

$$
\begin{equation*}
\tilde{u}=\frac{q}{\rho} . \tag{4}
\end{equation*}
$$

In this paper, we consider subsonic positive gas flow, that is we assume that

$$
\begin{equation*}
0<\tilde{u}<a \tag{5}
\end{equation*}
$$

Note that in the operation of the gas pipelines, there are strict upper bounds for the velocities in order to avoid noise pollution by pipeline vibrations that can be generated by the flow, see [22]. For sufficiently regular states, $\tilde{u}$ satisfies the quasilinear wave equation (see [14])

$$
\begin{equation*}
\tilde{u}_{t t}+2 \tilde{u} \tilde{u}_{t x}-\left(a^{2}-\tilde{u}^{2}\right) \tilde{u}_{x x}=\tilde{F}\left(\tilde{u}, \tilde{u}_{x}, \tilde{u}_{t}\right) . \tag{6}
\end{equation*}
$$

The lower order term is

$$
\begin{equation*}
\tilde{F}\left(\tilde{u}, \tilde{u}_{x}, \tilde{u}_{t}\right)=-2 \tilde{u}_{t} \tilde{u}_{x}-2 \tilde{u} \tilde{u}_{x}^{2}-\frac{3}{2} \theta \tilde{u}|\tilde{u}| \tilde{u}_{x}-\theta|\tilde{u}| \tilde{u}_{t} . \tag{7}
\end{equation*}
$$

We consider stationary states $\bar{u}$ of (6) that are solutions of the ordinary differential equation

$$
\bar{u}_{x}=\frac{\theta}{2} \frac{1}{\left(a^{2}-\bar{u}^{2}\right)}|\bar{u}| \bar{u}^{2}
$$

Obviously in the subsonic case these solutions are strictly increasing. These stationary states correspond to stationary states of the system (2), (3) and are discussed in detail in [10]. In fact with $\sigma \in\{-1,1\}$ that determines the direction of the flow and a real constant $c_{1}<-1$ the stationary states have the form

$$
\begin{equation*}
\bar{u}(x)=\frac{\sigma a}{\sqrt{-W_{-1}\left(-\exp \left(\sigma \theta x+c_{1}\right)\right)}} \tag{8}
\end{equation*}
$$

where $W_{-1}$ is the Lambert-W function (see [17, [5]), that is the inverse function of $x \mapsto x \exp (x)$ for $x \leq-1$. Thus $W_{-1}$ is defined on $\left(-\frac{1}{\mathrm{e}}, 0\right)$.

For positive gas flow we have $\sigma=1$. The representation (8) implies that for $\sigma=1$, the solution $\bar{u}(x)$ exists only for $x \leq L_{\text {crit }}$ with a critical length $L_{\text {crit }}$ and at $L_{\text {crit }}$, the flow becomes sonic that is $\bar{u}\left(L_{\text {crit }}\right)=a$ and the derivative blows up.

To stabilize the system governed by the quasilinear wave equation (6) locally around a desired stationary state $\bar{u}(x)$, at $x=0$ we use the Neumann boundary feedback law

$$
\begin{equation*}
\tilde{u}_{x}(t, 0)=\bar{u}_{x}(0)+k \tilde{u}_{t}(t, 0) \tag{9}
\end{equation*}
$$

with a feedback parameter $k \in(0, \infty)$. At $x=L$, the Dirichlet boundary condition is

$$
\begin{equation*}
\tilde{u}(t, L)=b(t, \omega)+\bar{u}(L) \tag{10}
\end{equation*}
$$

The feedback law (9) is similar to the feedback law for the stabilization of the linear wave equation that is studied for example in [15], [16] and [12]. Define

$$
\begin{equation*}
u=\tilde{u}-\bar{u} \tag{11}
\end{equation*}
$$

that is $u$ is the difference of the velocity and the stationary velocity. Our system stated in terms of $u$ is

$$
\left\{\begin{array}{l}
u(0, x)=\varphi(x), x \in[0, L]  \tag{12}\\
u_{t}(0, x)=\psi(x), x \in[0, L] \\
u_{t t}+2(\bar{u}+u) u_{t x}-\left(a^{2}-(\bar{u}+u)^{2}\right) u_{x x}=F\left(x, u, u_{x}, u_{t}\right) \text { on }[0, T] \times[0, L] \\
u_{x}(t, 0)=k u_{t}(t, 0), t \in[0, T] \\
u(t, L)=b(t, \omega), t \in[0, T]
\end{array}\right.
$$

where $F:=F\left(x, u, u_{x}, u_{t}\right)$ satisfies

$$
\begin{equation*}
F=\tilde{F}\left(u+\bar{u}, u_{x}+\bar{u}_{x}, u_{t}\right)-\frac{a^{2}-(\bar{u}+u)^{2}}{a^{2}-\bar{u}^{2}} \tilde{F}\left(\bar{u}, \bar{u}_{x}, 0\right) \tag{13}
\end{equation*}
$$

## 3 Well-posedness

In 19 a result about semi-global $C^{2}$-solutions of quasilinear wave equations is proved that we can apply to show the well-posedness of (12):

Theorem 3.1 Let $L>0, a>0, k>0$ and a subsonic stationary state $\bar{u} \in$ $C^{1}([0, L])$ be given such that $\bar{u}(x) \in(0, a)$. Choose $T>0$ arbitrarily large.

There exist constants $\varepsilon_{0}(T)>0$ and $C_{T}>0$, such that if the initial data $\left(u(0, x), u_{t}(0, x)\right)=(\varphi(x), \psi(x)) \in C^{2}([0, L]) \times C^{1}([0, L])$ and $b(\cdot, \omega)$ satisfy

$$
\begin{equation*}
\left.\max \left\{\|\varphi(x)\|_{C^{2}([0, L])}, \| \psi(x)\right)\left\|_{C^{1}([0, L])},\right\| b(\cdot, \omega) \|_{C^{2}([0, T])}\right\} \leq \varepsilon_{0}(T) \tag{14}
\end{equation*}
$$

and the $C^{2}$-compatibility conditions are satisfied at the points $(t, x)=(0,0)$ and $(0, L)$, then the initial-boundary problem (12) has a unique solution $u(t, x) \in$ $C^{2}([0, T] \times[0, L])$. Moreover the following a priori estimate holds:
$\left.\|u\|_{C^{2}([0, T] \times[0, L])} \leq C_{T} \max \left\{\|\varphi(x)\|_{C^{2}([0, L])}, \| \psi(x)\right)\left\|_{C^{1}([0, L])},\right\| b(\cdot, \omega) \|_{C^{2}([0, T])}\right\}$.

## 4 Exponential Decay

In order to show the exponential decay for the closed-loop system (12) with the quasilinear wave equation, we define
$E_{1}(t)=\int_{0}^{L} k\left[\left(a^{2}-(\bar{u}+u)^{2}\right) u_{x}^{2}+u_{t}^{2}\right]-2 \exp \left(-\frac{x}{L}\right)\left[(\bar{u}+u) u_{x}^{2}+u_{t} u_{x}\right] d x$.
Note that the first term of $E_{1}(t)$ is similar to the classical energy

$$
E_{\text {classic }}(t)=k \int_{0}^{L} a^{2} u_{x}^{2}+u_{t}^{2} d x
$$

The characteristic curves for the quasilinear wave equation in (12) have the nonconstant slopes $\lambda_{+}=\bar{u}+u+a$ and $\lambda_{-}=\bar{u}+u-a$, so for the product of the eigenvalues we have $\left|\lambda_{+} \lambda_{-}\right|=a^{2}-(\bar{u}+u)^{2}$. For the linear wave equation, this corresponds to the constant $|a(-a)|$. Therefore the constant $a^{2}$ in the definition of $E_{\text {classic }}(t)$ is replaced by the function $\left|\lambda_{+} \lambda_{-}\right|$in the definition of $E_{1}(t)$. The second term is added in the integral in $E_{1}(t)$ since in the quasilinear wave equation in (12), the mixed partial derivative $u_{t x}$ appears. The corresponding coefficient is given by the sum of the eigenvalues $\lambda_{+}+\lambda_{-}=2(\bar{u}+u)$. For the linear wave equation, this corresponds to the constant $a+(-a)=0$, therefore the mixed partial derivative does not appear. Exponential weights like $\exp \left(-\frac{x}{L}\right)$
have already been used to construct strict Lyapunov functions for hyperbolic pdes, see for example [4, 6.

The following lemma shows that $\sqrt{E_{1}}$ is equivalent to the $L^{2}-$ norm of $\left(u_{t}, u_{x}\right)$.
Lemma 4.1 Assume that

$$
\begin{equation*}
0 \leq \bar{u}+u \leq \frac{a}{2} \tag{17}
\end{equation*}
$$

and that $k>0$ is sufficiently large such that

$$
\begin{equation*}
M_{1}:=\min \left\{\frac{3}{4} k a^{2}-a-1, k-1\right\}>0 \tag{18}
\end{equation*}
$$

Define

$$
\begin{align*}
K_{1} & :=\frac{1+2 L^{2}}{M_{1}}  \tag{19}\\
K_{2} & :=\max \left\{k a^{2}+a+1, k+1\right\}>0 \tag{20}
\end{align*}
$$

Then we have the inequalities

$$
\begin{align*}
M_{1} \int_{0}^{L}\left(u_{t}^{2}+u_{x}^{2}\right) d x \leq E_{1}(t) & \leq K_{2} \int_{0}^{L}\left(u_{t}^{2}+u_{x}^{2}\right) d x  \tag{21}\\
\int_{0}^{L}\left(u_{t}^{2}+\left(1+2 L^{2}\right) u_{x}^{2}\right) d x & \leq K_{1} E_{1}(t) \tag{22}
\end{align*}
$$

Note that inequality (22) is used in the proof of Theorem 4.1 to obtain (61). Proof. Using Young's inequality we obtain

$$
\begin{aligned}
E_{1}(t) & \geq \int_{0}^{L} k \frac{3}{4} a^{2} u_{x}^{2}+k u_{t}^{2}-2 \frac{a}{2} u_{x}^{2}-u_{x}^{2}-u_{t}^{2} d x \\
& =\int_{0}^{L}\left(\frac{3}{4} k a^{2}-a-1\right) u_{x}^{2}+(k-1) u_{t}^{2} d x \\
& \geq M_{1} \int_{0}^{L} u_{x}^{2}+u_{t}^{2} d x \\
& \geq \frac{M_{1}}{1+2 L^{2}} \int_{0}^{L}\left(u_{t}^{2}+\left(1+2 L^{2}\right) u_{x}^{2}\right) d x
\end{aligned}
$$

and the first inequality in (21) and (22) follow. Using Young's inequality we also obtain

$$
\begin{aligned}
E_{1}(t) & \leq \int_{0}^{L} k a^{2} u_{x}^{2}+k u_{t}^{2}+2 \frac{a}{2} u_{x}^{2}+u_{x}^{2}+u_{t}^{2} d x \\
& =\int_{0}^{L}\left(k a^{2}+a+1\right) u_{x}^{2}+(k+1) u_{t}^{2} d x \\
& \leq K_{2} \int_{0}^{L} u_{x}^{2}+u_{t}^{2} d x
\end{aligned}
$$

and the second inequality in (21) follows.
For $t \geq T_{\text {period }}>0$ we consider the Lyapunov function

$$
\begin{equation*}
E(t)=\int_{t-T_{\text {period }}}^{t} E_{1}(\tau) d \tau \tag{23}
\end{equation*}
$$

By definition (23), $E(t)$ can be considered as a kind of moving horizon time average of $E_{1}(t)$. We consider the time average since the uncertain boundary data $b(\cdot, \omega)$ is distributed on the time interval and condition (1) allows large values of $|b(\cdot, \omega)|$ on short time intervals.

We consider the situation where on a finite time interval $[0, T]$, the noise in the customer behavior approaches zero with respect to the $H^{1}$-norm as in (11). We assume that during this process, the $C^{2}-$ norm $\|b(t, \omega)\|_{C^{2}(0, T)}$ is sufficiently small. Then due to Theorem 3.1, a semi-global $C^{2}$-solution exists for sufficiently small initial data that are $C^{2}$-compatible with the boundary conditions. Moreover, the a priori estimate (15) holds which implies that by further decreasing the norms of the initial and the boundary data, we can make the $C^{2}$-norm of the solution $u$ as small as desired.

In Theorem4.1 we state that if $k>0$ is sufficiently large under appropriate smallness conditions on $\bar{u}$ and $u$ and if the uncertain customer profile $b(\cdot, \omega)$ satisfies (1) with sufficiently large $\nu$ and $C_{\nu}$ sufficiently small, our Lyapunov function $E(t)$ defined in (23) decays exponentially fast as in (30) with a rate $\mu$ that is independent of $T$.

Theorem 4.1 Let $L>0$ and $\lambda \in\left(\frac{1}{2}, 1\right)$ be given. Assume that $k$ is sufficiently large such that

$$
\begin{equation*}
k \geq \max \left\{1, \frac{4}{3}\left(\frac{1}{a}+\frac{1}{a^{2}}\right), \frac{1}{\lambda a}\right\} \tag{24}
\end{equation*}
$$

Define

$$
\begin{equation*}
\mu=\frac{1}{4 \mathrm{e} L k} \tag{25}
\end{equation*}
$$

and the constant

$$
\begin{equation*}
C_{0}=12 k+4(k+1)\left(18+13 \theta+\frac{1}{a^{2}}(8+6 \theta)\right)+10 . \tag{26}
\end{equation*}
$$

Let a stationary state $\bar{u}(x)>0, \bar{u} \in C^{1}(0, L)$ be given. Assume that

$$
\begin{equation*}
\bar{u}(x) \leq \min \left\{1, \frac{1}{4 k \mathrm{e}},(1-\lambda) \frac{a}{2}, \frac{\mu}{C_{0} K_{1}}\right\}, \bar{u}_{x}(x) \leq \min \left\{1, \frac{\mu}{C_{0} K_{1}}\right\} \tag{27}
\end{equation*}
$$

with $K_{1}$ as defined in (19). Let $T_{\text {period }}>0$ and $T>T_{\text {period }}$ be given. Assume that the initial data of system (12) and $b(\cdot, \omega)$ satisfy (14) and the $C^{2}-$ compatibility conditions such that Theorem 3.1 implies that (12) has a $C^{2}$ solution on $[0, T] \times[0, L]$ that satisfies the a-priori estimate (15). Hence we can
assume that $\|\varphi(x)\|_{C^{2}(0, L)},\|\psi(x)\|_{C^{1}(0, L)}$ and $\|b(t, \omega)\|_{C^{2}(0, T)}$ are sufficiently small such that

$$
\begin{equation*}
|u| \leq \min \left\{\bar{u}(0),(1-\lambda) \frac{a}{2}, \frac{1}{4 k \mathrm{e}}\right\}, \max \left\{|u|,\left|u_{x}\right|,\left|u_{t}\right|\right\} \leq \min \left\{1, \frac{\mu}{C_{0} K_{1}}\right\} \tag{28}
\end{equation*}
$$

Assume that the uncertain function $b(t, \omega)$ satisfies (1) with $\nu>\mu$. Define

$$
\begin{equation*}
\delta=\nu-\mu>0 . \tag{29}
\end{equation*}
$$

Then for all $t \in\left(T_{\text {period }}, T\right)$ for the solution of (12) we have the inequality

$$
\begin{equation*}
E(t) \leq \exp \left(-\mu\left(t-T_{\text {period }}\right)\right)\left[E\left(T_{\text {period }}\right)+\frac{C_{g}}{\delta}\right] \tag{30}
\end{equation*}
$$

with $C_{g}>0$ defined by

$$
\begin{equation*}
C_{g}=\left(\frac{4}{3} \mathrm{e} a^{2} k^{2}+\frac{1}{2 \mathrm{e} K_{1} k}\right) C_{\nu} . \tag{31}
\end{equation*}
$$

In this sense $E(t)$ as defined in (23) decays exponentially with the rate $\mu$ that is independent of $T$. If we have

$$
\begin{equation*}
b(t, \omega)=0 \text { for all } t \geq T-T_{\text {period }}, \tag{32}
\end{equation*}
$$

then

$$
\begin{equation*}
\|u\|_{H^{1}\left(\left(T-T_{\text {period }}, T\right) \times(0, L)\right)}^{2} \leq K_{1} \exp \left(-\mu\left(T-T_{\text {period }}\right)\right)\left[E\left(T_{\text {period }}\right)+\frac{C_{g}}{\delta}\right] \tag{33}
\end{equation*}
$$

Remark 1 Assumption (1) means that the $H^{1}$-norm of the function that describes the noise in the customer behavior must decay exponentially fast. To guarantee the exponential decay of $E$ with the rate $\mu$ we assume that also the squared $H^{1}$-norm of $b(\cdot, \omega)$ decays exponentially with a rate $\nu$ that is greater than $\mu$. This condition holds if after a finite time, the customer behavior becomes almost stationary.

Remark 2 Since the conditions on $k$ do not depend on $T$, the decay rate $\mu$ as defined in (25) does not depend on $T$. Thus we can choose for example the time $T_{1 / 2}=\frac{1}{\mu} \ln \left(2 K_{1} K_{2}\right)+T_{\text {period }}$. Then $\exp \left(-\mu T_{1 / 2}\right)=\frac{1}{2 K_{1} K_{2}} \exp \left(-\mu T_{\text {period }}\right)$. For $t>T_{\text {period }}$, we introduce the notation $X(t)=H^{1}\left(\left(t-T_{\text {period }}, t\right) \times(0, L)\right)$.

If for all $t \geq T-T_{\text {period }}$, condition (32) holds and $T_{1 / 2}>T_{\text {period, }}$ (33) implies

$$
\|u\|_{X\left(T_{1 / 2}\right)}^{2} \leq \frac{1}{2}\left[\int_{0}^{T_{\text {period }}} \int_{0}^{L} u_{t}^{2}(\tau, x)+u_{x}^{2}(\tau, x) d x d \tau+\frac{1}{K_{2}} \frac{C_{g}}{\delta}\right]
$$

By a trace theorem (see [1], [18]) we have the inequality $\int_{0}^{L}|u(t, x)|^{2} d x \leq$ $C_{e}\|u\|_{X(t)}$ with an embedding constant $C_{e}$. Hence (33) yields an upper bound for $\|u(T, \cdot)\|_{L^{2}(0, L)}$.

Remark 3 The exponential decay in Theorem 4.1 can be interpreted as an exponential decay of the function

$$
H(t)=\int_{t-T_{\text {period }}}^{t} \int_{0}^{L} u^{2}+u_{x}^{2}+u_{t}^{2} d x d \tau=\|u\|_{H^{1}\left(\left(t-T_{\text {period }}, t\right) \times(0, L)\right)}^{2}
$$

In fact, for all $t \in\left(T_{\text {period }}, T\right)$ inequality (62) from the proof of Theorem 4.1 yields

$$
\begin{align*}
& H(t) \leq K_{1} \exp \left(-\mu\left(t-T_{\text {period }}\right)\right)\left[E\left(T_{\text {period }}\right)+\frac{C_{g}}{\delta}\right]+2 L C_{\nu} \exp (-\nu t)  \tag{34}\\
= & K_{1} \exp \left(-\mu\left(t-T_{\text {period }}\right)\right)\left[E\left(T_{\text {period }}\right)+\frac{C_{g}}{\delta}+2 \frac{L C_{\nu}}{K_{1}} \exp \left(-\delta t-\mu T_{\text {period }}\right)\right] .
\end{align*}
$$

On account of the trace theorem mentioned in Remark 2 and the definition of $H(t)$, (34) implies that the $L^{2}$-norm of the system state decays exponentially fast with the rate $\mu$.

Remark 4 It is interesting to compare the decay rate $\mu$ from (25) with the decay rate that is achieved for the system with the linear wave equation for $(t, x) \in(0, \infty) \times(0, L)$ that has already been studied in [15]:

$$
\left\{\begin{array}{l}
u(0, x)=\varphi(x) \\
u_{t}(0, x)=\psi(x) \\
u_{t t}-a^{2} u_{x x}=0 \\
u_{x}(t, 0)=k u_{t}(t, 0) \\
u(t, L)=0
\end{array}\right.
$$

The discussion in [14], (Chapter 5.2) implies that for $k>\frac{1}{a}, \varphi \in H^{1}((0, L))$ with $\varphi(L)=0$ and $\psi \in L^{2}((0, L))$ the classical energy $E_{\text {classic }}$ decays exponentially and the optimal rate is $\mu_{0}=\frac{a}{L} \ln \left(1+\frac{2}{a k-1}\right)$. Thus we have

$$
\frac{\mu_{0}}{\mu}=4 \mathrm{e} \ln \left(\left(1+\frac{2}{a k-1}\right)^{a k}\right)
$$

Hence $\lim _{(a k) \rightarrow \infty} \frac{\mu_{0}}{\mu}=8 e$. Thus asymptotically for large values of $(a k)$ the decay rate $\mu$ for the quasilinear system differs from the rate for the linear wave equation only by a multiplicative constant that is close to 8 e .

For the proof of Theorem 4.1, we use the following variant of Gronwall's Lemma, that we prove for the convenience of the reader.

Lemma 4.2 (Gronwall's Lemma) Let real numbers $\mu>0, \nu>0$ and $C_{g}>$ 0 be given such that $\nu>\mu$. Define

$$
\delta=\nu-\mu>0
$$

Assume that $U(t) \geq 0$ is a differentiable function that satisfies for all $t \in[0, T]$ the inequality

$$
\begin{equation*}
U^{\prime}(t) \leq-\mu U(t)+C_{g} \exp (-\nu t) \tag{35}
\end{equation*}
$$

Then for all $t \in[0, T]$ the function $U(t)$ satisfies the inequality

$$
\begin{equation*}
0 \leq U(t) \leq \exp (-\mu t)\left(U(0)+\frac{C_{g}}{\delta}\right) \tag{36}
\end{equation*}
$$

Proof. We define the auxiliary function

$$
\begin{equation*}
\mathcal{H}(t)=\exp (\mu t) U(t) \tag{37}
\end{equation*}
$$

Then we have $\mathcal{H}(0)=U(0)$. The product rule and (35) imply

$$
\begin{aligned}
\mathcal{H}^{\prime}(t) & =\mu \mathcal{H}(t)+\exp (\mu t) U^{\prime}(t) \\
& \leq \mu \mathcal{H}(t)+\exp (\mu t)\left[-\mu U(t)+C_{g} \exp (-\nu t)\right] \\
& =\mu \mathcal{H}(t)-\mu \mathcal{H}(t)+C_{g} \exp ((\mu-\nu) t) \\
& =C_{g} \exp (-\delta t)
\end{aligned}
$$

By integration the inequality $\mathcal{H}^{\prime}(t) \leq C_{g} \exp (-\delta t)$ yields

$$
\begin{aligned}
\mathcal{H}(t)-\mathcal{H}(0) & =\int_{0}^{t} \mathcal{H}^{\prime}(\tau) d \tau \\
& \leq \int_{0}^{t} C_{g} \exp (-\delta \tau) d \tau=C_{g} \frac{1}{\delta}\left(1-\mathrm{e}^{-\delta t}\right)
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
U(t) & =\mathrm{e}^{-\mu t} \mathcal{H}(t) \\
& \leq \mathrm{e}^{-\mu t}\left(\mathcal{H}(0)+C_{g} \frac{1}{\delta}\left(1-\mathrm{e}^{-\delta t}\right)\right) \\
& \leq \mathrm{e}^{-\mu t}\left(U(0)+\frac{C_{g}}{\delta}\right)
\end{aligned}
$$

and (36) follows.
In the subsequent analysis an upper bound for $F$ is used that is presented in the following lemma.
Lemma 4.3 Define

$$
\begin{equation*}
T_{L i}(t)=\max _{x \in[0, L]}\left\{|u(t, x)|,\left|u_{x}(t, x)\right|,\left|u_{t}(t, x)\right|,|\bar{u}(x)|,\left|\bar{u}_{x}(x)\right|\right\} \tag{38}
\end{equation*}
$$

Assume that

$$
\begin{equation*}
\bar{u}+u \geq 0,0<\bar{u} \leq \frac{a}{2}, T_{L i}(t) \leq 1 \tag{39}
\end{equation*}
$$

Then we have the equation

$$
\begin{equation*}
F=-2 u_{t}\left(u_{x}+\bar{u}_{x}\right)-\theta(u+\bar{u}) u_{t}-2 u\left(u_{x}+\bar{u}_{x}\right)^{2}-4 \bar{u} \bar{u}_{x} u_{x}-2 \bar{u} u_{x}^{2} \tag{40}
\end{equation*}
$$

$$
-\frac{3}{2} \theta u(u+2 \bar{u})\left(u_{x}+\bar{u}_{x}\right)-\frac{3}{2} \theta \bar{u}^{2} u_{x}-\frac{2 u \bar{u}+u^{2}}{a^{2}-\bar{u}^{2}}\left(2 \bar{u} \bar{u}_{x}^{2}+\frac{3}{2} \theta \bar{u}^{2} \bar{u}_{x}\right) .
$$

and the upper bound

$$
\begin{gather*}
\left|F\left(x, u(t, \cdot), u_{x}(t, \cdot), u_{t}(t, \cdot)\right)\right|  \tag{41}\\
\leq\left[18+13 \theta+\frac{1}{a^{2}}(8+6 \theta)\right] T_{L i}(t)\left(|u(t, x)|+\left|u_{x}(t, x)\right|+\left|u_{t}(t, x)\right|\right) .
\end{gather*}
$$

Proof. Equation (40) follows with the definition of $\tilde{F}$, using the assumptions $\bar{u}+u>0, \bar{u}>0$ from (39) to eliminate the absolute value brackets. From (40) we obtain the upper bound (41) using again the conditions from (39) that imply $T_{L i}(t)^{2} \leq T_{L i}(t)$.

Now we can proceed with the proof of Theorem 4.1 where we show that $E(t)$ satisfies a differential inequality of the type (35) and thus due to (29) we can apply Lemma 4.2.

Proof of Theorem 4.1. Define the exponential weight

$$
h_{2}(x)=\exp \left(-\frac{x}{L}\right), x \in[0, L] .
$$

The time-derivative of $E_{1}$ is given by the equation

$$
\begin{equation*}
\frac{d}{d t} E_{1}(t)=I_{1}+I_{2}+I_{3} \tag{42}
\end{equation*}
$$

with

$$
\begin{align*}
I_{1} & =\int_{0}^{L} h_{2 x}\left(a^{2}-(\bar{u}+u)^{2}\right) u_{x}^{2}+h_{2 x} u_{t}^{2} d x  \tag{43}\\
I_{2} & =\int_{0}^{L} 2 k\left(\bar{u}_{x}+u_{x}\right) u_{t}^{2}-2 k(\bar{u}+u) u_{t} u_{x}^{2}+4 k(\bar{u}+u)\left(\bar{u}_{x}+u_{x}\right) u_{x}  \tag{444}\\
& +2 k F u_{t}-2 h_{2} u_{t} u_{x}^{2}-2 h_{2}(\bar{u}+u)\left(\bar{u}_{x}+u_{x}\right) u_{x}^{2}-2 h_{2} F u_{x} d x \\
I_{3} & =\left[\left(a^{2}-(\bar{u}+u)^{2}\right)\left(2 k u_{x} u_{t}-h_{2} u_{x}^{2}\right)-\left(2 k(\bar{u}+u)+h_{2}\right) u_{t}^{2}\right]_{x=0}^{L} . \tag{45}
\end{align*}
$$

This can be seen as follows. With the notation

$$
\begin{equation*}
\hat{d}=a^{2}-(\bar{u}+u)^{2} \tag{46}
\end{equation*}
$$

we have $\hat{d}_{t}=-2(\bar{u}+u) u_{t}, \hat{d}_{x}=-2(\bar{u}+u)\left(\bar{u}_{x}+u_{x}\right)$ and

$$
E_{1}(t)=\int_{0}^{L} k\left(\hat{d} u_{x}^{2}+u_{t}^{2}\right)-2 h_{2}\left((\bar{u}+u) u_{x}^{2}+u_{t} u_{x}\right) d x
$$

Hence differentiation yields

$$
\begin{aligned}
\frac{d}{d t} E_{1}(t) & =\int_{0}^{L} 2 k\left[\left(u_{t t}-(\bar{u}+u) u_{x}^{2}\right) u_{t}+\hat{d} u_{x} u_{x t}\right] \\
& -2 h_{2}\left[u_{t} u_{x}^{2}+\left(u_{t t}+2(\bar{u}+u) u_{x t}\right) u_{x}+u_{t} u_{x t}\right] d x
\end{aligned}
$$

Now integration by parts for the term $\hat{d} u_{x} u_{x t}=\left(\hat{d} u_{x}\right)\left(u_{t}\right)_{x}$ yields the equation

$$
\begin{aligned}
\frac{d}{d t} E_{1}(t) & =\int_{0}^{L} 2 k\left[u_{t t}-\hat{d} u_{x x}-\hat{d}_{x} u_{x}-(\bar{u}+u) u_{x}^{2}\right] u_{t} \\
& -2 h_{2}\left[u_{t} u_{x}^{2}+\left(u_{t t}+2(\bar{u}+u) u_{x t}\right) u_{x}+u_{t} u_{x t}\right] d x+\left[2 k \hat{d} u_{x} u_{t}\right]_{x=0}^{L}
\end{aligned}
$$

Hence we get the equation

$$
\begin{aligned}
\frac{d}{d t} E_{1}(t) & =\int_{0}^{L} 2 k\left[\left(u_{t t}-\hat{d} u_{x x}\right) u_{t}-\hat{d}_{x} u_{x} u_{t}-(\bar{u}+u) u_{x}^{2} u_{t}\right] \\
& -2 h_{2}\left[u_{t} u_{x}^{2}+\left(u_{t t}+2(\bar{u}+u) u_{t x}\right) u_{x}+u_{t} u_{t x}\right] d x \\
& +\left[2 k \hat{d} u_{x} u_{t}\right]_{x=0}^{L}
\end{aligned}
$$

By the partial differential equation in (12) we have $u_{t t}-\hat{d} u_{x x}=F-2(\bar{u}+u) u_{t x}$ and obtain

$$
\begin{aligned}
\frac{d}{d t} E_{1}(t) & =\int_{0}^{L} 2 k\left[\left(F-2(\bar{u}+u) u_{t x}\right) u_{t}+2(\bar{u}+u)\left(\bar{u}_{x}+u_{x}\right) u_{t} u_{x}-(\bar{u}+u) u_{t} u_{x}^{2}\right] \\
& -2 h_{2}\left[\left(F+\hat{d} u_{x x}\right) u_{x}+u_{t} u_{x}^{2}+u_{t} u_{t x}\right] d x+\left[2 k \hat{d} u_{x} u_{t}\right]_{x=0}^{L}
\end{aligned}
$$

Using integration by parts we obtain the identities

$$
\begin{aligned}
\int_{0}^{L}-4 k(\bar{u}+u) u_{t} u_{t x} d x & =\int_{0}^{L}-2 k(\bar{u}+u)\left(u_{t}^{2}\right)_{x} d x \\
& =\left[-2 k u_{t}^{2}(\bar{u}+u)\right]_{x=0}^{L}+\int_{0}^{L} 2 k(\bar{u}+u)_{x} u_{t}^{2} d x
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{0}^{L}-2 & h_{2} \hat{d} u_{x} u_{x x}-2 h_{2} u_{t} u_{t x} d x=\int_{0}^{L}-h_{2} \hat{d}\left(u_{x}^{2}\right)_{x}-h_{2}\left(u_{t}\right)_{x}^{2} d x \\
& =\left[-h_{2} \hat{d} u_{x}^{2}-h_{2} u_{t}^{2}\right]_{x=0}^{L} \\
& +\int_{0}^{L} h_{2 x} \hat{d} u_{x}^{2}-2 h_{2}(\bar{u}+u)\left(\bar{u}_{x}+u_{x}\right) u_{x}^{2}+h_{2 x}\left(u_{t}\right)^{2} d x
\end{aligned}
$$

Using these identities we obtain equation (42). Here, $I_{3}$ contains all the terms coming from the boundary and $I_{1}=\int_{0}^{L} h_{2 x} \hat{d} u_{x}^{2}+h_{2 x} u_{t}^{2} d x$ contains all the terms where $h_{2 x}$ appears. The remaining terms appear in $I_{2}$.

We use the notation $\bar{u}_{0}=\bar{u}(0)$ and $\bar{u}_{L}=\bar{u}(L)$. We have $I_{3}=I_{3}^{L}-I_{3}^{0}$ with

$$
\begin{equation*}
I_{3}^{0}=\left(a^{2}-\left(\left(\bar{u}_{0}+u(t, 0)\right)+\frac{1}{k}\right)^{2}\right) u_{x}^{2}(t, 0) \tag{47}
\end{equation*}
$$

and

$$
\begin{aligned}
I_{3}^{L} & =\left[a^{2}-\left(\bar{u}_{L}+u(t, L)\right)^{2}\right]\left[2 k u_{x}(t, L) u_{t}(t, L)-\exp (-1) u_{x}(t, L)^{2}\right] \\
& -\left[2 k\left(\bar{u}_{L}+u(t, L)\right)+\exp (-1)\right] u_{t}(t, L)^{2}
\end{aligned}
$$

Due to (24) we have

$$
\frac{1}{k} \leq \lambda a
$$

Moreover, due to (27) and (28) we have

$$
\begin{equation*}
|\bar{u}+u| \leq \bar{u}+|u| \leq(1-\lambda) a \leq \frac{a}{2} \tag{48}
\end{equation*}
$$

Hence we have

$$
\left(\frac{1}{k}+(\bar{u}+u)\right)^{2} \leq a^{2}
$$

Thus we have

$$
\begin{equation*}
I_{3}^{0} \geq 0 \tag{49}
\end{equation*}
$$

Moreover, due to (48), the Cauchy-Schwarz inequality and (11) we have

$$
\begin{aligned}
& \int_{t-T_{\text {period }}}^{t}\left(a^{2}-\left(\bar{u}_{L}+u(\tau, L)\right)^{2}\right) 2 k u_{x}(\tau, L) b_{t}(\tau, \omega) d \tau \\
\leq & 2 a^{2} k \int_{t-T_{\text {period }}}^{t}\left|u_{x}(\tau, L)\right|\left|b_{t}(\tau, \omega)\right| d \tau \\
\leq & 2 a^{2} k \sqrt{\int_{t-T_{\text {period }}}^{t}\left(u_{x}(\tau, L)\right)^{2} d \tau} \sqrt{\int_{t-T_{\text {period }}}^{t}\left(b_{t}(\tau, \omega)\right)^{2} d \tau} \\
\leq & 2 a^{2} k \sqrt{C_{\nu}} \sqrt{\int_{t-T_{\text {period }}}^{t}\left(u_{x}(\tau, L)\right)^{2} d \tau} \exp \left(-\frac{\nu}{2} t\right) .
\end{aligned}
$$

Due to (48) we have

$$
\begin{equation*}
\frac{3}{4} a^{2} \leq a^{2}-(\bar{u}+u)^{2} \leq a^{2} \tag{50}
\end{equation*}
$$

Hence we obtain

$$
\begin{gather*}
\int_{t-T_{\text {period }}}^{t} I_{3}^{L} d \tau  \tag{51}\\
\leq 2 a^{2} k \sqrt{C_{\nu}} \frac{\sqrt{\int_{t-T_{\text {period }}}^{t}\left(u_{x}(\tau, L)\right)^{2} d \tau}}{\exp \left(\frac{\nu}{2} t\right)}-\frac{3 a^{2}}{4 \mathrm{e}} \int_{t-T_{\text {period }}}^{t}\left(u_{x}(\tau, L)\right)^{2} d \tau \\
=\frac{2 a^{2} k \sqrt{C_{\nu}}}{\exp \left(\frac{\nu}{2} t\right)}\left\|u_{x}(\cdot, L)\right\|_{L^{2}\left(t-T_{\text {period }}, t\right)}-\frac{3 a^{2}}{4 \mathrm{e}}\left\|u_{x}(\cdot, L)\right\|_{L^{2}\left(t-T_{\text {period }}, t\right)}^{2}
\end{gather*}
$$

Define the polynomial

$$
\begin{equation*}
p_{3}(z)=2 a^{2} k \sqrt{C_{\nu}} \exp \left(-\frac{\nu}{2} t\right) z-\frac{3 a^{2}}{4 \mathrm{e}} z^{2} \tag{52}
\end{equation*}
$$

The polynomial $p_{3}$ has the form $p_{3}(z)=\alpha z-\beta z^{2}$ with $\beta>0$. Hence for all $z \in(-\infty, \infty)$, we have the inequality

$$
\begin{equation*}
p_{3}(z) \leq p_{3}\left(\frac{\alpha}{2 \beta}\right)=\frac{\alpha^{2}}{4 \beta}=\frac{4}{3} \mathrm{e} a^{2} k^{2} C_{\nu} \exp (-\nu t) \tag{53}
\end{equation*}
$$

Hence (51) implies

$$
\begin{equation*}
\int_{t-T_{\text {period }}}^{t} I_{3}^{L} d \tau \leq \frac{4}{3} \mathrm{e} a^{2} k^{2} C_{\nu} \exp (-\nu t) \tag{54}
\end{equation*}
$$

Thus due to (49) we have

$$
\begin{equation*}
\int_{t-T_{\text {period }}}^{t} I_{3} d \tau \leq \frac{4}{3} \mathrm{e} a^{2} k^{2} C_{\nu} \exp (-\nu t) \tag{55}
\end{equation*}
$$

We have

$$
\begin{equation*}
I_{1} \leq-\frac{1}{2 \mathrm{e} L k} E_{1}(t) \tag{56}
\end{equation*}
$$

This can be seen as follows. We have

$$
\begin{aligned}
I_{1} & =-\frac{1}{L} \int_{0}^{L} \exp \left(-\frac{x}{L}\right)\left(a^{2}-(\bar{u}+u)^{2}\right) u_{x}^{2}+\exp \left(-\frac{x}{L}\right) u_{t}^{2} d x \\
& \leq-\frac{1}{2 k L \mathrm{e}} \int_{0}^{L} k\left[\left(a^{2}-(\bar{u}+u)^{2}\right) u_{x}^{2}+u_{t}^{2}\right] d x \\
& -\frac{1}{2 L} \int_{0}^{L} \exp \left(-\frac{x}{L}\right)\left(a^{2}-(\bar{u}+u)^{2}\right) u_{x}^{2}+\exp \left(-\frac{x}{L}\right) u_{t}^{2} d x
\end{aligned}
$$

Since $\bar{u}>0$ is strictly increasing and due to (28) we have

$$
\begin{equation*}
\bar{u}(x)+u(x) \geq \bar{u}(0)-|u(x)| \geq \bar{u}(0)-\bar{u}(0)=0 \tag{57}
\end{equation*}
$$

hence

$$
\begin{aligned}
I_{1} & \leq-\frac{1}{2 \mathrm{e} k L} \int_{0}^{L} k\left[\left(a^{2}-(\bar{u}+u)^{2}\right) u_{x}^{2}+u_{t}^{2}\right]-2 \exp \left(-\frac{x}{L}\right)\left[(\bar{u}+u) u_{x}^{2}+u_{t} u_{x}\right] d x \\
& +\frac{1}{2 \mathrm{e} k L} \int_{0}^{L} \exp \left(-\frac{x}{L}\right)\left[\left(u_{t}^{2}+u_{x}^{2}\right)-k \mathrm{e}\left[\left(a^{2}-(\bar{u}+u)^{2}\right) u_{x}^{2}+u_{t}^{2}\right]\right] d x
\end{aligned}
$$

Due to (24), we have $1-k \mathrm{e}<0$, hence

$$
I_{1} \leq-\frac{1}{2 \mathrm{e} k L} E_{1}(t)+\frac{1}{2 \mathrm{e} k L} \int_{0}^{L} \exp \left(-\frac{x}{L}\right)\left[1-k \mathrm{e}\left(a^{2}-(\bar{u}+u)^{2}\right)\right] u_{x}^{2} d x
$$

Due to (50) and (24) this yields

$$
\begin{aligned}
I_{1} & \leq-\frac{1}{2 \mathrm{e} k L} E_{1}(t)+\frac{1}{2 \mathrm{e} k L} \int_{0}^{L} \exp \left(-\frac{x}{L}\right)\left[1-k \mathrm{e} \frac{3}{4} a^{2}\right] u_{x}^{2} d x \\
& \leq-\frac{1}{2 \mathrm{e} k L} E_{1}(t)
\end{aligned}
$$

Thus we have shown (56).
From (27) we have $\bar{u} \in(0, a / 2)$ and (28) implies that

$$
\max \left\{|\bar{u}|,\left|\bar{u}_{x}\right|\right\} \leq \min \left\{1, \frac{\mu}{K_{1} C_{0}}\right\}
$$

with $C_{0}$ as defined in (26). Moreover (28) implies

$$
\max \left\{|u|,\left|u_{x}\right|,\left|u_{t}\right|\right\} \leq \min \left\{1, \frac{\mu}{K_{1} C_{0}}\right\}
$$

Hence for $T_{L i}(t)$ as defined in (38) we have

$$
\begin{equation*}
T_{L i}(t) \leq \min \left\{1, \frac{\mu}{K_{1} C_{0}}\right\} \tag{58}
\end{equation*}
$$

Hence (57) implies that (39) holds. Thus we can use (41) to derive an upper bound for $I_{2}$. Due to (58) we have

$$
\begin{aligned}
I_{2} & =\int_{0}^{L} 2 k\left(\bar{u}_{x}+u_{x}\right) u_{t}^{2}-2 k(\bar{u}+u) u_{t} u_{x}^{2}+4 k(\bar{u}+u)\left(\bar{u}_{x}+u_{x}\right) u_{x} u_{t} \\
& +2 k F u_{t}-2 h_{2} u_{t} u_{x}^{2}-2 h_{2}(\bar{u}+u)\left(\bar{u}_{x}+u_{x}\right) u_{x}^{2}-2 h_{2} F u_{x} d x \\
& \leq \int_{0}^{L} 4 k T_{L i}(t) u_{t}^{2}+4 k T_{L i}(t) u_{x}^{2}+8 k T_{L i}(t)\left(u_{x}^{2}+u_{t}^{2}\right)+2 T_{L i}(t) u_{x}^{2}+8 T_{L i}(t) u_{x}^{2} \\
& +2 k F u_{t}-2 h_{2} F u_{x} d x \\
& \leq \int_{0}^{L}(12 k+10) T_{L i}(t)\left(u_{t}^{2}+u_{x}^{2}\right)+2 k F u_{t}-2 h_{2} F u_{x} d x
\end{aligned}
$$

Hence (41) implies

$$
\begin{gathered}
I_{2} \leq \int_{0}^{L}(12 k+10) T_{L i}(t)\left(u_{t}^{2}+u_{x}^{2}\right)+ \\
+2 k\left[18+13 \theta+\frac{1}{a^{2}}(8+6 \theta)\right] T_{L i}(t)\left(\left|u u_{t}\right|+\left|u_{x} u_{t}\right|+\left|u_{t}^{2}\right|\right) \\
+2\left[18+13 \theta+\frac{1}{a^{2}}(8+6 \theta)\right] T_{L i}(t)\left(\left|u u_{x}\right|+\left|u_{x}^{2}\right|+\left|u_{t} u_{x}\right|\right) d x
\end{gathered}
$$

Using Young's inequality, this yields with the definition of $C_{0}$ in (26)

$$
\begin{gather*}
I_{0}^{L}\left[12 k+10+4(k+1)\left(18+13 \theta+\frac{1}{a^{2}}(8+6 \theta)\right)\right] T_{L i}(t)\left(u^{2}+u_{t}^{2}+u_{x}^{2}\right) d x \\
\leq C_{0} T_{L i}(t) \int_{0}^{L}\left(u^{2}+u_{t}^{2}+u_{x}^{2}\right) d x
\end{gather*}
$$

We have

$$
u(t, x)=u(t, L)-\int_{x}^{L} u_{x}(t, x) d x=b(t, \omega)-\int_{x}^{L} u_{x}(t, x) d x
$$

hence we have

$$
|u(t, x)| \leq|b(t, \omega)|+\sqrt{L}\left(\int_{0}^{L}\left|u_{x}(t, x)\right|^{2} d x\right)^{1 / 2}
$$

This implies with Young's inequality

$$
|u(t, x)|^{2} \leq 2|b(t, \omega)|^{2}+2 L \int_{0}^{L}\left|u_{x}(t, x)\right|^{2} d x
$$

Thus we obtain

$$
\begin{equation*}
\int_{0}^{L}|u(t, x)|^{2} d x \leq 2 L|b(t, \omega)|^{2}+2 L^{2} \int_{0}^{L}\left|u_{x}(t, x)\right|^{2} d x \tag{60}
\end{equation*}
$$

Hence (59) implies

$$
I_{2} \leq C_{0} T_{L i}(t)\left[\int_{0}^{L}\left(u_{t}^{2}+\left(1+2 L^{2}\right) u_{x}^{2}\right) d x+2 L|b(t, \omega)|^{2}\right]
$$

Due to (57) and (48), (17) holds. Condition (24) implies that (18) is valid. Thus we can apply (22) to obtain

$$
\begin{equation*}
I_{2} \leq C_{0} T_{L i}(t)\left[K_{1} E_{1}(t)+2 L|b(t, \omega)|^{2}\right] \tag{61}
\end{equation*}
$$

with the constant $K_{1}$ from Lemma 4.1. Thus (56) and (58) imply

$$
\begin{aligned}
I_{1}+I_{2} \leq & -\frac{1}{2 \mathrm{e} k L} E_{1}(t)+\mu E_{1}(t)+\frac{1}{2 \mathrm{e} K_{1} k}|b(t, \omega)|^{2} \\
& =-\mu E_{1}(t)+\frac{1}{2 \mathrm{e} K_{1} k}|b(t, \omega)|^{2}
\end{aligned}
$$

Thus we have

$$
\frac{d}{d t} E_{1}(\tau) \leq-\mu E_{1}(\tau)+I_{3}+\frac{1}{2 \mathrm{e} K_{1} k}|b(t, \omega)|^{2}
$$

This implies in turn due to (55) and (1)

$$
\begin{gathered}
E_{1}(t)-E_{1}\left(t-T_{\text {period }}\right)=\int_{t-T_{\text {period }}}^{t} \frac{d}{d t} E_{1}(\tau) d \tau \\
\leq-\mu \int_{t-T_{\text {period }}}^{t} E_{1}(\tau) d \tau+\left[\frac{4}{3} \mathrm{e} a^{2} k^{2} C_{\nu}+\frac{C_{\nu}}{2 \mathrm{e} K_{1} k}\right] \exp (-\nu t)
\end{gathered}
$$

Thus for $E(t)$ as defined in (23) for all $t \geq T_{\text {period }}$ we have the inequality

$$
\frac{d}{d t} E(t)=E_{1}(t)-E_{1}\left(t-T_{\text {period }}\right) \leq-\mu E(t)+C_{g} \exp (-\nu t)
$$

with the constant $C_{g}>0$ as defined in (31).
For $s \geq 0$, let $U(s)=E\left(T_{\text {period }}+s\right)$. By Lemma 4.2, this implies (30).
Due to (60) and (22), inequality (30) yields for $t \geq T_{\text {period }}$

$$
\begin{gather*}
\int_{t-T_{\text {period }}}^{t}\|u(\tau, \cdot)\|_{H^{1}(0, L)}^{2}+\left\|u_{t}(\tau, \cdot)\right\|_{L^{2}(0, L)}^{2} d \tau \\
\leq \int_{t-T_{\text {period }}}^{t} \int_{0}^{L}\left(1+2 L^{2}\right) u_{x}^{2}(\tau, x)+u_{t}^{2}(\tau, x) d x+2 L|b(\tau, \omega)|^{2} d \tau \\
\leq K_{1} \int_{t-T_{\text {period }}}^{t} E_{1}(\tau) d \tau+2 L C_{\nu} \exp (-\nu t) \\
\leq K_{1} \exp \left(-\mu\left(t-T_{\text {period }}\right)\right)\left(E\left(T_{\text {period }}\right)+\frac{C_{g}}{\delta}\right)+2 L C_{\nu} \exp (-\nu t) \tag{62}
\end{gather*}
$$

Analogously, (32) yields

$$
\begin{aligned}
& \int_{T-T_{\text {period }}}^{T}\|u(\tau, \cdot)\|_{H^{1}(0, L)}^{2}+\left\|u_{t}(\tau, \cdot)\right\|_{L^{2}(0, L)}^{2} d \tau \\
& \leq K_{1} \exp \left(-\mu\left(T-T_{\text {period }}\right)\right)\left(E\left(T_{\text {period }}\right)+\frac{C_{g}}{\delta}\right)
\end{aligned}
$$

and hence (33). Thus we have proved Theorem4.1.

## 5 Conclusions

In many applications, uncertainty in the boundary data occurs, for example if uncertain customer behavior influences the boundary data of the system. We have shown that also with such uncertain boundary data on parts of the boundary, a boundary-feedback law can lead to exponential decay for a system that is governed by a quasilinear hyperbolic equation, provided that the perturbations of the boundary data decay exponentially with a decay rate that is sufficiently large. To guarantee the exponential decay, the feedback parameter has to be sufficiently large and the desired stationary state has to be sufficiently small. Due to the nonlinearity of the system, the result is local, that is the initial state has to be sufficiently close to the desired stationary state. The proof is based upon a suitably defined Lyapunov function. Our results show that with the feedback controller the energy of the system in the sense of our Lyapunov function decays exponentially fast, even if there is some unknown input at parts of the boundary. The exponential decay of the Lyapunov function also implies that the $L^{2}$-norm of the system state decays exponentially fast.

If the boundary perturbations do not decrease exponentially fast to zero, is is not possible to achieve exponential decay of the system state. However, we expect that with the feedback law (9), also in this case the system state will decay with the same speed as the boundary perturbations. A detailed study of this situation is a project for future research. In such a situation it could be useful to consider the feedback law (9) in the integral form

$$
\tilde{u}(t, 0)=\tilde{u}(0,0)+\frac{1}{k} \int_{0}^{t} \tilde{u}_{x}(\tau, 0)-\bar{u}_{x}(0) d \tau
$$

The Riemann invariants of (22), (3) are $R_{ \pm}=-\frac{q}{\rho} \mp a \ln (\rho)$, see [11]. For the velocity this yields $\tilde{u}=-\frac{1}{2}\left(R_{+}+R_{-}\right)$. Hence the feedback law (9) is equivalent to the linear Riemann feedback

$$
\left(R_{+}\right)_{x}(t, 0)-k\left(R_{+}\right)_{t}(t, 0)=-2 \bar{u}_{x}(0)-\left(R_{-}\right)_{x}(t, 0)+k\left(R_{-}\right)_{t}(t, 0)
$$

Therefore we hope that our analysis is a motivation to consider this type of Riemann feedback laws in future studies.

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