TWO DOUBLE POSET POLYTOPES

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Abstract. To every poset P, Stanley (1986) associated two polytopes, the order polytope and the chain polytope, whose geometric properties reflect the combinatorial qualities of P. This construction allows for deep insights into combinatorics by way of geometry and vice versa. Malvenuto and Reutenauer (2011) introduced *double posets*, that is, (finite) sets equipped with two partial orders, as a generalization of Stanley's labelled posets. Many combinatorial constructions can be naturally phrased in terms of double posets. We introduce the *double order polytope* and the *double chain polytope* and we amply demonstrate that they geometrically capture double posets, i.e., the interaction between the two partial orders. We describe the facial structures, Ehrhart polynomials, and volumes of these polytopes in terms of the combinatorics of double posets. We also describe a curious connection to Geissinger's valuation polytopes and we characterize 2-level polytopes among our double poset polytopes.

Fulkerson's *anti-blocking* polytopes from combinatorial optimization subsume stable set polytopes of graphs and chain polytopes of posets. We determine the geometry of Minkowski- and Cayley sums of anti-blocking polytopes. In particular, we describe a canonical subdivision of Minkowski sums of anti-blocking polytopes that facilitates the computation of Ehrhart (quasi-)polynomials and volumes. This also yields canonical triangulations of double poset polytopes.

Finally, we investigate the affine semigroup rings associated to double poset polytopes. We show that they have quadratic Gröbner bases, which gives an algebraic description of the unimodular flag triangulations described in the first part.

1. Introduction

A (finite) **partially ordered set** (or **poset**, for short) is a finite set P together with a reflexive, transitive, and anti-symmetric relation $\preceq$. The notion of *partial order* pervades all of mathematics and the enumerative and algebraic combinatorics of posets is underlying in computations in virtually all areas. In 1986, Stanley [34] defined two convex polytopes for every poset P that, in quite different ways, geometrically capture combinatorial properties of P. The **order polytope** $\mathcal{O}(P)$ is set of all order preserving functions into the interval $[0,1]$. That is, all functions $f : P \to \mathbb{R}$ such that

$$0 \ \leq \ f(a) \ \leq \ f(b) \ \leq \ 1$$

for all $a, b \in P$ with $a \preceq b$. Hence, $\mathcal{O}(P)$ parametrizes functions on P and many properties of P are encoded in the boundary structure of $\mathcal{O}(P)$: faces of $\mathcal{O}(P)$ are in correspondence with quotients of P. In particular, the vertices of $\mathcal{O}(P)$ are in bijection to filters of P. But also metric and arithmetic properties of $\mathcal{O}(P)$ can be determined from P. The order polytope naturally has vertices in the lattice $\mathbb{Z}^P$ and its Ehrhart polynomial $\mathrm{Ehr}_{\mathcal{O}(P)}(n) = |n\mathcal{O}(P) \cap \mathbb{Z}^P|$,

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up to a shift, coincides with the order polynomial $\Omega_P(n)$; see Section 4.2 for details. A full-dimensional simplex with vertices in a lattice $\Lambda \subset \mathbb{R}^n$ is **unimodular** with respect to Λ if it has minimal volume. The **normalized volume** relative to Λ is the Euclidean volume scaled such that the volume of a unimodular simplex is 1. If the lattice is clear from the context, we denote the normalized volume by $\mathrm{Vol}(\mathcal{P})$. By describing a canonical triangulation of $\mathcal{O}(P)$ into unimodular simplices, Stanley showed that $\mathrm{Vol}(\mathcal{O}(P))$ is exactly the number of **linear extensions** of P, that is, the number $e(P)$ of refinements of $\preceq$ to a total order. We will review these results in more detail in Section 4.2. This bridge between geometry and combinatorics can, for example, be used to show that computing volume is *hard* (cf. [4]) and, conversely, geometric inequalities can be used on partially ordered sets; see [26, 34].

The **chain polytope** $\mathcal{C}(P)$ is the collection of functions $g : P \to \mathbb{R}_{\geq 0}$ such that

$$g(a_1) + g(a_2) + \cdots + g(a_k) \ \leq \ 1 \tag{1}$$

for all **chains** $a_1 \prec a_2 \prec \cdots \prec a_k$ in P. In contrast to the order polytope, $\mathcal{C}(P)$ does not determine P. In fact, $\mathcal{C}(P)$ is defined by the *comparability graph* of P and bears strong relations to so-called stable set polytopes of perfect graphs; see Section 3.2. Surprisingly, it is shown in [34] that the chain polytope and the order polytope have the same Ehrhart polynomial and hence $\mathrm{Vol}(\mathcal{C}(P)) = \mathrm{Vol}(\mathcal{O}(P)) = e(P)$, which shows that the number of linear extensions only depends on the comparability relation. Stanley's poset polytopes are very natural objects that appear in a variety of contexts in combinatorics and beyond; see [1, 25, 32, 11].

Inspired by Stanley's *labelled* posets, Malvenuto and Reutenauer [28] introduced double poset in the context of combinatorial Hopf algebras. A **double poset** $\mathbf{P}$ is a triple consisting of a finite ground set P and two partial order relations $\preceq_+$ and $\preceq_-$ on P. We will write $P_+ = (P, \preceq_+)$ and $P_- = (P, \preceq_-)$ to refer to the two underlying posets. If $\preceq_-$ is a total order, then this corresponds to labelled poset in the sense of Stanley [33], which is the basis for the rich theory of P-partitions. The combinatorial study of general double posets gained momentum in recent years with a focus on algebraic aspects; see, for example, [9, 10]. The goal of this paper is to build a bridge to geometry by introducing *two double poset polytopes* that, like the chain- and the order polytope, geometrically reflect the combinatorial properties of double posets and, in particular, the interaction between the two partial orders.

1.1. Double order polytopes.

For a double poset $\mathbf{P} = (P, \preceq_\pm)$, we define the **double order polytope** as

$$\mathcal{O}(\mathbf{P}) \ = \ \mathcal{O}(P, \preceq_+, \preceq_-) \ := \ \mathrm{conv}\big\{(2\mathcal{O}(P_+) \times \{1\}) \cup (-2\mathcal{O}(P_-) \times \{-1\})\big\}.$$

This is a $(|P|+1)$-dimensional polytope in $\mathbb{R}^P \times \mathbb{R}$. Its vertices are trivially in bijection to filters of P_+ and P_-. This is a lattice polytope with respect to $\mathbb{Z}^P \times \mathbb{Z}$ but we will mostly view $\mathcal{O}(\mathbf{P})$ as a lattice polytope with respect to the **affine lattice** $\mathbb{A} = 2\mathbb{Z}^P \times (2\mathbb{Z}+1)$. That is, up to a translation by $(\mathbf{0}, 1)$, $\mathcal{O}(\mathbf{P})$ is the polytope

$$2 \cdot \mathrm{conv}\big\{(\mathcal{O}(P_+) \times \{1\}) \cup (-\mathcal{O}(P_-) \times \{0\})\big\},$$

which is a lattice polytope with respect to $2\mathbb{Z}^P \times 2\mathbb{Z}$. In Section 2.2, we describe the facets of $\mathcal{O}(\mathbf{P})$ in terms of chains and cycles alternating between P_+ and P_- and, for the important case of *compatible* double posets, we completely determine the facial structure in Section 2.3 in terms of *double Birkhoff lattices* $\mathcal{J}(\mathbf{P}) := \mathcal{J}(P_+) \uplus \mathcal{J}(P_-)$. The double order polytope automatically has $2\mathcal{O}(P_+)$ and $-2\mathcal{O}(P_-)$ as facets. The non-trivial combinatorial structure is captured by the **reduced** double order polytope

$$\overline{\mathcal{O}}(\mathbf{P}) \ := \ \mathcal{O}(\mathbf{P}) \cap \{(f,t) : t = 0\} \ = \ \mathcal{O}(P_+) - \mathcal{O}(P_-),$$

which is a lattice polytope with respect to $\mathbb{Z}^P$ by our choice of embedding.

By placing $2\mathcal{O}(P_+)$ and $-2\mathcal{O}(P_-)$ at heights $+1$ and -1, respectively, we made sure that $\mathcal{O}(\mathbf{P})$ always contains the origin. Every poset $(P, \preceq)$ trivially induces a double poset $\mathbf{P}_\circ = (P, \preceq, \preceq)$ and for an induced double poset, $\mathcal{O}(\mathbf{P}_\circ)$ is centrally-symmetric and, up to a (lattice-preserving) shear, is the polytope

$$\mathcal{O}(\mathbf{P}_\circ) \ \cong \ \operatorname{conv}\big\{(2\mathcal{O}(P) \times \{1\}) \cup (2\mathcal{O}(P^{\mathrm{op}}) \times \{-1\})\big\},$$

where P^{op} is the poset with the opposite order. Geissinger [13] introduced a polytope associated to valuations on distributive lattices with values in $[0,1]$. In Section 2.4, we show a surprising connection between Geissinger's valuation polytopes and *polars* of the (reduced) double order polytopes of $\mathbf{P}_\circ$. We will review notions from the theory of double posets and emphasize their geometric counterparts.

1.2. **Double chain-, Hansen-, and anti-blocking polytopes.** The **double chain polytope** associated to a double poset $\mathbf{P}$ is the polytope

$$\mathcal{C}(\mathbf{P}) \ = \ \mathcal{C}(P, \preceq_+, \preceq_-) \ := \ \operatorname{conv}\big\{(2\mathcal{C}(P_+) \times \{1\}) \cup (-2\mathcal{C}(P_-) \times \{-1\})\big\}.$$

The **reduced** version $\overline{\mathcal{C}}(\mathbf{P}) := \mathcal{C}(P_+) - \mathcal{C}(P_-)$ is studied in Section 3 in the context of *anti-blocking* polytopes. According to Fulkerson [12], a full-dimensional polytope $\mathcal{P} \subseteq \mathbb{R}^n_{\geq 0}$ is **anti-blocking** if for any $q \in \mathcal{P}$, it contains all points $p \in \mathbb{R}^n$ with $0 \leq p_i \leq q_i$ for $i = 1, \dots, n$. It is obvious from (1) that chain polytopes are anti-blocking. Anti-blocking polytopes are important in combinatorial optimization and, for example, contain stable set polytopes of graphs. For two polytopes $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$, we define the **Cayley sum** as the polytope

$$\mathcal{P}_1 \boxplus \mathcal{P}_2 \ := \ \operatorname{conv}(\mathcal{P}_1 \times \{1\} \cup \mathcal{P}_2 \times \{-1\})$$

and we abbreviate $\mathcal{P}_1 \boxminus \mathcal{P}_2 := \mathcal{P}_1 \boxplus -\mathcal{P}_2$. Thus,

$$\mathcal{O}(\mathbf{P}) \ = \ 2\mathcal{O}(P_+) \boxminus 2\mathcal{O}(P_-) \qquad \text{and} \qquad \mathcal{C}(\mathbf{P}) \ = \ 2\mathcal{C}(P_+) \boxminus 2\mathcal{C}(P_-).$$

Section 3 is dedicated to a detailed study of the polytopes $\mathcal{P}_1 \boxminus \mathcal{P}_2$ as well as their sections $\mathcal{P}_1 - \mathcal{P}_2$ for anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n_{\geq 0}$. We completely determine the facets of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ in terms of $\mathcal{P}_1, \mathcal{P}_2$ in Section 3.1, which yields the combinatorics of $\mathcal{C}(\mathbf{P})$. In Section 3.3, we describe a canonical subdivision of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ and $\mathcal{P}_1 - \mathcal{P}_2$ for anti-blocking blocking polytopes $\mathcal{P}_1, \mathcal{P}_2$. Moreover, if $\mathcal{P}_1, \mathcal{P}_2$ have regular, unimodular, or flag triangulations, then so has $\mathcal{P}_1 \boxminus \mathcal{P}_2$ (for an appropriately chosen affine lattice). The canonical subdivision enables us to give explicit formulas for the volume and the Ehrhart (quasi-)polynomials of these classes of polytopes.

The chain polytope $\mathcal{C}(P)$ only depends on the comparability graph $G(P)$ of P and, more precisely, is the stable set polytope of $G(P)$. Thus, $\mathcal{C}(\mathbf{P})$ only depends on the **double graph** $(G(P_+), G(P_-))$. For a graph G, let $\mathcal{P}_G$ be its stable set polytope; see Section 3.1 for precise definitions. Lovász [27] characterized *perfect* graphs in terms of $\mathcal{P}_G$ and Hansen [17] studied the polytopes $\mathcal{H}(G) := 2\mathcal{P}_G \boxminus 2\mathcal{P}_G$. If G is perfect, then Hansen showed that the polar $\mathcal{H}(G)^\triangle$ is linearly isomorphic to $\mathcal{H}(\overline{G})$ where $\overline{G}$ is the complement graph of G. In Section 3.2, we generalize this result to all Cayley sums of anti-blocking polytopes.

1.3. **2-level polytopes and volume.** A full-dimensional polytope $\mathcal{P} \subset \mathbb{R}^n$ is called **2-level** if for any facet-defining hyperplane H there is $t \in \mathbb{R}^n$ such that $H \cup (t + H)$ contains all vertices of P. The class of 2-level polytopes plays an important role in, for example, the study of centrally-symmetric polytopes [30, 17], polynomial optimization [14, 15], statistics [37], and combinatorial optimization [31]. For example, Lovász [27] characterizes perfect graphs by the 2-levelness of their stable set polytopes and Hansen showed that $\mathcal{H}(G)$ is 2-level if G is perfect. In fact, we extend this to yet another characterization of perfect graphs in Corollary 3.11. This result implies that $\mathcal{C}(\mathbf{P}_\circ)$ is 2-level for double posets induced by posets. However, it is in general *not* true that $\mathcal{P}_1 \boxplus \mathcal{P}_2$ is 2-level if $\mathcal{P}_1$ and $\mathcal{P}_2$ are. A counterexample is the polytope $\Delta_{6,2} \boxplus \Delta_{6,4}$, where $\Delta_{n,k}$ is the (n,k)-hypersimplex. The starting point for this paper was the question for which double posets $\mathbf{P}$ the polytopes $\mathcal{O}(\mathbf{P})$ and $\mathcal{C}(\mathbf{P})$ are 2-level. Answers are given in Corollary 2.9, Proposition 2.10, and Corollary 3.11. A new class of 2-level polytopes comes from valuation polytopes; see Corollary 2.19. Sullivant [37, Thm. 2.4] showed that 2-level lattice polytopes $\mathcal{P}$ have the interesting property that any pulling triangulation that uses all lattice points in $\mathcal{P}$ is unimodular. Hence, for 2-level lattice polytopes, the normalized volume is the number of simplices. In particular, $\mathcal{O}(P)$ is 2-level and Stanley's canonical triangulation is a pulling triangulation. Stanley defined a piecewise linear homeomorphism between $\mathcal{O}(P)$ and $\mathcal{C}(P)$ whose domains of linearity are exactly the simplices of the canonical triangulation. Since this *transfer map* is lattice preserving, it follows that $\mathrm{Ehr}_{\mathcal{O}(P)}(n) = \mathrm{Ehr}_{\mathcal{C}(P)}(n)$, which also implies the volume result. In Section 4, we generalize this transfer map to a lattice preserving PL homeomorphism $\Psi_{\mathbf{P}} : \mathbb{R}^P \times \mathbb{R} \to \mathbb{R}^P \times \mathbb{R}$ for any compatible double poset $\mathbf{P}$. In particular, $\mathcal{C}(\mathbf{P})$ is mapped to $\mathcal{O}(\mathbf{P})$. This also transfers the canonical flag triangulation of $\mathcal{C}(\mathbf{P})$ to a canonical flag triangulation of $\mathcal{O}(\mathbf{P})$. Abstractly, the triangulation can be described in terms of a suitable subcomplex of the order complex of the double Birkhoff lattice $\mathcal{J}(\mathbf{P}) = \mathcal{J}(P_+) \uplus \mathcal{J}(P_-)$. In Section 4.2, we give explicit formulas for the Ehrhart polynomial and the volume of $\mathcal{O}(\mathbf{P})$ if $\mathbf{P}$ is compatible and for $\mathcal{C}(\mathbf{P})$ in general.

1.4. **Double Hibi rings.** Hibi [19] studied rings associated to finite posets that give posets an algebraic incarnation and that are called **Hibi rings**. In modern language, the Hibi ring $\mathbb{C}[\mathcal{O}(P)]$ associated to a poset P is the semigroup ring associated to $\mathcal{O}(P)$. Many properties of P posses an algebraic counterpart and, in particular, Hibi exhibited a quadratic Gröbner basis for the associated toric ideal. In Section 5, we introduce the **double Hibi rings** $\mathbb{C}[\mathcal{O}(\mathbf{P})]$ as suitable analogs for double posets, which are the semigroup rings associated to $\mathcal{O}(\mathbf{P})$. We construct a quadratic Gröbner basis for the cases of compatible double posets. Using a result by Sturmfels [36, Thm. 8.3], this shows the existence of a unimodular and flag triangulation of $\mathcal{O}(\mathbf{P})$ which coincides with the triangulation in Section 4. We also construct a quadratic Gröbner basis for the rings $\mathbb{C}[\mathcal{C}(\mathbf{P})]$ for arbitrary double posets and we remark on the algebraic implications for double posets.

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Contents

2. Double order polytopes

2.1. **Order polytopes.** Let $(P, \preceq)$ be a poset. We write $\widehat{P}$ for the poset obtained from P by adjoining a minimum $\widehat{0}$ and a maximum $\widehat{1}$. For an order relation $a \prec b$, we define a linear form $\ell_{a,b} : \mathbb{R}^P \to \mathbb{R}$ by

$$\ell_{a,b}(f) \ := \ f(a) - f(b)$$

for any $f \in \mathbb{R}^P$. Moreover, for $a \in P$, we define $\ell_{a,\widehat{1}}(f) := f(a)$ and $\ell_{\widehat{0},a}(f) := -f(a)$. With this notation, $f \in \mathbb{R}^P$ is contained in $\mathcal{O}(P)$ if and only if

$$\begin{aligned} \ell_{a,b}(f) \ &\leq \ 0 \quad \text{for all } a \prec b, \\ \ell_{\widehat{0},b}(f) \ &\leq \ 0 \quad \text{for all } b \in P, \text{ and} \\ \ell_{a,\widehat{1}}(f) \ &\leq \ 1 \quad \text{for all } a \in P. \end{aligned} \tag{2}$$

Every nonempty face F of $\mathcal{O}(P)$ is of the form

$$F \;=\; \mathcal{O}(P)^{\ell} \;:=\; \{f \in \mathcal{O}(P) : \ell(f) \geq \ell(f') \text{ for all } f' \in \mathcal{O}(P)\}$$

for some linear function $\ell \in (\mathbb{R}^P)^*$. Later, we want to identify ℓ with its vector of coefficients and thus we write

$$\ell(f) \;=\; \sum_{a \in P} \ell_a f(a).$$

Combinatorially, faces can be described using **face partitions**: To every face F is an associated collection $B_1, \ldots, B_m \subseteq \widehat{P}$ of nonempty and pairwise disjoint subsets that partition $\widehat{P}$. According to Stanley [34, Thm 1.2], a partition of $\widehat{P}$ is a (closed) face partition if and only if each $(B_i, \preceq)$ is a connected poset and $B_i \preceq' B_j :\Leftrightarrow p_i \preceq p_j$ for some $p_i \in B_i, p_j \in B_j$ is a partial order on $\{B_1, \ldots, B_m\}$. Of course, it is sufficient to remember the non-singleton parts and we define the **reduced** face partition of F as $\mathcal{B}(F) := \{B_i : |B_i| > 1\}$. The **normal cone** of a nonempty face $F \subseteq \mathcal{O}(P)$ is the polyhedral cone

$$N_P(F) \;:=\; \{\ell \in (\mathbb{R}^P)^* : F \subseteq \mathcal{O}(P)^{\ell}\}.$$

The following description of $N_P(F)$ follows directly from (2).

Proposition 2.1. *Let P be a finite poset and $F \subseteq \mathcal{O}(P)$ a nonempty face with reduced face partition $\mathcal{B} = \{B_1, \ldots, B_k\}$. Then*

$$N_P(F) \;=\; \operatorname{cone}\{\ell_{a,b} : [a,b] \subseteq B_i \textit{ for some } i = 1, \ldots, k\}.$$

We note the following simple but very useful consequence of this description.

Corollary 2.2. *Let $F \subseteq \mathcal{O}(P)$ be a nonempty face with reduced face partition $\mathcal{B} = \{B_1, \ldots, B_k\}$. Then for every $\ell \in \operatorname{relint} N_P(F)$ and $p \in P$ the following hold:*

(i) *if $p \in \min(B_i)$ for some i, then $\ell_p > 0$;*
(ii) *if $p \in \max(B_i)$ for some i, then $\ell_p < 0$;*
(iii) *if $p \notin \bigcup_i B_i$, then $\ell_p = 0$.*

The vertices of $\mathcal{O}(P)$ are exactly the indicator functions $\mathbf{1}_{\mathsf{J}} : P \to \{0,1\}$ where $\mathsf{J} \subseteq P$ is a filter. For a filter $\mathsf{J} \subseteq P$, we write $\widehat{\mathsf{J}} := \mathsf{J} \cup \{\widehat{1}\}$ for the filter induced in $\widehat{P}$.

Proposition 2.3. *Let $F \subseteq \mathcal{O}(P)$ be a face with (reduced) face partition $\mathcal{B} = \{B_1, \ldots, B_k\}$ and let $\mathsf{J} \subseteq P$ be a filter. Then $\mathbf{1}_{\mathsf{J}} \in F$ if and only if*

$$\widehat{\mathsf{J}} \cap B_i \;=\; \varnothing \quad \textit{or} \quad \widehat{\mathsf{J}} \cap B_i \;=\; B_i$$

for all $i = 1, \ldots, k$.

That is, $\mathbf{1}_{\mathsf{J}}$ belongs to F if and only if $\widehat{\mathsf{J}}$ does not separate any two comparable elements in B_i, for all i.

2.2. **Facets of double order polytopes.** Let $\mathbf{P} = (P, \preceq_{\pm})$ be a double poset. The double order polytope $\mathcal{O}(\mathbf{P})$ is a $(|P|+1)$-dimensional polytope in $\mathbb{R}^P \times \mathbb{R}$ with coordinates (f,t). It is obvious that the vertices of $\mathcal{O}(\mathbf{P})$ are exactly $(2\mathbf{1}_{\mathsf{J}_+}, 1), (-2\mathbf{1}_{\mathsf{J}_-}, -1)$ for filters $\mathsf{J}_+ \subseteq P_+$ and $\mathsf{J}_- \subseteq P_-$, respectively. To get the most out of our notational convention, for $\sigma \in \{-,+\}$ we define

$$-\sigma \;:=\; \begin{cases} - & \text{if } \sigma = + \\ + & \text{if } \sigma = -. \end{cases}$$

By construction, $2\mathcal{O}(P_+) \times \{1\}$ and $-2\mathcal{O}(P_-) \times \{-1\}$ are facets that are obtained by maximizing the linear function $\pm L_\varnothing(f,t) := \pm t$ over $\mathcal{O}(\mathbf{P})$. We call the remaining facets **vertical**, as they are of the form $F_+ \boxminus F_-$, where $F_\sigma \subset \mathcal{O}(P_\sigma)$ are certain nonempty proper faces for $\sigma = \pm$. The vertical facets are in bijection with the facets of the reduced double order polytope $\overline{\mathcal{O}}(\mathbf{P}) = \mathcal{O}(P_+) - \mathcal{O}(P_-)$.

More precisely, if $F \subset \mathcal{O}(\mathbf{P})$ is a facet, then there is a linear function $\ell \in (\mathbb{R}^P)^*$ such that $F = F_+ \boxminus F_-$ where $F_+ = \mathcal{O}(P_+)^\ell$ and $F_- = \mathcal{O}(P_-)^{-\ell}$. This linear function is necessarily unique up to scaling and hence the faces F_+, F_- are characterized by the property

$$\operatorname{relint} N_{P_+}(F_+) \ \cap\ \operatorname{relint} -N_{P_-}(F_-) \ =\ \mathbb{R}_{>0} \cdot \ell\,. \tag{3}$$

We will call a linear function ℓ **rigid** if it satisfies (3) for a pair of faces (F_+, F_-). Our next goal is to give an explicit description of all rigid linear functions for $\mathcal{O}(\mathbf{P})$ which then yields a characterization of vertical facets.

An **alternating chain** C of a double poset $\mathbf{P} = (P, \preceq_\pm)$ is a finite sequence of distinct elements

$$\widehat{0} \ =\ p_0 \ \prec_\sigma\ p_1 \ \prec_{-\sigma}\ p_2 \ \prec_\sigma\ \cdots \ \prec_{\pm\sigma}\ p_k \ =\ \widehat{1}, \tag{4}$$

where $\sigma \in \{\pm\}$. For an alternating chain C, we define a linear function ℓ_C by

$$\ell_C(f) \ :=\ \sigma \sum_{i=1}^{k-1} (-1)^i f(p_i).$$

Here, we severely abuse notation and interpret σ as ± 1. Note that $\ell_C \equiv 0$ if $k = 1$ and we call C a **proper** alternating chain if $k > 1$. An **alternating cycle** C of $\mathbf{P}$ is a sequence of length $2k$ of the form

$$p_0 \ \prec_\sigma\ p_1 \ \prec_{-\sigma}\ p_2 \ \prec_\sigma\ \cdots \ \prec_{-\sigma}\ p_{2k} \ =\ p_0,$$

where $\sigma \in \{\pm\}$ and $p_i \neq p_j$ for $0 \le i < j < 2k$. We similarly define a linear function associated to C by

$$\ell_C(f) \ :=\ \sigma \sum_{i=0}^{2k-1} (-1)^i f(p_i).$$

Note that it is possible that a sequence of elements $p_1, p_2, \ldots, p_k$ gives rise to two alternating chains, one starting with $\prec_+$ and one starting with $\prec_-$. On the other hand, every alternating cycle of length $2k$ yields k alternating cycles starting with $\prec_+$ and k alternating starting with $\prec_-$.

Proposition 2.4. *Let $\mathbf{P} = (P, \preceq_\pm)$ be a double poset. If ℓ is a rigid linear function for $\mathcal{O}(\mathbf{P})$, then $\ell = \mu \ell_C$ for some alternating chain or alternating cycle C and $\mu > 0$.*

Proof. Let $F_+ = \mathcal{O}(P_+)^\ell$ and $F_- = \mathcal{O}(P_-)^{-\ell}$ be the two faces for which (3) holds and let $\mathcal{B}_\pm = \{B_{\pm 1}, B_{\pm 2}, \ldots\}$ be the corresponding reduced face partitions. We define a directed bipartite graph $G = (V_+ \cup V_-, E)$ with nodes $V_+ = \{p \in P : \ell_p > 0\}$ and V_- accordingly. If $\widehat{1}$ is contained in some part of $\mathcal{B}_+$, then we add a corresponding node $\widehat{1}_+$ to V_- Consistently, we add a node $\widehat{1}_-$ to V_+ if $\widehat{1}$ it occurs in a part of $\mathcal{B}_-$. Note that $\widehat{0}_-$ and $\widehat{0}_+$ are distinct nodes. Similarly we add $\widehat{0}_+$ to V_+ and $\widehat{0}_-$ to V_- if they appear in $\mathcal{B}_+$ and $\mathcal{B}_-$, respectively. By Corollary 2.2, we have ensured that $\max(B_{+i}) \subseteq V_-$ and $\max(B_{-i}) \subseteq V_+$ for all i.

For $u \in V_+$ and $v \in V_-$, we add the directed edge $uv \in E$ if $u \prec_+ v$ and $[u,v]_{P_+} \subseteq B_{+i}$ for some i. Similarly, we add the directed edge $vu \in E$ if $v \prec_- u$ and $[v,u]_{P_-} \subseteq B_{-i}$ for some

i. We claim that every node u except for maybe the special nodes $\widehat{0}_\pm, \widehat{1}_\pm$ has an incoming and an outgoing edge. For example, if $u \in V_+$, then $\ell_u > 0$. By Corollary 2.2(iii), there is an i such that $u \in B_{+i}$ and by (ii), u is not a maximal element in B_{+i}. Thus, there is some $v \in \max(B_{+i})$ with $u \prec_+ v$ and uv is an edge. It follows that every longest path either yields an alternating cycle or a proper alternating chain.

For an alternating cycle $C = (p_0 \prec_+ \cdots \prec_- p_{2l})$, we observe that

$$\begin{aligned} \ell_C &= \ell_{p_0,p_1} + \ell_{p_2,p_3} + \cdots + \ell_{p_{2l-2},p_{2l-1}} \text{ and} \\ -\ell_C &= \ell_{p_1,p_2} + \ell_{p_3,p_4} + \cdots + \ell_{p_{2l-1},p_{2l}}. \end{aligned}$$

Since for every j, $[p_{2j}, p_{2j+1}]_{P_+}$ is contained in some part of $\mathcal{B}_+$, we conclude that $\ell_C \in N_{P_+}(F_+)$. Similarly, for all j, $[p_{2j-1}, p_{2j}]_{P_-}$ is contained in some part of $\mathcal{B}_-$, and hence $-\ell_C \in N_{P_-}(F_-)$. Assuming that ℓ is rigid then shows that $\ell = \mu\ell_C$ for some $\mu > 0$.

If G does not contain a cycle, then let $C = (p_0, p_1, \ldots, p_k)$ be a longest path in G. In particular $p_0 = \widehat{0}_\pm$ and $p_k = \widehat{1}_\pm$. The same reasoning applies and shows that $\ell_C \in N_{P_+}(F_+) \cap -N_{P_-}(F_-)$ and hence $\ell = \mu\ell_C$ for some $\mu > 0$. $\square$

In general, not every alternating chain or cycle gives rise to a rigid linear function. Let $(P, \preceq)$ be a poset that is not the antichain and define the double poset $\mathbf{P} = (P, \preceq, \preceq^{\mathrm{op}})$, where $\preceq^{\mathrm{op}}$ is the **opposite** order. In this case $\mathcal{O}(\mathbf{P})$ is, up to a shear, the ordinary prism over $\mathcal{O}(P, \preceq)$. Hence, the vertical facets of $\mathcal{O}(\mathbf{P})$ are prisms over the facets of $\mathcal{O}(P)$. It follows from (2) that these facets correspond to cover relations in P. Hence, every rigid ℓ is of the form $\ell = \mu\ell_{p,q}$ for cycles $p \prec_+ q \prec_- p$ where $p \prec q$ is a cover relation in P.

We call a double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$ **compatible** if $P_+ = (P, \preceq_+)$ and $P_- = (P, \preceq_-)$ have a common linear extension. Note that a double poset is compatible if and only if it does not contain alternating cycles. Following [28], we call a double poset $\mathbf{P}$ **special** if $\preceq_-$ is a total order. At the other extreme, we call $\mathbf{P}$ **anti-special** if $(P, \preceq_-)$ is an anti-chain. A **plane poset**, as defined in [9] is a double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$ such that distinct $a, b \in P$ are $\prec_+$-comparable if and only if they are not $\prec_-$-comparable. For two posets $(P_1, \preceq^1)$ and $(P_2, \preceq^2)$ one classically defines the **disjoint union** $\preceq_\uplus$ and the **ordinal sum** $\preceq_\oplus$ as the posets on $P_1 \uplus P_2$ as follows. For $a, b \in P_1 \uplus P_2$ set $a \preceq_\uplus b$ if $a, b \in P_i$ and $a \preceq^i b$ for some $i \in \{1, 2\}$. For the ordinal sum, $\preceq_\oplus$ restricts to $\preceq^1$ and $\preceq^2$ on P_1 and P_2 respectively and $p_1 \prec_\oplus p_2$ for all $p_1 \in P_1$ and $p_2 \in P_2$. The effect on order polytopes is $\mathcal{O}(P_1 \uplus P_2) = \mathcal{O}(P_1) \times \mathcal{O}(P_2)$ and $\mathcal{O}(P_1 \oplus P_2)$ is a *subdirect sum* in the sense of McMullen [29]. Malvenuto and Reutenauer [28] define the **composition** of two double posets $(P_1, \prec^1_\pm)$ and $(P'_2, \prec^2_\pm)$ as the double poset $(P, \preceq_\pm)$ such that $(P, \preceq_+) = (P_1, \prec^1_+) \uplus (P_2, \prec^2_+)$ and $(P, \preceq_-) = (P_1, \prec^1_-) \oplus (P_2, \prec^2_-)$.

The following is easily seen; for plane posets with the help of [9, Prop. 11].

Proposition 2.5. *Anti-special and plane posets are compatible. Moreover, the composition of two compatible double posets is a compatible double poset.*

This defining property of compatible double posets assures us that in an alternating chain $p_i \prec_\sigma p_j$ implies $i < j$ for any $\sigma \in \{\pm\}$. In particular, a compatible double poset does not have alternating cycles. This also shows the following.

Lemma 2.6. *Let $\mathbf{P} = (P, \preceq_\pm)$ be a compatible double poset. If $a_i \prec_\sigma a_{i+1} \prec_{-\sigma} \cdots \prec_{-\tau} a_j \prec_\tau a_{j+1}$ is part of an alternating chain with $\sigma, \tau \in \{\pm\}$ and $i < j$, then there is no $b \in P$ such that $a_i \prec_\sigma b \prec_\sigma a_{i+1}$ and $a_j \prec_\tau b \prec_\tau a_{j+1}$.*

For compatible double posets, we can give complete characterization of facets.

Theorem 2.7. *Let* $\mathbf{P}$ *a compatible double poset. A linear function* ℓ *is rigid if and only if* $\ell \in \mathbb{R}_{>0}\ell_C$ *for some alternating chain* C*. In particular, the facets of* $\mathcal{O}(\mathbf{P})$ *are in bijection with alternating chains.*

Proof. We already observed that $2\mathcal{O}(P_+) \times \{1\}$ and $-2\mathcal{O}(P_-) \times \{-1\}$ correspond to the improper alternating chains $\widehat{0} \prec_\sigma \widehat{1}$ for $\sigma = \pm$. By Proposition 2.4 it remains to show that for any proper alternating chain C the function ℓ_C is rigid. We only consider the case that C is an alternating chain of the form

$$\widehat{0} \;=\; p_0 \;\prec_+\; p_1 \;\prec_-\; p_2 \;\prec_+\; \cdots \;\prec_+\; p_{2k-1} \;\prec_-\; p_{2k} \;\prec_+\; p_{2k+1} \;=\; \widehat{1}.$$

The other cases can be treated analogously. Let $F_+ = \mathcal{O}(P_+)^{\ell_C}$ and and $F_- = \mathcal{O}(P_-)^{-\ell_C}$ be the corresponding faces with reduced face partitions $\mathcal{B}_\pm$. Define $O = \{p_1, p_3, \dots, p_{2k-1}\}$ and $E = \{p_2, p_4, \dots, p_{2k}\}$. Then for any set $A \subseteq P$, we observe that $\ell_C(\mathbf{1}_A) = |E \cap A| - |O \cap A|$. If J is a filter of P_+, then $p_{2i} \in \mathsf{J}$ implies $p_{2i+1} \in \mathsf{J}$ and hence $\ell_C(\mathbf{1}_\mathsf{J}) \le 1$ and thus $\mathbf{1}_\mathsf{J} \in F_+$ if and only if J does not separate p_{2j} and p_{2j+1} for $1 \le j \le k$. Likewise, a filter $\mathsf{J} \subseteq P_-$ is contained in F_- if and only if J does not separate p_{2j-1} and p_{2j} for $1 \le j \le k$. Lemma 2.6 implies that

$$\begin{aligned} \mathcal{B}_+ &= \{[p_0,p_1]_{P_+}, [p_2,p_3]_{P_+}, \dots, [p_{2k},p_{2k+1}]_{P_+}\} \text{ and} \\ \mathcal{B}_- &= \{[p_1,p_2]_{P_-}, [p_3,p_4]_{P_-}, \dots, [p_{2k-1},p_{2k}]_{P_-}\}. \end{aligned}$$

To show that ℓ_C is rigid pick a linear function $\ell(\phi) = \sum_{p \in P} \ell_p \phi(p)$ with $F_+ = \mathcal{O}(P_+)^\ell$ and $F_- = \mathcal{O}(P_-)^{-\ell}$. Since the elements in E and O are exactly the minimal and maximal elements of the parts in $\mathcal{B}_+$, it follows from Corollary 2.2 that $\ell_p > 0$ if $p \in E$, $\ell_p < 0$ for $p \in O$. By Lemma 2.6, it follows that if $q \in (p_i, p_{i+1})_{P_+}$, then q is not contained in a part of the reduced face partition $\mathcal{B}_-$ and vice versa. By Corollary 2.2(iii), it follows that $\ell_p = 0$ for $p \notin E \cup O$. Finally, $\ell_{p_i} + \ell_{p_{i+1}} = 0$ for all $1 \le i \le 2k$ by Proposition 2.1 and therefore $\ell = \mu \ell_C$ for some $\mu > 0$ finishes the proof. □

Example 1. Let $\mathbf{P} = (P, \preceq_\pm)$ be a compatible double poset with $|P| = n$.

(1) Let $\preceq_+ = \preceq_- = \preceq$ and $(P, \preceq)$ be the n-antichain. Then the only alternating chains are of the form $\widehat{0} \prec_\sigma a \prec_{-\sigma} \widehat{1}$ for $a \in P$. The double order polytope $\mathcal{O}(\mathbf{P})$ is the $(n+1)$-dimensional cube with vertices $\{0,2\}^n \times \{+1\}$ and $\{0,-2\}^n \times \{-1\}$.
(2) If $\preceq_+ = \preceq_- = \preceq$ and $(P, \preceq)$ is the n-chain $[n]$, then any alternating chain can be identified with an element in $\{-,+\}^{n+1}$. More precisely, $\mathcal{O}(\mathbf{P})$ is linearly isomorphic to the $(n+1)$-dimensional crosspolytope $C^\triangle_{n+1} = \operatorname{conv}\{\pm\mathbf{e}_1, \dots, \pm\mathbf{e}_{d+1}\}$.
(3) Let $(P, \preceq_+)$ be the n-chain and $(P, \preceq_-)$ be the n-antichain. Then any alternating chain is of the form $\widehat{0} \prec_\sigma a \prec_{-\sigma} \widehat{1}$ for $\sigma = \pm$ and any relation $a \prec_+ b$ be can be completed to a unique alternating chain. Thus, $\mathcal{O}(\mathbf{P})$ is a $(n+1)$-dimensional polytope with $2^n + n + 1$ vertices and $\binom{n}{2} + 2n + 2$ facets.
(4) The **comb** (see Figure 2) is the poset C_n on elements $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ such that $a_i \preceq a_j$ if $i \le j$ and $b_i \prec a_i$ for all $i, j \in [n]$. The n-comb has $2^{n+1} - 1$ filters and $3 \cdot 2^n - 2$ chains. Hence $\mathcal{O}(C_n, \preceq, \preceq)$ has $2^{n+2} - 2$ vertices and $3 \cdot 2^{n+1} - 4$ facets.
(5) Generally, let P_1, P_2 be two posets and denote by f_i and c_i the number of filters and chains of P_i for $i = 1, 2$. Let $\mathbf{P}_\circ$ be the trivial double poset induced by $P_1 \uplus P_2$. Then $\mathcal{O}(\mathbf{P}_\circ)$ has $2f_1f_2$ vertices and $2(c_1 + c_2) - 2$ facets.

Example 2. Consider the compatible 'XW'-double poset $\mathbf{P}_{XW}$ on five elements, whose Hasse diagrams are given in Figure 1. The polytope $\mathcal{O}(\mathbf{P}_{XW})$ is six-dimensional with face vector

$$f(\mathcal{O}(\mathbf{P}_{XW})) \ = \ (21, 112, 247, 263, 135, 28).$$

The facets correspond to the 28 alternating chains in $\widehat{\mathbf{P}}_{XW}$, which are shown in Figure 3.

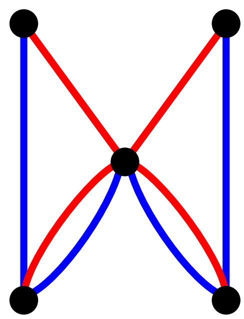

FIGURE 1. The 'XW'-double poset $\mathbf{P}_{XW}$. The red and blue lines are the Hasse diagram of P_+ and P_-, respectively. Striped lines are edges in both Hasse diagrams.

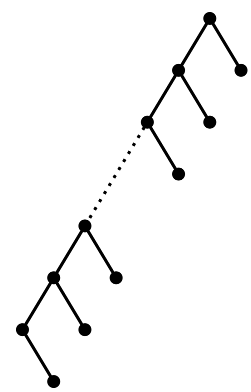

FIGURE 2. The comb C_n.

For two particular types of posets, we wish to determine the combinatorics of $\mathcal{O}(\mathbf{P})$ in more detail.

Example 3 (Dimension-2 posets)**.** For $n \geq 1$, let $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ be an ordered sequence of distinct numbers. We may define a partial order $\preceq_\pi$ on $[n]$ by $i \prec_\pi j$ if $i < j$ and $\pi_i < \pi_j$. Following Dushnik and Miller [8], these are, up to isomorphism, exactly the posets of order dimension 2. A chain in $P_\pi := ([n], \preceq_\pi)$ is a sequence $i_1 < i_2 < \cdots < i_k$ with $\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_k}$. Thus, chains in P_π are in bijection to **increasing subsequences** of π. Conversely, one checks that filters (via their minimal elements) are in bijection to decreasing subsequences. It follows from Theorem 2.7 that facets and vertices of $\mathcal{O}([n], \preceq_\pi, \preceq_\pi)$ are in 2-to-1 correspondence with increasing and decreasing sequences, respectively.

Example 4 (Plane posets)**.** Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a compatible double poset. We may assume that $P = \{p_1, \ldots, p_n\}$ are labelled such that $p_i \prec_\sigma p_j$ for $\sigma = +$ or $= -$ implies $i < j$. By [10, Prop. 15], $\mathbf{P}$ is a plane poset, if and only if there is a sequence of distinct numbers $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ such that for $p_i, p_j \in P$

$$\begin{aligned} p_i \prec_+ p_j \quad &\Longleftrightarrow \quad i < j \text{ and } \pi_i < \pi_j \text{ and} \\ p_i \prec_- p_j \quad &\Longleftrightarrow \quad i < j \text{ and } \pi_i > \pi_j. \end{aligned}$$

This is to say, P_+ is canonically isomorphic to $([n], \preceq_\pi)$ and P_- is canonically isomorphic to $([n], \preceq_{-\pi})$. It follows that alternating chains in $\mathbf{P}$ are in bijection to **alternating sequences**. That is, sequences $i_1 < i_2 < i_3 < \cdots < i_k$ such that $\pi_{i_1} < \pi_{i_2} > \pi_{i_3} < \cdots$. Hence, by Theorem 2.7, the facets of $\mathcal{O}(\mathbf{P})$ are in bijection to alternating sequences of π whereas the vertices are in bijection to increasing and decreasing sequences of π.

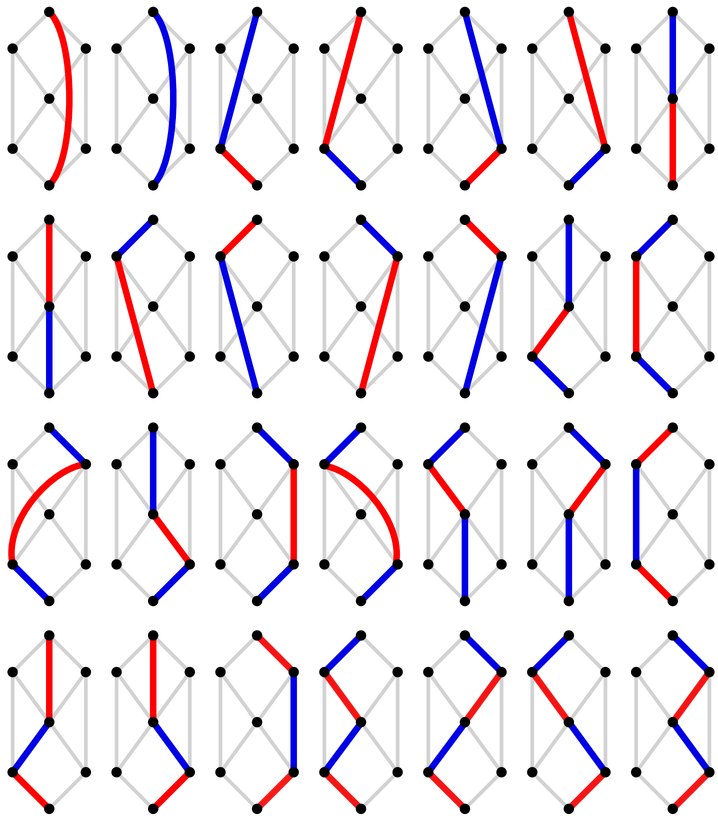

FIGURE 3. The 28 alternating chains in $\widehat{\mathbf{P}}_{XW}$.

As a consequence of the proof of Theorem 2.7 we can determine a facet-defining inequality description of double order polytopes. For an alternating chain C as in (4), let us write $\operatorname{sgn}(C) = \tau \in \{-,+\}$ if the last relation in C is $p_{k-1} \prec_\tau p_k = \widehat{1}$.

Corollary 2.8. *Let $\mathbf{P} = (P, \preceq_\pm)$ be a compatible double poset. Then $\mathcal{O}(\mathbf{P})$ is the set of points $(f,t) \in \mathbb{R}^P \times \mathbb{R}$ such that*

$$L_C(f,t) \ := \ \ell_C(f) - \operatorname{sgn}(C)\, t \ \leq \ 1$$

for all alternating chains C of $\mathbf{P}$.

Proof. Note that 0 is in the interior of $\mathcal{O}(\mathbf{P})$. Hence by Theorem 2.7 every facet-defining halfspace of $\mathcal{O}(\mathbf{P})$ is of the form $\{(\phi,t) : L(\phi,t) = \mu\ell_C + \beta t \leq 1\}$ for some alternating chain C and $\mu, \beta \in \mathbb{R}$ with $\mu > 0$. If C is an alternating chain with $\operatorname{sgn}(C) = +$, then the maximal value of ℓ_C over $2\mathcal{O}(P_+)$ is 2 and 0 over $-2\mathcal{O}(P_-)$. The values are exchanged for $\operatorname{sgn}(C) = -$. It then follows that $\mu = 1$ and $\beta = -\operatorname{sgn}(C)$. $\square$

With this, we can characterize the 2-level polytopes among compatible double order polytopes.

Corollary 2.9. *Let $\mathbf{P} = (P, \preceq_\pm)$ be a compatible double poset. Then $\mathcal{O}(\mathbf{P})$ is 2-level if and only if $\preceq_+ = \preceq_-$. In this case, the number of facets of $\mathcal{O}(\mathbf{P}_\circ)$ is twice the number of chains in $(P, \preceq)$.*

Proof. If $\preceq_+ = \preceq_- = \preceq$, then every alternating chain is a chain in P and conversely, every chain in $(P, \preceq)$ gives rise to exactly two distinct alternating chains in $(P, \preceq_\pm)$. In this case, it is straightforward to verify that the minimum of ℓ_C over $2\mathcal{O}(P)$ is -2 if $\operatorname{sgn}(C) = +$ and 0 otherwise. The claim now follows from Corollary 2.8 and together with Theorem 2.7 also yields the number of facets.

The converse follows from Proposition 2.10 by noting that if both $(P, \preceq_+, \preceq_-)$ and $(P, \preceq_-, \preceq_+)$ are compatible and tertispecial then $\preceq_+ = \preceq_-$. $\square$

In [16] a double poset $(P, \preceq_+, \preceq_-)$ is called **tertispecial** if a and b are $\prec_-$-comparable whenever $a \prec_+ b$ is a cover relation for $a, b \in P$.

Proposition 2.10. *Let $\mathbf{P} = (P, \preceq_\pm)$ be a double poset. If $\mathcal{O}(\mathbf{P})$ is 2-level, then $\mathbf{P}$ as well as $(P, \preceq_-, \preceq_+)$ are tertispecial.*

Proof. Let $\sigma = \pm$ and let $a \prec_\sigma b$ be a cover relation. The linear function $\ell_{a,b}$ is facet defining for $\mathcal{O}(P_\sigma)$ and hence yields a facet for $\mathcal{O}(\mathbf{P})$. If a, b are not comparable in $P_{-\sigma}$, then the filters $\varnothing, \{c \in P : c \succeq_- a\}$ and $\{c \in P : c \succeq_- b\}$ take three distinct values on $\ell_{a,b}$. $\square$

Let us remark that the number of facets of a given double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$ can be computed by the transfer-matrix method. Let us define the matrices $\eta^+, \eta^- \in \mathbb{R}^{\widehat{P} \times \widehat{P}}$ by

$$\eta^\sigma_{a,b} \ := \ \begin{cases} 1 & \text{if } a \prec_\sigma b \\ 0 & \text{otherwise} \end{cases}$$

for $a, b \in \widehat{P}$ and $\sigma = \pm$. Then $(\eta^+\eta^-)^k_{\hat{0},\hat{1}}$ is the number of alternating chains of $\mathbf{P}$ of length k starting with $\prec_+$ and ending with $\prec_-$. This shows the following.

Corollary 2.11. *Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a compatible double poset. Then the number of facets of $\mathcal{O}(\mathbf{P})$ is given by*

$$\left[(\mathrm{Id} - \eta^+\eta^-)^{-1}(\mathrm{Id} + \eta^+) + (\mathrm{Id} - \eta^-\eta^+)^{-1}(\mathrm{Id} + \eta^-)\right]_{\hat{0},\hat{1}}.$$

2.3. **Faces and embedded sublattices.** The **Birkhoff lattice** $\mathcal{J}(P)$ of a finite poset P is the distributive lattice given by the collection of filters of P ordered by inclusion. A subposet $\mathrm{L} \subseteq \mathcal{J}(P)$ is called an **embedded sublattice** if for any two filters $\mathsf{J}, \mathsf{J}' \in \mathcal{J}(P)$

$$\mathsf{J} \cup \mathsf{J}', \mathsf{J} \cap \mathsf{J}' \in \mathrm{L} \quad \text{if and only if} \quad \mathsf{J}, \mathsf{J}' \in \mathrm{L}.$$

For a subset $L \subseteq \mathcal{J}(P)$ of filters we write $F(L) := \operatorname{conv}(\mathbf{1}_\mathsf{J} : \mathsf{J} \in L)$. Embedded sublattices give an alternative way to characterize faces of $\mathcal{O}(P)$.

Theorem 2.12 ([38, Thm 1.1(f)])**.** *Let P be a poset and $\mathrm{L} \subseteq \mathcal{J}(P)$ a collection of filters. Then $F(\mathrm{L})$ is a face of $\mathcal{O}(P)$ if and only if L is an embedded sublattice.*

We will generalize this description to the case of double order polytopes. Throughout this section, let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a double poset. We define $\mathcal{J}(\mathbf{P}) := \mathcal{J}(P_+) \uplus \mathcal{J}(P_-)$. For any subset $\mathrm{L} \subseteq \mathcal{J}(\mathbf{P})$ we will denote by L_+ the set $\mathrm{L} \cap \mathcal{J}(P_+)$ and we define L_- accordingly. Moreover, we shall write

$$\overline{F}(\mathrm{L}) \ := \ \operatorname{conv}\left(\{(2\mathbf{1}_{\mathsf{J}_+}, +1) : \mathsf{J}_+ \in \mathrm{L}_+\} \cup \{(2\mathbf{1}_{\mathsf{J}_-}, -1) : \mathsf{J}_- \in \mathrm{L}_-\}\right) \ \subseteq \ \mathcal{O}(\mathbf{P}). \tag{5}$$

Thus, $\overline{F}(\mathrm{L}) = 2F(\mathrm{L}_+) \boxminus 2F(\mathrm{L}_-)$.

Theorem 2.13. *Let* $\mathbf{P} = (P, \preceq_+, \preceq_-)$ *be a compatible double poset and* $\mathrm{L} \subseteq \mathcal{J}(\mathbf{P})$*. Then* $\overline{F}(\mathrm{L})$ *is a face of* $\mathcal{O}(\mathbf{P})$ *if and only if*

(i) $\mathrm{L}_+ \subseteq \mathcal{J}(P_+)$ *and* $\mathrm{L}_- \subseteq \mathcal{J}(P_-)$ *are embedded sublattices and*
(ii) *for all filters* $\mathsf{J}_\sigma \subseteq \mathsf{J}'_\sigma \in \mathcal{J}(P_\sigma)$ *for* $\sigma = \pm$ *such that*

$$\mathsf{J}'_+ \setminus \mathsf{J}_+ \;=\; \mathsf{J}'_- \setminus \mathsf{J}_-$$

it holds that $\{\mathsf{J}_+, \mathsf{J}_-\} \subseteq \mathrm{L}$ *if and only if* $\{\mathsf{J}'_+, \mathsf{J}'_-\} \subseteq \mathrm{L}$.

We call a pair $\mathrm{L} = \mathrm{L}_+ \uplus \mathrm{L}_- \subseteq \mathcal{J}(\mathbf{P})$ of embedded sublattice **cooperating** if they satisfy condition (ii) of Theorem 2.13 above. We may also rephrase condition (ii) as follows.

Lemma 2.14. *Let* $\mathrm{L}_\sigma \subset \mathcal{J}(P_\sigma)$ *be an embedded sublattice for* $\sigma = \pm$*. Then* $\mathrm{L}_+, \mathrm{L}_-$ *are cooperating if and only if only if for any two filters* $\mathsf{J}_- \in \mathrm{L}_-, \mathsf{J}_+ \in \mathrm{L}_+$ *the following holds:*

(a) *For* $A \subseteq \min(\mathsf{J}_+) \cap \min(\mathsf{J}_-)$ *we have* $\mathsf{J}_- \setminus A \in \mathrm{L}_-$ *and* $\mathsf{J}_+ \setminus A \in \mathrm{L}_+$*, and*
(b) *for* $B \subseteq \max(P_+ \setminus \mathsf{J}_+) \cap \max(P_- \setminus \mathsf{J}_-)$ *we have* $\mathsf{J}_- \cup B \in \mathrm{L}_-$ *and* $\mathsf{J}_+ \cup B \in \mathrm{L}_+$*.*

Proof. It follows from the definition that for sets as stated, condition (ii) implies $\mathsf{J}_\sigma \setminus A, \mathsf{J}_\sigma \cup B \in \mathrm{L}_\sigma$ for $\sigma = \pm$. For the converse direction, let $\mathsf{J}_\sigma \subseteq \mathsf{J}'_\sigma$ such that $\mathsf{J}'_\sigma \in \mathrm{L}_\sigma$ for $\sigma = \pm$. Assume that $D := \mathsf{J}'_+ \setminus \mathsf{J}_+ = \mathsf{J}'_- \setminus \mathsf{J}_-$. Then $A := \min(D) \subseteq \min(\mathsf{J}'_+) \cap \min(\mathsf{J}'_-)$ and by (a), $\mathsf{J}'_\sigma \setminus A \in \mathrm{L}_\sigma$ for $\sigma = \pm$ and induction on $|D|$ yields the claim. □

Theorem 2.13 can be deduced from the description of facets in Theorem 2.7. We will give an alternative proof using Gröbner bases in Section 5. In conjunction with Theorem 5.2, we can read the dimension of $\overline{F}(\mathrm{L})$ from the cooperating pair L. In the case of order polytopes, the canonical triangulation (see Section 4) of $\mathcal{O}(P)$ yields the following.

Corollary 2.15. *Let* $F \subseteq \mathcal{O}(P)$ *be a face with corresponding embedded sublattice* $\mathrm{L} \subseteq \mathcal{J}(P)$*. Then* $\dim F = l(\mathrm{L}) - 1$ *where* $l(\mathrm{L})$ *is the length of a longest chain in* L.

Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a double poset and let $C_\sigma \subseteq \mathcal{J}(P_\sigma)$ be a chain of filters in $(P, \preceq_\sigma)$ for $\sigma = \pm$. The pair of chains $C = C_+ \uplus C_-$ is **non-interfering** if $\min(\mathsf{J}_+) \cap \min(\mathsf{J}_-) = \varnothing$ for any $\mathsf{J}_+ \in C_+$ and $\mathsf{J}_- \in C_-$. For $\mathrm{L} \subseteq \mathcal{J}(\mathbf{P})$, we denote by $cl(\mathrm{L})$ the maximum over $|C| = |C_+| + |C_-|$ where $C \subseteq \mathrm{L}$ is a pair of non-interfering chains.

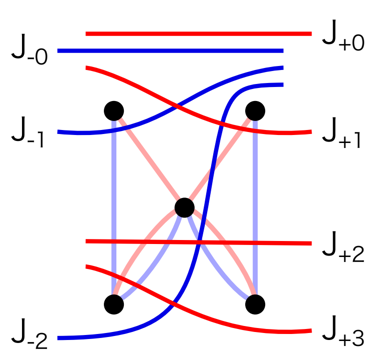

FIGURE 4. A maximal non-interfering set of filters in $\mathbf{P}_{XW}$. A red or blue curve denotes the filter consisting of all elements above the curve.

Corollary 2.16. *Let* $\mathbf{P}$ *be a compatible double poset and let* $\mathrm{L} \subseteq \mathcal{J}(\mathbf{P})$ *be a cooperating pair of embedded sublattices. Then* $\dim \overline{F}(\mathrm{L}) = cl(\mathrm{L}) - 1$.

As a consequence of Theorem 2.12, $[\mathbf{1}_{\mathsf{J}}, \mathbf{1}_{\mathsf{J}'}] \subseteq \mathcal{O}(P)$ is an edge if and only if $\mathsf{J} \subseteq \mathsf{J}'$ are filters of P such that $\mathsf{J}' \setminus \mathsf{J}$ is a connected poset. Of course, this description captures all the *horizontal* edges of $\mathcal{O}(\mathbf{P})$. The upcoming characterization of vertical edges follows from Theorem 2.7 but we supply a direct proof.

Corollary 2.17. *Let $\mathbf{P}$ be a compatible double poset and let $\mathsf{J}_+ \subseteq P_+$ and $\mathsf{J}_- \subseteq P_-$ be filters. Then $(2\mathbf{1}_{\mathsf{J}_+}, +1)$ and $(-2\mathbf{1}_{\mathsf{J}_-}, -1)$ are the endpoints of a vertical edge of $\mathcal{O}(\mathbf{P})$ if and only if $\mathbf{1}_{\mathsf{J}_+} - \mathbf{1}_{\mathsf{J}_-}$ is a vertex of $\overline{\mathcal{O}}(\mathbf{P})$ if and only if*

$$\min(\mathsf{J}_+) \cap \min(\mathsf{J}_-) \;=\; \varnothing \quad \textit{and} \quad \max(P_+ \setminus \mathsf{J}_+) \cap \max(P_- \setminus \mathsf{J}_-) \;=\; \varnothing.$$

Proof. The first equivalence follows from the fact that

$$\mathcal{O}(\mathbf{P}) \cap \{(\phi, t) : t = 0\} \;=\; (\mathcal{O}(P_+) - \mathcal{O}(P_-)) \times \{0\}$$

and $\mathbf{1}_{\mathsf{J}_+} - \mathbf{1}_{\mathsf{J}_-}$ is the midpoint between $(2\mathbf{1}_{\mathsf{J}_+}, +1)$ and $(-2\mathbf{1}_{\mathsf{J}_-}, -1)$.

Before we come to the second claim, let us note that the face partition of a vertex $\mathbf{1}_{\mathsf{J}}$ for a poset $(P, \preceq)$ is given by $\{\mathsf{J}, P \setminus \mathsf{J}\}$. Thus, if $\mathbf{1}_{\mathsf{J}_+} - \mathbf{1}_{\mathsf{J}_-}$ is a vertex of $\overline{\mathcal{O}}(\mathbf{P})$, then there is a linear function $\ell(f) = \sum_{a \in P} \ell_a f(a)$ such that $\mathcal{O}(P_+)^{\ell} = \{\mathbf{1}_{\mathsf{J}_+}\}$ and $\mathcal{O}(P_-)^{-\ell} = \{\mathbf{1}_{\mathsf{J}_-}\}$. Corollary 2.2 then yields that $\ell_a > 0$ for each $a \in \min(\mathsf{J}_+)$ and $\ell_a < 0$ for $a \in \min(\mathsf{J}_-)$. The same reasoning applies to $\max(P_+ \setminus \mathsf{J}_+)$ and $\max(P_- \setminus \mathsf{J}_-)$ and shows necessity.

Let $b \in \min(\mathsf{J}_+)$. If $b \notin \mathsf{J}_-$, then the linear function $\ell(f) := f(b)$ is maximized over $\mathcal{O}(P_+)$ at every filter that contains b and over $-\mathcal{O}(P_-)$ at every filter that does not contain b. If $b \in \mathsf{J}_-$, then, by assumption, $b \notin \min(\mathsf{J}_-)$ and there is some $p_2 \in \min(\mathsf{J}_-)$ with $p_2 \prec_- b$. Now, if $p_2 \in \mathsf{J}_+$, then there is $p_3 \in \min(\mathsf{J}_+)$ with $p_3 \prec_+ p_2$ and so on. Compatibility now assures us that we get a descending alternating chain of the form

$$\widehat{1} \;\succ_+\; b =: p_1 \;\succ_-\; p_2 \;\succ_+\; p_3 \;\succ_-\; \cdots \;\succ_\sigma\; p_k \;\succ_{-\sigma}\; a \;\succ_\sigma\; \widehat{0}$$

where $p_2, p_4, p_6, \ldots \in \min(\mathsf{J}_-) \cap \mathsf{J}_+$ and $p_1, p_3, p_5, \ldots \in \min(\mathsf{J}_+) \cap \mathsf{J}_-$ and $a \in \min(\mathsf{J}_{-\sigma}) \setminus \mathsf{J}_\sigma$. Consider the associated linear function

$$\ell(f) \;=\; f(p_0) - f(p_1) + f(p_2) - \cdots + (-1)^k f(p_k) + (-1)^{k+1} f(a)$$

for $f \in \mathbb{R}^P$. We claim that $\ell(\mathbf{1}_{\mathsf{J}'_+}) \leq 1$ for each filter $\mathsf{J}'_+ \subseteq P_+$ and with equality if $b \in \mathsf{J}'_+$. Indeed, if $p_{2i+1} \in \mathsf{J}'_+$, then $p_{2i} \in \mathsf{J}'_+$ for all $i \geq 1$. Conversely, $\ell(-\mathbf{1}_{\mathsf{J}'_-}) \leq 0 = \ell(-\mathbf{1}_{\mathsf{J}_-})$ for each filter $\mathsf{J}'_- \subseteq P_-$. This follows from the fact that $p_{2i} \in \mathsf{J}'_-$ implies $p_{2i-1} \in \mathsf{J}'_-$ for each $i \geq 1$.

For $a \in \max(P_+ \setminus \mathsf{J}_+)$ the situation is similar and we search for $b \in \max(P_- \setminus \mathsf{J}_-)$ with $a \prec_- b$ in the case that $a \notin \mathsf{J}_-$. This yields a linear function $\ell \in -N_{P_-}(\mathbf{1}_{\mathsf{J}_-})$ that is maximized over $\mathcal{O}(P_+)$ at filters $\mathbf{1}_{\mathsf{J}'_+}$ with $a \notin \mathsf{J}'_+$. Summing these linear functions for $b \in \min(\mathsf{J}_+)$ and $a \in \max(P_+ \setminus \mathsf{J}_+)$ yields a linear function ℓ^+ with $\mathcal{O}(P_+)^{\ell^+} = \{\mathbf{1}_{\mathsf{J}_+}\}$ and $\mathbf{1}_{\mathsf{J}_-} \in \mathcal{O}(P_-)^{-\ell^+}$.

Of course, the same reasoning applies to J_- instead of J_+ and it follows that $\ell^+ - \ell^-$ is uniquely maximized at $\mathbf{1}_{\mathsf{J}_+} - \mathbf{1}_{\mathsf{J}_-}$ over $\overline{\mathcal{O}}(\mathbf{P}) = \mathcal{O}(P_+) - \mathcal{O}(P_-)$. ☐

2.4. **Polars and valuation polytopes.** A real-valued **valuation** on a finite distributive lattice $(\mathcal{J}, \vee, \wedge)$ is a function $h : \mathcal{J} \to \mathbb{R}$ such that for any $a, b \in \mathcal{J}$,

$$h(a \vee b) \;=\; h(a) + h(b) - h(a \wedge b) \tag{6}$$

and $h(\widehat{0}) = 0$. Geissinger [13] studied the **valuation polytope**

$$\mathrm{Val}(\mathcal{J}) \;:=\; \{h : \mathcal{J} \to [0, 1] : h \text{ valuation}\}$$

and conjectured that its vertices are exactly the valuations with values in $\{0, 1\}$. This was shown by Dobbertin [7]. Not much is known about the valuation polytope and Stanley's '5-'-Exercise [35, Ex. 4.61(h)] challenges the reader to find interesting combinatorial properties of $\mathrm{Val}(\mathcal{J})$. In this section, we prove a curious relation between valuation polytopes and order polytopes.

It follows from Birkhoff's fundamental theorem (cf. [35, Sect. 3.4]) that any finite distributive lattice $\mathcal{J}$ is of the form $\mathcal{J} = \mathcal{J}(P)$, that is, it is the lattice of filters of some poset P. In particular, for every valuation $h : \mathcal{J}(P) \to \mathbb{R}$ there is a unique $h_0 : P \to \mathbb{R}$ such that

$$h(\mathsf{J}) \;=\; \sum_{a \in \mathsf{J}} h_0(a),$$

for every filter $\mathsf{J} \subseteq P$. Hence, $\mathrm{Val}(\mathcal{J})$ is linearly isomorphic to the $|P|$-dimensional polytope

$$\mathrm{Val}_0(P) \;:=\; \{h_0 : P \to \mathbb{R} : 0 \le h(\mathsf{J}) \le 1 \text{ for all filters } \mathsf{J} \subseteq P\}.$$

We denote by

$$S^{\triangle} \;=\; \{\ell \in (\mathbb{R}^n)^* : \ell(s) \le 1 \text{ for all } s \in S\}$$

the **polar** of a set $S \subset \mathbb{R}^d$. For a polytope $\mathcal{P} \subset \mathbb{R}^d$ we write $\mathrm{tprism}(\mathcal{P}) := \mathcal{P} \boxminus \mathcal{P} \subset \mathbb{R}^{d+1}$ for the **twisted prism** of $\mathcal{P}$.

Theorem 2.18. *For any finite poset P*

$$\mathcal{O}(\mathbf{P}_\circ)^{\triangle} \;=\; \mathcal{O}(P, \preceq, \preceq)^{\triangle} \;=\; \mathrm{tprism}(-\mathrm{Val}_0(P)).$$

Proof. For a chain $C = \{a_0 \prec a_1 \prec \cdots \prec a_k\}$ in P, we define

$$\ell'_C(f) \;:=\; \sum_{i=0}^{k} (-1)^{k-i} f(a_i)$$

and $L'_C(f, t) := \ell'_C(f) - t$. It follows from Corollary 2.8 and Corollary 2.9 that

$$\mathcal{O}(\mathbf{P}_\circ)^{\triangle} \;=\; \mathrm{conv}(\pm L'_C(f, t) : C \subseteq P \text{ chain}).$$

It is shown in Dobbertin [7, Theorem B] that

$$\mathrm{Val}_0(P) \;=\; \mathrm{conv}\left(\ell'_C : C \subseteq P \text{ chain}\right),$$

from which the claim follows. $\square$

As a direct consequence, we note the following.

Corollary 2.19. *Let P be a finite poset. Then* $\mathrm{tprism}(\mathrm{Val}(P))$ *is* 2*-level.*

Proof. Since $\mathcal{O}(\mathbf{P}_\circ)$ is centrally-symmetric and, by Corollary 2.9, 2-level, it follows that every vertex of $\mathcal{O}(\mathbf{P}_\circ)$ takes the values $+1$ or -1 on every facet-defining linear function. The vertices correspond to facet normals under polarity, which shows that $\mathcal{O}(\mathbf{P}_\circ)^{\triangle}$ is 2-level. Theorem 2.18 now yields the claim. $\square$

We can make the connection to valuations more transparent by considering valuations with values in $[-1, 1]$. Let $\mathrm{Val}^{\pm}(\mathcal{J}(P))$ denote the corresponding polytope, then

$$\mathrm{Val}_0^{\pm}(P) \;=\; \{h_0 : P \to \mathbb{R} : -1 \le h(\mathsf{J}) \le 1 \text{ for all filters } \mathsf{J} \subseteq P\} \;=\; (\mathcal{O}(P) \cup -\mathcal{O}(P))^{\triangle}. \tag{7}$$

Now, the convex hull of $\mathcal{O}(P) \cup -\mathcal{O}(P)$ is exactly the image of $\mathcal{O}(\mathbf{P}_\circ)$ under the projection $\pi : \mathbb{R}^P \times \mathbb{R} \to \mathbb{R}^P$ with $\pi(f,t) = \frac{1}{2}f$. Hence,

$$\mathrm{Val}_0^\pm(P) \;\cong\; \pi(\mathcal{O}(\mathbf{P}_\circ))^\triangle \;\cong\; \mathcal{O}(\mathbf{P}_\circ)^\triangle \cap \mathrm{im}(\pi^*) \;\cong\; \mathrm{tprism}(-2\mathrm{Val}_0(P)) \cap (\mathbb{R}^P \times \{0\}),$$

by Theorem 2.18. If we now view $\mathrm{tprism}(-\mathrm{Val}_0(P))$ as a Cayley sum, we obtain

Corollary 2.20. *For any poset P*

$$\mathrm{Val}_0^\pm(P) \;=\; \mathrm{Val}_0(P) - \mathrm{Val}_0(P).$$

A polytope $\mathcal{P}$ with vertices in a lattice $\Lambda \subset \mathbb{R}^n$ is **reflexive** if $\mathcal{P}^\triangle$ is a lattice polytope with respect to the dual lattice $\Lambda^\vee := \{\ell \in (\mathbb{R}^n)^* : \ell(x) \in \mathbb{Z} \text{ for all } x \in \Lambda\}$. For two polytopes $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^n$, write $\Gamma(\mathcal{P},\mathcal{Q}) := \mathrm{conv}(\mathcal{P} \cup -\mathcal{Q})$. Thus, $\Gamma(\mathcal{P},\mathcal{Q})$ is the projection of $\mathcal{P} \boxminus \mathcal{Q}$ onto the first n coordinates. The polytopes $\Gamma(\mathcal{O}(P),\mathcal{O}(P))$ where studied by Hibi, Matsuda, and Tsuchiya [22, 21] in the context of Gorenstein polytopes, i.e. lattice polytopes $\mathcal{P}$ such that $r\mathcal{P}$ is reflexive for some $r \in \mathbb{Z}_{>0}$. By taking polars, we obtain the following from (7) and Corollary 2.20.

Corollary 2.21. *For any poset P,*

$$\Gamma(\mathcal{O}(P),\mathcal{O}(P)) \;=\; (\mathrm{Val}_0(P) - \mathrm{Val}_0(P))^\triangle.$$

In particular, $\Gamma(\mathcal{O}(P),\mathcal{O}(P))$ is reflexive.

An explicit description of the face lattices of $\mathrm{Val}(P)$, $\mathrm{Val}^\pm(P)$ as well as $\Gamma(\mathcal{O}(P),\mathcal{O}(P))$ can be obtained from Theorem 2.13.

This theorem also yields information about the polars of $\mathcal{O}(P,\preceq_+,\preceq_-)$ for compatible double posets. For a poset that is not compatible, the next result shows that the origin is not contained in the interior of $\mathcal{O}(\mathbf{P})$ and hence the polar is not bounded.

Proposition 2.22. *Let $\mathbf{P}$ be a double poset. Then $\mathcal{O}(\mathbf{P})$ contains the origin in its interior if and only if $\mathbf{P}$ is compatible.*

Proof. If $\mathbf{P}$ is compatible, then Corollary 2.8 shows that $\mathbf{0}$ strictly satisfies all facet-defining inequalities. If $\mathbf{P}$ is not compatible, then it contains an alternating cycle C. It follows easily that $\ell_C \le 0$ on $\mathcal{O}(P_+)$ and $-\mathcal{O}(P_-)$ and hence $\mathcal{O}(\mathbf{P})$ is contained in the negative halfspace of $H = \{(f,t) : \ell_C(f) \le 0\}$. Moreover, $\mathbf{0} \in H \cap \mathcal{O}(\mathbf{P})$, which shows that $\mathbf{0} \notin \mathrm{relint}\,\mathcal{O}(\mathbf{P})$. □

3. Anti-blocking polytopes

3.1. **Anti-blocking polytopes and Minkowski sums.** A polytope $\mathcal{P} \subset \mathbb{R}^n_{\ge 0}$ is called **anti-blocking** if

$$q \in \mathcal{P} \;\text{ and }\; 0 \le p \le q \quad\Longrightarrow\quad p \in \mathcal{P}, \tag{8}$$

where $p \le q$ refers to componentwise order in $\mathbb{R}^n$. The notion of anti-blocking polyhedra was introduced by Fulkerson [12] in connection with min-max-relations in combinatorial optimization; our main reference for anti-blocking polytopes is Schrijver [31, Sect. 9.3]. In this section, we consider the Cayley sums

$$\mathcal{P} \boxminus \mathcal{Q} \;=\; \mathrm{conv}(\mathcal{P} \times \{1\} \cup (-\mathcal{Q}) \times \{-1\}),$$

where $\mathcal{P}$ and $\mathcal{Q}$ are anti-blocking polytopes. As before, we write $\mathrm{tprism}(\mathcal{P})$ for $\mathcal{P} \boxminus \mathcal{P}$. Our main source of examples will be the class of stable set polytopes: For a graph $G = (V,E)$, a **stable set** is a subset $S \subseteq V$ such that $\binom{S}{2} \cap E = \varnothing$. For simplicity, we will assume that $V = [n]$ and we write $\mathbf{1}_S \in \{0,1\}^n$ for the characteristic vector of a stable set S. The **stable set polytope** of G is the anti-blocking polytope

$$\mathcal{P}_G \ := \ \operatorname{conv}(\mathbf{1}_S : S \subseteq V \text{ stable set}) \ \subseteq \ \mathbb{R}^n.$$

The class of *perfect* graphs is particularly interesting in this respect. Lovász [27] characterized perfect graphs in terms of their stable set polytopes and we use his characterization as a definition of perfect graphs. A **clique** of a graph $G = (V, E)$ is a subset $C \subseteq V$ such that $\binom{C}{2} \subseteq E$. For a vector $x \in \mathbb{R}^n$ and a subset $J \subseteq [n]$, we write $x(J) = \sum_{j \in J} x_j$.

Theorem 3.1 ([27])**.** *A graph $G = ([n], E)$ is **perfect** if and only if*

$$\mathcal{P}_G \ = \ \{x \in \mathbb{R}^n : x \geq 0, x(C) \leq 1 \textit{ for all cliques } C \subseteq [n]\}.$$

In this language, we can express the *chain polytope* of a poset P as a stable set polytope: The **comparability graph** $G(P)$ of a poset $(P, \preceq)$ is the undirected graph with vertex set P and edge set $\{xy : x \prec y \text{ or } y \prec x\}$. Note that cliques in $G(P)$ are exactly the chains of P. For a poset $P = ([n], \preceq)$ the comparability graph $G(P)$ is perfect and hence

$$\mathcal{C}(P) \ = \ \{x \in \mathbb{R}^n : x \geq 0, x(C) \leq 1 \text{ for all chains } C \subseteq [n]\} \ = \ \mathcal{P}_{G(P)}.$$

If $\mathcal{P} \subset \mathbb{R}^n$ is an anti-blocking polytope, then there are $\mathbf{c}_1, \ldots, \mathbf{c}_r \in \mathbb{R}^n_{\geq 0}$ such that

$$\mathcal{P} \ = \ \{\mathbf{c}_1, \ldots, \mathbf{c}_r\}^{\downarrow} \ := \ \mathbb{R}^n_{\geq 0} \cap (\operatorname{conv}(\mathbf{c}_1, \ldots, \mathbf{c}_r) - \mathbb{R}^n_{\geq 0}). \tag{9}$$

The unique minimal such set, denoted by $V^{\downarrow}(P)$, is given by the minimal elements of the vertex set of $\mathcal{P}$ with respect to the partial order $\leq$. It also follows from (8) and the Minkowski–Weyl theorem that there is a minimal collection $\mathbf{d}_1, \ldots, \mathbf{d}_s \in \mathbb{R}^n_{\geq 0}$ such that

$$\mathcal{P} \ = \ \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \langle \mathbf{d}_i, \mathbf{x} \rangle \leq 1 \text{ for all } i = 1, \ldots, s\}$$

For a polytope $\mathcal{Q} \subseteq \mathbb{R}^n_{\geq 0}$, its **associated** anti-blocking polytope is the set

$$A(\mathcal{Q}) \ := \ \{\mathbf{d} \in \mathbb{R}^n_{\geq 0} : \langle \mathbf{d}, \mathbf{x} \rangle \leq 1 \text{ for all } \mathbf{x} \in \mathcal{Q}\}.$$

The following is the structure theorem for anti-blocking polytopes akin to the bipolar theorem for convex bodies.

Theorem 3.2 ([31, Thm. 9.4])**.** *Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional anti-blocking polytope with*

$$\mathcal{P} \ = \ \{\mathbf{c}_1, \ldots, \mathbf{c}_r\}^{\downarrow} \ = \ \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \langle \mathbf{d}_i, \mathbf{x} \rangle \leq 1 \textit{ for all } i = 1, \ldots, s\}$$

for some $\mathbf{c}_1, \ldots, \mathbf{c}_r, \mathbf{d}_1, \ldots, \mathbf{d}_s \in \mathbb{R}^n_{\geq 0}$. Then

$$A(\mathcal{P}) \ = \ \{\mathbf{d}_1, \ldots, \mathbf{d}_s\}^{\downarrow} \ = \ \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \langle \mathbf{c}_i, \mathbf{x} \rangle \leq 1 \textit{ for all } i = 1, \ldots, r\}.$$

In particular, $A(A(\mathcal{P})) = \mathcal{P}$.

Before we come to our first result regarding Cayley- and Minkowski-sums of anti-blocking polytopes, we note the following fact. We write $V(\mathcal{P})$ for the vertex set of a polytope $\mathcal{P}$.

Proposition 3.3. *Let $\mathcal{P}_1, \mathcal{P}_2$ be two full-dimensional anti-blocking polytopes. Then the vertices of $\operatorname{conv}(\mathcal{P}_1 \cup -\mathcal{P}_2)$ are exactly $(V(\mathcal{P}_1) \cup V(-\mathcal{P}_2)) \setminus \{\mathbf{0}\}$.*

For a polytope $\mathcal{P} \subset \mathbb{R}^n$ and a vector $\mathbf{c} \in \mathbb{R}^n$, we denote by $\mathcal{P}^{\mathbf{c}}$ the face of $\mathcal{P}$ that maximizes the linear function $\mathbf{x} \mapsto \langle \mathbf{c}, \mathbf{x} \rangle$.

Proof. It suffices to show that every $\mathbf{v} \in V(\mathcal{P}_1) \setminus \{\mathbf{0}\}$ is a vertex of $\operatorname{conv}(\mathcal{P}_1 \cup -\mathcal{P}_2)$. Let $\mathbf{c} \in \mathbb{R}^n$ such that $\mathcal{P}_1^{\mathbf{c}} = \{\mathbf{v}\}$. Since $\mathbf{v} \neq 0$, there is some $\mathbf{d} \in \mathbb{R}^n_{\geq 0}$ such that $\langle \mathbf{d}, \mathbf{u}_1 \rangle \leq 1$ for all $\mathbf{u}_1 \in \mathcal{P}_1$ and $\langle \mathbf{d}, \mathbf{v} \rangle = 1$. Hence, for any $\mu \geq 0$, $\mathcal{P}_1^{\mathbf{c}+\mu\mathbf{d}} = \{v\}$. Now, $\langle \mathbf{d}, -\mathbf{u}_2 \rangle \leq 0$ for all $\mathbf{u}_2 \in P_2$. In particular, for $\mu > 0$ sufficiently large,

$$\langle \mathbf{c} + \mu\mathbf{d}, \mathbf{u}_2 \rangle \ \leq\ \langle \mathbf{c}, \mathbf{u}_2 \rangle \ <\ \mu + \langle \mathbf{c}, \mathbf{v} \rangle \ =\ \langle \mathbf{c} + \mu\mathbf{d}, \mathbf{v} \rangle,$$

which shows that $\mathbf{v}$ uniquely maximizes $\langle \mathbf{c} + \mu\mathbf{d}, \mathbf{u} \rangle$ over $\operatorname{conv}(\mathcal{P}_1 \cup -\mathcal{P}_2)$. □

For $\mathbf{d} \in \mathbb{R}^n_{\geq 0}$ and $I \subseteq [n]$, we write $\mathbf{d}^{[I]}$ for the vector with

$$(\mathbf{d}^{[I]})_j \ =\ \begin{cases} d_j & \text{for } j \in I \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.4. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be full-dimensional anti-blocking polytopes. Then*

$$(\mathcal{P}_1 - \mathcal{P}_2)^{\triangle} \ =\ \operatorname{conv}(A(\mathcal{P}_1) \cup -A(\mathcal{P}_2)).$$

Moreover,

$$(2\mathcal{P}_1 \boxminus 2\mathcal{P}_2)^{\triangle} \ =\ -A(\mathcal{P}_2) \boxminus -A(\mathcal{P}_1).$$

Proof. Let us denote the right-hand side of the first equation by $\mathcal{Q}$. Note that $\langle \mathbf{u}_1, -\mathbf{v}_2 \rangle \leq 0$ for $\mathbf{u}_1 \in A(P_1)$ and $\mathbf{v}_2 \in P_2$. This shows that $\langle \mathbf{u}_1, \mathbf{v} \rangle \leq 1$ for all $\mathbf{v} \in P_1 - P_2$. By symmetry, this yields $\mathcal{Q} \subseteq (\mathcal{P}_1 - \mathcal{P}_2)^{\triangle}$.

For the converse, observe that every vertex of $\mathcal{Q}$ is of the form $\mathbf{d}^{[I]}$ with $\mathbf{d} \in V^{\downarrow}(A(\mathcal{P}_1)) \cup -V^{\downarrow}(A(\mathcal{P}_2))$. It follows that $\mathbf{z} \in \mathcal{Q}^{\triangle}$ if and only if $\langle \mathbf{d}^{[I]}, \mathbf{z} \rangle \leq 1$ for all $\mathbf{d} \in V^{\downarrow}(A(\mathcal{P}_1)) \cup -V^{\downarrow}(A(\mathcal{P}_2))$ and all $I \subseteq [n]$. For $\mathbf{z} \in \mathcal{Q}^{\triangle}$ write $\mathbf{z} = \mathbf{z}^1 - \mathbf{z}^2$ with $\mathbf{z}^1, \mathbf{z}^2 \geq 0$ and $\operatorname{supp}(\mathbf{z}^1) \cap \operatorname{supp}(\mathbf{z}^2) = \varnothing$, where for any $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$ we set $\operatorname{supp}(\mathbf{z}) := \{i : z_i \neq 0\}$. We claim that $\mathbf{z}^i \in \mathcal{P}_i$ for $i = 1, 2$. Indeed, let $I = \operatorname{supp}(\mathbf{z}^1)$. Then for any $\mathbf{d} \in V^{\downarrow}(\mathcal{P}_1)$ we have

$$\langle \mathbf{d}, \mathbf{z}^1 \rangle \ =\ \langle \mathbf{d}^{[I]}, \mathbf{z} \rangle \ \leq\ 1$$

and hence $\mathbf{z}^1 \in \mathcal{P}_1$. Applying the same argument to $\mathbf{z}^2$ shows that $\mathbf{z} \in \mathcal{P}_1 - \mathcal{P}_2$ and hence $(\mathcal{P}_1 - \mathcal{P}_2)^{\triangle} \subseteq \mathcal{Q}$.

For the second claim, note that any linear function on $\mathbb{R}^n \times \mathbb{R}$ that maximizes on a *vertical* facet of $2\mathcal{P}_1 \boxminus 2\mathcal{P}_2$ is of the form $\alpha_{\mathbf{d}} \langle \mathbf{d}, \mathbf{x} \rangle + \delta_{\mathbf{d}} t$ for $\mathbf{d}$ a vertex of $(\mathcal{P}_1 - \mathcal{P}_2)^{\triangle}$ and some $\alpha_{\mathbf{d}}, \delta_{\mathbf{d}} \in \mathbb{R}$ with $\alpha_{\mathbf{d}} > 0$. By the first claim and Proposition 3.3, it follows that $\mathbf{d} \in (V(A(\mathcal{P}_1)) \cup V(-A(\mathcal{P}_2))) \setminus \{\mathbf{0}\}$.

If $\mathbf{d} \in V(A(\mathcal{P}_1)) \setminus \{\mathbf{0}\}$, then $\langle \mathbf{d}, \mathbf{u}_1 \rangle \leq 1$ is tight for $\mathbf{u}_1 \in \mathcal{P}_1$ whereas $\langle \mathbf{d}, -\mathbf{u}_2 \rangle \leq 0$ is tight for $-\mathbf{u}_2 \in -\mathcal{P}_2$. Hence,

$$\langle \mathbf{d}, \mathbf{x} \rangle - t \ \leq\ 1$$

is the corresponding facet-defining halfspace. Similarly, if $-\mathbf{d} \in -V(A(\mathcal{P}_1)) \setminus \{\mathbf{0}\}$, then

$$\langle -\mathbf{d}, x \rangle + t \ \leq\ 1$$

is facet-defining. Together with the two horizontal facets $\langle \mathbf{0}, \mathbf{x} \rangle \pm t \leq 1$ this yields an inequality description of $(-A(\mathcal{P}_2) \boxminus -A(\mathcal{P}_1))^{\triangle}$, which proves the second claim. □

Theorem 3.4 together with Theorem 3.2 has a nice implication that was used in [30] in connection with Hansen polytopes.

Corollary 3.5. *For any full-dimensional anti-blocking polytope $\mathcal{P} \subset \mathbb{R}^n$, the polytope $\mathcal{P} \boxminus A(\mathcal{P})$ is linearly isomorphic to its polar $(\mathcal{P} \boxminus A(\mathcal{P}))^{\triangle}$. In particular, $\mathcal{P} \boxminus A(\mathcal{P})$ is self-dual.*

3.2. **Stable set polytopes of double graphs and double chain polytopes.** A **double graph** is a triple $\mathbf{G} = (V, E_+, E_-)$ consisting of a node set V with two sets of edges $E_+, E_- \subseteq \binom{V}{2}$. Again, we write $G_+ = (V, E_+)$ and $G_- = (V, E_-)$ to denote the two underlying ordinary graphs. The results of the preceding sections prompt the definition of **stable set polytope** of a double graph

$$\mathcal{P}_{\mathbf{G}} \ := \ 2\mathcal{P}_{G_+} \boxminus 2\mathcal{P}_{G_-}.$$

For a double graph $\mathbf{G}$, define the **complement graph** as $\overline{\mathbf{G}} = (V, E_-^c, E_+^c)$. Then Theorem 3.4 implies the following relation.

Corollary 3.6. *Let $\mathbf{G}$ be a perfect double graph. Then $\mathcal{P}_{\mathbf{G}}^{\triangle}$ is linearly isomorphic to $\mathcal{P}_{\overline{\mathbf{G}}}$.*

Proof. We have

$$\mathcal{P}_{\mathbf{G}}^{\triangle} = (2\mathcal{P}_{G_+} \boxminus 2\mathcal{P}_{G_-})^{\triangle} \ = \ -A(\mathcal{P}_{G_-}) \boxminus -A(\mathcal{P}_{G_+}) \ = \ -\mathcal{P}_{\overline{G}_-} \boxminus -\mathcal{P}_{\overline{G}_+} \ \cong \ \mathcal{P}_{\overline{\mathbf{G}}}. \qquad \square$$

In particular, a double poset $\mathbf{P} = (P, \preceq_\pm)$ gives rise to a double graph $\mathbf{G}(\mathbf{P}) = (G(P_+), G(P_-))$ and the double chain polytope of $\mathbf{P}$ is simply $\mathcal{C}(\mathbf{P}) = \mathcal{P}_{\mathbf{G}(\mathbf{P})}$, the **double chain polytope** of $\mathbf{P}$. Theorem 3.4 directly gives a facet description of the double chain polytope. Note that compatibility is not required.

Theorem 3.7. *Let $\mathbf{P}$ be a double poset and $\mathcal{C}(\mathbf{P})$ its double chain polytope. Then $(g, t) \in \mathbb{R}^P \times \mathbb{R}$ is contained in $\mathcal{C}(\mathbf{P})$ if and only if*

$$\sum_{a \in C_+} g(a) - t \ \leq \ 1 \qquad \textit{and} \qquad \sum_{a \in C_-} -g(a) + t \ \leq \ 1,$$

where $C_+ \subseteq P_+$ and $C_- \subseteq P_-$ ranges of all chains.

For the usual order- and chain polytope, Hibi and Li [20] showed that $\mathcal{O}(P)$ has at most as many facets as $\mathcal{C}(P)$ and equality holds if and only if P does not contain the 5-element poset with Hasse diagram 'X'. This is different in the case of double poset polytopes.

Corollary 3.8. *Let $(P, \preceq)$ be a poset. Then $\mathcal{O}(\mathbf{P}_\circ)$ and $\mathcal{C}(\mathbf{P}_\circ)$ have the same number of facets.*

Proof. Alternating chains in $\mathbf{P}_\circ$ are in bijection to twice the number of chains in P. $\square$

However, it is not true that $\mathcal{O}(\mathbf{P}_\circ)$ is always combinatorially isomorphic to $\mathcal{C}(\mathbf{P}_\circ)$.

Example 5. Let P be the 5-element poset with Hasse diagram 'X'. Then the face vectors of $\mathcal{O}(\mathbf{P}_\circ)$ and $\mathcal{C}(\mathbf{P}_\circ)$ are

$$\begin{aligned} f(\mathcal{O}(\mathbf{P}_\circ)) \ &= \ (16, 88, 204, 240, 144, 36) \\ f(\mathcal{C}(\mathbf{P}_\circ)) \ &= \ (16, 88, 222, 276, 162, 36). \end{aligned}$$

Hibi and Li [20] conjectured that $f(\mathcal{O}(P)) \leq f(\mathcal{C}(P))$ componentwise. Computations suggest that the same relation should hold for the double poset polytopes of induced double posets.

Conjecture 1. *Let $\mathbf{P} = (P, \preceq, \preceq)$ be a double poset induced by a poset $(P, \preceq)$. Then*

$$f_i(\mathcal{O}(\mathbf{P})) \ \leq \ f_i(\mathcal{C}(\mathbf{P}))$$

for $0 \leq i \leq |P|$.

An extension of the conjecture to general compatible double posets fails, as the following example shows.

Example 6. Let $\mathbf{A}_n$ be an **alternating chain** of length n, that is, $\mathbf{P}$ is a double poset on elements $a_1, a_2, \ldots, a_{n+1}$ with cover relations

$$a_1 \prec_+ a_2 \prec_- a_3 \prec_+ \cdots$$

It follows from Theorem 3.7 that the number of facets of $\mathcal{C}(\mathbf{A}_n)$ is $3n+4$. Since $\mathbf{A}_n$ is compatible, then by Theorem 2.7 the number of facets of $\mathcal{O}(\mathbf{A}_n)$ equals the number of alternating chains which is easily computed to be $\binom{n+3}{2}+1$. Thus, for $n \geq 3$, the alternating chains $\mathbf{A}_n$ fail Conjecture 1 for the number of facets. For $n=3$, we explicitly compute

$$\begin{aligned} f(\mathcal{O}(\mathbf{A}_3)) &= (21, 70, 95, 60, 16) \quad \text{and} \\ f(\mathcal{C}(\mathbf{A}_3)) &= (21, 67, 86, 51, 13). \end{aligned}$$

Every graph $G = (V, E)$ trivially gives rise to a double graph $\mathbf{G}_\circ = (V, E, E)$. Thus, the **Hansen polytope** of a graph G is the polytope $\mathcal{H}(G) = \mathcal{P}_{\mathbf{G}_\circ}$. Theorem 3.4 then yields a strengthening of the main result of Hansen [17]. Note that for the complement graph $\overline{G} = (V, E^c)$, it follows that a subset $S \subseteq V$ is a stable set of G if and only if S is a clique of $\overline{G}$ and vice versa.

Corollary 3.9 ([17, Thm. 4(c)])**.** *Let G be a perfect graph. Then $\mathcal{H}(G)$ is 2-level and $\mathcal{H}(G)^\triangle$ is affinely isomorphic to $\mathcal{H}(\overline{G})$.*

Proof. By Theorem 3.4 and Theorem 3.1

$$\mathcal{H}(G)^\triangle \;=\; -A(\mathcal{P}_G) \boxminus -A(\mathcal{P}_G) \;=\; -\mathcal{P}_{\overline{G}} \boxminus -\mathcal{P}_{\overline{G}} \;\cong\; \mathcal{H}(\overline{G}),$$

which proves the second claim. A vertex of $\mathcal{H}(G)^\triangle$ is of the form $\mathbf{d} = \pm(-\mathbf{1}_C, 1)$ for some clique C of G. Thus, for any vertex $\mathbf{v} = \pm(2\mathbf{1}_S, 1) \in \mathcal{H}(G)$, where S is a stable set of G, we compute $\langle \mathbf{d}, \mathbf{v} \rangle = \pm(1 - 2|S \cap C|) = \pm 1$. □

Example 7 (Double chain polytopes of dimension-two posets)**.** Following Example 3, let $\pi_+, \pi_- \in \mathbb{Z}^n$ be two integer sequences with associated posets P_{π_+} and P_{π_-} of order dimension two. Consider the double posets $\mathbf{P} = (P_{\pi_+}, P_{\pi_-})$ and $-\mathbf{P} = (P_{-\pi_-}, P_{-\pi_+})$. We have

$$\overline{\mathbf{G}(\mathbf{P})} \;=\; (\overline{G(P_{\pi_-})}, \overline{G(P_{\pi_+})}) \;=\; (G(P_{-\pi_-}), G(P_{-\pi_+})) \;=\; \mathbf{G}(-\mathbf{P})$$

and hence

$$\mathcal{C}(\mathbf{P})^\triangle \;\cong\; \mathcal{C}(-\mathbf{P})$$

by Corollary 3.6. However, it is not necessarily true that $\mathcal{O}(\mathbf{P})^\triangle \cong \mathcal{O}(-\mathbf{P})$, as can be checked for the double poset induced by the X-poset; cf. Example 5.

Example 8 (Double chain polytopes of plane posets)**.** Let $\mathbf{P}$ be a plane double poset. By the last example, the double chain polytope $\mathcal{C}(\mathbf{P})$ is linearly equivalent to its polar $\mathcal{C}(\mathbf{P})^\triangle$.

Among the 2-level polytopes, independence polytopes of perfect graphs play a distinguished role. The following observation, due to Samuel Fiorini (personal communication), characterizes 2-level anti-blocking polytopes.

Proposition 3.10. *Let $\mathcal{P}$ be a full-dimensional anti-blocking polytope. Then $\mathcal{P}$ is 2-level if and only if $\mathcal{P}$ is linearly isomorphic to $\mathcal{P}_G$ for some perfect graph G.*

Proof. The origin is a vertex of $\mathcal{P}$ and, since $\mathcal{P}$ is full-dimensional and anti-blocking, its neighbors are $\alpha_1\mathbf{e}_1, \ldots, \alpha_1\mathbf{e}_n$ are for some $\alpha_i > 0$. After a linear transformation, we can assume that $\alpha_1 = \cdots = \alpha_n = 1$. Since $\mathcal{P}$ is 2-level, $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : \langle \mathbf{d}_i, \mathbf{x} \rangle \leq 1 \text{ for } i = 1, \ldots, s\}$ where

$\mathbf{d}_i \in \{0,1\}^n$ for all $i = 1,\ldots,s$. Let $G = ([n], E)$ be the minimal graph with cliques $\operatorname{supp}(\mathbf{d}_i)$ for all $i = 1,\ldots,s$. That is, $E = \bigcup_i \binom{\operatorname{supp}(\mathbf{d}_i)}{2}$. We have $\mathcal{P}_G \subseteq \mathcal{P}$. Conversely, any vertex of $\mathcal{P}$ is of the form $\mathbf{1}_S$ for some $S \subseteq [n]$ and $\langle \mathbf{d}_i, \mathbf{1}_S\rangle = |\operatorname{supp}(\mathbf{d}_i) \cap S| \leq 1$ shows that $\mathcal{P} \subseteq \mathcal{P}_G$. □

This implies a characterization of the 2-level polytopes among Cayley sums of anti-blocking polytopes.

Corollary 3.11. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be full-dimensional anti-blocking polytopes. Then $\mathcal{P} = \mathcal{P}_1 \boxminus \mathcal{P}_2$ is 2-level if and only if $\mathcal{P}$ is affinely isomorphic to $\mathcal{H}(G)$ for some perfect graph G.*

Proof. Sufficiency is Hansen's result (Corollary 3.9). For necessity, observe that $\mathcal{P}_1$ and $\mathcal{P}_2$ are faces and hence have to be 2-level. By the proof of Proposition 3.10, we may assume that $\mathcal{P}_1 = \mathcal{P}_{G_1}$ for some perfect graph G_1 and $\mathcal{P}_2 = A\mathcal{P}_{G_2}$ for some perfect G_2 and a diagonal matrix $A \in \mathbb{R}^{n\times n}$ with diagonal entries $a_i > 0$ for $i \in [n]$. We will proceed in two steps: We first prove that A must be the identity matrix and then show that $G_1 = G_2$.

For every $i \in [n]$ the inequality $x_i \geq 0$ is facet-defining for $\mathcal{P}_1$. Hence it induces a facet-defining inequality for $\mathcal{P}$, which must be of the form

$$\ell_i \ := \ -b_i x_i + t \ \leq \ 1$$

for some $b_i > 0$, where t denotes the last coordinate in $\mathbb{R}^{n+1}$. Observe that ℓ_i takes the values 1 and $1 - b_i$ on the vertices $\{\mathbf{0}, \mathbf{e}_i\} \times \{1\}$ of the face $\mathcal{P}_1 \times \{1\}$. On the other hand, on $\{\mathbf{0}, -a_i\mathbf{e}_i\} \times \{-1\} \subset -\mathcal{P}_2 \times \{-1\}$, the values are -1 and $-1 + a_i b_i$. Now 2-levelness implies $a_i = 1$ and $b_i = 2$.

It now follows from Theorem 3.4 that the facet-defining inequalities for $\mathcal{P}$ are

$$\begin{aligned} 2\mathbf{1}_{C_1}(\mathbf{x}) - t \ &\leq \ 1 \text{ and} \\ -2\mathbf{1}_{C_2}(\mathbf{x}) + t \ &\leq \ 1, \end{aligned}$$

where C_1 and C_2 are cliques in G_1 and G_2, respectively. By 2-levelness each of these linear functions takes the values -1 and 1 on the vertices of P. This easily implies that every clique in G_1 must be a clique in G_2 and conversely. Hence $G_1 = G_2$. □

3.3. **Canonical Subdivisions.** We now turn to the canonical subdivisions of $\mathcal{P}_1 - \mathcal{P}_2$ and $\mathcal{P}_1 \boxminus \mathcal{P}_2$ for anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2$. A **subdivision** of $\mathcal{P} = \mathcal{P}_1 - \mathcal{P}_2$ is a collection of polytopes $\mathcal{Q}^1, \ldots, \mathcal{Q}^m \subseteq \mathcal{P}$ each of dimension $\dim \mathcal{P}$ such that $\mathcal{P} = \mathcal{Q}^1 \cup \cdots \cup \mathcal{Q}^m$ and $\mathcal{Q}^i \cap \mathcal{Q}^j$ is a face of both for all $i \neq j$. We call the subdivision **mixed** if each $\mathcal{Q}^i$ is of the form $\mathcal{Q}^i_1 - \mathcal{Q}^i_2$ where $\mathcal{Q}^i_j$ is a vertex-induced subpolytope of $\mathcal{P}_j$ for $j = 1, 2$. Finally, a mixed subdivision is **exact** if $\dim \mathcal{Q}^i = \dim \mathcal{Q}^i_1 + \dim \mathcal{Q}^i_2$. That is, $\mathcal{Q}^i$ is linearly isomorphic to the Cartesian product $\mathcal{Q}^i_1 \times \mathcal{Q}^i_2$. For a full-dimensional anti-blocking polytope $\mathcal{P} \subset \mathbb{R}^n$, every index set $J \subseteq [n]$ defines a distinct face $\mathcal{P}|_J := \{x \in \mathcal{P} : x_j = 0 \text{ for } j \notin J\}$. This is an anti-blocking polytope of dimension $|J|$. For disjoint $I, J \subseteq [n]$, the polytopes $\mathcal{P}_1|_I, \mathcal{P}_2|_J$ lie in orthogonal subspaces and $\mathcal{P}_1|_I - \mathcal{P}_2|_J$ is in fact a Cartesian product. In this case, the Cayley sum $\mathcal{P}_1|_I \boxminus \mathcal{P}_2|_J$ is called a **join** and denoted by $\mathcal{P}_1|_I * \mathcal{P}_2|_J$. As with the Cartesian product, the combinatorics of $\mathcal{P}_1|_I * \mathcal{P}_2|_J$ is completely determined by the combinatorics of $\mathcal{P}_1|_I$ and $\mathcal{P}_2|_J$.

Lemma 3.12. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be full-dimensional anti-blocking polytopes. Then $\mathcal{P}_1 - \mathcal{P}_2$ has a regular exact mixed subdivision with cells $\mathcal{P}_1|_J - \mathcal{P}_2|_{J^c}$ for all $J \subseteq [n]$. In particular, $\mathcal{P}_1 \boxminus \mathcal{P}_2$ has a regular subdivision into joins $\mathcal{P}_1|_J * \mathcal{P}_2|_{J^c}$ for all $J \subseteq [n]$.*

We call the subdivisions of Lemma 3.12 the **canonical subdivisions** of $\mathcal{P}_1 - \mathcal{P}_2$ and $\mathcal{P}_1 \boxminus \mathcal{P}_2$, respectively.

Proof. By the Cayley trick [6, Thm 9.2.18], it is suffices to prove only the first claim. The subdivision of $\mathcal{P}_1 - \mathcal{P}_2$ is very easy to describe: Let us first note that the polytopes $\mathcal{P}_1|_J - \mathcal{P}_2|_{J^c}$ for $J \subseteq [n]$ only meet in faces. Hence, we only need to verify that they cover $\mathcal{P}_1 - \mathcal{P}_2$. It suffices to show that for any point $\mathbf{x} \in \mathcal{P}_1 - \mathcal{P}_2$ with $x_i \neq 0$ for all i, there is a $J \subseteq [n]$ with $\mathbf{x} \in \mathcal{P}_1|_J - \mathcal{P}_2|_{J^c}$. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n_{\geq 0}$ with $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ and $\operatorname{supp}(\mathbf{x}_1) \cap \operatorname{supp}(\mathbf{x}_2) = \varnothing$. We claim that $\mathbf{x}_i \in \mathcal{P}_i$ for $i = 1, 2$. Indeed, if $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$ for some $\mathbf{y}_i \in \mathcal{P}_i$, then $0 \leq \mathbf{x}_i \leq \mathbf{y}_i$ and $\mathbf{x}_i \in \mathcal{P}_i$ by (8). In particular, $\mathbf{x}_1 \in \mathcal{P}_1|_J$ and $\mathbf{x}_2 \in \mathcal{P}_2|_{J^c}$ and therefore $\mathbf{x} \in \mathcal{P}_1|_J - \mathcal{P}_2|_{J^c}$.

To show regularity, let $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the linear function such that $\omega(\mathbf{e}_i, 0) = -\omega(0, \mathbf{e}_j) = 1$ for all $i, j = 1, \dots, n$. Then ω induces a mixed subdivision by picking for every point $\mathbf{x} \in \mathcal{P}_1 - \mathcal{P}_2$, the unique cell $F_1 - F_2$ such that $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ with $\mathbf{x}_i \in \operatorname{relint} F_i$ and $(\mathbf{x}_1, \mathbf{x}_2)$ minimizes ω over the set

$$\{(\mathbf{y}_1, \mathbf{y}_2) \in \mathcal{P}_1 \times \mathcal{P}_2 : \mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2\};$$

see Section 9.2.2 of de Loera *et al.* [6] for more details. If ω is not generic, one has to be careful as the minimizer is not necessarily unique but in our case, we observe that for any $\mathbf{y}_i \in \mathcal{P}_i$ with $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$ we have $\omega(\mathbf{y}_1, \mathbf{y}_2) > \omega(\mathbf{x}_1, \mathbf{x}_2)$ for all $(\mathbf{y}_i, \mathbf{y}_2) \neq (\mathbf{x}_1, \mathbf{x}_2)$ with $(\mathbf{x}_1, \mathbf{x}_2)$ defined above. □

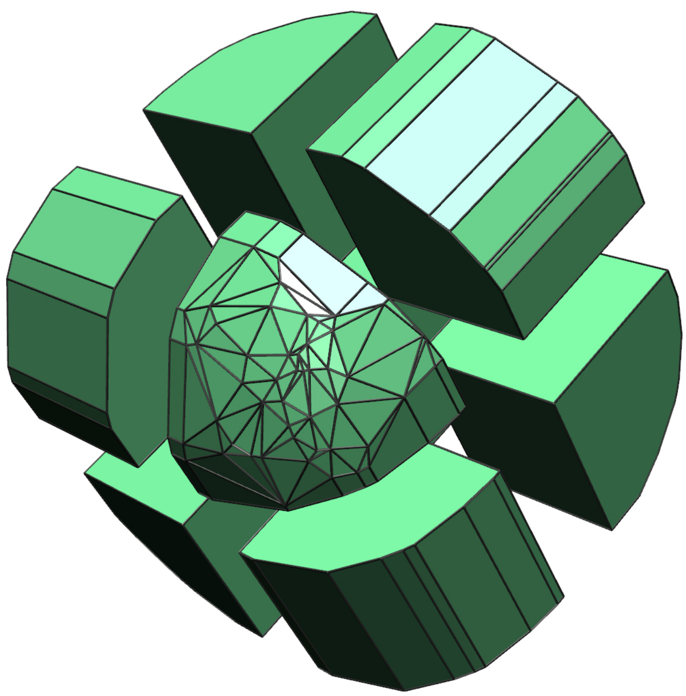

FIGURE 5. The canonical subdivision of $\mathcal{P}_1 - \mathcal{P}_2$ for two (random) anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^3_{\geq 0}$.

We define a **triangulation** of a polytope to be a subdivision into simplices without new vertices. For a polytope with vertices in an affine lattice $\mathbb{A}$, a triangulation is **unimodular** if each simplex is unimodular or, equivalently, has normalized volume $= 1$. A triangulation is **flag** if any minimal non-face is of dimension 1. This property implies that the underlying simplicial complex is completely determined by its graph.

Theorem 3.13. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be full-dimensional anti-blocking polytopes with subdivisions $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. For $J \subseteq [n]$, let $\mathcal{S}_i|_J := \{S \cap \mathcal{P}_i|_J : S \in \mathcal{S}_i\}$ be the restriction of $\mathcal{S}_i$ to $\mathcal{P}_i|_J$ for $i = 1, 2$. Then*

$$\mathcal{S} \ := \bigcup_{J \subseteq [n]} \mathcal{S}_1|_J * \mathcal{S}_2|_{J^c}$$

is a subdivision of $\mathcal{P}_1 \boxminus \mathcal{P}_2$. In particular,

(i) *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are regular, then $\mathcal{S}$ is regular.*
(ii) *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are unimodular triangulations with respect to Λ, then $\mathcal{S}$ is a unimodular triangulation with respect to the affine lattice $\Lambda \times (2\mathbb{Z} + 1)$.*
(iii) *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are flag, then $\mathcal{S}$ is flag.*

Note that (iii) also holds if the triangulations use more lattice points than just the vertices.

Proof. For the first claim, observe that $\mathcal{S}_i|_J$ is a subdivision of the face $\mathcal{P}_i|_J$. By [6, Thm 4.2.7], $\mathcal{S}_1|_J * \mathcal{S}_2|_{J^c}$ is a subdivision of $\mathcal{P}_1|_J * \mathcal{P}_2|_{J^c}$. Hence, $\mathcal{S}$ is a refinement of the canonical subdivision of Lemma 3.12.

If $\mathcal{S}_i$ is a regular subdivision of $\mathcal{P}_i$, then there are weights $\omega_i : V(\mathcal{P}_i) \to \mathbb{R}$ for $i = 1, 2$. By adding a constant weight to every vertex if necessary, we can assume that $\omega_1(\mathbf{v}_1) > 0$ and $\omega_2(\mathbf{v}_2) < 0$ for all $\mathbf{v}_1 \in V(\mathcal{P}_1)$ and $\mathbf{v}_2 \in V(\mathcal{P}_2)$. Again using the Cayley trick, it is easily seen that $\omega : V(\mathcal{P}_1 \boxminus \mathcal{P}_2) \to \mathbb{R}$ given by $\omega(\mathbf{v}_1, +1) := \omega_1(\mathbf{v}_1)$ and $\omega(\mathbf{v}_2, -1) := \omega_2(\mathbf{v}_2)$ induces $\mathcal{S}$.

Claim (ii) simply follows from the fact that the join of two unimodular simplices is unimodular.

For claim (iii), let $\sigma = \sigma_1 \uplus \sigma_2 \subseteq V(\mathcal{P}_1 \boxminus \mathcal{P}_2)$ be a minimal non-face. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are flag, it follows that $\sigma_1 \in \mathcal{S}_1$ and $\sigma_2 \in \mathcal{S}_2$. Thus, there vertices $v_i \in \sigma_i$ for $i = 1, 2$ such that $\operatorname{supp}(v_1) \cap \operatorname{supp}(v_2) \neq \varnothing$ but $\sigma \setminus \{v_i\}$ is a face for $i = 1$ and $i = 2$. But then $\{v_1, v_2\}$ is already a non-face and the claim follows. □

The theorem has some immediate consequences.

Corollary 3.14. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be two full-dimensional anti-blocking polytopes with vertices in a given lattice. If $\mathcal{P}_1, \mathcal{P}_2$ have unimodular triangulations, then $\mathcal{P}_1 - \mathcal{P}_2$ and $\Gamma(\mathcal{P}_1, \mathcal{P}_2) = \operatorname{conv}(\mathcal{P}_1 \cup -\mathcal{P}_2)$ also have unimodular triangulations.*

Proof. By Theorem 3.13 and the Cayley trick, $\mathcal{P}_1 - \mathcal{P}_2$ has a mixed subdivision into Cartesian products of unimodular simplices. Products of unimodular simplices are 2-level and, for example by [37, Thm. 2.4], have unimodular triangulations. The polytope $\operatorname{conv}(\mathcal{P}_1 \cup -\mathcal{P}_2)$ inherits a triangulation from the upper or lower hull of $\mathcal{P}_1 \boxminus \mathcal{P}_2$, which has a unimodular triangulation by Theorem 3.13. □

Corollary 3.15. *Let $\mathbf{G}$ be a perfect double graph. Then $\mathcal{P}_{\mathbf{G}}$, $\mathcal{P}_{G_+} - \mathcal{P}_{G_-}$, and $\Gamma(\mathcal{P}_{G_+}, \mathcal{P}_{G_-})$ have regular unimodular triangulations.*

Proof. By Theorem 3.1, both polytopes $\mathcal{P}_{G_+}$ and $\mathcal{P}_{G_-}$ are 2-level and by [37, Thm. 2.4] have unimodular triangulations. The result now follows from the previous corollary. □

3.4. **Lattice points and volume.** Lemma 3.12 directly implies a formula for the (normalized) volume of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ in terms of the volumes of the anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2$.

Corollary 3.16. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be full-dimensional anti-blocking polytopes. Then*

$$\mathrm{vol}(\mathcal{P}_1 - \mathcal{P}_2) \ = \ \sum_{J \subseteq [n]} \mathrm{vol}(\mathcal{P}_1|_J)\,\mathrm{vol}(\mathcal{P}_2|_{J^c}).$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ have unimodular triangulations with respect to a lattice Λ, then the normalized volume of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ with respect to the affine lattice $\Lambda \times (2\mathbb{Z}+1)$ is

$$\mathrm{Vol}(\mathcal{P}_1 \boxminus \mathcal{P}_2) \ = \ \sum_{J \subseteq [n]} \mathrm{Vol}(\mathcal{P}_1|_J)\,\mathrm{Vol}(\mathcal{P}_2|_{J^c}).$$

Proof. Both claims follow from Lemma 3.12. For the second statement, note that Theorem 3.13 yields that $\mathcal{P}_1 \boxminus \mathcal{P}_2$ has a unimodular triangulation and hence its normalized volume is the number of simplices of maximal dimension, which is the number in the right-hand side. □

If $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ are *rational* anti-blocking polytopes, then so are $2\mathcal{P}_1 \boxminus 2\mathcal{P}_2$ and $\mathcal{P}_1 - \mathcal{P}_2$. Our next goal is to determine their Ehrhart quasi-polynomials for a particular interesting case. We briefly recall the basics of Ehrhart theory; for more see, for example, [2, 3]. If $\mathcal{P} \subset \mathbb{R}^n$ is a d-dimensional polytope with rational vertex coordinates, then the function $\mathrm{Ehr}_{\mathcal{P}}(k) := |k\mathcal{P} \cap \mathbb{Z}^n|$ agrees with a quasi-polynomial of degree d. We will identify $\mathrm{Ehr}_{\mathcal{P}}(k)$ with this quasi-polynomial, called the **Ehrhart quasi-polynomial**. If $\mathcal{P}$ has its vertices in $\mathbb{Z}^n$, then $\mathrm{Ehr}_{\mathcal{P}}(k)$ is a polynomial of degree d. If $\mathcal{P}$ is full-dimensional, then the leading coefficient of $\mathrm{Ehr}_{\mathcal{P}}(k)$ is $\mathrm{vol}(\mathcal{P})$. We will need the following fundamental result of Ehrhart theory.

Theorem 3.17 (Ehrhart–Macdonald theorem)**.** *Let $\mathcal{P} \subset \mathbb{R}^n$ be a rational polytope of dimension d, then*

$$(-1)^d \mathrm{Ehr}_{\mathcal{P}}(-k) \ = \ |\,\mathrm{relint}(k\mathcal{P}) \cap \mathbb{Z}^n|.$$

We call an anti-blocking polytope $\mathcal{P} \subset \mathbb{R}^n$ **dual integral** if $A(\mathcal{P})$ has all vertices in $\mathbb{Z}^n$. By Theorem 3.2, this means that there are $\mathbf{d}_1, \dots, \mathbf{d}_s \in \mathbb{Z}^n_{\geq 0}$ such that

$$\mathcal{P} \ = \ \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \langle \mathbf{d}_i, \mathbf{x} \rangle \leq 1 \text{ for } i = 1, \dots, s\}.$$

Corollary 3.18. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be two full-dimensional rational anti-blocking polytopes. If $\mathcal{P}_1$ is dual integral, then*

$$\mathrm{Ehr}_{\mathcal{P}_1 - \mathcal{P}_2}(k) \ = \ \sum_{J \subseteq [n]} (-1)^{|J|} \mathrm{Ehr}_{\mathcal{P}_1|_J}(-k-1) \mathrm{Ehr}_{\mathcal{P}_2|_{J^c}}(k).$$

The Corollary is simply deduced from Theorem 3.17 and the following stronger assertion. For a set $S \subset \mathbb{R}^n$, let us write $E(S) := |S \cap \mathbb{Z}^n|$.

Theorem 3.19. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be two full-dimensional rational anti-blocking polytopes and assume that $\mathcal{P}_1$ is dual integral. For any $a, b \in \mathbb{Z}_{>0}$*

$$E(a\mathcal{P}_1 - b\mathcal{P}_2) \ = \ |(a\mathcal{P}_1 - b\mathcal{P}_2) \cap \mathbb{Z}^n| \ = \ \sum_{J \subseteq [n]} E(\mathrm{relint}((a+1)\mathcal{P}_1)) \cdot E(b\mathcal{P}_2).$$

Proof. It follows from Lemma 3.12 that for any $a, b \in \mathbb{Z}_{>0}$,

$$a\mathcal{P}_1 - b\mathcal{P}_2 \ = \ \bigcup_{J \subseteq [n]} (a\mathcal{P}_1|_J - b\mathcal{P}_2|_{J^c}).$$

For $J \subseteq [n]$, the cell $a\mathcal{P}_1|_J - b\mathcal{P}_2|_{J^c}$ is contained in the orthant $\mathbb{R}^J_{\geq 0} \times \mathbb{R}^{J^c}_{\leq 0}$. It is easy to see that

$$\mathbb{Z}^n \;=\; \biguplus_{J \subseteq [n]} \mathbb{Z}^J_{>0} \times \mathbb{Z}^{J^c}_{\leq 0}$$

is a partition and for each $J \subseteq [n]$

$$(a\mathcal{P}_1 - b\mathcal{P}_2) \cap (\mathbb{Z}^J_{>0} \times \mathbb{Z}^{J^c}_{\leq 0}) \;=\; (a\mathcal{P}_1|_J - b\mathcal{P}_2|_{J^c}) \cap (\mathbb{Z}^J_{>0} \times \mathbb{Z}^{J^c}_{\leq 0}) \;=\; (a\mathcal{P}_1|_J \cap \mathbb{Z}^J_{>0}) - (b\mathcal{P}_2|_{J^c} \cap \mathbb{Z}^{J^c}).$$

If $\mathcal{P}_1$ is dual integral, then $\mathcal{P}_1|_J$ is dual integral. Thus, for a fixed J, there are $\mathbf{d}_1, \dots, \mathbf{d}_s \in \mathbb{Z}^J_{\geq 0}$ such that

$$\begin{aligned} (a\mathcal{P}_1|_J \cap \mathbb{Z}^J_{>0}) \;&=\; \{\mathbf{x} \in \mathbb{Z}^J : \mathbf{x} > 0, \langle \mathbf{d}_i, \mathbf{x} \rangle \leq a\} \\ &=\; \{\mathbf{x} \in \mathbb{Z}^J : \mathbf{x} > 0, \langle \mathbf{d}_i, \mathbf{x} \rangle < a+1\} \;=\; \operatorname{relint}((a+1)\mathcal{P}_1|_J) \cap \mathbb{Z}^J. \end{aligned}$$

This proves the result. □

Clearly, it would be desirable to apply Corollary 3.18 to the case that $\mathcal{P}_1$ is a lattice polytope as well as dual integral.

Proposition 3.20. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional dual-integral anti-blocking polytope with vertices in $\mathbb{Z}^n$. Then $\mathcal{P} = \mathcal{P}_G$ for some perfect graph G.*

Proof. Let $\mathcal{P}$ be given by

$$\mathcal{P} \;=\; \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \langle \mathbf{d}_i, \mathbf{x} \rangle \leq 1 \text{ for } i = 1, \dots, s\}$$

for some $\mathbf{d}_1, \dots, \mathbf{d}_s \in \mathbb{Z}^n_{\geq 0}$. Since $\mathcal{P}$ is full-dimensional and a lattice polytope, it follows that $\mathbf{e}_1, \dots, \mathbf{e}_n \in \mathcal{P}$ and for any $1 \leq j \leq s$ we compute

$$0 \;\leq\; \langle \mathbf{d}_j, \mathbf{e}_i \rangle \;\leq\; 1$$

for all i and since the $\mathbf{d}_j$ are integer vectors, it follows that $\mathbf{d}_j = \mathbf{1}_{C_j}$ for some $C_j \subset [n]$. Consequently, the vertices of $\mathcal{P}$ are in $\{0,1\}^n$ and $\mathcal{P}$ is 2-level. By Proposition 3.10, $\mathcal{P} = \mathcal{P}_G$ for some perfect graph G. □

This severely limits the applicability of Corollary 3.18 to *lattice* anti-blocking polytopes. On the other hand, we do not know of many results regarding the Ehrhart polynomials or even volumes of stable set polytopes of perfect graphs; see also the next section.

Theorem 3.21. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be two full-dimensional rational anti-blocking polytopes such that $\mathcal{P}_1$ is dual integral. Then for $\mathcal{P} := 2\mathcal{P}_1 \boxminus 2\mathcal{P}_2$*

$$\operatorname{Ehr}_{\mathcal{P}}(k) \;=\; |k\mathcal{P} \cap \mathbb{Z}^{n+1}| \;=\; \sum_{J \subseteq [n]} (-1)^{|J|} \sum_{s=-k}^{k} \operatorname{Ehr}_{\mathcal{P}_1|_J}(s-k-1) \cdot \operatorname{Ehr}_{\mathcal{P}_2|_{J^c}}(k+s).$$

Proof. For $k > 0$,

$$k\mathcal{P} \;=\; \operatorname{conv}(2k\mathcal{P}_1 \times \{k\} \cup -2k\mathcal{P}_2 \times \{-k\}).$$

In particular, if $(\mathbf{p}, t)$ is a lattice point in $k\mathcal{P}$, then $-k \leq t \leq k$. For fixed t,

$$\{\mathbf{p} \in \mathbb{Z}^n : (\mathbf{p}, t) \in k\mathcal{P}\} \;=\; ((k-t)\mathcal{P}_1 - (k+t)\mathcal{P}_2) \cap \mathbb{Z}^n.$$

Theorems 3.19 and 3.17 then complete the proof. □

4. Triangulations and transfers

If $\mathbf{P} = \mathbf{P}_\circ = (P, \preceq, \preceq)$ is induced by a single poset, then Corollaries 2.9 and 3.11 assure us that $\mathcal{O}(\mathbf{P}_\circ)$ and $\mathcal{C}(\mathbf{P}_\circ)$ are 2-level and [37, Thm. 2.4] implies that both polytopes have unimodular triangulations with respect to the affine lattice $\mathbb{A} = 2\mathbb{Z}^P \times (2\mathbb{Z}+1)$. In this section we give explicit triangulations of the double chain polytope $\mathcal{C}(\mathbf{P})$ and, in the compatible case, of the double order polytope $\mathcal{O}(\mathbf{P})$. These triangulations will be *regular*, *unimodular*, and *flag*. To that end, we will generalize Stanley's approach [34] from poset polytopes to double poset polytopes. We put the triangulation to good use and explicitly compute the Ehrhart polynomial and the volume of $\mathcal{C}(\mathbf{P})$ and, in case that $\mathbf{P}$ is compatible, of $\mathcal{O}(\mathbf{P})$.

4.1. **Triangulations of double poset polytopes.** Recall from the introduction that for a poset $(P, \preceq)$, the order polytope $\mathcal{O}(P)$ parametrizes all order preserving maps $f : P \to [0,1]$. Any $f \in \mathcal{O}(P)$ induces a partial order $P_f = (P, \preceq_f)$ by $a \prec_f b$ if $a \prec b$ or, when a, b are incomparable, if $f(a) < f(b)$. Clearly, $\preceq_f$ refines $\preceq$ and hence $\mathcal{O}(P_f) \subseteq \mathcal{O}(P)$. Since filters in P_f are filters in P, $\mathcal{O}(P_f)$ is a vertex-induced subpolytope of $\mathcal{O}(P)$. If f is *generic*, that is, $f(a) \neq f(b)$ for all $a \neq b$, then $\preceq_f$ is a total order and $\mathcal{O}(P_f)$ is a unimodular simplex of dimension $|P|$. Stanley showed that the collection of all these simplices constitute a unimodular triangulation of $\mathcal{O}(P)$. More precisely, this canonical triangulation of $\mathcal{O}(P)$ is given by the **order complex** $\Delta(\mathcal{J}(P))$ of $\mathcal{J}(P)$, i.e., the collection of chains in the Birkhoff lattice of P ordered by inclusion. Since a collection of filters $\mathsf{J}_0, \dots, \mathsf{J}_k$ is *not* a chain if and only if $\mathsf{J}_i \not\subseteq \mathsf{J}_j$ and $\mathsf{J}_j \not\subseteq \mathsf{J}_i$ for some $0 \le i, j \le k$, the canonical triangulation is *flag*.

Stanley [34] elegantly *transferred* the canonical triangulation of $\mathcal{O}(P)$ to $\mathcal{C}(P)$ in the following sense. Define the **transfer map** $\phi_P : \mathcal{O}(P) \to \mathcal{C}(P)$ by

$$(\phi_P f)(b) \ := \ \min\{f(b) - f(a) : a \prec b\}, \tag{10}$$

for $f \in \mathcal{O}(P)$ and $b \in P$. This is a **piecewise linear** map and the domains of linearity are exactly the full-dimensional simplices $\mathcal{O}(P_f)$ for generic f. In particular, $\phi_P(\mathbf{1}_{\mathsf{J}}) = \mathbf{1}_{\min(\mathsf{J})}$ for any filter $\mathsf{J} \subseteq P$, which shows that ϕ_P maps $\mathcal{O}(P)$ into $\mathcal{C}(P)$. To show that ϕ_P is a PL homeomorphism of the two polytopes, Stanley gives an explicit inverse $\psi_P : \mathcal{C}(P) \to \mathcal{O}(P)$ by

$$(\psi_P g)(b) \ := \ \max\{g(a_0) + \cdots + g(a_{k-1}) + g(a_k) : a_0 \prec \cdots \prec a_{k-1} \prec a_k \preceq b\}, \tag{11}$$

for any $g \in \mathcal{C}(P)$. Note that our definition of ψ_P differs from that in [34] in that we do *not* require that the chain has to end in b. This will be important later. It can be easily checked that ψ_P is an inverse to ϕ_P. Hence, the simplices

$$\operatorname{conv}(\mathbf{1}_{\min(\mathsf{J}_0)}, \dots, \mathbf{1}_{\min(\mathsf{J}_k)}) \quad \text{for} \quad \{\mathsf{J}_0 \subseteq \cdots \subseteq \mathsf{J}_k\} \in \Delta(\mathcal{J}(P))$$

constitute a flag triangulation of $\mathcal{C}(P)$.

We will follow the same approach as Stanley but, curiously, it will be simpler to start with a triangulation of $\mathcal{C}(\mathbf{P})$. Recall from Section 2.3 that a pair of chains $C = C_+ \uplus C_-$ with $C_\sigma \subseteq \mathcal{J}(P_\sigma)$ is non-interfering if $\min(\mathsf{J}_+) \cap \min(\mathsf{J}_-) = \varnothing$ for any $\mathsf{J}_\sigma \in C_\sigma$ for $\sigma = \pm$.

Corollary 4.1. *Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a double poset. Then a regular triangulation of $\mathcal{C}(\mathbf{P})$ is given as follows: The $(k-1)$-dimensional simplices are in bijection to non-interfering pairs of chains $C = C_+ \uplus C_+ \subseteq \mathcal{J}(\mathbf{P})$ with $|C| = |C_+| + |C_-| = k$. Moreover, the triangulation is regular, unimodular (with respect to $\mathbb{A}$), and flag.*

Proof. The canonical triangulation of $\mathcal{C}(P_\sigma)$ is regular, unimodular, and flag for $\sigma = \pm$. As described above, its $(l_\sigma - 1)$-simplices are in bijection to chains $C_\sigma \subseteq \mathcal{J}(P_\sigma)$ of length $|C_\sigma| = l_\sigma$.

More precisely, the simplex corresponding to C_σ is given by

$$F(C_\sigma) = \mathrm{conv}(\mathbf{1}_{\min(\mathsf{J}_\sigma)} : \mathsf{J}_\sigma \in C_\sigma).$$

By Theorem 3.13 applied to $\mathcal{C}(\mathbf{P}) = 2\mathcal{C}(P_+) \boxminus 2\mathcal{C}(P_-)$, it follows that a unimodular and flag triangulation is given by the joins $2F(C_+) * 2F(C_-)$ for all chains $C_\sigma \subseteq \mathcal{J}(P_\sigma)$ such that $F(C_+)$ and $F(C_-)$ lie in complementary coordinate subspaces. This, however, is exactly the case when $\min(\mathsf{J}_+) \cap \min(\mathsf{J}_-) = \varnothing$ for all $\mathsf{J}_\sigma \in C_\sigma$ for $\sigma = \pm$. $\square$

Corollary 4.1 gives a canonical triangulation that combinatorially can be described as a subcomplex of $\Delta(\mathcal{J}(\mathbf{P})) = \Delta(\mathcal{J}(P_+)) * \Delta(\mathcal{J}(P_-))$, called the **non-interfering complex**

$$\Delta^{\mathrm{ni}}(\mathbf{P}) \;:=\; \{C : C = C_+ \uplus C_- \in \Delta(\mathcal{J}(\mathbf{P})),\, C \text{ non-interfering}\}.$$

Associating $\Delta(\mathcal{J}(P))$ to a poset P is very natural and can be motivated, for example, from an algebraic-combinatorial approach to the order polynomial (cf. [3]). It would be very interesting to know if the association $\mathbf{P}$ to $\Delta^{\mathrm{ni}}(\mathbf{P})$ is equally natural from a purely combinatorial perspective.

Given a double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$, we define a piecewise linear map $\Psi_{\mathbf{P}} : \mathbb{R}^P \to \mathbb{R}^P$ by

$$\Psi_{\mathbf{P}}(g) \;:=\; \psi_{P_+}(g) \;-\; \psi_{P_-}(-g), \tag{12}$$

for any $g \in \mathbb{R}^P$. Here, we use that ψ, as given in (11), is defined on all of $\mathbb{R}^P$ with the following important property: For $g \in \mathbb{R}^P$, let us write $g = g^+ - g^-$, where $g^+, g^- \in \mathbb{R}^P_{\geq 0}$ with disjoint supports. Then $\psi_{P_\sigma}(g) = \psi_{P_\sigma}(g^+)$ for $\sigma = \pm$. Thus,

$$\Psi_{\mathbf{P}}(g) \;=\; \psi_{P_+}(g^+) - \psi_{P_-}(g^-),$$

for any $g \in \mathbb{R}^P$. In particular, $\Psi_{\mathbf{P}}$ takes $\lambda\mathcal{C}(P_+) - \mu\mathcal{C}(P_-)$ into $\lambda\mathcal{O}(P_+) - \mu\mathcal{O}(P_-)$ for any $\lambda, \mu \geq 0$. Indeed, for any pair of antichains $A_\sigma \subseteq P_\sigma$, first observe that $\mathbf{1}_{A_+} - \mathbf{1}_{A_-} = \mathbf{1}_{A_+\setminus A_-} - \mathbf{1}_{A_-\setminus A_+}$. Hence, it suffices to assume that $A_+ \cap A_- = \varnothing$. We compute

$$\Psi_{\mathbf{P}}(\mathbf{1}_{A_+} - \mathbf{1}_{A_-}) \;=\; \mathbf{1}_{\mathsf{J}_+} - \mathbf{1}_{\mathsf{J}_-},$$

where for $\sigma = \pm$, $\mathsf{J}_\sigma \subseteq P_\sigma$ is the filter generated by A_σ. If $\mathbf{P}$ is a compatible double poset, then Corollary 2.17 implies that $\Psi_{\mathbf{P}}$ is a surjection on vertex sets.

Lemma 4.2. *Let $\mathbf{P} = (\mathcal{P}, \preceq_+, \preceq_-)$ be a compatible double poset. Then $\Psi_{\mathbf{P}} : \mathbb{R}^P \to \mathbb{R}^P$ is a lattice-preserving piecewise linear isomorphism.*

Proof. It follows directly from (12) that $\Psi_{\mathbf{P}}$ is piecewise linear. To show that $\Psi_{\mathbf{P}}$ is an isomorphism, we explicitly construct for $f \in \mathbb{R}^P$ a $g \in \mathbb{R}^P$ such that $\Psi_{\mathbf{P}}(g) = f$. Since $\mathbf{P}$ is compatible, we can assume that $P = \{a_1, \ldots, a_n\}$ such that $a_i \prec_+ a_j$ or $a_i \prec_- a_j$ implies $i < j$.

It follows from (12) that $\Psi_{\mathbf{P}}(g')(a_1) = g'(a_1)$ for any $g' \in \mathbb{R}^P$ and hence, we can set $g(a_1) := f(a_1)$. Now assume that g is already defined on $D_k := \{a_1, \ldots, a_k\}$ for some k. For $g' \in \mathbb{R}^P$ observe that

$$\psi_{P_+}(g')(a_{k+1}) \;=\; \max(g'(a_{k+1}), 0) + r$$

where $r = 0$ or $r = \psi_{P_+}(g')(a_i)$ for some $i \leq k$. Analogously,

$$\psi_{P_-}(-g')(a_{k+1}) \;=\; \max(-g'(a_{k+1}), 0) + s$$

where $s = 0$ or $s = \psi_{P_-}(-g')(a_j)$ for some $j \leq k$. Thus, we set

$$g(a_{k+1}) \;:=\; f(a_{k+1}) - r + s$$

This uniquely determines g by induction on k. To prove that $\Psi_{\mathbf{P}}$ is lattice-preserving, observe that by (12) we have $\Psi_{\mathbf{P}}(\mathbb{Z}^P) \subseteq \mathbb{Z}^P$. Moreover, if $f = \Psi_{\mathbf{P}}(g)$ with $f \in \mathbb{Z}^P$ and the above construction shows that $g \in \mathbb{Z}^P$. Hence, $\Psi_{\mathbf{P}}(\mathbb{Z}^P) \subseteq \mathbb{Z}^P$, which finishes the proof. □

Using the notation from (5) in Section 2.3, the lemma shows that

$$\{\overline{F}(C) : C \in \Delta^{\mathrm{ni}}(\mathbf{P})\} \tag{13}$$

is a realization of the flag simplicial complex $\Delta^{\mathrm{ni}}(\mathbf{P})$ by unimodular simplices inside $\mathcal{O}(\mathbf{P})$. Using Gröbner bases in Section 5, we will show the following result.

Theorem 4.3. *Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a compatible double poset. Then the map*

$$(g,t) \mapsto (\Psi_{\mathbf{P}}(g), t)$$

is a piecewise linear homeomorphism from $\mathbb{R}^P \times \mathbb{R}$ to itself that preserves the lattice $\mathbb{Z}^P \times \mathbb{Z}$. In particular, it maps $\mathcal{C}(\mathbf{P})$ to $\mathcal{O}(\mathbf{P})$ and hence $\Delta^{\mathrm{ni}}(\mathbf{P})$ is a regular, unimodular, and flag triangulation of $\mathcal{O}(\mathbf{P})$.

Proof. By the previous lemma, (13) is a realization of $\Delta^{\mathrm{ni}}(\mathbf{P})$ in $\mathcal{O}(\mathbf{P})$ without new vertices. Moreover, every maximal simplex contains the edge $e = \mathrm{conv}\{(\mathbf{0}, 1), (\mathbf{0}, -1)\}$. Hence, it suffices to show that for every maximal simplex in $\Delta^{\mathrm{ni}}(\mathbf{P})$, the supporting hyperplane of every facet not containing e is supporting for $\mathcal{O}(\mathbf{P})$.

Let $C = \{\mathsf{J}_{+0} \subset \cdots \subset \mathsf{J}_{+k}, \mathsf{J}_{-0} \subset \cdots \subset \mathsf{J}_{-l}\}$ be two maximal non-interfering chains. Set $A_{+i} := \min(\mathsf{J}_{+i})$ for $1 \le i \le k$ and $A_{-0}, \ldots, A_{-l}$ likewise. It follows that $P_1 = \bigcup A_{+i}$ and $P_2 = \bigcup A_{-j}$ give a partition of P. In particular, since C was maximal, we have that $\{a^+_{k-i-1}\} = A_{+i} \setminus A_{+(i-1)}$ and $P_1 = \{a^+_1, \ldots, a^+_k\}$. In particular, if $a^+_s \prec_+ a^+_t$, then $s < t$. The same argument yields $P_2 = \{a^-_1, \ldots, a^-_l\}$ and the labelling is a linear extension of $(P_2, \preceq_-)$.

We focus on P_1; the argument for P_2 is analogous. Pick the maximal chain D in $(P_1, \preceq_+)$ starting in a^+_k. Then $A_{+i} \cap D \neq \varnothing$ for all $i > 0$ and hence $\{(g,t) \in \mathbb{R}^{P_1} : \langle \mathbf{1}_D, g\rangle = 1\}$ is the hyperplane for the maximal simplex in the triangulation of $\mathcal{C}(P_1, \preceq_+)$ corresponding to $A_{+0}, \ldots, A_{+k}$ and not containing the origin. Thus, one of the two hyperplanes supporting a facet of the simplex in $\mathcal{C}(\mathbf{P})$ corresponding to C is given by $H := \{(g,t) \in \mathbb{R}^P \times \mathbb{R} : \ell(g,t) = 1\}$ where $\ell(g,t) = \langle \mathbf{1}_D, g\rangle - t$.

Now, $\Psi_{\mathbf{P}}$ is linear on the simplex C in $\mathcal{C}(\mathbf{P})$ and can be easily inverted. Since $\mathbf{P}$ is compatible, we can find a linear extension $\sigma : P \to \{1, \ldots, |P|\}$ that respects the constructed linear extensions on P_1 and P_2. On the image of C under $\Psi_{\mathbf{P}}$, the inverse is given by the linear transformation $T : \mathbb{R}^P \times \mathbb{R} \to \mathbb{R}^P \times \mathbb{R}$ with $T(f,t) = (f', t)$ and $f' : P \to \mathbb{R}$ is defined as follows. If $b \in P_1$, then by $f'(b) = f(b) - f(\bar{b})$, $\bar{b} \prec_+ b$ is a cover relation and $\sigma(\bar{b})$ is maximal. If $b \in P_2$, we choose $\bar{b}$ covered by b in with respect to $\preceq_-$. It can now be checked that $\ell \circ T = L_C$ for some alternating chain C. Thus H is supporting for $\mathcal{O}(\mathbf{P})$ and the map $\Psi_{\mathbf{P}}$ maps $\mathcal{C}(\mathbf{P})$ onto $\mathcal{O}(\mathbf{P})$. □

Theorem 4.3 does not extend to the non-compatible case as the following example shows.

Example 9. Consider the double poset $\mathbf{P} = ([2], \le, \ge)$, that is, P_+ is the 2-chain $\{1, 2\}$ and P_- is the opposite poset. Then $\mathcal{C}(P_+) = \mathcal{C}(P_-) = T := \{x \in \mathbb{R}^2 : x \ge 0, x_1 + x_2 \le 1\}$ and $\mathcal{C}(\mathbf{P})$ is a three-dimensional octahedron with volume $\frac{16}{3}$. Any triangulation of the octahedron has at least four simplices. In contrast, $\mathcal{O}(P_-) = \mathbf{1} - \mathcal{O}(P_+)$ and hence $\mathcal{O}(\mathbf{P})$ is linearly isomorphic to a prism over a triangle with volume 4. Any triangulation of the prism has exactly 3 tetrahedra.

4.2. **Volumes and Ehrhart polynomials.** The canonical subdivision of $\mathcal{O}(P)$ makes it easy to compute its volume. For a generic $f \in \mathcal{O}(P)$, there is a unique linear extension $\sigma : P \to \{1, 2, \dots, d\}$ where $d := |P|$ such that

$$\mathcal{O}(P_f) \;=\; \{h \in \mathbb{R}^P : 0 \le h(\sigma^{-1}(1)) \le \cdots \le h(\sigma^{-1}(d)) \le 1\}.$$

In particular, the full-dimensional simplex $\mathcal{O}(P_f)$ is unimodular relative to $\mathbb{Z}^P \subseteq \mathbb{R}^P$ and has volume $\mathrm{vol}(\mathcal{O}(P_f)) = \frac{1}{|P|!}$. If we denote by $e(P)$ the number of linear extensions of P, then Stanley [34] showed the following.

Corollary 4.4. $\mathrm{Vol}(\mathcal{O}(P)) = |P|! \cdot \mathrm{vol}(\mathcal{O}(P)) = e(P)$.

For the Ehrhart polynomial $\mathrm{Ehr}_{\mathcal{O}(P)}(n)$ of $\mathcal{O}(P)$ it suffices to interpret the lattice points in $n\mathcal{O}(P)$ for $n > 0$. Every point in $n\mathcal{O}(P) \cap \mathbb{Z}^P$ corresponds to an order preserving map $\phi : P \to [n+1]$. Counting order preserving maps is classical [35, Sect. 3.15]: the **order polynomial** $\Omega_P(n)$ of P counts the number of order preserving maps into n-chains. The **strict** order polynomial $\Omega^\circ_P(n)$ counts the number of strictly order preserving maps $f : P \to [n]$, that is, $f(a) < f(b)$ for $a \prec b$. The transfer map ϕ_P as well as its inverse ψ_P (given in (10) and (11), respectively) both take lattice points to lattice points and hence, together with Theorem 3.17, yield the following result.

Corollary 4.5. *Let P be a finite poset. Then for every $n > 0$*

$$\Omega_P(n+1) \;=\; \mathrm{Ehr}_{\mathcal{O}(P)}(n) \;=\; \mathrm{Ehr}_{\mathcal{C}(P)}(n)$$

and

$$(-1)^{|P|}\Omega^\circ_P(n-1) \;=\; \mathrm{Ehr}_{\mathcal{O}(P)}(-n) \;=\; \mathrm{Ehr}_{\mathcal{C}(P)}(-n).$$

In particular, $\mathrm{vol}(\mathcal{O}(P)) = \mathrm{vol}(\mathcal{C}(P))$.

This is an interesting result as it implies that the number of linear extensions of a poset P only depends on the comparability graph $G(P)$.

Theorem 4.6. *Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a double poset. Then $\overline{\mathcal{C}}(\mathbf{P})$ is a lattice polytope with respect to $\mathbb{Z}^P$ and*

$$\begin{aligned}\mathrm{Ehr}_{\overline{\mathcal{C}}(\mathbf{P})}(n-1) \;&=\; \sum_{P = P_1 \uplus P_2} \Omega^\circ_{(P_1,\preceq_+)}(n-1) \cdot \Omega_{(P_2,\preceq_-)}(n) \text{ and}\\ \mathrm{Vol}(\overline{\mathcal{C}}(\mathbf{P})) \;&=\; \sum_{P = P_1 \uplus P_2} \binom{|P|}{|P_1|} e(P_1, \preceq_+) \cdot e(P_2, \preceq_-).\end{aligned}$$

Proof. Since $\mathcal{C}(P) = \mathcal{P}_{G(P)}$ is a dual integral anti-blocking polytope, the first identity follows from Corollary 3.18 and Corollary 4.5. The second identity follows from Corollary 3.16 and Corollary 4.4. $\square$

Notice from Theorem 3.21 we can also deduce a closed formula for the Ehrhart polynomial of $\mathcal{C}(\mathbf{P})$ with respect to the lattice $\mathbb{Z}^P \times \mathbb{Z}$ and, by substituting $\frac{1}{2}k$ for k, also with respect to the affine lattice $\mathbb{A}$. These formulas are not very enlightening and instead we record the normalized volume. Note that the minimal Euclidean volume of a full-dimensional simplex with vertices in $\mathbb{A} = \mathbb{Z}^P \times (2\mathbb{Z} + 1)$ is $\frac{2^{|P|+1}}{(|P|+1)!}$.

Corollary 4.7. *Let* $\mathbf{P} = (P, \preceq_+, \preceq_-)$ *be a double poset. Then the normalized volume with respect to the affine lattice* $\mathbb{A} = 2\mathbb{Z}^P \times (2\mathbb{Z}+1)$ *is*

$$\mathrm{Vol}(\mathcal{C}(\mathbf{P})) \;=\; \sum_{P = P_1 \uplus P_2} e(P_1, \preceq_+) \cdot e(P_2, \preceq_-).$$

We leave it to the reader to give direct combinatorial interpretations of the volume and the Ehrhart polynomials for double posets.

It follows directly from (12) that $\Psi_{P_\pm} : \mathbb{R}^P \to \mathbb{R}^P$ maps lattice points to lattice points. If $\mathbf{P}$ is compatible, then the proof of Lemma 4.2 asserts that $\Psi_{P_\pm}$ is in fact lattice preserving. Hence, we record an analog to Corollary 4.5.

Corollary 4.8. *If* $\mathbf{P}$ *is a compatible double poset, then* $\mathcal{O}(\mathbf{P})$ *and* $\mathcal{C}(\mathbf{P})$ *have the same Ehrhart polynomials and hence the same volumes.*

The formulas of Theorem 4.6 are particularly simple when $\mathbf{P}$ is special or anti-special. We illustrate these cases at some simple examples.

Example 10. For the 'XW'-double poset we have

$$\mathrm{Vol}(\mathcal{O}(\mathbf{P}_{XW})) \;=\; \mathrm{Vol}(\mathcal{C}(\mathbf{P}_{XW})) \;=\; \tfrac{6!}{2^6}\,\mathrm{vol}(\mathcal{C}(\mathbf{P}_{XW})) \;=\; 128$$

and $\mathrm{Vol}(\overline{\mathcal{O}}(\mathbf{P}_{XW})) = \mathrm{Vol}(\overline{\mathcal{C}}(\mathbf{P}_{XW})) = 6!\,\mathrm{vol}(\overline{\mathcal{C}}(\mathbf{P}_{XW})) = 880$.

Example 11. As the following examples are all compatible, the given values also give the normalized volumes of the respective (reduced) double order polytopes.

(1) Let $\mathbf{P} = ([d], \leq, \leq)$ be the double chain on d elements. Then $\mathcal{C}(\mathbf{P})$ is a crosspolytope and $\mathrm{Vol}(\mathcal{C}(\mathbf{P})) = 2^d$ and it follows from Vandermonde's identity that

$$\mathrm{Vol}(\overline{\mathcal{C}}(\mathbf{P})) \;=\; d!\,\mathrm{vol}(\overline{\mathcal{C}}(\mathbf{P})) \;=\; \sum_{i=0}^{d} \binom{d}{i}^2 \;=\; \binom{2d}{d}.$$

(2) If $\mathbf{P}$ is the double anti-chain on d elements, then $\mathcal{C}(\mathbf{P})$ is isomorphic to $[0,2]^d \times [-1,1]$ and its normalized volume is

$$\mathrm{Vol}(\mathcal{C}(\mathbf{P})) \;=\; \tfrac{(d+1)!}{2^{d+1}}\,\mathrm{vol}(\mathcal{C}(\mathbf{P})) \;=\; \sum_{i=0}^{d} \binom{d}{i} i!(d-i)! \;=\; (d+1)!.$$

Likewise, $\overline{\mathcal{C}}(\mathbf{P})$ is isomorphic to $[-1,1]^d$, which can be decomposed into 2^d unit cubes. Consequently, its normalized volume is

$$\mathrm{Vol}(\overline{\mathcal{C}}(\mathbf{P})) \;=\; \sum_{i=0}^{d} \binom{d}{i}^2 i!(d-i)! \;=\; 2^d d!.$$

(3) Let $\mathbf{P}$ be the double poset such that P_+ is the d-chain and P_- is the d-antichain. Then

$$\mathrm{Vol}(\mathcal{C}(\mathbf{P})) \;=\; \sum_{i=0}^{d} \frac{d!}{i!}$$

is the number of choices of ordered subsets of a d-set. Moreover

$$\mathrm{Vol}(\overline{\mathcal{C}}(\mathbf{P})) \;=\; \sum_{i=0}^{d} \binom{d}{i}^2 i!$$

is the number of partial permutation matrices, i.e. 0/1-matrices of size d with at most one nonzero entry per row and column. Indeed, such a matrix is uniquely identified by an i-by-i permutation matrix and a choice of i rows and i columns in which it is embedded.

(4) For the comb C_n, the number of linear extensions is $e(C_n) = (2n-1)!!$. Let $\mathbf{P}$ be the double poset induced by the comb C_n. Then an induction argument shows that

$$\mathrm{Vol}(\mathcal{C}(\mathbf{P})) \;=\; 4^n\, n!.$$

It would be nice to have a bijective proof of this equality.

Let $\mathbf{P}_\circ = (P, \preceq, \preceq)$ be a compatible double poset induced by a poset $(P, \preceq)$. By Corollary 4.8, the polytopes $\mathcal{O}(\mathbf{P}_\circ)$ and $\mathcal{C}(\mathbf{P}_\circ)$ have the same normalized volume. Since both polytopes are 2-level, this means that the number of maximal simplices in any pulling triangulation of $\mathcal{O}(\mathbf{P}_\circ)$ and $\mathcal{C}(\mathbf{P}_\circ)$ coincides. From Theorem 2.18, we know that $\mathcal{O}(\mathbf{P}_\circ)^\triangle$ is the twisted prism over the valuation polytope associated to P. On the other hand, we know from Corollary 3.9 that $\mathcal{C}(\mathbf{P}_\circ)^\triangle$ is linearly isomorphic to the Hansen polytope $\mathcal{H}(\overline{G(P)})$. Moreover, $\mathcal{O}(\mathbf{P}_\circ)^\triangle$ and $\mathcal{C}(\mathbf{P}_\circ)^\triangle$ are both 2-level and it is enticing to conjecture that their normalized volumes also agree. Unfortunately, this is not the case. For the poset P on 5 elements whose Hasse diagram is the letter 'X', any pulling triangulation of $\mathcal{C}(\mathbf{P}_\circ)^\triangle$ has 324 simplices whereas for $\mathcal{O}(\mathbf{P}_\circ)^\triangle$ pulling triangulations have 320 simplices.

5. Gröbner bases and triangulations

5.1. **Double Hibi rings.** Hibi [19] associated to any finite poset $(P, \preceq)$ a ring $\mathbb{C}[\mathcal{O}(P)]$, nowadays called **Hibi ring**, that algebraically reflects many of the order-theoretic properties of P. The ring $\mathbb{C}[\mathcal{O}(P)]$ is defined as the graded subring of the polynomial ring $S = \mathbb{C}[t, s_a : a \in P]$ generated by the elements $t \cdot s^{\mathsf{J}}$, where

$$s^{\mathsf{J}} \;:=\; \prod_{a \in \mathsf{J}} s_a,$$

ranges over all filters $\mathsf{J} \subseteq P$. For example, Hibi showed that $\mathbb{C}[\mathcal{O}(P)]$ is a normal Cohen–Macaulay domain of dimension $|P| + 1$ and that $\mathbb{C}[\mathcal{O}(P)]$ is Gorenstein if and only if P is a graded poset. By definition, Hibi rings are toric and hence they have the following quotient description. Let $R = \mathbb{C}[x_{\mathsf{J}} : \mathsf{J} \in \mathcal{J}(P)]$ be the polynomial ring with variables indexed by filters and define the homogeneous ring map $\phi : R \to S$ by $\phi(x_{\mathsf{J}}) = t\, s_{\mathsf{J}}$. Then $\mathbb{C}[\mathcal{O}(P)] \cong R/\mathrm{I}_{\mathcal{O}(P)}$ where $\mathrm{I}_{\mathcal{O}(P)} = \ker \phi$ is a toric ideal.

Hibi elegantly described a reduced Gröbner basis of $\mathrm{I}_{\mathcal{O}(P)}$ in terms of $\mathcal{J}(P)$. Fix a total order $\leq$ on the variables of R such that $x_{\mathsf{J}} \leq x_{\mathsf{J}'}$ whenever $\mathsf{J} \subseteq \mathsf{J}'$ and let $\leq_{rev}$ denote the induced reverse lexicographic order on R. For $f \in R$, we write $\mathrm{in}_{\leq_{rev}}(f)$ for its leading term with respect to $\leq_{rev}$ and we will underline leading terms in what follows.

Theorem 5.1 ([18, Thm. 10.1.3])**.** *Let $(P, \preceq)$ be a finite poset. Then the collection*

$$\underline{x_{\mathsf{J}}\, x_{\mathsf{J}'}} \;-\; x_{\mathsf{J}\cap\mathsf{J}'}\, x_{\mathsf{J}\cup\mathsf{J}'} \qquad \text{with } \mathsf{J}, \mathsf{J}' \in \mathcal{J}(P) \text{ incomparable} \tag{14}$$

is a reduced Gröbner basis of $\mathrm{I}_{\mathcal{O}(P)}$.

The binomials (14) are called **Hibi relations**.

In light of the previous sections, the natural question that we will address now is regarding an algebraic counterpart of the Hibi rings for double posets. For a double poset $\mathbf{P} = (P, \preceq_+$

$, \preceq_-)$, we define the **double Hibi ring** $\mathbb{C}[\mathcal{O}(\mathbf{P})]$ as the subalgebra of the Laurent ring $\hat{S} := \mathbb{C}[t_-, t_+, s_a, s_a^{-1} : a \in P]$ spanned by the elements $t_+ \cdot s^{\mathsf{J}}$ for filters $\mathsf{J} \in \mathcal{J}(P_+)$ and $t_- \cdot (s^{\mathsf{J}})^{-1}$ for filters $\mathsf{J} \in \mathcal{J}(P_-)$. This is the affine semigroup ring associated to $\mathcal{O}(\mathbf{P})$ with respect to the affine lattice $\mathbb{A} = 2\mathbb{Z}^P \times (2\mathbb{Z}+1)$. Up to a translation by $(\mathbf{0}, 1)$ and the lattice isomorphism $2\mathbb{Z}^P \times 2\mathbb{Z} \cong \mathbb{Z}^P \times \mathbb{Z}$, the double Hibi ring $\mathbb{C}[\mathcal{O}(\mathbf{P})]$ is the affine semigroup ring of

$$\operatorname{conv}\big\{(\mathcal{O}(P_+) \times \{1\}) \cup (-\mathcal{O}(P_-) \times \{0\})\big\},$$

with respect to the usual lattice $\mathbb{Z}^P \times \mathbb{Z}$. In particular, the double Hibi ring $\mathbb{C}[\mathcal{O}(\mathbf{P})]$ is graded of Krull dimension $|P| + 1$. Moreover, since the double order polytope $\mathcal{O}(\mathbf{P})$ is reflexive by Corollary 2.8, it follows that $\mathbb{C}[\mathcal{O}(\mathbf{P})]$ is a Gorenstein domain for any compatible double poset $\mathbf{P}$. The rings $\mathbb{C}[\mathcal{O}(\mathbf{P})]$ as well as affine semigroup rings associated to the double chain polytopes $\mathcal{C}(\mathbf{P})$ as treated at the end of Section 5.2 were also considered by Hibi and Tsuchiya [24].

Set $\hat{R} := \mathbb{C}[x_{\mathsf{J}_+}, x_{\mathsf{J}_-} : \mathsf{J}_+ \in \mathcal{J}(P_+), \mathsf{J}_+ \in \mathcal{J}(P_+)]$ and define the monomial map $\hat{\phi} : \hat{R} \to \hat{S}$ by

$$\hat{\phi}(x_{\mathsf{J}_+}) \;=\; t_+ \, s^{\mathsf{J}_+} \quad \text{ and } \quad \hat{\phi}(x_{\mathsf{J}_-}) \;=\; t_- \, (s^{\mathsf{J}_-})^{-1}.$$

The corresponding toric ideal $\mathrm{I}_{\mathcal{O}(\mathbf{P})} = \ker \hat{\phi}$ is then generated by the binomials

$$\underline{x_{\mathsf{J}_{+1}} x_{\mathsf{J}_{+2}} \dots x_{\mathsf{J}_{+k_+}} \cdot x_{\mathsf{J}_{-1}} x_{\mathsf{J}_{-2}} \dots x_{\mathsf{J}_{-k_-}}} - x_{\mathsf{J}'_{+1}} x_{\mathsf{J}'_{+2}} \dots x_{\mathsf{J}'_{+k_+}} \cdot x_{\mathsf{J}'_{-1}} x_{\mathsf{J}'_{-2}} \dots x_{\mathsf{J}'_{-k_-}}, \tag{15}$$

for filters $\mathsf{J}_{+1}, \dots, \mathsf{J}_{+k_+}, \mathsf{J}'_{+1}, \dots, \mathsf{J}'_{+k_+} \in \mathcal{J}(P_+)$ and $\mathsf{J}_{-1}, \dots, \mathsf{J}_{-k_-}, \mathsf{J}'_{-1}, \dots, \mathsf{J}'_{-k_-} \in \mathcal{J}(P_-)$.

Again, fix a total order $\le$ on the variables of $\hat{R}$ such that for $\sigma = \pm$

- $x_{\mathsf{J}_\sigma} < x_{\mathsf{J}'_\sigma}$ for any filters $\mathsf{J}_\sigma, \mathsf{J}'_\sigma \in \mathcal{J}(P_\sigma)$ with $\mathsf{J}_\sigma \subset \mathsf{J}'_\sigma$, and
- $x_{\mathsf{J}_+} < x_{\mathsf{J}_-}$ for any filters $\mathsf{J}_+ \in \mathcal{J}(P_+)$ and $\mathsf{J}_- \in \mathcal{J}(P_-)$,

and denote by $\le_{rev}$ the reverse lexicographic term order on $\hat{R}$ induced by this order on the variables.

Theorem 5.2. *Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a compatible double poset. Then a Gröbner basis for $\mathrm{I}_{\mathcal{O}(\mathbf{P})}$ is given by the binomials*

$$\underline{x_{\mathsf{J}_\sigma} \, x_{\mathsf{J}'_\sigma}} \;-\; x_{\mathsf{J}_\sigma \cup \mathsf{J}'_\sigma} \, x_{\mathsf{J}_\sigma \cap \mathsf{J}'_\sigma} \tag{16}$$

for incomparable filters $\mathsf{J}_\sigma, \mathsf{J}'_\sigma \in \mathcal{J}(P_\sigma)$ and $\sigma = \pm$, and

$$\underline{x_{\mathsf{J}_+} \, x_{\mathsf{J}_-}} \;-\; x_{\mathsf{J}_+ \setminus A} \, x_{\mathsf{J}_- \setminus A} \tag{17}$$

for filters $\mathsf{J}_+ \in \mathcal{J}(P_+), \mathsf{J}_- \in \mathcal{J}(P_-)$ such that $A := \min(\mathsf{J}_+) \cap \min(\mathsf{J}_-) \neq \varnothing$.

It is clear that binomials of the form (16) and (17) are contained in $\mathrm{I}_{\mathcal{O}(\mathbf{P})}$ and hence it suffices to show that their leading terms generate $\mathrm{in}_{\le_{rev}}(\mathrm{I}_{\mathcal{O}(\mathbf{P})})$. For this, let us take a closer look at the combinatorics of $\hat{\phi}$. Let $\mathcal{G}$ be the collection of binomial given in (16) and (17) and let $f = \underline{m_1} - m_2$ be an element of the form (15). By reducing f by the binomial of (16), we can view f as a quadruple

$$\begin{array}{ll} \mathsf{J}_{+1} \subset \mathsf{J}_{+2} \subset \cdots \subset \mathsf{J}_{+k_+} & \qquad \mathsf{J}_{-1} \subset \mathsf{J}_{+2} \subset \cdots \subset \mathsf{J}_{-k_-} \\ \mathsf{J}'_{+1} \subset \mathsf{J}'_{+2} \subset \cdots \subset \mathsf{J}'_{+k_+} & \qquad \mathsf{J}'_{-1} \subset \mathsf{J}'_{+2} \subset \cdots \subset \mathsf{J}'_{-k_-}. \end{array} \tag{18}$$

From the definition of $\hat{\phi}$ it follows that such a quadruple defines a binomial in $\mathrm{I}_{\mathcal{O}(\mathbf{P})}$ if and only if for any $q \in P$

$$\max\{r : q \notin \mathsf{J}_{+r}\} - \max\{s : q \notin \mathsf{J}_{-s}\} \;=\; \max\{r : q \notin \mathsf{J}'_{+r}\} - \max\{s : q \notin \mathsf{J}'_{-s}\}. \tag{19}$$

and we note the following implication.

Lemma 5.3. *Let the collection of filters in (15) correspond to a binomial $f \in \mathrm{I}_{\mathcal{O}(\mathbf{P})}$ and let $q \in P$. Then there is some $1 \leq i \leq k_+$ such that $q \in \mathsf{J}_{+i} \setminus \mathsf{J}'_{+i}$ if and only if there is some $1 \leq j \leq k_-$ such that $q \in \mathsf{J}_{-j} \setminus \mathsf{J}'_{-j}$.*

Proof. If $q \in \mathsf{J}_{+i} \setminus \mathsf{J}'_{+i}$, then $\max\{r : q \notin \mathsf{J}_{+r}\} < i$ and $\max\{r : q \notin \mathsf{J}'_{+r}\} \geq i$ and (19) implies that $q \in \mathsf{J}_{-j} \setminus \mathsf{J}'_{-j}$ for some j. The other direction is identical. □

We call $q \in P$ **moving** if it satisfies one of the two equivalent conditions of Lemma 5.3.

Proof of Theorem 5.2. Let $f = \underline{m_1} - m_2 \in \mathrm{I}_{\mathcal{O}(\mathbf{P})}$ be a binomial represented by a collection of filters given by (18). If $k_- = 0$ or $k_+ = 0$, then the Hibi relations (16) for P_- or P_+ together with Theorem 5.1 yields the result. Thus, we assume that $k_-, k_+ > 0$ and we need to show that there are filters J_{+i} and J_{-j} such that $\min(\mathsf{J}_{+i}) \cap \min(\mathsf{J}_{-j}) \neq \varnothing$.

Observe that there is at least one moving element. Indeed, $\mathsf{J}_{+1} \not\subseteq \mathsf{J}'_{+1}$ and hence $\mathsf{J}_{+1} \setminus \mathsf{J}'_{+1} \neq \varnothing$. Otherwise, $x_{\mathsf{J}_{+1}} < x_{\mathsf{J}'_{+1}}$ and the reverse lexicographic term order $\leq_{rev}$ would not select m_1 as the lead term of f. Among all moving elements, choose q to be minimal with respect to $\preceq_+$ and $\preceq_-$. Since $\mathbf{P}$ is a compatible double poset, such a q exists. But then, if $q \in \mathsf{J}_{+i} \setminus \mathsf{J}'_{+i}$, then $q \in \min(\mathsf{J}_{+i})$. The same holds true for J_{-j} and shows that $\underline{m_1}$ is divisible by the leading term of a binomial of type (17). □

5.2. **Gröbner bases, faces, and triangulations.** In light of the regular and unimodular triangulation of $\mathcal{O}(P)$ given in [34] (and recalled in Section 4.1), the Hibi ring $\mathbb{C}[\mathcal{O}(P)]$ is exactly the affine semigroup ring associated to $\mathcal{O}(P)$. That is, $\mathbb{C}[\mathcal{O}(P)]$ is the standard graded $\mathbb{C}$-algebra associated to the normal affine semigroup

$$\{(f, k) \in \mathbb{Z}^P \times \mathbb{Z} : k \geq 0, f \in k\mathcal{O}(P)\}.$$

For a lattice polytope $\mathcal{P} \subset \mathbb{R}^n$, Sturmfels [36, Thm. 8.3] described a beautiful relationship between regular triangulations of $\mathcal{P}$ and radicals of initial ideals of the toric ideal $\mathrm{I}_{\mathcal{P}}$. It follows from Theorem 5.2 that $\mathrm{in}_{\leq_{rev}}(\mathrm{I}_{\mathcal{O}(\mathbf{P})})$ is a squarefree ideal generated by quadratic monomials. Appealing to [36, Thm. 8.3], this yields the following refinement of Theorem 4.3.

Corollary 5.4. *Let $\mathbf{P}$ be a compatible double poset. Then $\mathcal{O}(\mathbf{P})$ has a regular triangulation whose underlying simplicial complex is exactly $\Delta^{\mathrm{ni}}(\mathbf{P})$.*

Proof. The initial ideal $\mathrm{in}_{\leq_{rev}}(\mathrm{I}_{\mathcal{O}(\mathbf{P})})$ is already radical and Theorem 8.3 of [36] yields that $\mathrm{in}_{\leq_{rev}}(\mathrm{I}_{\mathcal{O}(\mathbf{P})})$ is the Stanley-Reisner ideal of a regular triangulation of $\mathcal{O}(\mathbf{P})$. Hence, a collection $C = C_1 \uplus C_2 \subseteq \mathcal{J}(\mathbf{P})$ forms a simplex in the triangulation of $\mathcal{O}(\mathbf{P})$ if and only if

$$\prod_{\mathsf{J}_+ \in C_+} x_{\mathsf{J}_+} \prod_{\mathsf{J}_- \in C_-} x_{\mathsf{J}_-} \notin \mathrm{in}_{\leq_{rev}}(\mathrm{I}_{\mathcal{O}(\mathbf{P})}).$$

Translating the conditions given in Theorem 5.2, this is the case if and only if $C_\sigma = C \cap \mathcal{J}(P_\sigma)$ is a chain of filters for $\sigma = \pm$ and C_+, C_- are non-interfering chains. This is exactly the definition of the flag complex $\Delta^{\mathrm{ni}}(\mathbf{P})$. □

Using the orbit-cone correspondence for affine toric varieties (see, for example, [5, Sect. 3.2]), we can give an algebraic perspective on Theorem 2.13. We are in a particularly nice situation as the polytopes we consider have unimodular triangulations and hence the affine semigroup rings are generated in degree 1 by the vertices of the underlying polytope.

Lemma 5.5. *Let $V \subset \Lambda$ be a finite set of lattice points and $\mathcal{P} = \operatorname{conv}(V)$ the corresponding lattice polytope. If $\mathrm{I} \subseteq \mathbb{C}[x_v : v \in V]$ is the toric ideal of the homogenization $\{(v, 1) : v \in V\} \subseteq \Lambda \times \mathbb{Z}$, then for any subset $U \subseteq V$, we have that $\operatorname{conv}(U)$ is a face of $\mathcal{P}$ with $\operatorname{conv}(U) \cap V = U$ if and only if*

$$f(\mathbf{1}_U) \;=\; 0 \qquad \textit{for all } f \in \mathrm{I}.$$

Proof of Theorem 2.13. Let $\mathrm{L} \subseteq \mathcal{J}(\mathbf{P})$. Then for $\sigma = \pm$ and $\mathsf{J}_\sigma, \mathsf{J}'_\sigma \in \mathcal{J}(P_\sigma)$ Lemma 5.5 and (16) of Theorem 5.2 states that

$$\mathsf{J}_\sigma, \mathsf{J}'_\sigma \in \mathrm{L}_\sigma \quad \Longleftrightarrow \quad \mathsf{J}_\sigma \cup \mathsf{J}'_\sigma, \mathsf{J}_\sigma \cap \mathsf{J}'_\sigma \in \mathrm{L}_\sigma.$$

That is, if and only if L_σ is an embedded Thus, L_σ is an embedded sublattice of $\mathcal{J}(P_\sigma)$. The same reasoning shows that the conditions imposed by (17) are equivalent to those of Lemma 2.14. □

We can also use Sturmfels' result in the other direction to find Gröbner bases. For a double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$ we may define the subring $\mathbb{C}[\mathcal{C}(\mathbf{P})] \subseteq \hat{R}$ generated by the monomials $t_+ s^{\min(\mathsf{J}_+)}$ and $t_+(s^{\min(\mathsf{J}_-)})^{-1}$ for filters $\mathsf{J}_+ \subseteq P_+$ and $\mathsf{J}_- \subseteq P_-$. The corresponding toric ideal $\mathrm{I}_{\mathcal{C}(\mathbf{P})}$ is contained in the ring $T = \mathbb{C}[x_{A_+}, x_{A_-}]$, where A_σ ranges over all anti-chains in P_σ for $\sigma = \pm$. Since $\mathcal{O}(\mathbf{P})$ is the stable set polytope of the perfect double graph $G(\mathbf{P})$, it follows from Corollary 4.1 that $\mathbb{C}[\mathcal{C}(\mathbf{P})]$ is the normal affine semigroup ring associated to the lattice polytope $\mathcal{C}(\mathbf{P})$. To describe a Gröbner basis for, we introduce the following notation. For $\sigma = \pm$ and two antichains $A, A' \subseteq P_\sigma$ define $A \sqcup A' \;:=\; \min(A \cup A')$ and

$$A \sqcap A' \;:=\; (A \cap A') \cup (\max(A \cup A') \setminus \min(A \cup A')).$$

For a subset $S \subseteq P$ and $\sigma = \pm$, we write $\langle S \rangle_\sigma := \{a \in P : a \succeq_\sigma s \text{ for some } s \in S\}$ for the filter in P_σ generated by S.

Theorem 5.6. *Let $\mathbf{P}$ be a double poset. Then a Gröbner basis for $\mathrm{I}_{\mathcal{C}(\mathbf{P})}$ is given by the binomials*

$$\underline{x_A\, x_{A'}} \;-\; x_{A \sqcup A'}\, x_{A \sqcap A'} \qquad \langle A \rangle_\sigma, \langle A \rangle_\sigma \in \mathcal{J}(P_\sigma) \textit{ incomparable}$$

for antichains $A, A' \subset P_\sigma$ for $\sigma = \pm$ and

$$\underline{x_{A_+}\, x_{A_-}} \;-\; x_{A_+ \setminus A_-} x_{A_- \setminus A_+} \qquad \textit{for antichains } A_\sigma \subseteq P_\sigma.$$

Proof. It is easy to verify that the given binomials are contained in $\mathrm{I}_{\mathcal{C}(\mathbf{P})}$. Moreover, the leading monomials are exactly the minimal non-faces of the unimodular triangulation of Corollary 4.1. The result now follows from Theorem 8.3 in [36]. □

Remark 1. Reformulated in the language of double posets, Hibi, Matsuda, and Tsuchiya [22, 21, 23] computed related Gröbner bases of the toric ideals associated with the polytopes $\Gamma(\mathcal{O}(P_+), \mathcal{O}(P_-))$ (in the compatible case), $\Gamma(\mathcal{C}(P_+), \mathcal{C}(P_-))$, and $\Gamma(\mathcal{O}(P_+), \mathcal{C}(P_-))$ for a double poset $\mathbf{P}$. See the paragraph before Corollary 2.21 for notation.

References

[1] F. Ardila, T. Bliem, and D. Salazar, *Gelfand–Tsetlin polytopes and Feigin–Fourier–Littelmann–Vinberg polytopes as marked poset polytopes*, J. Combin. Theory Ser. A, 118 (2011), pp. 2454–2462. 2

[2] M. Beck and S. Robins, *Computing the continuous discretely: Integer-point enumeration in polyhedra*, Undergraduate Texts in Mathematics, Springer, New York, 2007. 24

[3] M. Beck and R. Sanyal, *Combinatorial Reciprocity Theorems.* Book in preparation. Preliminary version available at http://math.sfsu.edu/beck/crt.html. 24, 27

[4] G. Brightwell and P. Winkler, *Counting linear extensions*, Order, 8 (1991), pp. 225–242. 2

[5] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, vol. 124 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2011. 33

[6] J. A. De Loera, J. Rambau, and F. Santos, *Triangulations: Structures for algorithms and applications*, vol. 25 of Algorithms and Computation in Mathematics, Springer-Verlag, Berlin, 2010. 22, 23

[7] H. Dobbertin, *About polytopes of valuations on finite distributive lattices*, Order, 2 (1985), pp. 193–198. 15

[8] B. Dushnik and E. W. Miller, *Partially ordered sets*, Amer. J. Math., 63 (1941), pp. 600–610. 10

[9] L. Foissy, *Algebraic structures on double and plane posets*, J. Algebraic Combin., 37 (2013), pp. 39–66. 2, 8

[10] L. Foissy, *Plane posets, special posets, and permutations*, Adv. Math., 240 (2013), pp. 24–60. 2, 10

[11] G. Fourier, *PBW-degenerated Demazure modules and Schubert varieties for triangular elements*, J. Combin. Theory Ser. A, 139 (2016), pp. 132–152. 2

[12] D. R. Fulkerson, *Blocking and anti-blocking pairs of polyhedra*, Mathematical Programming, 1, pp. 168–194. 3, 16

[13] L. Geissinger, *The face structure of a poset polytope*, in Proceedings of the Third Caribbean Conference on Combinatorics and Computing (Bridgetown, 1981), Univ. West Indies, Cave Hill Campus, Barbados, 1981, pp. 125–133. 3, 14

[14] J. Gouveia, P. A. Parrilo, and R. R. Thomas, *Theta bodies for polynomial ideals*, SIAM J. Optim., 20 (2010), pp. 2097–2118. 4

[15] F. Grande and R. Sanyal, *Theta rank, levelness, and matroid minors*, J. Combin. Theory Ser. B, 123 (2017), pp. 1–31. 4

[16] D. Grinberg, *Combinatorial reciprocity for monotone triangles.* 45 pages, `http://arxiv.org/abs/1509.08355`, September 2015. 12

[17] A. B. Hansen, *On a certain class of polytopes associated with independence systems*, Math. Scand., 41 (1977), pp. 225–241. 3, 4, 20

[18] J. Herzog and T. Hibi, *Monomial ideals*, vol. 260 of Graduate Texts in Mathematics, Springer-Verlag London, Ltd., London, 2011. 31

[19] T. Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, in Commutative algebra and combinatorics (Kyoto, 1985), vol. 11 of Adv. Stud. Pure Math., North-Holland, Amsterdam, 1987, pp. 93–109. 4, 31

[20] T. Hibi and N. Li, *Unimodular equivalence of order and chain polytopes*, Math. Scand., 118 (2016), pp. 5–12. 19

[21] T. Hibi and K. Matsuda, *Quadratic Gröbner bases of twinned order polytopes.* preprint arXiv:1505.04289, 2015. 16, 34

[22] T. Hibi, K. Matsuda, and A. Tsuchiya, *Gorenstein Fano polytopes arising from order polytopes and chain polytopes.* preprint arXiv:1507.03221, 2015. 16, 34

[23] ———, *Quadratic Gröbner bases arising from partially ordered sets.* preprint arXiv:1506.00802, 2015. 34

[24] T. Hibi and A. Tsuchiya, *Facets and volume of Gorenstein Fano polytopes.* Preprint, June 2016, 13 pages, arXiv:1606.03566. 32

[25] K. Jochemko and R. Sanyal, *Arithmetic of marked order polytopes, monotone triangle reciprocity, and partial colorings*, SIAM J. Discrete Math., 28 (2014), pp. 1540–1558. doi:10.1137/130944849. 2

[26] J. Kahn and N. Linial, *Balancing extensions via Brunn-Minkowski*, Combinatorica, 11 (1991), pp. 363–368. 2

[27] L. Lovász, *Normal hypergraphs and the perfect graph conjecture*, Discrete Math., 2 (1972), pp. 253–267. 3, 4, 17

[28] C. Malvenuto and C. Reutenauer, *A self paired Hopf algebra on double posets and a Littlewood-Richardson rule*, J. Combin. Theory Ser. A, 118 (2011), pp. 1322–1333. 2, 4, 8

[29] P. McMullen, *Constructions for projectively unique polytopes*, Discrete Math., 14 (1976), pp. 347–358. 8

[30] R. Sanyal, A. Werner, and G. M. Ziegler, *On Kalai's conjectures concerning centrally symmetric polytopes*, Discrete Comput. Geom., 41 (2009), pp. 183–198. 4, 18

[31] A. Schrijver, *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons, Ltd., Chichester, 1986. A Wiley-Interscience Publication. 4, 16, 17

[32] E. Soprunova and F. Sottile, *Lower bounds for real solutions to sparse polynomial systems*, Adv. Math., 204 (2006), pp. 116–151. 2

[33] R. P. Stanley, *Ordered structures and partitions*, American Mathematical Society, Providence, R.I., 1972. Memoirs of the American Mathematical Society, No. 119. 2

[34] ———, *Two poset polytopes*, Discrete Comput. Geom., 1 (1986), pp. 9–23. 1, 2, 6, 26, 29, 33

[35] ———, *Enumerative combinatorics. Volume 1*, vol. 49 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, second ed., 2012. 15, 29

[36] B. Sturmfels, *Gröbner bases and convex polytopes*, vol. 8 of University Lecture Series, American Mathematical Society, Providence, RI, 1996. 4, 33, 34

[37] S. Sullivant, *Compressed polytopes and statistical disclosure limitation*, Tohoku Math. J. (2), 58 (2006), pp. 433–445. 4, 23, 26

[38] D. G. Wagner, *Singularities of toric varieties associated with finite distributive lattices*, J. Algebraic Combin., 5 (1996), pp. 149–165. 12

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