

On the steady non-Newtonian fluids in domains with noncompact boundaries

Yang Jiaqi^{1,*} Yin Huicheng^{2,*}

1. Department of Mathematics and IMS, Nanjing University, Nanjing 210093, China
 2. School of Mathematical Sciences, Jiangsu Provincial Key Laboratory for Numerical Simulation of Large Scale Complex Systems, Nanjing Normal University, Nanjing 210023, China

Abstract

In this paper, we study the steady non-Newtonian fluids in a class of unbounded domains with noncompact boundaries. With respect to the resulting mathematical problems, we establish the global existence of solutions with arbitrary large flux under some suitable conditions, and meanwhile, show the uniqueness of the solutions when the flux is sufficiently small. Our results are an extension or an improvement of those obtained in some previous references.

Keywords. Non-Newtonian fluid, steady, noncompact boundary, Leray problem, Ladyzhenskaya-Solonnikov problem, Korn-type inequality.

2010 Mathematical Subject Classification. 35Q30, 35B30, 76D05, 76D07.

1 Introduction

Although the steady Navier-Stokes equations have been investigated extensively (see [2]- [10], [12], [14]-[18], [21]- [29] and the references therein), the global well-posedness of a flow in a domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with noncompact boundaries is still an interesting question for arbitrary fluxes. A special case is that the domain Ω is a distorted infinite cylinder or channel (see [10] and so on), namely, Ω can be described as follows (see Figure 1 and Figure 2 below):

$$\Omega = \bigcup_{i=0}^2 \Omega_i, \quad (1.1)$$

where Ω_0 is a smooth bounded subset of Ω , while Ω_1 and Ω_2 are disjoint regions which may be expressed in possibly different coordinate systems (x_1^1, \dots, x_d^1) and (x_1^2, \dots, x_d^2) by

$$\Omega_i = \{(x_1^i, \dots, x_d^i) \in \mathbb{R}^d : x_1^i > 0, (x_2^i, \dots, x_d^i) \in \Sigma_i(x_1^i)\}, \quad i = 1, 2, \quad (1.2)$$

here $\Sigma_i(x_1^i)$ represents the bounded cross section of Ω_i for fixed x_1^i .

*Yang Jiaqi (yjqmath@163.com) and Yin Huicheng (huicheng@nju.edu.cn, 05407@njnu.edu.cn) are supported by the NSFC (No. 11571177) and the Priority Academic Program Development of Jiangsu Higher Education Institutions.

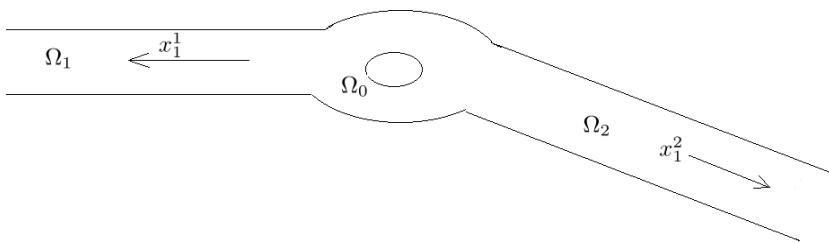


Figure 1. Domain Ω for $d = 2$

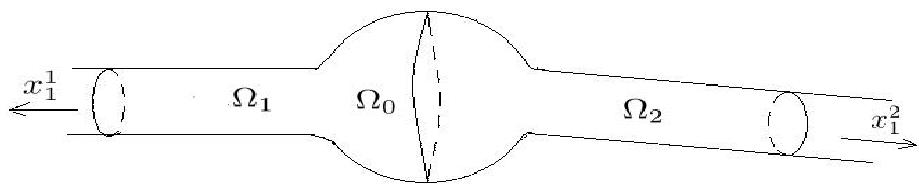


Figure 2. Domain Ω for $d = 3$

Owing to the incompressibility of the fluids and the vanishing property of the current velocity $\mathbf{v} = (v_1, \dots, v_d)$ on the boundary $\partial\Omega$, we deduce that the flux $\alpha_i \equiv \int_{\Sigma_i(x_1^i)} \mathbf{v} \cdot \mathbf{n} \, dS$ of velocity \mathbf{v} (\mathbf{n} stands

for the unit outer normal direction of $\Sigma_i(x_1^i)$) through $\Sigma_i(x_1^i)$ is a constant independent of the variable x_1^i , and α'_i 's ($i = 1, 2$) satisfy

$$\alpha_1 + \alpha_2 = 0. \quad (1.3)$$

When the cross section $\Sigma_i(x_1^i)$ is independent of x_1^i , which means that each outlet Ω_i is a semi-infinite strip for $d = 2$ or a semi-infinite straight cylinder for $d = 3$, respectively, $\Sigma_i(x_1^i)$ will be simply denoted by Σ_i . In this case, the classical Leray's problem (see [19]) is to study the well-posedness of the following steady flows:

$$\begin{cases} -\mu\Delta\mathbf{v} + \mathbf{v} \cdot \nabla\mathbf{v} + \nabla\pi = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0, & \text{on } \partial\Omega, \\ \int_{\Sigma_i} \mathbf{v} \cdot \mathbf{n} \, dS = \alpha_i & i = 1, 2, \\ \mathbf{v} \rightarrow \mathbf{v}_0^i & \text{as } x_1^i \rightarrow \infty \text{ in } \Omega_i \text{ for } i = 1, 2, \end{cases} \quad (1.4)$$

where \mathbf{v}_0^i stands for the velocity of the Poiseuille flow corresponding to the given constant α_i , which is determined by

$$\begin{cases} \mathbf{v}_0^i = v_0^i(z^i)\mathbf{e}_1, \\ \mu \sum_{j=2}^d \partial_{x_j^i}^2 v_0^i(z^i) = -C_i & \text{in } \Sigma_i, \\ v_0^i(z^i) = 0 & \text{on } \partial\Sigma_i \end{cases} \quad (1.5)$$

with $z^i = (x_2^i, \dots, x_d^i)$, $\mathbf{e}_1 = (1, 0, \dots, 0)$ and C_i being a constant uniquely determined by α_i .

Leray's problem (1.4)-(1.5) has been extensively studied. In [14], under the smallness assumption of the flux α_i ($i = 1, 2$), O.A.Ladyzhenskaya proved the existence of solution \mathbf{v} but the uniqueness argument was not given. In [2], C.J.Amick completed the proof of both existence and uniqueness when the flux α_i is sufficient small. Alternately, O.A.Ladyzhenskaya and V.A.Solonnikov in [18] considered problem (1.4) together with (1.5) under the weaker assumption that the section $\Sigma_i(x_1^i)$ is uniformly bounded with respect to the variable x_1^i instead of the straight outlet Σ_i in [2]. In this case, one cannot pose the condition of \mathbf{v} at infinity by Poiseuille flow since section $\Sigma_i(x_1^i)$ changes for different x_1^i . Consequently, the authors in [18] considered problem (1.4) in another way, which is called Ladyzhenskaya-Solonnikov Problem I (by prescribing a growth condition of $|\mathbf{v}|$ with respect to the distance along the direction of each outlet instead of condition (1.5)), and they established the global existence of \mathbf{v} for arbitrary large flux by utilizing a variant of Saint-Venant's principle. Furthermore, if the flux is sufficient small, they got the uniqueness of solution \mathbf{v} . In particular, if flux is small and both exits Ω_1 and Ω_2 are straight, then it has been shown that the solution \mathbf{v} to Ladyzhenskaya-Solonnikov Problem I tends to the corresponding Poiseuille solution of (1.5).

In [18], the authors also studied another problem for (1.4), i.e., Ladyzhenskaya -Solonnikov Problem II. At this time, the sections Ω_1 and Ω_2 of Ω are not uniformly bounded and admit some certain rates of "growth", i.e.,

$$\Omega_i = \{x = (x_1^i, y^i) \in \mathbb{R}^d : x_1^i > 0, |y^i| \equiv \sqrt{(x_2^i)^2 + \dots + (x_d^i)^2} < g_i(x_1^i)\}, \quad (1.6)$$

where $g_i(x_1^i)$ is a global Lipschitz function. Later, in a series of papers [24–27], K.Pileckas shows that the Ladyzhenskaya -Solonnikov Problem II is uniquely solvable if flux is small. Simultaneously, it is shown

in [24–27] that the decay rate of solution \mathbf{v} at infinity is related to the inverse power of the functions g_1 and g_2 .

The Navier-Stokes model of incompressible fluids is based on the Stokes-hypothesis which simplifies the relation between the stress tensor and the velocity. However, a number of experiments show that many other incompressible fluids, including bloods, cannot be described by this model. In the late 1960s, see [15, 16], O.A.Ladyzhenskaya started a systematic investigation on the well-posedness of the boundary value problems associated to certain generalized Newtonian models. In contrast to Newtonian flows, for non-Newtonian flows, the viscosity coefficient μ is no longer constant, it depends on the magnitude of $\mathcal{D}(\mathbf{v})$, i.e.,

$$\mu(\mathcal{D}(\mathbf{v})) = \mu_0 + \mu_1 |\mathcal{D}(\mathbf{v})|^{p-2}, \quad (1.7)$$

where $\mu_0 \geq 0$, $\mu_1 > 0$, $p > 1$, and $\mathcal{D}(\mathbf{v}) = (D_{ij}(\mathbf{v}))_{i,j=1}^d$ with $D_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$. In this case, the corresponding Leray's problem in the unbounded pipe domain Ω is described as follows

$$\begin{cases} -\operatorname{div}(\mu_0 \mathcal{D}(\mathbf{v}) + \mu_1 |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v})) + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \pi = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial \Omega, \\ \int_{\Sigma_i} \mathbf{v} \cdot \mathbf{n} \, dS = \alpha_i & i = 1, 2, \\ \mathbf{v} \rightarrow \mathbf{v}_{P_i} & \text{as } x_1^i \rightarrow \infty \text{ in } \Omega_i \text{ for } i = 1, 2, \end{cases} \quad (1.8)$$

where \mathbf{v}_{P_i} is the Hagen-Poiseuille flow, which satisfies

$$\begin{cases} \mathbf{v}_{P_i} = v_{P_i}(z^i) \mathbf{e}_1, \\ \mu_0 \Delta'_i v_{P_i} + \nabla'_i \cdot (\mu_1 |\mathcal{D}(v_{P_i})|^{p-2} \mathcal{D}(v_{P_i})) = -C_i, & \text{in } \Sigma_i \\ v_{P_i}(z^i) = 0, & \text{on } \partial \Sigma_i \end{cases} \quad (1.9)$$

here v_{P_i} is a scalar function, $z^i = (x_2^i, \dots, x_d^i)$, $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$, $\Delta'_i = \partial_{x_2^i}^2 + \dots + \partial_{x_d^i}^2$ and $\nabla'_i = (\partial_{x_2^i}, \dots, \partial_{x_d^i})$.

For the nonlinear equation systems in (1.8), O.A.Ladyzhenskaya [16] and J.L.Lions [20] proved the existence of the solution \mathbf{v} by the monotone operator theory in a bounded domain when $p \geq \frac{3d}{d+2}$. This result has been improved by some authors, in particular, in [5], the same result is established for $p > \frac{2d}{d+2}$. For noncompact boundaries, particularly, for piping-system, G.P.Galdi [8] proved that if $\mu_0 > 0$, $p > 2$, and flux is small, then problem (1.8) together with (1.9) has a unique weak solution \mathbf{v} . If $\mu_0 = 0$, by deriving some weighted energy estimates, E. Marušić-Paloka in [21] established the existence and uniqueness of the weak solution \mathbf{v} to problem (1.8) with (1.9) when $p > 2$ and the flux is small. Since the approach in [21] requires a detailed information about the dependence of v_{P_i} on the cross-sectional coordinates, where an explicit background solution is known, it seems that the resulting proof in [21] is only suitable for the case of a circular cross section. For arbitrary large flux, motivated by Ladyzhenskaya and Solonnikov's results in [18], the authors in [12] prove the existence and uniqueness of solution to the Ladyzhenskaya-Solonnikov Problem I for the non-Newtonian fluids when $p > 2$ and $\mu_0 = 0$. In this paper, we shall consider both Ladyzhenskaya-Solonnikov Problem I and Ladyzhenskaya-Solonnikov Problem II for the non-Newtonian fluids, and intend to establish some systematic results. Here we point out that the restriction of $p > 2$ when $\mu_0 = 0$ is essentially required in the proof of [12] (one can see the statements of lines 8-9 from below on pages 3874 in [12]: “As far as we know, the Leray problem for $p < 2$ (with small fluxes) is an open problem”), meanwhile only the corresponding Ladyzhenskaya-Solonnikov problem I is considered in [12]. We shall study problem (1.8) together with (1.9) for $p > 1$

and $\mu_0 \geqslant 0$ (when $1 < p < 2$, the condition $\mu_0 > 0$ will be needed). On the other hand, for the corresponding Ladyzhenskaya-Solonnikov Problem II of (1.8) (i.e., the outlets of Ω may be permitted to be unbounded), we shall establish both the existence and uniqueness of the solution \mathbf{v} for $\mu_0 > 0$ and $p > 1$ or $\mu_0 = 0$ and $2 < p \leq 3 - \frac{2}{d}$ (hence $d = 3$), especially, when the sections of Ω are uniformly bounded, the resulting conclusions also hold for $\mu_0 = 0$ and $p > 2$ (here we point out that this case has been solved in [12]).

Let us comment on the proofs of our results. For the case of $\mu_0 = 0$ in (1.8), if one wants to directly deal with the nonlinear term $\operatorname{div}(|D(\mathbf{v})|^{p-2}D(\mathbf{v}))$ for $p > 2$ and apply the integration by parts for equation (1.8) multiplying the solution \mathbf{v} to obtain a priori estimate of \mathbf{v} , then the regularities of $\mathbf{v} \in W^{2,l}$ and $\pi \in W^{1,l}$ for some positive number l in bounded domains are required as pointed out in [12]. However this regularity is not expected for the weak solutions of (1.8) if $p \neq 2$ as stated in [12] (see lines 15-16 of pages 3875). To overcome this kind of difficulty, the authors in [12] studied the following truncated modified problem

$$\begin{cases} -\operatorname{div}\left(\frac{1}{T}\mathcal{D}(\mathbf{v}^T) + \mu_1|\mathcal{D}(\mathbf{v}^T)|^{p-2}\mathcal{D}(\mathbf{v}^T)\right) + \mathbf{v}^T \cdot \nabla \mathbf{v}^T + \nabla \pi^T = 0 & \text{in } \Omega(T), \\ \operatorname{div} \mathbf{v}^T = 0 & \text{in } \Omega(T), \\ \mathbf{v}^T = 0 & \text{on } \partial\Omega(T), \end{cases} \quad (1.10)$$

where $\Omega(T) = \Omega_0 \cup \{x \in \Omega : 0 < x_1^1 < T, 0 < x_1^2 < T\}$. By deriving the uniform estimates of \mathbf{v}^T under the key assumption of $p > 2$ and applying a local version of the Minty trick, the authors in [12] proved the existence and uniqueness of solution \mathbf{v} to the Ladyzhenskaya-Solonnikov Problem I of (1.8) when $\mu_0 = 0$ and $p > 2$. We now state our ingredients for treating problem (1.8) in this paper. At first, we consider the following truncated modified problem instead of (1.8)

$$\begin{cases} -\operatorname{div}(\mu_0\mathcal{D}(\mathbf{v}^T) + \mu_1|\mathcal{D}(\mathbf{v}^T)|^{p-2}\mathcal{D}(\mathbf{v}^T)) + \mathbf{v}^T \cdot \nabla \mathbf{v}^T + \nabla \pi^T = 0 & \text{in } \Omega(T), \\ \operatorname{div} \mathbf{v}^T = 0 & \text{in } \Omega(T), \\ \mathbf{v}^T = 0 & \text{on } \partial\Omega(T). \end{cases} \quad (1.11)$$

As in [12] and [18], we assume that the velocity \mathbf{v}^T of (1.11) has the form $\mathbf{u}^T + \mathbf{a}$, where \mathbf{u}^T is the new unknown with zero flux, and \mathbf{a} is a specially constructed solenoidal field satisfying $\int_{\Sigma_i(x_1^i)} \mathbf{a} \cdot \mathbf{n} dS = \alpha_i$ and admitting some other “good” properties. To obtain a priori estimates of \mathbf{u}^T , we have to control the nonlinear term $\mathbf{u}^T \cdot \nabla \mathbf{u}^T \cdot \mathbf{a}$. If one only assumes that \mathbf{a} is bounded as in [12], then it follows from Young inequality and Poincaré inequality that only the following estimate for $p > 2$ is obtained

$$\begin{aligned} \left| \int_{\Omega_i(t)} \mathbf{u}^T \cdot \nabla \mathbf{u}^T \cdot \mathbf{a} dx \right| &\leqslant \varepsilon \int_{\Omega_i(t)} |\nabla \mathbf{u}^T|^p dx + c(\varepsilon) \int_{\Omega_i(t)} |\mathbf{u}^T|^{p'} dx \\ &\leqslant \varepsilon \int_{\Omega_i(t)} |\nabla \mathbf{u}^T|^p dx + c(\varepsilon) \int_{\Omega_i(t)} |\nabla \mathbf{u}^T|^{p'} dx \\ &\leqslant \varepsilon \int_{\Omega_i(t)} |\nabla \mathbf{u}^T|^p dx + \int_{\Omega_i(t)} (c(\varepsilon) + \varepsilon |\nabla \mathbf{u}^T|^p) dx \\ &\leqslant 2\varepsilon \int_{\Omega_i(t)} |\nabla \mathbf{u}^T|^p dx + c(\varepsilon)t, \end{aligned} \quad (1.12)$$

where $\Omega_i(t) = \{x \in \Omega_i : 0 < x_1^i < t\}$ and $c(\varepsilon) > 0$ stands for a generic constant depending on $\varepsilon > 0$. From (1.12), the authors in [12] obtained the crucial uniform estimate of $\int_{\Omega_i(t)} |\nabla \mathbf{u}^T|^p dx$ for the solution

\mathbf{v}^T to problem (1.10). To relax the restriction of power p and get the uniform control for solution \mathbf{v}^T of problem (1.11), we need more properties of \mathbf{a} and other interesting observations. Note that, for $\mu_0 > 0$, the leading term is $\int_{\Omega_i(t)} |\mathcal{D}(\mathbf{u}^T)|^2 dx$ in the energy estimate of \mathbf{u}^T (see (4.4) in §4), if one can find a field \mathbf{a} such that

$$\left| \int_{\Omega_i(t)} \mathbf{u}^T \cdot \nabla \mathbf{u}^T \cdot \mathbf{a} dx \right| \leq \varepsilon \int_{\Omega_i(t)} |\nabla \mathbf{u}^T|^2 dx, \quad (1.13)$$

then $\int_{\Omega_i(t)} |\nabla \mathbf{u}^T|^2 dx$ instead of $\int_{\Omega_i(t)} |\nabla \mathbf{u}^T|^p dx$ can be estimated; while, for $\mu_0 = 0$ and $p > 2$, the leading term is $\int_{\Omega_i(t)} |\mathcal{D}(\mathbf{u})|^p dx$ in the estimate of \mathbf{u}^T (see (4.4) in §4), if one can construct a vector field \mathbf{a} such that $|\mathbf{a}| \leq c |\Sigma_i(x_1^i)|^{-1}$ in $\Omega_i(t)$, then it follows from the Young inequality and Poincaré inequality that

$$\begin{aligned} & \left| \int_{\Omega_i(t)} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{a} dx \right| \\ & \leq \varepsilon \int_{\Omega_i(t)} |\mathbf{u}|^p |\Sigma_i(x_1^i)|^{\frac{p}{d-1}} dx + \varepsilon \int_{\Omega_i(t)} |\nabla \mathbf{u}|^p dx + c(\varepsilon) \int_{\Omega_i(t)} |\Sigma_i(x_1^i)|^{\frac{p(d-2)}{(p-2)(d-1)}} dx \\ & \leq 2\varepsilon \int_{\Omega_i(t)} |\nabla \mathbf{u}|^p dx + c(\varepsilon) \int_0^t |\Sigma_i(s)|^{1-\frac{p(d-2)}{(p-2)(d-1)}} ds, \end{aligned} \quad (1.14)$$

which derives the uniform estimate of $\int_{\Omega_i(t)} |\nabla \mathbf{u}^T|^p dx$ if $\int_0^t |\Sigma_i(s)|^{1-\frac{p(d-2)}{(p-2)(d-1)}} ds < \infty$. Thanks to [10] Lemma III.4.3 and [23] Lemma 2-Lemma 3, the aforementioned \mathbf{a} in (1.13) and (1.14) can be found. On the other hand, for Ladyzhenskaya-Solonnikov Problem I and II, the condition $\int_{\Omega_i(t)} |\mathcal{D}(\mathbf{u})|^p dx \leq c \int_0^t |\Sigma_i(s)|^{1-\frac{dp}{d-1}} ds$ should be required (see problem (2.4) and problem (2.5) in §2). Hence, by (1.14) we require such an inequality

$$\int_0^t |\Sigma_i(s)|^{1-\frac{p(d-2)}{(p-2)(d-1)}} ds \leq c \int_0^t |\Sigma_i(s)|^{1-\frac{dp}{d-1}} ds. \quad (1.15)$$

In the case of $\mu_0 = 0$, for Ladyzhenskaya-Solonnikov Problem I, (1.15) is automatically satisfied for any $p > 2$ since $\Sigma_i(x_1^i)$ is bounded, while for Ladyzhenskaya-Solonnikov Problem II, (1.15) is satisfied only for $2 < p \leq 3 - \frac{2}{d}$ ($d = 3$). Based on the uniform estimates of \mathbf{u}^T , inspired by [10] and [22], through choosing some suitable test functions and taking some delicate analysis on the resulting nonlinear terms, we can show $\mathbf{v}^T \rightarrow \mathbf{v}$ a.e. in any compact subset of Ω by establishing the uniform interior estimates of solution \mathbf{v}^T to (1.11). From this, together with some methods introduced in [18] for treating the Newtonian fluids and involved analysis on the resulting nonlinear terms in non-Newtonian fluids, we eventually complete the proofs on the existence and uniqueness of solution \mathbf{v} to the related Ladyzhenskaya-Solonnikov Problem I and Ladyzhenskaya-Solonnikov Problem II of (1.8) under some suitable conditions.

Our paper is organized as follows. In §2, the detailed descriptions on the resulting Ladyzhenskaya-Solonnikov Problems for the non-Newtonian flows are given. In §3, we present some preliminary conclusions which will be applied to prove our main results in subsequent sections. In §4, we establish the existence of the solutions \mathbf{v}^T to the bounded truncated problem corresponding to (1.8). In §5, we study the interior regularity of solutions \mathbf{v}^T obtained in §4. Based on §4 and §5, we shall complete the proofs on Ladyzhenskaya-Solonnikov Problem I and Ladyzhenskaya-Solonnikov Problem II of (1.8) in §6 and §7 respectively.

2 Descriptions of Ladyzhenskaya-Solonnikov Problems for non-Newtonian fluids

We focus on the following non-Newtonian fluid problem in the domain Ω with noncompact boundaries (see Figure 3 and Figure 4 below):

$$\begin{cases} -\operatorname{div}(\mu_0 \mathcal{D}(\mathbf{v}) + \mu_1 |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v})) + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \pi = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial\Omega, \\ \int_{\Sigma_i(x_1^i)} \mathbf{v} \cdot \mathbf{n} \, dS = \alpha_i & \text{with } \sum_{i=1}^N \alpha_i = 0, \end{cases} \quad (2.1)$$

where

$$\Omega = \Omega_0 \cup \left(\bigcup_{i=1}^N \Omega_i \right),$$

and

$$\Omega_i = \{x \in \mathbb{R}^n : x_1^i > 0, y^i = (x_2^i, \dots, x_d^i) \in \Sigma_i(x_1^i)\}.$$

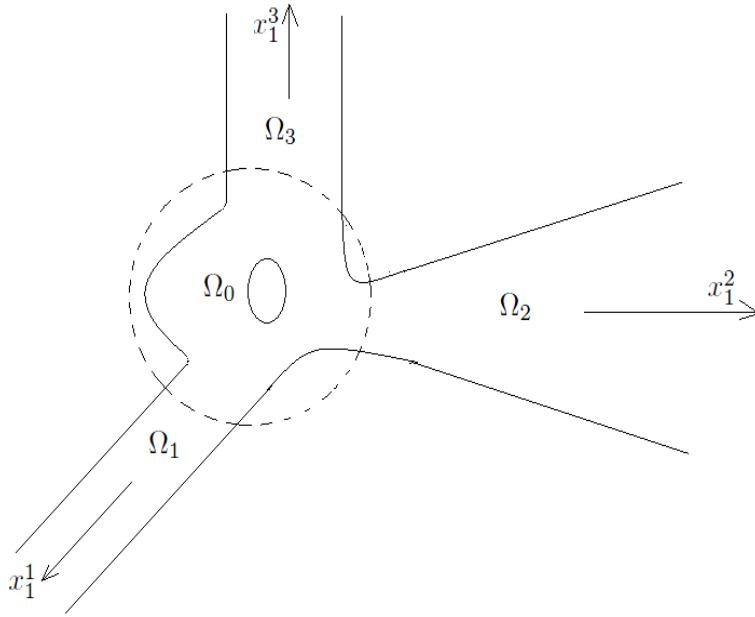


Figure 3. Domain Ω for $d = 2$.

Suppose that \mathbf{v} is a solenoidal field and $\mathbf{v} = 0$ holds on $\partial\Omega_i \setminus \{x^i = (x_1^i, y^i) : x_1^i = 0, y^i \in \Sigma_i(0)\}$.

Then

$$|\alpha_i|^p = \left| \int_{\Sigma_i(t)} \mathbf{v} \cdot \mathbf{n} \, dS \right|^p \leqslant |\Sigma_i(t)|^{p-1} \int_{\Sigma_i(t)} |\mathbf{v}|^p \, dS \leqslant c |\Sigma_i(t)|^{\frac{dp}{d-1}-1} \int_{\Sigma_i(t)} |\nabla' \mathbf{v}|^p \, dS,$$

where $c > 0$ stands for a generic constant. This means

$$|\alpha_i|^p \int_0^t |\Sigma_i(s)|^{1-\frac{dp}{d-1}} ds \leqslant c \int_{\Omega_i(t)} |\nabla' \mathbf{v}|^p dx,$$

where $\Omega_i(t) = \{x^i \in \Omega_i : 0 < x_1^i < t\}$ for $1 \leq i \leq N$. Hence, if $\alpha_i \neq 0$ and

$$I_i(t) \equiv \int_0^t |\Sigma_i(s)|^{1-\frac{dp}{d-1}} ds \rightarrow +\infty \text{ as } t \rightarrow +\infty, \quad (2.2)$$

then

$$Q_i(t) \equiv \int_{\Omega_i(t)} |\nabla \mathbf{v}|^p dx \rightarrow +\infty \text{ as } t \rightarrow +\infty. \quad (2.3)$$

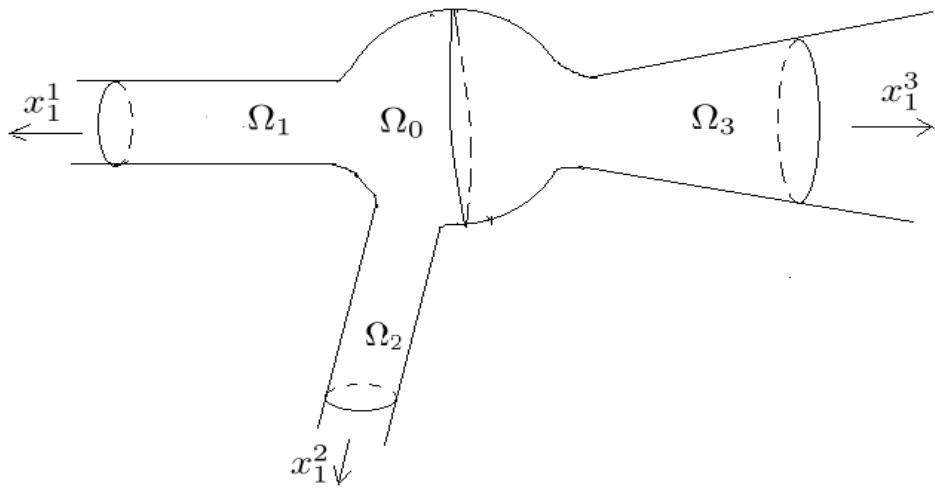


Figure 4. Domain Ω for $d = 3$.

From (2.2) and (2.3), it is natural to consider the following two problems

Ladyzhenskaya – Solonnikov Problem I. Suppose that there are two positive constant c_1 and

c_2 such that $c_1 < |\Sigma_i(t)| < c_2$ for $1 \leq i \leq N$. We look for a pair vector field (\mathbf{v}, π) to fulfill

$$\begin{cases} -\operatorname{div}(\mu_0 \mathcal{D}(\mathbf{v}) + \mu_1 |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v})) + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \pi = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial\Omega, \\ \int_{\Sigma_i(x_1^i)} \mathbf{v} \cdot \mathbf{n} dS = \alpha_i & \text{with } \sum_{i=1}^N \alpha_i = 0, \\ \sup_{t>0} t^{-1} Q_i(t) < \infty & \text{for } 1 \leq i \leq N, \end{cases} \quad (2.4)$$

where $Q_i(t)$ is defined in (2.3) for $\mu_0 = 0$, and $Q_i(t)$ is defined as $Q_i(t) \equiv \int_{\Omega_i(t)} (|\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}|^p) dx$ for $\mu_0 > 0$.

Ladyzhenskaya – Solonnikov Problem II. Suppose that $I_i(\infty) = \infty$ for $1 \leq i \leq m$, while $I_i(\infty) < \infty$ for $m+1 \leq i \leq N$, we look for a pair vector field (\mathbf{v}, π) such that

$$\begin{cases} -\operatorname{div}(\mu_0 \mathcal{D}(\mathbf{v}) + \mu_1 |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v})) + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \pi = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial\Omega, \\ \int_{\Sigma_i(x_1^i)} \mathbf{v} \cdot \mathbf{n} dS = \alpha_i & \text{with } \sum_{i=1}^N \alpha_i = 0, \\ \sup_{t>0} I_i^{-1}(t) Q_i(t) < \infty & \text{for } 1 \leq i \leq N, \end{cases} \quad (2.5)$$

where $I_i(t)$ is defined in (2.2) for $\mu_0 = 0$, and $I_i(t)$ is defined as $I_i(t) \equiv \int_0^t (|\Sigma_i(s)|^{-\frac{d+1}{d-1}} + |\Sigma_i(s)|^{1-\frac{dp}{d-1}}) ds$ for $\mu_0 > 0$.

In subsequent sections, we shall focus on the studies on these two problems above. The obtained results will be stated in Theorem 6.1-Theorem 6.2 and Theorem 7.3-Theorem 7.4 respectively. In addition, for notational convenience, we introduce some function spaces as follows:

$$\begin{aligned} D^{1,p}(\Omega) &= \{\mathbf{u} \in L^1_{loc}(\Omega) : \nabla \mathbf{u} \in L^p(\Omega)\}, \\ D_0^{1,p}(\Omega) &= \{ \text{completion of } C_0^\infty(\Omega) \text{ in the semi-norm } \|\nabla \mathbf{u}\|_{p,\Omega} \equiv (\int_\Omega |\nabla \mathbf{u}|^p dx)^{\frac{1}{p}} \}, \\ D^{-1,p'}(\Omega) &= (D^{1,p}(\Omega))', \quad D_0^{-1,p'}(\Omega) = (D_0^{1,p}(\Omega))', \\ \mathcal{D}(\Omega) &= \{\mathbf{u} \in C_0^\infty(\Omega) : \nabla \cdot \mathbf{u} = 0\}, \\ \mathcal{D}_0^{1,p}(\Omega) &= \{\text{completion of } \mathcal{D}(\Omega) \text{ in the semi-norm } \|\nabla \mathbf{u}\|_{p,\Omega}\}. \end{aligned}$$

3 Preliminary results

In this part, some preliminary results will be listed so that we can apply them to study the described problems in §2. It follows from the proof of Appendix of [18] that we have

Lemma 3.1. Let $\omega = \{x = (x_1, y) \in \mathbb{R}^d : t_1 < x_1 < t_2, y \in \Sigma(x_1)\}$ and $\mathbf{u}|_{\{x:t_1 < x_1 < t_2, y \in \partial\Sigma(x_1)\}} = 0$. Then

$$\|\mathbf{u}\|_{r,\omega} \leq c(d, r, q) \max\{1, (t_2 - t_1)^{-\frac{1}{d}} \max_{x_1 \in [t_1, t_2]} |\Sigma(x_1)|^{\frac{1}{d(d-1)}}\} |\omega|^{\frac{1}{d} + \frac{1}{r} - \frac{1}{q}} \|\nabla \mathbf{u}\|_{q,\omega},$$

where $1 \leq r \leq \frac{dq}{d-q}$, if $q < d$; $1 \leq r < \infty$, if $q = d$; $1 \leq r \leq \infty$, if $q > d$.

The following result will play a crucial role in estimating pressure π in problems (2.4)-(2.5).

Lemma 3.2. (Theorem III.3.3 of [10]) *For a bounded Lipschitzian domain $\omega \subset \mathbb{R}^d$, suppose that*

$$f \in L^p(\omega) \text{ and } \int_{\omega} f dx = 0, \quad (3.1)$$

where $1 < p < \infty$. Then one can find a vector field \mathbf{w} such that

$$\begin{cases} \nabla \cdot \mathbf{w} = 0, \\ \mathbf{w} \in W_0^{1,p}(\omega), \\ \|\mathbf{w}\|_{1,p} \leq M(\omega) \|f\|_p, \end{cases} \quad (3.2)$$

where $M(\omega) > 0$ is a constant depending only on the Lebesgue's measure $|\omega|$ of domain ω .

Remark 3.1. Assume that ω is a star-shaped domain with respect to a ball B with the radius R_0 . Then it follows from Theorem III.3.1 of [10] that the positive constant $M(\omega)$ in (3.2) satisfies $M(\omega) \leq c(d, p) \left(\frac{\text{diam}(\omega)}{R_0} \right)^d (1 + \frac{\text{diam}(\omega)}{R_0})$. This property will be useful in order to solve problem (2.5).

Remark 3.2. If $f \in L^p(\omega) \cap L^r(\omega)$ with $1 < p, r < \infty$, then one can find a vector field $\mathbf{w} \in W^{1,p}(\omega) \cap W^{1,r}(\omega)$ (see Remark III.3.12 of [10]) such that

$$\|\mathbf{w}\|_{1,p} \leq M(\omega) \|f\|_p \quad \text{and} \quad \|\mathbf{w}\|_{1,r} \leq M(\omega) \|f\|_r.$$

Next, we list some results, whose proofs can be found in Lemma 2.3 of [18] or Lemma 3.1 of [12].

Lemma 3.3. Let δ be a fixed constant with $\delta \in (0, 1)$ and $t_0 < T$. In addition, we suppose that $\Psi(\tau)$ is a monotonically increasing function, equal to zero for $\tau = 0$ and equal to infinity for $\tau = \infty$.

(i) Assume that the nondecreasing, nonnegative smooth functions $z(t)$ and $\varphi(t)$, not identically equal to zero, satisfy the following inequalities for all $t \in [t_0, T]$,

$$z(t) \leq \Psi(z'(t)) + (1 - \delta)\varphi(t), \quad (3.3)$$

and

$$\varphi(t) \geq \delta^{-1}\Psi(\varphi'(t)). \quad (3.4)$$

If

$$z(T) \leq \varphi(T), \quad (3.5)$$

then for all $t \in [t_0, T]$,

$$z(t) \leq \varphi(t). \quad (3.6)$$

(ii) Assume that inequalities (3.3) and (3.4) are fulfilled for all $t \geq t_0$. Then (3.6) holds for $t \geq t_0$ if

$$\liminf_{t \rightarrow \infty} \frac{z(t)}{\varphi(t)} < 1, \quad (3.7)$$

or if $z(t)$ has an order of growth for $t \rightarrow \infty$, less than the order of growth of the positive solutions to the equation

$$\tilde{z}(t) = \delta^{-1}\Psi(\tilde{z}'(t)). \quad (3.8)$$

(iii) Assume that the nonidentical zero nonnegative functions $z(t)$, satisfying the homogenous inequality

$$z(t) \leq \delta^{-1}\Psi(z'(t)) \text{ for } t \geq t_0, \quad (3.9)$$

increases unboundedly for $t \rightarrow \infty$. If $\delta^{-1}\Psi(\tau) \leq c_0\tau^m$ holds for $m > 1$ and $\tau \geq \tau_1$, then

$$\liminf_{t \rightarrow \infty} t^{-\frac{m}{m-1}} z(t) > 0; \quad (3.10)$$

if, however, $\delta^{-1}\Psi(\tau) \leq c_0\tau$ holds for $\tau \geq \tau_1$, then

$$\liminf_{t \rightarrow \infty} z(t) \exp(-\frac{t}{c_0}) > 0. \quad (3.11)$$

The following Korn-type inequality can be referred in Theorem 3.2 of [13] or Theorem 1 of [9].

Lemma 3.4. Let K be a cone in \mathbb{R}^d and $p > 1$. If $\int_K |\mathcal{D}(\mathbf{u})|^p dx < +\infty$, then there is a skew-symmetric matrix A with constant coefficients such that

$$\int_K |\nabla(\mathbf{u}(x) - Ax)|^p dx \leq C \int_K |\mathcal{D}(\mathbf{u})(x)|^p dx, \quad (3.12)$$

where the positive constant C does not depend on the function \mathbf{u} itself.

Remark 3.3. If $\int_K |\nabla \mathbf{u}|^p dx < +\infty$, then $A = 0$ holds in (3.12).

Finally, we state a conclusion as follows, whose proof can be found in [10] Lemma III.4.3, and [23] Lemma 2-Lemma 3.

Lemma 3.5. Assume that the domain Ω and the numbers α_i ($1 \leq i \leq N$) are defined in problem (2.4) or problem (2.5). Let $\alpha = \max_{1 \leq i \leq N} |\alpha_i|$. Then for any fixed $\varepsilon > 0$, there exists a smooth divergence-free vector field $\mathbf{a}(x, \varepsilon)$ which vanish in a neighborhood of $\partial\Omega \cap \partial\Omega_i$ ($1 \leq i \leq N$), and which satisfies

- (i) $|\mathbf{a}| \leq c(\varepsilon)\alpha|\Sigma_i(t)|^{-1}$ and $|\nabla \mathbf{a}| \leq c(\varepsilon)\alpha|\Sigma_i(t)|^{-\frac{d}{d-1}}$ for $x \in \Omega_i(t)$ and $1 \leq i \leq N$.
- (ii) $\int_{\Sigma_i(t)} \mathbf{a} \cdot \mathbf{n} dS = \alpha_i$ for $1 \leq i \leq N$.
- (iii) $\int_{\Omega_0} \mathbf{a}^2 \mathbf{w}^2 dx \leq \varepsilon\alpha^2 \int_{\Omega_0} |\nabla \mathbf{w}|^2 dx$ for any $\mathbf{w} \in \mathcal{D}(\Omega)$.
- (iv) $\int_{\Omega_i(t_2) \setminus \Omega_i(t_1)} \mathbf{a}^2 \mathbf{w}^2 dx \leq \varepsilon\alpha^2 \int_{\Omega_i(t_2) \setminus \Omega_i(t_1)} |\nabla \mathbf{w}|^2 dx$ for any $\mathbf{w} \in \mathcal{D}(\Omega)$, $t_2 > t_1 > 0$, and $1 \leq i \leq N$.

4 Existence of solutions to problems (2.4) and (2.5) in bounded truncated domains

In this part, for the following problem in the bounded domain $\Omega(T) = \Omega_0 \cup \{x \in \Omega : 0 < x_1^1 < T, \dots, 0 < x_1^d < T\}$

$$\begin{cases} -\operatorname{div}(\mu_0 \mathcal{D}(\mathbf{v}^T) + \mu_1 |\mathcal{D}(\mathbf{v}^T)|^{p-2} \mathcal{D}(\mathbf{v}^T)) + \mathbf{v}^T \cdot \nabla \mathbf{v}^T + \nabla \pi^T = 0 & \text{in } \Omega(T), \\ \operatorname{div} \mathbf{v}^T = 0 & \text{in } \Omega(T), \\ \mathbf{v}^T = 0 & \text{on } \partial\Omega(T), \end{cases} \quad (4.1)$$

we intend to find a weak solution $(\mathbf{v}^T, \pi^T) = (\mathbf{u}^T + \mathbf{a}, \pi^T)$ such that

$$\begin{aligned} & \mu_0(\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a}), \mathcal{D}(\psi)) + \mu_1(|\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a})|^{p-2}(\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\psi)) \\ &= (\mathbf{u}^T \cdot \nabla \psi, \mathbf{u}^T) + (\mathbf{u}^T \cdot \nabla \psi, \mathbf{a}) + (\mathbf{a} \cdot \nabla \psi, \mathbf{u}^T) + (\mathbf{a} \cdot \nabla \psi, \mathbf{a}), \forall \psi \in \mathcal{D}(\Omega(T)), \end{aligned} \quad (4.2)$$

where the vector value function \mathbf{a} is given in Lemma 3.5.

Theorem 4.1. *Let $\mu_0 > 0$ and $p > 1$ or $\mu_0 = 0$ and $p > 2$. Then there is a vector field \mathbf{u}^T such that (4.2) holds, and $\mathbf{u}^T \in \mathcal{D}_0^{1,2}(\Omega(T)) \cap \mathcal{D}_0^{1,p}(\Omega(T))$, if $\mu_0 > 0$; $\mathbf{u}^T \in \mathcal{D}_0^{1,p}(\Omega(T))$, if $\mu_0 = 0$.*

Proof. Although the proof of Theorem 4.1 is standard as in [8]- [9] and [12], where the authors treated problem (4.2) for different vector value function \mathbf{a} , we still give out the detailed proof for the sake of completeness.

Case I. $\mu_0 > 0, p > 1$

Let $\{\psi_k^T\}$ be a basis in $\mathcal{D}_0^{1,2}(\Omega(T))$. We look for a series $\{c_{km}^T\}$ such that $\mathbf{u}_m^T = \sum_{i=1}^m c_{km}^T \psi_k^T$ satisfies

$$\begin{aligned} & \mu_0(\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a}), \mathcal{D}(\psi_k^T)) + \mu_1(|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2}(\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\psi_k^T)) \\ &= (\mathbf{u}_m^T \cdot \nabla \psi_k^T, \mathbf{u}_m^T) + (\mathbf{u}_m^T \cdot \nabla \psi_k^T, \mathbf{a}) + (\mathbf{a} \cdot \nabla \psi_k^T, \mathbf{u}_m^T) + (\mathbf{a} \cdot \nabla \psi_k^T, \mathbf{a}) \quad \text{for } k = 1, \dots, m. \end{aligned} \quad (4.3)$$

Multiplying both sides of (4.3) by c_{km}^T and summing over k yield

$$\begin{aligned} & \mu_0 \|\mathcal{D}\mathbf{u}_m^T\|_2^2 + \mu_0(\mathcal{D}(\mathbf{a}), \mathcal{D}(\mathbf{u}_m^T)) + \mu_1(|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2}(\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})) \\ & - \mu_1(|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2}(\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\mathbf{a})) \\ &= (\mathbf{u}_m^T \cdot \nabla \mathbf{u}_m^T, \mathbf{a}) + (\mathbf{a} \cdot \nabla \mathbf{u}_m^T, \mathbf{a}). \end{aligned} \quad (4.4)$$

Using Schwarz inequality we get

$$\mu_0(\mathcal{D}(\mathbf{a}), \mathcal{D}(\mathbf{u}_m^T)) \geq -\frac{\mu_0}{2} \|\mathcal{D}(\mathbf{u}_m^T)\|_2^2 - \frac{\mu_0}{2} \|\mathcal{D}(\mathbf{a})\|_2^2. \quad (4.5)$$

In addition,

$$\|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})\|_p^p \geq \frac{1}{2^{p-1}} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p - \|\mathcal{D}(\mathbf{a})\|_p^p. \quad (4.6)$$

We also notice that, by Hölder inequality and Young inequality,

$$\begin{aligned} & \left| \mu_1(|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2}(\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\mathbf{a})) \right| \\ & \leq \frac{\mu_1}{2^p} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p + c \|\mathcal{D}(\mathbf{a})\|_p^p, \end{aligned} \quad (4.7)$$

where and below $c > 0$ denotes by a generic positive constant. By Lemma 3.5 (iii) and (iv) we have that for any fixed $\varepsilon > 0$,

$$\left| \int_{\Omega(T)} \mathbf{u}_m^T \cdot \nabla \mathbf{u}_m^T \cdot \mathbf{a} dx \right| \leq c \int_{\Omega(T)} \mathbf{a}^2 |\mathbf{u}_m^T|^2 dx + \frac{\varepsilon}{2} \int_{\Omega(T)} |\nabla \mathbf{u}_m^T|^2 dx \leq \varepsilon \int_{\Omega(T)} |\nabla \mathbf{u}_m^T|^2 dx. \quad (4.8)$$

On the other hand, by Hölder inequality and Korn inequality,

$$|(\mathbf{a} \cdot \nabla \mathbf{u}_m^T, \mathbf{a})| \leq c \|\mathcal{D}(\mathbf{u}_m^T)\|_2 \|\mathbf{a}\|_4^2 \leq \frac{\mu_0}{4} \|\mathcal{D}(\mathbf{u}_m^T)\|_2^2 + c \|\mathbf{a}\|_4^4. \quad (4.9)$$

Set $\mathbf{c}_m^T = (c_{1m}^T, \dots, c_{mm}^T) \in \mathbb{R}^m$. To obtain a solution of (4.3), we define a function $P : \mathbb{R}^m \rightarrow \mathbb{R}^m$, whose components are

$$\begin{aligned} P_k(\mathbf{c}_m^T) &= \mu_0 (\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a}), \mathcal{D}(\psi_k^T)) + \mu_1 \left(|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\psi_k^T) \right) \\ &\quad - (\mathbf{u}_m^T \cdot \nabla \psi_k^T, \mathbf{u}_m^T) + (\mathbf{u}_m^T \cdot \nabla \psi_k^T, \mathbf{a}) + (\mathbf{a} \cdot \nabla \psi_k^T, \mathbf{u}_m^T) + (\mathbf{a} \cdot \nabla \psi_k^T, \mathbf{a}), \quad k = 1, \dots, m. \end{aligned}$$

It follows from (4.5)-(4.9) and Korn inequality that

$$\begin{aligned} P(\mathbf{c}_m^T) \cdot \mathbf{c}_m^T &\geq \frac{\mu_0}{2} \|\mathcal{D}(\mathbf{u}_m^T)\|_2^2 - \frac{\mu_0}{2} \|\mathcal{D}(\mathbf{a})\|_2^2 + \frac{\mu_1}{2^{p-1}} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p - \mu_1 \|\mathcal{D}(\mathbf{a})\|_p^p \\ &\quad - \frac{\mu_1}{2^p} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p - c \|\mathcal{D}(\mathbf{a})\|_p^p - \varepsilon \|\nabla \mathbf{u}_m^T\|_2^2 - \frac{\mu_0}{4} \|\mathcal{D}(\mathbf{u}_m^T)\|_2^2 - c \|\mathbf{a}\|_4^4 \\ &\geq \frac{\mu_0}{8} \|\mathcal{D}(\mathbf{u}_m^T)\|_2^2 + \frac{\mu_1}{2^p} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p - c \|\mathcal{D}(\mathbf{a})\|_2^2 - c \|\mathcal{D}(\mathbf{a})\|_p^p - c \|\mathbf{a}\|_4^4 > 0 \end{aligned}$$

provided that $\|\mathcal{D}(\mathbf{u}_m^T)\|_2$ is large enough. This, together with Lemma I.4.3 of [20], yields that there exists $\bar{\mathbf{c}}_m^T \in \mathbb{R}^m$ such that $P(\bar{\mathbf{c}}_m^T) = 0$. Hence we find a solution of Equation (4.3) for any fixed $m \in \mathbb{N}$. Moreover, by (4.4)-(4.9) we get

$$\begin{aligned} &\frac{\mu_0}{2} \|\mathcal{D}(\mathbf{u}_m^T)\|_2^2 + \frac{\mu_1}{2^{p-1}} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p \\ &\leq \frac{\mu_0}{2} \|\mathcal{D}(\mathbf{a})\|_2^2 + \mu_1 \|\mathcal{D}(\mathbf{a})\|_p^p + \frac{\mu_1}{2^p} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p + c \|\mathcal{D}(\mathbf{a})\|_p^p + \varepsilon \|\nabla \mathbf{u}_m^T\|_2^2 \\ &\quad + \frac{\mu_0}{4} \|\mathcal{D}(\mathbf{u}_m^T)\|_2^2 + c \|\mathbf{a}\|_4^4, \end{aligned}$$

and by Korn inequality,

$$\|\nabla \mathbf{u}_m^T\|_2^2 + \|\nabla \mathbf{u}_m^T\|_p^p \leq c(\|\nabla \mathbf{a}\|_2^2 + \|\nabla \mathbf{a}\|_p^p + \|\mathbf{a}\|_4^4). \quad (4.10)$$

From (4.10), we obtain that there is a vector field \mathbf{u}^T and a subsequence of $\{\mathbf{u}_m^T\}$, which is still denoted by $\{\mathbf{u}_m^T\}$, such that

$$\begin{aligned} \mathbf{u}_m^T &\rightharpoonup \mathbf{u}^T \text{ in } \mathcal{D}_0^{1,2}(\Omega(T)), \\ \mathbf{u}_m^T &\rightharpoonup \mathbf{u}^T \text{ in } \mathcal{D}_0^{1,p}(\Omega(T)) \end{aligned} \quad (4.11)$$

and

$$\mathbf{u}_m^T \rightarrow \mathbf{u}^T \text{ in } L^2(\Omega(T)). \quad (4.12)$$

Meanwhile, by (4.10)

$$\begin{aligned} &\| |\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})) \|_{p'}^{p'} \\ &= \|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})\|_p^p \\ &\leq 2^{p-1} (\|\mathcal{D}(\mathbf{u}_m^T)\|_p^p + \|\mathcal{D}(\mathbf{a})\|_p^p) \\ &\leq c(\|\nabla \mathbf{a}\|_2^2 + \|\nabla \mathbf{a}\|_p^p + \|\mathbf{a}\|_4^4), \end{aligned}$$

which means that one can find a vector function $\mathbf{G}^T \in L^{p'}(\Omega(T))$ such that

$$\mu_1 |\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})) \rightharpoonup \mathbf{G}^T \text{ in } L^{p'}(\Omega(T)). \quad (4.13)$$

Thus, by (4.11)-(4.13) we arrive at

$$\begin{aligned} \mu_0 (\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a}), \mathcal{D}(\psi_k^T)) + (\mathbf{G}^T, \mathcal{D}(\psi_k^T)) \\ = (\mathbf{u}^T \cdot \nabla \psi_k^T, \mathbf{u}^T) + (\mathbf{u}^T \cdot \nabla \psi_k^T, \mathbf{a}) + (\mathbf{a} \cdot \nabla \psi_k^T, \mathbf{u}^T) + (\mathbf{a} \cdot \nabla \psi_k^T, \mathbf{a}). \end{aligned} \quad (4.14)$$

To prove \mathbf{u}^T is a weak solution of (4.2), one should establish that for any $\varphi \in \mathcal{D}(\Omega(T))$,

$$(\mathbf{G}^T, \mathcal{D}(\varphi)) = \left(\mu_1 |\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\varphi) \right). \quad (4.15)$$

In fact, multiplying both sides of (4.14) with c_{km}^T and summing over k yield

$$\begin{aligned} \mu_0 (\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a}), \mathcal{D}(\mathbf{u}_m^T)) + (\mathbf{G}^T, \mathcal{D}(\mathbf{u}_m^T)) \\ = (\mathbf{u}^T \cdot \nabla \mathbf{u}_m^T, \mathbf{u}^T) + (\mathbf{u}^T \cdot \nabla \mathbf{u}_m^T, \mathbf{a}) + (\mathbf{a} \cdot \nabla \mathbf{u}_m^T, \mathbf{u}^T) + (\mathbf{a} \cdot \nabla \mathbf{u}_m^T, \mathbf{a}). \end{aligned} \quad (4.16)$$

Subtracting (4.16) by (4.4) and then passing to limit as $m \rightarrow \infty$, we get

$$\lim_{m \rightarrow \infty} (\mu_0 \mathcal{D}(\mathbf{u}_m^T) + \mathcal{S}(\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\mathbf{u}_m^T)) = \mu_0 \|\mathcal{D}(\mathbf{u}^T)\|_2^2 + (G^T, \mathcal{D}(\mathbf{u}^T)), \quad (4.17)$$

where and below $\mathcal{S}(D) \equiv \mu_1 |D|^{p-2} D$ for the tensor D . Since for any pair of tensors D and C , we have the monotonicity property

$$(\mathcal{S}(D) - \mathcal{S}(C)) \cdot (D - C) \geq 0.$$

This yields that for any $\Phi \in \mathcal{D}_0^{1,p}(\Omega(T)) \cap \mathcal{D}_0^{1,2}(\Omega(T))$,

$$(\mu_0 (\mathcal{D}(\mathbf{u}_m^T) - \mathcal{D}(\Phi)) + \mathcal{S}(\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})) - \mathcal{S}(\mathcal{D}(\Phi) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\mathbf{u}_m^T) - \mathcal{D}(\Phi)) \geq 0. \quad (4.18)$$

Together with (4.17), we have that for $m \rightarrow \infty$,

$$(\mu_0 (\mathcal{D}(\mathbf{u}^T) - \mathcal{D}(\Phi)) + G^T - \mathcal{S}(\mathcal{D}(\Phi) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\mathbf{u}^T) - \mathcal{D}(\Phi)) \geq 0.$$

Choosing $\Phi = \mathbf{u}^T - \varepsilon \varphi$ with $\varepsilon > 0$ and $\varphi \in \mathcal{D}(\Omega(T))$. Then

$$(\varepsilon \mu_0 \mathcal{D}(\varphi) + G^T - \mu_1 |\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a}) - \varepsilon \mathcal{D}(\varphi)|^{p-2} (\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a}) - \varepsilon \mathcal{D}(\varphi)), \mathcal{D}(\varphi)) \geq 0. \quad (4.19)$$

Let $\varepsilon \rightarrow 0$, we arrive at

$$(\mathbf{G}^T - \mu_1 |\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\varphi)) \geq 0. \quad (4.20)$$

If φ is replaced by $-\varphi$ in (4.20), then

$$(\mathbf{G}^T - \mu_1 |\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(-\varphi)) \leq 0. \quad (4.21)$$

Combining (4.18) with (4.19) yields

$$(\mathbf{G}^T, \mathcal{D}(\varphi)) = (\mu_1 |\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\varphi)). \quad (4.22)$$

Thus, \mathbf{u}^T is a weak solution of (4.2).

Case II. $\mu_0 = 0, p > 2$

As in Case I, let $\{\psi_k^T\}$ be a basis in $\mathcal{D}_0^{1,p}(\Omega(T))$ and set $\mathbf{u}_m^T = \sum_{i=1}^m c_{km}^T \psi_k^T$. Then for $k = 1, \dots, m$,

$$\begin{aligned} \mu_1 \left(|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\psi_k^T) \right) \\ = (\mathbf{u}_m^T \cdot \nabla \psi_k^T, \mathbf{u}_m^T) + (\mathbf{u}_m^T \cdot \nabla \psi_k^T, \mathbf{a}) + (\mathbf{a} \cdot \nabla \psi_k^T, \mathbf{u}_m^T) + (\mathbf{a} \cdot \nabla \psi_k^T, \mathbf{a}). \end{aligned} \quad (4.23)$$

Multiplying both sides of (4.23) by c_{km}^T and summing over k yield

$$\begin{aligned} \mu_1 \left(|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a}) \right) \\ - \mu_1 \left(|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\mathbf{a}) \right) = (\mathbf{u}_m^T \cdot \nabla \mathbf{u}_m^T, \mathbf{a}) + (\mathbf{a} \cdot \nabla \mathbf{u}_m^T, \mathbf{a}). \end{aligned} \quad (4.24)$$

Note that

$$\|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})\|_p^p \geq \frac{1}{2^{p-1}} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p - \|\mathcal{D}(\mathbf{a})\|_p^p \quad (4.25)$$

and

$$\begin{aligned} & \left| \mu_1 \left(|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\mathbf{a}) \right) \right| \\ & \leq \frac{\mu_1}{2^{p+1}} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p + c \|\mathcal{D}(\mathbf{a})\|_p^p. \end{aligned} \quad (4.26)$$

In addition, it follows from Lemma 3.5 (i), Poincaré inequality and Young's inequality that

$$\begin{aligned} \left| \int_{\Omega(T)} \mathbf{u}_m^T \cdot \nabla \mathbf{u}_m^T \cdot \mathbf{a} dx \right| & \leq \varepsilon \int_{\Omega(T)} |\mathbf{a}|^{\frac{p}{d-1}} |\mathbf{u}_m^T|^p + \varepsilon \int_{\Omega(T)} |\nabla \mathbf{u}_m^T|^p dx + c \int_{\Omega(T)} |\mathbf{a}|^{\frac{p(d-2)}{(p-2)(d-1)}} dx \\ & \leq c\varepsilon \|\nabla \mathbf{u}_m^T\|_p^p + c \|\mathbf{a}\|_{\frac{p(d-2)}{(p-2)(d-1)}}^{\frac{p(d-2)}{(p-2)(d-1)}}. \end{aligned} \quad (4.27)$$

As in (4.9), we have

$$|(\mathbf{a} \cdot \nabla \mathbf{u}_m^T, \mathbf{a})| \leq \|\mathcal{D}(\mathbf{u}_m^T)\|_p \|\mathbf{a}\|_{2p'}^{2p'} \leq \frac{\mu_1}{2^p} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p + c \|\mathbf{a}\|_{2p'}^{2p'}. \quad (4.28)$$

Similarly to Case I, set $\mathbf{c}_m^T = (c_{1,m}^T, \dots, c_{m,m}^T) \in \mathbb{R}^m$ and define a function $P : \mathbb{R}^m \rightarrow \mathbb{R}^m$ as follows

$$\begin{aligned} P_k(\mathbf{c}_m^T) & = \mu_1 \left(|\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\psi_k^T) \right) \\ & - (\mathbf{u}_m^T \cdot \nabla \psi_k^T, \mathbf{u}_m^T) + (\mathbf{u}_m^T \cdot \nabla \psi_k^T, \mathbf{a}) + (\mathbf{a} \cdot \nabla \psi_k^T, \mathbf{u}_m^T) + (\mathbf{a} \cdot \nabla \psi_k^T, \mathbf{a}) \quad \text{for } k = 1, \dots, m. \end{aligned}$$

Then, by (4.25)-(4.28) and Korn inequality, we arrive at

$$\begin{aligned} P(\mathbf{c}_m^T) \cdot \mathbf{c}_m^T & \geq \frac{\mu_1}{2^{p-1}} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p - \mu_1 \|\mathcal{D}(\mathbf{a})\|_p^p - \frac{\mu_1}{2^{p+1}} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p \\ & - c \|\mathcal{D}(\mathbf{a})\|_p^p - c\varepsilon \|\nabla \mathbf{u}_m^T\|_p^p - c \|\mathbf{a}\|_{\frac{p(d-2)}{(p-2)(d-1)}}^{\frac{p(d-2)}{(p-2)(d-1)}} - \frac{\mu_1}{2^p} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p - c \|\mathbf{a}\|_{2p'}^{2p'} \\ & \geq \frac{\mu_1}{2^{p+1}} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p - c \|\mathcal{D}(\mathbf{a})\|_p^p - c \|\mathbf{a}\|_{2p'}^{2p'} - c \|\mathbf{a}\|_{\frac{p(d-2)}{(p-2)(d-1)}}^{\frac{p(d-2)}{(p-2)(d-1)}} > 0 \end{aligned}$$

for sufficiently large $\|\mathcal{D}(\mathbf{u}_m^T)\|_p$. From this, we then obtain the existence of solution to Equation (4.3) for any fixed $m \in \mathbb{N}$. Moreover, by (4.24)-(4.28) we get

$$\begin{aligned} & \frac{\mu_1}{2^{p-1}} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p \\ & \leq \frac{\mu_1}{2^{p+1}} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p + c \|\mathcal{D}(\mathbf{a})\|_p^p + c\varepsilon \|\nabla \mathbf{u}_m^T\|_p^p \\ & \quad + c \|\mathbf{a}\|_{\frac{p(d-2)}{(p-2)(d-1)}}^{\frac{p(d-2)}{(p-2)(d-1)}} + \frac{\mu_1}{2^p} \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p + c \|\mathbf{a}\|_{2p'}^{2p'} \end{aligned}$$

and

$$\|\nabla \mathbf{u}_m^T\|_p^p \leq c(\|\nabla \mathbf{a}\|_p^p + \|\mathbf{a}\|_{\frac{p(d-2)}{(p-2)(d-1)}}^{\frac{p(d-2)}{(p-2)(d-1)}} + \|\mathbf{a}\|_{2p'}^{2p'}). \quad (4.29)$$

Based on (4.29), we know that there is a vector field \mathbf{u}^T and a subsequence of $\{\mathbf{u}_m^T\}$, which we still denote by $\{\mathbf{u}_m^T\}$, such that

$$\mathbf{u}_m^T \rightharpoonup \mathbf{u}^T \text{ in } \mathcal{D}_0^{1,p}(\Omega(T)) \quad (4.30)$$

and

$$\mathbf{u}_m^T \rightarrow \mathbf{u}^T \text{ in } L^p(\Omega(T)). \quad (4.31)$$

Meanwhile, by (4.29),

$$\begin{aligned} & \| |\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})) \|_{p'}^{p'} \\ & = \| \mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a}) \|_p^p \leq 2^p (\|\mathcal{D}(\mathbf{u}_m^T)\|_p^p + \|\mathcal{D}(\mathbf{u}_m^T)\|_p^p) \\ & \leq c (\|\nabla \mathbf{a}\|_p^p + \|\mathbf{a}\|_{\frac{p(d-2)}{(p-2)(d-1)}}^{\frac{p(d-2)}{(p-2)(d-1)}} + \|\mathbf{a}\|_{2p'}^{2p'}), \end{aligned}$$

which means that one can find $\mathbf{G}^T \in L^{p'}(\Omega(T))$ such that

$$\mu_1 |\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}_m^T) + \mathcal{D}(\mathbf{a})) \rightharpoonup \mathbf{G}^T \text{ in } L^{p'}(\Omega(T)). \quad (4.32)$$

Thus, by (4.30)-(4.32) we can obtain

$$(\mathbf{G}^T, \mathcal{D}(\psi_k^T)) = (\mathbf{u}^T \cdot \nabla \psi_k^T, \mathbf{u}^T) + (\mathbf{u}^T \cdot \nabla \psi_k^T, \mathbf{a}) + (\mathbf{a} \cdot \nabla \psi_k^T, \mathbf{u}^T) + (\mathbf{a} \cdot \nabla \psi_k^T, \mathbf{a}). \quad (4.33)$$

Completely analogous to the proof in Case I, one can prove that for any $\varphi \in \mathcal{D}(\Omega(T))$,

$$(\mathbf{G}^T, \mathcal{D}(\varphi)) = (\mu_1 |\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}^T) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\varphi)). \quad (4.34)$$

Namely, \mathbf{u}^T is a weak solution of (4.2). □

5 Interior regularity of weak solutions

In this part, we will establish the uniform interior estimates of weak solution \mathbf{u} to the steady non-Newtonian fluid equations in (2.4) or (2.5).

Theorem 5.1. *Let Ω be any domain in \mathbb{R}^d , and $\Omega' \subset\subset \Omega$. Suppose that $\mathbf{v} = \mathbf{u} + \mathbf{a}$ is a weak solution to steady non-Newtonian fluid equations in Ω , which satisfies for any $\psi \in \mathcal{D}(\Omega)$,*

$$\begin{aligned} \mu_0(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}), \mathcal{D}(\psi)) + \mu_1(|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2}(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\psi)) \\ = (\mathbf{u} \cdot \nabla \psi, \mathbf{u}) + (\mathbf{u} \cdot \nabla \psi, \mathbf{a}) + (\mathbf{a} \cdot \nabla \psi, \mathbf{u}) + (\mathbf{a} \cdot \nabla \psi, \mathbf{a}), \end{aligned} \quad (5.1)$$

where the vector value function \mathbf{a} is given in Lemma 3.5. Let $\Omega'_{4r} \equiv \{x : d(x, \Omega') < 4r\} \subset\subset \Omega$ for any fixed number $r > 0$. Then we have that:

If $p > 2$ and $\mu_0 \geq 0$, then $\nabla \mathbf{u} \in W^{\kappa, p}(\Omega')$ and

$$\|\nabla \mathbf{u}\|_{\kappa, p, \Omega'} \leq c(\kappa, r, |\Omega'_{4r}|, \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}),$$

where $\kappa \in [0, \frac{2\hat{\theta}}{p}]$ with $\hat{\theta} = \min\{p', \theta\}$, and $\theta = \frac{p(d+2)-3d}{2p}$, if $p < d$; θ is an arbitrary constant less than 1, if $d \leq p < 3$ (only for $d = 2$); $\theta = 1$, if $p \geq 3$.

If $1 < p < 2$ and $\mu_0 > 0$, then $\nabla \mathbf{u} \in W^{\frac{2\theta}{p}-\varepsilon, p}(\Omega') \cap W^{\theta-\varepsilon, 2}(\Omega')$ for any $\varepsilon > 0$, and

$$\|\nabla \mathbf{u}\|_{\theta-\varepsilon, p, \Omega'} + \|\nabla \mathbf{u}\|_{\frac{2\theta}{p}-\varepsilon, p, \Omega'} \leq c(\varepsilon, r, |\Omega'_{4r}|, \|\nabla \mathbf{u}\|_{2, \Omega'_{4r}}),$$

where $\theta = \frac{1}{4}$, if $d = 3$; θ is an arbitrary constant less than 1, if $d = 2$.

Proof. By $\Omega' \subset\subset \Omega$, then $\text{dist}(\partial\Omega', \partial\Omega) > 0$. For any ρ with $0 < \rho < \text{dist}(\partial\Omega', \partial\Omega)$, set

$$\Omega'_\rho = \{x \in \Omega : \text{dist}(x, \Omega') < \rho\}. \quad (5.2)$$

Let $0 < r < \frac{1}{4}\text{dist}(\partial\Omega', \partial\Omega)$ and choose a cutoff function η such that $\eta = 1$ in Ω'_r , $\eta = 0$ in $\mathbb{R}^3 \setminus \Omega'_{2r}$, $0 \leq \eta \leq 1$ and $|\nabla \eta| < \frac{1}{r}$, $|\nabla^2 \eta| < \frac{1}{r^2}$ in Ω'_{2r} . Next we study the following functional $F: D_0^{1,p}(\Omega'_{4r}) \rightarrow \mathbb{R}$, where

$$\begin{aligned} F(\psi) = & \mu_0(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}), \mathcal{D}(\psi))_{\Omega'_{4r}} + \mu_1(|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2}(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\psi))_{\Omega'_{4r}} \\ & - (\mathbf{u} \cdot \nabla \psi, \mathbf{u})_{\Omega'_{4r}} - (\mathbf{u} \cdot \nabla \psi, \mathbf{a})_{\Omega'_{4r}} - (\mathbf{a} \cdot \nabla \psi, \mathbf{u})_{\Omega'_{4r}} - (\mathbf{a} \cdot \nabla \psi, \mathbf{a})_{\Omega'_{4r}}. \end{aligned} \quad (5.3)$$

Case I. $\mu_0 \geq 0, p > 2$

In this case, we just only treat the case of $\mu_0 = 0$ since the treatment for $\mu_0 > 0$ is easier. Since \mathbf{u} satisfies (5.1), it follows that $\ker F = D_0^{1,p}(\Omega'_{4r})$. Then according to De Rham Theorem, we know that there exists a function $\pi \in L^{p'}(\Omega'_{4r})$ such that for any $\psi \in D_0^{1,p}(\Omega'_{4r})$,

$$F(\psi) = (\pi, \nabla \cdot \psi). \quad (5.4)$$

Without loss of generality, $\int_{\Omega'_{4r}} \pi dx = 0$ can be assumed. By Lemma 3.2 we can find a vector filed $\widehat{\psi} \in D_0^{1,p}(\Omega'_{4r})$ such that $\nabla \cdot \widehat{\psi} = |\pi|^{p'-2}\pi - \frac{1}{|\Omega'_{4r}|} \int_{\Omega'_{4r}} |\pi|^{p'-2}\pi dx$ and $\|\nabla \widehat{\psi}\|_{p, \Omega'_{4r}} \leq c(|\Omega'_{4r}|) \|\pi\|_{p', \Omega'_{4r}}^{p'-1}$. Combining (5.3) with (5.4), in terms of $\int_{\Omega'_{4r}} \pi dx = 0$, we have that

$$\|\pi\|_{p', \Omega'_{4r}}^{p'} = F(\widehat{\psi}) \leq c(|\Omega'_{4r}|) \|\pi\|_{p', \Omega'_{4r}}^{p'-1} \|F\|_{D_0^{-1, p'}(\Omega'_{4r})}.$$

In addition,

$$\begin{aligned} \|F\|_{D_0^{-1, p'}(\Omega'_{4r})} & \leq \mu_1 \|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})\|_{p, \Omega'_{4r}}^{p-1} + \|\mathbf{u}\|_{2p', \Omega'_{4r}}^2 + 2\|\mathbf{a}\|_\infty \|\mathbf{u}\|_{p', \Omega'_{4r}} + \|\mathbf{a}\|_{2p', \Omega'_{4r}}^2 \\ & \leq c(|\Omega'_{4r}|) (1 + \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}^{p-1} + \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}^2 + \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}). \end{aligned}$$

Hence we have

$$\|\pi\|_{p', \Omega'_{4r}} \leq c(|\Omega'_{4r}|)(1 + \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}^{p-1} + \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}^2 + \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}). \quad (5.5)$$

Set $\Delta_{\lambda, k} \mathbf{u} = \mathbf{u}(x + \lambda \mathbf{e}_k) - \mathbf{u}(x)$. Let $\Delta_\lambda \mathbf{u}$ stand for $\Delta_{\lambda, k} \mathbf{u}$ for any k ($1 \leq k \leq d$). Choosing $\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})$ as a test function, we then get that from (5.4),

$$\begin{aligned} & \mu_1 \int_{\Omega'_{3r}} |\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) : \mathcal{D}(\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx \\ &= - \int_{\Omega'_{3r}} \mathbf{u} \cdot \nabla \mathbf{u} \cdot (\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx + \int_{\Omega'_{3r}} \pi \nabla \cdot (\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx \\ & \quad - \int_{\Omega'_{3r}} \mathbf{u} \cdot \nabla \mathbf{a} \cdot (\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx - \int_{\Omega'_{3r}} \mathbf{a} \cdot \nabla \mathbf{u} \cdot (\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx \\ & \quad - \int_{\Omega'_{3r}} \mathbf{a} \cdot \nabla \mathbf{a} \cdot (\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx. \end{aligned} \quad (5.6)$$

A direct computation yields

$$\begin{aligned} & \mu_1 \int_{\Omega'_{3r}} |\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) : \mathcal{D}(\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx \\ &= p \mu_1 \int_{\Omega'_{3r}} |\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) : \Delta_{-\lambda} \text{Sym}(\Delta_\lambda \mathbf{u} \otimes \eta^{p-1} \nabla \eta) dx \\ & \quad + \mu_1 \int_{\Omega'_{3r}} \Delta_\lambda(|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}))) : (\eta^p \Delta_\lambda \mathcal{D}(\mathbf{u})) dx \\ &= \mu_1 \int_{\Omega'_{3r}} |\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) : \Delta_{-\lambda} \text{Sym}(\Delta_\lambda \mathbf{u} \otimes \eta^{p-1} \nabla \eta) dx \\ & \quad + \mu_1 \int_{\Omega'_{3r}} \eta^p \Delta_\lambda(|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}))) : \Delta_\lambda(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) dx \\ & \quad - \mu_1 \int_{\Omega'_{3r}} \eta^p \Delta_\lambda(|\mathcal{D}(\mathbf{u})|^{p-2} \mathcal{D}(\mathbf{u})) : \Delta_\lambda(\mathcal{D}(\mathbf{a})) dx, \end{aligned} \quad (5.7)$$

where $\text{Sym}(D) \equiv D + D^t$ for a given second order tensor D . Since

$$(|D|^{p-2} D - |C|^{p-2} C) \cdot (D - C) \geq \delta(|D| + |C|)^{p-2} |D - C|^2 \quad (5.8)$$

holds for any pair of tensors D and C , where $\delta > 0$ is some suitable constant, we have

$$\begin{aligned} & \mu_1 \int_{\Omega'_{3r}} \eta^p \Delta_\lambda(|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}))) : \Delta_\lambda(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) dx \\ & \geq \frac{\delta}{2^{p-1}} \mu_1 \int_{\Omega'_{3r}} \eta^p |\Delta_\lambda(\mathcal{D}(\mathbf{u}))|^p dx - \delta \mu_1 \int_{\Omega'_{3r}} \eta^p |\Delta_\lambda(\mathcal{D}(\mathbf{a}))|^p dx. \end{aligned} \quad (5.9)$$

Collecting (5.6)-(5.9) yields

$$\begin{aligned}
& \frac{\delta}{2^{p-1}} \mu_1 \int_{\Omega'_{3r}} \eta^p |\Delta_\lambda(\mathcal{D}(\mathbf{u}))|^p dx - \delta \mu_1 \int_{\Omega'_{3r}} \eta^p |\Delta_\lambda(\mathcal{D}(\mathbf{a}))|^p dx \\
& \leq -p \mu_1 \int_{\Omega'_{3r}} |\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) : \Delta_{-\lambda} Sym(\Delta_\lambda \mathbf{u} \otimes \eta^{p-1} \nabla \eta) dx \\
& \quad + \mu_1 \int_{\Omega'_{3r}} \eta^p \Delta_\lambda(|\mathcal{D}(\mathbf{u})|^{p-2} \mathcal{D}(\mathbf{u})) : \Delta_\lambda(\mathcal{D}(\mathbf{a})) dx \\
& \quad + \int_{\Omega'_{3r}} \pi \nabla \cdot (\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx \\
& \quad - \int_{\Omega'_{3r}} \Phi \cdot (\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx \\
& \quad - \int_{\Omega'_{3r}} \mathbf{u} \cdot \nabla \mathbf{u} \cdot (\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx \\
& \equiv I_1 + \dots + I_5,
\end{aligned} \tag{5.10}$$

where $\Phi = -\mathbf{u} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{u} - \mathbf{a} \cdot \nabla \mathbf{a}$. We now focus on the treatments on I_i ($1 \leq i \leq 5$) in (5.10). At first, it is well known that $\|\Delta_{\lambda,k} u\|_{p,\Omega'} \leq |\lambda| \|\partial_k u\|_{p,\Omega}$ for $0 < |\lambda| < dist(\Omega', \partial\Omega)$ (see Chapter 7 of [11]). For the term I_1 , we have

$$|I_1| \leq p \mu_1 \|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})\|_{p,\Omega'_{3r}}^{p-1} \|\Delta_{-\lambda} Sym(\Delta_\lambda \mathbf{u} \otimes \eta^{p-1} \nabla \eta)\|_{p,\Omega'_{3r}}.$$

Note that

$$\begin{aligned}
& \|\Delta_{-\lambda} Sym(\Delta_\lambda \mathbf{u} \otimes \eta^{p-1} \nabla \eta)\|_{p,\Omega'_{3r}}^p \\
& = \int_{\Omega'_{3r}} |\Delta_{-\lambda}[\eta^{p-1}(\partial_j \eta \Delta_\lambda u_i + \partial_i \eta \Delta_\lambda u_j)]|^p dx \\
& \leq 2^{p-1} |\lambda|^p \left(\int_{\Omega'_{3r}} (p-1) |\partial_k \eta (\partial_j \eta \Delta_\lambda u_i + \partial_i \eta \Delta_\lambda u_j)|^p dx \right. \\
& \quad \left. + \int_{\Omega'_{3r}} |\eta|^p |\partial_{jk} \eta \Delta_\lambda u_i + \partial_j \eta \Delta_\lambda \partial_k u_i + \partial_{ik} \eta \Delta_\lambda u_j + \partial_i \eta \Delta_\lambda \partial_k u_j|^p dx \right) \\
& \leq c |\lambda|^p \left(\frac{|\lambda|^p}{r^2} \int_{\Omega'_{4r}} |\nabla \mathbf{u}|^p dx + \frac{1}{r} \int_{\Omega'_{3r}} |\eta \nabla(\Delta_\lambda \mathbf{u})|^p dx \right).
\end{aligned}$$

In addition, it follows from $\eta \nabla(\Delta_\lambda \mathbf{u}) = \nabla(\eta \Delta_\lambda \mathbf{u}) - (\nabla \eta) \cdot \Delta_\lambda \mathbf{u}$ and Korn inequality that

$$\begin{aligned}
& \left(\int_{\Omega'_{3r}} |\eta \nabla(\Delta_\lambda \mathbf{u})|^p dx \right)^{1/p} \\
& \leq c \left(\int_{\Omega'_{3r}} |\mathcal{D}(\eta \Delta_\lambda \mathbf{u})|^p dx \right)^{1/p} + \frac{c}{r} \left(\int_{\Omega'_{3r}} |\Delta_\lambda \mathbf{u}|^p dx \right)^{1/p} \\
& \leq c \frac{|\lambda|}{r} \left(\int_{\Omega'_{4r}} |\nabla \mathbf{u}|^p dx \right)^{1/p} + c \left(\int_{\Omega'_{3r}} |\eta \mathcal{D}(\Delta_\lambda \mathbf{u})|^p dx \right)^{1/p}.
\end{aligned}$$

Hence, we arrive at

$$\|\Delta_{-\lambda} \text{Sym}(\Delta_\lambda \mathbf{u} \otimes \eta^{p-1} \nabla \eta)\|_{p, \Omega'_{3r}} \leq c \frac{\lambda^2}{r^2} \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}} + c \frac{|\lambda|}{r} \left(\int_{\Omega'_{4r}} |\eta \mathcal{D}(\Delta_\lambda \mathbf{u})|^p dx \right)^{1/p}.$$

Therefore,

$$\begin{aligned} |I_1| &\leq c \|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})\|_{p, \Omega'_{3r}}^{p-1} \left(\frac{\lambda^2}{r^2} \|\nabla \mathbf{u}\|_{p, \Omega'_{3r}} + \frac{|\lambda|}{r} \left(\int_{\Omega'_{3r}} |\eta \mathcal{D}(\Delta_\lambda \mathbf{u})|^p dx \right)^{1/p} \right) \\ &\leq c(r, |\Omega'_{4r}|, \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}) |\lambda|^{p'} + \varepsilon \int_{\Omega'_{3r}} |\eta \mathcal{D}(\Delta_\lambda \mathbf{u})|^p dx. \end{aligned} \quad (5.11)$$

While

$$\begin{aligned} |I_2| &\leq \mu_1 \| \mathcal{D}(\mathbf{u}) \|_{p', \Omega'_{3r}}^{p-2} \|\mathcal{D}(\mathbf{u})\|_{p', \Omega'_{3r}} \|\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathcal{D}(\mathbf{a}))\|_{p, \Omega'_{3r}} \\ &\leq c(r, |\Omega'_{4r}|, \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}) |\lambda|^2. \end{aligned} \quad (5.12)$$

On the other hand,

$$\begin{aligned} \left| \int_{\Omega'_{3r}} \pi \nabla \cdot (\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx \right| &= p \left| \int_{\Omega'_{3r}} \pi \Delta_{-\lambda}(\eta^{p-1} \partial_i \eta \Delta_\lambda u_i) dx \right| \\ &\leq p \|\pi\|_{p', \Omega'_{3r}} |\lambda| \|\partial_k(\eta^{p-1} \partial_i \eta \Delta_\lambda u_i)\|_{p, \Omega'_{3r}} \\ &\leq c \|\pi\|_{p', \Omega'_{3r}} |\lambda| \left(\frac{1}{r^2} \|\Delta_\lambda u_i\|_{p, \Omega'_{3r}} + \frac{1}{r} \|\eta \Delta_\lambda \nabla \mathbf{u}\|_{p, \Omega'_{3r}} \right) \\ &\leq \frac{c}{r} \|\pi\|_{p', \Omega'_{3r}} |\lambda| \left(\frac{1}{r} |\lambda| \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}} + \|\eta \Delta_\lambda \mathcal{D}(\mathbf{u})\|_{p, \Omega'_{3r}} \right) \\ &\leq \frac{c}{r^2} \lambda^2 \|\pi\|_{p', \Omega'_{3r}} \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}} + \frac{c}{\varepsilon r^2} \lambda^{p'} \|\pi\|_{p', \Omega'_{3r}}^{p'} + \varepsilon \|\eta \Delta_\lambda \mathcal{D}(\mathbf{u})\|_{p, \Omega'_{3r}}^p. \end{aligned}$$

Then we arrive at

$$|I_3| \leq c(|\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}) |\lambda|^{p'} + \varepsilon \|\eta \Delta_\lambda \mathcal{D}(\mathbf{u})\|_p^p. \quad (5.13)$$

Since

$$\|\Phi\|_{p', \Omega'_{4r}} \leq \|\mathbf{u} \cdot \nabla \mathbf{a}\|_{p', \Omega'_{4r}} + \|\mathbf{a} \cdot \nabla \mathbf{u}\|_{p', \Omega'_{4r}} + \|\mathbf{a} \cdot \nabla \mathbf{a}\|_{p', \Omega'_{4r}} \leq c(|\Omega'_{4r}|)(1 + \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}),$$

one has

$$\begin{aligned}
|I_4| &\leq \|\Phi\|_{p',\Omega'_{3r}} \|\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})\|_{p,\Omega'_{3r}} \\
&\leq \|\Phi\|_{p',\Omega'_{3r}} |\lambda| \left(\int_{\Omega'_{3r}} |\partial_k(\eta^p \Delta_\lambda \mathbf{u})|^p dx \right)^{1/p} \\
&\leq \|\Phi\|_{p',\Omega'_{3r}} |\lambda| \left(\left(\int_{\Omega'_{3r}} |p\eta^{p-1} \nabla \eta \Delta_\lambda \mathbf{u}|^p \right)^{1/p} + \left(\int_{\Omega'_{3r}} |\eta^p \partial_k \Delta_\lambda \mathbf{u}|^p dx \right)^{1/p} \right) \\
&\leq c \|\Phi\|_{p',\Omega'_{3r}} |\lambda| \left(\frac{1}{r} |\lambda| \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}} + \left(\int_{\Omega'_{3r}} |\eta \nabla \Delta_\lambda \mathbf{u}|^p dx \right)^{1/p} \right) \\
&\leq \frac{c}{r} \lambda^2 \|\Phi\|_{p',\Omega'_{3r}} \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}} + c \|\Phi\|_{p',\Omega'_{3r}} |\lambda| \left(\frac{1}{r} |\lambda| \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}} + \left(\int_{\Omega'_{3r}} |\eta \Delta_\lambda(\mathcal{D}(\mathbf{u}))|^p dx \right)^{1/p} \right) \\
&\leq \frac{c}{r} \lambda^2 \|\Phi\|_{p',\Omega'_{3r}} \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}} + c |\lambda|^{p'} \|\Phi\|_{p',\Omega'_{3r}}^{p'} + \varepsilon \int_{\Omega'_{3r}} |\eta \Delta_\lambda(\mathcal{D}(\mathbf{u}))|^p dx.
\end{aligned}$$

Hence, we have

$$|I_4| \leq c(|\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}}) |\lambda|^{p'} + \varepsilon \|\eta \Delta_\lambda(\mathcal{D}(\mathbf{u}))\|_{p,\Omega'_{3r}}^p. \quad (5.14)$$

Finally, we start to deal with I_5 . Note that

$$\begin{aligned}
-I_5 &= \int_{\Omega'_{3r}} u_i \partial_i u_j \Delta_{-\lambda}(\eta^p \Delta_\lambda u_j) dx = \int_{\Omega'_{3r}} \Delta_\lambda(u_i \partial_i u_j) \eta^p \Delta_\lambda u_j dx \\
&= \int_{\Omega'_{3r}} \Delta_\lambda u_i (\partial_i u_j)(x + \lambda e_k) \eta^p \Delta_\lambda u_j dx + \int_{\Omega'_{3r}} u_i \Delta_\lambda(\partial_i u_j) \eta^p \Delta_\lambda u_j dx. \\
&\equiv I_{5,1} + I_{5,2}.
\end{aligned} \quad (5.15)$$

It follows from a direct computation that

$$|I_{5,1}| \leq c \|\Delta_\lambda \mathbf{u}\|_{2p',\Omega'_{3r}}^2 \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}}.$$

If $p < d$, we set $\theta = \frac{(d+2)p-3d}{2p}$. Then

$$\|\Delta_\lambda \mathbf{u}\|_{2p',\Omega'_{2r}}^2 \leq \|\Delta_\lambda \mathbf{u}\|_{p^*,\Omega'_{3r}}^{2(1-\theta)} \|\Delta_\lambda \mathbf{u}\|_{p,\Omega'_{3r}}^{2\theta} \leq |\lambda|^{2\theta} \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}}^2,$$

where $p^* = \frac{dp}{d-p}$.

If $d \leq p < 3$, by $W^{1,d} \hookrightarrow L^q$ for any $q < \infty$, we then have that for any $\theta < 1$,

$$\|\Delta_\lambda \mathbf{u}\|_{2p',\Omega'_{3r}}^2 \leq \|\Delta_\lambda \mathbf{u}\|_{q(\theta),\Omega'_{3r}}^{2(1-\theta)} \|\Delta_\lambda \mathbf{u}\|_{p,\Omega'_{3r}}^{2\theta} \leq c(|\Omega'_{4r}|) |\lambda|^{2\theta} \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}}^2.$$

If $p \geq 3$, due to $2p' \leq p$, one then has

$$\|\Delta_\lambda \mathbf{u}\|_{2p',\Omega'_{3r}}^2 \leq c(|\Omega'_{4r}|) \|\Delta_\lambda \mathbf{u}\|_{p,\Omega'_{3r}}^2 \leq c(|\Omega'_{4r}|) |\lambda|^2 \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}}^2.$$

Hence, we have

$$|I_{5,1}| \leq |\lambda|^{2\theta} \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}}^3. \quad (5.16)$$

In addition,

$$|I_{5,2}| = \frac{p}{2} \left| \int_{\Omega'_{3r}} u_i (\Delta_\lambda u_j)^2 \eta^{p-1} \partial_i \eta dx \right| \leq \frac{c(|\Omega'_{4r}|)}{r} \|\Delta_\lambda \mathbf{u}\|_{2p', \Omega'_{3r}}^2 \|\nabla \mathbf{u}\|_{p, \Omega'_{3r}} \leq \frac{c(|\Omega'_{4r}|)}{r} |\lambda|^{2\theta} \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}^3. \quad (5.17)$$

Combining (5.16) with (5.17) yields

$$|I_5| \leq c(|\Omega'_{4r}|) \left(1 + \frac{1}{r}\right) |\lambda|^{2\theta} \|\nabla \mathbf{u}\|_{\Omega'_{4r}}^3 \leq c(|\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}) |\lambda|^{2\theta}. \quad (5.18)$$

Collecting all above estimates on I_i ($1 \leq i \leq 5$) and setting $\hat{\theta} = \min\{\frac{p'}{2}, \theta\}$, we eventually obtain

$$\|\Delta_\lambda \mathcal{D}(u)\|_{p, \Omega'} \leq c(|\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}) |\lambda|^{\frac{2\hat{\theta}}{p}}.$$

Thus, by the characterization of the fractional order Sobolev space (see [1] or [30]), we have that for any $\kappa \in [0, \frac{2\hat{\theta}}{p})$,

$$\nabla \mathbf{u} \in W^{\kappa, p}(\Omega')$$

and

$$\|\nabla \mathbf{u}\|_{\kappa, p, \Omega'} \leq c(\kappa, r, |\Omega'_{4r}|, \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}).$$

Case II. $\mu_0 > 0, 1 < p < 2$

As in Case I, we can find a function $\pi \in L^2(\Omega'_{4r})$ such that for any $\psi \in D_0^{1,2}(\Omega'_{4r})$,

$$F(\psi) = (\pi, \nabla \cdot \psi) \quad (5.19)$$

and

$$\|\pi\|_{2, \Omega'_{4r}} \leq \|F\|_{D_0^{-1,2}(\Omega'_{4r})} \leq c(|\Omega'_{4r}|) (1 + \|\nabla \mathbf{u}\|_{2, \Omega'_{4r}} + \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}^{p-1} + \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}^2 + \|\nabla \mathbf{u}\|_{p, \Omega'_{4r}}).$$

Choosing $\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})$ as a test function in (5.19) yields that

$$\begin{aligned} & \mu_0 \int_{\Omega'_{3r}} \mathcal{D}(\mathbf{u}) : \mathcal{D}(\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})) dx \\ & + \mu_1 \int_{\Omega'_{3r}} |\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) : \mathcal{D}(\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})) dx \\ & = - \int_{\Omega'_{3r}} \mathbf{u} \cdot \nabla \mathbf{u} \cdot (\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})) dx + \int_{\Omega'_{3r}} \pi \nabla \cdot (\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})) dx \\ & - \int_{\Omega'_{3r}} \mathbf{u} \cdot \nabla \mathbf{a} \cdot (\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})) - \int_{\Omega'_{3r}} \mathbf{a} \cdot \nabla \mathbf{u} \cdot (\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})) dx \\ & - \int_{\Omega'_{3r}} \mathbf{a} \cdot \nabla \mathbf{a} \cdot (\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})) dx + \mu_0 \int_{\Omega'_{3r}} \Delta \mathbf{a} : \Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u}) dx. \end{aligned} \quad (5.20)$$

It follows from a direct computation that

$$\mu_0 \int_{\Omega'_{3r}} \mathcal{D}(\mathbf{u}) : \mathcal{D}(\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})) dx$$

$$= 2\mu_0 \int_{\Omega'_{3r}} \Delta_\lambda(\mathcal{D}(\mathbf{u})) : Sym(\Delta_\lambda \mathbf{u} \otimes \eta \nabla \eta) dx + \mu_0 \int_{\Omega'_{3r}} \eta^2 (\Delta_\lambda \mathcal{D}(\mathbf{u}))^2 dx.$$

Then we have

$$\begin{aligned} & \mu_0 \int_{\Omega'_{3r}} \eta^2 (\Delta_\lambda \mathcal{D}(\mathbf{u}))^2 dx \\ & \leq \frac{2\mu_0}{r} \left(\int_{\Omega'_{3r}} \eta^2 (\Delta_\lambda \mathcal{D}(\mathbf{u}))^2 dx \right)^{1/2} \left(\int_{\Omega'_{3r}} |\Delta_\lambda \mathbf{u}|^2 dx \right)^{1/2} + \mu_0 \int_{\Omega'_{3r}} \mathcal{D}(\mathbf{u}) \cdot \mathcal{D}(\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})) dx \\ & \leq \frac{2\mu_0 \varepsilon}{r} \int_{\Omega'_{3r}} \eta^2 (\Delta_\lambda \mathcal{D}(\mathbf{u}))^2 dx + \frac{\lambda^2}{4\varepsilon} \int_{\Omega'_{4r}} |\nabla \mathbf{u}|^2 dx + \mu_0 \int_{\Omega'_{3r}} \mathcal{D}(\mathbf{u}) \cdot \mathcal{D}(\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})) dx. \end{aligned} \quad (5.21)$$

In addition

$$\begin{aligned} & \mu_1 \int_{\Omega'_{3r}} |\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) : \mathcal{D}(\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})) dx \\ & = 2\mu_1 \int_{\Omega'_{3r}} |\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) \Delta_{-\lambda} Sym(\Delta_\lambda \mathbf{u} \otimes \eta \nabla \eta) dx \\ & \quad + \mu_1 \int_{\Omega'_{3r}} \eta^2 \Delta_\lambda (|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) \Delta_\lambda (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}))) dx \\ & \quad - \mu_1 \int_{\Omega'_{3r}} \eta^2 |\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) \Delta_{-\lambda} (\Delta_\lambda (\mathcal{D}(\mathbf{a}))) dx \end{aligned}$$

It follows from (5.8) that

$$\begin{aligned} & \mu_1 \int_{\Omega'_{3r}} |\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2} (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) : \mathcal{D}(\Delta_{-\lambda}(\eta^2 \Delta_\lambda \mathbf{u})) dx \\ & \geq \frac{\delta}{2} I(\mathbf{u}) - \delta I(\mathbf{a}) - 2\mu_1 \|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})\|_{p, \Omega'_{3r}}^{p-1} \|\Delta_{-\lambda} Sym(\Delta_\lambda \mathbf{u} \otimes \eta \nabla \eta)\|_{p, \Omega'_{3r}} \\ & \quad - \mu_1 \|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})\|_{p, \Omega'_{3r}}^{p-1} \|\Delta_{-\lambda} (\Delta_\lambda (\mathcal{D}(\mathbf{a})))\|_{p, \Omega'_{3r}}, \end{aligned} \quad (5.22)$$

where

$$I(\mathbf{u}) = \int_{\Omega'_{3r}} \eta^2 (|(1 + \mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}))(x + \lambda e_k)| + |(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}))(x)|)^{p-2} |\Delta_\lambda \mathcal{D}(\mathbf{u})|^2 dx,$$

and

$$I(\mathbf{a}) = \int_{\Omega'_{3r}} \eta^2 (|(1 + (\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}))(x + \lambda e_k)| + |(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}))(x)|)^{p-2} |\Delta_\lambda \mathcal{D}(\mathbf{a})|^2 dx.$$

From (5.20)-(5.22), set $\varepsilon = \frac{r}{4}$, one has

$$\begin{aligned}
& \frac{\mu_0}{2} \int_{\Omega'_{3r}} \eta^2 (\Delta_\lambda \mathcal{D}(\mathbf{u}))^2 dx + \frac{\delta}{2} I(u) \\
& \leq \frac{\lambda^2}{r} \int_{\Omega'_{4r}} |\nabla \mathbf{u}|^2 dx + \delta I(\mathbf{a}) \\
& \quad + 2\mu_1 \|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})\|_{p,\Omega'_{3r}}^{p-1} \|\Delta_{-\lambda} Sym(\Delta_\lambda \mathbf{u} \otimes \eta \nabla \eta)\|_{p,\Omega'_{3r}} \\
& \quad + \mu_1 \|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})\|_{p,\Omega'_{3r}}^{p-1} \|\Delta_{-\lambda}(\Delta_\lambda(\mathcal{D}(\mathbf{a})))\|_{p,\Omega'_{3r}} \\
& \quad + \int_{\Omega'_{3r}} \pi \nabla \cdot (\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx \\
& \quad + \int_{\Omega'_{3r}} \Phi \cdot (\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx \\
& \quad - \int_{\Omega'_{3r}} \mathbf{u} \cdot \nabla \mathbf{u} \cdot (\Delta_{-\lambda}(\eta^p \Delta_\lambda \mathbf{u})) dx \\
& \equiv \frac{\lambda^2}{r} \int_{\Omega'_{4r}} |\nabla \mathbf{u}|^2 dx + \delta I(\mathbf{a}) + I_1 + \dots + I_5,
\end{aligned} \tag{5.23}$$

where $\Phi = \mu_0 \Delta \mathbf{a} - \mathbf{u} \cdot \nabla \mathbf{a} - \mathbf{a} \cdot \nabla \mathbf{u} - \mathbf{a} \cdot \nabla \mathbf{a}$. Similarly to the treatment in Case I, we have

$$\|\Delta_{-\lambda} Sym(\Delta_\lambda \mathbf{u} \otimes \eta \nabla \eta)\|_{p,\Omega'_{3r}} \leq c \frac{\lambda^2}{r^2} \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}} + c \frac{|\lambda|}{r} \left(\int_{\Omega'_{3r}} |\eta \mathcal{D}(\Delta_\lambda \mathbf{u})|^p dx \right)^{1/p}.$$

Since

$$\begin{aligned}
& \int_{\Omega'_{3r}} |\eta \mathcal{D}(\Delta_\lambda \mathbf{u})|^p dx \\
& \leq \int_{\Omega'_{3r}} \left(1 + |(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}))(x + \lambda e_k)| + |(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}))(x)| \right)^{\frac{(p-2)p}{2}} |\eta \Delta_\lambda \mathcal{D}(\mathbf{u})|^p \\
& \quad \times \left(1 + |(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}))(x + \lambda e_k)| + |(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}))(x)| \right)^{\frac{(2-p)p}{2}} dx \\
& \leq 2I(\mathbf{u})^{\frac{p}{2}} \left(\int_{\Omega'_{4r}} (1 + |\mathcal{D}(\mathbf{u})| + |\mathcal{D}(\mathbf{a})|)^p dx \right)^{\frac{2-p}{2}},
\end{aligned} \tag{5.24}$$

we get

$$\begin{aligned}
|I_1| & \leq 2\mu_1 \|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})\|_{p,\Omega'_{3r}}^{p-1} \left(\frac{\lambda^2}{r^2} \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}} + \frac{|\lambda|}{r} \left(\int_{\Omega'_{3r}} |\eta \mathcal{D}(\Delta_\lambda \mathbf{u})|^p dx \right)^{1/p} \right) \\
& \leq c(|\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}}) \lambda^2 + c \frac{|\lambda|}{r} I(\mathbf{u})^{\frac{1}{2}} \left(\int_{\Omega'_{4r}} (1 + |\mathcal{D}(\mathbf{u})| + |\mathcal{D}(\mathbf{a})|)^p dx \right)^{\frac{1}{2}} \\
& \leq c(|\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}}) \lambda^2 + \frac{\delta}{2} I(\mathbf{u}).
\end{aligned} \tag{5.25}$$

It is easy to get that

$$|I(\mathbf{a})| \leq c(|\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}}) \lambda^2$$

and

$$|I_2| \leq c(|\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{p, \Omega_{4r}}) \lambda^2.$$

In addition,

$$\begin{aligned} |I_3| &= 2 \left| \int_{\Omega'_{3r}} \pi \Delta_{-\lambda} (\eta \partial_i \eta \Delta_\lambda u_i) dx \right| \\ &\leq 2 \|\pi\|_{2, \Omega'_{3r}} |\lambda| \|\partial_k (\eta \partial_i \eta \Delta_\lambda u_i)\|_{2, \Omega'_{3r}} \\ &\leq \|\pi\|_{2, \Omega'_{3r}} |\lambda| \left(\frac{1}{r^2} \|\Delta_\lambda u_i\|_{2, \Omega'_{3r}} + \frac{1}{r} \|\eta \Delta_\lambda \nabla \mathbf{u}\|_{2, \Omega'_{3r}} \right) \\ &\leq \|\pi\|_{2, \Omega'_{3r}} |\lambda| \left(\frac{1}{r^2} \|\Delta_\lambda u_i\|_{2, \Omega'_{3r}} + \frac{|\lambda|}{r^2} \|\nabla \mathbf{u}\|_{2, \Omega'_{4r}} + \frac{1}{r} \int_{\Omega'_{3r}} |\eta \Delta_\lambda \mathcal{D}(\mathbf{u})|^2 dx \right) \\ &\leq \frac{c}{r^2} \|\pi\|_{2, \Omega'_{3r}} \|\nabla \mathbf{u}\|_{2, \Omega'_{4r}} \lambda^2 + \frac{c}{\varepsilon r^2} \|\pi\|_{2, \Omega'_{3r}}^2 \lambda^2 + \varepsilon \int_{\Omega'_{3r}} |\eta \Delta_\lambda \mathcal{D}(\mathbf{u})|^2 dx \\ &\leq c(|\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{2, \Omega'_{4r}}) \lambda^2 + \varepsilon \int_{\Omega'_{3r}} |\eta \Delta_\lambda \mathcal{D}(\mathbf{u})|^2 dx. \end{aligned}$$

And

$$\begin{aligned} |I_4| &\leq \|\Phi\|_{2, \Omega'_{3r}} \|\Delta_{-\lambda} (\eta^2 \Delta_\lambda \mathbf{u})\|_{2, \Omega'_{3r}} \\ &\leq \|\Phi\|_{2, \Omega'_{3r}} |\lambda| \left(\int_{\Omega'_{3r}} |\partial_k (\eta^2 \Delta_\lambda \mathbf{u})|^2 dx \right)^{1/2} \\ &\leq \|\Phi\|_{2, \Omega'_{3r}} |\lambda| \left(\left(\int_{\Omega'_{3r}} (\eta |\nabla \eta| |\Delta_\lambda \mathbf{u}|)^2 dx \right)^{1/2} + \left(\int_{\Omega'_{3r}} |\eta^2 \partial_k \Delta_\lambda \mathbf{u}|^2 dx \right)^{1/2} \right) \\ &\leq \|\Phi\|_{2, \Omega'_{3r}} |\lambda| \left(\frac{1}{r} |\lambda| \|\nabla \mathbf{u}\|_{2, \Omega'_{3r}} + \left(\int_{\Omega'_{3r}} |\eta \nabla \Delta_\lambda \mathbf{u}|^2 dx \right)^{1/2} \right) \\ &\leq \frac{1}{r} \lambda^2 \|\Phi\|_{2, \Omega'_{3r}} \|\nabla \mathbf{u}\|_{2, \Omega'_{3r}} + \|\Phi\|_{2, \Omega'_{3r}} |\lambda| \left(\frac{1}{r} |\lambda| \|\nabla \mathbf{u}\|_{2, \Omega'_{3r}} + \left(\int_{\Omega'_{3r}} |\eta \Delta_\lambda \mathcal{D}(\mathbf{u})|^2 dx \right)^{1/2} \right) \\ &\leq c(|\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{2, \Omega'_{4r}}) \lambda^2 + \varepsilon \int_{\Omega'_{3r}} |\eta \Delta_\lambda \mathcal{D}(\mathbf{u})|^2 dx. \end{aligned}$$

Finally, we start to deal with the term I_5 . It follows from a direct computation that

$$\begin{aligned} -I_5 &= \sum_{i,j=1}^d \int_{\Omega'_{3r}} u_i \partial_i u_j \Delta_{-\lambda} (\eta^2 \Delta_\lambda u_j) dx = \sum_{i,j=1}^d \int_{\Omega'_{3r}} \Delta_\lambda (u_i \partial_i u_j) \eta^2 \Delta_\lambda u_j dx \\ &= \sum_{i,j=1}^d \int_{\Omega'_{3r}} \Delta_\lambda u_i (\partial_i u_j) (x + \lambda e_k) \eta^2 \Delta_\lambda u_j dx + \sum_{i,j=1}^d \int_{\Omega'_{3r}} u_i \Delta_\lambda (\partial_i u_j) \eta^2 \Delta_\lambda u_j dx \\ &\equiv I_{5,1} + I_{5,2}. \end{aligned} \tag{5.26}$$

Obviously,

$$|I_{5,1}| \leq \|\Delta_\lambda \mathbf{u}\|_{4, \Omega'_{3r}}^2 \|\nabla \mathbf{u}\|_{2, \Omega'_{4r}}. \tag{5.27}$$

If $d = 2$, by $W^{1,2} \hookrightarrow L^q$ for any $q < \infty$, we then have that for any $\theta < 1$,

$$\|\Delta_\lambda \mathbf{u}\|_{4, \Omega'_{3r}}^2 \leq \|\Delta_\lambda \mathbf{u}\|_{q(\theta), \Omega'_{3r}}^{2(1-\theta)} \|\Delta_\lambda \mathbf{u}\|_{2, \Omega'_{3r}}^{2\theta} \leq c(|\Omega'_{4r}|) |\lambda|^{2\theta} \|\nabla \mathbf{u}\|_{2, \Omega'_{4r}}^2.$$

If $d = 3$, we set $\theta = \frac{1}{4}$. Then

$$\|\Delta_\lambda \mathbf{u}\|_{4,\Omega'_{3r}}^2 \leq \|\Delta_\lambda \mathbf{u}\|_{6,\Omega'_{3r}}^{2(1-\theta)} \|\Delta_\lambda \mathbf{u}\|_{2,\Omega'_{3r}}^{2\theta} \leq c(|\Omega'_{4r}|) |\lambda|^{2\theta} \|\nabla \mathbf{u}\|_{2,\Omega'_{4r}}^2$$

and

$$|I_{5,1}| \leq c(|\Omega'_{4r}|) |\lambda|^{2\theta} \|\nabla \mathbf{u}\|_{2,\Omega'_{4r}}^3. \quad (5.28)$$

On the other hand,

$$\begin{aligned} |I_{5,2}| &= \frac{p}{2} \left| \int_{\Omega'_{3r}} u_i (\Delta_\lambda u_j)^2 \eta^{p-1} \partial_i \eta dx \right| \leq \frac{c}{r} \|\Delta_\lambda \mathbf{u}\|_{4,\Omega'_{3r}}^2 \|\mathbf{u}\|_{2,\Omega'_{3r}} \\ &\leq c(|\Omega'_{4r}|, r) \|\Delta_\lambda \mathbf{u}\|_{4,\Omega'_{3r}}^2 \|\nabla \mathbf{u}\|_{2,\Omega'_{4r}}. \end{aligned}$$

Hence, we can obtain that

$$|I_5| \leq c(\varepsilon, |\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}}) |\lambda|^{2\theta}. \quad (5.29)$$

Collecting all above estimates on I_i ($1 \leq i \leq 4$), we eventually get

$$\|\Delta_\lambda \mathcal{D}(\mathbf{u})\|_{2,\Omega'} \leq c(|\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{2,\Omega'_{4r}}) |\lambda|^\theta$$

and

$$\|\Delta_\lambda \mathcal{D}(\mathbf{u})\|_{p,\Omega'} \leq c(|\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{2,\Omega'_{4r}}) |\lambda|^{\frac{2\theta}{p}}.$$

Thus, we have that for any $\varepsilon > 0$,

$$\nabla \mathbf{u} \in W^{\frac{2\theta}{p}-\varepsilon,p}(\Omega') \cap W^{\theta-\varepsilon,2}(\Omega')$$

and

$$\|\nabla \mathbf{u}\|_{\theta-\varepsilon,p,\Omega'} + \|\nabla \mathbf{u}\|_{\frac{2\theta}{p}-\varepsilon,p,\Omega'} \leq c(\varepsilon, |\Omega'_{4r}|, r, \|\nabla \mathbf{u}\|_{p,\Omega'_{4r}}).$$

□

6 Solvability of Ladyzhenskaya-Solonnikov Problem I (2.4)

To solve Ladyzhenskaya-Solonnikov Problem I (2.4) under some suitable conditions, based on Sections 4-5, we take the following three parts.

6.1 Part 1. Uniform estimate of $\|\mathbf{u}^T\|_{p,\Omega(t)}$

In what follows, for convenience and without loss of generality, we assume that

$$\Omega = \{x : x_1 \in \mathbb{R}, x' \in \Sigma(x_1)\}$$

and

$$\Omega(t) = \{x \in \Omega : -t < x_1 < t\}.$$

Taking the inner product of (4.1)₁ with \mathbf{u}^T and integrating by parts over $\Omega(t)$ yield

$$\begin{aligned} \mu_0 \|\mathcal{D}(\mathbf{v}^T)\|_{2,\Omega(t)}^2 + \mu_1 \|\mathcal{D}(\mathbf{v}^T)\|_{p,\Omega(t)}^p \\ = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \end{aligned} \quad (6.1)$$

with

$$\begin{aligned}
I_1 &= \mu_0 \int_{\Sigma(t)} \mathbf{u}^T \cdot \mathcal{D}(\mathbf{v}^T) \cdot \mathbf{e}_1 dS + \mu_1 \int_{\Sigma(t)} \mathbf{u}^T \cdot (|\mathcal{D}(\mathbf{v}^T)|^{p-2} \mathcal{D}(\mathbf{v}^T)) \cdot \mathbf{e}_1 dS, \\
I_2 &= -\mu_0 \int_{\Sigma(-t)} \mathbf{u}^T \cdot \mathcal{D}(\mathbf{v}^T) \cdot \mathbf{e}_1 dS - \mu_1 \int_{\Sigma(-t)} \mathbf{u}^T \cdot (|\mathcal{D}(\mathbf{v}^T)|^{p-2} \mathcal{D}(\mathbf{v}^T)) \cdot \mathbf{e}_1 dS, \\
I_3 &= - \int_{\Omega(t)} \mathbf{v}^T \cdot \nabla \mathbf{v}^T \cdot \mathbf{u}^T dx, \\
I_4 &= - \int_{\Sigma(t)} \pi^T u_1^T dS + \int_{\Sigma(-t)} \pi^T u_1^T dS, \\
I_5 &= \mu_0 \int_{\Omega(t)} \mathcal{D}(\mathbf{v}^T) : \mathcal{D}(\mathbf{a}) dx, \\
I_6 &= \mu_1 \int_{\Omega(t)} |\mathcal{D}(\mathbf{v}^T)|^{p-2} \mathcal{D}(\mathbf{v}^T) : \mathcal{D}(\mathbf{a}) dx.
\end{aligned}$$

Next we deal with the terms I_i for $1 \leq i \leq 6$ in (6.1). We still divide the related process into the following two cases:

Case I. $\mu_0 > 0, p > 1$

Using the Hölder and Poincaré inequality we get that

$$\begin{aligned}
\left| \int_{\eta-1}^{\eta} I_1 dt \right| &= \left| \mu_0 \int_{\omega_{\eta}^+} \mathbf{u}^T \cdot \mathcal{D}(\mathbf{v}^T) \cdot \mathbf{e}_1 dx + \mu_1 \int_{\omega_{\eta}^+} \mathbf{u}^T \cdot (|\mathcal{D}(\mathbf{v}^T)|^{p-2} \mathcal{D}(\mathbf{v}^T)) \cdot \mathbf{e}_1 dx \right| \\
&\leq \mu_0 \|\mathbf{u}^T\|_{2,\omega_{\eta}^+} \|\mathcal{D}(\mathbf{v}^T)\|_{2,\omega_{\eta}^+} + \mu_1 \|\mathbf{u}^T\|_{p,\omega_{\eta}^+} \|\mathcal{D}(\mathbf{v}^T)\|_{p,\omega_{\eta}^+}^{p-1} \\
&\leq \mu_0 \|\mathbf{u}^T\|_{2,\omega_{\eta}^+}^2 + \mu_0 \|\mathcal{D}(\mathbf{v}^T)\|_{2,\omega_{\eta}^+}^2 + \mu_1 \|\mathbf{u}^T\|_{p,\omega_{\eta}^+}^p + \mu_1 \|\mathcal{D}(\mathbf{v}^T)\|_{p,\omega_{\eta}^+}^p \\
&\leq c \|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^+}^2 + c \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^+}^p + c\alpha,
\end{aligned} \tag{6.2}$$

where $\omega_{\eta}^+ = \{x \in \Omega : \eta - 1 < x_1 < \eta\}$, and $\alpha = \max_{1 \leq i \leq N} |\alpha_i|$. Similarly, we have

$$\left| \int_{\eta-1}^{\eta} I_2 dt \right| \leq c \|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^-}^2 + c \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^-}^p + c\alpha, \tag{6.3}$$

where $\omega_{\eta}^- = \{x \in \Omega : -\eta < x_1 < -\eta + 1\}$.

Next, we estimate I_3 . Note that

$$\begin{aligned}
\int_{\Omega(t)} \mathbf{v}^T \cdot \nabla \mathbf{v}^T \cdot \mathbf{u}^T dx &= \int_{\Sigma(t)} (\mathbf{a} \cdot \mathbf{u}^T)(\mathbf{a} \cdot \mathbf{e}_1) dS - \int_{\Sigma(-t)} (\mathbf{a} \cdot \mathbf{u}^T)(\mathbf{a} \cdot \mathbf{e}_1) dS \\
&\quad + \int_{\Sigma(t)} \frac{(\mathbf{u}^T)^2}{2} \mathbf{v}^T \cdot \mathbf{e}_1 dS - \int_{\Sigma(-t)} \frac{(\mathbf{u}^T)^2}{2} \mathbf{v}^T \cdot \mathbf{e}_1 dS \\
&\quad - \int_{\Omega(t)} \mathbf{u}^T \cdot \nabla \mathbf{u}^T \cdot \mathbf{a} dx + \int_{\Omega(t)} \mathbf{a} \cdot \nabla \mathbf{a} \cdot \mathbf{u}^T dx.
\end{aligned} \tag{6.4}$$

By Hölder inequality and Lemma 3.1 we arrive at

$$\begin{aligned} & \left| \int_{\eta-1}^{\eta} \int_{\Sigma(t)} (\mathbf{a} \cdot \mathbf{u}^T)(\mathbf{a} \cdot \mathbf{e}_1) dy dt + \int_{\eta-1}^{\eta} \int_{\Sigma(t)} \frac{(\mathbf{u}^T)^2}{2} \mathbf{v}^T \cdot \mathbf{e}_1 dy dt \right| \\ & \leq \| \mathbf{a} \|_{4,\omega_{\eta}^{+}}^2 \| \mathbf{u}^T \|_{2,\omega_{\eta}^{+}} + \| \mathbf{u}^T \|_{4,\omega_{\eta}^{+}}^2 \| \mathbf{v}^T \|_{2,\omega_{\eta}^{+}} \\ & \leq \| \mathbf{a} \|_{4,\omega_{\eta}^{+}}^2 \| \mathbf{u}^T \|_{2,\omega_{\eta}^{+}} + \| \mathbf{u}^T \|_{4,\omega_{\eta}^{+}}^2 \| \mathbf{u}^T \|_{2,\omega_{\eta}^{+}} + \| \mathbf{a} \|_{2,\omega_{\eta}^{+}} \| \mathbf{u}^T \|_{4,\omega_{\eta}^{+}}^2 \\ & \leq c(\| \mathcal{D}(\mathbf{u}^T) \|_{2,\omega_{\eta}^{+}}^3 + \| \mathcal{D}(\mathbf{u}^T) \|_{2,\omega_{\eta}^{+}}^2 + \| \mathcal{D}(\mathbf{u}^T) \|_{2,\omega_{\eta}^{+}}). \end{aligned}$$

Similarly,

$$\begin{aligned} & \left| \int_{\eta-1}^{\eta} \int_{\Sigma(-t)} (\mathbf{a} \cdot \mathbf{u}^T)(\mathbf{a} \cdot \mathbf{e}_1) dy dt + \int_{\eta-1}^{\eta} \int_{\Sigma(-t)} \frac{\mathbf{u}^2}{2} \mathbf{v}^T \cdot \mathbf{e}_1 dy dt \right| \\ & \leq c(\| \mathcal{D}(\mathbf{u}^T) \|_{2,\omega_{\eta}^{-}}^3 + \| \mathcal{D}(\mathbf{u}^T) \|_{2,\omega_{\eta}^{-}}^2 + \| \mathcal{D}(\mathbf{u}^T) \|_{2,\omega_{\eta}^{-}}). \end{aligned} \quad (6.5)$$

In addition, by Lemma 3.5 (iii) and (iv) we can arrive at

$$\left| \int_{\Omega(t)} \mathbf{u}^T \cdot \nabla \mathbf{u}^T \cdot \mathbf{a} dx \right| \leq c(\varepsilon) \int_{\Omega(t)} |\mathbf{a}|^2 |\mathbf{u}^T|^2 dx + \frac{\varepsilon}{2} \int_{\Omega(t)} |\nabla \mathbf{u}^T|^2 dx \leq \varepsilon \int_{\Omega(t)} |\nabla \mathbf{u}^T|^2 dx.$$

Note that

$$\left| \int_{\Omega(t)} \mathbf{a} \cdot \nabla \mathbf{a} \cdot \mathbf{u}^T dx \right| \leq \varepsilon \|\mathbf{u}\|_{2,\Omega(t)}^2 + c\|\mathbf{a} \cdot \nabla \mathbf{a}\|_{2,\Omega(t)}^2 \leq c\varepsilon \|\nabla \mathbf{u}^T\|_{2,\Omega(t)}^2 + c\alpha t. \quad (6.6)$$

Hence,

$$\begin{aligned} \left| \int_{\eta-1}^{\eta} I_3 dt \right| & \leq \varepsilon \int_{\eta-1}^{\eta} \left(\int_{\Omega(t)} |\nabla \mathbf{u}^T|^2 dx \right) dt + c(\| \nabla \mathbf{u}^T \|_{p,\omega_{\eta}^{+}}^3 + \| \nabla \mathbf{u}^T \|_{p,\omega_{\eta}^{+}}^2 + \| \nabla \mathbf{u}^T \|_{p,\omega_{\eta}^{+}}) \\ & \quad + c(\| \nabla \mathbf{u}^T \|_{p,\omega_{\eta}^{-}}^3 + \| \nabla \mathbf{u}^T \|_{p,\omega_{\eta}^{-}}^2 + \| \nabla \mathbf{u}^T \|_{p,\omega_{\eta}^{-}}) + c\alpha\eta. \end{aligned} \quad (6.7)$$

Next, we treat the term I_4 . By $\int_{\Sigma(\pm t)} u_1^T dS = 0$ for any $t > 0$, then from Remark 3.2, we can find a vector $\mathbf{w} \in W_0^{1,p}(\omega_{\eta}^{\pm}) \cap W_0^{1,2}(\omega_{\eta}^{\pm})$ such that $\operatorname{div} \mathbf{w} = u_1^T$ in ω_{η}^{\pm} . A direct computation yields

$$\begin{aligned} & \left| \int_{\omega_{\eta}^{\pm}} \pi^T u_1^T dx \right| = \left| \int_{\omega_{\eta}^{\pm}} \pi^T \operatorname{div} \mathbf{w} dx \right| \\ & = \left| \int_{\omega_{\eta}^{\pm}} (\mu_0 \mathcal{D}(\mathbf{v}^T) + \mu_1 |\mathcal{D}(\mathbf{v}^T)|^{p-2} \mathcal{D}(\mathbf{v}^T)) : \mathcal{D}(\mathbf{w}) dx - \int_{\omega_{\eta}^{\pm}} \mathbf{v}^T \cdot \nabla \mathbf{w} \cdot \mathbf{v}^T dx \right| \\ & \leq \mu_0 \|\mathcal{D}(\mathbf{v}^T)\|_{2,\omega_{\eta}^{\pm}} \|\mathcal{D}(\mathbf{w})\|_{2,\omega_{\eta}^{\pm}} + \mu_1 \|\mathcal{D}(\mathbf{v}^T)\|_{p,\omega_{\eta}^{\pm}}^{p-1} \|\mathcal{D}(\mathbf{w})\|_{p,\omega_{\eta}^{\pm}} + \|\mathbf{v}^T\|_{4,\omega_{\eta}^{\pm}}^2 \|\nabla \mathbf{w}\|_{2,\omega_{\eta}^{\pm}} \\ & \leq c\|\mathcal{D}(\mathbf{v}^T)\|_{2,\omega_{\eta}^{\pm}} \|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{\pm}} + c\|\mathcal{D}(\mathbf{v}^T)\|_{p,\omega_{\eta}^{\pm}}^{p-1} \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^{\pm}} + c\|\mathbf{v}^T\|_{4,\omega_{\eta}^{\pm}}^2 \|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{\pm}} \\ & \leq c\|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{\pm}}^2 + c\|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{\pm}} + c\|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^{\pm}}^p + c\|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^{\pm}} + c\|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{\pm}}^3. \end{aligned}$$

This means that

$$\begin{aligned} \left| \int_{\eta-1}^{\eta} I_4 dt \right| & \leq c(\|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{+}} + \|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{+}}^2 + \|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{+}}^3 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^{+}} + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^{+}}^p) \\ & \quad + c(\|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{-}} + \|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{-}}^2 + \|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{-}}^3 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^{-}} + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^{-}}^p). \end{aligned} \quad (6.8)$$

Since

$$\begin{aligned} & \left| \mu_0 \int_{\Omega(t)} \mathcal{D}(\mathbf{v}^T) : \mathcal{D}(\mathbf{a}) dx + \mu_1 \int_{\Omega(t)} |\mathcal{D}(\mathbf{v}^T)|^{p-2} \mathcal{D}(\mathbf{v}^T) : \mathcal{D}(\mathbf{a}) dx \right| \\ & \leq \varepsilon \|\mathcal{D}(\mathbf{v}^T)\|_{2,\Omega(t)}^2 + \varepsilon \|\mathcal{D}(\mathbf{v}^T)\|_{p,\Omega(t)}^p + c \|\mathcal{D}(\mathbf{a})\|_{2,\Omega(t)}^2 + c \|\mathcal{D}(\mathbf{a})\|_{p,\Omega(t)}^p \\ & \leq \varepsilon \|\mathcal{D}(\mathbf{u}^T)\|_{2,\Omega(t)}^2 + \varepsilon \|\mathcal{D}(\mathbf{u}^T)\|_{p,\Omega(t)}^p + c\alpha t, \end{aligned}$$

we arrive at

$$\int_{\eta-1}^{\eta} (|I_5| + |I_6|) dt \leq \varepsilon \int_{\eta-1}^{\eta} \|\mathcal{D}(\mathbf{u}^T)\|_{2,\Omega(t)}^2 dt + \varepsilon \int_{\eta-1}^{\eta} \|\mathcal{D}(\mathbf{u}^T)\|_{p,\Omega(t)}^p dt + c\alpha\eta, \quad (6.9)$$

Collecting all above estimates yields

$$\begin{aligned} & \int_{\eta-1}^{\eta} (\|\nabla \mathbf{u}^T\|_{2,\Omega(t)}^2 + \|\nabla \mathbf{u}^T\|_{p,\Omega(t)}^p) dt \\ & \leq c(\|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^+}^3 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^+}^p) + c(\|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^-}^3 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^-}^p) + c\alpha\eta + c\alpha. \end{aligned} \quad (6.10)$$

Let $y(t) = \int_{\Omega(t)} (|\nabla \mathbf{u}^T|^2 + |\nabla \mathbf{u}^T|^p) dx$ and $z(\eta) = \int_{\eta-1}^{\eta} y(t) dt$. Then from (6.10)

$$z(\eta) \leq c_3(z'(\eta) + z'(\eta)^{\frac{3}{2}}) + c_4\alpha\eta + c_5\alpha. \quad (6.11)$$

To apply Lemma 3.3 (i), we set $\Psi(\tau) = c_3(\tau + \tau^{\frac{3}{2}})$, $\delta = \frac{1}{2}$, $t_0 = 1$, and $\varphi(\eta) = 2c_4\alpha\eta + 2c_5\alpha$, where $c_5 > 0$ satisfies

$$c_4\alpha + c_5\alpha \geq \Psi(2c_4\alpha). \quad (6.12)$$

In addition, it follows from the proof of Theorem 4.1 that

$$z(T) \leq \varphi(T). \quad (6.13)$$

Therefore, according (6.11)-(6.13) and Lemma 3.3 (i), we arrive at

$$y(\eta-1) \leq z(\eta) \leq \varphi(\eta), \quad (6.14)$$

which means that for any $t \in [1, T]$,

$$y(t) \leq 2c_4\alpha(t+1) + 2c_5\alpha. \quad (6.15)$$

Case II. $\mu_0 = 0, p > 2$

As in Case I, by the same calculation we can get

$$\left| \int_{\eta-1}^{\eta} I_1 dx_1 \right| + \left| \int_{\eta-1}^{\eta} I_2 dx_1 \right| \leq c \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^+}^p + c\alpha. \quad (6.16)$$

Next, we estimate I_3 as in Case I. Note that

$$\begin{aligned} & \left| \int_{\eta-1}^{\eta} \int_{\Sigma(t)} (\mathbf{a} \cdot \mathbf{u}^T)(\mathbf{a} \cdot \mathbf{e}_1) dy dt + \int_{\eta-1}^{\eta} \int_{\Sigma(t)} \frac{(\mathbf{u}^T)^2}{2} \mathbf{v}^T \cdot \mathbf{e}_1 dy dt \right| \\ & \leq \|\mathbf{a}\|_{2p',\omega_{\eta}^+}^2 \|\mathbf{u}^T\|_{p,\omega_{\eta}^+} + \|\mathbf{u}^T\|_{2p',\omega_{\eta}^+}^2 \|\mathbf{v}^T\|_{p,\omega_{\eta}^+} \\ & \leq c(\|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^+}^3 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^+}^2 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^+}). \end{aligned}$$

Similarly,

$$\begin{aligned} & \left| \int_{\eta-1}^{\eta} \int_{\Sigma(-t)} (\mathbf{a} \cdot \mathbf{u}^T) (\mathbf{a} \cdot \mathbf{e}_1) dy dt + \int_{\eta-1}^{\eta} \int_{\Sigma(-t)} \frac{(\mathbf{u}^T)^2}{2} \mathbf{v}^T \cdot \mathbf{e}_1 dy dt \right| \\ & \leq c(\|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^-}^3 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^-}^2 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^-}). \end{aligned}$$

In addition,

$$\begin{aligned} \left| \int_{\Omega(t)} \mathbf{u}^T \cdot \nabla \mathbf{u}^T \cdot \mathbf{a} dx \right| & \leq \varepsilon \int_{\Omega(t)} |\mathbf{u}^T|^p dx + \varepsilon \int_{\Omega(t)} |\nabla \mathbf{u}^T|^p dx + c \int_{\Omega(t)} |\mathbf{a}|^{\frac{p}{p-2}} dx \\ & \leq \varepsilon \int_{\Omega(t)} |\nabla \mathbf{u}^T|^p dx + c\alpha t \end{aligned}$$

and

$$\begin{aligned} \left| \int_{\Omega(t)} \mathbf{a} \cdot \nabla \mathbf{a} \cdot \mathbf{u}^T dx \right| & \leq c \int_{\Omega(t)} |\mathbf{a} \cdot \nabla \mathbf{a}|^{p'} dx + \varepsilon \int_{\Omega(t)} |\mathbf{u}^T|^p dx \\ & \leq \varepsilon \int_{\Omega(t)} |\nabla \mathbf{u}^T|^p dx + c\alpha t. \end{aligned}$$

Hence it follows from the expression of I_3 and the estimates above that

$$\begin{aligned} \left| \int_{\eta-1}^{\eta} I_3 dt \right| & \leq \varepsilon \int_{\eta-1}^{\eta} \left(\int_{\Omega(t)} |\nabla \mathbf{u}^T|^p dx \right) dt + c(\|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^+}^3 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^+}^2 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^+}) \\ & \quad + c(\|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^-}^3 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^-}^2 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^-}) + c\alpha\eta. \end{aligned} \tag{6.17}$$

In addition, we can find a function $\mathbf{w} \in W_0^{1,p}(\omega_{\eta}^{\pm})$ such that $\operatorname{div} \mathbf{w} = u_1^T$ in ω_{η}^{\pm} . Then we have that

$$\begin{aligned} \left| \int_{\omega_{\eta}^{\pm}} \pi^T u_1^T dx \right| & = \left| \mu_1 \int_{\omega_{\eta}^{\pm}} |\mathcal{D}(\mathbf{v}^T)|^{p-2} \mathcal{D}(\mathbf{v}^T) : \mathcal{D}(\mathbf{w}) dx - \int_{\omega_{\eta}^{\pm}} \mathbf{v}^T \cdot \nabla \mathbf{w} \cdot \mathbf{v}^T dx \right| \\ & \leq \|\mathcal{D}(\mathbf{v}^T)\|_{p,\omega_{\eta}^{\pm}}^{p-1} \|\mathcal{D}(\mathbf{w})\|_{p,\omega_{\eta}^{\pm}} + \|\mathbf{v}^T\|_{4,\omega_{\eta}^{\pm}}^2 \|\nabla \mathbf{w}\|_{2,\omega_{\eta}^{\pm}} \\ & \leq \|\mathcal{D}(\mathbf{v}^T)\|_{p,\omega_{\eta}^{\pm}}^p + \|\mathcal{D}(\mathbf{w})\|_{p,\omega_{\eta}^{\pm}}^p + \|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{\pm}}^3 + \|\mathbf{a}\|_{4,\omega_{\eta}^{\pm}}^2 \|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^{\pm}} \\ & \leq c(\|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^{\pm}}^p + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^{\pm}}^3) + c\alpha. \end{aligned}$$

This yields

$$\left| \int_{\eta-1}^{\eta} I_4 dt \right| \leq c(\|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^+}^p + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^+}^3) + c(\|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^-}^p + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^-}^3) + c\alpha. \tag{6.18}$$

Finally, due to

$$\begin{aligned} & \left| \mu_1 \int_{\Omega(t)} |\mathcal{D}(\mathbf{v}^T)|^{p-2} \mathcal{D}(\mathbf{v}^T) : \mathcal{D}(\mathbf{a}) dx \right| \\ & \leq \varepsilon \|\mathcal{D}(\mathbf{v}^T)\|_{p,\Omega(t)}^p + c \|\mathcal{D}(\mathbf{a})\|_{p,\Omega(t)}^p \\ & \leq \varepsilon \|\mathcal{D}(\mathbf{u}^T)\|_{p,\Omega(t)}^p + c\alpha t, \end{aligned}$$

we arrive at

$$\int_{\eta-1}^{\eta} (|I_5| + |I_6|) dt \leq \varepsilon \int_{\eta-1}^{\eta} \|\mathcal{D}(\mathbf{u}^T)\|_{p,\Omega(t)}^p dt + c\alpha\eta, \quad (6.19)$$

Collecting all above estimates yields

$$\int_{\eta-1}^{\eta} \|\nabla \mathbf{u}^T\|_{p,\Omega(t)}^p dt \leq c(\|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^+}^3 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^+}^p) + c(\|\nabla \mathbf{u}^T\|_{2,\omega_{\eta}^-}^3 + \|\nabla \mathbf{u}^T\|_{p,\omega_{\eta}^-}^p) + c\alpha\eta + c\alpha. \quad (6.20)$$

Let $y(t) = \int_{\Omega(t)} |\nabla \mathbf{u}^T|^p dx$ and $z(\eta) = \int_{\eta-1}^{\eta} y(t) dt$. Then

$$z(\eta) \leq c_6(z'(\eta) + z'(\eta)^{\frac{3}{p}}) + c_7\alpha\eta + c_8\alpha. \quad (6.21)$$

To apply Lemma 3.3 (i), we set $\Psi(\tau) = c_6(\tau + \tau^{\frac{3}{p}})$, $\delta = \frac{1}{2}$, $t_0 = 1$, and $\varphi(\eta) = 2c_7\alpha\eta + 2c_8\alpha$, where c_8 satisfies

$$c_7\alpha + c_8\alpha \geq \Psi(2c_7\alpha). \quad (6.22)$$

In addition, it follows from the proof of Theorem 4.1 that

$$z(T) \leq \varphi(T). \quad (6.23)$$

Therefore, according to (6.22)-(6.24) and Lemma 3.3 (i), we arrive at

$$y(\eta - 1) \leq z(\eta) \leq \varphi(\eta), \quad (6.24)$$

which means that for any $t \in [1, T]$,

$$y(t) \leq 2c_7\alpha(t + 1) + 2c_8\alpha. \quad (6.25)$$

6.2 Part 2. Existence of solution v to problem (2.4)

Theorem 6.1. *Let $\mu_0 > 0$ and $p > 1$ or $\mu_0 = 0$ and $p > 2$. Then problem (2.4) at least has a weak solution.*

Remark 6.1. *Here point out that in the case of $\mu_0 = 0$ and $p > 2$, Theorem 6.1 has been proved in [12] by different methods.*

Proof. We just only treat the case of $\mu_0 = 0$ and $p > 2$, the treatment for $\mu_0 > 0$ and $p > 1$ is similar. Let $T = k$ and $\mathbf{u}^k = 0$ in $\Omega \setminus \Omega_k$. By (6.25) and a diagonalization process, we obtain a subsequence $\{\mathbf{u}^k\}$, which is still denoted by $\{\mathbf{u}^k\}$, and a vector filed $\mathbf{u} \in W_{loc}^{1,p}(\bar{\Omega})$ such that for any $t > 0$,

$$\begin{aligned} \mathbf{u}^k &\rightharpoonup \mathbf{u} \quad \text{in } W^{1,p}(\Omega(t)), \\ \mathbf{u}^k &\rightarrow \mathbf{u} \quad \text{in } L^p(\Omega(t)). \end{aligned} \quad (6.26)$$

Next we show that for all compact subset $K \subset\subset \Omega$,

$$|\mathcal{D}(\mathbf{u}^k) + \mathcal{D}(\mathbf{a})|^{p-2}(\mathcal{D}(\mathbf{u}^k) + \mathcal{D}(\mathbf{a})) \rightharpoonup |\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2}(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})) \quad \text{in } L^{p'}(K). \quad (6.27)$$

In fact, from Theorem 5.1 and (6.25), we have that there is a constant $c(K)$ which is independent of k

$$\|\nabla \mathbf{u}^k\|_{\kappa,p,K} \leq c(K). \quad (6.28)$$

Thus, by the Rellich-Kondrachov theorem we have that there exist a subsequence, which we still denote $\{\mathbf{u}^k\}$, and $\bar{\mathbf{u}} \in W_{loc}^{1,p}(\Omega)$ such that

$$\mathbf{u}^k \rightarrow \bar{\mathbf{u}} \quad \text{in } W^{1,p}(K). \quad (6.29)$$

Hence,

$$\nabla \mathbf{u}^k \rightarrow \nabla \bar{\mathbf{u}} \quad \text{in } L^p(K). \quad (6.30)$$

On the other hand,

$$\nabla \mathbf{u}^k \rightharpoonup \nabla \mathbf{u} \quad \text{in } L^p(K). \quad (6.31)$$

Hence we have $\nabla \bar{\mathbf{u}} = \nabla \mathbf{u}$ and

$$\mathcal{D}(\mathbf{u}^k) \rightarrow \mathcal{D}(\mathbf{u}) \quad \text{a.e. in K.} \quad (6.32)$$

Note that

$$\|\mathcal{D}\mathbf{u}^k + \mathcal{D}\mathbf{a}|^{p-2}(\mathcal{D}\mathbf{u}^k + \mathcal{D}\mathbf{a})\|_{p',K} \leq \|\mathcal{D}\mathbf{u}^k + \mathcal{D}\mathbf{a}\|_{p,K}^{p-1} \leq c(K). \quad (6.33)$$

Therefore, according to Lemma I.1.3 of [20], we have

$$|\mathcal{D}\mathbf{u}^k + \mathcal{D}\mathbf{a}|^{p-2}(\mathcal{D}\mathbf{u}^k + \mathcal{D}\mathbf{a}) \rightharpoonup |\mathcal{D}\mathbf{u} + \mathcal{D}\mathbf{a}|^{p-2}(\mathcal{D}\mathbf{u} + \mathcal{D}\mathbf{a}) \quad \text{in } L^{p'}(K). \quad (6.34)$$

From (6.26) and (6.34), let $k \rightarrow \infty$, one derives that for all $\psi \in \mathcal{D}(\Omega)$

$$\begin{aligned} \mu_0(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a}), \mathcal{D}(\psi)) + \mu_1(|\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})|^{p-2}(\mathcal{D}(\mathbf{u}) + \mathcal{D}(\mathbf{a})), \mathcal{D}(\psi)) \\ = (\mathbf{u} \cdot \nabla \psi, \mathbf{u}) + (\mathbf{u} \cdot \nabla \psi, \mathbf{a}) + (\mathbf{a} \cdot \nabla \psi, \mathbf{u}) + (\mathbf{a} \cdot \nabla \psi, \mathbf{a}). \end{aligned} \quad (6.35)$$

This means that \mathbf{u} is a weak solution of problem (2.4).

6.3 Part 3. Uniqueness

At first, as in [18], we show that the dissipation of the solution \mathbf{v} to Problem (2.4) is distributed uniformly along Ω .

Lemma 6.1. *Assume that \mathbf{v} is a solution of problem (2.4). Then there exists a fixed constant $c_9 > 0$ such that if $\mu_0 > 0$, $p > 1$,*

$$\int_{\tau}^{\tau+1} dx_1 \int_{\Sigma(x_1)} (|\nabla \mathbf{v}^T|^2 + |\nabla \mathbf{v}^T|^p) dy \leq c_9 \alpha; \quad (6.36)$$

and if $\mu_0 = 0$, $p > 2$,

$$\int_{\tau}^{\tau+1} dx_1 \int_{\Sigma(x_1)} |\nabla \mathbf{v}^T|^p dy \leq c_9 \alpha. \quad (6.37)$$

Proof. Let

$$\Omega_{\tau}(t) = \{x \in \Omega : \tau - t < x_1 < \tau + t\},$$

$$\begin{aligned} y_{\tau}(t) &= \int_{\Omega_{\tau}(t)} (|\nabla \mathbf{u}^T|^2 + |\nabla \mathbf{u}^T|^p) dx, & \text{if } \mu_0 > 0, p > 1, \\ y_{\tau}(t) &= \int_{\Omega_{\tau}(t)} |\nabla \mathbf{u}^T|^p dx, & \text{if } \mu_0 = 0, p > 2, \end{aligned}$$

and

$$z_\tau(\eta) = \int_{\eta-1}^{\eta} y_\tau(t) dt \quad \text{for } \eta \geq 1.$$

Since

$$z_\tau(\tau) \leq y_\tau(\tau) \leq \varphi(2\tau + 1),$$

similarly to the proof of (6.15) or (6.25), we have that for any $\eta \in [1, \tau]$,

$$z_\tau(\eta) \leq \varphi(2\eta + 1),$$

where the definition of $\varphi(\eta)$ is given in (6.11) or (6.22). Therefore,

$$y_\tau\left(\frac{1}{2}\right) \leq \varphi(2) \leq c\alpha,$$

while, by Lemma 3.5, the same inequalities also hold for the vector \mathbf{a} , hence Lemma 6.1 is proved. \square

The following result comes from Lemma 4.2 of [21] (one can see also Proposition 4.3 of [12]).

Lemma 6.2. *Assume that \mathbf{v} is a divergence free vector field in $W_{loc}^{1,p}(\bar{\Omega})$ vanishing on $\partial\Omega$ and satisfying (6.34) for $\mu_0 = 0$ and $p > 2$. If*

$$|\mathcal{D}(\mathbf{v})(x_1, y)| \geq c|y|^{\frac{1}{p-1}} \tag{6.38}$$

holds for some positive number $c > 0$, then there is a constant $C > 0$ such that for all $\mathbf{w} \in \mathcal{D}_{loc}^{1,p}(\Omega)$ and $t > 0$,

$$|(\mathbf{w} \cdot \nabla \mathbf{v}, \mathbf{w})_{\Omega(t)}| \leq C\alpha^{\frac{1}{p}} \|\mathcal{D}(\mathbf{v})\|^{\frac{1}{p-1}} \|\mathcal{D}(\mathbf{w})\|_{2,\Omega(t)}^2, \tag{6.39}$$

where the number α has been defined in Lemma 3.5.

Theorem 6.2. *Assume that the flux $|\alpha_i|$ ($1 \leq i \leq N$) is sufficient small and there exists a solution \mathbf{v} satisfying (6.38) to problem (2.4). Then the solution \mathbf{v} of problem (2.4) is unique for $p > 2$ and $\mu_0 = 0$. When $\mu_0 > 0$, even if the assumption (6.38) on \mathbf{v} is removed, then the solution \mathbf{v} of problem (2.4) exists uniquely for $p > 1$.*

Remark 6.2. *Here point out that in the case of $\mu_0 = 0$ and $p > 2$, Theorem 6.2 has been proved in [12].*

Proof. Assume that \mathbf{v}_1 and \mathbf{v}_2 are the solutions of problem (2.4). Let $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2$. Then

$$\begin{aligned} -\mu_0 \operatorname{div}(\mathcal{D}(\mathbf{w})) - \mu_1 \operatorname{div}(\mathcal{D}(\mathbf{w}) + \mathcal{D}(\mathbf{v}_2))^{p-2} (\mathcal{D}(\mathbf{w}) + \mathcal{D}(\mathbf{v}_2)) - |\mathcal{D}(\mathbf{v}_2)|^{p-2} \mathcal{D}(\mathbf{v}_2) \\ + \mathbf{w} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}_2 + \mathbf{v}_2 \cdot \nabla \mathbf{w} + \nabla(\pi_1 - \pi_2) = 0, \end{aligned} \tag{6.40}$$

From (6.40) and the proof of (6.15) and (6.25), we just need to estimate $(\mathbf{w} \cdot \nabla \mathbf{v}_2, \mathbf{w})_{\Omega(t)}$.

If $\mu_0 = 0$, $p > 2$, since \mathbf{v}_2 satisfies (6.38), then according to Lemma 6.2, we have

$$(\mathbf{w} \cdot \nabla \mathbf{v}_2, \mathbf{w})_{\Omega(t)} \leq C\alpha^{\frac{1}{p}} \|\mathcal{D}(\mathbf{v}_2)\|^{\frac{p-2}{2}} \|\mathcal{D}(\mathbf{w})\|_{2,\Omega(t)}^2. \tag{6.41}$$

If $\mu_0 > 0$, without loss of generality, we set $t = k$, by (6.36) we can get that

$$|(\mathbf{w} \cdot \nabla \mathbf{v}_2, \mathbf{w})_{\Omega(k)}| = \left| \sum_{j=-k}^{k-1} \int_j^{j+1} \int_{\Sigma(x_1)} \mathbf{w} \cdot \nabla \mathbf{v}_2 \cdot \mathbf{w} dx \right|$$

$$\leq \sum_{j=-k}^{k-1} \left(\int_j^{j+1} \int_{\Sigma(x_1)} |\nabla \mathbf{v}_2|^2 dx \right)^{1/2} \left(\int_j^{j+1} \int_{\Sigma(x_1)} |\mathbf{w}|^4 dx \right)^{1/2} \leq C\alpha^{\frac{1}{2}} \|\nabla \mathbf{w}\|_{2,\Omega(t)}^2. \quad (6.42)$$

By (6.41) and (6.42), similarly to the proof of (6.11) or (6.21), we have

$$z(\eta) \leq \Psi(z'(\eta)), \quad (6.43)$$

where

$$z(\eta) = \int_{\eta-1}^{\eta} y(t) dt \quad \text{with} \quad y(t) = \int_{\Omega(t)} (|\nabla \mathbf{w}^T|^2 + |\nabla \mathbf{w}^T|^p) dx, \quad \text{if } \mu_0 > 0, p > 1,$$

and

$$z(\eta) = \int_{\eta-1}^{\eta} y(t) dt \quad \text{with} \quad y(t) = \int_{\Omega(t)} |\nabla \mathbf{w}^T|^p dx, \quad \text{if } \mu_0 = 0, p > 2,$$

and the definition of function $\Psi(\tau)$ is given in (6.12) or (6.22). If $z(t)$ is not identically zero, it then follows from Lemma 3.3 (iii) that when $\mu_0 = 0$,

$$\begin{aligned} \liminf_{t \rightarrow \infty} t^{\frac{-3}{3-p}} z(t) &> 0 & \text{if } p < 3, \\ \liminf_{t \rightarrow \infty} e^{-t} z(t) &> 0 & \text{if } p \geq 3; \end{aligned}$$

when $\mu_0 > 0$,

$$\liminf_{t \rightarrow \infty} t^{-3} z(t) > 0.$$

These contradict with (2.4)₅. Hence, $z \equiv 0$, and further $\mathbf{v}_1 = \mathbf{v}_2$. This completes the proof of Theorem 6.1. \square

Remark 6.3. When Ω admits straight outlets, then the corresponding solution \mathbf{v} of Leray Problem (1.8) exactly satisfies (6.38) (see [21]). Therefore, in this case, if the flux is sufficiently small, then the solution of problem (2.4) coincides with the solution of Leray Problem (1.8).

7 Solvability of Ladyzhenskaya-Solonnikov Problem II (2.5)

To solve Ladyzhenskaya-Solonnikov Problem II (2.5) under some suitable conditions, based on Sections 4-5, we will take the following two parts.

7.1 Part 1. Existence of solution \mathbf{v} to problem (2.5)

In the following, we will assume that

$$\Omega_i = \{x^i = (x_1^i, y^i) : 0 < x_1^i < \infty, |y^i| < g_i(x_1^i)\}. \quad (7.1)$$

Hence it follows from the definition of $I_i(t)$ in (2.5) that

$$\begin{aligned} I_i(t) &= \int_0^t g_i^{-(1+d)}(s) ds + \int_0^t g_i^{(1-p)d-1}(s) ds, & \text{if } \mu_0 > 0; \\ I_i(t) &= \int_0^t g_i^{(1-p)d-1}(s) ds, & \text{if } \mu_0 = 0. \end{aligned} \quad (7.2)$$

As in [18], we suppose that some outlets are “narrow”, namely,

$$I_i(\infty) = \infty, \quad i = 1, \dots, m, \quad (7.3)$$

meanwhile other outlets are “wide”, that is,

$$I_i(\infty) < \infty, \quad i = m+1, \dots, N. \quad (7.4)$$

In addition, we assume that

$$|g'_i(t)| \leq (2\beta)^{-1} \text{ and } g_i(t) \geq g_0 > 0. \quad (7.5)$$

In this case, $g_i(t)$ has the properties as follows

$$\begin{aligned} g_i(t) &\leq g_i(0) + (2\beta)^{-1}t, \\ t - \beta g_i(t) &\geq \frac{1}{2}t - \beta g_i(0), \\ t - \beta g_i(t) &\geq 0 \quad \text{for } t \geq t^* = 2\beta \max_{i=1,\dots,m} g_i(0), \\ \frac{1}{2}g_i(t) &\leq g(s) \leq \frac{3}{2}g_i(t) \quad \text{for } s \in [t - \beta g_i(t), t] \text{ and } t \geq t^*. \end{aligned}$$

Let

$$\Omega(t) = \Omega_0 \cup \left(\cup_{i=1}^m \Omega_i(h_i(t)) \right) \cup \left(\cup_{i=m+1}^N \Omega_i \right),$$

where $\Omega_i(s) = \{x \in \Omega_i : 0 < x_1^i < s\}$ for $1 \leq i \leq N$, and the function $h_i(t)$ ($1 \leq i \leq m$) is determined by the equations

$$\begin{cases} h'_i(t) = g_i^{\frac{7-d}{3}}(h_i(t)), & \text{if } \mu_0 > 0; \\ h'_i(t) = g_i^{\frac{2p+3}{3}-(1-\frac{p}{3})d}(h_i(t)), & \text{if } \mu_0 = 0. \end{cases} \quad (7.6)$$

Actually, $h_i(t)$ ($1 \leq i \leq m$) is the inverse function of $k_i(t) = \int_0^t g_i^{\frac{d-7}{3}}(s)ds$ (for the case of $\mu_0 > 0$) or $k_i(t) = \int_0^t g_i^{-\frac{2p+3}{3}+(1-\frac{p}{3})d}(s)ds$ (for the case of $\mu_0 = 0$) for $t > 0$. It is worth noting that we need $h_i(t) \rightarrow \infty$ as $t \rightarrow \infty$. If $p \geq \frac{4+2d}{3d}$, since $\frac{7-d}{3} \leq (p-1)d+1$ and $I_i(\infty) = \infty$, automatically, we have that $h_i(t) \rightarrow \infty$ as $t \rightarrow \infty$. While, if $1 < p < \frac{4+2d}{3d}$, we will assume that

$$\begin{aligned} h_i(t) &\rightarrow \infty \text{ as } t \rightarrow \infty, \\ \text{or } k_i(t) &= \int_0^t g_i^{\frac{d-7}{3}}(s)ds \rightarrow \infty \text{ as } t \rightarrow \infty. \end{aligned} \quad (7.7)$$

Since $h_i(t) \rightarrow \infty$ as $t \rightarrow \infty$, we can introduce the truncated domains

$$\Omega(t, R) = \Omega_0 \cup \left(\cup_{i=1}^m \Omega_i(h_i(t)) \right) \cup \left(\cup_{i=m+1}^N \Omega_i(R) \right)$$

and the “truncating” functions $\zeta(x, t)$

$$\zeta(x, t) = \begin{cases} \beta & \text{for } x \in \Omega_0 \cup \left(\cup_{i=1}^m \Omega_i(h_i(t) - \beta g_i(h_i(t))) \right) \cup \left(\cup_{i=m+1}^N \Omega_i \right), \\ 0 & \text{for } x \in \cup_{i=1}^m (\Omega_i \setminus \Omega_i(h_i(t))), \\ \frac{h_i(t) - x_1^i}{g_i(h_i(t))} & \text{for } x \in \omega_i(t), \end{cases} \quad (7.8)$$

where $\omega_i(t) = \{x \in \Omega : h_i(t) - \beta g_i(h_i(t)) < x_1^i < h_i(t)\}$ for $1 \leq i \leq m$. It is easy to check that

$$|\nabla_x \zeta| \leq [g_i(h_i(t))]^{-1} \quad (7.9)$$

and

$$\frac{\partial \zeta(x, t)}{\partial t} = \frac{h'_i(t)}{g_i(h_i(t))} \left[1 - \frac{h_i(t) - x_1^i}{g_i(h_i(t))} g'_i(h_i(t)) \right] \geq \frac{1}{2} \frac{h'(t)}{g_i(h_i(t))}. \quad (7.10)$$

This yields

$$\begin{aligned} \frac{\partial \zeta(x, t)}{\partial t} &\geq \frac{1}{2} g_i^{\frac{4-d}{3}}(h_i(t)), & \text{if } \mu_0 > 0; \\ \frac{\partial \zeta(x, t)}{\partial t} &\geq \frac{1}{2} g_i^{\frac{2p}{3} - (1-\frac{p}{3})d}(h_i(t)), & \text{if } \mu_0 = 0. \end{aligned} \quad (7.11)$$

We now set

$$\begin{aligned} Q(t) &= \int_{\Omega(t)} (|\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}|^p) \zeta(x, t) dx, & \text{if } \mu_0 > 0; \\ Q(t) &= \int_{\Omega(t)} |\nabla \mathbf{v}|^p \zeta(x, t) dx, & \text{if } \mu_0 = 0, \end{aligned} \quad (7.12)$$

and

$$\begin{aligned} \phi(t) &= \sum_{i=1}^m \int_0^{h_i(t)} (g^{-(d+1)}(s) + g^{(1-p)d-1}(s)) ds, & \text{if } \mu_0 > 0; \\ \phi(t) &= \sum_{i=1}^m \int_0^{h_i(t)} g^{(1-p)d-1}(s) ds, & \text{if } \mu_0 = 0. \end{aligned} \quad (7.13)$$

Then we have

Theorem 7.1. if $\mu_0 > 0$, $p \geq \frac{4+2d}{3d}$ or $\mu_0 = 0$, $2 < p \leq 3 - \frac{2}{d}$, under the assumptions (7.3)-(7.5) and (7.7), there exists at least one solution \mathbf{u} satisfying (2.5)₁ – (2.5)₄, moreover;

$$Q(t) \leq \bar{c}_{10} \alpha \phi(t) + \bar{c}_{11} \alpha, \quad \text{for any } t \geq \bar{t} = \max_{i=1,\dots,m} k_i(t^*), \quad (7.14)$$

where $\bar{c}_{10} > 0$ and $\bar{c}_{11} > 0$ are some constants, and the definitions of $Q(t)$ and $\phi(t)$ are given in (7.12) and (7.13) respectively. When $1 < p < \frac{4+2d}{3d}$, if we assume that $g_i(t)$ satisfies the condition

$$|g'_i(t) g_i^{\frac{4-pd}{2}}(t)| < \gamma, \quad (7.15)$$

where $\gamma > 0$ is a sufficient small constant, then the corresponding conclusion (7.14) still holds.

Proof. Let $\mathbf{v}^{T,R} = \mathbf{u}^{T,R} + \mathbf{a}$ be the solution of problem (2.5) in the bounded domains $\Omega(T, R)$. Note that the existence of $\mathbf{v}^{T,R}$ is ensured by Theorem 4.1. Moreover, since $|\mathbf{a}(x_1, x')| \leq c g_i^{-(d-1)}(x_1)$ and $|\mathbf{a}(x_1, x')| \leq c g_i^{-d}(x_1)$ for $x \in \Omega_i$, if $\mu_0 > 0$, by (4.10) we have

$$\|\nabla \mathbf{u}^{T,R}\|_{2,\Omega(T,R)}^2 + \|\nabla \mathbf{u}^{T,R}\|_{p,\Omega(T,R)}^p \leq c \alpha \phi(T).$$

Let $R \rightarrow \infty$, we can find a vector function \mathbf{u}^T such that $\mathbf{v}^T = \mathbf{u}^T + \mathbf{a}$ is a solution of Problem (2.5) in $\Omega(T)$ and

$$\|\nabla \mathbf{u}^T\|_{2,\Omega(T)}^2 + \|\nabla \mathbf{u}^T\|_{p,\Omega(T)}^p \leq c\alpha\phi(T).$$

If $\mu_0 = 0$, by (4.29) we obtain that

$$\|\nabla \mathbf{u}^T\|_{p,\Omega(T)}^p \leq c\left(\|\nabla \mathbf{a}\|_{p,\Omega(T)}^p + \|\mathbf{a}\|_{\frac{p(d-2)}{(p-2)(d-1)},\Omega(T)}^{\frac{p(d-2)}{(p-2)(d-1)}} + \|\mathbf{a}\|_{2p',\Omega(T)}^{2p'}\right).$$

Since $2 < p \leq 3 - \frac{2}{d}$, a direct calculation derives

$$\|\nabla \mathbf{a}\|_{p,\Omega(T)}^p + \|\mathbf{a}\|_{\frac{p(d-2)}{(p-2)(d-1)},\Omega(T)}^{\frac{p(d-2)}{(p-2)(d-1)}} + \|\mathbf{a}\|_{2p',\Omega(T)}^{2p'} \leq c\alpha\phi(T).$$

Therefore,

$$\|\nabla \mathbf{u}^T\|_{p,\Omega(T)}^p \leq c\alpha\phi(T).$$

Next we derive (7.14). Multiplying the equation in (2.5) by $\mathbf{u}(x)\zeta(x,t)$ and integrating over $\Omega(t)$ yield

$$\int_{\Omega(t)} (\mu_0 \mathcal{D}(\mathbf{v}) + \mu_1 |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v})) : \mathcal{D}(\zeta \mathbf{v}) dx + \int_{\Omega(t)} \mathbf{v} \cdot \nabla \mathbf{v} \cdot \zeta \mathbf{u} dx + \int_{\Omega(t)} \nabla \pi \cdot \zeta \mathbf{u} dx = 0, \quad (7.16)$$

where and below the superscript T of \mathbf{u} is omitted for notational convenience. It is easy to check that

$$\int_{\Omega(t)} \mathcal{D}(\mathbf{v}) : \mathcal{D}(\zeta \mathbf{u}) dx = \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{u})|^2 dx + \int_{\Omega(t)} \zeta \mathcal{D}(\mathbf{a}) : \mathcal{D}(\mathbf{u}) dx + \int_{\Omega(t)} \mathcal{D}(\mathbf{v}) : \text{Sym}(\mathbf{u} \otimes \nabla \zeta) dx$$

and

$$\begin{aligned} \int_{\Omega(t)} |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v}) : \mathcal{D}(\zeta \mathbf{u}) dx &= \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v}) : \mathcal{D}(\mathbf{u}) dx \\ &\quad + \int_{\Omega(t)} |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v}) : \text{Sym}(\mathbf{u} \otimes \nabla \zeta) dx. \end{aligned}$$

In addition, by integrating by parts we deduce

$$\begin{aligned} \int_{\Omega(t)} \mathbf{v} \cdot \nabla \mathbf{v} \cdot \zeta \mathbf{u} dx &= \int_{\Omega(t)} \mathbf{v} \cdot \nabla \mathbf{u} \cdot \zeta \mathbf{u} dx + \int_{\Omega(t)} \mathbf{v} \cdot \nabla \mathbf{a} \cdot \zeta \mathbf{u} dx \\ &= -\frac{1}{2} \int_{\Omega(t)} \mathbf{u}^2 \mathbf{v} \cdot \nabla \zeta dx + \int_{\Omega(t)} \mathbf{v} \cdot \nabla \mathbf{a} \cdot \zeta \mathbf{u} dx \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega(t)} \mathbf{v} \cdot \nabla \mathbf{a} \cdot \zeta \mathbf{u} dx &= \int_{\Omega(t)} \mathbf{u} \cdot \nabla \mathbf{a} \cdot \zeta \mathbf{u} dx + \int_{\Omega(t)} \mathbf{a} \cdot \nabla \mathbf{a} \cdot \zeta \mathbf{u} dx \\ &= - \int_{\Omega(t)} (\mathbf{u} \cdot \nabla \zeta)(\mathbf{u} \cdot \mathbf{a}) dx - \int_{\Omega(t)} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{a} \zeta dx - \int_{\Omega(t)} (\mathbf{a} \cdot \nabla \zeta)(\mathbf{u} \cdot \mathbf{a}) dx - \int_{\Omega(t)} \mathbf{a} \cdot \nabla \mathbf{u} \cdot \mathbf{a} \zeta dx. \end{aligned}$$

Meanwhile,

$$\int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v}) : \mathcal{D}(\mathbf{u}) dx - \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{a})|^{p-2} \mathcal{D}(\mathbf{a}) : \mathcal{D}(\mathbf{u}) dx \geq \delta \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{u})|^p dx,$$

and

$$\left| \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{a})|^{p-2} \mathcal{D}(\mathbf{a}) : \mathcal{D}(\mathbf{u}) dx \right| \leq c \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{a})|^p dx + \varepsilon \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{u})|^p dx,$$

which yields

$$\int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v}) : \mathcal{D}(\mathbf{u}) dx \geq c \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{u})|^p dx - \frac{\delta}{2} \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{a})|^p dx.$$

Hence,

$$\mu_0 \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{u})|^2 dx + \frac{\delta}{2} \mu_1 \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{u})|^p dx \leq c \mu_1 \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{a})|^p dx + I_1 + \dots + I_9, \quad (7.17)$$

where

$$\begin{aligned} I_1 &= -\mu_0 \int_{\Omega(t)} \zeta \mathcal{D}(\mathbf{u}) : \mathcal{D}(\mathbf{a}) dx, \\ I_2 &= -\mu_0 \int_{\Omega(t)} \mathcal{D}(\mathbf{v}) : \text{Sym}(\mathbf{u} \otimes \nabla \zeta) dx, \\ I_3 &= -\mu_1 \int_{\Omega(t)} |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v}) : \text{Sym}(\mathbf{u} \otimes \nabla \zeta) dx, \\ I_4 &= \frac{1}{2} \int_{\Omega(t)} \mathbf{u}^2 \mathbf{v} \cdot \nabla \zeta dx, \\ I_5 &= \int_{\Omega(t)} (\mathbf{u} \cdot \nabla \zeta)(\mathbf{u} \cdot \mathbf{a}) dx, \\ I_6 &= \int_{\Omega(t)} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{a} \zeta dx, \\ I_7 &= \int_{\Omega(t)} (\mathbf{a} \cdot \nabla \zeta)(\mathbf{u} \cdot \mathbf{a}) dx, \\ I_8 &= \int_{\Omega(t)} \mathbf{a} \cdot \nabla \mathbf{u} \cdot \mathbf{a} \zeta dx, \\ I_9 &= \int_{\Omega(t)} \pi \mathbf{u} \cdot \nabla \zeta dx. \end{aligned}$$

Next, we treat the terms I_i ($1 \leq i \leq 9$) in two cases as follows:

Case I. $\mu_0 > 0, p > 1$

It is easy to check that

$$|I_1| \leq \varepsilon \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{u})|^2 dx + c \int_{\Omega(t)} \zeta |\mathcal{D}(\mathbf{a})|^2 dx.$$

By Hölder and Poincaré inequality, we have

$$|I_2| \leq \varepsilon \int_{\Omega(t)} |\mathcal{D}(\mathbf{v})|^2 dx + \int_{\Omega(t)} ||\mathbf{u}| \nabla \zeta|^2 dx$$

$$\leq 2\varepsilon \int_{\Omega(t)} |\mathcal{D}(\mathbf{u})|^2 dx + \int_{\omega_i(t)} |\mathcal{D}(\mathbf{u})|^2 dx + 2\varepsilon \int_{\Omega(t)} |\mathcal{D}(\mathbf{a})|^2 dx.$$

Similarly,

$$\begin{aligned} |I_3| &\leq \varepsilon \int_{\Omega(t)} |\mathcal{D}(\mathbf{v})|^p dx + \int_{\Omega(t)} \|\mathbf{u}|\nabla\zeta\|^p dx \\ &\leq 2^{p-1}\varepsilon \int_{\Omega(t)} |\mathcal{D}(\mathbf{u})|^p dx + \int_{\omega_i(t)} |\mathcal{D}(\mathbf{u})|^p dx + 2^{p-1}\varepsilon \int_{\Omega(t)} |\mathcal{D}(\mathbf{a})|^p dx. \end{aligned}$$

Using Hölder inequality and Lemma 3.1, we deduce that

$$\begin{aligned} |I_4| &= \left| \frac{1}{2} \int_{\Omega(t)} \mathbf{u}^2 \mathbf{u} \cdot \nabla \zeta dx + \frac{1}{2} \int_{\Omega(t)} \mathbf{u}^2 \mathbf{a} \cdot \nabla \zeta dx \right| \\ &\leq c \sum_{i=1}^m g_i^{-1}(h_i(t)) \|\mathbf{u}\|_{3,\omega_i(t)}^3 + c \sum_{i=1}^m g_i^{-d}(h_i(t)) \|\mathbf{u}\|_{2,\omega_i(t)}^2 \\ &\leq c \sum_{i=1}^m g_i^{\frac{4-d}{2}}(h_i(t)) \|\nabla \mathbf{u}\|_{2,\omega_i(t)}^3 + c \sum_{i=1}^m g_i^{2-d}(h_i(t)) \|\nabla \mathbf{u}\|_{2,\omega_i(t)}^2. \end{aligned}$$

By Poincaré inequality, we have

$$|I_5| \leq \sum_{i=1}^m g_i^{-d}(h_i(t)) \|\mathbf{u}\|_{2,\omega_i(t)}^2 \leq c \sum_{i=1}^m g_i^{2-d}(h_i(t)) \|\nabla \mathbf{u}\|_{2,\omega_i(t)}^2.$$

It follows from Lemma 3.1 and Lemma 3.5 (iii) and (iv) that

$$\begin{aligned} |I_6| &\leq \int_{\Omega(t)} \mathbf{a}^2 \mathbf{u}^2 \zeta dx + \varepsilon \int_{\Omega(t)} |\nabla \mathbf{u}|^2 \zeta dx \\ &\leq \beta \int_{\Omega_0 \cup \left(\cup_{i=1}^m \Omega_i(h_i(t) - \beta g_i(h(t))) \right) \cup (\cup_{i=m+1}^N \Omega_i)} \mathbf{a}^2 \mathbf{u}^2 dx + \beta \sum_{i=1}^m \int_{\omega_i(t)} \mathbf{a}^2 \mathbf{u}^2 dx + \varepsilon \int_{\Omega(t)} |\nabla \mathbf{u}|^2 \zeta dx \\ &\leq 2\varepsilon \int_{\Omega(t)} |\nabla \mathbf{u}|^2 \zeta dx + \varepsilon \sum_{i=1}^m \int_{\omega_i(t)} |\nabla \mathbf{u}|^2 dx. \end{aligned}$$

By Schwartz inequality and Lemma 3.1, we arrive at

$$\begin{aligned} |I_7| &\leq \sum_{i=1}^m \|\mathbf{u}|\nabla\zeta\|_{2,\omega_i(t)}^2 + \sum_{i=1}^m \|\mathbf{a}\|_{4,\omega_i(t)}^2 \\ &\leq c \sum_{i=1}^m \|\nabla \mathbf{u}\|_{2,\omega_i(t)}^2 + \sum_{i=1}^m \|\mathbf{a}\|_{4,\omega_i(t)}^4. \end{aligned}$$

On the other hand, it is easy to get

$$|I_8| \leq \varepsilon \|\zeta \nabla \mathbf{u}\|_{2,\Omega(t)}^2 + \|\mathbf{a}\|_{4,\Omega(t)}^4.$$

Finally, we estimate I_9 . By Lemma 3.2, we can find $\mathbf{w} \in W_0^{1,2}(\omega_i(t)) \cap W_0^{1,p}(\omega_i(t))$ such that $\operatorname{div} \mathbf{w} = u_1$ (see Remark 3.2), and there is a constant $M(d, p) > 0$ (see Remark 3.1) such that

$$\begin{aligned} \|\nabla \mathbf{w}\|_{2,\omega_i(t)} &\leq M(d, p) \|u_1\|_{2,\omega_i(t)}, \\ \|\nabla \mathbf{w}\|_{p,\omega_i(t)} &\leq M(d, p) \|u_1\|_{p,\omega_i(t)}. \end{aligned} \tag{7.18}$$

In this case,

$$\begin{aligned}
I_9 &= g_i^{-1}(h_i(t)) \int_{\omega_i(t)} \pi u_1 dx \\
&= -g_i^{-1}(h_i(t)) \int_{\omega_i(t)} \mathbf{w} \cdot \nabla \pi dx \\
&= g^{-1}(h_i(t)) \int_{\omega_i(t)} [-\operatorname{div}(\mu_0 \mathcal{D}(\mathbf{v}) + \mu_1 |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v})) \cdot \mathbf{w} + \mathbf{v} \cdot \nabla \mathbf{v} \cdot \mathbf{w}] dx \\
&= g^{-1}(h_i(t)) \int_{\omega_i(t)} [(\mu_0 \mathcal{D}(\mathbf{v}) + \mu_1 |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v})) \cdot \nabla \mathbf{w} + \mathbf{v} \cdot \nabla \mathbf{v} \cdot \mathbf{w}] dx.
\end{aligned} \tag{7.19}$$

Note that by (7.18), one has

$$\left| \int_{\Omega_i(t)} \mathcal{D}(\mathbf{v}) \cdot \nabla \mathbf{w} dx \right| \leq \|\mathcal{D}(\mathbf{v})\|_{2,\omega_i(t)} \|u_1\|_{2,\omega_i(t)}$$

and

$$\left| \int_{\omega_i(t)} |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v}) \cdot \nabla \mathbf{w} dx \right| \leq \|\mathcal{D}(\mathbf{v})\|_{p,\omega_i(t)}^{p-1} \|\mathcal{D}(\mathbf{w})\|_{p,\omega_i(t)} \leq \|\mathcal{D}(\mathbf{v})\|_{p,\omega_i(t)}^{p-1} \|u_1\|_{p,\omega_i(t)}.$$

In addition, by Lemma 3.1,

$$\|u_1\|_{q,\omega_i(t)} \leq c g_i(h_i(t)) \|\nabla \mathbf{u}\|_{q,\omega_i(t)} \quad \text{for any } q > 1. \tag{7.20}$$

Consequently, we have

$$\begin{aligned}
&\left| g^{-1}(h_i(t)) \int_{\omega_i(t)} (\mu_0 \mathcal{D}(\mathbf{v}) + \mu_1 |\mathcal{D}(\mathbf{v})|^{p-2} \mathcal{D}(\mathbf{v})) \cdot \nabla \mathbf{w} dx \right| \\
&\leq c \|\nabla \mathbf{u}\|_{2,\omega_i(t)}^2 + c \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^p + c \|\nabla \mathbf{a}\|_{2,\omega_i(t)}^2 + c \|\nabla \mathbf{a}\|_{p,\omega_i(t)}^p.
\end{aligned} \tag{7.21}$$

On the other hand, by Lemma 3.1 and (7.18),

$$\begin{aligned}
&\left| \int_{\omega_i(t)} \mathbf{v} \cdot \nabla \mathbf{v} \cdot \mathbf{w} dx \right| = \left| \int_{\omega_i(t)} \mathbf{v} \cdot \nabla \mathbf{w} \cdot \mathbf{v} dx \right| \\
&\leq 2 \|\mathbf{u}\|_{4,\omega_i(t)}^2 \|\nabla \mathbf{w}\|_{2,\omega_i(t)} + 2 \|\mathbf{a}\|_{4,\omega_i(t)}^2 \|\nabla \mathbf{w}\|_{2,\omega_i(t)} \\
&\leq c g_i^{\frac{4-d}{2}}(h_i(t)) \|\nabla \mathbf{u}\|_{2,\omega_i(t)}^2 \|u_1\|_{2,\omega_i(t)} + c \|\mathbf{a}\|_{4,\omega_i(t)}^2 \|u_1\|_{2,\omega_i(t)}.
\end{aligned} \tag{7.22}$$

Therefore,

$$\begin{aligned}
|I_9| &= \left| \sum_{i=1}^m \int_{\Omega_i(t)} \pi \mathbf{u} \cdot \nabla \zeta dx \right| \leq c \sum_{i=1}^m \left(\|\nabla \mathbf{u}\|_{2,\omega_i(t)}^2 + \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^p \right. \\
&\quad \left. + g_i^{\frac{4-d}{2}}(h_i(t)) \|\nabla \mathbf{u}\|_{2,\omega_i(t)}^3 + \|\nabla \mathbf{a}\|_{2,\omega_i(t)}^2 + \|\nabla \mathbf{a}\|_{4,\omega_i(t)}^4 + \|\nabla \mathbf{a}\|_{p,\omega_i(t)}^p \right).
\end{aligned} \tag{7.23}$$

Collecting the estimates on I_i ($1 \leq i \leq 9$), together with Korn inequality and (7.17), we have

$$\begin{aligned} \|\nabla \mathbf{u}\|_{2,\Omega(t)}^2 + \|\nabla \mathbf{u}\|_{p,\Omega(t)}^p &\leq c \sum_{i=1}^m \left(\|\nabla \mathbf{u}\|_{2,\omega_i(t)}^2 + g_i^{\frac{4-d}{2}}(h_i(t)) \|\nabla \mathbf{u}\|_{2,\omega_i(t)}^3 + \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^p \right) \\ &\quad + c \|\nabla \mathbf{a}\|_{2,\Omega(t)}^2 + c \|\nabla \mathbf{a}\|_{p,\Omega(t)}^p + c \|\mathbf{a}\|_{4,\Omega(t)}^4. \end{aligned} \quad (7.24)$$

In addition, it is easy to get

$$\frac{dy(t)}{dt} \geq \sum_{i=1}^m \frac{1}{2} g_i^{\frac{4-d}{3}}(h_i(t)) \int_{\omega_i(t)} (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{u}|^p) dx, \quad (7.25)$$

where $y(t) = \int_{\Omega(t)} (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{u}|^p) \zeta(x, t) dx$. Hence

$$\sum_{i=1}^m g_i^{\frac{4-d}{2}}(h_i(t)) \|\nabla \mathbf{u}\|_{\omega_i(t)}^3 \leq c \left(\frac{dy(t)}{dt} \right)^{3/2} \quad (7.26)$$

and

$$\sum_{i=1}^m (\|\nabla \mathbf{u}\|_{2,\omega_i(t)}^2 + \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^p) \leq g_0^{\frac{d-4}{3}} \frac{dy(t)}{dt}. \quad (7.27)$$

Since

$$\|\nabla \mathbf{a}\|_{2,\Omega(t)}^2 + \|\mathbf{a}\|_{4,\Omega(t)}^4 + \|\nabla \mathbf{a}\|_{p,\Omega(t)}^p \leq c \alpha \sum_{i=1}^m \int_0^{h_i(t)} (g_i^{-(d+1)}(s) + g_i^{(1-p)d-1}(s)) ds, \quad (7.28)$$

we have

$$y(t) \leq c[y'(t) + y'^{3/2}(t)] + c\alpha\phi(t). \quad (7.29)$$

Note that

$$\frac{d\phi(t)}{dt} = \sum_{i=1}^m \left(\frac{h'_i(t)}{g_i^{d+1}(h_i(t))} + \frac{h'_i(t)}{g_i^{(p-1)d+1}(h_i(t))} \right) = \sum_{i=1}^m \left(g_i^{\frac{4(1-d)}{3}}(h_i(t)) + g_i^{\frac{4}{3}+(\frac{2}{3}-p)d}(h_i(t)) \right). \quad (7.30)$$

If $p \geq \frac{4+2d}{3d}$, we then have $\frac{d\phi(t)}{dt} \leq m(g_0^{\frac{4(1-d)}{3}} + g_0^{\frac{4}{3}+(\frac{2}{3}-p)d}) \equiv c_{12}$. Thus, if we set $\Psi(t) = c_{13}(t + t^{\frac{3}{2}})$, $\varphi(t) = 2c_{10}\alpha\phi(t) + 2c_{11}\alpha$ and $\delta = \frac{1}{2}$, $t_0 = \bar{t}$, where c_{11} satisfies

$$c_{10}\phi(\bar{t})\alpha + c_{11}\alpha \geq \Psi(2c_{10}c_{12}\alpha),$$

then it is easy to check that all the conditions of Lemma 3.3 are satisfied. Hence, by Lemma 3.3 (i) we have

$$y(t) \leq 2c_{10}\alpha\phi(t) + 2c_{11}\alpha. \quad (7.31)$$

If $1 < p < \frac{4+2d}{3d}$, we then have $\frac{d\phi(t)}{dt} \leq mg_0^{\frac{4(1-d)}{3}} + \sum_{i=1}^m g_i^{\frac{4}{3}+(\frac{2}{3}-p)d}(h_i(t))$. Therefore,

$$(\phi'(t))^{\frac{3}{2}} \leq cg_0^{2(1-d)} + c \sum_{i=1}^m g_i^{2+(1-\frac{3}{2}p)d}(h_i(t)).$$

By (7.15), we have

$$\begin{aligned}
& g_i^{2+(1-\frac{3}{2}p)d}(h_i(t)) \\
&= g_0^{2+(1-\frac{3}{2}p)d} + \int_0^{h_i(t)} \frac{d}{dt} g_i^{2+(1-\frac{3}{2}p)d}(s) ds \\
&= g_0^{2+(1-\frac{3}{2}p)d} + (2 + (1 - \frac{3}{2}p)d) \int_0^{h_i(t)} g'_i(s) g_i^{1+(1-\frac{3}{2}p)d}(s) ds \\
&\leq g_0^{2+(1-\frac{3}{2}p)d} + c\gamma \int_0^{h_i(t)} g_i^{(1-p)d-1}(s) ds \\
&\leq c\gamma\phi(t) + c.
\end{aligned}$$

Thus, if γ is sufficient, then all the conditions of Lemma 3.3 (i) are satisfied. Hence, by Lemma 3.3 (i) we arrive at

$$y(t) \leq 2c_{10}\alpha\phi(t) + 2c_{11}\alpha. \quad (7.32)$$

Finally, since $\int_{\Omega(t)} (|\nabla \mathbf{a}|^2 + |\nabla \mathbf{a}|^p) \zeta(x, t) dx \leq c\alpha\phi(t)$, we get

$$Q(t) \equiv \int_{\Omega(t)} (|\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}|^p) \zeta(x, t) dx \leq \bar{c}_{10}\alpha\phi(t) + \bar{c}_{11}\alpha.$$

Case II. $\mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}$

In this case $I_1 = I_2 = 0$. By completely analogous treatments in Case I, we can obtain

$$\begin{aligned}
|I_3| &\leq \varepsilon \int_{\Omega(t)} |\mathcal{D}(\mathbf{u})|^p dx + \int_{\omega_i(t)} |\mathcal{D}(\mathbf{u})|^p dx + \int_{\Omega(t)} |\mathcal{D}(\mathbf{a})|^p dx. \\
|I_4| &\leq c \sum_{i=1}^m (g_i^{2-(\frac{3}{p}-1)d}(h_i(t)) \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^3 + g_i^{2(1-\frac{d}{p})}(h_i(t)) \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^2), \\
|I_5| &\leq c \sum_{i=1}^m g_i^{2(1-\frac{d}{p})}(h_i(t)) \|\nabla \mathbf{u}\|_{2,\omega_i(t)}^2, \\
|I_7| &\leq c \sum_{i=1}^m \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^p + \sum_{i=1}^m \|\mathbf{a}\|_{2p',\omega_i(t)}^{2p'}, \\
|I_8| &\leq \varepsilon \|\zeta \nabla \mathbf{u}\|_{p,\Omega(t)}^p + \|\mathbf{a}\|_{2p',\Omega(t)}^{2p'}, \\
|I_9| &\leq c \sum_{i=1}^m \left(g_i^{2-(1+\frac{1}{p})d}(h_i(t)) \|\nabla \mathbf{u}\|_{p,\omega_i(t)} + \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^p + g_i^{2-(\frac{3}{p}-1)d}(h_i(t)) \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^3 \right. \\
&\quad \left. + \|\mathbf{a}\|_{2p',\Omega_i(t)}^{2p'} + \|\nabla \mathbf{a}\|_{p,\Omega_i(t)}^p \right).
\end{aligned}$$

For I_6 , by Young inequality and Lemma 3.1, we get that

$$\begin{aligned} |I_6| &= \left| \int_{\Omega(t)} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{a} \zeta dx \right| \\ &\leq \varepsilon \int_{\Omega(t)} \zeta |\mathbf{a}|^{\frac{p}{d-1}} |\mathbf{u}|^p dx + \varepsilon \int_{\Omega(t)} \zeta |\nabla \mathbf{u}|^p dx + c \int_{\Omega(t)} |\mathbf{a}|^{\frac{(d-2)p}{(d-1)(p-2)}} dx. \\ &\leq c\varepsilon \int_{\Omega(t)} \zeta |\nabla \mathbf{u}|^p dx + c \sum_{i=1}^m \int_{\omega_i(t)} |\nabla \mathbf{u}|^p dx + c \int_{\Omega(t)} |\mathbf{a}|^{\frac{(d-2)p}{(d-1)(p-2)}} dx. \end{aligned}$$

As in Case I, based on the estimates on I_i ($1 \leq i \leq 9$), we have

$$\begin{aligned} \|\nabla \mathbf{u}\|_{p,\Omega(t)}^p &\leq c \sum_{i=1}^m \left(g_i^{2-(1+\frac{1}{p})d}(h_i(t)) \|\nabla \mathbf{u}\|_{p,\omega_i(t)} + g_i^{2(1-\frac{d}{p})}(h_i(t)) \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^2 \right. \\ &\quad \left. + g_i^{2-(\frac{3}{p}-1)d}(h_i(t)) \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^3 + \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^p \right) + c \|\nabla \mathbf{a}\|_{p,\Omega(t)}^p \\ &\quad + c \|\mathbf{a}\|_{2p',\Omega(t)}^{2p'} + c \|\mathbf{a}\|_{\frac{(d-2)p}{(d-1)(p-2)},\Omega(t)}^{\frac{(d-2)p}{(d-1)(p-2)}}. \end{aligned} \quad (7.33)$$

In addition, it is easy to get

$$\frac{dy(t)}{dt} \geq \frac{1}{2} \sum_{i=1}^m g_i^{\frac{2p}{3}-(1-\frac{p}{3})d}(h_i(t)) \int_{\omega_i(t)} |\nabla \mathbf{u}|^p dx, \quad (7.34)$$

where $y(t) = \int_{\Omega(t)} |\nabla \mathbf{u}|^p \zeta(x,t) dx$. Hence

$$\sum_{i=1}^m g_i^{2-(\frac{3}{p}-1)d}(h_i(t)) \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^3 \leq c \left(\frac{dy(t)}{dt} \right)^{3/p}, \quad (7.35)$$

$$\sum_{i=1}^m g_i^{2(1-\frac{d}{p})} \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^2 \leq g_0^{\frac{2}{3}(1-d)} \sum_{i=1}^m g_i^{\frac{4}{3}-(\frac{2}{p}-\frac{2}{3})d} \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^2 \leq c \left(\frac{dy(t)}{dt} \right)^{2/p}, \quad (7.36)$$

and

$$\sum_{i=1}^m \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^p \leq g_0^{-\frac{4}{3}+(\frac{2}{p}-\frac{2}{3})d} \left(\frac{dy(t)}{dt} \right). \quad (7.37)$$

By virtue of

$$\|\nabla \mathbf{a}\|_{2p',\Omega(t)}^{2p'} + \|\mathbf{a}\|_{\frac{(d-2)p}{(d-1)(p-2)},\Omega(t)}^{\frac{(d-2)p}{(d-1)(p-2)}} + \|\nabla \mathbf{a}\|_{p,\Omega(t)}^p \leq c\alpha \sum_{i=1}^m \int_0^{h_i(t)} g_i^{(1-p)d-1}(s) ds, \quad (7.38)$$

we have

$$y(t) \leq c[y'(t) + y'^{2/p} + y'^{3/p}(t)] + c\alpha\phi(t). \quad (7.39)$$

Note that

$$\frac{d\phi(t)}{dt} = \sum_{i=1}^m \frac{h'_i(t)}{g_i^{(p-1)d+1}(h_i(t))} = \sum_{i=1}^m g_i^{\frac{2p}{3}(1-d)}(h_i(t)), \quad (7.40)$$

we then have $\frac{d\phi(t)}{dt} \leq mg_0^{\frac{2p}{3}(1-d)} \equiv c_{14}$. Thus, if we set $\Psi(t) = c_{15}(t + t^{\frac{2}{p}} + t^{\frac{3}{p}})$, $\varphi(t) = 2c_{10}\alpha\phi(t) + 2c_{11}\alpha$ and $\delta = \frac{1}{2}$, $t_0 = \bar{t}$, where c_{11} satisfies

$$c_{10}\alpha\phi(\bar{t}) + c_{11}\alpha \geq \Psi(2c_{10}c_{14}\alpha),$$

then it is easy to check that all the conditions of Lemma 3.3 are fulfilled. Hence, by Lemma 3.3 (i) we have

$$y(t) \leq 2c_{10}\alpha\phi(t) + 2c_{11}\alpha. \quad (7.41)$$

Finally, since $\int_{\Omega(t)} |\nabla \mathbf{a}|^p \zeta(x, t) dx \leq c\alpha\phi(t)$, we get

$$Q(t) \equiv \int_{\Omega(t)} (|\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}|^p) \zeta(x, t) dx \leq \bar{c}_{10}\alpha\phi(t) + \bar{c}_{11}\alpha.$$

From (7.32) and (7.41), completely similar to the proof of Part 2 in §6, we can establish the existence of the solution \mathbf{v} to problem (2.5)₁ – (2.5)₄, here the details are omitted. Thus, Theorem 7.1 is shown. \square

Remark 7.1. If there is a constant $c > 0$ such that $|g_i(t)| < c$ for $1 \leq i \leq N$, then Theorem 7.1 is also true for $\mu_0 = 0$ and $p > 2$. Actually, from the proof of Theorem 7.1, in the case of $\mu_0 = 0$ and $p > 2$, we just only need to treat the term $\int_{\Omega(t)} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{a} \zeta dx$ since the other terms can be estimated analogously. For $|g_i(t)| < c$, by $h_i(t) \sim t$ and $\phi(t) \sim t$ one has

$$\begin{aligned} \left| \int_{\Omega(t)} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{a} \zeta dx \right| &\leq c\alpha \int_{\Omega(t)} |\nabla \mathbf{u}|^2 \zeta dx + c\alpha \int_{\Omega(t)} |\mathbf{u}|^2 \zeta dx \\ &\leq c\alpha \int_{\Omega(t)} |\nabla \mathbf{u}|^2 \zeta dx + c\alpha \sum_{i=1}^m \int_{\omega_i(t)} |\nabla \mathbf{u}|^2 dx \\ &\leq \varepsilon \int_{\Omega(t)} |\nabla \mathbf{u}|^p \zeta dx + c \sum_{i=1}^m \int_{\omega_i(t)} |\nabla \mathbf{u}|^p + c\alpha t + c\alpha \\ &\leq \varepsilon \int_{\Omega(t)} |\nabla \mathbf{u}|^p \zeta dx + c \sum_{i=1}^m \int_{\omega_i(t)} |\nabla \mathbf{u}|^p + c\alpha\phi(t) + c\alpha. \end{aligned}$$

Hence the crucial estimate (7.14) still holds.

Remark 7.2. If $1 < p < \frac{4+2d}{3d}$, in order to get the existence of solution to problem (2.5), we need both the conditions (7.7) and (7.15). It is worth noting that there are some g'_i 's such that conditions (7.7) and (7.15) are satisfied. For examples, choosing $g_i(t) = \gamma(t+1)^\alpha$, then it is easy to check that (7.7) and (7.15) are satisfied when $0 \leq \alpha \leq \frac{2}{6-pd}$ and γ is sufficient small.

Theorem 7.2. Let \mathbf{v} be a weak solution of the system (2.5)₁ – (2.5)₄. In addition, we assume that

$$\begin{aligned} \liminf_{t \rightarrow \infty} t^{-3} \int_{\Omega_i(t)} (|\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}|^p) dx &= 0, & \text{if } \mu_0 > 0, p > 1; \\ \liminf_{t \rightarrow \infty} t^{-\frac{3}{3-p}} \int_{\Omega_i(t)} |\nabla \mathbf{v}|^p dx &= 0, & \text{if } \mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}. \end{aligned} \quad (7.42)$$

Then

$$\begin{aligned} \int_{\Omega_i(t)} (|\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}|^p) dx &\leq c_{15}\alpha\phi_i(t) + c_{16}\alpha, & \text{if } \mu_0 > 0, p > 1; \\ \int_{\Omega_i(t)} |\nabla \mathbf{v}|^p dx &\leq c_{15}\alpha\phi_i(t) + c_{16}\alpha, & \text{if } \mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}, \end{aligned} \quad (7.43)$$

where

$$\begin{aligned}\phi_i(t) &= \int_0^t (g_i^{-(d+1)}(s) + g_i^{(1-p)d-1}(s))ds, && \text{if } \mu_0 > 0, p > 1; \\ \phi_i(t) &= \int_0^t g_i^{(1-p)d-1}(s)ds, && \text{if } \mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}.\end{aligned}$$

Proof. Let $t \geq \bar{t}_i$, where \bar{t}_i is a constant such that $h_i(\bar{t}_i) \geq 4\beta g_i(0)$. Then we can define that

$$\hat{\zeta}(x^i, t) = \begin{cases} \frac{h_i(t) - x_1^i}{g_i(h_i(t))} & \text{for } x_1^i \in [h_i(t) - \beta g_i(h_i(t)), h_i(t)], \\ g_i^{-1}(0)x_1^i & \text{for } x_1^i \in [0, \beta g_i(0)], \\ \beta & \text{for } x_1^i \in [\beta g_i(0), h_i(t) - \beta g_i(h_i(t))] \end{cases} \quad (7.44)$$

and

$$\begin{aligned}\hat{y}_i(t) &= \int_{\hat{\Omega}_i(t)} (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{u}|^p) \hat{\zeta} dx, && \text{if } \mu_0 > 0, p > 1, \\ \hat{y}_i(t) &= \int_{\hat{\Omega}_i(t)} |\nabla \mathbf{u}|^p \hat{\zeta} dx, && \text{if } \mu_0 = 0, 2 < p \leq 3 - \frac{2}{d},\end{aligned}$$

where $\hat{\Omega}_i(t) = \Omega_i(h_i(t))$. Similarly to the proof of Theorem 7.1, we can obtain that if $\mu_0 > 0, p > 1$,

$$\begin{aligned}\hat{y}_i(t) &\leq c \|\nabla \mathbf{u}\|_{2,\omega_i(t)}^2 + c \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^p + c g_i^{\frac{4-d}{2}}(h_i(t)) \|\nabla \mathbf{u}\|_{2,\omega_i(t)}^3 \\ &\quad + c \|\nabla \mathbf{a}\|_{2,\hat{\Omega}_i(t)}^2 + c \|\mathbf{a}\|_{4,\hat{\Omega}_i(t)}^4 + c \|\nabla \mathbf{a}\|_{p,\hat{\Omega}_i(t)}^p; \quad (7.45)\end{aligned}$$

if $\mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}$,

$$\begin{aligned}\hat{y}_i(t) &\leq c \left(g_i^{2-(1+\frac{1}{p})d}(h_i(t)) \|\nabla \mathbf{u}\|_{p,\omega_i(t)} + g_i^{2(1-\frac{d}{p})}(h_i(t)) \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^2 \right. \\ &\quad \left. + g_i^{2-(\frac{3}{p}-1)d}(h_i(t)) \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^3 + \|\nabla \mathbf{u}\|_{p,\omega_i(t)}^p \right) \\ &\quad + c \left(\|\nabla \mathbf{a}\|_{p,\hat{\Omega}_i(t)}^p + \|\mathbf{a}\|_{2p',\hat{\Omega}_i(t)}^{2p'} + \|\mathbf{a}\|_{\frac{(d-2)p}{(d-1)(p-2)},\hat{\Omega}_i(t)}^{\frac{(d-2)p}{(d-1)(p-2)}} \right).\end{aligned} \quad (7.46)$$

Hence

$$\begin{aligned}\hat{y}_i(t) &\leq c(\hat{y}'_i(t) + (\hat{y}'_i(t))^{3/2}) + c\alpha\hat{\phi}_i(t), && \text{if } \mu_0 > 0, p > 1; \\ \hat{y}_i(t) &\leq c(\hat{y}'_i(t) + (\hat{y}'_i(t))^{2/p} + (\hat{y}'_i(t))^{3/p}) + c\alpha\hat{\phi}_i(t), && \text{if } \mu_0 = 0, 2 < p \leq 3 - \frac{2}{d},\end{aligned}$$

where $\hat{\phi}_i(t) = \phi_i(h_i(t))$. Similarly to the proof of Theorem 7.1, setting $\Psi(\tau) = c(\tau + \tau^{\frac{3}{2}})$, if $\mu_0 > 0$; $\Psi(\tau) = c(\tau + \tau^{\frac{2}{p}} + \tau^{\frac{3}{p}})$, if $\mu_0 = 0$, and taking $\varphi_i(t) = 2c\alpha\hat{\phi}_i(t) + 2c\alpha$. By virtue of (7.42), from Lemma 3.3 (ii), we can arrive at

$$\hat{y}_i(t) \leq 2c\alpha\hat{\phi}_i(t) + 2c\alpha, \quad \text{for } t \geq \bar{t}_i. \quad (7.47)$$

By (7.47), we have that:

if $\mu_0 > 0, p > 1$,

$$\beta \int_{\beta g_i(0)}^{x_1 - \beta g(x_1)} ds \int_{\Sigma_i(s)} (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{u}|^p) dy \leq c\alpha \int_0^{x_1} (g_i^{-(d+1)}(s) + g_i^{(1-p)d-1}(s)) ds + c\alpha; \quad (7.48)$$

if $\mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}$,

$$\beta \int_{\beta g_i(0)}^{x_1 - \beta g(x_1)} ds \int_{\Sigma_i(s)} |\nabla \mathbf{u}|^p dy \leq c\alpha \int_0^{x_1} g_i^{(1-p)d-1}(s) ds + c\alpha. \quad (7.49)$$

Meanwhile, if $\mu_0 > 0, p > 1$,

$$\begin{aligned} & \int_{x_1 - \beta g(x_1)}^{x_1} (g_i^{-(d+1)}(s) + g_i^{(1-p)d-1}(s)) ds \\ & \leq \max_{[x_1 - \beta g(x_1), x_1]} (g_i^{-(d+1)}(s) + g_i^{(1-p)d-1}(s)) \cdot \beta g_i(x_1) \\ & \leq c(g_i^{-d}(x_1) + g_i^{(1-p)d}(x_1)) \\ & \leq c(g_0^{-d} + g_0^{(1-p)d}); \end{aligned}$$

if $\mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}$,

$$\begin{aligned} & \int_{x_1 - \beta g(x_1)}^{x_1} g_i^{(1-p)d-1}(s) ds \\ & \leq \max_{[x_1 - \beta g(x_1), x_1]} g_i^{(1-p)d-1}(s) \cdot \beta g_i(x_1) \\ & \leq c g_i^{(1-p)d}(x_1) \\ & \leq c g_0^{(1-p)d}. \end{aligned}$$

Therefore, if $\mu_0 > 0, p > 1$,

$$\int_0^t ds \int_{\Sigma(s)} (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{u}|^p) dy \leq c_{15}\alpha \phi_i(t) + c_{16}\alpha;$$

if $\mu_0 = 0, 2 \leq p \leq 3 - \frac{2}{d}$,

$$\int_0^t ds \int_{\Sigma(s)} |\nabla \mathbf{u}|^p dy \leq c_{15}\alpha \phi_i(t) + c_{16}\alpha.$$

Since $\int_{\Omega_i(t)} |\nabla \mathbf{a}|^p \zeta(x, t) dx \leq c\alpha \phi_i(t)$, we have that (7.43) is satisfied. \square

From (7.14), it is easy to check that the solutions of Theorem 7.1 satisfy (7.42). Hence, we can get the following result:

Theorem 7.3. *we assume that all the conditions of Theorem 7.1 are satisfied, then problem (2.5) has at least one weak solution.*

7.2 Part 2. Uniqueness

At first, we establish the following result:

Lemma 7.1. Assume that $g_i(t)$ satisfies the conditions

$$\begin{aligned} |g'_i(t)g_i^{\frac{4}{3}-\frac{d}{3}}(t)| &< \tilde{\gamma}, & \text{if } \mu_0 > 0, p \geq \frac{4+2d}{3p}; \\ |g'_i(t)g_i^{\frac{2p}{3}+(\frac{p}{3}-1)d}(t)| &< \tilde{\gamma}, & \text{if } \mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}, \end{aligned} \quad (7.50)$$

where $\tilde{\gamma} > 0$ is a sufficient small constant. If \mathbf{v} satisfies (7.43), then we have

$$\begin{aligned} \int_{\hat{\omega}_i(t)} (|\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}|^p) dx &\leq c_{17}\alpha(g_i^{\frac{4}{3}(1-d)}(t) + g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(t)), & \text{if } \mu_0 > 0, p \geq \frac{4+2d}{3p}; \\ \int_{\hat{\omega}_i(t)} |\nabla \mathbf{v}|^p dx &\leq c_{17}\alpha g_i^{\frac{2}{3}(1-d)}(t), & \text{if } \mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}, \end{aligned} \quad (7.51)$$

where $\hat{\omega}_i(t) = \{x \in \Omega_i : t - \beta g_i(t) < x_1^i < t\}$ and $t \geq 4\bar{t} + 6\beta g_i(0)$.

Proof. For fixed t , we set

$$\hat{t} = t - \beta g_i(t), \quad t_1 = \hat{t} - \beta g_i(\hat{t}), \quad t_2 - \beta g_i(t_2) = t,$$

then $\bar{t} \leq t_1 < \hat{t} < t < t_2$.

Case I. $\mu_0 > 0, p \geq \frac{4+2d}{3p}$

Introduce the functions $m_1(\tau)$ and $m_2(\tau)$ such that

$$\frac{dm_1(\tau)}{d\tau} = g_i^{\frac{7-d}{3}}(t_1 - m_1(\tau)), \quad \frac{dm_2(\tau)}{d\tau} = g_i^{\frac{7-d}{3}}(t_2 + m_2(\tau))$$

and $m_1(0) = m_2(0) = 0$. In addition, constructing the following truncating function

$$\chi(x^i, \tau) = \begin{cases} \frac{x_1^i - t_1 + m_1(\tau)}{g_i(t_1 - m_1(\tau))}, & \text{for } x_1^i \in [t_1 - m_1(\tau), t_1 - m_1(\tau) + \beta g_i(t_1 - m_1(\tau))], \\ \beta, & \text{for } x_1^i \in [t_1 - m_1(\tau) + \beta g_i(t_1 - m_1(\tau)), t_2 + m_2(\tau) - \beta g_i(t_2 + m_2(\tau))], \\ \frac{t_2 + m_2(\tau) - x_1^i}{g_i(t_2 + m_2(\tau))}, & \text{for } x_1^i \in [t_2 + m_2(\tau) - \beta g_i(t_2 + m_2(\tau)), t_2 + m_2(\tau)]. \end{cases}$$

Set

$$\Omega_i(\tau; t) = \{x \in \Omega_i : t_1 - m_1(\tau) < x_1^i < t_2 + m_2(\tau)\}.$$

As the proof of Theorem 7.1, we multiply the equation in (2.5) by $\mathbf{u}\chi$ and integrate over $\Omega_i(\tau; t)$ to get

$$z(\tau) \leq c_{18}[z'(\tau) + (z'(\tau))^{3/2}] + c_{19}\alpha \int_{t_1 - m_1(\tau)}^{t_2 + m_2(\tau)} (g_i^{-(d+1)}(s) + g^{(1-p)d-1}(s)) ds \quad \text{for } \tau \in [0, \tau_1], \quad (7.52)$$

where

$$z(\tau) = \int_{\Omega_i(\tau;t)} (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{u}|^p) \chi(x, \tau) dx,$$

and τ_1 is a constant such that $\bar{t} = t_1 - m_1(\tau_1)$. In addition,

$$\begin{aligned} & 1 + \int_0^{\bar{t}} (g_i^{-(d+1)}(s) + g_i^{(1-p)d-1}(s)) ds \\ & \leqslant 1 + \int_0^{\bar{t}} (g_0^{-(d+1)} + g_0^{(1-p)d-1}) ds \\ & \leqslant (1 + \bar{t}(g_0^{-(d+1)} + g_0^{(1-p)d-1})) \frac{(g_i(0) + \beta^{-1}\bar{t})^{d+1} + (g_i(0) + \beta^{-1}\bar{t})^{(p-1)d+1}}{\bar{t}} \\ & \quad \times \int_{\bar{t}}^{2\bar{t}} (g_i^{-(d+1)}(s) + g_i^{(1-p)d-1}(s)) ds. \end{aligned}$$

This, together with (7.43) and $2\bar{t} < t < t_2 + m_2(\tau)$, yields

$$z(\tau_1) \leqslant c_{20}\alpha \int_{t_1-m_1(\tau_1)}^{t_2+m_2(\tau_1)} (g_i^{-(d+1)}(s) + g_i^{(1-p)d-1}(s)) ds. \quad (7.53)$$

Set

$$\varphi(\tau) = c_{21}\alpha \int_{t_1-m_1(\tau)}^{t_2+m_2(\tau)} (g_i^{-(d+1)}(s) + g_i^{(1-p)d-1}(s)) ds + c_{22}\alpha(g_i^{\frac{4}{3}(1-d)}(t) + g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(t)),$$

where $c_{21} = \max\{2c_{19}, c_{20}\}$, and c_{22} is a sufficiently large constant. We now prove

$$\varphi(\tau) \geqslant 2c_{18}(\varphi'(\tau) + (\varphi'(\tau))^{3/2}). \quad (7.54)$$

At first, it is easy to get

$$\begin{aligned} \varphi'(\tau) &= c_{21}\alpha \left(\frac{m'_2(\tau)}{g_i^{d+1}(t_2 + m_2(\tau))} + \frac{m'_1(\tau)}{g_i^{d+1}(t_1 - m_1(\tau))} \right) \\ &\quad + c_{21}\alpha \left(\frac{m'_2(\tau)}{g_i^{(p-1)d+1}(t_2 + m_2(\tau))} + \frac{m'_1(\tau)}{g_i^{(p-1)d+1}(t_1 - m_1(\tau))} \right) \\ &= c_{21}\alpha(g_i^{\frac{4}{3}(1-d)}(t_2 + m_2(\tau)) + g_i^{\frac{4}{3}(1-d)}(t_1 - m_1(\tau)) \\ &\quad + g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(t_2 + m_2(\tau)) + g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(t_1 - m_1(\tau))) \\ &\equiv c_{21}\alpha A(\tau). \end{aligned}$$

Since $p \geqslant \frac{4+2d}{3p}$, we have $A(\tau) \leqslant 2(g_0^{\frac{4}{3}(1-d)} + g_0^{\frac{4}{3}-(p-\frac{2}{3})d})$. Thus $(\varphi'(\tau))^{3/2} \leqslant \sqrt{2}(c_{21}\alpha)^{\frac{3}{2}}(g_0^{\frac{2}{3}(1-d)} + g_0^{\frac{2}{3}-(\frac{p}{2}-\frac{1}{3})d})A(\tau)$. On the other hand, by virtue of (7.50),

$$\begin{aligned} A(\tau) &= 2g_i^{\frac{4}{3}(1-d)}(t) - \int_{t_1-m_1(\tau)}^t \frac{d}{ds}g_i^{\frac{4}{3}(1-d)}(s) ds + \int_t^{t_1+m_2(\tau)} \frac{d}{ds}g_i^{\frac{4}{3}(1-d)}(s) ds \\ &\quad + 2g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(t) - \int_{t_1-m_1(\tau)}^t \frac{d}{ds}g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(s) ds + \int_t^{t_1+m_2(\tau)} \frac{d}{ds}g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(s) ds \end{aligned}$$

$$\leq 2(g_i^{\frac{4}{3}(1-d)}(t) + g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(t)) + c_{22}\tilde{\gamma} \int_{t_1-m_1(\tau)}^{t_2+m_2(\tau)} (g_i^{-(d+1)}(s) + g_i^{(1-p)d-1}(s))ds.$$

Hence, if $\tilde{\gamma}$ is sufficient small, then (7.54) holds. This yields for $\tau \in [0, \tau_1]$,

$$z(\tau) \leq \varphi(\tau).$$

Choosing $\tau = 0$, we then have

$$\begin{aligned} & \beta \int_{t-\beta g(t)}^t dx_1 \int_{\Sigma_i(x_1)} (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{u}|^p) dy \\ & \leq z(0) = \int_{\Omega_i(0;t)} (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{u}|^p) \chi(x, 0) dx \\ & \leq \varphi(0) = c_{20}\alpha \int_{t_1}^{t_2} (g_i^{-(d+1)}(s) + g_i^{(1-p)d-1}(s)) ds + c_{21}\alpha (g_i^{\frac{4}{3}(1-d)}(t) + g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(t)). \end{aligned}$$

Assume that $|g'_i(s)| \leq \gamma_1$ for $s \in [t_1, t_2]$. From this, we have

$$t_2 \leq t + \frac{\beta}{1 - \beta\gamma_1} g_i(t) \text{ and } t_1 \geq t - \beta(2 + \beta\gamma_1) g_i(t).$$

Hence, if $c_{23} = \beta[2 + \beta\gamma_1 + \frac{1}{1 - \beta\gamma_1}] < \gamma_1^{-1}$, then

$$\begin{aligned} & \int_{t_1}^{t_2} (g_i^{-(d+1)}(s) + g_i^{(1-p)d-1}(s)) ds \\ & \leq \frac{c_{23}}{(1 - c_{23}\gamma_1)^{d+1}} g_i^{-d}(t) + \frac{c_{23}}{(1 - c_{23}\gamma_1)^{(p-1)d+1}} g_i^{(1-p)d}(t) \\ & \leq c_{24} (g_i^{\frac{4}{3}(1-d)}(t) + g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(t)). \end{aligned}$$

Therefore

$$\int_{\hat{\omega}_i(t)} (|\nabla \mathbf{u}|^2 + |\nabla \mathbf{u}|^p) dx \leq c\alpha (g_i^{\frac{4}{3}(1-d)}(t) + g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(t)).$$

Since $\int_{\hat{\omega}_i(t)} (|\nabla \mathbf{a}|^2 + |\nabla \mathbf{a}|^p) dx \leq c\alpha (g_i^{\frac{4}{3}(1-d)}(t) + g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(t))$, we have that

$$\int_{\hat{\omega}_i(t)} (|\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}|^p) dx \leq c_{17}\alpha (g_i^{\frac{4}{3}(1-d)}(t) + g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(t)).$$

Case II. $\mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}$

Choosing functions $m_1(\tau)$ and $m_2(\tau)$ such that

$$\frac{dm_1(\tau)}{d\tau} = g_i^{\frac{2p+3}{3} - (1 - \frac{p}{3})d}(t_1 - m_1(\tau)), \quad \frac{dm_2(\tau)}{d\tau} = g_i^{\frac{2p+3}{3} - (1 - \frac{p}{3})d}(t_2 + m_2(\tau))$$

and $m_1(0) = m_2(0) = 0$. Similarly to the proof of Theorem 7.1, we multiply the equation in (2.5) by $\mathbf{u}\chi$ and integrate over $\Omega_i(\tau; t)$ to get

$$z(\tau) \leq c_{25}[z'(\tau) + (z'(\tau))^{2/p} + (z'(\tau))^{3/p}] + c_{26}\alpha \int_{t_1-m_1(\tau)}^{t_2+m_2(\tau)} g^{(1-p)d-1}(s) ds \quad \text{for } \tau \in [0, \tau_1], \tag{7.55}$$

where

$$z(\tau) = \int_{\Omega_i(\tau; t)} |\nabla \mathbf{u}|^p \chi(x, \tau) dx.$$

Note that

$$\begin{aligned} 1 + \int_0^{\bar{t}} g_i^{(1-p)d-1}(s) ds &\leq 1 + \int_0^{\bar{t}} g_0^{(1-p)d-1} ds \\ &\leq (1 + \bar{t} g_0^{(1-p)d-1}) \frac{(g_i(0) + \beta^{-1} \bar{t})^{(p-1)d+1}}{\bar{t}} \int_{\bar{t}}^{2\bar{t}} g_i^{(1-p)d-1}(s) ds. \end{aligned}$$

By this inequality and (7.43), we have

$$z(\tau_1) \leq c_{27} \alpha \int_{t_1 - m_1(\tau_1)}^{t_2 + m_2(\tau_1)} g_i^{(1-p)d-1}(s) ds. \quad (7.56)$$

Set

$$\varphi(\tau) = c_{29} \alpha \int_{t_1 - m_1(\tau)}^{t_2 + m_2(\tau)} g_i^{(1-p)d-1}(s) ds + c_{30} \alpha g_i^{\frac{2p}{3}(1-d)}(t),$$

where $c_{29} = \max\{2c_{26}, c_{27}\}$, and c_{30} is a sufficiently large constant. Next we prove

$$\varphi(\tau) \geq 2c_{25}(\varphi'(\tau) + (\varphi'(\tau))^{2/p} + (\varphi'(\tau))^{3/p}). \quad (7.57)$$

It follows from a direct computation that

$$\begin{aligned} \varphi'(\tau) &= c_{29} \alpha \left(\frac{m'_2(\tau)}{g_i^{(p-1)d+1}(t_2 + m_2(\tau))} + \frac{m'_1(\tau)}{g_i^{(p-1)d+1}(t_1 - m_1(\tau))} \right) \\ &= c_{29} \alpha \left(g_i^{\frac{2p}{3}(1-d)}(t_2 + m_2(\tau)) + g_i^{\frac{2p}{3}(1-d)}(t_1 - m_1(\tau)) \right) \\ &\equiv c_{29} \alpha A(\tau). \end{aligned}$$

Together with $A(\tau) \leq 2g_0^{\frac{2p}{3}(1-d)}$, this yields $(\varphi'(\tau))^{3/p} \leq 2^{\frac{3}{p}-1} (c_{29} \alpha)^{\frac{3}{p}} g_0^{\frac{2(3-p)}{3p}(1-d)} A(\tau)$. Moreover, by (7.50)

$$\begin{aligned} A(\tau) &= 2g_i^{\frac{2p}{3}(1-d)}(t) - \int_{t_1 - m_1(\tau)}^t \frac{d}{ds} g_i^{\frac{2p}{3}(1-d)}(s) ds + \int_t^{t_1 + m_2(\tau)} \frac{d}{ds} g_i^{\frac{2p}{3}(1-d)}(s) ds \\ &\leq 2g_i^{\frac{2p}{3}(1-d)}(t) + c_{31} \tilde{\gamma} \int_{t_1 - m_1(\tau)}^{t_2 + m_2(\tau)} g_i^{(1-p)d-1}(s) ds. \end{aligned}$$

Hence, if $\tilde{\gamma}$ is sufficient small, then (7.57) holds. Therefore,

$$z(\tau) \leq \varphi(\tau), \quad \text{for } \tau \in [0, \tau_1].$$

Choosing $\tau = 0$, we have

$$\begin{aligned} &\beta \int_{t-\beta g(t)}^t dx_1 \int_{\Sigma_i(x_1)} |\nabla \mathbf{u}|^p dy \\ &\leq z(0) = \int_{\Omega_i(0; t)} |\nabla \mathbf{u}|^p \chi(x, 0) dx \end{aligned}$$

$$\leq \varphi(0) = c_{29}\alpha \int_{t_1}^{t_2} g_i^{(1-p)d-1}(s)ds + c_{30}\alpha g_i^{\frac{2p}{3}(1-d)}(t).$$

Assume that $|g'_i(s)| \leq \gamma_1$ for $s \in [t_1, t_2]$. Then

$$t_2 \leq t + \frac{\beta}{1 - \beta\gamma_1}g(t) \text{ and } t_1 \geq t - \beta(2 + \beta\gamma_1)g_i(t).$$

Hence, if $c_{32} = \beta[2 + \beta\gamma_1 + \frac{1}{1-\beta\gamma_1}] \leq \gamma_1^{-1}$, then

$$\int_{t_1}^{t_2} g_i^{(1-p)d-1}(s)ds \leq \frac{c_{32}}{(1 - c_{32}\gamma_1)^{(p-1)d}} g_i^{(1-p)d}(t) \leq c_{33} g_i^{\frac{2p}{3}(1-d)}(t).$$

Therefore,

$$\int_{\hat{\omega}_i(t)} |\nabla \mathbf{u}|^p dx \leq c\alpha g_i^{\frac{2p}{3}(1-d)}(t).$$

It is easy to check that $\int_{\hat{\omega}_i(t)} |\nabla \mathbf{a}|^p dx \leq c\alpha g_i^{\frac{4}{3}-(p-\frac{2}{3})d}(t)$, hence

$$\int_{\hat{\omega}_i(t)} |\nabla \mathbf{v}|^p dx \leq c_{17}\alpha g_i^{\frac{2p}{3}(1-d)}(t).$$

Collecting all the estimates above, we complete the proof of Lemma 7.1. \square

Next, let \mathbf{v} be a divergence free vector field in $W_{loc}^{1,p}(\bar{\Omega})$, and we assume that there exists a constant ν satisfying $\nu(p-2) \leq \frac{p-2}{p-1} + (\frac{2}{3} + \frac{1}{p})d - \frac{8}{3}$ such that

$$|\mathcal{D}(\mathbf{v})(x^i)| \geq cg_i^{-\nu}(x_1^i)|y^i|^{\frac{1}{p-1}} \quad \text{for } x^i \in \Omega_i, \quad (7.58)$$

where $y^i = (x_2^i, \dots, x_d^i)$ and $1 \leq i \leq N$.

Theorem 7.4. *Let $d = 3$, we assume that $g_i(t)$ satisfies (7.50) and α is sufficiently small, and there exists a solution \mathbf{v} satisfying (7.58) to problem (2.5). Then the solution \mathbf{v} of problem (2.5) is unique for $\mu_0 = 0$, $2 < p \leq 3 - \frac{2}{d}$. When $\mu_0 > 0$, even if the assumption (7.58) on \mathbf{u} is removed, the solution \mathbf{v} of problem (2.5) exists uniquely for $p \geq \frac{16-d}{3d}$.*

Remark 7.3. *If $\mathbf{v}(x_1, y)$ is the Hagen-Poiseuille flow in pipes of circular cross section with radius $g_i(t)$, then it follows from (7.15) of [28] that*

$$|\mathcal{D}(\mathbf{v})(x_1, y)| \geq cg_i^{-\frac{3p-2}{p-1}}(t)|y|^{\frac{1}{p-1}} \quad \text{for some positive number } c > 0. \quad (7.59)$$

This means that the assumption (7.58) is reasonable under some cases.

Proof. By the assumptions in Theorem 7.4, from Lemma 7.1, since $d = 3$, one has

$$\begin{aligned} \int_{\hat{\omega}_i(t)} |\nabla \mathbf{v}_2|^2 dx &\leq c\alpha(g_i^{\frac{4}{3}(1-d)}(t) + g_i^{\frac{4}{3}-(p-\frac{2}{3})d}) \leq c\alpha g_i^{d-4}(t), \quad \text{if } \mu_0 > 0, p \geq \frac{16-d}{3d}; \\ \int_{\hat{\omega}_i(t)} |\nabla \mathbf{v}_2|^p dx &\leq c\alpha g_i^{\frac{2p}{3}(1-d)}(t), \quad \text{if } \mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}. \end{aligned} \quad (7.60)$$

Set $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2$. Then

$$\begin{aligned} & -\mu_0 \operatorname{div}(\mathcal{D}(\mathbf{w})) - \mu_1 \operatorname{div}\left(|\mathcal{D}(\mathbf{w}) + \mathcal{D}(\mathbf{v}_2)|^{p-2} (\mathcal{D}(\mathbf{w}) + \mathcal{D}(\mathbf{v}_2)) - |\mathcal{D}(\mathbf{v}_2)|^{p-2} \mathcal{D}(\mathbf{v}_2)\right) \\ & \quad + \mathbf{w} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}_2 + \mathbf{v}_2 \cdot \nabla \mathbf{w} + \nabla(\pi_1 - \pi_2) = 0. \end{aligned} \quad (7.61)$$

From the proof of Theorem 7.1, we just need to estimate $(\mathbf{w} \cdot \nabla \mathbf{v}_2, \mathbf{w})_{\hat{\omega}_i(t)}$. While, by (7.59)₁ and Lemma 3.1 if $\mu_0 > 0$,

$$\begin{aligned} & \left| \int_{\hat{\omega}_i(t)} \mathbf{w} \cdot \nabla \mathbf{v}_2 \cdot \mathbf{w} dx \right| \\ & \leq \|\mathbf{w}\|_{4,\hat{\omega}_i(t)}^2 \|\nabla \mathbf{v}_2\|_{2,\hat{\omega}_i(t)} \\ & \leq g_i^{2-\frac{1}{2}d}(t) \|\nabla \mathbf{w}\|_{2,\hat{\omega}_i(t)}^2 \|\nabla \mathbf{v}_2\|_{2,\hat{\omega}_i(t)} \\ & \leq c\alpha^{\frac{1}{2}} \|\nabla \mathbf{w}\|_{2,\hat{\omega}_i(t)}^2; \end{aligned} \quad (7.62)$$

if $\mu_0 = 0$, $2 < p \leq 3 - \frac{2}{d}$,

$$\begin{aligned} & \left| \int_{\hat{\omega}_i(t)} \mathbf{w} \cdot \nabla \mathbf{v}_2 \cdot \mathbf{w} dx \right| \\ & \leq \|\mathbf{w}\|_{2p',\hat{\omega}_i(t)}^2 \|\nabla \mathbf{v}_2\|_{p,\hat{\omega}_i(t)} \\ & \leq g_i^{2+(\frac{p-1}{p}-\frac{2}{r})d}(t) \|\nabla \mathbf{w}\|_{r,\hat{\omega}_i(t)}^2 \|\nabla \mathbf{v}_2\|_{p,\hat{\omega}_i(t)}. \end{aligned} \quad (7.63)$$

Meanwhile, if $1 \leq r < \frac{2(d-1)(p-1)}{dp-(d+1)}$, by (7.58)

$$\begin{aligned} & \int_{\hat{\omega}_i(t)} |\nabla \mathbf{w}|^r dx \\ & \leq \int_{\hat{\omega}_i(t)} |\mathcal{D}(\mathbf{v}_2)|^{\frac{(p-2)r}{2}} |\mathcal{D}(\mathbf{w})|^r |\mathcal{D}(\mathbf{v}_2)|^{-\frac{(p-2)r}{2}} dx \\ & \leq \left(\int_{\hat{\omega}_i(t)} |\mathcal{D}(\mathbf{v}_2)|^{(p-2)} |\mathcal{D}(\mathbf{w})|^2 dx \right)^{\frac{r}{2}} \left(\int_{\hat{\omega}_i(t)} |\mathcal{D}(\mathbf{v}_2)|^{-\frac{(p-2)r}{2-r}} dx \right)^{\frac{2-r}{2}} \\ & \leq c g_i^{\frac{\nu r(p-2)}{2} + \frac{(2-r)}{2}}(t) \|\mathcal{D}(\mathbf{v}_2)|^{p-2} \mathcal{D}(\mathbf{w})\|_{2,\hat{\omega}_i(t)}^r \left(\int_0^{g_i(t)} s^{d-2-\frac{(p-2)r}{(p-1)(2-r)}} ds \right)^{\frac{2-r}{2}} \\ & \leq c g_i^{\frac{d(2-r)}{2} - \frac{(p-2)r}{2(p-1)} + \frac{\nu r(p-2)}{2}}(t) \|\mathcal{D}(\mathbf{v}_2)|^{p-2} \mathcal{D}(\mathbf{w})\|_{2,\hat{\omega}_i(t)}^r. \end{aligned} \quad (7.64)$$

Hence, we have

$$\left| \int_{\hat{\omega}_i(t)} \mathbf{w} \cdot \nabla \mathbf{v}_2 \cdot \mathbf{w} dx \right| \leq c\alpha^{\frac{1}{p}} g_i^\theta(t) \|\mathcal{D}(\mathbf{v}_2)|^{p-2} \mathcal{D}(\mathbf{w})\|_{2,\hat{\omega}_i(t)}^2, \quad (7.65)$$

where $\theta = -\frac{p-2}{p-1} - (\frac{2}{3} + \frac{1}{p})d + \frac{8}{3} + \nu(p-2) \leq 0$. Therefore,

$$\left| \int_{\hat{\omega}_i(t)} \mathbf{w} \cdot \nabla \mathbf{v}_2 \cdot \mathbf{w} dx \right| \leq c\alpha \|\mathcal{D}(\mathbf{v}_2)|^{p-2} \mathcal{D}(\mathbf{w})\|_{2,\hat{\omega}_i(t)}^2. \quad (7.66)$$

From (7.61) and (7.65), similarly to the proof of Theorem 7.1, if α is sufficiently small, we can get

$$\begin{aligned} y(t) &\leq c[y'(t) + (y'(t))^{\frac{3}{2}}], & \text{if } \mu_0 > 0, p \geq \frac{16-d}{3d}, \\ y(t) &\leq c[y'(t) + (y'(t))^{\frac{2}{p}} + (y'(t))^{\frac{3}{p}}], & \text{if } \mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}, \end{aligned} \quad (7.67)$$

where $y(t) = \int_{\Omega(t)} (|\nabla \mathbf{w}|^2 + |\nabla \mathbf{w}|^p) dx$, if $\mu_0 > 0$; $y(t) = \int_{\Omega(t)} |\nabla \mathbf{w}|^p dx$, if $\mu_0 = 0$. If $y(t)$ is not identically zero, it then follows from Lemma 3.3 (iii) that

$$\liminf_{t \rightarrow \infty} t^{-3} y(t) > 0, \text{ if } \mu_0 > 0, p \geq \frac{16-d}{3d}; \liminf_{t \rightarrow \infty} t^{-\frac{3}{3-p}} y(t) > 0, \text{ if } \mu_0 = 0, 2 < p \leq 3 - \frac{2}{d}.$$

These contradict with (2.5)₅. Hence, $y(t) \equiv 0$, and further $\mathbf{v}_1 = \mathbf{v}_2$. \square

References

- [1] R. A. Adams, J. Fournier, Sobolev spaces, Second edition, Pure and Applied Mathematics (Amsterdam), 140, Elsevier/Academic Press, Amsterdam, 2003.
- [2] C. J. Amick, Steady solutions of the Navier-Stokes equations in unbounded channel and pipes, Ann. Scuola Norm. Pisa (4) 4, 473-513 (1977)
- [3] C. J. Amick, On Leray's problem of steady Navier-Stokes flow past a body in the plane, Acta Math. 161, 71-130 (1988)
- [4] Dong Hongjie, R. M. Strain, On partial regularity of steady-state solutions to the 6D Navier-Stokes equations, Indiana Univ. Math. J. 61, no. 6, 2211-2229 (2012)
- [5] J. Frehse, J. Málek, M. Steinhauer, On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method, SIAM J. Math. Anal. 34, 1064-1083 (2003)
- [6] J. Frehse, M. Ruzicka, Existence of regular solutions to the stationary Navier-Stokes equations, Math. Ann. 302, no. 4, 699-717 (1995)
- [7] J. Frehse, M. Ruzicka, Existence of regular solutions to the steady Navier-Stokes equations in bounded six-dimensional domains, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 23, no. 4, 701-719 (1996)
- [8] G. P. Galdi, Mathematical problems in classical and non-Newtonian fluid mechanics. In: Galdi, G.P., Robertson, A.M., Rannacher, R., Turek, S.: Hemodynamical Flows: Modeling, Analysis and Simulation (Oberwolfach Seminars), Vol. 37, Birkhaeuser, Basel, 2008.
- [9] G. P. Galdi, C. R. Grisanti, Existence and regularity of steady flows for shear-thinning liquids in exterior two-dimensional, Arch. Ration. Mech. Anal. 200, no. 2, 533-559 (2011)
- [10] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems, Springer, 2011.
- [11] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

- [12] J. D. Gilberlandio, M. S. Marcelo, Steady flow for shear thickening fluids with arbitrary fluxes, *J. Differential Equations* 252, 3873-3898 (2012)
- [13] V. A. Kondrat'ev, O. A. Olenik, Boundary value problems for a system in elasticity theory in unbounded domains. Korn inequalities, *Russ. Math. Surv.* 43, 65-119 (1988)
- [14] O. A. Ladyzhenskaya, Investigation of the Navier-Stokes equation for a stationary flow of an incompressible fluid, *Uspehi Mat. Nauk.* 14 (3), 75-97 (1959) (in Russian)
- [15] O. A. Ladyzenskaja, New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problem for them, *Trudy Mat. Inst. Steklov.* 102, 80-103; English Transl.: *Proc. Steklov Inst. Math.*, 102, 95-118 (1967)
- [16] O. A. Ladyzenskaja, The mathematical theory of viscous incompressible flow, Second English edition, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2, Gordon and Breach, Science Publishers, New York-London-Paris, 1969.
- [17] O. A. Ladyzenskaja, V. A. Solonnikov, The solvability of boundary value and initial-boundary value problems for the Navier-Stokes equations in domains with noncompact boundaries (Russian), *Vestnik Leningrad. Univ. no. 13, Mat. Meh. Astronom. vyp. 3*, 39-47 (1977)
- [18] O. A. Ladyzhenskaya, V.A. Solonnikov, Determination of the solutions of boundary value problems for steady-state Stokes and Navier-Stokes equations in domains having an unbounded Dirichlet integral, *Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov. (LOMI)* 96, 117-160 (1980); English transl.: *J. Soviet Math.* 21, 728-761 (1983)
- [19] J. Leray, Étude de diverses équations intégrales non linéaire et de quelques problèmes que pose l'hydrodynamique, *J. Math. Pures Appl.* 12, 1-82 (1933)
- [20] J. L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969.
- [21] E. Marušić-Paloka, Steady flow of a non-Newtonian fluid in unbounded channels and pipes, *Math. Models Methods Appl. Sci.* 10 (9), 1425-1445 (2000)
- [22] J. Naumann, J. Wolf, Interior differentiability of weak solutions to the equations of stationary motion of a class of non-Newtonian fluids, *J. Math. Fluid Mech.* 7, 298-313 (2005)
- [23] K. Pileckas, On the existence of solutions of Navier-Stokes equations that have infinite dissipation of energy in a class of domains with a noncompact boundary. (Russian) Boundary value problems of mathematical physics and related questions in the theory of functions, 13, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 110, 180-202, 245-246 (1981)
- [24] K. Pileckas, Weighted L^q -solvability for the steady Stokes system in domains with noncompact boundaries, *Math. Models and Methods in Appl. Sci.* 6, 97-136 (1996)
- [25] K. Pileckas, Classical solvability and uniform estimates for the steady Stokes system in domains with noncompact boundaries, *Math. Models and Methods in Appl. Sci.* 6, 151-186 (1996)

- [26] K. Pileckas, Recent advances in the theory of Stokes and Navier-Stokes equations in domains with non-compact boundaries, Mathematical Theory in Fluid Mechanics, Galdi, G.P., Málek J., and Nečas, J., Eds., Pitman Research Notes in Mathematics Series, Longman Scientific and Technical, Vol. 354, 30-85 (1996)
- [27] K. Pileckas, Strong solutions of the steady nonlinear Navier-Stokes system in domains with exits at infinity, *Rend. Sem. Mat. Padova*, 97, 235-267 (1997)
- [28] A.M.Robertson, Review of relevant continuum mechanics. Hemodynamical flows, 1-62, Oberwolfach Semin., 37, Birkhäuser, Basel, 2008.
- [29] M. Struwe, Regular solutions of the stationary Navier-Stokes equations on \mathbb{R}^5 , *Math. Ann.* 302, no. 4, 719-741 (1995)
- [30] H. Triebel, Theory of function spaces, Monographs in Mathematics, 78, Birkhäuser Verlag, Basel, 1983.