NONUNIFORM DEPENDENCE ON INITIAL DATA FOR COMPRESSIBLE GAS DYNAMICS: THE CAUCHY PROBLEM ON $\mathbb{R}^2$

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ABSTRACT. The Cauchy problem for the two dimensional compressible Euler equations with data in the Sobolev space $H^s(\mathbb{R}^2)$ is known to have a unique solution of the same Sobolev class for a short time, and the data-to-solution map is continuous. We prove that the data-to-solution map on the plane is not uniformly continuous on any bounded subset of Sobolev class functions.

1. INTRODUCTION

In this paper, we consider the Cauchy problem for the two-dimensional compressible Euler equations with data in the Sobolev space $H^s(\mathbb{R}^2)$. The problem can be written in the form

$$
\begin{cases}
\rho_t + \rho_0 u_x + (\rho u)_x + \rho_0 v_y + (\rho v)_y = 0 \\
u_t + u u_x + v u_y + h_x + \frac{h_0+h}{\rho_0+\rho}\rho_x = 0 \\
v_t + u v_x + v v_y + h_y + \frac{h_0+h}{\rho_0+\rho}\rho_y = 0 \\
h_t + u h_x + v h_y + (\gamma-1)(h_0+h)(u_x+v_y) = 0
\end{cases}
\tag{1.1}
$$

$$
\rho|_{t=0} = \phi_1, \quad u|_{t=0} = \phi_2, \quad v|_{t=0} = \phi_3, \quad h|_{t=0} = \phi_4,
\tag{1.2}
$$

where $\gamma > 1$, $\rho_0 > 0$ and $h_0 > 0$ are constant. In order to arrive at this from the standard form of the equations for ideal compressible gas dynamics (see for example Majda, [34, pp 3-4]), we have written the density as $\rho_0 + \rho$ and have replaced the pressure p by a multiple of the internal energy, $h_0 + h = p/(\rho_0 + \rho)$. The velocity components are u and v. We have also written the system in nonconservative form, as we are considering only classical solutions in this paper. The purpose of the constants ρ_0 and h_0 is to allow us to work with a state variable $U = (\rho, u, v, h)$ whose components lie in the Sobolev space, $H^s(\mathbb{R}^2) = H^s$, defined as

$$
H^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) : \|\mathcal{F}^{-1}\left((1+|\xi|^2)^{s/2}\widehat{f}\right)\|_{L^2(\mathbb{R}^2)} < \infty \right\}.
$$

Pointwise restrictions on the initial data (see discussion following the statement of Theorem 4) allow us to stay a positive distance from a vacuum state.

Local in time well-posedness in the sense of Hadamard for the system (1.1) (in d space dimensions) is well known when $s > 1 + d/2$. The idea of the proof goes back to Gårding [17], Leray, [31], Lax [30] and Kato [25]; a modern version can be found in Taylor's monograph, [40]. For a more detailed exposition of the background and for alternative proofs, see Majda [34] or

Serre [38]. In particular, if the initial data is in the Sobolev space H^s, for any $s > 1 + d/2$, then there exists a unique solution for some time interval which depends upon the H^s norm of the initial data, and the solution depends continuously on the initial condition. In addition, the solution size (in H^s) is bounded by twice the size of the initial condition for some period of time. Classical solutions to the compressible Euler equations do not exist globally in time. Indeed, it has been shown that even for almost constant initial data, there is generally a critical time, T^C, at which the classical (H^s) solution breaks down [34]. This breakdown is characterized by the formation of shock waves; that is, as $t \nearrow T^C$,

$$\limsup_{t \to T^C} \|u_t\|_{L^\infty} + \|\nabla u\|_{L^\infty} = \infty.$$

Weak solutions for quasilinear systems in conservation form (the standard form for (1.1)) have been extensively studied in a single space dimension, where there is a complete well-posedness theory for data of small total variation (and in some cases small oscillation). Excellent monographs by the originators of this theory can be found in Bressan [4] and Dafermos [13].

An outstanding open problem in multidimensional hyperbolic conservation laws is to develop a theory of weak solutions for times after the formation of a shock wave. This is an active area of current research. Čanić, Keyfitz, Jegdić and co-authors (for example, [5, 6, 23] and the recent [24]) have looked at self-similar solutions of two-dimensional problems, as have Chen, Feldman and co-authors, [7, 8] for example). There is also interesting work by Shu-Xing Chen, [9] and other papers, and by Elling, see [15] and references there. An intriguing line of research concerns ill-posedness of multi-dimensional problems of the type of (1.1) in spaces other than H^s; Rauch [37], following Brenner [3], identified key points of this issue, first identified by Littman [32]; and Dafermos [12] and Lopes [33] have followed it up. Yet another question, that concerns the proper definition of weak solutions, is raised by the "wild" weak solutions of De Lellis and Székelyhidi [14]. While this does not seem to bear on the question we tackle here, which concerns classical solutions, it is worth mentioning, both as a note about well-posedness, and as evidence of the relationship between the compressible and incompressible gas dynamics equations, which we exploit in this paper.

The compressible Euler equations can be reduced, in the zero Mach number limit, to the incompressible Euler equations (see [34] or [35] for details on the asymptotic analysis),

$$\begin{aligned}
u_t + uu_x + vu_y + p_x &= 0 \\
v_t + uv_x + vv_y + p_y &= 0 \\
u_x + v_y &= 0,
\end{aligned} \tag{1.3}$$

where p is pressure. Global in time well-posedness is also an important question for the incompressible Euler equations. For a summary of the open questions, we refer the reader to Constantin [10] and Fefferman [16]. For local well-posedness and related results see Majda–Bertozzi [35]. Because the incompressible system is not hyperbolic, analysis of the two problems – (1.1) and (1.3) – has proceeded along rather different lines. This paper finds a rather striking connection.

A point of departure for our analysis is the proof of the non-uniform continuity of the data-to-solution map for the incompressible Euler equations recently established by Himonas and Misiołek [21]. In particular, in dimensions two and three they found solutions for periodic data and for Sobolev space data, for which the data-to-solution map was not uniformly continuous. In the non-periodic (full plane) case, their method used a technique of high-low frequency approximate solutions developed by Koch and Tzvetkov [29] for the one-dimensional Benjamin-Ono equation. Our main result is to show that, in a similar way, dependence on the initial data is not better than continuous for classical solutions of the compressible Euler equations. We state our result as follows. (Here we assume the standard restriction on s, $s > d/2 + 1$.)

Theorem 1. *For $s > 2$, the data-to-solution map for the system* (1.1) *is not uniformly continuous from any bounded subset of* $(H^s(\mathbb{R}^2))^4$ *to the solution space* $C([-T, T]; (H^s(\mathbb{R}^2))^4)$.

Our proof of non-uniform dependence of the data-to-solution map uses a method similar to that of [21] and [29]: construction of high-low frequency approximate solutions. We formulate a different way of defining the low frequency terms. In particular, Koch-Tzvetkov and Himonas-Misiołek use an L^2 energy estimate, while we use an energy estimate in H^σ, $\sigma < s - 1$. We are able to do this by sidestepping the construction of some low frequency exact solutions to the compressible Euler equations. The strategy in this paper is to find estimates in the H^σ norm for σ near s. We find that the low frequency residual terms actually *help* to give the desired estimates by allowing for a crucial cancellation. These convenient cancellations, obtained in our construction, simplify technical difficulties created by the more complicated system of equations. The construction of approximate solutions and demonstration of non-uniformity were first carried out in the ideal compressible gas dynamics system (1.1) for periodic data. This result is in the companion paper of Keyfitz and Tığlay [28] along with the description of the flow for the approximate solutions that we use.

Continuity properties of the data-to-solution map for a variety of equations have been studied by many other authors. In particular, the first result of this type was shown by Kato [25] for Burgers' equation, $u_t + (u^2)_x = 0$. Kato showed that the data-to-solution map is not Hölder continuous from any bounded subset of H^s to H^s, when $s > 3/2$.

The idea of using high-frequency approximate solutions has also been employed extensively in the context of dispersive equations. For example, both Christ, Colliander and Tao [11] and Kenig, Ponce and Vega [27] used a similar method of high-frequency approximate solutions to show ill-posedness of some defocusing dispersive equations. This methodology was also adapted by Himonas and Kenig [19] for the Camassa-Holm (CH) equation on the circle, and by Himonas, Kenig and Misiołek [20] for the CH equation in the non-periodic case. For additional related results concerning the continuity of data-to-solution maps, we refer the reader to Bona and Tzvetkov [2], Holmes [22], Molinet, Saut and Tzvetkov [36] and the references contained therein.

In the next section, we give some preliminary results and notation which we shall use throughout our proof. Section 3 gives the proof of non-uniform dependence.

2. Preliminary Results and Notation

This section summarizes background needed in the rest of the paper. The operator $\Lambda^s f$ is defined by the formula

$$\widehat{\Lambda^s f}(\xi, \eta) = (1 + \xi^2 + \eta^2)^{s/2} \widehat{f}(\xi, \eta) \, ,$$

where f is a test function. Here s may be any positive real number; in order to use the standard existence theorems for classical solutions of (1.1), we take $s > d/2 + 1 = 2$. The notation $\widehat{}$ stands for the usual Fourier transform. The Sobolev space H^s is a Hilbert space equipped with inner product and norm given by

$$\|f\|_s^2 = \langle \Lambda^s f, \Lambda^s f \rangle_{L^2} \, .$$

We will frequently employ the following Sobolev embedding theorem (see, for instance, Taylor [40, p 272]).

Theorem 2 (Sobolev embedding). *If $s > k + 1$, then H^s is continuously embedded into C^k: $H^s \hookrightarrow C^k$. Specifically,*

$$H^s \subset \{f \in C^k : D^\alpha f(x_1, x_2) \to 0 \ as \ |(x_1, x_2)| \to \infty, |\alpha| \le k\}, \tag{2.1}$$

and the inclusion is continuous; for some constant $C(s, k)$ we have

$$\|f\|_{C^k} \doteq \sum_{|\alpha| \le k} \|D^\alpha f\|_{L^\infty} \le C(s, k) \|f\|_s \, . \tag{2.2}$$

We will also liberally employ the following classical product estimate (see for instance Taylor [40, p 66]).

Lemma 1. *If $s > 0$ and $f, g \in L^\infty \cap H^s$, we have the estimate*

$$\|fg\|_s \le C(s) \left[\|f\|_{L^\infty} \|g\|_s + \|f\|_s \|g\|_{L^\infty} \right] \, .$$

This, combined with the Sobolev embedding theorem, implies that H^s is a Banach algebra whenever $s > 1$; in other words, for $f, g \in H^s$, the product $fg \in H^s$. Moreover, we have the algebra estimate

$$\|fg\|_s \le C(s) \|f\|_s \|g\|_s \, . \tag{2.3}$$

For any test function f, the commutator operator $[\Lambda^s, f]$ applied to a test function g is

$$[\Lambda^s, f] g = \Lambda^s(fg) - f \Lambda^s g \, . \tag{2.4}$$

The following commutator estimate can be found in Kato and Ponce [26].

Theorem 3 (Kato-Ponce commutator estimate). *If $s \ge 0$, and $f \in Lip \cap H^s$ and $g \in L^\infty \cap H^{s-1}$, then*

$$\|[\Lambda^s, f] g\|_{L^2} \le C(s) \left(\|\partial f\|_{L^\infty} \|\Lambda^{s-1} g\|_{L^2} + \|\Lambda^s f\|_{L^2} \|g\|_{L^\infty} \right) \, . \tag{2.5}$$

In the proof of Lemma 7 we need a simple interpolation estimate:

Proposition 1. *If $u \in H^\tau$ and $\sigma < s < \tau$ then*

$$\|u\|_s \leq \|u\|_\sigma^\alpha \|u\|_\tau^\beta, \quad where \quad \alpha = \frac{\tau - s}{\tau - \sigma}, \quad \beta = \frac{s - \sigma}{\tau - \sigma}.$$

Proof. We write

$$\|u\|_s^2 = \int \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 \, d\xi = \int \left[\left(1 + |\xi|^2\right)^\sigma |\hat{u}(\xi)|^2\right]^\alpha \left[\left(1 + |\xi|^2\right)^\tau |\hat{u}(\xi)|^2\right]^\beta \, d\xi$$

and apply Hölder's inequality with $p = 1/\alpha$ and $q = 1/\beta$. $\square$

Finally, owing to the nature of the nonlinearities in (1.1), we need the following reciprocal estimate. It was proved by Kato, [25, Lemma 2.13], for functions in "uniformly local" Sobolev spaces (which generalize our construction of coefficients of the form $\rho_0 + \rho$), and for integer values of $s > 2$. We provide a sketch of the proof in the delicate case when $1 < s < 2$; the larger value of s are straightforward.

Lemma 2. *If $s > 1$, $h \in H^s(\mathbb{R}^2)$, $g \in H^s(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)$ and $b > 0$ is a constant such that $g + b > \frac{1}{2}b$, then*

$$\left\|\frac{h}{g + b}\right\|_s \leq C(s, b) \left(1 + \|g\|_{C^1}^s + \|g\|_s^s\right) \|h\|_s.$$

Proof. In the case $s = 1 + \gamma$ with $\gamma \in (0, 1)$, the integer parts of the norm satisfy this bound as in Kato, [25, Lemma 2.13]. The fractional portion of the norm (see [39, page 155] for instance, for this form of the Sobolev norm) is

$$\sup_{|\alpha|=1} \|D^\alpha \left(h(g + b)^{-1}\right)\|_{\dot{H}^\gamma}^2 \equiv \sup_{|\alpha|=1} \int_{\mathbb{R}^4} \frac{|D^\alpha \left(h(g + b)^{-1}\right)(\mathbf{x}) - D^\alpha \left(h(g + b)^{-1}\right)(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{2\gamma+2}} d\mathbf{x} \, d\mathbf{y},$$

where $\mathbf{x}$ and $\mathbf{y}$ are points in $\mathbb{R}^2$ and $\dot{H}^\gamma$ is the homogeneous Sobolev space. Consider $D^\alpha = \partial_1$ (the partial derivative with respect to the first component) so that

$$\partial_1 \left(h(g + b)^{-1}\right) = (g + b)^{-1}\partial_1 h - h(g + b)^{-2}\partial_1 g. \tag{2.6}$$

Estimating the first term on the right hand side of (2.6) is a straightforward calculation after breaking the integral into the following two pieces

$$\|(g + b)^{-1}\partial_1 h\|_{\dot{H}^\gamma}^2 \leq 2 \int \frac{1}{|g(\mathbf{x}) + b|^2} \frac{|\partial_1 h(\mathbf{x}) - \partial_1 h(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{2\gamma+2}} d\mathbf{x} d\mathbf{y}$$

$$+ 2 \int \frac{|g(\mathbf{x}) - g(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{2\gamma+2}} \frac{1}{|g(\mathbf{x}) + b|^2} d\mathbf{x} \frac{|\partial_1 h(\mathbf{y})|^2}{|g(\mathbf{y}) + b|^2} d\mathbf{y}. \tag{2.7}$$

The first integral is bounded by an application of Hölder's inequality, while the second term additionally requires the Sobolev embedding theorem and the following calculus estimate:

$$\sup_{\mathbf{y} \in \mathbb{R}^2} \int \frac{|g(\mathbf{x}) - g(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{2\gamma+2}} d\mathbf{x} \leq C(\gamma)\|g\|_{C^1}^2.$$

This estimate is obtained by splitting the domain of integration into two pieces, $|x - y| < 1$ and $|x - y| \geq 1$, and then applying the mean value theorem. Returning to equation (2.6), the second term on the right hand side is bounded by Lemma 1 and the Sobolev embedding theorem:

$$\|h(g+b)^{-2}\partial_1 g\|_\gamma \leq \|(g+b)^{-2}\partial_1 g\|_\gamma \|h\|_{L^\infty} + \|(g+b)^{-2}\partial_1 g\|_{L^\infty}\|h\|_\gamma$$

$$\leq \left(\frac{4}{b^2}\|g\|_s + \|(g+b)^{-2}\partial_1 g\|_{\dot{H}^\gamma} \right) \|h\|_s + \frac{4}{b^2}\|g\|_{C^1}\|h\|_s.$$

We bound $\|(g+b)^{-2}\partial_1 g\|_{\dot{H}^\gamma}$ in the same way as $\|(g+b)^{-1}\partial_1 h\|_{\dot{H}^\gamma}$. The same estimates hold for ∂_2. $\qquad\square$

3. Proof of Nonuniform Dependence

We write the compressible Euler system (1.1) in the form

$$U_t + A(U)U_x + B(U)U_y = 0, \tag{3.1}$$

with $U = (\rho, u, v, h)^T$ and

$$A(U) = \begin{pmatrix} u & \rho_0 + \rho & 0 & 0 \\ \frac{h_0 + h}{\rho_0 + \rho} & u & 0 & 1 \\ 0 & 0 & u & 0 \\ 0 & (\gamma - 1)(h_0 + h) & 0 & u \end{pmatrix}, \quad B(U) = \begin{pmatrix} v & 0 & \rho_0 + \rho & 0 \\ 0 & v & 0 & 0 \\ \frac{h_0 + h}{\rho_0 + \rho} & 0 & v & 1 \\ 0 & 0 & (\gamma - 1)(h_0 + h) & v \end{pmatrix}.$$

3.1. **Symmetrized system.** The system (3.1) is symmetrizable; that is, it can be written as

$$A_0 U_t + A_1(U)U_x + B_1(U)U_y = 0,$$

where the matrices A_0, A_1, B_1 are symmetric and A_0 is positive definite. We can choose

$$A_0(U) = \begin{pmatrix} \frac{h_0 + h}{\rho_0 + \rho} & 0 & 0 & 0 \\ 0 & \rho_0 + \rho & 0 & 0 \\ 0 & 0 & \rho_0 + \rho & 0 \\ 0 & 0 & 0 & \frac{\rho_0 + \rho}{(\gamma - 1)(h_0 + h)} \end{pmatrix};$$

$$A_1(U) = \begin{pmatrix} \frac{h_0 + h}{\rho_0 + \rho}u & h_0 + h & 0 & 0 \\ h_0 + h & (\rho_0 + \rho)u & 0 & \rho_0 + \rho \\ 0 & 0 & (\rho_0 + \rho)u & 0 \\ 0 & \rho_0 + \rho & 0 & \frac{(\rho_0 + \rho)u}{(\gamma - 1)(h_0 + h)} \end{pmatrix};$$

$$B_1(U) = \begin{pmatrix} \frac{h_0 + h}{\rho_0 + \rho}v & 0 & h_0 + h & 0 \\ 0 & (\rho_0 + \rho)v & 0 & 0 \\ h_0 + h & 0 & (\rho_0 + \rho)v & \rho_0 + \rho \\ 0 & 0 & \rho_0 + \rho & \frac{(\rho_0 + \rho)v}{(\gamma - 1)(h_0 + h)} \end{pmatrix}.$$

3.2. **Approximate solutions.** Our strategy, following the template laid out by Himonas and Misiołek [21], is to use two sequences ($\omega = \pm 1$) of approximate solutions:

$$U^{\omega,n} = \begin{pmatrix} \rho^{\omega,n} \\ u^{\omega,n} \\ v^{\omega,n} \\ h^{\omega,n} \end{pmatrix} = \begin{pmatrix} 0 \\ u_1 + u_2 \\ v_1 + v_2 \\ 0 \end{pmatrix}. \tag{3.2}$$

The approximate solutions contain low frequency functions u_1, v_1 and high frequency functions u_2 and v_2. (Our notation suppresses, for clarity, the dependence of the u_i and v_i on n and ω.) The high frequency functions are defined for a constant $\delta > 0$ as

$$u_2 = n^{-\delta-s-1}\partial_y S, \text{ and } v_2 = -n^{-\delta-s-1}\partial_x S, \tag{3.3}$$

where S is a stream function, given by

$$S(x,y,t) = \psi(n^{-\delta}x)\psi(n^{-\delta}y)\sin(ny+\omega t),$$

for a compactly supported nonnegative cutoff function ψ which equals one on $[-2,2]$. Expanding u_2 and v_2 gives

$$u_2 = n^{-2\delta-s-1}\psi(n^{-\delta}x)\psi'(n^{-\delta}y)\sin(ny+\omega t) + n^{-\delta-s}\psi(n^{-\delta}x)\psi(n^{-\delta}y)\cos(ny+\omega t)$$
$$v_2 = -n^{-2\delta-s-1}\psi'(n^{-\delta}x)\psi(n^{-\delta}y)\sin(ny+\omega t). \tag{3.4}$$

The low frequency functions, u_1 and v_1, are

$$u_1 = \omega n^{-1}\varphi_1\left(n^{-\delta}x\right)\varphi_2'\left(n^{-\delta}y\right), \text{ and } v_1 = -\omega n^{-1}\varphi_1'\left(n^{-\delta}x\right)\varphi_2\left(n^{-\delta}y\right), \tag{3.5}$$

where φ_1' and φ_2 are also smooth compactly supported functions; φ_1' is identically 1 on the support of ψ' and $\varphi_2 \equiv 1$ on $\operatorname{supp}\psi$. The following cancellation holds.

Lemma 3. *For u and v defined in (3.2), (3.4), and (3.5), we have $\partial_x u^{\omega,n} + \partial_y v^{\omega,n} = 0$.*

Proof. We have

$$u_{1,x} = \frac{\omega}{n^{1+\delta}}\phi_1'\left(\frac{x}{n^\delta}\right)\phi_2'\left(\frac{y}{n^\delta}\right) = -v_{1,y}.$$

Considering the high frequency terms, we see from (3.3)

$$\partial_x u_2 + \partial_y v_2 = n^{-\delta-s-1}\partial_x\partial_y S - n^{-\delta-s-1}\partial_y\partial_x S = 0. \qquad \square$$

As a result of the definition, the approximate solutions satisfy

$$U_t^{\omega,n} + A(U^{\omega,n})U_x^{\omega,n} + B(U^{\omega,n})U_y^{\omega,n} = R,$$

where

$$R = \begin{pmatrix} 0 \\ R_2 \\ R_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_t u^{\omega,n} + u^{\omega,n}\partial_x u^{\omega,n} + v^{\omega,n}\partial_y u^{\omega,n} \\ \partial_t v^{\omega,n} + u^{\omega,n}\partial_x v^{\omega,n} + v^{\omega,n}\partial_y v^{\omega,n} \\ 0 \end{pmatrix}. \tag{3.6}$$

Denote the inner product of two vectors, V and W, by $\langle V, W \rangle = \sum \langle V_i, W_i \rangle_{L^2}$, and for any vector U denote

$$\|U\|_\sigma^2 = \langle \Lambda^\sigma U, \Lambda^\sigma U \rangle = \|\rho\|_\sigma^2 + \|u\|_\sigma^2 + \|v\|_\sigma^2 + \|h\|_\sigma^2. \tag{3.7}$$

Let $U_{\omega,n} = (\rho_{\omega,n}, u_{\omega,n}, v_{\omega,n}, h_{\omega,n})^T$ be the actual solution to the Cauchy problem corresponding to equation (3.1), with the same data:

$$U_{\omega,n}(x,y,0) = U^{\omega,n}(x,y,0) = \left(0, u_1(x,y,0) + u_2(x,y,0), v_1(x,y,0) + v_2(x,y,0), 0\right),$$

again with $\omega = 1$ or -1.

The actual solution is unique, and exists on a time interval which depends only upon the size (in the H^s norm) of the initial data and on its distance from the boundary of the region of state space (called G in the statement below) where the system is hyperbolic. We quote the following theorem found in [34].

Theorem 4 (Majda, Theorem 2.1). *Assume* $U(\cdot, 0) = U_0 \in H^s$, $s > d/2 + 1$ *and* $U_0(x) \in G_1$, $\bar{G}_1 \subset\subset G$. *Then there is a time interval* $[0, T]$ *with* $T > 0$, *so that the equations* (1.1) *have a unique classical solution* $U \in C([0,T]; (H^s)^4) \cap C^1([0,T]; (H^{s-1})^4)$, *and* $U(x,t) \in G_2$, $\overline{G}_2 \subset\subset G$, *for* $(x,t) \in \mathbb{R}^2 \times [0,T]$; *here* $T = T(\|U_0\|_s, G_1)$.

In our coordinate system, $G = \{\rho > -\rho_0\}$. Having specified values for $\rho_0 > 0$ and h_0, we might choose, for example, data to lie in a bounded set

$$G_1 = \left\{ -\frac{1}{4}\rho_0 < \rho(\cdot, 0) < M_\rho, |u(\cdot, 0)| < M_u, |v(\cdot, 0)| < M_u, -\frac{1}{4}h_0 < h(\cdot, 0) < M_h \right\},$$

and then take G_2 to be

$$G_2 = \left\{ -\frac{1}{2}\rho_0 < \rho(\cdot, 0) < 2M_\rho, |u(\cdot, 0)| < 2M_u, |v(\cdot, 0)| < M_u, -\frac{1}{2}h_0 < h(\cdot, 0) < 2M_h \right\},$$

where M_ρ, M_u and M_h are positive numbers. The significant bound, which we need throughout, is the lower bound on ρ in G_2. Additionally, continuous dependence on the data yields the following H^s solution size estimate.

Theorem 5 ([34], Theorem 2.2). *Assume* $U|_{t=0} \in H^s$, $s > 2$ *and* $U|_{t=0} \in G_1$. *There exists a* T^*, $0 < T^* \leq T$ *such that*

$$\sup_{t \in [0,T^*]} \|U\|_s \leq 2\|U|_{t=0}\|_s.$$

In what follows, we take T^* to be the value given by this theorem.

We obtain the proof of Theorem 1 by showing the following properties of the corresponding solutions:

(1) Boundedness of initial data (proved in Section 3.3):

$$\|U^{\omega,n}(\cdot, 0)\|_s = \|U_{\omega,n}(\cdot, 0)\|_s = \|(\rho_{\omega,n}(0), u_{\omega,n}(0), v_{\omega,n}(0), h_{\omega,n}(0))\|_s \leq C, \tag{3.8}$$

uniformly in n.

(2) Convergence of initial data (Section 3.3): for $\delta < 1/2$,

$$\lim_{n \to \infty} \|U_{1,n}(\cdot, 0) - U_{-1,n}(\cdot, 0)\|_s = 0. \tag{3.9}$$

(3) Uniformity of approximation of $U^{\omega,n}$ to actual solution $U_{\omega,n}$ (Section 3.4):

$$\|U_{\omega,n}(\cdot, t) - U^{\omega,n}(\cdot, t)\|_s \le Cn^{-\varepsilon}, \qquad 0 < t < T^*, \tag{3.10}$$

for some $\varepsilon > 0$.

(4) Non-uniformity of divergence of $U_{1,n}$ and $U_{-1,n}$ from each other in time (Section 3.7):

$$\|U_{1,n}(\cdot, t) - U_{-1,n}(\cdot, t)\|_s > |\sin(t)|, \qquad 0 < t. \tag{3.11}$$

The following estimates can be found in the appendix of [21].

Lemma 4. *Let $\sigma \ge 0$, $\delta > 0$ and $n \gg 1$. For any Schwarz function $\psi \in \mathcal{S}(\mathbb{R})$ we have*

$$\|\psi(n^{-\delta}\cdot)\|_{H^\sigma(\mathbb{R})} \le n^{\delta/2}\|\psi\|_{H^\sigma(\mathbb{R})}. \tag{3.12}$$

For any constant $a \in \mathbb{R}$ we have

$$\|\psi(n^{-\delta}\cdot)\sin(n\cdot+a)\|_{H^\sigma(\mathbb{R})} + \|\psi(n^{-\delta}\cdot)\cos(n\cdot+a)\|_{H^\sigma(\mathbb{R})} \approx n^{\sigma+\delta/2}\|\psi\|_{L^2(\mathbb{R})}. \tag{3.13}$$

The notation $\approx$ means that the expression on the left is bounded above and below by constants independent of σ, δ and n. Note that the L^2 bound implies an H^σ bound. From this lemma we obtain bounds on the approximate solutions:

Lemma 5. *For $s - 2 < \sigma < s - 1$ and $0 < \delta < 1$, we have*

$$\|U^{\omega,n}\|_{\sigma+1} \le Cn^{\sigma-s+1},$$

where C depends on the norms of the functions ψ, ϕ_1 and ϕ_2.

Proof. The nonzero terms in $U^{\omega,n}$ are u_1, u_2, v_1 and v_2. Since u_1 and v_1 are products (in x and y) of terms of the form $\psi(n^{-\delta}\cdot)$, we have, from Lemma 4, and with C a generic constant,

$$\|u_1\|_{\sigma+1} = n^{-1}\|\phi_1(n^{-\delta}x)\|_{\sigma+1}\|\phi_2'(n^{-\delta}y)\|_{\sigma+1} \le n^{-1}n^{\delta/2}\|\phi_1\|_{\sigma+1}n^{\delta/2}\|\phi_2'\|_{\sigma+1} = Cn^{-1+\delta}.$$

A similar bound holds for v_1, which has the same structure. Note that these bounds are valid for any σ. On the other hand,

$$\begin{aligned}
\|u_2\|_{\sigma+1} \le &n^{-2\delta-s-1}\|\psi(n^{-\delta}x)\|_{\sigma+1}\|\psi'(n^{-\delta}y)\sin(ny+\omega t)\|_{\sigma+1} \\
&+ n^{-\delta-s}\|\psi(n^{-\delta}x)\|_{\sigma+1}\|\psi(n^{-\delta}y)\cos(ny+\omega t)\|_{\sigma+1} \\
\le &n^{-2\delta-s-1}n^{\delta/2}\|\psi\|_{\sigma+1}n^{\sigma+1+\delta/2}\|\psi'\|_{\sigma+1} + n^{-\delta-s}n^{\delta/2}\|\psi\|_{\sigma+1}n^{\sigma+1+\delta/2}\|\psi\|_{\sigma+1} \\
\le &Cn^{-\delta+\sigma-s} + Cn^{\sigma-s+1},
\end{aligned}$$

while v_2, which has the structure of the first term in u_2, satisfies

$$\|v_2\|_{\sigma+1} \le Cn^{-\delta+\sigma-s}.$$

Now, δ is a positive number and $\sigma < s - 1$, so all the exponents of n are negative if $\delta < 1$. To bound the low frequency terms by the high frequency terms we need $-1 + \delta < \sigma - s + 1$, or

$\delta < \sigma - s + 2$, and provided $\sigma > s - 2$, as we have assumed, it is possible to achieve this with $\delta > 0$. $\qquad\square$

Lemma 5 implies a bound on the actual solution, using Theorem 5.

Corollary 1. *If $t \leq T^*$, then*

$$\|U_{\omega,n}\|_{\sigma+1} \leq Cn^{\sigma+1-s}$$

where T^*, as in Theorem 5, is the time to doubling of the initial norm, and $\sigma < s - 1$.

3.3. **(1) Boundedness and (2) Convergence of the Initial Data.** From Lemma 4, we have $\|U^{\omega,n}(\cdot,0)\|_s \leq Cn^{-1+\delta} + C$. For any δ with $0 < \delta < 1$ we have $\|U^{\omega,n}(0)\|$ bounded, uniformly in n.

To see that the difference in the initial data for $\omega = \pm 1$ converges to zero in H^s, we calculate $U^{1,n} - U^{-1,n}$ at $t = 0$, noting that the oscillatory terms cancel at $t = 0$, leaving only

$$\begin{pmatrix} \rho_{1,n}(0) - \rho_{-1,n}(0) \\ u_{1,n}(0) - u_{-1,n}(0) \\ v_{1,n}(0) - v_{-1,n}(0) \\ h_{1,n}(0) - h_{-1,n}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2n^{-1}\varphi_1\left(n^{-\delta}x\right)\varphi_2'\left(n^{-\delta}y\right) \\ 2n^{-1}\varphi_1'\left(n^{-\delta}x\right)\varphi_2\left(n^{-\delta}y\right) \\ 0 \end{pmatrix}.$$

This tends to zero in H^s by the first estimate in Lemma 4, for any $\delta \in (0,1)$, as in the first estimate in the proof of Lemma 5.

3.4. **(3) Uniformity of the Approximation.** In this subsection, we denote the actual solutions by U. Let $\mathcal{E} = U - U^{\omega,n} = (E, F, G, H)^T$ be the error, the difference between the actual and approximate solutions. The main result of this section is

Theorem 6. *For $\max\{1, s-2\} < \sigma < s - 1$, $\mathcal{E}$ satisfies*

$$\frac{d}{dt}\|\mathcal{E}\|_\sigma \lesssim n^{\sigma+1-s}\|\mathcal{E}\|_\sigma + n^{\delta-2}.$$

Furthermore, we have on the time interval of existence

$$\|\mathcal{E}\|_\sigma \lesssim n^{\delta-3+s-\sigma}.$$

Proof. An equation for the error (the symmetric form of the system is useful here) is

$$A_0(U^{\omega,n})\mathcal{E}_t + A_1(U^{\omega,n})\mathcal{E}_x + B_1(U^{\omega,n})\mathcal{E}_y + C(U)\mathcal{E} + A_0(U^{\omega,n})R = 0 \qquad (3.14)$$

where

$$C(U) = A_0(U^{\omega,n}) \begin{pmatrix} u_x + v_y & \rho_x & \rho_y & 0 \\ -\frac{h_0\rho_x}{\rho\rho_0+\rho_0^2} & u_x & u_y & \frac{\rho_x}{\rho+\rho_0} \\ -\frac{h_0\rho_y}{\rho\rho_0+\rho_0^2} & v_x & v_y & \frac{\rho_y}{\rho+\rho_0} \\ 0 & h_x & h_y & (\gamma-1)(u_x+v_y) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{(u_x+v_y)h_0}{\rho_0} & \frac{h_0}{\rho_0}\rho_x & \frac{h_0}{\rho_0}\rho_y & 0 \\ -\frac{h_0\rho_x}{\rho+\rho_0} & \rho_0 u_x & \rho_0 u_y & \frac{\rho_0\rho_x}{\rho+\rho_0} \\ -\frac{h_0\rho_y}{\rho+\rho_0} & \rho_0 v_x & \rho_0 v_y & \frac{\rho_0\rho_y}{\rho+\rho_0} \\ 0 & \frac{\rho_0}{(\gamma-1)h_0}h_x & \frac{\rho_0}{(\gamma-1)h_0}h_y & \frac{\rho_0(u_x+v_y)}{h_0} \end{pmatrix}.$$

We write $C(U)\mathcal{E}$ as

$$C(U)\mathcal{E} = \begin{pmatrix} \frac{(u_x+v_y)h_0}{\rho_0}E + \frac{h_0}{\rho_0}\rho_x F + \frac{h_0}{\rho_0}\rho_y G \\ -\frac{h_0\rho_x}{\rho+\rho_0}E + \rho_0 u_x F + \rho_0 u_y G + \frac{\rho_0\rho_x}{\rho+\rho_0}H \\ -\frac{h_0\rho_y}{\rho+\rho_0}E + \rho_0 v_x F + \rho_0 v_y G + \frac{\rho_0\rho_y}{\rho+\rho_0}H \\ \frac{\rho_0}{(\gamma-1)h_0}h_x F + \frac{\rho_0}{(\gamma-1)h_0}h_y G + \frac{\rho_0(u_x+v_y)}{h_0}H \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}.$$

We apply the operator Λ^σ, where $\sigma > 1$ and $s - 2 < \sigma < s - 1$, to the left hand side of (3.14) and then take the inner product with $\Lambda^\sigma \mathcal{E}$ to obtain

$$\langle \Lambda^\sigma (A_0(U^{\omega,n})\mathcal{E}_t), \Lambda^\sigma \mathcal{E} \rangle = -\langle \Lambda^\sigma (C(U)\mathcal{E}), \Lambda^\sigma \mathcal{E} \rangle \tag{3.15}$$

$$-\langle \Lambda^\sigma (\mathrm{diag}(A_1(U^{\omega,n}))\mathcal{E}_x + \mathrm{diag}(B_1(U^{\omega,n}))\mathcal{E}_y), \Lambda^\sigma \mathcal{E} \rangle \tag{3.16}$$

$$-\langle \Lambda^\sigma (A_R(U^{\omega,n})\mathcal{E}_x + B_R(U^{\omega,n})\mathcal{E}_y), \Lambda^\sigma \mathcal{E} \rangle \tag{3.17}$$

$$-\langle \Lambda^\sigma A_0(U^{\omega,n})R, \Lambda^\sigma \mathcal{E} \rangle, \tag{3.18}$$

where $\mathrm{diag}(A)$ denotes the diagonal part of a matrix A and $A_R = A - \mathrm{diag}(A)$. We now bound the terms on the right hand side.

Estimate for (3.15). We have

$$\langle \Lambda^\sigma C(U)\mathcal{E}, \Lambda^\sigma \mathcal{E} \rangle = \langle \Lambda^\sigma C_1, \Lambda^\sigma E \rangle + \langle \Lambda^\sigma C_2, \Lambda^\sigma F \rangle + \langle \Lambda^\sigma C_3, \Lambda^\sigma G \rangle + \langle \Lambda^\sigma C_4, \Lambda^\sigma H \rangle. \tag{3.19}$$

These terms are all estimated in a similar way. The Cauchy-Schwarz inequality yields

$$|\langle \Lambda^\sigma C_1, \Lambda^\sigma E \rangle| \le \|\Lambda^\sigma C_1\|_{L_2}\|\Lambda^\sigma E\|_{L_2} = \|C_1\|_\sigma\|E\|_\sigma \le \|C_1\|_\sigma\|\mathcal{E}\|_\sigma$$

and so on for the other three terms. To estimate $\|C_i\|_\sigma$, we note that all of the C_i are of the form

$$C_i = a_1 E + a_2 F + a_3 G + a_4 H, \tag{3.20}$$

where, up to constant multiples, each a_j consists of a derivative, or sum of derivatives, of components of U, in some cases divided by $\rho + \rho_0$. So, taking C_2 as an example, and looking at the first summand, we have

$$\|a_1 E\|_\sigma = h_0 \left\|\frac{\rho_x}{\rho+\rho_0}E\right\|_\sigma \le h_0 \left\|\frac{\rho_x}{\rho+\rho_0}\right\|_\sigma \|E\|_\sigma \le h_0 \left\|\frac{\rho_x}{\rho+\rho_0}\right\|_\sigma \|\mathcal{E}\|_\sigma, \tag{3.21}$$

where we have used the algebra property, Lemma 2.3. Now we use Lemma 2 to obtain

$$\left\|\frac{\rho_x}{\rho + \rho_0}\right\|_\sigma \leq C(\sigma, \rho_0)\big(1 + \|\rho\|_\sigma^\sigma\big)\|\rho_x\|_\sigma \leq C\|\rho\|_{\sigma+1},$$

since $\|\rho_x\|_\sigma \leq \|\rho\|_{\sigma+1}$, and from Corollary 1 we can absorb all the other factors into a constant that depends on σ, ρ_0 and on the H^σ bound on ρ. Finally, estimating $\|\rho\|_{\sigma+1} \leq Cn^{\sigma-s+1}$ as in Corollary 1, and treating the other terms in (3.20) in the same way as (3.21), we have

$$\langle \Lambda^\sigma C(U)\mathcal{E}, \Lambda^\sigma \mathcal{E}\rangle \leq Cn^{\sigma-s+1}\|\mathcal{E}\|_\sigma^2,$$

with a constant C that depends upon ρ_0, h_0, γ and σ. (Since $\|U\|_{\sigma+1}$ decreases with n, we can eliminate the dependence of the constant on U.)

Estimate of (3.16). We have (up to a sign)

$$(3.16) = \langle \Lambda^\sigma \left(\text{diag}(A_1(U^{\omega,n}))\mathcal{E}_x + \text{diag}(B_1(U^{\omega,n}))\mathcal{E}_y\right), \Lambda^\sigma \mathcal{E}\rangle$$

$$= \left\langle \Lambda^\sigma \left(\frac{h_0}{\rho_0}u^{\omega,n}E_x + \frac{h_0}{\rho_0}v^{\omega,n}E_y\right), \Lambda^\sigma E\right\rangle_{L^2} + \langle \Lambda^\sigma (\rho_0 u^{\omega,n}F_x + \rho_0 v^{\omega,n}F_y), \Lambda^\sigma F\rangle_{L^2}$$

$$+ \langle \Lambda^\sigma (\rho_0 u^{\omega,n}G_x + \rho_0 v^{\omega,n}G_y), \Lambda^\sigma G\rangle_{L^2} + \left\langle \Lambda^\sigma \left(\frac{\rho_0 u^{\omega,n}}{(\gamma-1)h_0}H_x + \frac{\rho_0 v^{\omega,n}}{(\gamma-1)h_0}H_y\right), \Lambda^\sigma H\right\rangle_{L^2}.$$

The eight terms in this expression are similar to each other; we show how the first is estimated. Ignoring the constant h_0/ρ_0, consider

$$I_1 \equiv \langle \Lambda^\sigma \left(u^{\omega,n}E_x\right), \Lambda^\sigma E\rangle_{L^2} = \int_{\mathbb{R}^2} \Lambda^\sigma \left(u^{\omega,n}E_x\right) \Lambda^\sigma E \, dxdy.$$

This can be written as (recall equation (2.4) for the definition of the commutator)

$$I_1 = \int_{\mathbb{R}^2} \left([\Lambda^\sigma, (u^{\omega,n})] E_x + (u^{\omega,n}) \Lambda^\sigma \partial_x E\right) \Lambda^\sigma E \, dxdy.$$

We split this integral into two pieces, and apply the Cauchy-Schwarz estimate to the first term, to obtain

$$I_1 \leq \|[\Lambda^\sigma, (u^{\omega,n})] E_x\|_{L^2} \|E\|_\sigma + \left|\int_{\mathbb{R}^2} u^{\omega,n}\Lambda^\sigma \partial_x E \Lambda^\sigma E \, dxdy\right|.$$

Now, the Kato-Ponce commutator estimate, (2.5), applied to the first factor gives

$$\|[\Lambda^\sigma, (u^{\omega,n})] E_x\|_{L^2} \leq C(\sigma) \left(\|u_x^{\omega,n}\|_{L^\infty}\|\Lambda^{\sigma-1}E_x\|_{L_2} + \|\Lambda^\sigma u^{\omega,n}\|_{L_2}\|E_x\|_{L^\infty}\right)$$

$$\leq C(\sigma) \left(\|u^{\omega,n}\|_{\sigma+1}\|\Lambda^\sigma E\|_{L_2} + \|u^{\omega,n}\|_\sigma\|E_x\|_{L^\infty}\right),$$

using the Sobolev embedding theorem, Theorem 2, which applies here since $\sigma + 1 > 2$. Since we can replace $\|u^{\omega,n}\|_\sigma$ by $\|u^{\omega,n}\|_{\sigma+1}$, and, using the same Sobolev embedding, replace $\|E_x\|_{L^\infty}$ by $\|E\|_\sigma$, we obtain

$$\|[\Lambda^\sigma, (u^{\omega,n})] E_x\|_{L^2} \leq C(s)\|u^{\omega,n}\|_{\sigma+1}\|E\|_\sigma^2.$$

For the second term, integration by parts followed by Hölder's inequality yields

$$\left| \int_{\mathbb{R}^2} u^{\omega,n} \Lambda^\sigma \partial_x E \Lambda^\sigma E \, dxdy \right| = \left| \frac{1}{2} \int_{\mathbb{R}^2} u_x^{\omega,n} (\Lambda^\sigma E)^2 \, dxdy \right| \leq \frac{\|u_x^{\omega,n}\|_{L^\infty}}{2} \int_{\mathbb{R}^2} (\Lambda^\sigma E)^2 \, dxdy$$
$$= \frac{1}{2} \|u_x^{\omega,n}\|_{L^\infty} \|E\|_\sigma^2 \,,$$

and we get a bound similar to the first term, so that

$$I_1 \leq C \, \|u^{\omega,n}\|_{\sigma+1} \|E\|_\sigma^2 \leq C n^{\sigma+1-s} \|E\|_\sigma^2 \,,$$

from Corollary 1, with $C = C(\sigma)$. Proceeding the same way with the other seven terms, we obtain

$$|\langle \Lambda^\sigma \left(\mathrm{diag}(A_1(U^{\omega,n})) \mathcal{E}_x + \mathrm{diag}(B_1(U^{\omega,n})) \mathcal{E}_y \right), \Lambda^\sigma \mathcal{E} \rangle| \leq C n^{\sigma+1-s} \|\mathcal{E}\|_\sigma^2,$$

with the constant depending on on ρ_0, h_0, γ and σ.

Estimate of (3.17). Inserting the off-diagonal elements of A_1 and B_1 from Section 3.1 (note that they are all constant since $h^{\omega,n} = 0 = \rho^{\omega,n}$), we have

$$-(3.17) = \langle \Lambda^\sigma \left(h_0 F_x + h_0 G_y \right), \Lambda^\sigma E \rangle_{L^2} + \langle \Lambda^\sigma \left(h_0 E_x + \rho_0 H_x \right), \Lambda^\sigma F \rangle_{L^2}$$
$$+ \langle \Lambda^\sigma \left(h_0 E_y + \rho_0 H_y \right), \Lambda^\sigma G \rangle_{L^2} + \langle \Lambda^\sigma \left(\rho_0 F_x + \rho_0 G_y \right), \Lambda^\sigma H \rangle_{L^2} \,.$$

Writing the above as an integral and rearranging terms gives

$$-(3.17) = \int_{\mathbb{R}^2} h_0 \left(\partial_x (\Lambda^\sigma E) \Lambda^\sigma F + \Lambda^\sigma E \, \partial_x (\Lambda^\sigma F) + \partial_y (\Lambda^\sigma E) \Lambda^\sigma G + \Lambda^\sigma E \, \partial_y (\Lambda^\sigma G) \right) dx \, dy$$
$$+ \int_{\mathbb{R}^2} \rho_0 \left(\partial_x (\Lambda^\sigma F) \Lambda^\sigma H + \Lambda^\sigma F \, \partial_x (\Lambda^\sigma H) + \partial_y (\Lambda^\sigma G) \Lambda^\sigma H + \Lambda^\sigma G \, \partial_y (\Lambda^\sigma H) \right) dx \, dy$$
$$= h_0 \int_{\mathbb{R}^2} \partial_x (\Lambda^\sigma E \, \Lambda^\sigma F) + \partial_y (\Lambda^\sigma E \, \Lambda^\sigma G) \, dx \, dy$$
$$+ \rho_0 \int_{\mathbb{R}^2} \partial_x (\Lambda^\sigma F \, \Lambda^\sigma H) + \partial_y (\Lambda^\sigma G \, \Lambda^\sigma H) \, dx \, dy$$

and therefore they all integrate to zero.

Estimate of (3.18). Since A_0 is diagonal and $A_0(U^{\omega,n})$ is constant, we have

$$-(3.18) = \langle \Lambda^\sigma A_0(U^{\omega,n}) R, \Lambda^\sigma \mathcal{E} \rangle = \langle \rho_0 \Lambda^\sigma R_2, \Lambda^\sigma F \rangle + \langle \rho_0 \Lambda^\sigma R_3, \Lambda^\sigma G \rangle$$

The Cauchy-Schwarz inequality yields

$$|(3.18)| \leq \rho_0 \|R\|_\sigma \|\mathcal{E}\|_\sigma \,.$$

Combining the estimates for (3.15), (3.16), (3.17) and (3.18) we have

$$\langle \Lambda^\sigma \left(A_0(U^{\omega,n}) \mathcal{E}_t \right), \Lambda^\sigma \mathcal{E} \rangle \leq C n^{\sigma+1-s} \|\mathcal{E}\|_{H^\sigma}^2 + C \|R\|_\sigma \|\mathcal{E}\|_\sigma, \tag{3.22}$$

where the constants depend upon ρ_0, h_0, γ and σ.

We show that the residue R satisfies the following estimate.

Proposition 2. *If* $\max\{1, s-2\} < \sigma < s-1$, *then* $\|R\|_\sigma \lesssim n^{\delta-2}$.

Proof. (3.6) the nonzero components of R are

$$\begin{pmatrix} R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} \partial_t u^{\omega,n} + u^{\omega,n}\partial_x u^{\omega,n} + v^{\omega,n}\partial_y u^{\omega,n} \\ \partial_t v^{\omega,n} + u\partial_x v^{\omega,n} + v^{\omega,n}\partial_y v^{\omega,n} \end{pmatrix} .$$

3.5. Estimating R_2. We have (omitting the superscripts for brevity)

$$R_2 = u_t + uu_x + vu_y = (u_1 + u_2)_t + (u_1 + u_2)(u_1 + u_2)_x + (v_1 + v_2)(u_1 + u_2)_y$$

$$= \overbrace{u_{2,t}}^{\boxed{0}} + \overbrace{u_1 u_{1,x}}^{\boxed{1}} + \overbrace{u_1 u_{2,x}}^{\boxed{2}} + \overbrace{u_2 u_{1,x}}^{\boxed{3}} + \overbrace{u_2 u_{2,x}}^{\boxed{4}} + \overbrace{v_1 u_{1,y}}^{\boxed{5}} + \overbrace{v_1 u_{2,y}}^{\boxed{6}} + \overbrace{v_2 u_{1,y}}^{\boxed{7}} + \overbrace{v_2 u_{2,y}}^{\boxed{8}} .$$

Now, three of these terms are zero by design, since $\operatorname{supp} u_2 = \operatorname{supp} v_2 = \operatorname{supp} S$ and $\phi_2' = 0 = \phi_2''$ for $y \in \operatorname{supp} S$:

$$\boxed{2} \equiv \frac{\omega}{n}\phi_1\left(\frac{x}{n^\delta}\right)\phi_2'\left(\frac{y}{n^\delta}\right)\frac{1}{n^{\delta+s+1}}\partial_{xy}S = 0$$

$$\boxed{3} \equiv u_2 u_{1,x} = \frac{1}{n^{\delta+s+1}}\partial_y S \frac{\omega}{n^{1+\delta}}\phi_1'\left(\frac{x}{n^\delta}\right)\phi_2'\left(\frac{y}{n^\delta}\right) = 0$$

$$\boxed{7} \equiv v_2 u_{1,y} = -\frac{1}{n^{\delta+s+1}}\partial_x S\left(-\frac{\omega}{n^{1+\delta}}\right)\phi_1\left(\frac{x}{n^\delta}\right)\phi_2''\left(\frac{y}{n^\delta}\right) = 0;$$

and another term takes a simpler form, since $\phi_1' \equiv 1 \equiv \phi_2$ on the support of S:

$$\boxed{6} \equiv v_1 u_{2,y} = -\frac{\omega}{n}\phi_1'\left(\frac{x}{n^\delta}\right)\phi_2\left(\frac{y}{n^\delta}\right)\frac{1}{n^{\delta+s+1}}\partial_y S = -\frac{\omega}{n^{\delta+s+2}}\partial_y^2 S .$$

From the form of the low-frequency and high-frequency terms, it is clear that differentiation of u_1 or v_1 with respect to either x or y improves the result by a factor of $n^{-\delta}$, as does differentiation of S with respect to x; however, differentiation of S with respect to y introduces a term with an additional multiplicative factor of n. The amplitudes of the low- and high-frequency terms have been balanced so that the largest contributions due to this, in $\boxed{0}$ and $\boxed{6}$, cancel each other. This is exhibited in the proof of

Lemma 6 (Crucial cancellation). *If $\varphi_1' \equiv 1$ on $\operatorname{supp}\psi'$ and $\varphi_2 \equiv 1$ on $\operatorname{supp}\psi$, then*

$$u_{2,t} + v_1 u_{2,y} \equiv \frac{1}{n^{\delta+s+1}}\partial_y\left(\partial_t - \frac{\omega}{n}\partial_y\right)S$$

$$= -\frac{\omega}{n^{2+2\delta+s}}\psi\left(\frac{x}{n^\delta}\right)\left[\frac{1}{n^\delta}\psi''\left(\frac{y}{n^\delta}\right)\sin(ny+\omega t) + n\psi'\left(\frac{y}{n^\delta}\right)\cos(ny+\omega t)\right] , \quad (3.23)$$

and hence

$$\left\|\boxed{0} + \boxed{6}\right\|_\sigma \leq Cn^{\sigma-\delta-s-1} .$$

Proof. Using $S(x, y, t) = \psi(n^{-\delta}x)\psi(n^{-\delta}y)\sin(ny + \omega t)$, we calculate

$$
\begin{aligned}
\left(\partial_t - \frac{\omega}{n}\partial_y\right) S &= \psi\left(\frac{x}{n^\delta}\right)\left(\partial_t - \frac{\omega}{n}\partial_y\right)\left(\psi\left(\frac{y}{n^\delta}\right)\sin(ny + \omega t)\right) \\
&= \psi\left(\frac{x}{n^\delta}\right)\left[\sin(ny + \omega t)\left(\partial_t - \frac{\omega}{n}\partial_y\right)\psi\left(\frac{y}{n^\delta}\right) + \psi\left(\frac{y}{n^\delta}\right)\left(\partial_t - \frac{\omega}{n}\partial_y\right)\sin(ny + \omega t)\right] \\
&= \psi\left(\frac{x}{n^\delta}\right)\left[\sin(ny + \omega t)\left(\partial_t - \frac{\omega}{n}\partial_y\right)\psi\left(\frac{y}{n^\delta}\right)\right] \\
&= -\frac{\omega}{n^{1+\delta}}\psi\left(\frac{x}{n^\delta}\right)\psi'\left(\frac{y}{n^\delta}\right)\sin(ny + \omega t).
\end{aligned}
$$

From this we obtain (3.23). Now it is a direct application of estimate (3.13) to complete the proof. $\square$

To complete the estimate for the H^σ norm of R_2, we estimate the norms of S and its derivatives. From

$$
S = \psi\left(\frac{x}{n^\delta}\right)\psi\left(\frac{y}{n^\delta}\right)\sin(ny + \omega t)
$$

and Lemma 4 we have $\|S\|_\sigma \lesssim n^{\sigma+\delta}$. Since differentiation with respect to x scales the expression by $n^{-\delta}$ and differentiation with respect to y scales it by n (where we ignore the lower order contribution), we have

$$
\|\partial_x S\|_\sigma \lesssim n^\sigma, \quad \|\partial_y S\|_\sigma \lesssim n^{\sigma+\delta+1}, \quad \|\partial_x \partial_y S\|_\sigma \lesssim n^{\sigma+1}, \quad \|\partial_y^2 S\|_\sigma \lesssim n^{\sigma+\delta+2}. \tag{3.24}
$$

We also note the H^σ bounds on u_1 and v_1 and their derivatives:

$$
\|u_1\|_\sigma = \left\|\frac{\omega}{n}\phi_1\left(\frac{x}{n^\delta}\right)\phi_2'\left(\frac{x}{n^\delta}\right)\right\|_\sigma \lesssim n^{\delta-1}, \quad \|u_{1,x}\|_\sigma \lesssim \frac{1}{n},
$$

and the same bounds hold for v_1 and for the y derivatives. With this we can find the remaining bounds for R_2:

$$
\left\|\boxed{1}\right\|_\sigma = \|u_1 u_{1,x}\|_\sigma \lesssim n^{\delta-2},
$$

$$
\left\|\boxed{4}\right\|_\sigma = \|u_2 u_{2,x}\|_\sigma = \frac{1}{(n^{\delta+s+1})^2}\|S_y S_{xy}\|_\sigma \lesssim n^{-2s+2\sigma-\delta},
$$

$$
\left\|\boxed{5}\right\|_\sigma = \|v_1 u_{1,y}\|_\sigma \lesssim n^{\delta-2},
$$

$$
\left\|\boxed{8}\right\|_\sigma = \|v_2 u_{2,y}\|_\sigma = \frac{1}{(n^{\delta+s+1})^2}\|S_x S_{yy}\|_\sigma \lesssim n^{-2s+2\sigma-\delta}.
$$

Combining this with Lemma 6, we find the H_σ norm of R_2 to be bounded by n^α where

$$
\alpha = \max\{\delta - 2, -2(s - \sigma) - \delta, \sigma - \delta - s - 1\}.
$$

Since $\sigma < s - 1$, if we now choose $\delta \ll 1$, the largest exponent is $\delta - 2$, so we have

$$
\|R_2\|_\sigma \lesssim n^{\delta-2}. \tag{3.25}
$$

3.6. **Estimating R_3.** This goes the same way. (Again we omit the superscripts.)

$$R_3 = v_t + uv_x + vv_y = (v_1 + v_2)_t + (u_1 + u_2)(v_1 + v_2)_x + (v_1 + v_2)(v_1 + v_2)_y$$

$$= \overbrace{v_{2,t}}^{\boxed{0}} + \overbrace{u_1 v_{1,x}}^{\boxed{1}} + \overbrace{u_1 v_{2,x}}^{\boxed{2}} + \overbrace{u_2 v_{1,x}}^{\boxed{3}} + \overbrace{u_2 v_{2,x}}^{\boxed{4}} + \overbrace{v_1 v_{1,y}}^{\boxed{5}} + \overbrace{v_1 v_{2,y}}^{\boxed{6}} + \overbrace{v_2 v_{1,y}}^{\boxed{7}} + \overbrace{v_2 v_{2,y}}^{\boxed{8}} \,.$$

Because u_1, $v_{1,x}$ and $v_{1,y}$ are zero on the support of S, we find that the terms $\boxed{2}$, $\boxed{3}$ and $\boxed{7}$ are again zero and (since v_1 is constant on supp S), $\boxed{6}$ reduces to $-\omega v_{2,y}/n$. This again gives us a cancellation between the highest order terms in $\boxed{0}$ and $\boxed{6}$ (we do not actually need it in the case of R_3 since the largest terms are already smaller by a factor of n). Specifically, using the identity in the proof of Lemma 6,

$$\boxed{0} + \boxed{6} = v_{2,t} + v_1 v_{2,y} = v_{2,t} - \frac{\omega}{n} v_{2,y} = -\frac{1}{n^{\delta+s+1}} \partial_x \left(\partial_t - \frac{\omega}{n} \partial_y \right) S$$

$$= -\frac{1}{n^{\delta+s+1}} \partial_x \left(-\frac{\omega}{n^{1+\delta}} \psi \left(\frac{x}{n^\delta} \right) \psi' \left(\frac{y}{n^\delta} \right) \sin(ny + \omega t) \right)$$

$$= \frac{1}{n^{3\delta+s+2}} \psi' \left(\frac{x}{n^\delta} \right) \psi' \left(\frac{y}{n^\delta} \right) \sin(ny + \omega t) \,,$$

and so

$$\left\| \boxed{0} + \boxed{6} \right\|_\sigma \lesssim n^{-2\delta-s+\sigma-2} \,.$$

The estimates for the remaining terms are straightforward, as in the estimates for R_2. We use (3.24) and we need also $\|S_{xx}\|_\sigma \lesssim n^{\sigma-\delta}$:

$$\left\| \boxed{1} \right\|_\sigma = \|u_1 v_{1,x}\|_\sigma \lesssim n^{\delta-2} \,,$$

$$\left\| \boxed{4} \right\|_\sigma = \|u_2 v_{2,x}\|_\sigma = \frac{1}{(n^{\delta+s+1})^2} \|S_y S_{xx}\|_\sigma \lesssim n^{-2s+2\sigma-2\delta-1} \,,$$

$$\left\| \boxed{5} \right\|_\sigma = \|v_1 v_{1,y}\|_\sigma \lesssim n^{\delta-2} \,,$$

$$\left\| \boxed{8} \right\|_\sigma = \|v_2 v_{2,y}\|_\sigma = \frac{1}{(n^{\delta+s+1})^2} \|S_x S_{xy}\|_\sigma \lesssim n^{-2s+2\sigma-2\delta-1} \,.$$

Once again, the largest exponent is $\delta - 2$, and so

$$\|R_3\|_\sigma \lesssim n^{\delta-2} \,. \tag{3.26}$$

Combining estimates (3.25) and (3.26) completes the proof of Proposition 2.   □

To complete the proof of Theorem 6, first notice that from the definition $A_0(U^{\omega,n}) \geq cI$ for some positive constant c, and $A_0(U^{\omega,n})$ is a constant matrix. Therefore, the L^2 inner product $\langle A_0(U^{\omega,n})V, V \rangle$ defines an equivalent norm. Thus,

$$\frac{d}{dt} \|\mathcal{E}\|_\sigma^2 = \frac{d}{dt} \langle \Lambda^\sigma \mathcal{E}, \Lambda^\sigma \mathcal{E} \rangle \approx \frac{d}{dt} \langle A_0(U^{\omega,n}) \Lambda^\sigma \mathcal{E}, \Lambda^\sigma \mathcal{E} \rangle \tag{3.27}$$

Applying the derivative, we have

$$\frac{d}{dt}\langle A_0(U^{\omega,n})\Lambda^\sigma\mathcal{E}, \Lambda^\sigma\mathcal{E}\rangle = 2\langle A_0(U^{\omega,n})\Lambda^\sigma\mathcal{E}_t, \Lambda^\sigma\mathcal{E}\rangle + \langle A_0(U^{\omega,n})_t\Lambda^\sigma\mathcal{E}, \Lambda^\sigma\mathcal{E}\rangle \qquad (3.28)$$

$$= 2\left\langle \Lambda^\sigma\left(A_0(U^{\omega,n})\mathcal{E}_t\right), \Lambda^\sigma\mathcal{E}\right\rangle$$

(since $A_0(U^{\omega,n})$ is constant). This quantity was estimated in Section 3.4; substituting inequality (3.22) and applying Proposition 2 we have

$$2\|\mathcal{E}\|_\sigma \frac{d}{dt}\|\mathcal{E}\|_\sigma \approx \frac{d}{dt}\langle A_0(U^{\omega,n})\Lambda^\sigma\mathcal{E}, \Lambda^\sigma\mathcal{E}\rangle \lesssim n^{\sigma+1-s}\|\mathcal{E}\|_{H^\sigma}^2 + n^{\delta-2}\|\mathcal{E}\|_\sigma . \qquad (3.29)$$

Dividing by $\|\mathcal{E}\|_\sigma$ in (3.29) gives the first inequality stated in the theorem. We apply Grönwall's inequality, [18, p 24]. The Grönwall estimate for

$$z'(t) \le az(t) + b, \quad z(0) = 0,$$

is

$$z(t) \le \frac{b}{a}\left(e^{at} - 1\right).$$

Since here $a \simeq n^{\sigma-s+1} < C$, the upper bound for e^{at} is a constant that depends only on T^*, the time interval on which we are tracking the solution, and with $b \simeq n^{\delta-2}$, then b/a gives the estimate in the theorem. $\square$

This completes the proof of the uniformity (in n) of the approximation of $U^{\omega,n}$ to the actual solution, $U_{\omega,n}$, for $\omega = 1$ and $\omega = -1$. For ε in equation (3.10) we have $3 - (s - \sigma) - \delta > 2$.

3.7. **Nonuniform convergence.** We are now prepared to complete the proof of nonuniform convergence, the final item, (4), in the program. We use a fact we proved in [28]: For a range of $\tau > s$, (specifically $s < \tau \le \lfloor s \rfloor + 1$, where $\lfloor \cdot \rfloor$ is the greatest integer function), the error in the H^τ norm is bounded by

$$\|\mathcal{E}\|_\tau \lesssim n^{\tau-s} . \qquad (3.30)$$

This uses the form of $\mathcal{E}$ and the bound in Lemma 4. Interpolation yields an estimate for the error in the s norm.

Lemma 7. *For any $s > 2$ and $n \gg 1$, there exists an $\varepsilon > 0$ such that*

$$\|\mathcal{E}\|_s \lesssim n^{-\varepsilon}. \qquad (3.31)$$

Proof. From Proposition 1 we have

$$\|\mathcal{E}\|_s \le \|\mathcal{E}\|_\sigma^\alpha\|\mathcal{E}\|_\tau^\beta, \quad \text{where} \quad \alpha = \frac{\tau - s}{\tau - \sigma}, \quad \beta = \frac{s - \sigma}{\tau - \sigma} . \qquad (3.32)$$

Using Theorem 6 and (3.30) we find

$$\|\mathcal{E}\|_s \lesssim \left(n^{\delta-3+s-\sigma}\right)^\alpha\left(n^{\tau-s}\right)^\beta = n^{(\tau-s)(\delta-3+2s-2\sigma)/(\tau-\sigma)}. \qquad (3.33)$$

By choosing $\max\{1, s - 3/2 + \delta\} < \sigma < s - 1$, we obtain $\delta - 3 + 2s - 2\sigma < 0$, which completes the proof. $\square$

We are now ready to give the proof of Theorem 1.

Proof. We estimate the difference between two actual solutions by the triangle inequality

$$\|U_{1,n} - U_{-1,n}\|_s \geq \|U^{1,n} - U^{-1,n}\|_s - \|U^{1,n} - U_{1,n}\|_s - \|U^{-1,n} - U_{-1,n}\|_s. \qquad (3.34)$$

From Lemma 7, the last two terms tend to zero as $n \to \infty$, and therefore, tracking the terms that do not tend to zero as $n \to \infty$,

$$\begin{aligned}
\|U_{1,n} - U_{-1,n}\|_s &\geq \liminf_{n\to\infty} \|U^{1,n} - U^{-1,n}\|_s \\
&\geq \lim_{n\to\infty} \|n^{-\delta-s}\psi(n^{-\delta}x)\psi(n^{-\delta}y)\left(\cos(ny+t) - \cos(ny-t)\right)\|_s \\
&= \lim_{n\to\infty} \|n^{-\delta-s}\psi(n^{-\delta}x)\psi(n^{-\delta}y)\cos(ny)\|_s|\sin(t)| \approx |\sin(t)|. \qquad \square
\end{aligned}$$

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