Linear quadratic stochastic control problems with stochastic terminal constraint*

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Abstract

We study a linear quadratic optimal control problem with stochastic coefficients and a terminal state constraint, which may be in force merely on a set with positive, but not necessarily full probability. Under such a partial terminal constraint, the usual approach via a coupled system of a backward stochastic Riccati equation and a linear backward equation breaks down. As a remedy, we introduce a family of auxiliary problems parametrized by the supersolutions to this Riccati equation alone. The target functional of these problems dominates the original constrained one and allows for an explicit description of both the optimal control policy and the auxiliary problem's value in terms of a suitably constructed optimal signal process. This suggests that, for the minimal supersolution of the Riccati equation, the minimizers of the auxiliary problem coincide with those of the original problem, a conjecture that we see confirmed in all examples understood so far.

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1 Introduction

Linear quadratic stochastic optimal control problems (stochastic LQ problems in short) represent an important class of stochastic control problems and are very well studied in the literature, cf., e.g., the book by Yong and Zhou [25], Chapter 6, for an overview. A prototype of a stochastic LQ problem with linear quadratic cost functional is given by the so-called *optimal follower* or *optimal tracking* problem where one seeks to minimize a cost criterion of the following form: For a deterministic time horizon $T > 0$, for a predictable target process $(\xi_t)_{0\leq t\leq T}$ as well as progressively measurable, nonnegative processes $(\nu_t)_{0\leq t\leq T}$ and $(\kappa_t)_{0\leq t\leq T}$, for random variables η and Ξ_T known at time T and $x \in \mathbb{R}$, find a control u with state process

$$X_t^u = x + \int_0^t u_s ds \quad (0 \leq t \leq T)$$

which minimizes the objective

$$J^\eta(u) \triangleq \mathbb{E}\left[\int_0^T (X_t^u - \xi_t)^2 \nu_t dt + \int_0^T \kappa_t u_t^2 dt + \eta(X_T^u - \Xi_T)^2\right]. \tag{1}$$

The interpretation of such an LQ problem is the following: The first term in (1) measures the overall quadratic deviation of the controlled state process X^u from the target process ξ weighted with a stochastic weight process ν. The second term in (1) measures the incurred tracking effort in terms of running quadratic costs which are imposed on the control u with stochastic cost process κ. The third term in (1) implements a penalization on the quadratic deviation of the controlled state X_T^u from the final target position Ξ_T at terminal time T with nonnegative random penalization parameter η.

It is well known in the literature that the optimal control to such a stochastic LQ problem as well as its optimal value is typically characterized by two coupled backward stochastic differential equations (BSDEs): A backward stochastic *Riccati* differential equation (BSRDE) of the form

$$dc_t = \left(\frac{c_t^2}{\kappa_t} - \nu_t\right) dt - dN_t \quad \text{on } [0,T] \text{ with } c_T = \eta \tag{2}$$

and a linear BSDE of the form

$$db_t = \left(\frac{c_t}{\kappa_t} b_t - \nu_t \xi_t\right) dt + dM_t \quad \text{on } [0,T] \text{ with } b_T = \eta \Xi_T, \tag{3}$$

where N and M denote suitable càdlàg martingales (cf., e.g., Kohlmann and Tang [15], Section 5.1).

A number of interesting challenges arise when one allows the terminal penalization parameter η to take the value infinity with positive (not necessarily full) probability. It is then intuitively sensible to interpret the "blow up" of η as a *stochastic terminal state* constraint of the form

$$X_T^u = \Xi_T \quad \text{a.e. on the set } \{\eta = +\infty\} \tag{4}$$

on all controlled processes X^u that produce a finite value in (1). Mathematically, it is less obvious how to tackle this delicate "partial" constraint and how to compute the optimal control as well as the optimal value. Indeed, note that the involved BS(R)DEs in (2) and (3) will both now exhibit with positive probability a *singularity at final time* in this case. The possibly singular BSRDE in (2) does not pose a serious problem; see Kruse and Popier [16] and Popier [21]. In contrast, the singularity in the terminal condition of the linear BSDE in (3) is rather unpleasant because it also involves the desired target position Ξ_T, leaving the terminal condition $b_T = \eta\Xi_T$ depend solely on the sign of Ξ_T on the very set $\{\eta = +\infty\}$ where this random variable has to be matched by the state processes' terminal value X_T^u.

As a consequence, the classical solution approach cannot be followed directly. Instead we introduce a family of auxiliary target functionals

$$J^c(u) \triangleq \limsup_{\tau \uparrow T} \mathbb{E}\left[\int_0^\tau (X_t^u - \xi_t)^2 \nu_t dt + \int_0^\tau \kappa_t u_t^2 dt + c_\tau (X_\tau^u - \hat{\xi}_\tau^c)^2\right]$$

parametrized by supersolutions c of the BSRDE (2) and where $\hat{\xi}_\tau^c$ is an *optimal signal process* constructed as a judiciously chosen average of future target positions $(\xi_t)_{t\geq\tau}$ and Ξ_T. The target functional J^c avoids the singularity at time T by a "truncation in time" focussing on shorter time horizons $\tau < T$ at which we impose a "classical" finite terminal penalization. This penalization is chosen in such a way that the corresponding optimizers can be extended consistently to the full interval $[0, T)$ as $\tau \uparrow T$. In fact, the corresponding auxiliary minimization problems turn out to be solvable in a very satisfactory way: As already observed in a much simpler setting in Bank et al. [4], we can give necessary and sufficient conditions for the domain $\{J^c < \infty\}$ to be nonempty and we can also describe explicitly the optimal control in feedback form

$$\hat{u}_t^c = \frac{c_t}{\kappa_t}(\hat{\xi}_t^c - X_t^{\hat{u}^c}) \quad (0 \leq t < T),$$

revealing that one should always push the controlled process towards the optimal signal $\hat{\xi}^c$ with time-varying urgency given by the ratio c_t/κ_t. We can even show how the regularity and predictability of the targets ξ and Ξ_T as reflected in the signal process $\hat{\xi}^c$ and its quadratic variation determine the problem's value.

We show that for the considered supersolutions c of the BSRDE (2) we have $J^c(u) \geq J^\eta(u)$. This leads us to the conjecture that for the *minimal* supersolution $c = c^{\min}$ (whose existence is guaranteed under mild conditions; see Kruse and Popier [16]) the minimizers of these functionals are the same. While we have to leave the proof of this conjecture for future research that allows one to better control singular BSRDE supersolutions, we do verify the validity of our conjecture in the examples we found in the literature.

Stochastic control problems, referred to as optimal liquidation problems in the literature, with almost sure (i.e., $\eta \equiv +\infty$ almost surely) and deterministic terminal state constraint (targeting the terminal position $\Xi_T = 0$), where the cost functional is allowed to be quadratic in X^u and u (that is, $\xi \equiv 0$ in (1)) have already been studied in, e.g., Schied [24], Ankirchner et al. [3] and, in a more general BSPDE framework, in Graewe et al. [12]; allowing the penalization parameter η to take the value infinity with positive probability has been investigated in Kruse and Popier [16]. Ankirchner and Kruse [2], still within this context of optimal liquidation, allow the objective functional to be additionally linear in the control u. They also incorporate a specific nonzero stochastic terminal state constraint where the random target position Ξ_T is gradually revealed up to terminal time T. A general class of stochastic control problems including LQ problems with terminal states being constrained to a convex set were studied by Ji and Zhou [13]. However, to the best of our knowledge, stochastic linear quadratic control problems with $\xi \neq 0$ and possible stochastic terminal state constraint $\Xi_T \neq 0$ as considered in the present paper have not yet been investigated.

The analysis of the stochastic LQ problem in (1) above is especially motivated by optimal trading and hedging problems in Mathematical Finance. In this framework the state process X^u denotes an agent's position in some risky asset that she trades at a turnover rate u. She wants her position to be as close as possible to a given target strategy ξ but simultaneously seeks to minimize the induced quadratic transaction costs which are levied on her transactions due to, e.g., stochastic price impact as measured by κ. The weight process ν captures stochastic volatility, that is, the risk of her open trading position due to random market fluctuations. Finally, in case of a possible but not necessarily almost sure occurrence of specific market conditions, encoded by the event set $\{\eta = +\infty\}$, she may be required to drive her position X^u imperatively towards a predetermined random value Ξ_T at maturity T (e.g., to respect specific requirements of contractual or regulatory nature concerning her risky asset position). Otherwise, a penalization depending on the deviation of X^u_T from the target position Ξ_T is implemented. We refer to, e.g., Rogers and Singh [23], Naujokat and Westray [19], Almgren and Li [1], Frei and Westray [10], Cartea and Jaimungal [8], Cai et al. [7], Bank et al. [4] and, for asymptotic considerations, to Chan and Sircar [9]. Note, however, that the above cited papers may neither allow for an arbitrary predictable target strategy ξ nor for stochastic price impact κ and stochastic volatility ν. In particular, none of them consider a stochastic terminal state constraint like (4) above with general random target position Ξ_T.

The rest of the paper is organized as follows. In Section 2 we formulate the general stochastic LQ problem with stochastic terminal state constraint. Our auxiliary control problem and its solution are presented in Section 3. Its relation to the original LQ problem is discussed and exemplarily illustrated in Section 4. The proofs are deferred to Section 5 and an appendix collects a few BSDE-results which may be of independent interest.

2 A stochastic LQ problem with stochastic terminal state constraint

We fix a finite deterministic time horizon $T > 0$ and a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathbb{P})$ satisfying the usual conditions of right continuity and completeness. We let $(\kappa_t)_{0\le t\le T}$ and $(\nu_t)_{0\le t\le T}$ denote two progressively measurable, nonnegative processes such that

$$\int_0^T \left(\nu_t + \frac{1}{\kappa_t}\right) dt < \infty \quad \mathbb{P}\text{-a.s.} \tag{5}$$

Moreover, we are given a predictable target process $(\xi_t)_{0\le t\le T}$ satisfying

$$\mathbb{E}\left[\int_0^T |\xi_t|\nu_t dt\right] < \infty \quad \text{and} \quad \int_0^T \xi_t^2 \nu_t dt < \infty \quad \mathbb{P}\text{-a.s.}, \tag{6}$$

a random terminal target position $\Xi_T \in L^0(\mathbb{P}, \mathscr{F}_{T-})$ as well as an $\mathscr{F}_{T-}$-measurable penalization parameter η taking values in $[0, +\infty]$. We further assume that

$$\mathbb{P}\left[\eta = 0\,, \int_t^T \nu_u du = 0 \,\middle|\, \mathscr{F}_t\right] < 1 \quad \mathbb{P}\text{-a.s. for all } t \in [0,T). \tag{7}$$

For such ν, κ, ξ, Ξ_T and η, one can formulate the following stochastic linear quadratic optimal control problem: Find a control u from the class of processes

$$\mathscr{U} \triangleq \left\{ u \text{ progressively measurable s.t. } \int_0^T |u_t|\, dt < \infty \ \mathbb{P}\text{-a.s.} \right\} \tag{8}$$

such that, for given $x \in \mathbb{R}$, the controlled state process

$$X_t^u \triangleq x + \int_0^t u_s ds \quad (0 \le t \le T) \tag{9}$$

minimizes the objective functional

$$J^\eta(u) \triangleq \mathbb{E}\left[\int_0^T (X_t^u - \xi_t)^2 \nu_t dt + \int_0^T \kappa_t u_t^2 dt + \eta 1_{\{0 \le \eta < \infty\}} (X_T^u - \Xi_T)^2\right] \tag{10}$$

over the set of all constrained policies

$$\mathscr{U}^\Xi \triangleq \left\{ u \in \mathscr{U} \text{ satisfying } X_T^u = \Xi_T \ \mathbb{P}\text{-a.s. on } \{\eta = +\infty\} \right\}. \tag{11}$$

In short, we are interested in the stochastic LQ problem

$$J^\eta(u) \to \min_{u \in \mathscr{U}^\Xi}, \tag{12}$$

where the controller seeks to keep the controlled process X^u close to a given target process ξ in such a way that deviations from the final target position Ξ_T are also minimized. On $\{\eta = +\infty\}$, the final target position has to be reached a.s. as incorporated in the set of admissible strategies $\mathscr{U}^\Xi$ in (11).

Remark 2.1.

1. In case where the random penalization parameter η is finite almost surely, the optimization problem in (12) does not include a terminal state constraint and boils down to a classical stochastic optimal control problem which is well studied in the literature; c.f., e.g., Kohlmann and Tang [15].

2. The dynamic condition (7) is very natural for our optimal tracking problem in (12). It means that at any time $t < T$ some penalization for deviating from the targets ξ and Ξ_T remains conceivable, even conditionally on $\mathscr{F}_t$, so that the controller has to stay alert all the way until the end.

3. The mild integrability conditions in (5), (6) and (8) ensure that the stochastic LQ problem in (12) is well defined along with some processes to be introduced shortly.

Mathematically, the stochastic terminal state constraint

$$X^u_T = \Xi_T \quad \text{a.e. on } \{\eta = +\infty\} \tag{13}$$

in the set of allowed controls $\mathscr{U}^\Xi$ in (11) entails technical difficulties. For instance, it is far from obvious under what conditions we have $\mathscr{U}^\Xi \neq \varnothing$ or whether $J^\eta(u) < \infty$ for some $u \in \mathscr{U}^\Xi$. Also, as explained in the introduction, the usual solution approach via BSDEs does not accommodate this partial constraint.

As a possible remedy, instead of tackling the constrained stochastic LQ problem posed in (12), we propose to formulate a suitable variant of this problem. Specifically, we will introduce a family of stochastic control problems

$$J^c(u) \to \min_{u \in \mathscr{U}^c} \tag{14}$$

with set of admissible controls

$$\mathscr{U}^c \triangleq \{u \in \mathscr{U} \text{ satisfying } J^c(u) < \infty\} \tag{15}$$

and target functional J^c which are parametrized by supersolutions $c \triangleq (c_t)_{0 \le t < T}$ to a certain singular backward stochastic differential equation (BSDE) to be described below in Section 3.1. These auxiliary problems will dominate the stochastic LQ problem stated in (12) in the sense that, for all parametrizations c, we have

$$J^c(u) \geq J^\eta(u) \quad \text{for all } u \in \mathscr{U}^c \tag{16}$$

and

$$\mathscr{U}^c \subseteq \mathscr{U}^\Xi \tag{17}$$

(cf. Lemma 4.1 below). We will show in Section 3.3 that our auxiliary problems in (14) can be solved in a very satisfactory way: In terms of ξ, Ξ_T and the parameter process c, we provide necessary and sufficient conditions which ensure that $\mathscr{U}^c \neq \varnothing$ and describe explicitly the optimal policy $\hat{u}^c$ for (14) as well as the associated minimal costs $J^c(\hat{u}^c)$. In view of (16) and (17), we thus obtain both explicit candidate strategies for the general constrained stochastic LQ problem formulated in (12) as well as conditions which guarantee existence of controls entailing finite costs in the latter.

To link these problems to the original problem (12) it is natural to consider "small" solutions to the BSDE. In fact, we conjecture that for the *minimal supersolution* $c^{\min}$ of the BSDE we have

$$\arg\min_{\mathscr{U}^\Xi} J^\eta = \arg\min_{\mathscr{U}^{c^{\min}}} J^{c^{\min}}. \tag{18}$$

While we cannot prove this conjecture in full generality, we provide in Section 4 evidence for its validity in certain settings. These include the case of bounded coefficients, but also some singular cases where, possibly, $\mathbb{P}[\eta = +\infty] > 0$.

3 An auxiliary control problem

In this section, we will formulate and solve our auxiliary stochastic LQ problem (14) for fixed c. The process c will be a supersolution to a BSRDE which we discuss in Section 3.1. In Section 3.2, we will introduce our target functional J^c whose minimizer $\hat{u}^c$ is derived in Section 3.3 along with the optimal costs $J^c(\hat{u}^c)$.

3.1 Connection between stochastic LQ problems and BSRDEs

It is well known in the literature that the solution to stochastic LQ problems like (12) is intimately related to backward stochastic Riccati differential equations (BSRDEs): For (12), the Riccati dynamics take the form

$$dc_t = \left(\frac{c_t^2}{\kappa_t} - \nu_t\right) dt - dN_t \quad \text{on } [0, T) \tag{19}$$

for some càdlàg martingale $(N_t)_{0\le t<T}$; cf., e.g., Bismut [5, 6]. Moreover, the recent papers by, e.g., Ankirchner et al. [3], Kruse and Popier [16], Graewe et al. [12] or Graewe and Horst [11] have shown that a terminal state constraint as (13) in the LQ problem typically leads to a singular terminal condition for the corresponding BSRDE of the form

$$\liminf_{t\uparrow T} c_t \ge \eta \quad \mathbb{P}\text{-a.s.} \tag{20}$$

This motivates us to let $c = (c_t)_{0\le t<T}$ denote from now on an $(\mathscr{F}_t)_{0\le t<T}$-adapted, càdlàg semimartingale with BSRDE dynamics (19) and terminal condition (20). In addition, we will assume that

$$\int_{[0,T)} \frac{d[c]_t}{c_{t-}^2} < \infty \quad \text{on the set } \{\eta = +\infty\}, \tag{21}$$

where $[c]$ denotes the quadratic variation process of c (cf., e.g., Protter [22], Chapter II.6, for the quadratic variation process of càdlàg semimartingales).

Remark 3.1.

1. As usual the dynamics in (19) have to be understood in the sense that the pair (c, N) satisfies

$$c_s = c_t - \int_s^t \left(\frac{c_u^2}{\kappa_u} - \nu_u\right) du + \int_s^t dN_u \quad (0 \le s \le t < T). \tag{22}$$

In particular, the dynamics in (19) are only required to hold on $[0, T-\varepsilon]$ for every $\varepsilon > 0$, that is, strictly before T. So, more precisely, we will say that (c, N) is a supersolution of the BSRDE (19) with terminal condition η if (22) and (20) hold true.

2. For bounded coefficients ν, κ, $1/\kappa$, η, Kohlmann and Tang [15] prove within a Brownian framework existence and uniqueness of c with dynamics in (19) such that $\lim_{t\uparrow T} c_t = \eta$ exists $\mathbb{P}$-a.s. For the fully singular case $\eta \equiv +\infty$ $\mathbb{P}$-a.s. and again within a Brownian framework, existence of a minimal solution (under suitable integrability conditions on the processes $(\nu_t)_{0\le t\le T}$ and $(\kappa_t)_{0\le t\le T}$) to the above BSRDE in (19) with singular terminal condition $\liminf_{t\uparrow T} c_t = +\infty$ $\mathbb{P}$-a.s. are provided in Ankirchner et al. [3]; cf. also Graewe et al. [12]. For the present partially singular setup, Kruse and Popier [16] provide sufficient conditions (including suitable integrability conditions on $(\kappa_t)_{0\le t\le T}$ and $(\nu_t)_{0\le t\le T}$) for the existence of a minimal supersolution $(c_t^{\min})_{0\le t\le T}$ to the above

BSRDE in (19) with terminal condition (20) in the sense that $c_t^{\min} \leq c_t$ for all $t \in [0,T)$ and all processes c satisfying likewise (19) and (20). Existence of actual solutions c with $\lim_{t\uparrow T} c_t = \eta$ is only known under additional assumptions on η; see Popier [21].

3. The additional integrability condition (21) on the "blow up" set $\{\eta = +\infty\}$ is implicitly shown to hold true in Popier [20] in a Brownian framework for constant coefficients $\nu \equiv 0$ and $\kappa \equiv 1$; see Theorem 2 and Proposition 3 in [20]. We require this integrability condition (21) in our proof of Lemma 3.3 below whose result crucially feeds into our solution presented in Section 3.3. We will therefore briefly discuss exemplarily in the appendix sufficient conditions on $(\kappa_t)_{0\leq t\leq T}$, $(\nu_t)_{0\leq t\leq T}$ and η under which property (21) does hold true in the more generic setting of Kruse and Popier [16].

As a consequence of (19), (20) and (21), let us first ascertain that c is strictly positive on $[0,T)$, a result which is crucial for our approach below and which follows immediately from Lemma 5.4 in the appendix.

Lemma 3.2. *For all $t \in [0,T)$ we have $c_t > 0$ if (7) holds true.* ☐

Next, the BSRDE supersolution c gives rise to the following auxiliary process

$$L_t \triangleq L_t^c \triangleq c_t \exp\left(-\int_0^t \frac{c_u}{\kappa_u} du\right) \quad (0 \leq t < T). \tag{23}$$

Lemma 3.3. *Granted (7) holds true, the process $(L_t)_{0\leq t<T}$ is a strictly positive càdlàg supermartingale. In particular,*

$$L_T \triangleq \lim_{t\uparrow T} L_t \geq 0 \quad \text{exists } \mathbb{P}\text{-a.s.} \tag{24}$$

and the extended process $(L_t)_{0\leq t\leq T}$ is a supermartingale on $[0,T]$. Moreover, we have $\{\eta > 0\} \subset \{L_T > 0\}$ up to a $\mathbb{P}$-null set.

Proof. Since $c_t > 0$ $\mathbb{P}$-a.s. for all $0 \leq t < T$ by Lemma 3.2, it is immediate from (23) that also $L_t > 0$ $\mathbb{P}$-a.s. for all $0 \leq t < T$. Integration by parts and using the Riccati dynamics of c in (19) yields that L satisfies the stochastic differential equation

$$L_0 = c_0, \quad dL_t = L_{t-}\left(-\frac{\nu_t}{c_{t-}}dt - \frac{1}{c_{t-}}dN_t\right) \quad \text{on } [0,T). \tag{25}$$

Since N is a càdlàg local martingale on $[0, T)$, we obtain from (25) that the process L is a strictly positive càdlàg supermartingale on $[0, T)$. Hence, it follows by the (super-)martingale convergence theorem (see, e.g., Karatzas and Shreve [14], Chapter 1.3, Problem 3.16) that the limit $L_T \triangleq \lim_{t \uparrow T} L_t$ exists $\mathbb{P}$-a.s. and extends the process L to a càdlàg supermartingale on all of $[0, T]$. Moreover, appealing to the definiton of L in (23) and the terminal condition $\liminf_{t \uparrow T} c_t \geq \eta$ of the process c in (20), we have $\{0 < \eta < \infty\} \subset \{L_T > 0\}$. Concerning the "blow up" set $\{\eta = +\infty\}$, observe that we may write

$$L_t = c_0 e^{X_t - \frac{1}{2}[X]^{\mathbf{c}}_t} \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s} \quad (0 \leq t < T), \tag{26}$$

where $X_t \triangleq -\int_0^t \frac{\nu_s}{c_{s-}} ds - \int_0^t \frac{1}{c_{s-}} dN_s$ and where $[X]^{\mathbf{c}}$ denotes the continuous part of its quadratic variation (cf., e.g., Protter [22], Theorem II.37). Note that $L_s > 0$ $\mathbb{P}$-a.s. for all $0 \leq s < T$ implies $\Delta X_s > -1$ for all $0 \leq s < T$. Moreover, applying Taylor's formula, it holds for all $0 \leq t < T$ that

$$\sum_{s \leq t} \left| \log\left((1 + \Delta X_s) e^{-\Delta X_s} \right) \right| \leq \frac{1}{2} \int_{[0,T)} \frac{1}{c^2_{s-}} d[c]_s < +\infty$$

a.e. on the set $\{\eta = +\infty\}$ by virtue of condition (21). This implies that the product of the jumps in (26) will converge to a strictly positive limit as $t \uparrow T$ on $\{\eta = +\infty\}$. Concerning the limiting behaviour of the exponential $\exp(X_t - \frac{1}{2}[X]^{\mathbf{c}}_t)$ in (26) for $t \uparrow T$, observe that once more condition (21) prevents the limiting value from becoming 0 on $\{\eta = +\infty\}$. Indeed, the local martingale $\int_0^t dN_s / c_{s-}$ cannot explode as $t \uparrow T$ for those paths along which its quadratic variation $\int_0^t d[c]_s / c^2_{s-}$ remains bounded on $[0, T)$ (cf., e.g., Protter [22], Chapter V.2, for more details). □

3.2 Auxiliary target functional

Let us assume that the terminal target position Ξ_T is bounded, or, more generally, that it satisfies

$$\Xi_T L_T \in L^1(\mathscr{F}_{T-}, \mathbb{P}), \tag{27}$$

where $L_T = L^c_T = \lim_{t \uparrow T} c_t e^{-\int_0^t c_u / \kappa_u du}$ as in (24). Recalling the integrability requirement (6) for the running target ξ, let us now introduce the key object for our approach, the optimal signal process $\hat{\xi}$ which is given by the càdlàg semimartingale

$$\hat{\xi}_t \triangleq \hat{\xi}^c_t \triangleq \frac{1}{L_t}\mathbb{E}\left[\Xi_T L_T + \int_t^T \xi_r e^{-\int_0^r \frac{c_u}{\kappa_u}du}\nu_r dr \,\middle|\, \mathscr{F}_t\right] \quad (0 \leq t < T). \tag{28}$$

The optimal signal process $\hat{\xi}$ can be viewed as a weighted average of expected future targets ξ and Ξ_T; see our discussion in Remark 3.5 and the representation of $\hat{\xi}$ in (37) below. Our motivation for introducing $\hat{\xi}$ becomes very apparent when reviewing known results in the literature to the stochastic LQ problem in (12) with bounded coefficients; see Section 4.1 below. Observe that $\hat{\xi}$ remains unspecified for $t = T$. In fact, we can readily deduce from Lemma 3.3 and the integrability conditions (6) and (27)

$$\exists \lim_{t\uparrow T} \hat{\xi}_t = \Xi_T \quad \text{on the set } \{L_T > 0\} \supset \{\eta > 0\}. \tag{29}$$

On the set $\{L_T = 0\} \subset \{\eta = 0\}$, though, this convergence may fail (without harm as it turns out).

Given the optimal signal process $(\hat{\xi}_t)_{0\leq t<T}$, we are now in a position to introduce the auxiliary LQ target functional

$$J^c(u) \triangleq \limsup_{\tau\uparrow T} \mathbb{E}\left[\int_0^\tau (X^u_t - \xi_t)^2 \nu_t dt + \int_0^\tau \kappa_t u_t^2 dt + c_\tau (X^u_\tau - \hat{\xi}^c_\tau)^2\right], \tag{30}$$

where the limes superior is taken over all sequences of stopping times $(\tau^n)_{n=1,2,\ldots}$ converging to terminal time T strictly from below. Introducing the set of admissible controls

$$\mathscr{U}^c = \{u \in \mathscr{U} \text{ satisfying } J^c(u) < +\infty\} \tag{31}$$

as in (15), we will solve completely the auxiliary optimization problem

$$J^c(u) \to \min_{u\in\mathscr{U}^c} \tag{32}$$

in the next section.

3.3 Explicit solution to the auxiliary problem

As it turns out, the optimal control to our auxiliary stochastic LQ problem in (32) and its corresponding optimal value are explicitly computable and fully characterized by the processes c and $\hat{\xi}^c$. In terms of these, we can also characterize when the set of admissible controls $\mathscr{U}^c$ defined in (31) is nonempty. In fact, it follows from our analysis below that $\mathscr{U}^c \neq \varnothing$ if and only if

$$\mathbb{E}\left[\int_0^T (\xi_t - \hat{\xi}^c_t)^2 \nu_t dt\right] < +\infty \quad \text{and} \quad \mathbb{E}\left[\int_{[0,T)} c_t d[\hat{\xi}^c]_t\right] < +\infty, \tag{33}$$

where $[\hat{\xi}^c]$ denotes the quadratic variation process of the semimartingale $\hat{\xi}^c$ of (28). In particular, (33) is necessary and sufficient for well-posedness of the LQ problem in (32):

Theorem 3.4. *Let (5), (6), and (7) hold true. In addition, suppose that c follows the Riccati dynamics (19) with terminal condition (20) and satisfies the integrability conditions (21) and (27).*

Then we have $\mathscr{U}^c \neq \varnothing$ if and only if (33) is satisfied. In this case, the optimal control $\hat{u}^c \in \mathscr{U}^c$ for the auxiliary problem (32) with controlled process $\hat{X}^c_\cdot \triangleq X^{\hat{u}^c}_\cdot$ is given by the feedback law

$$\hat{u}^c_t = \frac{c_t}{\kappa_t}\left(\hat{\xi}^c_t - \hat{X}^c_t\right) \quad (0 \leq t < T), \tag{34}$$

and the minimal costs are

$$J^c(\hat{u}^c) = c_0(x - \hat{\xi}^c_0)^2 + \mathbb{E}\left[\int_0^T (\xi_t - \hat{\xi}^c_t)^2 \nu_t dt\right] + \mathbb{E}\left[\int_{[0,T)} c_t d[\hat{\xi}^c]_t\right]. \tag{35}$$

The proof of Theorem 3.4 is deferred to Section 5 below. Observe that the feedback law of the optimal control in (34) prescribes a reversion towards the optimal signal process $\hat{\xi}^c_t$ rather than towards the current target position ξ_t. The reversion speed is controlled by the ratio c_t/κ_t. In particular, on the "blow-up" set $\{\eta = +\infty\}$ the optimizer reverts with stronger and stronger urgency towards the optimal signal $\hat{\xi}^c$ and hence to the ultimate target position Ξ_T due to (29). This result generalizes the insights from the constant coefficient case with almost sure terminal state constraint which are presented in Bank et al. [4].

Under the integrability conditions (33), the optimal costs $J^c(\hat{u}^c)$ in (35) of the optimizer $\hat{u}^c$ in (34) are obviously finite. Actually, they nicely separate into three intuitively appealing terms making transparent how the regularity and predictability of the targets ξ and Ξ_T determine the auxiliary problem's optimal value. The first term represents the costs due to a possibly suboptimal initial position x. The second term shows how the regularity of the target process ξ feeds into the overall costs: Targets which are poorly approximated by the optimal signal process $\hat{\xi}^c$ in the $L^2(\mathbb{P}\otimes\nu_t dt)$-sense produce higher costs. Finally, the third term reveals the importance of the optimal signal's quadratic variation process $[\hat{\xi}^c]$. Referring to the definition of $\hat{\xi}^c$ in (28) (cf. also the representation in (37) below), the quadratic variation $[\hat{\xi}^c]$ can be viewed as a measure for the strength of the fluctuations in the assessment of the average future target positions of ξ, the terminal position Ξ_T and the random variable L_T which involves the outcome of the penalization parameter η at time T. In this sense, the second integrability condition in (33) can be interpreted as encoding a condition on the predictability of the final stochastic target position Ξ_T as well as the random penalization parameter η. Loosely speaking, it ensures that the outcome of the final position Ξ_T as well as the "blow-up" event $\{\eta = +\infty\}$ on which Ξ_T has to be matched by controls in $\mathscr{U}^c$ are not allowed to come as "too big a surprise" at final time T; see also our discussion in Section 4.2 below.

Remark 3.5 (Interpretation of the optimal signal). Let us present a way to interpret our optimal signal process $\hat{\xi}$ defined in (28). For ease of presentation and to avoid unnecessary technicalities, let us assume here that the convergence in (24) also holds in $L^1(\mathbb{P})$, that $\mathbb{E}[L_T] > 0$ and that $0 < \nu \in L^1(\mathbb{P}\otimes dt)$ (these assumptions merely simplify the justification of the representation in (38) below; cf. Lemma 5.3 in Section 5). Then, by defining the *weight process* $(w_t)_{0\le t<T}$ via

$$w_t \triangleq \frac{\mathbb{E}[L_T|\mathscr{F}_t]}{L_t} \qquad (0 \le t < T) \tag{36}$$

as well as the measure $\mathbb{Q} \ll \mathbb{P}$ on $(\Omega, \mathscr{F}_T)$ via

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{L_T}{\mathbb{E}[L_T]},$$

we may write

$$\begin{aligned}\hat{\xi}_t &= \frac{1}{L_t}\mathbb{E}\left[\Xi_T L_T + \int_t^T \xi_r e^{-\int_0^r \frac{c_u}{\kappa_u}du}\nu_r dr \,\middle|\, \mathscr{F}_t\right] \\ &= w_t\,\mathbb{E}_{\mathbb{Q}}[\Xi_T|\mathscr{F}_t] + (1-w_t)\,\mathbb{E}\left[\int_t^T \xi_r \frac{e^{-\int_t^r \frac{c_u}{\kappa_u}du}}{(1-w_t)c_t}\nu_r dr \middle|\mathscr{F}_t\right]\end{aligned} \tag{37}$$

for all $0 \leq t < T$. Recall that the process $(L_t)_{0\leq t<T}$ is a strictly positive supermartingale by virtue of Lemma 3.3. Consequently, the weight process satisfies

$$0 \leq w_t < 1 \quad \mathbb{P}\text{-a.s. for all } 0 \leq t < T,$$

where the strict inequality follows from Lemma 5.3 below because we assumed $\nu > 0$ here for simplicity. Moreover, the same lemma gives the identity

$$\mathbb{E}\left[\int_t^T \frac{e^{-\int_t^r \frac{c_u}{\kappa_u}du}}{(1-w_t)c_t}\nu_r dr \middle| \mathscr{F}_t\right] = 1 \quad d\mathbb{P}\otimes dt\text{-a.e. on } \Omega\times[0,T). \tag{38}$$

That is, loosely speaking, the optimal signal process $\hat{\xi}$ in (37) is a convex combination of a weighted average of expected future target positions of ξ and the expected terminal position Ξ_T, computed under the auxiliary measure $\mathbb{Q}$. The weight shifts gradually towards the ultimate target position Ξ_T as $t \uparrow T$, provided that $L_T > 0$. Indeed, by definition of the weight process in (36), martingale convergence theorem and the convergence of the process L in Lemma 3.3, we have

$$\exists \lim_{t\uparrow T} w_t = 1 \quad \text{on the set } \{L_T > 0\}.$$

4 Discussion and illustration

Let us return to the initial stochastic LQ problem (12) with target functional (10) and stochastic terminal state constraint (13) and discuss how it relates to our auxiliary LQ problem (32). Observe that, for the latter, we tackle and resolve the delicate partial terminal state constraint $X_T^u = \Xi_T$ on $\{\eta = +\infty\}$ incorporated in the set of admissible policies $\mathscr{U}^{\Xi}$ in (11) by performing a *truncation in time* in the auxiliary objective functional J^c in (30). Specifically, we replace the original target functional J^η of problem (12) by a properly chosen limit of stochastic LQ target functionals with strictly shorter time horizon $\tau < T$ at which we impose a finite terminal penalization term $c_\tau(X_\tau^u - \hat{\xi}_\tau^c)^2$. In fact, the optimal signal process $\hat{\xi}^c$ turns out to be the proper key ingredient for choosing these penalizations in a *time consistent* manner; see Remark 5.2 below. Moreover, in light of $\liminf_{t\uparrow T} c_t \geq \eta$ in (20) and $\lim_{t\uparrow T} \hat{\xi}_t^c = \Xi_T$ on $\{0 < \eta \leq +\infty\}$ in (29), for any $\tau < T$ the penalty $c_\tau(X_\tau^u - \hat{\xi}_\tau^c)^2$ can be viewed as a proxy of the terminal penalty $\eta 1_{\{0\leq\eta<+\infty\}}(X_T^u - \Xi_T)^2$ in J^η. Indeed, appealing to Fatou's Lemma, monotone convergence as well as (20) and (29), we readily obtain the following:

Lemma 4.1. *It holds that $J^c(u) \geq J^\eta(u)$ for all $u \in \mathscr{U}^c$ and all processes c satisfying* (19), (20) *and* (27). *In particular, we have*

$$X_T^u = \Xi_T \quad \text{on the set } \{\eta = +\infty\} \text{ for all } u \in \mathscr{U}^c,$$

that is, $\mathscr{U}^c \subseteq \mathscr{U}^\Xi$. □

In light of this lemma, it appears very natural for our auxiliary LQ problem in (32) to consider parameter processes c which are *minimal* supersolutions. In fact, this motivates our conjecture (18) that

$$\arg\min_{\mathscr{U}^\Xi} J^\eta = \arg\min_{\mathscr{U}^{c^{\min}}} J^{c^{\min}}$$

holds true for the minimal supersolution $c^{\min}$ to the BSRDE in (19) with terminal condition (20). In the following paragraphs of this section, we provide evidence for the validity of this conjecture. Specifically, we will show how our approach via the auxiliary LQ problem in (32) with $c^{\min}$ allows us to recover existing results in the literature to specific variants of the stochastic LQ problem with stochastic terminal state constraint posed in (12). We will also discuss possible approaches to prove the conjecture (18) based on the insights from Section 3.3; see the end of Section 4.3 and also Remark 5.2 in Section 5.

4.1 Bounded coefficients

In case where η is bounded along with the processes $(\nu_t)_{0\leq t\leq T}$, $(\kappa_t)_{0\leq t\leq T}$ and $(\xi)_{0\leq t\leq T}$, our conjecture in (18) holds true. Indeed, under this conditions and within a Brownian framework, Kohlmann and Tang [15] provide existence and uniqueness of a (minimal supersolution) $c^{\min}$ to the stochastic Riccati equation (19) such that $\lim_{t\uparrow T} c_t^{\min} = \eta$ holds true $\mathbb{P}$-a.s. They show that the optimal control $\hat{u}^{c^{\min}}$ in (34) from our Theorem 3.4 solves the LQ problem in (12) with objective functional J^η (over the set of unconstrained policies $\mathscr{U}^\Xi$, recall Remark 2.1, 1.)); see Kohlmann and Tang [15], Theorem 5.2. Obviously, our necessary and sufficient integrability conditions stated in (33) are satisfied in this case. Note, though, that in [15], Section 5.1, the optimal control $\hat{u}^{c^{\min}}$ is characterized in terms of both the process $c^{\min}$ and the solution process b to the linear BSDE

$$db_t = \left(\frac{c_t^{\min}}{\kappa_t} b_t - \nu_t \xi_t \right) dt + dM_t \quad \text{on } [0,T] \text{ with } b_T = \eta \Xi_T, \tag{39}$$

with some càdlàg (local) martingale $(M_t)_{0\le t\le T}$. More precisely, the optimal control is described by the feedback law

$$\hat{u}_t^{c^{\min}} = -\frac{1}{\kappa_t}\left(c_t^{\min}\hat{X}_t^{c^{\min}} - b_t\right) = \frac{c_t^{\min}}{\kappa_t}\left(\frac{b_t}{c_t^{\min}} - \hat{X}_t^{c^{\min}}\right) \quad (0 \le t \le T).$$

That is, in that setting without terminal constraints, our signal process $\hat{\xi}^{c^{\min}}$ of (28) coincides with the ratio $b/c^{\min}$ and so the solution b to the linear BSDE is an equivalent substitute for this signal process. In contrast, in case where $\{\eta = +\infty, \Xi_T \neq 0\}$ has positive probability, the terminal condition $b_T = \eta\Xi_T$ becomes problematic in the sense that the linear BSDE loses all information on Ξ_T except its sign. Our signal process $\hat{\xi}^{c^{\min}}$, however, still makes sense in this rather natural case. Note that $c^{\min}\hat{\xi}^{c^{\min}}$ still satisfies the linear BSDE dynamics in (39) on $[0,T)$ (see equation (61) below) but this product may not have a sensible terminal value on $\{\eta = 0\} \cup \{\eta = +\infty\}$. Fortunately, as our analysis shows the optimal signal process always makes sense when needed. In particular its possible lack of a terminal value on $\{\eta = 0\}$ is without harm for our approach to the optimization problem. It thus can be viewed as a convenient substitute for the no longer operative linear BSDE above.

4.2 Constant coefficients

In case of constant coefficients $\nu_t \equiv \nu \in \mathbb{R}_+$, $\kappa_t \equiv \kappa \in \mathbb{R}_+$ and $\eta \in [0,+\infty]$ the stochastic Riccati differential equation in (19) boils down to a deterministic *ordinary Riccati differential equation* on $[0,T]$ of the form

$$c_t' = \frac{c_t^2}{\kappa} - \nu \quad \text{subject to } c_T = \eta$$

with explicitly available deterministic (minimal super-)solutions

$$c_t^{\min} = \begin{cases} \sqrt{\nu\kappa}\,\frac{\sqrt{\nu\kappa}\sinh\left(\sqrt{\nu/\kappa}\,(T-t)\right)+\eta\cosh\left(\sqrt{\nu/\kappa}\,(T-t)\right)}{\eta\sinh\left(\sqrt{\nu/\kappa}\,(T-t)\right)+\sqrt{\nu\kappa}\cosh\left(\sqrt{\nu/\kappa}\,(T-t)\right)} & 0 \le \eta < +\infty \\ \sqrt{\nu\kappa}\coth(\sqrt{\nu}(T-t)/\sqrt{\kappa}), & \eta = +\infty \end{cases}, \tag{40}$$

for all $0 \le t < T$. As a consequence, the process L given in (23) is also just deterministic and the optimal signal process $\hat{\xi}^{c^{\min}}$ in (28) can be computed explicitly (up to the conditional expectation). Again, our conjecture in (18) holds true. Indeed, our optimal control $\hat{u}^{c^{\min}}$ from (34) provided in Theorem 3.4 coincides with the optimal solution of the stochastic LQ problem in (12) with objective functional J^0 and J^∞, respectively, derived in Bank et al. [4], Theorems 3.1 and 3.2. Therein, our first integrability condition in (33) is satisfied as soon as the target process ξ belongs to $L^2(\mathbb{P} \otimes dt)$ and $\Xi_T \in L^2(\mathbb{P}, \mathscr{F}_{T-})$. The second integrability condition in (33) simplifies to a condition on the terminal position Ξ_T which is equivalent to

$$\int_0^T \frac{\mathbb{E}[(\Xi_T - \mathbb{E}[\Xi_T | \mathscr{F}_s])^2]}{(T-s)^2} ds < \infty;$$

see Remark 2.1 and Lemma 5.4 in [4]. It thus reveals that the ultimate target position Ξ_T has to become known "fast enough" for the optimally controlled process $\hat{X}^{c^{\min}}$ in order to reach it at terminal time T with finite expected costs; c.f. also Ankirchner and Kruse [2] who confine themselves to stochastic terminal state constraints of the form $\Xi_T = \int_0^T \lambda_t dt$ for some progressively measurable and suitably integrable process $(\lambda_t)_{0 \leq t \leq T}$ which are gradually revealed as $t \uparrow T$. Related results of this nature are also provided in Lü et al. [18].

For the general case with stochastic coefficients $\nu = (\nu_t)_{0 \leq t \leq T}$, $\kappa = (\kappa_t)_{0 \leq t \leq T}$ and random $\eta \in [0, \infty]$, similar effects are to be expected concerning the final target position Ξ_T *and* the "blow up" event $\{\eta = +\infty\}$. As, in general, all these coefficients can be rather intricately intertwined among each other, it seems difficult to give conditions on these that ensure $\mathscr{U}^\Xi \neq \varnothing$ and are more succinct than our conditions in (33).

4.3 Special case: Vanishing targets

In the special case $\xi \equiv \Xi_T \equiv 0$ $\mathbb{P}$-a.s., where obviously $\hat{\xi} \equiv 0$ and the integrability conditions in (33) hold trivially, our conjecture in (18) holds true as well. Indeed, Kruse and Popier [16] derive under sufficient integrability conditions on $(\kappa_t)_{0 \leq t \leq T}$ and $(\nu_t)_{0 \leq t \leq T}$ existence of a minimal supersolution $c^{\min}$ to the Riccati BSDE in (19) with terminal condition $\liminf_{t \uparrow T} \geq \eta \in [0, +\infty]$ (recall (20)). The minimal supersolution $c^{\min}$ is constructed via the monotone limit $c_t^{\min} \triangleq \lim_{n \uparrow \infty} c_t^{(n)}$ for all $t \in [0, T)$, where $c^{(n)}$ denotes the unique (minimal super-)solution with Riccati dynamics (19) satisfying the terminal condition $\lim_{t \uparrow T} c_T^{(n)} = \eta \wedge n$ for some constant $n > 0$. They show that the optimal control $\hat{u}^{c^{\min}}$ with state process (9) to the stochastic LQ problem in (12) with $\xi \equiv \Xi_T \equiv 0$ is given as in (34) of our Theorem 3.4; see [16], Theorem 3. That is, since $\hat{\xi}^{c^{\min}} \equiv 0$, the optimal control with controlled process $\hat{X}^{c^{\min}} \triangleq X^{\hat{u}^{c^{\min}}}$ is simply given by

$$\hat{u}_t^{c^{\min}} = -\frac{c_t^{\min}}{\kappa_t} X_t^{\hat{u}^{c^{\min}}} = -\frac{xL_t}{\kappa_t} \quad (0 \leq t \leq T) \tag{41}$$

and the corresponding optimal costs in (35) simplify dramatically to

$$J^{c^{\min}}(\hat{u}^{c^{\min}}) = c_0^{\min} x^2. \tag{42}$$

In fact, in order to tackle the partial state constraint $\Xi_T = 0$ on the set $\{\eta = +\infty\}$, Kruse and Popier [16] proceed via a *truncation in space*. Specifically, they introduce a family of unconstrained variants of problem (12) (with $\xi \equiv \Xi_T \equiv 0$) with objective functionals

$$J^{(n)}(u) \triangleq \mathbb{E}\left[\int_0^T (X_t^u)^2 \nu_t dt + \int_0^T \kappa_t u_t^2 dt + (\eta \wedge n)(X_T^u)^2\right], \tag{43}$$

where the random penalization parameter η is replaced by truncated versions $\eta \wedge n$. Then the corresponding optimal controls $\hat{u}_t^{(n)} = -c_t^{(n)} X_t^{\hat{u}^{(n)}}/\kappa_t$ and the corresponding optimal costs $J^{(n)}(\hat{u}^{(n)}) = c_0^{(n)} x^2$ clearly satisfy

$$\begin{aligned} J^\eta(\hat{u}^\eta) &\leq J^{c^{\min}}(\hat{u}^{c^{\min}}) = c_0^{\min} x^2 = \lim_{n \uparrow \infty} c_0^{(n)} x_0^2 \\ &= \lim_{n \uparrow \infty} J^{(n)}(\hat{u}^{(n)}) = \lim_{n \uparrow \infty} J^{c^{(n)}}(\hat{u}^{c^{(n)}}) \leq J^\eta(\hat{u}^\eta), \end{aligned} \tag{44}$$

where $\hat{u}^\eta$ denotes the optimizer of problem (12) (with $\xi \equiv \Xi_T \equiv 0$). It follows that equality holds everywhere and, by uniqueness of optimizers, $\hat{u}^\eta = \hat{u}^{c^{\min}}$ as conjectured in (18).

For the general case $\xi \neq 0$ and $\Xi_T \neq 0$, one could likewise introduce as above in (43) a family of unconstrained variants of problem (12) with objective functionals

$$J^{(n)}(u) = \mathbb{E}\left[\int_0^T (X_t^u - \xi_t)^2 \nu_t dt + \int_0^T \kappa_t u_t^2 dt + (\eta \wedge n)(X_T^u - \Xi_T)^2\right].$$

Recall from the discussion in Section 4.1 that this stochastic LQ problem is fully characterized by the solution processes $c^{(n)}$ and $b^{(n)}$ satisfying the Riccati BSDE in (19) and the linear BSDE in (39) with terminal conditions $c_T^{(n)} = \eta \wedge$

n and $b_T^{(n)} = (\eta \wedge n)\Xi_T$, respectively. In addition, under sufficient conditions (e.g. boundedness as discussed in Section 4.1) which guarantee (33) as well as the convergence

$$\limsup_{\tau \uparrow T} \mathbb{E}\left[c_\tau^{(n)}(X_\tau^{\hat{u}^{c^{(n)}}} - \xi_\tau^{c^{(n)}})^2\right] = \mathbb{E}\left[(\eta \wedge n)(X_T^{\hat{u}^{c^{(n)}}} - \Xi_T)^2\right],$$

our Theorem 3.4 applies in this context and it holds that

$$\begin{aligned} J^{(n)}(\hat{u}^{(n)}) &= c_0^{(n)}(x - \hat{\xi}_0^{c^{(n)}})^2 \\ &\quad + \mathbb{E}\left[\int_0^T (\xi_t - \hat{\xi}_t^{c^{(n)}})^2 \nu_t dt\right] + \mathbb{E}\left[\int_{[0,T)} c_t^{(n)} d[\hat{\xi}^{c^{(n)}}]_t\right] \end{aligned} \tag{45}$$

for the optimal control $\hat{u}^{(n)}$. As in Kruse and Popier [16], one could then try to pass to the limit $n \uparrow \infty$. However, passing to the limit is not as straightforward in (45) as it is in (44) where we relied heavily on $\xi \equiv 0$, $\Xi_T \equiv 0$. Indeed, for convergence of (45), a suitable convergence of our signal processes $\hat{\xi}^{c^{(n)}}$ would be required which seems to be out of reach with the current knowledge of singular BSRDEs and so a full proof of our conjecture (18) by this approach has to be left for future research.

5 Proofs

Throughout this section we work under the assumptions of our main result, Theorem 3.4. Its verification relies on a completion of squares argument similar to Kohlmann and Tang [15] (cf. also Yong and Zhou [25] for this method in solving LQ problems). The following lemma summarizes the key identity for our verification and illustrates again the usefulness of our signal process $\hat{\xi}$.

Lemma 5.1. *Suppose the assumptions of Theorem 3.4 hold true. Then for all progressively measurable, $\mathbb{P}$-a.s. locally $L^2([0,T), \kappa_t dt)$-integrable processes u, the cost process*

$$C_t(u) \triangleq \int_0^t (X_s^u - \xi_s)^2 \nu_s ds + \int_0^t \kappa_s u_s^2 ds + c_t(X_t^u - \hat{\xi}_t)^2 \quad (0 \le t < T)$$

is a nonnegative, càdlàg local submartingale. It allows for the decomposition

$$C_t(u) = c_0(x - \hat{\xi}_0)^2 + A_t(u) + M_t(u) \quad (0 \le t < T), \tag{46}$$

where

$$A_t(u) \triangleq \int_0^t (\xi_s - \hat{\xi}_s)^2 \nu_s ds + \int_0^t c_s d[\hat{\xi}]_s$$
$$+ \int_0^t \kappa_s \left(u_s - \frac{c_s}{\kappa_s} \left(\hat{\xi}_s - X_s^u \right) \right)^2 ds \quad (0 \leq t < T) \tag{47}$$

is a right continuous, nondecreasing, adapted process and where

$$M_t(u) \triangleq \int_0^t (\hat{\xi}_{s-}^2 - (X_{s-}^u)^2) dN_s + 2 \int_0^t \frac{c_{s-}}{L_{s-}} (\hat{\xi}_{s-} - X_{s-}^u) d\tilde{M}_s \tag{48}$$

with

$$\tilde{M}_t \triangleq \mathbb{E} \left[\Xi_T L_T + \int_0^T \xi_s e^{-\int_0^s \frac{c_u}{\kappa_u} du} \nu_s ds \,\middle|\, \mathscr{F}_t \right] \quad (0 \leq t < T) \tag{49}$$

is a local martingale on $[0, T)$*.*

Proof. First, note that by (5) and $u \in L^2([0,T), \kappa_t dt)$ locally $\mathbb{P}$-a.s., the cost process $(C_t(u))_{0 \leq t < T}$ in (46) is well defined along with X^u. Let us expand

$$c_t (X_t^u - \hat{\xi}_t)^2 = c_t (X_t^u)^2 - 2 X_t^u c_t \hat{\xi}_t + c_t \hat{\xi}_t^2 \quad (0 \leq t < T)$$

and then apply Itô's formula to each of the resulting three terms. This will be prepared by computing the dynamics of the processes $\hat{\xi}$, $c\hat{\xi}$ and $c\hat{\xi}^2$, respectively, in the following steps 1, 2 and 3. In step 4 we put everything together and derive our main identity (46).

Step 1: We start with computing the dynamics of our optimal signal process $(\hat{\xi}_t)_{0 \leq t < T}$ defined in (28). For ease of notation, let us define the process

$$Y_t \triangleq \int_0^t \xi_r e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr \quad (0 \leq t \leq T).$$

Observe that $Y_T \in L^1(\mathbb{P})$ due to (6). Moreover, recall that $\Xi_T L_T \in L^1(\mathscr{F}_{T-}, \mathbb{P})$ by (27) so that (49) defines a càdlàg martingale on $[0, T]$. By the definition of $\hat{\xi}$ in (28), we can now express $\hat{\xi}$ in terms of Y and $\tilde{M}$ via

$$\hat{\xi}_t = \frac{1}{L_t} \mathbb{E} \left[\Xi_T L_T + \int_t^T \xi_r e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr \,\middle|\, \mathscr{F}_t \right] = \frac{1}{L_t} \left(\tilde{M}_t - Y_t \right) \tag{50}$$

for all $0 \leq t < T$. Next, recall the dynamics of L on $[0, T)$ in (25) and note that

$$\Delta L_t = -\frac{L_{t-}}{c_{t-}} \Delta N_t \quad \text{and} \quad [L]_t^c = \int_0^t \frac{L_{s-}^2}{c_{s-}^2} d[N]_s^c, \tag{51}$$

where $[L]^c$ and $[N]^c$ denote the path-by-path continuous parts of the quadratic variations of $[L]$ and $[N]$, respectively (cf., e.g., Protter [22], Chapter II.6, for more details). Hence, applying Itô's formula as in, e.g., [22], Theorem II.32, we obtain

$$\begin{aligned}\frac{1}{L_t} = \frac{1}{L_0} - \int_0^t \frac{1}{L_{s-}^2} dL_s + \int_0^t \frac{1}{L_{s-}^3} d[L]_s^c \\ + \sum_{s \leq t} \left(\frac{1}{L_s} - \frac{1}{L_{s-}} + \frac{1}{L_{s-}^2} \Delta L_s \right).\end{aligned} \tag{52}$$

Using (51), the summands in the sum in (52) above can be written as

$$\frac{L_{s-} - L_s}{L_s L_{s-}} - \frac{\Delta N_s}{L_{s-} c_{s-}} = \frac{\Delta N_s}{c_{s-}} \frac{L_{s-} - L_s}{L_s L_{s-}} = \frac{(\Delta N_s)^2}{L_s c_{s-}^2} = \frac{(\Delta N_s)^2}{L_{s-} c_{s-} c_s},$$

where we also used $\Delta c_s = -\Delta N_s$ and thus the identity $1/L_s = c_{s-}/(L_{s-} c_s)$ in the last equality. Hence, together with the dynamics of L in (25) and $[L]^c$ in (51) we can rewrite (52) as

$$\begin{aligned}\frac{1}{L_t} = \frac{1}{L_0} + \int_0^t \frac{\nu_s}{L_{s-} c_{s-}} ds + \int_0^t \frac{1}{L_{s-} c_{s-}} dN_s \\ + \int_0^t \frac{1}{L_{s-} c_{s-}^2} d[N]_s^c + \sum_{s \leq t} \frac{(\Delta N_s)^2}{L_{s-} c_{s-} c_s}.\end{aligned} \tag{53}$$

Now, integrating by parts in (50) and then using the dynamics of $1/L$ in (53) gives us

$$\begin{aligned}\hat{\xi}_t &= \hat{\xi}_0 + \int_0^t \frac{1}{L_{s-}} (d\tilde{M}_s - dY_s) + \int_0^t \hat{\xi}_{s-} L_{s-} d\left(\frac{1}{L_s}\right) + \left[\frac{1}{L}, \tilde{M}\right]_t \\ &= \hat{\xi}_0 - \int_0^t (\xi_s - \hat{\xi}_{s-}) \frac{\nu_s}{c_{s-}} ds + \int_0^t \frac{1}{L_{s-}} d\tilde{M}_s + \int_0^t \frac{\hat{\xi}_{s-}}{c_{s-}} dN_s \\ &\quad + \int_0^t \frac{\hat{\xi}_{s-}}{c_{s-}^2} d[N]_s^c + \sum_{s \leq t} \frac{\hat{\xi}_{s-}}{c_{s-} c_s} (\Delta N_s)^2 + \left[\frac{1}{L}, \tilde{M}\right]_t,\end{aligned} \tag{54}$$

where the quadratic covariation can be computed as

$$\left[\frac{1}{L},\tilde{M}\right]_t = \int_0^t \frac{1}{L_{s-}c_{s-}} d[\tilde{M},N]_s^c \\ + \sum_{s\le t}\left(\frac{\Delta\tilde{M}_s\Delta N_s}{L_{s-}c_{s-}} + \frac{(\Delta N_s)^2\Delta\tilde{M}_s}{L_{s-}c_{s-}c_s}\right). \tag{55}$$

Collecting all the sums in (54) together with those in (55) yields

$$\sum_{s\le t}\frac{\Delta N_s}{L_{s-}c_{s-}c_s}\left(c_s\Delta\tilde{M}_s + \Delta N_s\Delta\tilde{M}_s + \hat{\xi}_{s-}L_{s-}\Delta N_s\right) \\ = \sum_{s\le t}\frac{\Delta N_s}{L_{s-}c_{s-}c_s}\left(\hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-}L_{s-}c_s\right), \tag{56}$$

where we used the fact that $\Delta N_s = -\Delta c_s$ as well as

$$\Delta\tilde{M}_s = \tilde{M}_s - \tilde{M}_{s-} = \hat{\xi}_s L_s - \hat{\xi}_{s-}L_{s-} \tag{57}$$

due to the representation of $\hat{\xi}$ in (50) and the continuity of Y. Plugging back (56) into (54) finally gives us

$$\hat{\xi}_t = \hat{\xi}_0 - \int_0^t(\xi_s - \hat{\xi}_{s-})\frac{\nu_s}{c_{s-}}ds + \int_0^t \frac{1}{L_{s-}}d\tilde{M}_s + \int_0^t \frac{\hat{\xi}_{s-}}{c_{s-}}dN_s \\ + \int_0^t \frac{\hat{\xi}_{s-}}{c_{s-}^2}d[N]_s^c + \int_0^t \frac{1}{L_{s-}c_{s-}}d[\tilde{M},N]_s^c \\ + \sum_{s\le t}\frac{\Delta N_s}{L_{s-}c_{s-}c_s}\left(\hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-}L_{s-}c_s\right). \tag{58}$$

Step 2: Let us now compute the dynamics of $c\hat{\xi}$. Again, integration by parts, together with the dynamics of $\hat{\xi}$ in (58), yields

$$c_t\hat{\xi}_t = c_0\hat{\xi}_0 + \int_0^t c_{s-}d\hat{\xi}_s + \int_0^t \hat{\xi}_{s-}dc_s + \left[c,\hat{\xi}\right]_t \\ = c_0\hat{\xi}_0 - \int_0^t \xi_s\nu_s ds + \int_0^t \hat{\xi}_{s-}\frac{c_s^2}{\kappa_s}ds + \int_0^t \frac{c_{s-}}{L_{s-}}d\tilde{M}_s \\ + \int_0^t \frac{\hat{\xi}_{s-}}{c_{s-}}d[N]_s^c + \int_0^t \frac{1}{L_{s-}}d[\tilde{M},N]_s^c \\ + \sum_{s\le t}\frac{\Delta N_s}{L_{s-}c_s}\left(\hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-}L_{s-}c_s\right) + \left[c,\hat{\xi}\right]_t. \tag{59}$$

The quadratic covariation in (59) can be computed as

$$\begin{aligned}\left[c,\hat{\xi}\right]_t = &-\int_0^t \frac{1}{L_{s-}} d[\tilde{M},N]_s^c - \int_0^t \frac{\hat{\xi}_{s-}}{c_{s-}} d[N]_s^c \\ &- \sum_{s\le t} \frac{\Delta N_s \Delta \tilde{M}_s}{L_{s-}} - \sum_{s\le t} \frac{\hat{\xi}_{s-}(\Delta N_s)^2}{c_{s-}} \\ &- \sum_{s\le t} \frac{(\Delta N_s)^2}{L_{s-}c_{s-}c_s}\left(\hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s\right). \end{aligned} \tag{60}$$

The sums of the jumps in the quadratic covariation in (60) can be rewritten (using again the identity in (57) as well as the fact that $\Delta c_s = -\Delta N_s$) as

$$\begin{aligned}&-\sum_{s\le t} \frac{\Delta N_s}{L_{s-}c_{s-}c_s}\left(\Delta \tilde{M}_s c_s c_{s-} + \hat{\xi}_{s-}\Delta N_s L_{s-} c_s + \Delta N_s \hat{\xi}_s L_s c_{s-} - \Delta N_s \hat{\xi}_{s-} L_{s-} c_s\right) \\ &= -\sum_{s\le t} \frac{\Delta N_s}{L_{s-}c_s}\left(\hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s\right).\end{aligned}$$

With this observation, plugging back the quadratic covariation in (60) into (59), we simply get

$$c_t\hat{\xi}_t = c_0\hat{\xi}_0 - \int_0^t \xi_s \nu_s ds + \int_0^t \hat{\xi}_{s-}\frac{c_s^2}{\kappa_s} ds + \int_0^t \frac{c_{s-}}{L_{s-}} d\tilde{M}_s. \tag{61}$$

Step 3: Next, we compute the dynamics of $c\hat{\xi}^2$. Application of integration by parts together with the dynamics of $\hat{\xi}$ in (58) yields

$$\begin{aligned}\hat{\xi}_t^2 &= \hat{\xi}_0^2 + 2\int_0^t \hat{\xi}_{s-} d\hat{\xi}_s + [\hat{\xi}]_t \\ &= \hat{\xi}_0^2 - 2\int_0^t \hat{\xi}_{s-}(\xi_s - \hat{\xi}_{s-})\frac{\nu_s}{c_{s-}} ds + 2\int_0^t \frac{\hat{\xi}_{s-}}{L_{s-}} d\tilde{M}_s + 2\int_0^t \frac{\hat{\xi}_{s-}^2}{c_{s-}} dN_s \\ &\quad + 2\int_0^t \frac{\hat{\xi}_{s-}^2}{c_{s-}^2} d[N]_s^c + 2\int_0^t \frac{\hat{\xi}_{s-}}{L_{s-}c_{s-}} d[\tilde{M},N]_s^c \\ &\quad + 2\sum_{s\le t} \frac{\hat{\xi}_{s-}\Delta N_s}{L_{s-}c_{s-}c_s}\left(\hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s\right) + [\hat{\xi}]_t.\end{aligned}$$

Consequently, using once more integration by parts, we obtain

$$
\begin{aligned}
c_t\hat{\xi}_t^2 &= c_0\hat{\xi}_0^2 + \int_0^t c_{s-}d\hat{\xi}_s^2 + \int_0^t \hat{\xi}_{s-}^2 dc_s + [c,\hat{\xi}^2]_t \\
&= c_0\hat{\xi}_0^2 - 2\int_0^t \hat{\xi}_{s-}(\xi_s - \hat{\xi}_{s-})\nu_s ds + 2\int_0^t \frac{c_{s-}\hat{\xi}_{s-}}{L_{s-}} d\tilde{M}_s + 2\int_0^t \hat{\xi}_{s-}^2 dN_s \\
&\quad + 2\int_0^t \frac{\hat{\xi}_{s-}^2}{c_{s-}} d[N]_s^c + 2\int_0^t \frac{\hat{\xi}_{s-}}{L_{s-}} d[\tilde{M},N]_s^c \\
&\quad + 2\sum_{s\le t} \frac{\hat{\xi}_{s-}\Delta N_s}{L_{s-}c_s}\left(\hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-}L_{s-}c_s\right) + \int_0^t c_{s-}d[\hat{\xi}]_s \\
&\quad + \int_0^t \hat{\xi}_{s-}^2 \frac{c_s^2}{\kappa_s} ds - \int_0^t \hat{\xi}_{s-}^2 \nu_s ds - \int_0^t \hat{\xi}_{s-}^2 dN_s + [c,\hat{\xi}^2]_t. \qquad (62)
\end{aligned}
$$

The final quadratic covariation in (62) can be computed as

$$
\begin{aligned}
[c,\hat{\xi}^2]_t = &-2\int_0^t \frac{\hat{\xi}_{s-}}{L_{s-}} d[\tilde{M},N]_s^c - 2\sum_{s\le t}\frac{\hat{\xi}_{s-}}{L_{s-}}\Delta\tilde{M}_s\Delta N_s \\
&-2\int_0^t \frac{\hat{\xi}_{s-}^2}{c_{s-}} d[N]_s^c - 2\sum_{s\le t}\frac{\hat{\xi}_{s-}^2}{c_{s-}}(\Delta N_s)^2 \\
&-2\sum_{s\le t}\frac{\hat{\xi}_{s-}(\Delta N_s)^2}{L_{s-}c_{s-}c_s}\left(\hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-}L_{s-}c_s\right) + \int_0^t \Delta c_s d[\hat{\xi}]_s. \qquad (63)
\end{aligned}
$$

Observe that the sum of jumps in (63) can be rewritten as

$$
\begin{aligned}
&-2\sum_{s\le t}\frac{\hat{\xi}_{s-}\Delta N_s}{L_{s-}c_{s-}c_s}\Big(\Delta\tilde{M}_s c_s c_{s-} + \hat{\xi}_{s-}\Delta N_s c_s L_{s-} \\
&\qquad\qquad + \Delta N_s\hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-}L_{s-}c_s\Delta N_s\Big) \\
&= -2\sum_{s\le t}\frac{\hat{\xi}_{s-}\Delta N_s}{L_{s-}c_s}\left(\hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-}L_{s-}c_s\right),
\end{aligned}
$$

where we used once more the identity in (57) and $\Delta c_s = -\Delta N_s$. With this observation, plugging back (63) into (62), we finally obtain

$$
\begin{aligned}
c_t\hat{\xi}_t^2 &= c_0\hat{\xi}_0^2 - 2\int_0^t \hat{\xi}_{s-}\xi_s\nu_s ds + \int_0^t \hat{\xi}_{s-}^2\nu_s ds + 2\int_0^t \frac{c_{s-}\hat{\xi}_{s-}}{L_{s-}} d\tilde{M}_s \\
&\quad + \int_0^t \hat{\xi}_{s-}^2 dN_s + \int_0^t c_s d[\hat{\xi}]_s + \int_0^t \hat{\xi}_{s-}^2 \frac{c_s^2}{\kappa_s} ds. \qquad (64)
\end{aligned}
$$

Step 4: Let us now put together all the computations from the preceding steps. Specifically, let u be a progressively measurable, $\mathbb{P}$-a.s. locally $L^2([0,T),\kappa_t dt)$-integrable process with corresponding controlled process X^u. Due to our computations in (61) and (64) as well as the fact that X^u is continuous and of finite variation, we get for all $0 \leq t < T$ that

$$
\begin{aligned}
&c_t(X_t^u - \hat{\xi}_t)^2 = c_t(X_t^u)^2 - 2X_t^u c_t\hat{\xi}_t + c_t\hat{\xi}_t^2 \\
&= c_0(x-\hat{\xi}_0)^2 + \int_0^t c_s d[\hat{\xi}]_s - \int_0^t (X_s^u)^2\nu_s ds + 2\int_0^t X_s^u\nu_s\xi_s ds \\
&\quad - 2\int_0^t c_s u_s(\hat{\xi}_s - X_s^u)ds + \int_0^t \frac{c_s^2}{\kappa_s}(X_s^u - \hat{\xi}_s)^2 ds - 2\int_0^t \hat{\xi}_s\xi_s\nu_s ds + \int_0^t \hat{\xi}_s^2\nu_s ds \\
&\quad + \int_0^t (\hat{\xi}_{s-}^2 - (X_{s-}^u)^2) dN_s + 2\int_0^t \frac{c_{s-}}{L_{s-}}\left(\hat{\xi}_{s-} - X_{s-}^u\right) d\tilde{M}_s. \qquad (65)
\end{aligned}
$$

Observe that the last two stochastic integrands sum up to $M_t(u)$ defined in (48). Furthermore, two completions of squares in the third line of (65) yield

$$
\begin{aligned}
&c_t(X_t^u - \hat{\xi}_t)^2 \\
&= c_0(x-\hat{\xi}_0)^2 + \int_0^t c_s d[\hat{\xi}]_s - \int_0^t (X_s^u)^2\nu_s ds + 2\int_0^t X_s^u\nu_s\xi_s ds \\
&\quad + \int_0^t \kappa_s\left(u_s - \frac{c_s}{\kappa_s}\left(\hat{\xi}_s - X_s^u\right)\right)^2 ds + \int_0^t (\xi_s - \hat{\xi}_s)^2\nu_s ds \\
&\quad - \int_0^t \kappa_s u_s^2 ds - \int_0^t \xi_s^2\nu_s ds + M_t(u) \\
&= c_0(x-\hat{\xi}_0)^2 + \int_0^t c_s d[\hat{\xi}]_s - \int_0^t (X_s^u - \xi_s)^2\nu_s ds \\
&\quad + \int_0^t \kappa_s\left(u_s - \frac{c_s}{\kappa_s}\left(\hat{\xi}_s - X_s^u\right)\right)^2 ds + \int_0^t (\xi_s - \hat{\xi}_s)^2\nu_s ds \\
&\quad - \int_0^t \kappa_s u_s^2 ds + M_t(u)
\end{aligned}
$$

Consequently, we can write

$$
\begin{aligned}
0 \leq C_t(u) = & \int_0^t (X_s^u - \xi_s)^2 \nu_s ds + \int_0^t \kappa_s u_s^2 ds + c_t (X_t^u - \hat{\xi}_t)^2 \\
= & \, c_0 (x - \hat{\xi}_0)^2 + \int_0^t c_s d[\hat{\xi}]_s + \int_0^t (\xi_s - \hat{\xi}_s)^2 \nu_s ds \\
& + \int_0^t \kappa_s \left(u_s - \frac{c_s}{\kappa_s} \left(\hat{\xi}_s - X_s^u \right) \right)^2 ds + M_t(u) \\
= & \, c_0 (x - \hat{\xi}_0)^2 + A_t(u) + M_t(u) \quad (0 \leq t < T)
\end{aligned} \tag{66}
$$

with $(A_t(u))_{0 \leq t < T}$ as defined in (47). Finally, observe that the process $(A_t(u))_{0 \leq t < T}$ is a right continuous, nondecreasing, adapted process and that $(M_t(u))_{0 \leq t < T}$ is a càdlàg local martingale because $\tilde{M}$ and N are local martingales on $[0, T)$ and all integrands in (48) are left continuous (cf., e.g., Protter [22], Theorem III.33). Consequently, we have that $(C_t(u))_{0 \leq t < T}$ is a nonnegative, càdlàg local submartingale. $\square$

We are now ready to give the proof of our main Theorem 3.4:

Proof of Theorem 3.4: First, let us assume that $\mathscr{U}^c \neq \varnothing$. For any $u \in \mathscr{U}^c$ we can consider the corresponding cost process $C_t(u) = c_0(x - \hat{\xi}_0)^2 + A_t(u) + M_t(u)$, $0 \leq t < T$, as in (46) of Lemma 5.1 above. Let $(\tau^n)_{n=1,2,\ldots}$ be a localizing sequence of stopping times for the local martingale $(M_t(u))_{0 \leq t < T}$ such that $\tau^n \uparrow T$ $\mathbb{P}$-a.s. strictly from below as $n \to \infty$ and $(M_{t \wedge \tau^n}(u))_{0 \leq t < T}$ is a uniformly integrable martingale for each n (cf., e.g., Protter [22], Chapter I.6, for more details). Then it holds by definition of our performance functional J in (30) that

$$
\begin{aligned}
\infty > J(u) &\triangleq \limsup_{\tau\uparrow T} \mathbb{E}[C_\tau(u)] \\
&\geq c_0(x-\hat{\xi}_0)^2 + \limsup_{n\to\infty} \{\mathbb{E}[A_{\tau^n}(u)] + \mathbb{E}[M_{\tau^n}(u)]\} \\
&= c_0(x-\hat{\xi}_0)^2 \\
&\quad + \mathbb{E}\left[\int_0^T (\xi_s - \hat{\xi}_s)^2 \nu_s ds + \int_0^T c_s d[\hat{\xi}]_s \right. \\
&\qquad\qquad \left. + \int_0^T \kappa_s \left(u_s - \frac{c_s}{\kappa_s}\left(\hat{\xi}_s - X_s^u\right)\right)^2 ds\right] \\
&\geq c_0(x-\hat{\xi}_0)^2 + \mathbb{E}\left[\int_0^T (\xi_s - \hat{\xi}_s)^2 \nu_s ds\right] + \mathbb{E}\left[\int_{[0,T)} c_s d[\hat{\xi}]_s\right], \quad (67)
\end{aligned}
$$

where we used monotone convergence and applied Doob's Optional Sampling Theorem as, e.g., in Protter [22], Theorem I.16, in order to get $\mathbb{E}[M_{\tau^n}(u)] = 0$ for all $n \geq 1$. In particular, the computations in (67) show that (33) necessarily holds true if $\mathscr{U}^c \neq \varnothing$ (as assumed for now). In other words, setting

$$
v \triangleq c_0(x-\hat{\xi}_0)^2 + \mathbb{E}\left[\int_0^T (\xi_s - \hat{\xi}_s)^2 \nu_s ds\right] + \mathbb{E}\left[\int_{[0,T)} c_s d[\hat{\xi}]_s\right] < \infty, \qquad (68)
$$

we have for all $u \in \mathscr{U}^c$ the lower bound

$$
J(u) \geq v. \qquad (69)
$$

Now, let us define the control $\hat{u}$ with corresponding controlled process $\hat{X} \triangleq X^{\hat{u}}$ via the feedback law

$$
\hat{u}_t = \frac{c_t}{\kappa_t}(\hat{\xi}_t - \hat{X}_t) \quad (0 \leq t < T).
$$

Observe that $\hat{u}$ is a progressively measurable process and locally dt-integrable and locally $\kappa_t dt$-square-integrable on $[0,T)$ due to (5). In particular, $\hat{X}_T = x + \int_0^T \hat{u}_t dt$ exists $\mathbb{P}$-a.s and we can invoke Lemma 5.1. We denote by $C_t(\hat{u}) = c_0(x-\hat{\xi}_0)^2 + M_t(\hat{u}) + A_t(\hat{u})$, $0 \leq t < T$, the corresponding cost process from this lemma. We will now show that $\hat{u} \in \mathscr{U}^c$ and that $\hat{u}$ attains the lower bound in (69), i.e.,

$$
J(\hat{u}) = v
$$

finishing our verification argument. Indeed, first note that, by choice of $\hat{u}$, we have

$$A_t(\hat{u}) = \int_0^t (\xi_s - \hat{\xi}_s)^2 \nu_s ds + \int_0^t c_s d[\hat{\xi}]_s \quad (0 \leq t < T),$$

whence, in particular,

$$v = c_0(x - \hat{\xi}_0)^2 + \mathbb{E}[A_{T-}(\hat{u})] < \infty.$$

Next, since $M(\hat{u})$ is a local martingale on $[0, T)$ by virtue of Lemma 5.1 above, we can fix a localizing sequence of stopping times $(\hat{\tau}^n)_{n=1,2,\ldots}$ such that $\hat{\tau}^n \uparrow T$ $\mathbb{P}$-a.s. strictly from below for $n \to \infty$ and such that $(M_{t\wedge\hat{\tau}^n}(\hat{u}))_{0\leq t<T}$ is a uniformly integrable martingale for each n. Then, for any stopping time $\tau < T$, applying Fatou's Lemma and once more Doob's Optional Sampling Theorem yields

$$\begin{aligned}
\mathbb{E}[C_\tau(\hat{u})] &= \mathbb{E}[\liminf_{n\to\infty} C_{\tau\wedge\hat{\tau}^n}(\hat{u})] \leq \liminf_{n\to\infty} \mathbb{E}[C_{\tau\wedge\hat{\tau}^n}(\hat{u})] \\
&= c_0(x - \hat{\xi}_0)^2 + \liminf_{n\to\infty} \{\mathbb{E}[A_{\tau\wedge\hat{\tau}^n}(\hat{u})] + \mathbb{E}[M_{\tau\wedge\hat{\tau}^n}(\hat{u})]\} \\
&= c_0(x - \hat{\xi}_0)^2 + \liminf_{n\to\infty} \mathbb{E}[A_{\tau\wedge\hat{\tau}^n}(\hat{u})] \\
&= c_0(x - \hat{\xi}_0)^2 + \mathbb{E}[A_\tau(\hat{u})] \leq c_0(x - \hat{\xi}_0)^2 + \mathbb{E}[A_{T-}(\hat{u})] = v,
\end{aligned}$$

where we also used monotone convergence as well as the fact that $(A(\hat{u})_t)_{0\leq t<T}$ is an increasing process. Hence, it holds that

$$J(\hat{u}) = \limsup_{\tau\uparrow T} \mathbb{E}[C_\tau(\hat{u})] \leq v < \infty \tag{70}$$

and thus $\hat{u} \in \mathscr{U}^c$. In particular, due to (69), we actually have $J(\hat{u}) = v$ as desired.

Finally, let us assume that (33) is satisfied. Then, it follows from (68) and (70) that $\hat{u} \in \mathscr{U}^c$, i.e., $\mathscr{U}^c \neq \varnothing$. In other words, condition (33) is not only necessary but also sufficient for $\mathscr{U}^c \neq \varnothing$. □

Remark 5.2. Let us briefly comment on a few insights offered by the preceding proof. The argument rests on the key identity (46) of Lemma 5.1. For bounded coefficients $\kappa, 1/\kappa, \nu, \eta$ and bounded targets ξ, Ξ_T, the theory of BSRDEs readily allows one to deduce that $M(u)$ of (48) is a true martingale for any control u with finite expected costs; see, e.g., Kohlmann and Tang

[15]. From the key identity (46) it then transpires that the control $\hat{u}^c$ of (34) minimizes

$$\mathbb{E}\left[\int_0^\tau (X_t^u - \xi_t)^2 \nu_t dt + \int_0^\tau \kappa_t u_t^2 dt + c_\tau (X_\tau^u - \hat{\xi}_\tau^c)^2\right]$$

simultaneously for all stopping times $\tau \leq T$. In that sense, the above terminal penalizations $c_\tau(X_\tau^u - \hat{\xi}_\tau^c)^2$ are consistent replacements for $\eta(X_T^u - \Xi_T)^2$ for these problems with shorter time horizons.

When coefficients are unbounded, particularly when $\mathbb{P}[\eta = +\infty] > 0$, it is quite possible that $M(u)$ is a strict local martingale and so the preceding argument breaks down. Still, being bounded from below by an integrable random variable under condition (33), $M(u)$ is a supermartingale; its martingale property thus turns out to hinge on the control in $L^1(\mathbb{P})$ of $c_{\tau^n}(X_{\tau^n}^u - \hat{\xi}_{\tau^n}^c)^2$ along a suitable sequence of stopping times $\tau^n \uparrow T$. This control does not seem to be available in the BSRDE literature at present, leaving our conjecture (18) still open at this point.

Observe, though, that, taking the $\limsup_{\tau\uparrow T}$ of the above expectations, the formulation in our auxiliary target functional $J^c(u)$ of (30) avoids these issues and thus allows us to solve at least these closely related auxiliary problems.

The final lemma justifies the interpretation in Remark 3.5:

Lemma 5.3. *Let us assume that* $\lim_{t\uparrow T} L_t = L_T$ *in* $L^1(\mathbb{P})$ *and that the local càdlàg martingale* $(N_t)_{0\leq t<T}$ *in* (19) *satisfies* $\mathbb{E}[[N]_t^{1/2}] < \infty$ *for all* $0 \leq t < T$. *Then we have the representation*

$$c_t = \mathbb{E}\left[L_T e^{\int_0^t \frac{c_u}{\kappa_u} du} + \int_t^T e^{-\int_t^r \frac{c_u}{\kappa_u} du} \nu_r dr \,\middle|\, \mathscr{F}_t\right] \quad (0 \leq t < T). \tag{71}$$

Moreover, on $\{\mathbb{P}[\int_t^T \nu_r dr = 0 \mid \mathscr{F}_t] < 1\}$ *we have the identity*

$$\mathbb{E}\left[\int_t^T \frac{e^{-\int_t^r \frac{c_u}{\kappa_u} du}}{(1-w_t)c_t} \nu_r dr \middle| \mathscr{F}_t\right] = 1, \tag{72}$$

and the weight process $w_t = \mathbb{E}[L_T|\mathscr{F}_t]/L_t$ *of* (36) *satisfies* $0 \leq w_t < 1$.

Proof. Recall the dynamics of the process $(L_t)_{0\leq t<T}$ in (25), i.e.,

$$L_t = c_0 - \int_0^t e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr - \int_0^t e^{-\int_0^r \frac{c_u}{\kappa_u} du} dN_r \quad (0 \leq t < T).$$

Hence, for all $0 \le t \le s < T$ we may write

$$L_s - L_t = -\int_t^s e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr - \int_t^s e^{-\int_0^r \frac{c_u}{\kappa_u} du} dN_r. \tag{73}$$

Observe that the stochastic integral in (73) is a martingale on $[0, T)$ by our integrability assumption on $[N]_t^{1/2}$. Thus, taking conditional expectations in (73) yields

$$\mathbb{E}[L_s | \mathscr{F}_t] - L_t = -\mathbb{E}\left[\int_t^s e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr \,\middle|\, \mathscr{F}_t \right] \quad (0 \le t \le s < T). \tag{74}$$

Passing to the limit $s \uparrow T$ in (74) we obtain, due to monotone convergence and due to the assumption that L_s converges in $L^1(\mathbb{P})$ to L_T, the representation

$$L_t = \mathbb{E}\left[L_T + \int_t^T e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr \,\middle|\, \mathscr{F}_t \right] \quad (0 \le t < T). \tag{75}$$

In other words, using that $L_t = c_t e^{-\int_0^t \frac{c_u}{\kappa_u} du}$, we can write

$$c_t = \mathbb{E}\left[L_T e^{\int_0^t \frac{c_u}{\kappa_u} du} + \int_t^T e^{-\int_t^r \frac{c_u}{\kappa_u} du} \nu_r dr \,\middle|\, \mathscr{F}_t \right] \quad (0 \le t < T)$$

as desired for (71). Finally, by definition of the weight process $(w_t)_{0 \le t < T}$ in (36) together with the identity in (71), we can write

$$\begin{aligned} w_t = \frac{\mathbb{E}[L_T | \mathscr{F}_t]}{L_t} &= \frac{e^{\int_0^t \frac{c_u}{\kappa_u} du}}{c_t} \mathbb{E}[L_T \mid \mathscr{F}_t] = \frac{1}{c_t} \mathbb{E}\left[e^{\int_0^t \frac{c_u}{\kappa_u} du} L_T \,\middle|\, \mathscr{F}_t \right] \\ &= \frac{1}{c_t} \left(c_t - \mathbb{E}\left[\int_t^T e^{-\int_t^r \frac{c_u}{\kappa_u} du} \nu_r dr \,\middle|\, \mathscr{F}_t \right] \right) \\ &= 1 - \frac{1}{c_t} \mathbb{E}\left[\int_t^T e^{-\int_t^r \frac{c_u}{\kappa_u} du} \nu_r dr \,\middle|\, \mathscr{F}_t \right] \quad \text{for all } 0 \le t < T, \end{aligned} \tag{76}$$

which yields our claim (72). In particular, representation (76) also reveals that $0 \le w_t < 1$ $\mathbb{P}$-a.s. on $\{\mathbb{P}[\int_t^T \nu_r dr = 0 \mid \mathscr{F}_t] < 1\}$ for all $0 \le t < T$. $\square$

Appendix

In this appendix, we collect some results on the BSRDE in (19) with terminal condition (20) which may be of independent interest for the theory of BSDEs. First, let us provide lower estimates for a minimal supersolution $c^{\min}$ to the Riccati BSDE in (19) with terminal condition (20).

Lemma 5.4. *Let $(\nu_t)_{0\le t\le T}$, $(\kappa_t)_{0\le t\le T}$ satisfy* (5) *and let $c^{\min}$ denote a minimal supersolution to* (19) *with terminal condition* (20)*. Then for all $t\in[0,T)$ we have*

$$c_t^{\min} \ge \mathbb{E}\left[\frac{1}{\int_t^T \frac{1}{\kappa_s}ds + \frac{1}{\eta}}\,\middle|\,\mathscr{F}_t\right] \ge 0 \quad \mathbb{P}\text{-}a.s. \tag{77}$$

with strict inequality holding true in the first estimate on $\{\mathbb{P}[\int_t^T \nu_s ds = 0|\mathscr{F}_t] < 1\}$ and strict inequality in the second estimate on $\{\mathbb{P}[\eta = 0|\mathscr{F}_t] < 1\}$. In particular, any supersolution c of (19) *and* (20) *will be strictly positive throughout $[0,T)$ if* (7) *holds true.*

Proof. We will adopt the same idea as in the proof of Lemma 11 in Popier [20] in the case $\kappa\equiv 1$ (and $\nu\equiv 0$). For all $n\ge 1$ we define the processes

$$\Gamma_t^n \triangleq \mathbb{E}\left[\frac{1}{\int_t^T \frac{1}{\kappa_s}ds + \frac{1}{\eta\wedge n}}\,\middle|\,\mathscr{F}_t\right] \quad (0\le t\le T).$$

Note that Γ^n is well defined because the term in the conditional expectation is bounded by n. Moreover, we have pathwise the identity

$$\frac{1}{\int_t^T \frac{1}{\kappa_s}ds + \frac{1}{\eta\wedge n}} = \eta\wedge n - \int_t^T \frac{1}{\kappa_s}\left(\frac{1}{\int_s^T \frac{1}{\kappa_u}du + \frac{1}{\eta\wedge n}}\right)^2 ds \quad (0\le t\le T).$$

Thus, the process Γ^n verifies

$$\begin{aligned}\Gamma_t^n &= \mathbb{E}\left[\eta\wedge n - \int_t^T \frac{1}{\kappa_s}\left(\frac{1}{\int_s^T \frac{1}{\kappa_u}du + \frac{1}{\eta\wedge n}}\right)^2 ds\,\middle|\,\mathscr{F}_t\right]\\ &= \mathbb{E}\left[\eta\wedge n - \int_t^T \frac{1}{\kappa_s}\left((\Gamma_s^n)^2 + U_s^n\right) ds\,\middle|\,\mathscr{F}_t\right] \quad (0\le t\le T)\end{aligned}$$

with adapted process U^n given by

$$U_s^n \triangleq \mathbb{E}\left[\left(\frac{1}{\int_s^T \frac{1}{\kappa_u}du + \frac{1}{\eta\wedge n}}\right)^2\,\middle|\,\mathscr{F}_s\right] - (\Gamma_s^n)^2 \quad (0\le s\le T).$$

Observe that

$$d\Gamma_t^n = \left(\frac{(\Gamma_t^n)^2}{\kappa_t} + \frac{U_t^n}{\kappa_t}\right) + dM_t^n, \quad \Gamma_T^n = \eta\wedge n,$$

for some càdlàg local martingale $(M_t^n)_{0\le t\le T}$. Moreover, since $U_t^n \ge 0$ for all $0 \le t \le T$ due to Jensen's inequality, we have

$$-\frac{y^2}{\kappa_t} - \frac{U_t^n}{\kappa_t} \le -\frac{y^2}{\kappa_t} \le -\frac{y^2}{\kappa_t} + \nu_t \quad (y \in \mathbb{R}).$$

Thus, classical comparison results as in Kruse and Popier [17], Proposition 4, together with the construction of the minimal supersolution $(c_t^{\min})_{0\le t<T}$ via a truncation procedure in [16], finally yields that for all $t \in [0,T)$ we have

$$c_t^{\min} \ge \mathbb{E}\left[\frac{1}{\int_t^T \frac{1}{\kappa_s} ds + \frac{1}{\eta\wedge n}} \,\middle|\, \mathscr{F}_t\right] \ge 0 \quad \mathbb{P}\text{-a.s.}$$

In fact on $\{\mathbb{P}[\int_t^T \nu_s ds = 0|\mathscr{F}_t] < 1\}$ comparison is strict in the first of these estimates. Moreover, letting $n \to \infty$ we conclude (77) where "> 0" holds on $\{\mathbb{P}[\eta = 0|\mathscr{F}_t] < 1\}$. ☐

Finally, let us briefly discuss the integrability condition in (21) for the minimal supersolution $(c_t^{\min})_{0\le t<T}$ with Riccati dynamics (19) satisfying the terminal condition (20). This condition is not regularly discussed in the BSRDE literature and thus calls for a verification in some sufficiently generic setting. So let us place ourselves in the context of Kruse and Popier [16] and therein restrict ourselves to a Brownian framework. It follows from Proposition 3 and Remark 4 as well as Corollary 1 in [16] with $p = 2$ that for any $t \in [0,T)$ we have the upper estimates

$$c_t^{\min} \le \frac{1}{(T-t)^2}\mathbb{E}\left[\int_t^T (\kappa_s + (T-s)^2\nu_s) ds \,\middle|\, \mathscr{F}_t\right] \quad \mathbb{P}\text{-a.s.} \tag{78}$$

In addition to that, observe that also the lower estimates derived in Lemma 5.4 hold true.

For simplicity, let us further confine ourselves to the following additional assumptions on $(\nu_t)_{0\le t\le T}$, $(\kappa_t)_{0\le t\le T}$ and η: We assume that the process $(\kappa_t)_{0\le t\le T}$ is bounded from below and above, i.e., it holds that

$$0 < k \le \kappa_t \le K < \infty \quad (0 \le t \le T) \tag{79}$$

for some constants $k, K \in \mathbb{R}$. Moreover, we assume that $\nu \in L^1(\mathbb{P} \otimes dt)$ with

$$\frac{1}{T-t}\mathbb{E}\left[\int_t^T (T-s)^2 \nu_s ds \,\middle|\, \mathscr{F}_t\right] \le C \qquad (0 \le t < T) \tag{80}$$

for some constant $C < \infty$. Finally, we assume that there exists a constant $\varepsilon > 0$ such that

$$\mathbb{P}\left[\varepsilon \leq \eta \leq +\infty\right] = 1. \tag{81}$$

Observe that condition (81) implies in particular that $c_t > 0$ $\mathbb{P}$-a.s. for all $t \in [0, T]$ by virtue of Lemma 5.4.

Lemma 5.5. *Under the conditions (79), (80), and (81) the minimal supersolution $c \triangleq (c_t^{\min})_{0 \leq t < T}$ to the BSRDE in (19) on $[0, T)$ with terminal condition (20) satisfies*

$$\int_0^T \frac{d\langle c\rangle_t}{c_t^2} < \infty \quad \text{on the set } \{\eta = +\infty\},$$

i.e., condition (21) holds true.

Proof. We extend the proof of Proposition 10 in Popier [20] done for the specific case $\kappa \equiv 1$ and $\nu \equiv 0$ to our more general setting by using the upper and lower bounds of the process $(c_t)_{0 \leq t < T}$ in (78) and (77). First, note that conditions (79) and (81) imply for the lower bound in (77) that

$$c_t \geq \frac{k\varepsilon}{(T-t)\varepsilon + k} \quad (0 \leq t < T). \tag{82}$$

Concerning the upper bound in (78), we obtain due to (79) and (80)

$$c_t \leq \frac{K + \text{const}}{T-t} \quad (0 \leq t < T). \tag{83}$$

Since the process c is bounded from below on $[0, T]$, we can apply Itô's formula on $[0, T-\delta]$ for some $0 < \delta < T$ to the process $\sqrt{(T-t)c_t}$. Using the BSRDE dynamics of c in (19), we obtain

$$\begin{aligned}
0 &\leq \sqrt{(T-t)c_t} \\
&= \sqrt{Tc_0} + \int_0^t \left(\frac{\sqrt{T-s}}{2\sqrt{c_s}} \left(\frac{c_s^2}{\kappa_s} - \nu_s \right) - \frac{\sqrt{c_s}}{2\sqrt{T-s}} \right) ds \\
&\quad - \frac{1}{8} \int_0^t \frac{\sqrt{T-s}}{c_s^{3/2}} d\langle c\rangle_s - \frac{1}{2} \int_0^t \frac{\sqrt{T-s}}{\sqrt{c_s}} dN_s \\
&= \sqrt{Tc_0} + \frac{1}{2} \int_0^t \sqrt{T-s} \frac{\sqrt{c_s}}{\kappa_s} \left(c_s - \frac{\nu_s \kappa_s}{c_s} - \frac{\kappa_s}{T-s} \right) ds \\
&\quad - \frac{1}{8} \int_0^t \frac{\sqrt{T-s}}{c_s^{3/2}} d\langle c\rangle_s - \frac{1}{2} \int_0^t \frac{\sqrt{T-s}}{\sqrt{c_s}} dN_s \qquad (0 \leq t \leq T-\delta)
\end{aligned}$$

and hence

$$
\begin{aligned}
&\frac{1}{8}\int_0^{T-\delta}\frac{\sqrt{T-s}}{c_s^{3/2}}d\langle c\rangle_s+\frac{1}{2}\int_0^{T-\delta}\frac{\sqrt{T-s}}{\sqrt{c_s}}dN_s\\
&\le\sqrt{Tc_0}+\frac{1}{2}\int_0^{T-\delta}\sqrt{T-s}\frac{\sqrt{c_s}}{\kappa_s}\left(c_s-\frac{\nu_s\kappa_s}{c_s}-\frac{\kappa_s}{T-s}\right)ds
\end{aligned}\tag{84}
$$

for all $0<\delta<T$. Observe that due to the bounds on c in (82) and (83) and κ in (79) as well as the integrability assumption on ν, i.e., $\nu\in L^1(\mathbb{P}\otimes dt)$, it holds for all $0<\delta<T$ that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{T-\delta}\sqrt{T-s}\frac{\sqrt{c_s}}{\kappa_s}\left|c_s-\frac{\nu_s\kappa_s}{c_s}-\frac{\kappa_s}{T-s}\right|ds\right]\\
&\le\text{const}\,\mathbb{E}\left[\int_0^{T-\delta}\left|c_s-\frac{\nu_s\kappa_s}{c_s}-\frac{\kappa_s}{T-s}\right|ds\right]\\
&\le\text{const}\left(\mathbb{E}\left[\int_0^{T-\delta}c_s ds\right]+\mathbb{E}\left[\int_0^{T-\delta}\frac{\nu_s\kappa_s}{c_s}ds\right]+\mathbb{E}\left[\int_0^{T-\delta}\frac{\kappa_s}{T-s}ds\right]\right)<\infty.
\end{aligned}
$$

Hence, by using the upper bound on c in (78) and Fubini's Theorem, we can compute

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{T-\delta}\left(c_s-\frac{\nu_s\kappa_s}{c_s}-\frac{\kappa_s}{T-s}\right)ds\right]\le\mathbb{E}\left[\int_0^{T-\delta}\left(c_s-\frac{\kappa_s}{T-s}\right)ds\right]\\
&\le\mathbb{E}\left[\int_0^{T-\delta}\left(\frac{1}{(T-s)^2}\mathbb{E}\left[\int_s^T(\kappa_u+(T-u)^2\nu_u)du\,\middle|\,\mathscr{F}_s\right]-\frac{\kappa_s}{T-s}\right)ds\right]\\
&\le\mathbb{E}\left[\int_0^{T-\delta}\frac{1}{(T-s)^2}\left(\int_s^T\kappa_u du\right)ds-\int_0^{T-\delta}\frac{\kappa_s}{T-s}ds\right]\\
&\quad+\mathbb{E}\left[\int_0^{T-\delta}\frac{1}{(T-s)^2}\left(\int_s^T(T-u)^2\nu_u du\right)ds\right].
\end{aligned}\tag{85}
$$

Using once more Fubini's Theorem and the fact that $\kappa_t\le K$ for all $0\le t\le T$, we get for the first expectation in (85) the estimate

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{T-\delta}\frac{1}{(T-s)^2}\left(\int_s^T\kappa_u du\right)ds-\int_0^{T-\delta}\frac{\kappa_s}{T-s}ds\right]\\
&=\mathbb{E}\left[\int_0^{T-\delta}\frac{\kappa_u}{T-u}du+\int_{T-\delta}^T\frac{\kappa_u}{\delta}du-\frac{1}{T}\int_0^T\kappa_u du-\int_0^{T-\delta}\frac{\kappa_s}{T-s}ds\right]\\
&\le K.
\end{aligned}\tag{86}
$$

Concerning the second expectation in (85), application of Fubini's Theorem yields

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{T-\delta} \frac{1}{(T-s)^2}\left(\int_s^T (T-u)^2 \nu_u du\right) ds\right] \\
&\leq \mathbb{E}\left[\int_0^{T-\delta} (T-u)\nu_u du + \delta \int_{T-\delta}^T \nu_u du\right]. \qquad (87)
\end{aligned}
$$

Consequently, taking expectation in (84) and using that the stochastic integral with respect to N in (84) is a true martingale on $[0, T-\delta]$ due to (82) and (80), we obtain together with the estimates in (86) and (87) the upper bound

$$
\begin{aligned}
&\frac{1}{8}\mathbb{E}\left[\int_0^{T-\delta} \frac{\sqrt{T-s}}{c_s^{3/2}} d\langle c \rangle_s\right] \\
&\leq \sqrt{Tc_0} + \text{const}\left(K + \mathbb{E}\left[\int_0^{T-\delta} (T-u)\nu_u du + \delta \int_{T-\delta}^T \nu_u du\right]\right).
\end{aligned}
$$

Passing to the limit $\delta \downarrow 0$ we get with monotone convergence

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{T} \frac{\sqrt{T-s}}{c_s^{3/2}} d\langle c \rangle_s\right] \\
&\leq 8\left(\sqrt{Tc_0} + \text{const}\left(K + \mathbb{E}\left[\int_0^{T} (T-u)\nu_u du\right]\right)\right) < \infty, \qquad (88)
\end{aligned}
$$

due to $\nu \in L^1(\mathbb{P} \otimes dt)$. Now, using (77), observe that we can further estimate the process $(c_t)_{0 \leq t < T}$ from below by

$$
\begin{aligned}
c_s &\geq \mathbb{E}\left[\frac{1}{\int_s^T \frac{1}{\kappa_u} du + \frac{1}{\eta}} \,\middle|\, \mathscr{F}_s\right] \geq \mathbb{E}\left[\frac{1}{\int_s^T \frac{1}{\kappa_u} du + \frac{1}{\eta}} 1_{\{\eta = +\infty\}} \,\middle|\, \mathscr{F}_s\right] \\
&= \mathbb{E}\left[\frac{1}{\int_s^T \frac{1}{\kappa_u} du} 1_{\{\eta = +\infty\}} \,\middle|\, \mathscr{F}_s\right] \geq \frac{k}{T-s}\mathbb{E}\left[1_{\{\eta = +\infty\}} \,\middle|\, \mathscr{F}_s\right].
\end{aligned}
$$

Plugging back this lower bound into the left hand side of (88) and using optional projection, we get

$$\begin{aligned}\infty > \mathbb{E}\left[\int_0^T \frac{\sqrt{T-s}}{c_s^{3/2}} d\langle c\rangle_s\right] &= \mathbb{E}\left[\int_0^T \frac{\sqrt{T-s}}{c_s^2}\sqrt{c_s}\, d\langle c\rangle_s\right] \\ &\geq \sqrt{k}\,\mathbb{E}\left[\int_0^T \frac{1}{c_s^2}\mathbb{E}\left[1_{\{\eta=+\infty\}} \mid \mathscr{F}_s\right]\, d\langle c\rangle_s\right] = \sqrt{k}\,\mathbb{E}\left[\int_0^T \frac{1}{c_s^2} 1_{\{\eta=+\infty\}}\, d\langle c\rangle_s\right] \\ &= \sqrt{k}\,\mathbb{E}\left[1_{\{\eta=+\infty\}}\left(\int_0^T \frac{1}{c_s^2}\, d\langle c\rangle_s\right)\right],\end{aligned}$$

which yields the desired result. □

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