# EXISTENCE OF WEAK SOLUTIONS TO STATIONARY MEAN-FIELD GAMES THROUGH VARIATIONAL INEQUALITIES 

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#### Abstract

Here, we consider stationary monotone mean-field games (MFGs) and study the existence of weak solutions. First, we introduce a regularized problem that preserves the monotonicity. Next, using variational inequality techniques, we prove the existence of solutions to the regularized problem. Then, using Minty's method, we establish the existence of solutions for the original MFG. Finally, we examine the properties of these weak solutions in several examples. Our methods provide a general framework to construct weak solutions to stationary MFGs with local, nonlocal, or congestion terms.


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## 1. Introduction

In the last years, mean-field games (MFGs) have become an active area of research in both the mathematics 41, 42, 43 and engineering communities 37, 38. In spite of substantial progress, many questions remain open. Some of the more fundamental issues regard the existence and uniqueness of solutions. Various authors have attempted to answer these questions through explicit solutions and transformations [3, 4, 5, 35, 36, a priori estimates [18, 21, 22, 23, 24, 25, 27, 28, 29, 32, 45, penalization ideas [19, 20, random-variable techniques [14, 15, 33, 40, 48, weak and renormalized solutions 46, 47, and variational methods [8, 9, 10, 11. Here, we develop a new approach to investigate the existence of weak solutions to stationary MFGs using variational inequalities.

We consider MFGs given by the system of partial differential equations

$$
\left\{\begin{array}{l}
-u-H\left(x, D u, D^{2} u, m, h(\boldsymbol{m})\right)=0  \tag{1.1}\\
m-\operatorname{div}\left(m D_{p} H\left(x, D u, D^{2} u, m, h(\boldsymbol{m})\right)\right)+\left(m D_{M_{i j}} H\left(x, D u, D^{2} u, m, h(\boldsymbol{m})\right)\right)_{x_{i} x_{j}}=1
\end{array}\right.
$$

For convenience, we take periodic boundary conditions for (1.1) and work in the $d$-dimensional torus, $\mathbb{T}^{d}, d \in \mathbb{N}$. Moreover, $i, j \in\{1, \ldots, d\}$ and we use the Einstein convention on repeated indices. In (1.1), the Hamiltonian

$$
\begin{align*}
H: \mathbb{T}^{d} \times \mathbb{R}^{d} \times \mathbb{R}^{d \times d} \times \mathbb{E} \times \mathbb{R} & \rightarrow \mathbb{R} \\
(x, p, M, m, \theta) & \mapsto H(x, p, M, m, \theta) \tag{1.2}
\end{align*}
$$

satisfies the regularity and growth assumptions discussed in Section 2 The set $\mathbb{E}$ is either $\mathbb{R}^{+}$ or $\mathbb{R}_{0}^{+}$, and $h: \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right) \rightarrow C\left(\mathbb{T}^{d}\right)$ is a non-local operator also examined later. Here, $\mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$ stands for the space of positive measures on $\mathbb{T}^{d}$, absolutely continuous with respect to the Lebesgue measure. We use a boldface font to denote elements of $\mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$ and represent their densities with the same non-boldface letter.

Our assumptions encompass a broad class of Hamiltonians that includes the following important examples:

$$
\begin{align*}
& H(x, p, M, m, \theta):=\bar{H}(x, p, m)-V(x)-g(m, \theta)-\sigma(x) \sum_{i=1}^{d} M_{i i}  \tag{1.3}\\
& h(\boldsymbol{m}):=c_{1} \zeta * \boldsymbol{m}+c_{2} \zeta *\left((\zeta * \boldsymbol{m})^{\bar{\alpha}}\right)
\end{align*}
$$

[^0]where
\[

$$
\begin{equation*}
\bar{H}(x, p, m):=a(x)\left(1+|p|^{2}\right)^{\frac{\gamma}{2}} \quad \text { or } \quad \bar{H}(x, p, m):=a(x) \frac{|p|^{2}}{2 m^{\tau}} \tag{1.4}
\end{equation*}
$$

\]

and

$$
\begin{equation*}
g(m, \theta):=m^{\alpha}+\theta \quad \text { or } \quad g(m, \theta):=\ln (m)+\theta \tag{1.5}
\end{equation*}
$$

with $V, \sigma, \zeta, a: \mathbb{T}^{d} \rightarrow \mathbb{R}$ smooth, periodic functions such that $\sigma \geqslant 0, \inf _{\mathbb{T}^{d}} \zeta \geqslant 0, \inf _{\mathbb{T}^{d}} a>0$, $c_{1}, c_{2} \geqslant 0, \gamma>1,0 \leqslant \tau<1$, and $\bar{\alpha}, \alpha>0$.

MFGs model problems with large numbers of competing rational agents who seek to optimize an individual utility, see, for example, the lectures in 44]. In the time-dependent case, these games are given by a (time-dependent) Hamilton-Jacobi equation coupled with a transport or Fokker-Planck equation. The stationary case captures ergodic equilibria and corresponds to the long-time limit of time-dependent MFGs [12, 13]. For up-to-date developments on MFGs, we refer the reader to the recent monographs [6, 26], the survey paper 31], and the notes (7]. For numerical aspects, we recommend [1] and the references therein.

Uniformly elliptic MFGs and their corresponding weak solutions were introduced in 41. The systematic study of the regularity theory for these MFGs was developed in [21, 27, 29, 45. Those references establish the existence of classical solutions for local MFGs with logarithmic nonlinearities and power-like nonlinearities with certain growth conditions. A particular stationary congestion model was considered in [18. Little is known about the general stationary congestion problem. Finally, we point out that some results for time-dependent problems relying on the variational structure of MFGs [8, 9, 10] and some weak solution methods 46, 47] may be extended to the stationary case. However, to the best of our knowledge, this has not been pursued in the literature.

In the MFGs literature, there are several gaps in the existence of solutions that we try to address here. First, to obtain smooth solutions of local MFGs, state-of-the-art methods 45] require growth conditions in the non-linearities. These conditions seem to be of a technical nature rather than of a fundamental nature. Second, regarding weak solutions, the uniformly parabolic case with subquadratic or quadratic Hamiltonians is well understood 46, 47. In contrast, the degenerate parabolic case and the uniformly parabolic with superquadratic Hamiltonians case are well understood only for variational problems [8, 9, 10. Moreover, we expect analogous results to hold for degenerate elliptic problems. Unfortunately, variational MFGs are a restricted class of problems that is unstable under perturbations. For example, adding a small, non-local perturbation to a variational MFG should not change the theory substantially, but it destroys the variational structure. Third, apart from the preliminary results in [18] and the short-time problems in 32, 34, little is known about weak or classical solutions of congestion models.

Here, we present a unified approach to studying these problems and construct weak solutions based on monotone operator methods. Monotonicity assumptions are central in MFG theory and are at the heart of the uniqueness proof by Lasry and Lions [44; also, the numerical methods in [2] (for stationary MFGs) and [30] (for finite state MFGs) rely on monotonicity ideas. Monotone operator methods have several advantages. First, monotonicity is stable under perturbations. Second, there is a well-developed theory of weak solutions that, when combined with the Minty method, makes it possible to consider various limit problems. Finally, our approach to MFGs through monotonicity answers the earlier existence questions and suggests new computational approaches.

Next, we put forward the basic definitions. A weak solution to (1.1) is a pair $(m, u) \in$ $\mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right) \times \mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right)$ with $m \geqslant 0$ that satisfies the variational inequality

$$
\left\langle\left[\begin{array}{l}
\eta  \tag{1.6}\\
v
\end{array}\right]-\left[\begin{array}{c}
m \\
u
\end{array}\right], A\left[\begin{array}{c}
\eta \\
v
\end{array}\right]\right\rangle_{\mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right) \times \mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right), C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)} \geqslant 0
$$

for all $(\eta, v) \in C^{\infty}\left(\mathbb{T}^{d} ; \mathbb{E}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$, where

$$
A\left[\begin{array}{l}
\eta  \tag{1.7}\\
v
\end{array}\right]:=\left[\begin{array}{c}
-v-H\left(x, D v, D^{2} v, \eta, h(\boldsymbol{\eta})\right) \\
\eta-\operatorname{div}\left(\eta D_{p} H\left(x, D v, D^{2} v, \eta, h(\boldsymbol{\eta})\right)\right)+\left(\eta D_{M_{i j}} H\left(x, D v, D^{2} v, \eta, h(\boldsymbol{\eta})\right)\right)_{x_{i} x_{j}}-1
\end{array}\right]
$$

Any solution to (1.1) is a weak solution. Moreover, suppose that $H$ is regular and let $(m, u) \in$ $C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$ with $m>0$ a weak solution. A straightforward argument shows that $(m, u)$ solves (1.1). To see this, fix a pair $(\psi, \phi) \in C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$. Then, for all sufficiently small $\epsilon>0, m+\epsilon \psi>0$; taking $(\eta, v)=(m+\epsilon \psi, u+\epsilon \phi)$ in (1.6), dividing by $\epsilon$, and letting $\epsilon \rightarrow 0^{+}$ yield

$$
\left\langle\left[\begin{array}{l}
\psi \\
\phi
\end{array}\right], A\left[\begin{array}{l}
m \\
u
\end{array}\right]\right\rangle_{\mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right) \times \mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right), C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)} \geqslant 0
$$

Because $\phi$ and $\psi$ are arbitrary, we conclude that $(m, u)$ solves (1.1).
Our goal is to prove the existence of weak solutions and to study their properties. Our main result is the following.

Theorem 1.1. Suppose that Assumptions (h1), (g1), (g2), and (H1) (H3) are satisfied. Then, there exists a weak solution, $(m, u) \in \mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right) \times \mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right)$, with $m \geqslant 0$ to (1.1). Moreover, $(\boldsymbol{m}, u) \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right)$ and $\int_{\mathbb{T}^{d}} m \mathrm{~d} x=1$.

This foregoing theorem gives the existence of solutions for a minimal set of assumptions and is a substantial improvement on prior results. First, the theorem is valid for degenerate elliptic MFGs. These are technically challenging since various analytical techniques for first-order MFGs do not apply, and the regularizing effects due to ellipticity are mild to non-existent. In this case, prior results only apply to problems with a variational structure. Second, the theorem holds for congestion MFGs. These were studied before for a particular problem in [18]; little is known about the existence of solutions for general stationary problems. In the time-dependent setting, only the short-time problem has been considered in the literature 32, 34 .

While Theorem 1.1 gives the existence of solutions for MFGs, these solutions have low regularity. In Section [5 we consider the degenerate diffusion case and investigate further properties of the weak solutions constructed in Theorem 1.1 Under appropriate conditions, we prove that such solutions are, in the sense of distributions, subsolutions of the first equation in (1.1) (Proposition 5.1), relaxed solutions of the second equation in (1.1) (Proposition 5.4), and relaxed supersolutions of the first equation in (1.1) (Proposition 5.5). The quadratic case is examined in detail in Section 6, where we establish higher integrability and Sobolev estimates for $m$. Moreover, under appropriate assumptions, we prove that the second equation in (1.1) is satisfied pointwise in $\mathbb{T}^{d}$ (Theorem 6.2) and that the first equation in (1.1) is satisfied pointwise in the set where $m$ is positive (Corollary 6.3).

This paper is structured as follows. Our main assumptions are discussed in Section 2 Then, in Section 3, we begin the study of (1.1). First, we introduce a regularized problem that involves two positive, small parameters. Next, we prove a priori estimates for solutions to the regularized problem. Combining these estimates with a continuation argument, we show the existence of solutions to the regularized problem. In Section [4 we establish further uniform estimates with respect to the parameters that allow us to pass the regularized problem to the limit as these parameters tend to zero. Then, this limiting procedure enables us to prove Theorem 1.1 by using the Minty device. Finally, as mentioned before, Sections 5 and 6 are devoted to establishing further properties of weak solutions for specific Hamiltonians. We conclude this paper in Section 7 with some final remarks.

## 2. MAin assumptions

Here, we discuss the main assumptions of this paper. In our choice of assumptions, we have attempted to balance generality with brevity and clarity. Naturally, this requires some compromises in illustrating the core ideas, and several extensions and variations of our results are possible though not included in our discussion. Nevertheless, our assumptions encompass a broad range of examples, including variational mean-field games with power and logarithmic nonlinearities, congestion problems, elliptic, degenerate elliptic, and first-order Hamiltonians with both sub- and superquadratic growth.

We recall that $\mathbb{T}^{d}$ represents the $d$-dimensional torus, which we identify with the quotient space $\mathbb{R}^{d} / \mathbb{Z}^{d}$. As before, $\mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$ is the space of positive measures on $\mathbb{T}^{d}$, absolutely continuous with respect to the Lebesgue measure. Moreover, elements of $\mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$ are denoted with
boldface font while their densities are represented with the same letter in a non-boldface font. For instance, if $\boldsymbol{m} \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$, then $m \in L^{1}\left(\mathbb{T}^{d}\right)$ represents the corresponding density function. Similarly, if $m \in L^{1}\left(\mathbb{T}^{d}\right)$ is such that $m \geqslant 0$ a.e. in $\mathbb{T}^{d}$ and $\mathcal{L}^{d}$ represents the $d$-dimensional Lebesgue measure, then $m:=m \mathcal{L}^{d}{ }_{\left\lfloor\mathbb{T}^{d}\right.}$ denotes the corresponding measure in $\mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$. We also recall that the set $\mathbb{E}$ is either $\mathbb{R}^{+}$or $\mathbb{R}_{0}^{+}$.

To avoid longer expressions, we often omit some explicit dependencies on the space variable. For example, we write $H\left(x, D u, D^{2} u, m, h(\boldsymbol{m})\right)$ in place of $H\left(x, D u(x), D^{2} u(x), m(x), h(\boldsymbol{m})(x)\right)$; similarly, we write $g(x, m, h(\boldsymbol{m}))$ in place of $g(x, m(x), h(\boldsymbol{m})(x))$, and so on.

In what follows,

$$
\begin{aligned}
h: \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right) & \rightarrow C\left(\mathbb{T}^{d}\right) \\
\boldsymbol{m} & \mapsto h(\boldsymbol{m})
\end{aligned}
$$

is a (possibly nonlinear) operator. In most of our statements, we suppose that:
(h1) For all $\kappa>\frac{d}{2}+1$ :
(a) $\left\{h(\boldsymbol{m}): m \in W^{\kappa, 2}\left(\mathbb{T}^{d}\right)\right\} \subset W^{\kappa, 2}\left(\mathbb{T}^{d}\right)$;
(b) $m \in W^{\kappa, 2}\left(\mathbb{T}^{d}\right) \mapsto h(\boldsymbol{m}) \in W^{\kappa, 2}\left(\mathbb{T}^{d}\right)$ defines a Fréchet differentiable map.

If $h$ satisfies the Assumption (h1), then for all $m_{0} \in W^{\kappa, 2}\left(\mathbb{T}^{d}\right)$, there exists a bounded linear operator, $\mathfrak{H}_{m_{0}} \in \mathcal{L}\left(W^{\kappa, 2}\left(\mathbb{T}^{d}\right) ; W^{\kappa, 2}\left(\mathbb{T}^{d}\right)\right)$, such that, for all $m \in W^{\kappa, 2}\left(\mathbb{T}^{d}\right)$,

$$
\begin{align*}
& \left\|h(\boldsymbol{m})-h\left(\boldsymbol{m}_{0}\right)\right\|_{W^{\kappa, 2}\left(\mathbb{T}^{d}\right)} \\
& \quad \leqslant\left\|\tilde{H}_{m_{0}}\right\|_{\mathcal{L}\left(W^{\kappa, 2}\left(\mathbb{T}^{d}\right) ; W^{\kappa, 2}\left(\mathbb{T}^{d}\right)\right)}\left\|m-m_{0}\right\|_{W^{\kappa, 2}\left(\mathbb{T}^{d}\right)}+o\left(\left\|m-m_{0}\right\|_{W^{\kappa, 2}\left(\mathbb{T}^{d}\right)}\right) . \tag{2.1}
\end{align*}
$$

Moreover, taking $m_{0}=0$ in (2.1), we get

$$
\begin{equation*}
\|h(\boldsymbol{m})\|_{W^{\kappa, 2}\left(\mathbb{T}^{d}\right)} \leqslant C_{0}\left(1+\|m\|_{W^{\kappa, 2}\left(\mathbb{T}^{d}\right)}\right) \tag{2.2}
\end{equation*}
$$

for some positive constant $C_{0}=C_{0}\left(\kappa, \mathbb{T}^{d},\left\|\mathfrak{H}_{0}\right\|_{\mathcal{L}\left(W^{\kappa, 2}\left(\mathbb{T}^{d}\right) ; W^{\kappa, 2}\left(\mathbb{T}^{d}\right)\right)},\|\mathfrak{H}(\mathbf{0})\|_{W^{\kappa, 2}\left(\mathbb{T}^{d}\right)}\right)$.
Examples of operators satisfying the previous assumption are those in (1.3).
To describe the behavior of the Hamiltonian (1.2) in the variables $(x, m, \theta)$, we introduce a continuous function, $g: \mathbb{T}^{d} \times \mathbb{E} \times \mathbb{R} \rightarrow \mathbb{R}$, and the following assumptions:
(g1) (a) There exist functions, $g_{1}$ and $g_{2}$, such that, for all $(x, m, \theta) \in \mathbb{T}^{d} \times \mathbb{E} \times \mathbb{R}$,

$$
g(x, m, \theta)=g_{1}(x, m, \theta)+g_{2}(x, m, \theta)
$$

and, for all $\boldsymbol{m} \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$,

$$
g_{2}(x, m(x), h(\boldsymbol{m})(x)) \geqslant 0
$$

(b) There exists a constant, $c>0$, such that, for all $\boldsymbol{m} \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$,

$$
\int_{\mathbb{T}^{d}} g_{2}(x, m, h(\boldsymbol{m})) \mathrm{d} x \leqslant c\left(1+\int_{\mathbb{T}^{d}} m \mathrm{~d} x\right)
$$

(c) For all $\delta>0$, there exists a constant, $C_{\delta}>0$, such that, for all $\boldsymbol{m} \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$,

$$
\max \left\{\int_{\mathbb{T}^{d}} g_{1}(x, m, h(\boldsymbol{m})) \mathrm{d} x, \int_{\mathbb{T}^{d}} m \mathrm{~d} x\right\} \leqslant \delta \int_{\mathbb{T}^{d}} m g_{1}(x, m, h(\boldsymbol{m})) \mathrm{d} x+C_{\delta}
$$

(d) There exists $R \geqslant 0$ such that, for all $\boldsymbol{m} \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$,

$$
\int_{\mathbb{T}^{d}} m g_{1}(x, m, h(\boldsymbol{m})) \mathrm{d} x \geqslant-R
$$

(g2) If $\left(\boldsymbol{m}_{j}\right)_{j \in \mathbb{N}} \subset \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$ is such that

$$
\sup _{j \in \mathbb{N}} \int_{\mathbb{T}^{d}} m_{j} g_{1}\left(x, m_{j}, h\left(\boldsymbol{m}_{j}\right)\right) \mathrm{d} x<+\infty
$$

then there exists a subsequence of $\left(m_{j}\right)_{j \in \mathbb{N}}$ that converges weakly in $L^{1}\left(\mathbb{T}^{d}\right)$.
The functions $g$ in (1.5) with $h$ as in (1.3) satisfy Assumptions (g1) and (g2) for $g_{1}(x, m, \theta):=$ $m^{\alpha}$ or $g_{1}(x, m, \theta):=\ln (m)$, and $g_{2}(x, m, \theta):=\theta$.

Next, we enumerate hypotheses on the Hamiltonian. Recall that $M_{1}: M_{2}=\operatorname{tr}\left(M_{1} M_{2}\right)$ whenever $M_{1}$ and $M_{2}$ are symmetric matrices.
(H1) There exist constants, $\gamma>1,0 \leqslant \tau<1, C_{1}>0, C_{2}>0, C_{3}>0$, and $C_{4}>0$, and a continuous function, $g: \mathbb{T}^{d} \times \mathbb{E} \times \mathbb{R} \rightarrow \mathbb{R}$, such that:
(a) for all $(x, p, M, m, \theta) \in \mathbb{T}^{d} \times \mathbb{R}^{d} \times \mathbb{R}^{d \times d} \times \mathbb{E} \times \mathbb{R}$,

$$
\begin{aligned}
& -H(x, p, M, m, \theta)+D_{p} H(x, p, M, m, \theta) \cdot p+D_{M} H(x, p, M, m, \theta): M \\
& \quad \geqslant \frac{1}{C_{1}} m^{-\tau}|p|^{\gamma}+C_{2} g(x, m, \theta)-C_{1}
\end{aligned}
$$

(b) for all $(\boldsymbol{m}, u) \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right)$ with $D^{2} u$ a measurable function,

$$
\begin{aligned}
- & C_{3} \int_{\mathbb{T}^{d}} g(x, m, h(\boldsymbol{m})) \mathrm{d} x+\frac{1}{C_{1}} \int_{\mathbb{T}^{d}}|D u|^{\gamma} m^{-\tau} \mathrm{d} x-C_{1} \\
& \leqslant \int_{\mathbb{T}^{d}} H\left(x, D u, D^{2} u, m, h(\boldsymbol{m})\right) \mathrm{d} x \\
& \leqslant-C_{4} \int_{\mathbb{T}^{d}} g(x, m, h(\boldsymbol{m})) \mathrm{d} x+C_{1}\left(1+\int_{\mathbb{T}^{d}}|D u|^{\gamma} m^{-\tau} \mathrm{d} x\right) .
\end{aligned}
$$

(H2) The map $(x, p, M, m, \theta) \mapsto H(x, p, M, m, \theta)$ from $\mathbb{T}^{d} \times \mathbb{R}^{d} \times \mathbb{R}^{d \times d} \times \mathbb{E} \times \mathbb{R}$ to $\mathbb{R}$ is of class $C^{\infty}$.
(H3) The operator, $A$, defined in (1.7) is monotone with respect to the $L^{2} \times L^{2}$-inner product. More precisely, for all $(\eta, v),(\bar{\eta}, \bar{v}) \in C^{\infty}\left(\mathbb{T}^{d} ; \mathbb{E}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$, we have

$$
\left(A\left[\begin{array}{l}
\eta \\
v
\end{array}\right]-A\left[\begin{array}{l}
\bar{\eta} \\
\bar{v}
\end{array}\right],\left[\begin{array}{l}
\eta \\
v
\end{array}\right]-\left[\begin{array}{l}
\bar{\eta} \\
\bar{v}
\end{array}\right]\right)_{L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)} \geqslant 0 .
$$

Examples of Hamiltonians satisfying (H1)-(H3) are given by (1.3)-(1.5). Virtually all nonlinear Hamiltonians in the MFGs literature satisfy these assumptions, and Theorem 1.1 holds under them.

As mentioned in the Introduction, we also investigate further properties of the weak solutions constructed in Theorem 1.1. This study requires additional assumptions on the Hamiltonian. The precise statements of the additional assumptions are given in Section 5 (see Assumption (H4)) and in Section 6 (see Assumption (H5)).

## 3. Regularized mean-Field game

To construct solutions to (1.1), we introduce the regularized problem

$$
\left[\begin{array}{c}
-u-H+\varepsilon_{1}\left(m+\Delta^{2 p} m\right)+\varepsilon_{2}(m-\Delta m)+\beta_{\varepsilon_{1}}(m)  \tag{3.1}\\
m-\operatorname{div}\left(m D_{p} H\right)+\left(m D_{M_{i j}} H\right)_{x_{i} x_{j}}-1+\varepsilon_{1}\left(u+\Delta^{2 p} u\right)+\varepsilon_{2}(u-\Delta u)
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right],
$$

where $\varepsilon_{1}, \varepsilon_{2}>0, p \in \mathbb{N}$ satisfies $2 p-4>\frac{d}{2}+1$, and $\beta_{\varepsilon_{1}}:(0,+\infty) \rightarrow(-\infty, 0]$ is a nondecreasing, $C^{\infty}$ function satisfying $\beta_{\varepsilon_{1}}(s)=0$ if $s \geqslant \varepsilon_{1}$ and $\beta_{\varepsilon_{1}}(s)=-\frac{1}{s q}$ if $0<s \leqslant \frac{\varepsilon_{1}}{2}$, where $q>d$. Moreover, the function $H$ (and, analogously, $D_{p} H$ and $D_{M_{i j}} H$ ) is identified with the map

$$
x \mapsto H\left(x, D u(x), D^{2} u(x), m(x), h(\boldsymbol{m})(x)\right) .
$$

Here, we prove the following result.
Proposition 3.1. Suppose that Assumptions (h1), (g1), and (H1)-(H3) hold. Then, for all $\varepsilon_{1}, \varepsilon_{2} \in(0,1)$, (3.1) admits a $C^{\infty}$ solution $(m, u)$ with $\inf _{\mathbb{T}^{d}} m>0$.

We postpone the proof of this proposition to the end of this section because we need to establish certain a priori bounds first. These bounds rely on two main effects: the term $\left(m+\Delta^{2 p} m\right)$ gives regularity; the penalization term $\beta_{\varepsilon_{1}}$ forces $m$ to be strictly positive. After proving these bounds, we use the continuation method to show the existence of a solution.
3.1. The perturbed problem. Fix $\varepsilon_{1}, \varepsilon_{2} \in(0,1), \mu \in[0,1], q>d$, and $p \in \mathbb{N}$ such that $2 p-4>\frac{d}{2}+1$. Assume that $H$ satisfies Assumption (H1), and define

$$
H_{\mu}(x, p, M, m, \theta):=(1-\mu) H(x, p, M, m, \theta)+\mu \widetilde{H}(p, m),
$$

where

$$
\widetilde{H}(p, m):=\frac{\left(1+|p|^{2}\right)^{\frac{\gamma}{2}}}{\gamma m^{\tau}}-m
$$

Set $\varepsilon:=\left(\varepsilon_{1}, \varepsilon_{2}\right)$, and define the operator $A_{\mu}^{\varepsilon}: D\left(A_{\mu}^{\varepsilon}\right) \subset L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right) \rightarrow L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)$, where

$$
\begin{equation*}
D\left(A_{\mu}^{\varepsilon}\right):=\left\{(m, u) \in W^{4 p, 2}\left(\mathbb{T}^{d}\right) \times W^{4 p, 2}\left(\mathbb{T}^{d}\right): \inf _{\mathbb{T}^{d}} m>0\right\} \tag{3.2}
\end{equation*}
$$

by

$$
A_{\mu}^{\varepsilon}\left[\begin{array}{c}
m  \tag{3.3}\\
u
\end{array}\right]:=\left[\begin{array}{c}
-u-H_{\mu}+\varepsilon_{1}\left(m+\Delta^{2 p} m\right)+\varepsilon_{2}(m-\Delta m)+\beta_{\varepsilon_{1}}(m) \\
m-\operatorname{div}\left(m D_{p} H_{\mu}\right)+\left(m D_{M_{i j}} H_{\mu}\right)_{x_{i} x_{j}}-1+\varepsilon_{1}\left(u+\Delta^{2 p} u\right)+\varepsilon_{2}(u-\Delta u)
\end{array}\right]
$$

for $(m, u) \in D\left(A_{\mu}^{\varepsilon}\right)$. As before, the function $H_{\mu}$ (and, analogously, $D_{p} H_{\mu}$ and $D_{M_{i j}} H_{\mu}$ ) is identified with the map

$$
x \mapsto H_{\mu}\left(x, D u(x), D^{2} u(x), m(x), h(\boldsymbol{m})(x)\right)
$$

The parameter $\varepsilon_{1}$ controls the main regularizing effects; the parameter $\varepsilon_{2}$ is a technical device that is used later to improve the regularity of the weak solutions.

Remark 3.2. Observe that $\tilde{H}$ satisfies Assumptions (H2) and (H3). Moreover, if $\tilde{g}_{1}(x, m, \theta):=$ $m$ and $\tilde{g}_{2}(x, m, \theta):=0$, then $\tilde{g}:=\tilde{g}_{1}+\tilde{g}_{2}$ satisfies Assumptions (g1) and (g2). Furthermore, Assumption (H1) is also satisfied, where $C_{1}$ is replaced by some positive constant that depends on $\gamma, \widetilde{C}_{1}=\widetilde{C}_{1}(\gamma)$, and with $C_{2}=C_{3}=C_{4}=1$. Without loss of generality, we may assume that $\widetilde{C}_{1}=C_{1}$; otherwise, we simply replace both $C_{1}$ and $\widetilde{C}_{1}$ by $\max \left\{C_{1}, \widetilde{C}_{1}\right\}$ and relabel the constants conveniently.

Recall that $W^{k, 2}\left(\mathbb{T}^{d}\right)$ is an algebra for $k>\frac{d}{2}$. Moreover, since $2 p-4>\frac{d}{2}+1$, Morrey's theorem yields that the embedding $W^{2 p-4,2}\left(\mathbb{T}^{d}\right) \hookrightarrow C^{\lambda}\left(\mathbb{T}^{d}\right)$ is continuous for some $\lambda \in(0,1)$. In particular, there exists a positive constant, $C$, depending only on $p, d$, and $\lambda$, such that, for all $w \in W^{2 p-4,2}\left(\mathbb{T}^{d}\right)$,

$$
\begin{equation*}
\|w\|_{L^{\infty}\left(\mathbb{T}^{d}\right)} \leqslant C\|w\|_{W^{2 p-4,2}\left(\mathbb{T}^{d}\right)} . \tag{3.4}
\end{equation*}
$$

Finally, consider the problem of finding $\left(m_{\mu}^{\varepsilon}, u_{\mu}^{\varepsilon}\right) \in D\left(A_{\mu}^{\varepsilon}\right)$ satisfying the perturbed meanfield game

$$
A_{\mu}^{\varepsilon}\left[\begin{array}{c}
m_{\mu}^{\varepsilon}  \tag{3.5}\\
u_{\mu}^{\varepsilon}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] .
$$

Note that (3.5) with $\mu=0$ reduces to (3.1).
We set

$$
\begin{equation*}
\Lambda^{\varepsilon}:=\left\{\mu \in[0,1]: \exists\left(m_{\mu}^{\varepsilon}, u_{\mu}^{\varepsilon}\right) \in D\left(A_{\mu}^{\varepsilon}\right) \cap C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right) \text { solving (3.5) }\right\} . \tag{3.6}
\end{equation*}
$$

Our goal is to establish the equality

$$
\begin{equation*}
\Lambda^{\varepsilon}=[0,1] \tag{3.7}
\end{equation*}
$$

To this end, next, we begin by proving a priori estimates for classical solutions of (3.5). These are essential to the application of the continuation method. In Subsection 3.3 we use this method to show that $\Lambda^{\varepsilon}$ is a nonempty set, relatively closed and open in $[0,1]$. Consequently, (3.7) holds.
3.2. A priori estimates for classical solutions to the perturbed mean-field game. We start by establishing preliminary a priori estimates for classical solutions of (3.5). These estimates involve constants whose main feature is their independence of any particular choice of solutions to problem (3.5). To simplify the notation, we introduce some nomenclature to specify these constants. We say that a constant depends only on the problem data if it is a function of the dimension, $d$, of $p$, and of the constants in Assumptions (h1), (g1), and (H1). We say that a constant depends only on the problem data and on $\varepsilon$ if it is a function of $d$, of $p$, of the constants in Assumptions (h1), (g1), and (H1), of $\lambda$, of $\varepsilon_{1}$, of $\varepsilon_{2}$, and of the $L^{\infty}$-estimates of $\beta_{\varepsilon_{1}}(\cdot)$ and its derivatives on a compact subset of $(0, \infty)$ that depends only on $\varepsilon_{1}$, on $\varepsilon_{2}$, and on a constant that depends only on the problem data. We stress that these constants do not depend on the choice of solutions to (3.1) nor on $\mu$.

Lemma 3.3. Suppose that Assumptions (g1) and (H1) hold. Let $\varepsilon \in(0,1)^{2}$ and $\mu \in[0,1]$. Assume that $\left(m_{\mu}^{\varepsilon}, u_{\mu}^{\varepsilon}\right) \in C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$ solves (3.5) in $\mathbb{T}^{d}$. Suppose that $m_{\mu}^{\varepsilon}>0$ in $\mathbb{T}^{d}$. Then, there exists a positive constant, $C$, that depends only on the problem data such that

$$
\begin{align*}
\int_{\mathbb{T}^{d}} & \left|D u_{\mu}^{\varepsilon}\right|^{\gamma}\left(m_{\mu}^{\varepsilon}\right)^{1-\tau} \mathrm{d} x+\int_{\mathbb{T}^{d}}\left|D u_{\mu}^{\varepsilon}\right|^{\gamma}\left(m_{\mu}^{\varepsilon}\right)^{-\tau} \mathrm{d} x+\int_{\mathbb{T}^{d}}-\beta_{\varepsilon_{1}}\left(m_{\mu}^{\varepsilon}\right) \mathrm{d} x \\
& +\int_{\mathbb{T}^{d}} \beta_{\varepsilon_{1}}\left(m_{\mu}^{\varepsilon}\right)\left(m_{\mu}^{\varepsilon}-\varepsilon_{1}\right) \mathrm{d} x+(1-\mu) \int_{\mathbb{T}^{d}} m_{\mu}^{\varepsilon} g_{2}\left(x, m_{\mu}^{\varepsilon}, h\left(\boldsymbol{m}_{\mu}^{\varepsilon}\right)\right) \mathrm{d} x \\
& +\varepsilon_{1} \int_{\mathbb{T}^{d}}\left[\left(u_{\mu}^{\varepsilon}\right)^{2}+\left(\Delta^{p} u_{\mu}^{\varepsilon}\right)^{2}+\left(m_{\mu}^{\varepsilon}\right)^{2}+\left(\Delta^{p} m_{\mu}^{\varepsilon}\right)^{2}\right] \mathrm{d} x \\
& +\varepsilon_{2} \int_{\mathbb{T}^{d}}\left(\left(u_{\mu}^{\varepsilon}\right)^{2}+\left|D u_{\mu}^{\varepsilon}\right|^{2}+\left(m_{\mu}^{\varepsilon}\right)^{2}+\left|D m_{\mu}^{\varepsilon}\right|^{2}\right) \mathrm{d} x \leqslant C \tag{3.8}
\end{align*}
$$

Proof. In this proof, to simplify the notation, we drop the dependence on $\varepsilon$ and $\mu$ of $\boldsymbol{m}_{\mu}^{\varepsilon}, m_{\mu}^{\varepsilon}$, and $u_{\mu}^{\varepsilon}$. Accordingly, we simply write $\boldsymbol{m}, m$, and $u$ in place of $\boldsymbol{m}_{\mu}^{\varepsilon}, m_{\mu}^{\varepsilon}$, and $u_{\mu}^{\varepsilon}$, respectively.

By the statement of the lemma, the two following equalities hold in $\mathbb{T}^{d}$ :

$$
\begin{align*}
& -u-H_{\mu}+\varepsilon_{1}\left(m+\Delta^{2 p} m\right)+\varepsilon_{2}(m-\Delta m)+\beta_{\varepsilon_{1}}(m)=0  \tag{3.9}\\
& m-\operatorname{div}\left(m D_{p} H_{\mu}\right)+\left(m D_{M_{i j}} H_{\mu}\right)_{x_{i} x_{j}}-1+\varepsilon_{1}\left(u+\Delta^{2 p} u\right)+\varepsilon_{2}(u-\Delta u)=0 \tag{3.10}
\end{align*}
$$

where $H_{\mu}, D_{p} H_{\mu}$, and $D_{M_{i j}} H_{\mu}$ are computed at $\left(x, D u(x), D^{2} u(x), m(x), h(\boldsymbol{m})(x)\right)$.
We multiply (3.9) by $\left(m-\varepsilon_{1}-1\right)$ and (3.10) by $u$. Next, adding the resulting expressions, integrating over $\mathbb{T}^{d}$, and integrating by parts yield

$$
\begin{align*}
& \int_{\mathbb{T}^{d}} m\left(-H_{\mu}+D_{p} H_{\mu} \cdot D u+D_{M} H_{\mu}: D^{2} u\right) \mathrm{d} x+\left(\varepsilon_{1}+1\right) \int_{\mathbb{T}^{d}} H_{\mu} \mathrm{d} x+\int_{\mathbb{T}^{d}}-\beta_{\varepsilon_{1}}(m) \mathrm{d} x \\
& \quad+\int_{\mathbb{T}^{d}} \beta_{\varepsilon_{1}}(m)\left(m-\varepsilon_{1}\right) \mathrm{d} x+\varepsilon_{1} \int_{\mathbb{T}^{d}}\left[u^{2}+\left(\Delta^{p} u\right)^{2}+m^{2}+\left(\Delta^{p} m\right)^{2}\right] \mathrm{d} x \\
& \quad+\varepsilon_{2} \int_{\mathbb{T}^{d}}\left(u^{2}+|D u|^{2}+m^{2}+|D m|^{2}\right) \mathrm{d} x \\
& \quad=-\varepsilon_{1} \int_{\mathbb{T}^{d}} u \mathrm{~d} x+\left(\varepsilon_{1}+\varepsilon_{2}\right)\left(\varepsilon_{1}+1\right) \int_{\mathbb{T}^{d}} m \mathrm{~d} x \leqslant \frac{\varepsilon_{1}}{2} \int_{\mathbb{T}^{d}} u^{2} \mathrm{~d} x+\frac{\varepsilon_{1}+\varepsilon_{2}}{2} \int_{\mathbb{T}^{d}} m^{2} \mathrm{~d} x+\frac{5}{2} . \tag{3.11}
\end{align*}
$$

Let $c, C_{1}, C_{2}$, and $C_{3}$ be as in Assumptions g1b) and (H1) and as in Remark 3.2 let $\mathfrak{c}:=\max \left\{C_{1}, C_{3}\right\}$, and set

$$
\begin{aligned}
F_{\mu}^{\varepsilon}:= & \frac{1}{\mathfrak{c}} \int_{\mathbb{T}^{d}}|D u|^{\gamma} m^{1-\tau} \mathrm{d} x+\frac{1}{\mathfrak{c}} \int_{\mathbb{T}^{d}}|D u|^{\gamma} m^{-\tau} \mathrm{d} x \\
& +(1-\mu) C_{2} \int_{\mathbb{T}^{d}} m g_{2}(x, m, h(\boldsymbol{m})) \mathrm{d} x+\int_{\mathbb{T}^{d}}-\beta_{\varepsilon_{1}}(m) \mathrm{d} x \\
& +\int_{\mathbb{T}^{d}} \beta_{\varepsilon_{1}}(m)\left(m-\varepsilon_{1}\right) \mathrm{d} x+\frac{\varepsilon_{1}}{2} \int_{\mathbb{T}^{d}}\left[u^{2}+\left(\Delta^{p} u\right)^{2}+m^{2}+\left(\Delta^{p} m\right)^{2}\right] \mathrm{d} x \\
& +\frac{\varepsilon_{2}}{2} \int_{\mathbb{T}^{d}}\left(u^{2}+|D u|^{2}+m^{2}+|D m|^{2}\right) \mathrm{d} x
\end{aligned}
$$

Due to (g1b) and the definition of $\beta_{\varepsilon_{1}}(\cdot)$, the integral terms defining $F_{\mu}^{\varepsilon}$ are nonnegative.
Using in (3.11) the Assumptions (H1a) and (H1b) for $H$ and $\widetilde{H}$, it follows that

$$
\begin{align*}
F_{\mu}^{\varepsilon}+ & (1-\mu) C_{2} \int_{\mathbb{T}^{d}} m g_{1}(x, m, h(\boldsymbol{m})) \mathrm{d} x+\mu \int_{\mathbb{T}^{d}} m^{2} \mathrm{~d} x \\
\leqslant & \frac{5}{2}+2 \mathfrak{c}+(\mathfrak{c}+2) \int_{\mathbb{T}^{d}} m \mathrm{~d} x+2 \mathfrak{c} \int_{\mathbb{T}^{d}} g_{2}(x, m, h(\boldsymbol{m})) \mathrm{d} x \\
& +C_{3}\left(\varepsilon_{1}+1\right)(1-\mu) \int_{\mathbb{T}^{d}} g_{1}(x, m, h(\boldsymbol{m})) \mathrm{d} x \\
\leqslant & \mathfrak{c}_{1}+\mathfrak{c}_{2} \mu \int_{\mathbb{T}^{d}} m \mathrm{~d} x+\mathfrak{c}_{2}(1-\mu) \int_{\mathbb{T}^{d}} m \mathrm{~d} x+\mathfrak{c}_{3}(1-\mu) \int_{\mathbb{T}^{d}} g_{1}(x, m, h(\boldsymbol{m})) \mathrm{d} x \tag{3.12}
\end{align*}
$$

where $\mathfrak{c}_{1}:=\frac{5}{2}+2 \mathfrak{c}+2 c \mathfrak{c}, \mathfrak{c}_{2}:=\mathfrak{c}+2+2 c \mathfrak{c}$, and $\mathfrak{c}_{3} \in\left\{C_{3}, 2 C_{3}\right\}$, and where we also used g1a) and g1b. Using now (g1c) with $\delta:=\frac{C_{2}}{2\left(\mathfrak{c}_{2}+\mathfrak{c}_{3}\right)}$, we conclude that

$$
\begin{equation*}
F_{\mu}^{\varepsilon}+\frac{(1-\mu) C_{2}}{2} \int_{\mathbb{T}^{d}} m g_{1}(x, m, h(\boldsymbol{m})) \mathrm{d} x+\frac{\mu}{2} \int_{\mathbb{T}^{d}} m^{2} \mathrm{~d} x \leqslant \mathfrak{c}_{1}+\frac{\mathfrak{c}_{2}^{2}}{2}+\left(\mathfrak{c}_{2}+\mathfrak{c}_{3}\right) C_{\delta} \tag{3.13}
\end{equation*}
$$

Finally, by (g1d, we obtain

$$
0 \leqslant F_{\mu}^{\varepsilon} \leqslant \mathfrak{c}_{1}+\frac{\mathfrak{c}_{2}^{2}}{2}+\left(\mathfrak{c}_{2}+\mathfrak{c}_{3}\right) C_{\delta}+\frac{C_{2}}{2} R
$$

and this completes the proof of (3.8).
Remark 3.4. If $\mu=0$, then (3.13) and the condition $F_{0}^{\varepsilon} \geqslant 0$ yield

$$
\int_{\mathbb{T}^{d}} m_{0}^{\varepsilon} g_{1}\left(x, m_{0}^{\varepsilon}, h\left(\boldsymbol{m}_{0}^{\varepsilon}\right)\right) \mathrm{d} x \leqslant C
$$

where $C$ is a positive constant that depends only on the problem data. Moreover, using (g1c) with $\delta=\frac{C_{2}}{2 \mathbf{c}_{2}}$ in (3.12) first, and then (g1d) together with the condition $F_{0}^{\varepsilon} \geqslant 0$, it follows that

$$
\int_{\mathbb{T}^{d}}-g_{1}\left(x, m_{0}^{\varepsilon}, h\left(\boldsymbol{m}_{0}^{\varepsilon}\right)\right) \mathrm{d} x \leqslant C
$$

where $C$ is a positive constant that depends only on the problem data. These two estimates are uniform in $\varepsilon$ and play a major role in the study of the limit of (3.1) as $\varepsilon_{1} \rightarrow 0^{+}$and $\varepsilon_{2} \rightarrow 0^{+}$.

In what follows, if $\varepsilon \in(0,1)^{2}$ and $\mu \in \Lambda^{\varepsilon}$, where $\Lambda^{\varepsilon}$ is the set defined in (3.6), then $\left(m_{\mu}^{\varepsilon}, u_{\mu}^{\varepsilon}\right)$ represents an arbitrary solution to (3.5), which belongs to $D\left(A_{\mu}^{\varepsilon}\right) \cap C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$. The next lemma provides a uniform bound with respect to $\mu \in[0,1]$ on the infimum of such solutions.

Lemma 3.5. Assume that Assumptions (g1) and (H1) hold. Let $\varepsilon \in(0,1)^{2}$ be such that $\Lambda^{\varepsilon} \neq \emptyset$, where $\Lambda^{\varepsilon}$ is the set defined in (3.6). Then,

$$
\begin{equation*}
\inf _{\mu \in \Lambda^{\varepsilon}} \inf _{\mathbb{T}^{d}} m_{\mu}^{\varepsilon}>0 \tag{3.14}
\end{equation*}
$$

Proof. We begin by proving that

$$
\begin{equation*}
L^{\varepsilon_{1}}:=\sup _{\mu \in \Lambda^{\varepsilon}}\left\|m_{\mu}^{\varepsilon}\right\|_{W^{1, \infty}\left(\mathbb{T}^{d}\right)}<+\infty . \tag{3.15}
\end{equation*}
$$

Because the integral terms in (3.8) are non-negative, we deduce that

$$
\sup _{\mu \in \Lambda^{\varepsilon}}\left[\varepsilon_{1} \int_{\mathbb{T}^{d}}\left(\left(u_{\mu}^{\varepsilon}\right)^{2}+\left(\Delta^{p} u_{\mu}^{\varepsilon}\right)^{2}+\left(m_{\mu}^{\varepsilon}\right)^{2}+\left(\Delta^{p} m_{\mu}^{\varepsilon}\right)^{2}\right) \mathrm{d} x\right] \leqslant C
$$

for some positive constant, $C$, that depends only on the problem data. From this estimate and using integration by parts, we conclude that

$$
\begin{equation*}
\sup _{\mu \in \Lambda^{\varepsilon}}\left[\left\|u_{\mu}^{\varepsilon}\right\|_{W^{2 p, 2}\left(\mathbb{T}^{d}\right)}+\left\|m_{\mu}^{\varepsilon}\right\|_{W^{2 p, 2}\left(\mathbb{T}^{d}\right)}\right] \leqslant C \tag{3.16}
\end{equation*}
$$

for some positive constant, $C$, that depends only on the problem data and on $\varepsilon$. Hence, using (3.4), $u_{\mu}^{\varepsilon}, m_{\mu}^{\varepsilon} \in C^{4, \lambda}\left(\mathbb{T}^{d}\right)$ with

$$
\begin{equation*}
\sup _{\mu \in \Lambda^{\varepsilon}}\left[\left\|u_{\mu}^{\varepsilon}\right\|_{W^{4, \infty}\left(\mathbb{T}^{d}\right)}+\left\|m_{\mu}^{\varepsilon}\right\|_{W^{4, \infty}\left(\mathbb{T}^{d}\right)}\right] \leqslant C \tag{3.17}
\end{equation*}
$$

for some positive constant, $C$, that depends only on the problem data and on $\varepsilon$. In particular, (3.15) holds.

We now establish (3.14). By contradiction, assume that $\inf _{\mu \in \Lambda^{\varepsilon}} \inf _{\mathbb{T}^{d}} m_{\mu}^{\varepsilon}=0$. Set $\bar{m}_{\mu}^{\varepsilon}:=$ $\inf _{\mathbb{T}^{d}} m_{\mu}^{\varepsilon}$ and let $\left(\mu_{n}\right)_{n \in \mathbb{N}} \subset \Lambda^{\varepsilon}$ be such that

$$
\lim _{n \rightarrow \infty} \bar{m}_{\mu_{n}}^{\varepsilon}=0
$$

Note that $\bar{m}_{\mu_{n}}^{\varepsilon}>0$ by the definition of $D\left(A_{\mu_{n}}^{\varepsilon}\right)$ (see (3.2)). Because $m_{\mu_{n}}^{\varepsilon}$ is continuous in $\mathbb{T}^{d}$, there exists $\bar{x}_{\mu_{n}}^{\varepsilon} \in \mathbb{T}^{d}$ satisfying $m_{\mu_{n}}^{\varepsilon}\left(\bar{x}_{\mu_{n}}^{\varepsilon}\right)=\bar{m}_{\mu_{n}}^{\varepsilon}$. Let $Q\left(\bar{x}_{\mu_{n}}^{\varepsilon}, \bar{m}_{\mu_{n}}^{\varepsilon}\right)$ be the cube centered at $\bar{x}_{\mu_{n}}^{\varepsilon}$ and with side length $m_{\mu_{n}}^{\varepsilon}$. For all sufficiently large $n \in \mathbb{N}, 0<\bar{m}_{\mu_{n}}^{\varepsilon} \leqslant \min \left\{\frac{\varepsilon_{1}}{4}, \frac{\varepsilon_{1}}{2 L^{\varepsilon} 1}\right\}$. Therefore, for all such $n \in \mathbb{N}$ and for a.e. $x \in Q\left(\bar{x}_{\mu_{n}}^{\varepsilon}, \bar{m}_{\mu_{n}}^{\varepsilon}\right)$, we have

$$
0<m_{\mu_{n}}^{\varepsilon}(x) \leqslant \frac{\bar{m}_{\mu_{n}}^{\varepsilon} L^{\varepsilon_{1}}}{2}+\bar{m}_{\mu_{n}}^{\varepsilon} \leqslant \frac{\varepsilon_{1}}{2}
$$

Consequently, using (3.8), the non-negativeness of $-\beta_{\varepsilon_{1}}\left(m_{\mu_{n}}^{\varepsilon}\right)$, and the identity $\beta_{\varepsilon_{1}}(s)=-s^{-q}$ if $0<s \leqslant \frac{\varepsilon_{1}}{2}$, we obtain

$$
\begin{aligned}
C & \geqslant \int_{\mathbb{T}^{d}}-\beta_{\varepsilon_{1}}\left(m_{\mu_{n}}^{\varepsilon}\right) \mathrm{d} x \geqslant \int_{Q\left(\bar{x}_{\mu_{n}}^{\varepsilon}, \bar{m}_{\mu_{n}}^{\varepsilon}\right)} \frac{1}{\left(m_{\mu_{n}}^{\varepsilon}\right)^{q}} \mathrm{~d} x \\
& \geqslant \frac{\left(\bar{m}_{\mu_{n}}^{\varepsilon}\right)^{d}}{\left(\frac{\bar{m}_{\mu_{n}}^{\varepsilon} L_{1}}{2}+\bar{m}_{\mu_{n}}^{\varepsilon}\right)^{q}}=\left(\frac{2}{L^{\varepsilon_{1}}+2}\right)^{q} \frac{1}{\left(\bar{m}_{\mu_{n}}^{\varepsilon}\right)^{q-d}},
\end{aligned}
$$

where $C$ is a positive constant that depends only on the problem data. Because $q>d$ and $\lim _{n \rightarrow \infty} \bar{m}_{\mu_{n}}^{\varepsilon}=0$, we have a contradiction when we let $n \rightarrow \infty$ in the estimate above.

Remark 3.6. Under the conditions of the previous lemma, the estimates in (3.14) and (3.17) imply that

$$
\underline{\mathrm{m}}^{\varepsilon}:=\inf _{\mu \in \Lambda^{\varepsilon}} \inf _{\mathbb{T}^{d}} m_{\mu}^{\varepsilon} \quad \text { and } \quad \bar{m}^{\varepsilon}:=\sup _{\mu \in \Lambda^{\varepsilon}} \sup _{\mathbb{T}^{d}} m_{\mu}^{\varepsilon}
$$

satisfy

$$
\begin{equation*}
0<\underline{m}^{\varepsilon} \leqslant \bar{m}^{\varepsilon} \leqslant C \tag{3.18}
\end{equation*}
$$

where $C$ is a positive constant that depends only on the problem data and on $\varepsilon$.
Next, we prove uniform bounds in $\mu \in[0,1]$ for classical solutions to (3.5) with respect to the norm $\|\cdot\|_{W^{k, \infty}\left(\mathbb{T}^{d}\right)}$. These bounds rely on the previous two lemmas and play an important role in Subsection 3.3, where we establish the existence of classical solutions to (3.5).

Lemma 3.7. Suppose that Assumptions (h1), (g1), (H1), and (H2) are satisfied. Let $\varepsilon \in$ $(0,1)^{2}$ be such that $\Lambda^{\varepsilon} \neq \emptyset$, where $\Lambda^{\varepsilon}$ is the set defined in (3.6). Then, for all $k \in \mathbb{N}_{0}$, there exists a positive constant, $C$, that depends only on the problem data and on $\varepsilon$, such that

$$
\sup _{\mu \in \Lambda^{\varepsilon}}\left(\left\|m_{\mu}^{\varepsilon}\right\|_{W^{k, \infty}\left(\mathbb{T}^{d}\right)}+\left\|u_{\mu}^{\varepsilon}\right\|_{W^{k, \infty}\left(\mathbb{T}^{d}\right)}\right) \leqslant C
$$

Proof. To simplify the notation, we drop the dependence on $\varepsilon$ and $\mu$ of $\boldsymbol{m}_{\mu}^{\varepsilon}, m_{\mu}^{\varepsilon}$, and $u_{\mu}^{\varepsilon}$. Accordingly, we write $\boldsymbol{m}, m$, and $u$ in place of $\boldsymbol{m}_{\mu}^{\varepsilon}, m_{\mu}^{\varepsilon}$, and $u_{\mu}^{\varepsilon}$, respectively. Moreover, $C$ is a positive constant that depends only on the problem data and on $\varepsilon$ and whose value may change from one expression to another.

Arguing as in the proof of Lemma 3.5 and using the hypothesis, we conclude that $m$, $u \in W^{4, \infty}\left(\mathbb{T}^{d}\right) \cap W^{2 p, 2}\left(\mathbb{T}^{d}\right)$, that (3.16) and (3.17) hold, and that

$$
\begin{align*}
& m+\Delta^{2 p} m=f_{1}(m, u)  \tag{3.19}\\
& u+\Delta^{2 p} u=f_{2}(m, u) \tag{3.20}
\end{align*}
$$

where

$$
\begin{align*}
f_{1}(m, u):= & \frac{1}{\varepsilon_{1}}\left[u+H_{\mu}\left(x, D u, D^{2} u, m, h(\boldsymbol{m})\right)-\beta_{\varepsilon_{1}}(m)-\varepsilon_{2}(m-\Delta m)\right]  \tag{3.21}\\
f_{2}(m, u):= & \frac{1}{\varepsilon_{1}}\left[-m+\operatorname{div}\left(m D_{p} H_{\mu}\left(x, D u, D^{2} u, m, h(\boldsymbol{m})\right)\right)\right. \\
& \left.\quad-\left(m D_{M_{i j}} H_{\mu}\left(x, D u, D^{2} u, m, h(\boldsymbol{m})\right)\right)_{x_{i} x_{j}}+1-\varepsilon_{2}(u-\Delta u)\right] . \tag{3.22}
\end{align*}
$$

By (2.2), (3.4), and (3.16), we also have that

$$
\begin{equation*}
\|h(\boldsymbol{m})\|_{L^{\infty}\left(\mathbb{T}^{d}\right)} \leqslant C\|h(\boldsymbol{m})\|_{W^{2 p-4,2}\left(\mathbb{T}^{d}\right)} \leqslant C\left(1+\|m\|_{W^{2 p-4,2}\left(\mathbb{T}^{d}\right)}\right) \leqslant C \tag{3.23}
\end{equation*}
$$

Hence, when

$$
K:=\left\{\left(x, D u(x), D^{2} u(x), m(x), h(\boldsymbol{m})(x)\right): x \in \mathbb{T}^{d}\right\}
$$

is thus defined, the estimates (3.17), (3.18), and (3.23) yield that $\bar{K}$ is a compact subset of $\mathbb{T}^{d} \times \mathbb{R}^{d} \times \mathbb{R}^{d \times d} \times \mathbb{E} \times \mathbb{R}$ and there exists a ball, $B(0, C)$, centered at the origin and of radius $C$ such that $\bar{K} \subset B(0, C)$. Using in addition Assumption (H2), it follows that, for all $\kappa \in \mathbb{N}_{0}$,

$$
\begin{equation*}
\left\|H_{\mu}\right\|_{W^{\kappa, \infty}(\bar{K})} \leqslant\|H\|_{W^{\kappa, \infty}(\bar{K})}+\|\tilde{H}\|_{W^{\kappa, \infty}(\bar{K})} \leqslant C \tag{3.24}
\end{equation*}
$$

Next, we recall that if $q>\frac{d}{2}$, then $W^{q, 2}\left(\mathbb{T}^{d}\right)$ is an algebra. Thus, because $p$ is such that $2 p-4>\frac{d}{2}+1$, from Remark (3.6] (3.16), (2.2), and (3.24), it follows that

$$
\begin{aligned}
& f_{1}(m, u) \in W^{2,2}\left(\mathbb{T}^{d}\right), \quad\left\|f_{1}(m, u)\right\|_{W^{2,2}\left(\mathbb{T}^{d}\right)} \leqslant c=c\left(C,\left\|\beta_{\varepsilon_{1}}\right\|_{W^{2,2}\left(\left[\underline{m}^{\varepsilon} \cdot \bar{m}^{\varepsilon}\right]\right)}\right) \\
& f_{2}(m, u) \in L^{2}\left(\mathbb{T}^{d}\right), \quad\left\|f_{2}(m, u)\right\|_{L^{2}\left(\mathbb{T}^{d}\right)} \leqslant \bar{c}=\bar{c}(C)
\end{aligned}
$$

Therefore, (3.19), (3.20), and the elliptic regularity theory yield

$$
\begin{aligned}
& m \in W^{4 p+2,2}\left(\mathbb{T}^{d}\right) \text { with }\|m\|_{W^{4 p+2,2}\left(\mathbb{T}^{d}\right)} \leqslant c_{1}=c_{1}(c), \\
& u \in W^{4 p, 2}\left(\mathbb{T}^{d}\right) \text { with }\|u\|_{W^{4 p, 2}\left(\mathbb{T}^{d}\right)} \leqslant \bar{c}_{1}=\bar{c}_{1}(\bar{c}) .
\end{aligned}
$$

Then, going back to (3.21) and (3.22), and arguing as above, we have

$$
\begin{aligned}
& f_{1}(m, u) \in W^{4 p-2,2}\left(\mathbb{T}^{d}\right), \quad\left\|f_{1}(m, u)\right\|_{W^{4 p-2,2}\left(\mathbb{T}^{d}\right)} \leqslant c_{2}=c_{2}\left(c_{1}, \bar{c}_{1},\left\|\beta_{\varepsilon_{1}}\right\|_{W^{4 p-2,2}\left(\left[\underline{m}^{\varepsilon} \cdot \bar{m}^{\varepsilon}\right]\right)}\right) \\
& f_{2}(m, u) \in W^{4 p-4,2}\left(\mathbb{T}^{d}\right), \quad\left\|f_{2}(m, u)\right\|_{W^{4 p-4,2}\left(\mathbb{T}^{d}\right)} \leqslant \bar{c}_{2}=\bar{c}_{2}\left(c_{1}, \bar{c}_{1}\right)
\end{aligned}
$$

Arguing as before, we conclude that

$$
\begin{aligned}
& m \in W^{8 p-2,2}\left(\mathbb{T}^{d}\right) \text { with }\|m\|_{W^{8 p-2,2}\left(\mathbb{T}^{d}\right)} \leqslant c_{3}=c_{3}\left(c_{2}, \bar{c}_{2}\right) \\
& u \in W^{8 p-4,2}\left(\mathbb{T}^{d}\right) \text { with }\|u\|_{W^{8 p-4,2}\left(\mathbb{T}^{d}\right)} \leqslant \bar{c}_{3}=\bar{c}_{3}\left(c_{2}, \bar{c}_{2}\right)
\end{aligned}
$$

The conclusion follows by iterating this "bootstrap" argument and using the Sobolev embedding theorem.
3.3. Existence of classical solutions to the perturbed mean-field game. Here, we prove that $\Lambda^{\varepsilon}=[0,1]$ for all $\varepsilon \in(0,1)^{2}$. Consequently, by the definition of $\Lambda^{\varepsilon}$ in (3.6), we have that (3.1) has a solution, $\left(u^{\varepsilon}, m^{\varepsilon}\right)$, with $m^{\varepsilon}>0$. To prove that $\Lambda^{\varepsilon}=[0,1]$, we show that $\Lambda^{\varepsilon}$ is a nonempty set, relatively closed and open in $[0,1]$. These properties are established in the next three lemmas.

Lemma 3.8. Let $\Lambda^{\varepsilon}$ be the set defined in (3.6). Then, $\Lambda^{\varepsilon} \neq \emptyset$.
Proof. To prove the lemma, we show that $1 \in \Lambda^{\varepsilon}$. We set

$$
f(c):=-c-\frac{1}{\gamma\left(1-c\left(\varepsilon_{1}+\varepsilon_{2}\right)\right)^{\tau}}+\left(\varepsilon_{1}+\varepsilon_{2}\right)\left(1-c\left(\varepsilon_{1}+\varepsilon_{2}\right)\right)+\beta_{\varepsilon_{1}}\left(1-c\left(\varepsilon_{1}+\varepsilon_{2}\right)\right)
$$

Recall that $\gamma \geqslant 1>0, \tau \geqslant 0, \beta_{\varepsilon_{1}}(s)=0$ if $s \geqslant \varepsilon_{1}$, and $\lim _{s \rightarrow 0^{+}} \beta_{\varepsilon_{1}}(s)=-\infty$. Consequently, we have that $\lim _{c \rightarrow-\infty} f(c)=+\infty$ and $\lim _{c \rightarrow\left(\frac{1}{\varepsilon_{1}+\varepsilon_{2}}\right)^{-}} f(c)=-\infty$. Therefore, there exists $\bar{c} \in\left(-\infty, \frac{1}{\varepsilon_{1}+\varepsilon_{2}}\right)$ such that $f(\bar{c})=0$.

Set $m_{1}^{\varepsilon}:=1-\bar{c}\left(\varepsilon_{1}+\varepsilon_{2}\right)$ and $u_{1}^{\varepsilon}:=\bar{c}$. Then, $\left(m_{1}^{\varepsilon}, u_{1}^{\varepsilon}\right) \in D\left(A_{1}^{\varepsilon}\right) \cap C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$ and
$A_{1}^{\varepsilon}\left[\begin{array}{c}m_{1}^{\varepsilon} \\ u_{1}^{\varepsilon}\end{array}\right]=\left[\begin{array}{c}-\bar{c}-\frac{1}{\gamma\left(1-\bar{c}\left(\varepsilon_{1}+\varepsilon_{2}\right)\right)^{\tau}}+\left(\varepsilon_{1}+\varepsilon_{2}\right)\left(1-\bar{c}\left(\varepsilon_{1}+\varepsilon_{2}\right)\right)+\beta_{\varepsilon_{1}}\left(1-\bar{c}\left(\varepsilon_{1}+\varepsilon_{2}\right)\right) \\ 1-\bar{c}\left(\varepsilon_{1}+\varepsilon_{2}\right)+\varepsilon_{1} \bar{c}+\varepsilon_{2} \bar{c}-1\end{array}\right]=\left[\begin{array}{l}0 \\ 0\end{array}\right]$.

This completes the proof.
Lemma 3.9. Suppose that Assumptions (h1), (g1), (H1), and (H2) hold. Let $\Lambda^{\varepsilon}$ be the set defined in (3.6). Then, $\Lambda^{\varepsilon}$ is a closed subset of $[0,1]$.

Proof. Let $\left\{\mu_{n}\right\}_{n \in \mathbb{N}} \subset \Lambda^{\varepsilon}$ be a sequence converging to some $\mu_{0} \in[0,1]$. We claim that $\mu_{0} \in \Lambda^{\varepsilon}$. To see this, let $\left\{\left(m_{\mu_{n}}^{\varepsilon}, u_{\mu_{n}}^{\varepsilon}\right)\right\}_{n \in \mathbb{N}} \subset D\left(A_{\mu_{n}}^{\varepsilon}\right) \cap C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$ be such that

$$
A_{\mu_{n}}^{\varepsilon}\left[\begin{array}{c}
m_{\mu_{n}}^{\varepsilon}  \tag{3.25}\\
u_{\mu_{n}}^{\varepsilon}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] .
$$

By Lemma 3.7

$$
\sup _{n \in \mathbb{N}}\left(\left\|m_{\mu_{n}}^{\varepsilon}\right\|_{W^{4 p+1,2}\left(\mathbb{T}^{d}\right)}+\left\|u_{\mu_{n}}^{\varepsilon}\right\|_{W^{4 p+1,2}\left(\mathbb{T}^{d}\right)}\right) \leqslant C
$$

for some positive constant, $C$, that depends only on the problem data and on $\varepsilon$. Hence, by the Rellich-Kondrachov theorem, we can find $\left(m^{\varepsilon}, u^{\varepsilon}\right) \in W^{4 p+1,2}\left(\mathbb{T}^{d}\right) \times W^{4 p+1,2}\left(\mathbb{T}^{d}\right)$ such that, up to a (not relabeled) subsequence, $\left(m_{\mu_{n}}^{\varepsilon}, u_{\mu_{n}}^{\varepsilon}\right)_{n \in \mathbb{N}}$ converges to $\left(m^{\varepsilon}, u^{\varepsilon}\right)$ weakly in $W^{4 p+1,2}\left(\mathbb{T}^{d}\right)$ and strongly in $W^{4 p, 2}\left(\mathbb{T}^{d}\right)$ and

$$
D^{\alpha} m_{\mu_{n}}^{\varepsilon}(x) \rightarrow D^{\alpha} m^{\varepsilon}(x) \text { and } D^{\alpha} u_{\mu_{n}}^{\varepsilon}(x) \rightarrow D^{\alpha} u^{\varepsilon}(x)
$$

as $n \rightarrow \infty$ for a.e. $x \in \mathbb{T}^{d}$ and for all $\alpha \in \mathbb{N}_{0}^{d}$ with $|\alpha| \leqslant 4 p$. Using (2.1) with $\boldsymbol{m}$ and $\boldsymbol{m}_{0}$ replaced by $\boldsymbol{m}_{\mu_{n}}^{\varepsilon}$ and $\boldsymbol{m}^{\varepsilon}$, respectively, up to a further (not relabeled) subsequence, we also have that for a.e. $x \in \mathbb{T}^{d}$ and for all $\alpha \in \mathbb{N}_{0}^{d}$ with $|\alpha| \leqslant 4 p$,

$$
D^{\alpha} h\left(\boldsymbol{m}_{\mu_{n}}^{\varepsilon}\right)(x) \rightarrow D^{\alpha} h\left(\boldsymbol{m}^{\varepsilon}\right)(x)
$$

as $n \rightarrow \infty$. Moreover, using Lemma 3.7 once more and in view of Lemma 3.5. $\left(m^{\varepsilon}, u^{\varepsilon}\right) \in$ $W^{4 p+1,2}\left(\mathbb{T}^{d}\right) \times W^{4 p+1,2}\left(\mathbb{T}^{d}\right) \cap C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\inf _{\mathbb{T}^{d}} m^{\varepsilon}>0$. By Assumption (H2) and because $\beta_{\varepsilon_{1}}(\cdot)$ is smooth, passing (3.25) to the limit as $n \rightarrow \infty$, we conclude that

$$
A_{\mu_{0}}^{\varepsilon}\left[\begin{array}{c}
m^{\varepsilon} \\
u^{\varepsilon}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

Hence, $\mu_{0} \in \Lambda^{\varepsilon}$.
Lemma 3.10. Under Assumptions (h1), (g1), and (H1) (H3), the set $\Lambda^{\varepsilon}$ defined in (3.6) is an open subset of $[0,1]$.

Proof. Fix $\mu_{0} \in \Lambda^{\varepsilon}$. We want to show that there exists a neighborhood of $\mu_{0}$ in $[0,1]$ contained in $\Lambda^{\varepsilon}$. That will be a consequence of the implicit function theorem in Banach spaces (see, for example, [16]).

Fix a solution, $\left(m_{0}, u_{0}\right) \in D\left(A_{\mu}^{\varepsilon}\right) \cap C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$, to (3.5) with $\mu=\mu_{0}$.
In this proof, to simplify the notation, we omit the dependence on $\varepsilon$ and $\mu_{0}$.
To use the implicit function theorem, we have to prove that the Fréchet derivative, $\mathcal{L}$ : $W^{4 p, 2}\left(\mathbb{T}^{d}\right) \times W^{4 p, 2}\left(\mathbb{T}^{d}\right) \rightarrow L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)$, of

$$
\left[\begin{array}{c}
m \\
u
\end{array}\right] \mapsto A_{\mu_{0}}^{\varepsilon}\left[\begin{array}{c}
m \\
u
\end{array}\right]
$$

at $\left(m_{0}, u_{0}\right)$ is an isomorphism from $W^{4 p, 2}\left(\mathbb{T}^{d}\right) \times W^{4 p, 2}\left(\mathbb{T}^{d}\right)$ to $L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)$.
Given $(\eta, v) \in W^{4 p, 2}\left(\mathbb{T}^{d}\right) \times W^{4 p, 2}\left(\mathbb{T}^{d}\right)$, we have that

$$
\mathcal{L}\left[\begin{array}{l}
\eta \\
v
\end{array}\right]=\left[\begin{array}{c}
-v-\ell_{1}(\eta, v)+\ell_{2}(\eta) \\
\eta-\operatorname{div}\left(\eta D_{p} H_{\mu_{0}}+m_{0} \ell_{3}(\eta, v)\right)+\left(\eta D_{M_{i j}} H_{\mu_{0}}+m_{0} \ell_{4}(\eta, v)\right)_{x_{i} x_{j}}+\ell_{5}(v)
\end{array}\right]
$$

where, recalling that $\mathfrak{H}_{m_{0}}$ is the Fréchet derivative of $m \in W^{4 p, 2}\left(\mathbb{T}^{d}\right) \mapsto h(\boldsymbol{m}) \in W^{4 p, 2}\left(\mathbb{T}^{d}\right)$ at $m_{0}$ (see (2.1)),

$$
\begin{aligned}
\ell_{1}(\eta, v) & :=D_{p} H_{\mu_{0}} \cdot D v+D_{M} H_{\mu_{0}}: D^{2} v+D_{m} H_{\mu_{0}} \eta+D_{\theta} H_{\mu_{0}} \mathfrak{H}_{m_{0}}(\eta), \\
\ell_{2}(\eta) & :=\varepsilon_{1}\left(\eta+\Delta^{2 p} \eta\right)+\varepsilon_{2}(\eta-\Delta \eta)+\beta_{\varepsilon}^{\prime}\left(m_{0}\right) \eta \\
\ell_{3}(\eta, v) & :=D_{p p}^{2} H_{\mu_{0}} D v+D_{M_{i j} p}^{2} H_{\mu_{0}} D_{i j}^{2} v+D_{m p}^{2} H_{\mu_{0}} \eta+D_{\theta p}^{2} H_{\mu_{0}} \mathfrak{H}_{m_{0}}(\eta), \\
\ell_{4}(\eta, v) & :=D_{p M_{i j}}^{2} H_{\mu_{0}} \cdot D v+D_{M M_{i j}}^{2} H_{\mu_{0}}: D^{2} v+D_{m M_{i j}}^{2} H_{\mu_{0}} \eta+D_{\theta M_{i j}} H_{\mu_{0}} \mathfrak{H}_{m_{0}}(\eta),
\end{aligned}
$$

$$
\ell_{5}(v):=\varepsilon_{1}\left(v+\Delta^{2 p} v\right)+\varepsilon_{2}(v-\Delta v)
$$

and the argument of $H_{\mu_{0}}$ and all its partial derivatives is $\left(x, D u_{0}, D^{2} u_{0}, m_{0}, h\left(\boldsymbol{m}_{0}\right)\right)$.
Let

$$
K_{0}:=\left\{\left(x, D u_{0}(x), D^{2} u_{0}(x), m_{0}(x), h\left(\boldsymbol{m}_{0}\right)(x)\right): x \in \mathbb{T}^{d}\right\}
$$

Arguing as in Lemma 3.7, for all $\kappa \in \mathbb{N}_{0}$, we have

$$
\begin{equation*}
\left\|H_{\mu}\right\|_{W^{\kappa, \infty}\left(\bar{K}_{0}\right)} \leqslant\|H\|_{W^{\kappa, \infty}\left(\bar{K}_{0}\right)}+\|\widetilde{H}\|_{W^{\kappa, \infty}\left(\bar{K}_{0}\right)} \leqslant C \tag{3.26}
\end{equation*}
$$

for some positive constant, $C$, that depends only on the problem data and on $\varepsilon$.
Note also that $\mathcal{L}(\eta, v) \in L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)$ because $W^{4 p-4,2}\left(\mathbb{T}^{d}\right)$ is an algebra, $\inf _{\mathbb{T}^{d}} m_{0}>0$, and in view of (2.2) and (3.26). We will use the Lax-Milgram theorem to prove that $\mathcal{L}$ defines an isomorphism.

Fix $\bar{z}=\left(\bar{z}_{1}, \bar{z}_{2}\right) \in L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)$. Next, we show that there exists a unique $\bar{w}=(\bar{\eta}, \bar{v}) \in$ $W^{4 p, 2}\left(\mathbb{T}^{d}\right) \times W^{4 p, 2}\left(\mathbb{T}^{d}\right)$ such that $\mathcal{L} \bar{w}=\bar{z}$.

Define a map $L: W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right) \rightarrow \mathbb{R}$ by $L[w]:=(\bar{z}, w)_{L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)}$ for $w=(\eta, v) \in$ $W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right)$. Clearly, $L$ is linear and bounded in $W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right)$.

Now, let $B: W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right) \rightarrow \mathbb{R}$ be the bilinear form defined for $w_{1}=\left(\eta_{1}, v_{1}\right)$, $w_{2}=\left(\eta_{2}, v_{2}\right) \in W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right)$ by

$$
\begin{aligned}
B\left[w_{1}, w_{2}\right]:=\int_{\mathbb{T}^{d}}[ & -v_{1} \eta_{2}+\eta_{1} v_{2}-\ell_{1}\left(\eta_{1}, v_{1}\right) \eta_{2}+m_{0} D_{p} H_{\mu_{0}} \cdot D v_{2}+\eta_{1} \ell_{3}\left(\eta_{1}, v_{1}\right) \cdot D v_{2} \\
& +\eta_{1} D_{M} H_{\mu_{0}}: D^{2} v_{2}+m_{0} \ell_{4}\left(\eta_{1}, v_{1}\right) D_{i j}^{2} v_{2} \\
& \left.+\varepsilon_{1}\left(\eta_{1} \eta_{2}+\Delta^{p} \eta_{1} \Delta^{p} \eta_{2}+v_{1} v_{2}+\Delta^{p} v_{1} \Delta^{p} v_{2}\right)\right)+\beta_{\varepsilon_{1}}^{\prime}\left(m_{0}\right) \eta_{1} \eta_{2} \\
& \left.+\varepsilon_{2}\left(\eta_{1} \eta_{2}+\nabla \eta_{2} \nabla \eta_{2}+v_{1} v_{2}+\nabla v_{1} \nabla v_{2}\right)\right] \mathrm{d} x
\end{aligned}
$$

We start by observing that if $w_{1}=\left(\eta_{1}, v_{1}\right) \in W^{4 p, 2}\left(\mathbb{T}^{d}\right) \times W^{4 p, 2}\left(\mathbb{T}^{d}\right)$ and $w_{2}=\left(\eta_{2}, v_{2}\right) \in$ $W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right)$, then

$$
\begin{equation*}
B\left[w_{1}, w_{2}\right]=\left(\mathcal{L} w_{1}, w_{2}\right)_{L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)} \tag{3.27}
\end{equation*}
$$

Moreover, for $w=(\eta, v) \in W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right)$,

$$
\begin{aligned}
B[w, w]= & B_{1}[w, w]+\int_{\mathbb{T}^{d}} \varepsilon_{1}\left(\eta^{2}+\left(\Delta^{p} \eta\right)^{2}+v^{2}+\left(\Delta^{p} v\right)^{2}\right)+\beta_{\varepsilon_{1}}^{\prime}\left(m_{0}\right) \eta^{2} \mathrm{~d} x \\
& +\int_{\mathbb{T}^{d}} \varepsilon_{1}\left(\eta^{2}+|\nabla \eta|^{2}+v^{2}+|\nabla v|^{2}\right) \mathrm{d} x
\end{aligned}
$$

where

$$
\begin{aligned}
B_{1}[w, w]:=\int_{\mathbb{T}^{d}}[ & -\ell_{1}(\eta, v) \eta+m_{0} D_{p} H_{\mu_{0}} \cdot D v+\eta \ell_{3}(\eta, v) \cdot D v+\eta D_{M} H_{\mu_{0}}: D^{2} v \\
& \left.+m_{0} \ell_{4}(\eta, v) D_{i j}^{2} v\right] \mathrm{d} x
\end{aligned}
$$

We claim that

$$
\begin{equation*}
B_{1}[w, w] \geqslant 0 . \tag{3.28}
\end{equation*}
$$

Assume that (3.28) holds. Then, using integration by parts, Gagliardo-Nirenberg's interpolation inequalities, and the fact that $m_{0}>0$ and $\beta_{\varepsilon_{1}}^{\prime}\left(m_{0}\right) \geqslant 0$ in $\mathbb{T}^{d}$, it follows that there exists a positive constant, $C=C\left(\varepsilon, p, d, \mathbb{T}^{d}\right)$, such that

$$
B[w, w] \geqslant C\|w\|_{W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right)}^{2}
$$

Hence, $B[\cdot, \cdot]$ is coercive in $W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right)$.
Next, we prove that (3.28) is a consequence of the monotonicity of $H$ and $\widetilde{H}$ encoded in Assumption (H3). In fact, let $\widetilde{A}$ and $A_{\mu_{0}}$ be the operators associated with $\widetilde{H}$ and $H_{\mu_{0}}$, respectively (see (1.7)). Then, $A_{\mu_{0}}=\left(1-\mu_{0}\right) A+\mu_{0} \widetilde{A}$. Since both $A$ and $\widetilde{A}$ satisfy Assumption (H3), the same holds for $A_{\mu_{0}}$. Consequently,

$$
0 \leqslant \lim _{\delta \rightarrow 0^{+}} \frac{1}{\delta^{2}}\left(A_{\mu_{0}}^{\varepsilon}\left[\begin{array}{c}
m_{0}+\delta \eta \\
u_{0}+\delta v
\end{array}\right]-A_{\mu_{0}}^{\varepsilon}\left[\begin{array}{c}
m_{0} \\
u_{0}
\end{array}\right],\left[\begin{array}{c}
\delta \eta \\
\delta v
\end{array}\right]\right)_{L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)}=B_{1}[w, w]
$$

which proves (3.28).

We also have that $B[\cdot, \cdot]$ is bounded in $W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right)$; more precisely,

$$
\left|B\left[w_{1}, w_{2}\right]\right| \leqslant C\left\|w_{1}\right\|_{W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right)}\left\|w_{2}\right\|_{W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right)}
$$

for some positive constant, $C$, that depends only on the problem data and on $\varepsilon$ in view of Remark 3.6. Lemma 3.7, (2.2), and (3.26).

By the Lax-Milgram theorem, there exists a unique $\bar{w}=(\bar{\eta}, \bar{v}) \in W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right)$ such that

$$
\begin{equation*}
B[\bar{w}, w]=L[w]=(\bar{z}, w)_{L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)} \tag{3.29}
\end{equation*}
$$

for all $w \in W^{2 p, 2}\left(\mathbb{T}^{d}\right) \times W^{2 p, 2}\left(\mathbb{T}^{d}\right)$. From (3.27), the proof is complete once we show that $\bar{w} \in W^{4 p, 2}\left(\mathbb{T}^{d}\right) \times W^{4 p, 2}\left(\mathbb{T}^{d}\right)$.

Equality (3.29) yields that $\bar{\theta} \in W^{2 p, 2}\left(\mathbb{T}^{d}\right)$ and $\bar{v} \in W^{2 p, 2}\left(\mathbb{T}^{d}\right)$ are weak solutions of

$$
\varepsilon\left(\bar{\eta}+\Delta^{2 p} \bar{\eta}\right)=\bar{v}+\ell_{1}(\bar{\eta}, \bar{v})-\ell_{2}(\bar{\eta})+\bar{z}_{1}=: g_{1}[\bar{\eta}, \bar{v}]
$$

and

$$
\begin{aligned}
\varepsilon\left(\bar{v}+\Delta^{2 p} \bar{v}\right)= & -\bar{\eta}+\operatorname{div}\left(m_{0} D_{p} H_{\mu_{0}}+\bar{\eta} \ell_{3}(\bar{\eta}, \bar{v})\right) \\
& -\left(\bar{\eta} D_{M_{i j}} H_{\mu_{0}}+m_{0} \ell_{4}(\bar{\eta}, \bar{v})\right)_{x_{i} x_{j}}-\ell_{5}(\bar{v})+\bar{z}_{2}=: g_{2}[\bar{\eta}, \bar{v}]
\end{aligned}
$$

respectively, where the argument of $H_{\mu_{0}}$ and all its partial derivatives is $\left(x, D u_{0}, D^{2} u_{0}, m_{0}, h\left(\boldsymbol{m}_{0}\right)\right.$ ). Arguing as above, we have that $g_{1}[\bar{\eta}, \bar{v}], g_{2}[\bar{\eta}, \bar{v}] \in L^{2}\left(\mathbb{T}^{d}\right)$; hence, by the elliptic regularity theory, it follows that $\bar{\eta} \in W^{4 p, 2}\left(\mathbb{T}^{d}\right)$ and $\bar{v} \in W^{4 p, 2}\left(\mathbb{T}^{d}\right)$.

As a result of the previous lemmas, we obtain (3.7):
Corollary 3.11. Assume that assumptions (h1), (g1), and (H1) -(H3) are satisfied and let $\Lambda^{\varepsilon}$ be the set defined in (3.6). Then, $\Lambda^{\varepsilon}=[0,1]$.

The proof of Proposition 3.1 is now a simple matter.

Proof of Proposition 3.1. By Corollary 3.11. $0 \in \Lambda^{\varepsilon}$. Therefore, the claim in the statement holds.

## 4. Existence of weak solutions

Our next goal is to establish the existence of weak solutions to (1.1). Because the parameter $\mu$ does not play any role in this section, we set $\mu=0$ and $A^{\varepsilon}=A_{0}^{\varepsilon}$, and we recall that (3.1) corresponds to (3.5) with $\mu=0$. We proceed in three steps. First, we prove that the operator $A^{\varepsilon}$ is monotone. Next, we prove estimates for solutions of (3.1) that are uniform in $\varepsilon$. Finally, combining the monotonicity and these estimates, we use Minty's method to obtain a weak solution to (1.1).

Lemma 4.1. Suppose that Assumption (H3) holds. Then, the operator $A^{\varepsilon}$ given by (3.2) (3.3) (with $\mu=0$ ) is monotone in $L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)$. More precisely, for all $\left(m_{1}, u_{1}\right),\left(m_{2}, u_{2}\right) \in$ $D\left(A^{\varepsilon}\right) \cap C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$, we have that

$$
\left(A^{\varepsilon}\left[\begin{array}{l}
m_{1} \\
u_{1}
\end{array}\right]-A^{\varepsilon}\left[\begin{array}{l}
m_{2} \\
u_{2}
\end{array}\right],\left[\begin{array}{l}
m_{1} \\
u_{1}
\end{array}\right]-\left[\begin{array}{l}
m_{2} \\
u_{2}
\end{array}\right]\right)_{L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)} \geqslant 0
$$

Proof. Let $\left(m_{1}, u_{1}\right),\left(m_{2}, u_{2}\right) \in D\left(A^{\varepsilon}\right) \cap C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$. Using the definition of $A^{\varepsilon}$ and $A$ (see (3.3) and (1.7)), we obtain

$$
\begin{aligned}
&\left(A^{\varepsilon}\left[\begin{array}{l}
m_{1} \\
u_{1}
\end{array}\right]-A^{\varepsilon}\left[\begin{array}{c}
m_{2} \\
u_{2}
\end{array}\right],\left[\begin{array}{c}
m_{1} \\
u_{1}
\end{array}\right]-\left[\begin{array}{c}
m_{2} \\
u_{2}
\end{array}\right]\right)_{L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)} \\
&=\left(A\left[\begin{array}{c}
m_{1} \\
u_{1}
\end{array}\right]-A\left[\begin{array}{c}
m_{2} \\
u_{2}
\end{array}\right],\left[\begin{array}{c}
m_{1} \\
u_{1}
\end{array}\right]-\left[\begin{array}{c}
m_{2} \\
u_{2}
\end{array}\right]\right)_{L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)} \\
&+\int_{\mathbb{T}^{d}}\left(\beta_{\varepsilon_{1}}\left(m_{1}\right)-\beta_{\varepsilon_{1}}\left(m_{2}\right)\right)\left(m_{1}-m_{2}\right) \mathrm{d} x \\
&+\varepsilon_{1} \int_{\mathbb{T}^{d}}\left[\left(m_{1}-m_{2}\right)^{2}+\left(\Delta^{p} m_{1}-\Delta^{p} m_{2}\right)^{2}+\left(u_{1}-u_{2}\right)^{2}+\left(\Delta^{p} u_{1}-\Delta^{p} u_{2}\right)^{2}\right] \mathrm{d} x \\
&+\varepsilon_{2} \int_{\mathbb{T}^{d}}\left[\left(m_{1}-m_{2}\right)^{2}+\left(D m_{1}-D m_{2}\right)^{2}+\left(u_{1}-u_{2}\right)^{2}+\left(D u_{1}-D u_{2}\right)^{2}\right] \mathrm{d} x
\end{aligned}
$$

The conclusion follows because $\beta_{\varepsilon_{1}}(\cdot)$ is a non-decreasing function and $A$ satisfies (H3).
Next, we give bounds for solutions of (3.1).
Proposition 4.2. Suppose that Assumptions (g1) and (H1) hold. Then, there exists a positive constant, $C$, such that, for any $\varepsilon \in(0,1)^{2}$ and any solution $\left(u^{\varepsilon}, m^{\varepsilon}\right) \in D\left(A^{\varepsilon}\right)$ to (3.1), we have

$$
\begin{gather*}
\int_{\mathbb{T}^{d}}\left|D u^{\varepsilon}\right|^{\gamma} \mathrm{d} x \leqslant C  \tag{4.1}\\
\left|\int_{\mathbb{T}^{d}} u^{\varepsilon} \mathrm{d} x\right| \leqslant C  \tag{4.2}\\
\int_{\mathbb{T}^{d}}\left|\beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right)\right| \mathrm{d} x \leqslant C  \tag{4.3}\\
\int_{\mathbb{T}^{d}} m^{\varepsilon} g_{1}\left(x, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right) \mathrm{d} x \leqslant C \tag{4.4}
\end{gather*}
$$

and

$$
\begin{equation*}
\int_{\mathbb{T}^{d}}-g_{1}\left(x, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right) \mathrm{d} x \leqslant C \tag{4.5}
\end{equation*}
$$

Proof. To simplify the notation, $C$ represents a positive constant depending only on the problem data and whose value may change from one expression to another.

Estimates (4.4) and (4.5) follow from Remark 3.4
Next, we prove (4.1) and (4.3). Because all integral terms in (3.8) are nonnegative, $0 \leqslant \tau<1$, $m^{\varepsilon}>0$, and $\beta_{\varepsilon_{1}}(\cdot) \leqslant 0$, it follows from Lemma 3.3 with $\mu=0$ that

$$
C \geqslant \int_{\mathbb{T}^{d}}-\beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right) \mathrm{d} x=\int_{\mathbb{T}^{d}}\left|\beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right)\right| \mathrm{d} x
$$

and

$$
\begin{aligned}
C & \geqslant \int_{\mathbb{T}^{d}}\left|D u^{\varepsilon}\right|^{\gamma}\left(m^{\varepsilon}\right)^{1-\tau} \mathrm{d} x+\int_{\mathbb{T}^{d}}\left|D u^{\varepsilon}\right|^{\gamma}\left(m^{\varepsilon}\right)^{-\tau} \mathrm{d} x \\
& \geqslant \int_{\left\{x \in \mathbb{T}^{d} \cdot m^{\varepsilon}(x) \geqslant 1\right\}}\left|D u^{\varepsilon}\right|^{\gamma} \mathrm{d} x+\int_{\left\{x \in \mathbb{T}^{d} \cdot m^{\varepsilon}(x)<1\right\}}\left|D u^{\varepsilon}\right|^{\gamma} \mathrm{d} x \\
& =\int_{\mathbb{T}^{d}}\left|D u^{\varepsilon}\right|^{\gamma} \mathrm{d} x .
\end{aligned}
$$

Therefore, (4.1) and (4.3) hold.
Finally, we show (4.2). Integrating the first equation in (3.1) over $\mathbb{T}^{d}$, we obtain

$$
\begin{align*}
\int_{\mathbb{T}^{d}} u^{\varepsilon} \mathrm{d} x= & -\int_{\mathbb{T}^{d}} H\left(x, D u^{\varepsilon}, D^{2} u^{\varepsilon}, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right) \mathrm{d} x+\int_{\mathbb{T}^{d}} \beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right) \mathrm{d} x \\
& +\left(\varepsilon_{1}+\varepsilon_{2}\right) \int_{\mathbb{T}^{d}} m^{\varepsilon} \mathrm{d} x \tag{4.6}
\end{align*}
$$

By (3.8) with $\mu=0$ and (4.3), we have that

$$
\begin{align*}
& \int_{\mathbb{T}^{d}}\left|\beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right)\right| \mathrm{d} x+\left(\varepsilon_{1}+\varepsilon_{2}\right) \int_{\mathbb{T}^{d}}\left|m^{\varepsilon}\right| \mathrm{d} x \\
& \quad \leqslant C+2+\left(\varepsilon_{1}+\varepsilon_{2}\right) \int_{\mathbb{T}^{d}}\left(m^{\varepsilon}\right)^{2} \mathrm{~d} x \leqslant C . \tag{4.7}
\end{align*}
$$

Next, we estimate the upper and lower bounds of $\int_{\mathbb{T}^{d}} H \mathrm{~d} x$ in (H1b). According to (g1a), (4.5), and (3.8) with $\mu=0$, we have that

$$
\begin{equation*}
-C_{4} \int_{\mathbb{T}^{d}} g\left(x, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right) \mathrm{d} x+C_{1}\left(1+\int_{\mathbb{T}^{d}}\left|D u^{\varepsilon}\right|^{\gamma}\left(m^{\varepsilon}\right)^{-\tau} \mathrm{d} x\right) \leqslant C \tag{4.8}
\end{equation*}
$$

Similarly, from (g1a), g1b, (g1c) with $\delta=1$, and (4.4), we get

$$
\begin{align*}
& C_{3} \int_{\mathbb{T}^{d}} g\left(x, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right) \mathrm{d} x-\frac{1}{C_{1}} \int_{\mathbb{T}^{d}}\left|D u^{\varepsilon}\right|^{\gamma}\left(m^{\varepsilon}\right)^{-\tau} \mathrm{d} x+C_{1} \\
& \quad \leqslant C_{3}(1+c) \max \left\{\int_{\mathbb{T}^{d}} g_{1}(x, m, h(\boldsymbol{m})) \mathrm{d} x, \int_{\mathbb{T}^{d}} m \mathrm{~d} x\right\}+C_{3} c+C_{1} \leqslant C \tag{4.9}
\end{align*}
$$

Hence, in view of (H1b), (4.8) and (4.9) yield

$$
\begin{equation*}
\left|-\int_{\mathbb{T}^{d}} H\left(x, D u^{\varepsilon}, D^{2} u^{\varepsilon}, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right) \mathrm{d} x\right| \leqslant C \tag{4.10}
\end{equation*}
$$

Consequently, owing to (4.6), (4.7), and (4.10), we conclude that (4.2) holds.

Next, we present the proof of Theorem 1.1 .

Proof of Theorem 1.1. As mentioned at the beginning of this section, we set $A^{\varepsilon}=A_{0}^{\varepsilon}$. In view of Corollary 3.11 with $\mu=0$, there exists $\left(m^{\varepsilon}, u^{\varepsilon}\right) \in D\left(A^{\varepsilon}\right) \cap C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$ such that

$$
A^{\varepsilon}\left[\begin{array}{l}
m^{\varepsilon}  \tag{4.11}\\
u^{\varepsilon}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

pointwise in $\mathbb{T}^{d}$.
As usual, we assume that $\varepsilon_{1}$ and $\varepsilon_{2}$ take values on fixed sequences of positive numbers converging to zero. To simplify the notation, we write $\varepsilon \rightarrow 0^{+}$instead of $\varepsilon \rightarrow\left(0^{+}, 0^{+}\right)$.

We now proceed in two steps.
Step 1. In this step, we prove that there exists $(\boldsymbol{m}, u) \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right)$ with $\int_{\mathbb{T}^{d}} \overline{m \mathrm{~d} x}=1$ such that, up to a not relabeled subsequence, $\left(m^{\varepsilon}, u^{\varepsilon}\right) \rightharpoonup(m, u)$ weakly in $L^{1}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right)$ as $\varepsilon=\left(\varepsilon_{1}, \varepsilon_{2}\right) \rightarrow 0^{+}$.

From (4.4) and Assumption (g2), it follows that there exists $m \in L^{1}\left(\mathbb{T}^{d}\right)$ such that, up to a not relabeled subsequence, $m^{\varepsilon} \rightharpoonup m$ weakly in $L^{1}\left(\mathbb{T}^{d}\right)$ as $\varepsilon \rightarrow 0^{+}$. Because $m^{\varepsilon}$ is nonnegative in $\mathbb{T}^{d}$, we conclude that $\boldsymbol{m} \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$.

Next, we observe that $\left(u^{\varepsilon}\right)_{\varepsilon}$ is a bounded sequence in $W^{1, \gamma}\left(\mathbb{T}^{d}\right)$ by the Poincaré-Wirtinger inequality, (4.1), and (4.21). Therefore, there is $u \in W^{1, \gamma}\left(\mathbb{T}^{d}\right)$ satisfying, up to a not relabeled further subsequence, $u^{\varepsilon} \rightharpoonup u$ weakly in $W^{1, \gamma}\left(\mathbb{T}^{d}\right)$ as $\varepsilon \rightarrow 0^{+}$.

Finally, integrating the second equation in (4.11) over $\mathbb{T}^{d}$ and letting $\varepsilon \rightarrow 0^{+}$, we conclude that $\int_{\mathbb{T}^{d}} m \mathrm{~d} x=1$. This completes Step 1.

Step 2. In this step, we show that $(m, u)$ satisfies (1.6).

We apply a variation of Minty's device (see, for instance, [17, 39]). Let $(\eta, v) \in C^{\infty}\left(\mathbb{T}^{d}\right) \times$ $C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\inf _{\mathbb{T}^{d}} \eta>0$. By Lemma 4.1 and (4.11), we have

$$
\begin{aligned}
0 \leqslant & \left(A^{\varepsilon}\left[\begin{array}{l}
\eta \\
v
\end{array}\right],\left[\begin{array}{l}
\eta \\
v
\end{array}\right]-\left[\begin{array}{c}
m^{\varepsilon} \\
u^{\varepsilon}
\end{array}\right]\right)_{L^{2}\left(\mathbb{T}^{d}\right) \times L^{2}\left(\mathbb{T}^{d}\right)} \\
= & \int_{\mathbb{T}^{d}}\left(-v-H\left(x, D v, D^{2} v, \eta, h(\boldsymbol{\eta})\right)\right)\left(\eta-m^{\varepsilon}\right) \mathrm{d} x \\
& +\int_{\mathbb{T}^{d}}\left(\varepsilon_{1}\left(\eta+\Delta^{2 p} \eta\right)+\varepsilon_{2}(\eta-\Delta \eta)+\beta_{\varepsilon_{1}}(\eta)\right)\left(\eta-m^{\varepsilon}\right) \mathrm{d} x \\
& +\int_{\mathbb{T}^{d}}\left(\eta-\operatorname{div}\left(\eta D_{p} H\left(x, D v, D^{2} v, \eta, h(\boldsymbol{\eta})\right)\right)\right. \\
& \left.\quad+\left(\eta D_{M_{i j}} H\left(x, D v, D^{2} v, \eta, h(\boldsymbol{\eta})\right)\right)_{x_{i} x_{j}}-1\right)\left(v-u^{\varepsilon}\right) \mathrm{d} x \\
& +\int_{\mathbb{T}^{d}}\left(\varepsilon_{1}\left(v+\Delta^{2 p} v\right)+\varepsilon_{2}(v-\Delta v)\right)\left(v-u^{\varepsilon}\right) \mathrm{d} x
\end{aligned}
$$

Letting $\varepsilon \rightarrow 0^{+}$and using (H2), (h1), the convergence results from Step 1, and the convergence $\varepsilon_{1}\left(\eta+\Delta^{2 p} \eta\right)+\varepsilon_{2}(\eta-\Delta \eta)+\beta_{\varepsilon_{1}}(\eta)+\varepsilon_{1}\left(v+\Delta^{2 p} v\right)+\varepsilon_{2}(v-\Delta v) \rightarrow 0$ uniformly in $\mathbb{T}^{d}$ as $\varepsilon \rightarrow 0^{+}$, which holds since $\inf _{\mathbb{T}^{d}} \eta>0$, we obtain

$$
\begin{align*}
\int_{\mathbb{T}^{d}}[ & \left.-v-H\left(x, D v, D^{2} v, \eta, h(\boldsymbol{\eta})\right)\right](\eta-m) \mathrm{d} x \\
& +\int_{\mathbb{T}^{d}}\left[\eta-\operatorname{div}\left(\eta D_{p} H\left(x, D v, D^{2} v, \eta, h(\boldsymbol{\eta})\right)\right.\right.  \tag{4.12}\\
& \quad+\left(\eta D_{M_{i j}} H\left(x, D v, D^{2} v, \eta, h(\boldsymbol{\eta})\right)_{x_{i} x_{j}}-1\right](v-u) \mathrm{d} x \geqslant 0
\end{align*}
$$

This concludes the proof in the $\mathbb{E}=\mathbb{R}^{+}$case because $(m, u) \in L^{1}\left(\mathbb{T}^{d}\right) \times L^{1}\left(\mathbb{T}^{d}\right)$; thus, the left-hand side of (4.12) coincides with

$$
\left\langle\left[\begin{array}{l}
\eta \\
v
\end{array}\right]-\left[\begin{array}{l}
m \\
u
\end{array}\right], A\left[\begin{array}{l}
\eta \\
v
\end{array}\right]\right\rangle_{\mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right) \times \mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right), C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)} .
$$

In the $\mathbb{E}=\mathbb{R}_{0}^{+}$case, we proceed as follows. Let $\tilde{\eta} \in C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\tilde{\eta} \geqslant 0$ and for $\delta>0$, set $\eta_{\delta}:=\tilde{\eta}+\delta$; using (4.12) with $\eta$ replaced by $\eta_{\delta}$ and letting $\delta \rightarrow 0^{+}$, by (H2), (2.1), and (3.4), we conclude that (4.12) holds for all $(\eta, v) \in C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\eta \geqslant 0$. This concludes Step 2, as well as the proof of Theorem 1.1

We end this section with a corollary to the previous results.
Corollary 4.3. Suppose that Assumptions (h1), (g1), (g2), and (H1) -(H3) are satisfied. Then, there exist $(\boldsymbol{m}, u) \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right)$ with $\int_{\mathbb{T}^{d}} m \mathrm{~d} x=1$ and a sequence of positive numbers, $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$, convergent to zero such that:
(i) $(m, u)$ is a weak solution to (1.1).
(ii) $(m, u)$ is the weak limit in $L^{1}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right)$ of a sequence $\left(m_{\varepsilon_{j}}, u_{\varepsilon_{j}}\right)_{j \in \mathbb{N}} \subset D\left(A^{\varepsilon_{j}}\right)$, where each $\left(m_{\varepsilon_{j}}, u_{\varepsilon_{j}}\right)$ satisfies (3.1) with $\varepsilon=\varepsilon_{j}$.

Proof. We established (ii) in the first step of the proof of Theorem 1.1 and (i) in the second step of the same proof.

## 5. Properties of weak solutions

Here, we examine some properties of weak solutions. To illustrate our methods, we consider the degenerate diffusion case, which corresponds to Hamiltonians of the form (5.1) below. We show that the weak solutions given by Theorem 1.1 are subsolutions of the first equation in (1.1), the Hamilton-Jacobi equation. Next, we recover an analog to the second equation in (1.1), the Fokker-Planck equation. Finally, we show that in the set where $m$ is positive, the Hamiton-Jacobi equation in (1.1) holds in a relaxed sense.

The main assumption of this section is the following.
(H4) The Hamiltonian, $H$, can be written as

$$
\begin{equation*}
H(x, p, M, m, \theta)=H_{0}(x, p, m, \theta)-\sum_{i, j=1}^{d} a_{i j}(x) M_{i j} \tag{5.1}
\end{equation*}
$$

where $H_{0}$ is a real-valued $C^{\infty}$ function and $\left(a_{i j}\right)_{1 \leqslant i, j \leqslant d}$ is a $C^{\infty}$ matrix-valued function.
(a) Let $\gamma$ be as in Assumption (H1). For all $\varphi \in C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\varphi \geqslant 0$, the functional

$$
(m, u) \in L^{1}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right) \mapsto \int_{\mathbb{T}^{d}} \varphi H_{0}(x, D u, m, h(\boldsymbol{m})) \mathrm{d} x
$$

is sequentially weakly lower semicontinuous in $L^{1}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right)$.
(b) Let $\gamma>1$ be as in Assumption (H1) with $\tau=0$ and $g$ satisfying (g1a). There exist constants, $C_{5}>0$ and $\alpha>0$, such that, for all $(x, p, m, \theta) \in \mathbb{T}^{d} \times \mathbb{R}^{d} \times \mathbb{E} \times \mathbb{R}$,
(i) $\left|D_{p} H_{0}(x, p, m, \theta)\right| \leqslant C_{5}\left(1+|p|^{\gamma-1}\right)$;
(ii) $g_{1}(x, m, \theta) \geqslant \frac{1}{C_{5}} m^{\alpha}-C_{5}$.
(c) Let $\gamma>1$ and $\alpha>0$ be as in Assumption (H4b). Let $L_{0}$ be the Lagrangian associated with $H_{0}$, defined for $(x, v, m, \theta) \in \mathbb{T}^{d} \times \mathbb{R}^{d} \times \mathbb{E} \times \mathbb{R}$ by

$$
L_{0}(x, v, m, \theta)=\sup _{p \in \mathbb{R}^{d}}\left\{-p \cdot v-H_{0}(x, p, m, \theta)\right\}
$$

For all $(x, m, \theta) \in \mathbb{T}^{d} \times \mathbb{E} \times \mathbb{R}$, the map $p \mapsto H_{0}(x, p, m, \theta)$ is convex; moreover, the functional
$(m, J) \in L^{\alpha+1}\left(\mathbb{T}^{d}\right) \times L^{\frac{(\alpha+1) \gamma}{(\alpha+1) \gamma-\alpha}}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right) \mapsto \int_{\mathbb{T}^{d}} m L_{0}\left(x, \frac{J}{m}, m, h(\boldsymbol{m})\right) \mathrm{d} x$
is sequentially weakly lower semicontinuous in $L^{\alpha+1}\left(\mathbb{T}^{d}\right) \times L^{\frac{(\alpha+1) \gamma}{(\alpha+1) \gamma-\alpha}}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)$.
We observe that the preceding assumption holds in the case given by (1.3) with quadratic Hamiltonian for $\tau=0$, and with $g(m, \theta)=m^{\alpha}+\theta$, where $\alpha>0$. In general, Assumptions (H4a) and (H4c) depend on convexity properties of the maps $(p, m, \theta) \in \mathbb{R}^{d} \times \mathbb{E} \times \mathbb{R} \mapsto H_{0}(x, p, m, \theta)$ and $(v, m, \theta) \in \mathbb{R}^{d} \times \mathbb{E} \times \mathbb{R} \mapsto m L_{0}\left(x, \frac{v}{m}, m, \theta\right)$, respectively.
5.1. Subsolution property. We begin by examining the Hamilton-Jacobi equation in (1.1). As previously stated, we suppose that $H$ satisfies (5.1). We say that $(\boldsymbol{m}, u) \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right) \times$ $W^{1, \gamma}\left(\mathbb{T}^{d}\right)$ is a subsolution in the sense of distributions (i.e., in $\left.\mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right)\right)$ of the Hamilton-Jacobi equation in (1.1) if for all $\varphi \in C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\varphi \geqslant 0$, we have

$$
\begin{equation*}
\int_{\mathbb{T}^{d}}\left[\varphi(x)\left(u+H_{0}(x, D u, m, h(\boldsymbol{m}))\right)-\left(\varphi(x) a_{i j}(x)\right)_{x_{i} x_{j}} u\right] \mathrm{d} x \leqslant 0 \tag{5.2}
\end{equation*}
$$

Next, we show that the weak solutions given by Theorem 1.1 are subsolutions in $\mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right)$.
Proposition 5.1. Suppose that Assumptions (h1), (g1), (g2), (H1)-H3), and (H4a) hold. Let $\left(m^{\varepsilon}, u^{\varepsilon}\right) \in D\left(A^{\varepsilon}\right)$ be a solution of (3.1). Let $(m, u) \in L^{1}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right)$ with $m \geqslant 0$ and $\int_{\mathbb{T}^{d}} m \mathrm{~d} x=1$ be a weak solution of (1.1) obtained as a weak sublimit of $\left(m^{\varepsilon}, u^{\varepsilon}\right)_{\varepsilon}$ in $L^{1}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right)$. Then, $(m, u)$ is a subsolution in the sense of distributions of the HamiltonJacobi equation in (1.1).

Remark 5.2. We observe that there exist $\left(m^{\varepsilon}, u^{\varepsilon}\right)$ and $(m, u)$ as in the statement of Proposition 5.1 in view of Corollary 3.11 with $\mu=0$ and Corollary 4.3.

Proof. To simplify the notation, we do not distinguish $\left(m^{\varepsilon}, u^{\varepsilon}\right)_{\varepsilon}$ from its subsequence that converges to $(m, u)$.

Take $\varphi \in C^{\infty}\left(\mathbb{T}^{d}\right), \varphi \geqslant 0$. Multiplying the first equation in (3.1) by $\varphi$ and integrating over $\mathbb{T}^{d}$ gives

$$
\begin{aligned}
& \int_{\mathbb{T}^{d}}\left[\varphi\left(u^{\varepsilon}+H_{0}\left(x, D u^{\varepsilon}, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right)\right)-\left(\varphi a_{i j}\right)_{x_{i} x_{j}} u^{\varepsilon}\right] \mathrm{d} x-\varepsilon_{1} \int_{\mathbb{T}^{d}} m^{\varepsilon}\left(\varphi+\Delta^{2 p} \varphi\right) \mathrm{d} x \\
& \quad-\varepsilon_{2} \int_{\mathbb{T}^{d}} m^{\varepsilon}(\varphi-\Delta \varphi) \mathrm{d} x=\int_{\mathbb{T}^{d}} \beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right) \varphi \mathrm{d} x \leqslant 0
\end{aligned}
$$

since $\beta_{\varepsilon_{1}}(\cdot) \leqslant 0$.
Because $m^{\varepsilon} \rightharpoonup m$ in $L^{1}\left(\mathbb{T}^{d}\right)$, we have

$$
\lim _{\varepsilon \rightarrow 0^{+}}\left[\varepsilon_{1} \int_{\mathbb{T}^{d}} m^{\varepsilon}\left(\varphi+\Delta^{2 p} \varphi\right) \mathrm{d} x+\varepsilon_{2} \int_{\mathbb{T}^{d}} m^{\varepsilon}(\varphi-\Delta \varphi) \mathrm{d} x\right]=0
$$

Next, since $u^{\varepsilon} \rightarrow u$ in $L^{\gamma}\left(\mathbb{T}^{d}\right)$, we have

$$
\lim _{\varepsilon \rightarrow 0^{+}} \int_{\mathbb{T}^{d}}\left[\varphi u^{\varepsilon}-\left(\varphi a_{i j}\right)_{x_{i} x_{j}} u^{\varepsilon}\right] \mathrm{d} x=\int_{\mathbb{T}^{d}}\left[\varphi u-\left(\varphi a_{i j}\right)_{x_{i} x_{j}} u\right] \mathrm{d} x .
$$

Finally, Assumption (H4a), together with the weak convergences $m^{\varepsilon} \rightharpoonup m$ in $L^{1}\left(\mathbb{T}^{d}\right)$ and $u^{\varepsilon} \rightharpoonup u$ in $W^{1, \gamma}\left(\mathbb{T}^{d}\right)$, gives

$$
\liminf _{\varepsilon \rightarrow 0^{+}} \int_{\mathbb{T}^{d}} \varphi H_{0}\left(x, D u^{\varepsilon}, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right) \mathrm{d} x \geqslant \int_{\mathbb{T}^{d}} \varphi H_{0}(x, D u, m, h(\boldsymbol{m})) \mathrm{d} x
$$

Consequently, we have (5.2).
5.2. Fokker-Planck equation. To pursue our analysis further, we consider the case without congestion, $\tau=0$, a natural growth condition on $D_{p} H_{0}$, and a power-like growth in $g_{1}$. This case corresponds to Assumption (H4b). We begin by examining the integrability of the drift in the Fokker-Planck equation.

Proposition 5.3. Suppose that Assumptions (g1), (H1), and H4b) hold. Let $q=\frac{(\alpha+1) \gamma}{(\alpha+1) \gamma-\alpha}>$ 1. Then, there exists a positive constant, $C$, such that, for any $\varepsilon \in(0,1)^{2}$ and any solution $\left(u^{\varepsilon}, m^{\varepsilon}\right) \in D\left(A^{\varepsilon}\right)$ to (3.1), we have

$$
\left\|m^{\varepsilon}\right\|_{L^{\alpha+1}\left(\mathbb{T}^{d}\right)}+\left\|m^{\varepsilon} D_{p} H\left(\cdot, D u^{\varepsilon}, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right)\right\|_{L^{q}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)} \leqslant C .
$$

Proof. By (4.4), Assumption (H4(b)iil, and Assumption (g1c) with $\delta=1, m^{\varepsilon}$ is bounded in $L^{\alpha+1}\left(\mathbb{T}^{d}\right)$ independently of $\varepsilon$. In particular, because $q<\alpha+1,\left(m^{\varepsilon}\right)_{\varepsilon}$ is bounded in $L^{q}\left(\mathbb{T}^{d}\right)$. Then, in view of Assumptions (5.1) and (H4(b)i), it suffices to prove that $m^{\varepsilon}\left|D u^{\varepsilon}\right|^{\gamma-1}$ is bounded in $L^{q}\left(\mathbb{T}^{d}\right)$ independently of $\varepsilon$.

Set $s=\gamma(\alpha+1)$ and $r=\frac{\gamma}{\gamma-1}$. Observe that $m^{\varepsilon}\left|D u^{\varepsilon}\right|^{\gamma-1}=\left[\left(m^{\varepsilon}\right)^{\alpha+1}\right]^{\frac{1}{s}}\left(\left|D u^{\varepsilon}\right|^{\gamma} m^{\varepsilon}\right)^{\frac{1}{r}}$ and $\frac{1}{q}=\frac{1}{r}+\frac{1}{s}$. As we proved before, $\left(m^{\varepsilon}\right)^{\alpha+1}$ is bounded in $L^{1}\left(\mathbb{T}^{d}\right)$ independently of $\varepsilon$. Moreover, from Lemma 3.3 with $\tau=0$, we have that $\left|D u^{\varepsilon}\right|^{\gamma} m^{\varepsilon}$ is bounded in $L^{1}\left(\mathbb{T}^{d}\right)$ independently of $\varepsilon$. The result then follows by Hölder's inequality.

Proposition 5.4. Suppose that Assumptions (h1), (g1), (g2), (H1) (H3), and (H4b) hold. Let $\left(u^{\varepsilon}, m^{\varepsilon}\right) \in D\left(A^{\varepsilon}\right)$ be a solution of (3.1). Let $(m, u) \in L^{1}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right)$ with $m \geqslant 0$ and $\int_{\mathbb{T}^{d}} m \mathrm{~d} x=1$ be a weak solution to (1.1) obtained as a weak sublimit of $\left(m^{\varepsilon}, u^{\varepsilon}\right)_{\varepsilon}$ in $L^{1}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right)$. Let $q=\frac{(\alpha+1) \gamma}{(\alpha+1) \gamma-\alpha}$ and set $J^{\varepsilon}=m^{\varepsilon} D_{p} H\left(x, D u^{\varepsilon}, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right)$. Then, there exists $J \in L^{q}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)$ such that, up to a not relabeled subsequence, $J^{\varepsilon} \rightharpoonup J$ in $L^{q}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)$. Moreover,

$$
\begin{equation*}
m-\operatorname{div}(J)-\left(a_{i j}(x) m\right)_{x_{i} x_{j}}=1 \tag{5.3}
\end{equation*}
$$

in $\mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right)$.
Proof. By Proposition 5.3, up to a not relabeled subsequence, we have $J^{\varepsilon} \rightharpoonup J$ in $L^{q}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)$ for some $J \in L^{q}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)$. The result follows by convergence in $\mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right)$ together with the second equation in (3.1).
5.3. Supersolution property. Here, we revisit the Hamilton-Jacobi equation in (1.1). From the results in Subsection 5.1. we know that weak solutions are subsolutions. Next, we prove the converse result; that is, weak solutions solve, in a relaxed sense, the Hamilton-Jacobi in the set where $m$ is positive.

Proposition 5.5. Suppose that Assumptions (h1), (g1), (g2), (H1) (H3), H4b), and (H4c) hold. Let $q=\frac{(\alpha+1) \gamma}{(\alpha+1) \gamma-\alpha}$. Let $\left(u^{\varepsilon}, m^{\varepsilon}\right),(u, m)$, and $J$ be as in Proposition 5.4. Then, for all
$\varphi \in C^{\infty}\left(\mathbb{T}^{d}\right)$, we have

$$
\begin{equation*}
\int_{\mathbb{T}^{d}}\left[J D \varphi-H_{0}(x, D \varphi, m, h(\boldsymbol{m})) m-u\right] \mathrm{d} x \leqslant 0 . \tag{5.4}
\end{equation*}
$$

Remark 5.6. We regard (5.4) as a weak version of

$$
\begin{equation*}
\int_{\mathbb{T}^{d}} m\left[-u-H_{0}(x, D u, m, h(\boldsymbol{m}))+a_{i j}(x) u_{x_{i} x_{j}}\right] \mathrm{d} x \leqslant 0 . \tag{5.5}
\end{equation*}
$$

To see this, we observe that if $u$ is regular enough, then (5.4) for $\varphi=u$ yields

$$
\begin{equation*}
\int_{\mathbb{T}^{d}}\left[J D u-H_{0}(x, D u, m, h(\boldsymbol{m})) m-u\right] \mathrm{d} x \leqslant 0 \tag{5.6}
\end{equation*}
$$

Then, due to (5.3), we get

$$
\int_{\mathbb{T}^{d}}(J D u-u) \mathrm{d} x=\int_{\mathbb{T}^{d}} m\left(-u+a_{i j}(x) u_{x_{i} x_{j}}\right) \mathrm{d} x .
$$

Combining the previous identity with (5.6) gives (5.5).
Proof of Proposition 5.5. To simplify the notation, we do not relabel the weakly convergent subsequence of $\left(m^{\varepsilon}, u^{\varepsilon}, J^{\varepsilon}\right)_{\varepsilon}$ to $(m, u, J)$ in $L^{1}\left(\mathbb{T}^{d}\right) \times W^{1, \gamma}\left(\mathbb{T}^{d}\right) \times L^{q}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)$. Note that in view of Proposition 5.3, we also have $m^{\varepsilon} \rightharpoonup m$ weakly in $L^{\alpha+1}\left(\mathbb{T}^{d}\right)$.

Next, we observe that

$$
\begin{aligned}
& \int_{\mathbb{T}^{d}} m^{\varepsilon}\left[-u^{\varepsilon}-H_{0}\left(x, D u^{\varepsilon}, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right)+a_{i j}(x) u_{x_{i} x_{j}}^{\varepsilon}\right] \mathrm{d} x \\
& +\int_{\mathbb{T}^{d}} m^{\varepsilon}\left[\varepsilon_{1}\left(m^{\varepsilon}+\Delta^{2 p} m^{\varepsilon}\right)+\varepsilon_{2}\left(m^{\varepsilon}-\Delta m^{\varepsilon}\right)+\beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right)\right] \mathrm{d} x \\
& +\int_{\mathbb{T}^{d}} u^{\varepsilon}\left[m^{\varepsilon}-\operatorname{div}\left(m^{\varepsilon} D_{p} H_{0}\right)-\left(m^{\varepsilon} a_{i j}\right)_{x_{i} x_{j}}-1+\varepsilon_{1}\left(u^{\varepsilon}+\Delta^{2 p} u^{\varepsilon}\right)+\varepsilon_{2}\left(u^{\varepsilon}-\Delta u^{\varepsilon}\right)\right] \mathrm{d} x=0 .
\end{aligned}
$$

Integrating the terms multiplied by $\varepsilon_{1}$ and $\varepsilon_{2}$ by parts, and taking into account their sign, we conclude that

$$
\begin{aligned}
& \int_{\mathbb{T}^{d}} m^{\varepsilon}\left[-u^{\varepsilon}-H_{0}\left(x, D u^{\varepsilon}, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right)+a_{i j}(x) u_{x_{i} x_{j}}^{\varepsilon}+\beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right)\right] \mathrm{d} x \\
& +\int_{\mathbb{T}^{d}} u^{\varepsilon}\left[m^{\varepsilon}-\operatorname{div}\left(m^{\varepsilon} D_{p} H_{0}\right)-\left(m^{\varepsilon} a_{i j}\right)_{x_{i} x_{j}}-1\right] \mathrm{d} x \leqslant 0 .
\end{aligned}
$$

By Proposition 4.2, we have that $\left\|\beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right)\right\|_{L^{1}\left(\mathbb{T}^{d}\right)}$ is bounded independently of $\varepsilon$. Accordingly, we find

$$
\lim _{\varepsilon \rightarrow 0^{+}} \int_{\mathbb{T}^{d}} m^{\varepsilon} \beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right) \mathrm{d} x=0
$$

because $\left|m^{\varepsilon} \beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right)\right| \leqslant \varepsilon_{1}\left|\beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right)\right|$ by definition of $\beta_{\varepsilon_{1}}(\cdot)$. Hence,

$$
\liminf _{\varepsilon \rightarrow 0^{+}}\left(\int_{\mathbb{T}^{d}} m^{\varepsilon}\left(-H_{0}\left(x, D u^{\varepsilon}, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right)\right) \mathrm{d} x+\int_{\mathbb{T}^{d}} u^{\varepsilon}\left[-\operatorname{div}\left(m^{\varepsilon} D_{p} H_{0}\right)-1\right] \mathrm{d} x\right) \leqslant 0
$$

Recalling that $J^{\varepsilon}=m^{\varepsilon} D_{p} H_{0}\left(x, D u^{\varepsilon}, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right)$, the definition of the Legendre transform and Assumption ( H 4 c ) give

$$
\liminf _{\varepsilon \rightarrow 0^{+}} \int_{\mathbb{T}^{d}} m^{\varepsilon} L_{0}\left(x, \frac{-J^{\varepsilon}}{m^{\varepsilon}}, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right) \mathrm{d} x \leqslant \lim _{\varepsilon \rightarrow 0^{+}} \int_{\mathbb{T}^{d}} u^{\varepsilon} \mathrm{d} x=\int_{\mathbb{T}^{d}} u \mathrm{~d} x
$$

where we used the convergence $u^{\varepsilon} \rightarrow u$ in $L^{\gamma}\left(\mathbb{T}^{d}\right)$. Then, because $-J^{\varepsilon} \rightharpoonup-J$ in $L^{q}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)$ and $m^{\varepsilon} \rightharpoonup m$ in $L^{\alpha+1}\left(\mathbb{T}^{d}\right)$, Assumption ( $(\underline{\mathrm{H}} \mathrm{C})$ implies that

$$
\begin{equation*}
\int_{\mathbb{T}^{d}} m L_{0}\left(x, \frac{-J}{m}, m, h(\boldsymbol{m})\right) \mathrm{d} x \leqslant \int_{\mathbb{T}^{d}} u \mathrm{~d} x . \tag{5.7}
\end{equation*}
$$

On the other hand, using the definition of the Legendre transform again, we have

$$
\begin{equation*}
m L_{0}\left(x, \frac{-J}{m}, m, h(\boldsymbol{m})\right) \geqslant J D \varphi-H_{0}(x, D \varphi, m, h(\boldsymbol{m})) m \tag{5.8}
\end{equation*}
$$

for any $\varphi \in C^{\infty}\left(\mathbb{T}^{d}\right)$. Consequently, (5.7) and (5.8) yield (5.4).

## 6. Improved Regularity

In this section, we consider the quadratic case for either first-order problems or elliptic problems with constant coefficients, encoded in the Assumption (H5) below. We get improved regularity for $m$ and $u$, including higher integrability and Sobolev estimates for $m$. Further, we show that the Fokker-Plank equation holds pointwise, not just in the sense of distributions. Finally, we prove that the Hamilton-Jacobi equation holds pointwise in the set where $m$ is positive.

The main assumption of this section is the following.
(H5) The Hamiltonian, $H$, is of the form

$$
\begin{equation*}
H(x, p, M, m, \theta)=\frac{|p|^{2}}{2}-V(x, m, \theta)-\sigma^{2} \sum_{i=1}^{d} M_{i i} \tag{6.1}
\end{equation*}
$$

where $\sigma \in \mathbb{R}$ and $V: \mathbb{T}^{d} \times \mathbb{E} \times \mathbb{R} \rightarrow \mathbb{R}$ is a $C^{\infty}$ function.
(a) There exist constants, $c_{1}>0, c_{2}>0$, and $c_{3}>0$, and a continuous function, $g: \mathbb{T}^{d} \times \mathbb{E} \times \mathbb{R} \rightarrow \mathbb{R}$, such that:
(i) For all $(x, m, \theta) \in \mathbb{T}^{d} \times \mathbb{E} \times \mathbb{R}$,

$$
V(x, m, \theta) \geqslant c_{1} g(x, m, \theta)-\frac{1}{c_{2}}
$$

(ii) For all $\boldsymbol{m} \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$,

$$
\int_{\mathbb{T}^{d}} V(x, m, h(\boldsymbol{m})) \mathrm{d} x \leqslant c_{2}+c_{3} \int_{\mathbb{T}^{d}} g(x, m, h(\boldsymbol{m})) \mathrm{d} x .
$$

(iii) For all $\eta, \bar{\eta} \in C^{\infty}\left(\mathbb{T}^{d} ; \mathbb{E}\right)$,

$$
\int_{\mathbb{T}^{d}}[V(x, \eta, h(\boldsymbol{\eta}))-V(x, \bar{\eta}, h(\overline{\boldsymbol{\eta}}))](\eta-\bar{\eta}) \mathrm{d} x \geqslant 0
$$

(b) There exist constants, $\alpha>0$ and $\kappa_{1}>0$, such that:
(i) For all $(x, m, \theta) \in \mathbb{T}^{d} \times \mathbb{E} \times \mathbb{R}$, the function $g$ in (H5a) satisfies (g1a) with

$$
g_{1}(x, m, \theta) \geqslant \frac{1}{\kappa_{1}} m^{\alpha}-\kappa_{1} .
$$

(ii) For all $\eta \in C^{\infty}\left(\mathbb{T}^{d} ; \mathbb{E}\right)$,

$$
\int_{\mathbb{T}^{d}}(V(x, \eta, h(\boldsymbol{\eta})))_{x_{i}} \eta_{x_{i}} \mathrm{~d} x \geqslant \frac{1}{\kappa_{1}} \int_{\mathbb{T}^{d}} \eta^{\alpha-1}|D \eta|^{2} \mathrm{~d} x-\kappa_{1} .
$$

(c) Let $\alpha>0$ and $\kappa_{1}>0$ be as in (H5b). For all $(x, m, \theta) \in \mathbb{T}^{d} \times \mathbb{E} \times \mathbb{R}$, we have

$$
V(x, m, \theta) \leqslant \kappa_{1}\left(1+m^{\alpha}+|\theta|\right) .
$$

We observe that a Hamiltonian satisfying (H5a) is a particular case of a Hamiltonian for which (H1)-(H3) hold (with $\gamma=2$ and $\tau=0$ ).

In some cases, we will also require the operator $h$ to satisfy the following condition:
(h2) There exists a positive constant, $C$, such that, for all $\boldsymbol{m}, \overline{\boldsymbol{m}} \in \mathcal{M}_{a c}\left(\mathbb{T}^{d}\right)$, we have

$$
\|h(\boldsymbol{m})-h(\overline{\boldsymbol{m}})\|_{L^{1}\left(\mathbb{T}^{d}\right)} \leqslant C\|m-\bar{m}\|_{L^{1}\left(\mathbb{T}^{d}\right)}
$$

Examples of operators satisfying the Assumptions (h1) and (h2) are those in (1.3).
For Hamiltonians as in (6.1), (1.1) takes the form

$$
\left\{\begin{array}{l}
-u-\frac{|D u|^{2}}{2}+V(x, m, h(\boldsymbol{m}))+\sigma^{2} \Delta u=0  \tag{6.2}\\
m-\operatorname{div}(m D u)-\sigma^{2} \Delta m-1=0
\end{array}\right.
$$

First, we consider the regularized problem (3.1) or, equivalently, (3.5) with $\mu=0$ and prove additional estimates that are uniform in $\varepsilon \in(0,1)^{2}$.

Lemma 6.1. Suppose that Assumptions (g1), H5a), and H5b hold. Fix $\varepsilon \in(0,1)^{2}$. Let $\left(m^{\varepsilon}, u^{\varepsilon}\right) \in C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\inf _{\mathbb{T}^{d}} m^{\varepsilon}>0$ in $\mathbb{T}^{d}$ be a solution of (3.1). Then, there exists a positive constant, $C$, independent of $\varepsilon$ such that

$$
\begin{aligned}
& \left\|u^{\varepsilon}\right\|_{W^{1,2}\left(\mathbb{T}^{d}\right)}+\left\|\left(m^{\varepsilon}\right)^{\frac{\alpha+1}{2}}\right\|_{W^{1,2}\left(\mathbb{T}^{d}\right)}+\left\|\beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right)\right\|_{L^{1}\left(\mathbb{T}^{d}\right)}+\left\|\left(m^{\varepsilon}\right)^{\frac{1}{2}} D u^{\varepsilon}\right\|_{L^{2}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)} \\
& +\left\|\left(m^{\varepsilon}\right)^{\frac{1}{2}} D^{2} u^{\varepsilon}\right\|_{L^{2}\left(\mathbb{T}^{d} ; \mathbb{M}_{d \times d}\right)}+\left\|\left(m^{\varepsilon}\right)^{\frac{\alpha+1}{2}} D u^{\varepsilon}\right\|_{W^{1,1}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)}+\left\|\sqrt{\varepsilon_{1}} u^{\varepsilon}\right\|_{W^{2 p+1,2}\left(\mathbb{T}^{d}\right)} \\
& +\left\|\sqrt{\varepsilon_{1}} m^{\varepsilon}\right\|_{W^{2 p+1,2}\left(\mathbb{T}^{d}\right)}+\left\|\sqrt{\varepsilon_{2}} u^{\varepsilon}\right\|_{W^{2,2}\left(\mathbb{T}^{d}\right)}+\left\|\sqrt{\varepsilon_{2}} m^{\varepsilon}\right\|_{W^{2,2}\left(\mathbb{T}^{d}\right)} \leqslant C .
\end{aligned}
$$

Proof. Here, to simplify the notation, we write $\boldsymbol{m}, m$, and $u$ in place of $\boldsymbol{m}^{\varepsilon}, m^{\varepsilon}$, and $u^{\varepsilon}$, respectively. Moreover, $C$ represents a positive constant independent of $\varepsilon$ and whose value may change from one line to another. Also, we do not use the Einstein convention on repeated indices, and we specify all the sums.

Since $(m, u)$ is a solution to (3.1) and $H$ is given by (6.1), we have

$$
\begin{align*}
& -u-\frac{|D u|^{2}}{2}+V(x, m, h(\boldsymbol{m}))+\sigma^{2} \Delta u+\varepsilon_{1}\left(m+\Delta^{2 p} m\right)+\beta_{\varepsilon_{1}}(m)+\varepsilon_{2}(m-\Delta m)=0  \tag{6.3}\\
& m-\operatorname{div}(m D u)-\sigma^{2} \Delta m-1+\varepsilon_{1}\left(u+\Delta^{2 p} u\right)+\varepsilon_{2}(u-\Delta u)=0 \tag{6.4}
\end{align*}
$$

pointwise in $\mathbb{T}^{d}$.
Next, we recall that under Assumption (H5a), the Hamiltonian $H$ in (6.1) satisfies Assumption (H1) with $\gamma=2, \tau=0$, and $g$ given by (H5a). Because $g$ also satisfies (g1), Proposition 4.2 and Lemma 3.3 with $\gamma=2, \tau=0$, and $\mu=0$ give the bounds

$$
\begin{gather*}
\|D u\|_{L^{2}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)}+\left|\int_{\mathbb{T}^{d}} u \mathrm{~d} x\right| \leqslant C  \tag{6.5}\\
\int_{\mathbb{T}^{d}} m g_{1}(x, m, h(\boldsymbol{m})) \mathrm{d} x \leqslant C \tag{6.6}
\end{gather*}
$$

and

$$
\left\|\beta_{\varepsilon_{1}}(m)\right\|_{L^{1}\left(\mathbb{T}^{d}\right)}+\left\|m^{\frac{1}{2}} D u\right\|_{L^{2}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)} \leqslant C
$$

The estimate (6.5) together with the Poincaré-Wirtinger inequality yields

$$
\begin{equation*}
\|u\|_{W^{1,2}\left(\mathbb{T}^{d}\right)} \leqslant C \tag{6.7}
\end{equation*}
$$

Combining (6.6), H5(b)i], and (g1c) with $\delta=1$, we get

$$
\begin{equation*}
\left\|m^{\frac{\alpha+1}{2}}\right\|_{L^{2}\left(\mathbb{T}^{d}\right)} \leqslant C \tag{6.8}
\end{equation*}
$$

After differentiating (6.3) twice with respect to $x_{j}, j \in\{1, \ldots, d\}$, multiplying by $m$, and integrating over $\mathbb{T}^{d}$, the resulting equality becomes

$$
\begin{aligned}
- & \int_{\mathbb{T}^{d}} u_{x_{j} x_{j}} m \mathrm{~d} x-\int_{\mathbb{T}^{d}} \sum_{i=1}^{d}\left|u_{x_{i} x_{j}}\right|^{2} m \mathrm{~d} x+\int_{\mathbb{T}^{d}} \sum_{i=1}^{d}\left(m u_{x_{i}}\right)_{x_{i}} u_{x_{j} x_{j}} \mathrm{~d} x \\
& -\int_{\mathbb{T}^{d}}\left(V(x, m, h(\boldsymbol{m}))_{x_{j}} m_{x_{j}} \mathrm{~d} x\right. \\
& +\sigma^{2} \int_{\mathbb{T}^{d}} \Delta u m_{x_{j} x_{j}} \mathrm{~d} x-\varepsilon_{1} \int_{\mathbb{T}^{d}}\left|m_{x_{j}}\right|^{2} \mathrm{~d} x-\varepsilon_{1} \int_{\mathbb{T}^{d}}\left|\Delta^{p} m_{x_{j}}\right|^{2} \mathrm{~d} x \\
& -\int_{\mathbb{T}^{d}} \beta_{\varepsilon_{1}}^{\prime}(m)\left|m_{x_{j}}\right|^{2} \mathrm{~d} x-\varepsilon_{2} \int_{\mathbb{T}^{d}}\left|m_{x_{j}}\right|^{2} \mathrm{~d} x-\varepsilon_{2} \int_{\mathbb{T}^{d}}\left|D m_{x_{j}}\right|^{2} \mathrm{~d} x=0
\end{aligned}
$$

using integration by parts. In view of (6.4), we have that

$$
\sum_{i=1}^{d}\left(m u_{x_{i}}\right)_{x_{i}}=m-\sigma^{2} \Delta m-1+\varepsilon_{1} u+\varepsilon_{1} \Delta^{2 p} u+\varepsilon_{2} u-\varepsilon_{2} \Delta u
$$

Using this identity in the preceding equality, summing over $j \in\{1, \ldots, d\}$, and using H5(b)ii), we conclude that

$$
\begin{aligned}
& \int_{\mathbb{T}^{d}}\left|D^{2} u\right|^{2} m \mathrm{~d} x+\frac{1}{\kappa_{1}} \int_{\mathbb{T}^{d}} m^{\alpha-1}|D m|^{2} \mathrm{~d} x+\varepsilon_{1} \int_{\mathbb{T}^{d}}|D u|^{2} \mathrm{~d} x+\varepsilon_{1} \int_{\mathbb{T}^{d}}\left|\Delta^{p} D u\right|^{2} \mathrm{~d} x \\
& \quad+\varepsilon_{1} \int_{\mathbb{T}^{d}}|D m|^{2} \mathrm{~d} x+\varepsilon_{1} \int_{\mathbb{T}^{d}}\left|\Delta^{p} D m\right|^{2} \mathrm{~d} x+\varepsilon_{2} \int_{\mathbb{T}^{d}}|D u|^{2} \mathrm{~d} x+\varepsilon_{2} \int_{\mathbb{T}^{d}}\left|D^{2} u\right|^{2} \mathrm{~d} x
\end{aligned}
$$

$$
\begin{equation*}
+\varepsilon_{2} \int_{\mathbb{T}^{d}}|D m|^{2} \mathrm{~d} x+\varepsilon_{2} \int_{\mathbb{T}^{d}}\left|D^{2} m\right|^{2} \mathrm{~d} x+\int_{\mathbb{T}^{d}} \beta_{\varepsilon_{1}}^{\prime}(m)|D m|^{2} \mathrm{~d} x \leqslant \kappa_{1} \tag{6.9}
\end{equation*}
$$

Because $\left|D\left(m^{\frac{\alpha+1}{2}}\right)\right|^{2}=\frac{(\alpha+1)^{2}}{4} m^{\alpha-1}|D m|^{2}$ and because all terms on the left-hand side of the inequality in (6.9) are non-negative, it follows that

$$
\begin{equation*}
\left\|m^{\frac{1}{2}} D^{2} u\right\|_{L^{2}\left(\mathbb{T}^{d} ; \mathbb{M}_{d \times d}\right)}+\left\|m^{\frac{\alpha+1}{2}}\right\|_{W^{1,2}\left(\mathbb{T}^{d}\right)} \leqslant C \tag{6.10}
\end{equation*}
$$

where we also take into account (6.8). Moreover, invoking the Poincaré-Wirtinger inequality once more and using integration by parts, we get

$$
\left\|\sqrt{\varepsilon_{1}} u\right\|_{W^{2 p+1,2}\left(\mathbb{T}^{d}\right)}+\left\|\sqrt{\varepsilon_{1}} m\right\|_{W^{2 p+1,2}\left(\mathbb{T}^{d}\right)}+\left\|\sqrt{\varepsilon_{2}} u\right\|_{W^{2,2}\left(\mathbb{T}^{d}\right)}+\left\|\sqrt{\varepsilon_{2}} m\right\|_{W^{2,2}\left(\mathbb{T}^{d}\right)} \leqslant C
$$

Finally, we observe that

$$
\left\|m^{\frac{\alpha+1}{2}} D u\right\|_{W^{1,1}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)} \leqslant C
$$

is a consequence of (6.7) and (6.10).
The next result refines Theorem 1.1 and improves the result in Proposition 5.4.
Theorem 6.2. Suppose that Assumptions (h1), (h2), (g1), and (H5) hold. Then, there exists a weak solution, $(m, u) \in \mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right) \times \mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right)$, with $m \geqslant 0$ to (6.2). Moreover, $\left(m^{\frac{\alpha+1}{2}}, u\right) \in$ $W^{1,2}\left(\mathbb{T}^{d}\right) \times W^{1,2}\left(\mathbb{T}^{d}\right), \int_{\mathbb{T}^{d}} m \mathrm{~d} x=1$,

$$
\begin{equation*}
-u-\frac{|D u|^{2}}{2}+V(x, m, h(\boldsymbol{m}))+\sigma^{2} \Delta u \geqslant 0 \tag{6.11}
\end{equation*}
$$

in the sense of distributions, and

$$
\begin{equation*}
m-\operatorname{div}(m D u)-\sigma^{2} \Delta m-1=0 \tag{6.12}
\end{equation*}
$$

a.e. $i n \mathbb{T}^{d}$.

Proof. By Corollary 3.11, for each $\varepsilon=\left(\varepsilon_{1}, \varepsilon_{2}\right) \in(0,1)^{2}$, there exists $\left(m^{\varepsilon}, u^{\varepsilon}\right) \in C^{\infty}\left(\mathbb{T}^{d}\right) \times$ $C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\inf _{\mathbb{T}^{d}} m^{\varepsilon}>0$ satisfying (6.3)-(6.4).

We assume that $\varepsilon_{1}$ and $\varepsilon_{2}$ take values on fixed sequences of positive numbers converging to zero. We proceed with the proof in four steps.

Step 1. Here, we establish preliminary convergences of ( $m^{\varepsilon}, u^{\varepsilon}$ ) first as $\varepsilon_{1} \rightarrow 0^{+}$, and then as $\varepsilon_{2} \rightarrow 0^{+}$.

We recall that the embeddings $W^{1,2}\left(\mathbb{T}^{d}\right) \hookrightarrow L^{q}\left(\mathbb{T}^{d}\right)$ and $W^{2,2}\left(\mathbb{T}^{d}\right) \hookrightarrow L^{r}\left(\mathbb{T}^{d}\right)$ are compact for all $q \in\left[1,2^{*}\right)$ and $r \in\left[1,\left(2^{*}\right)^{*}\right)$ and continuous for all $q \in\left[1,2^{*}\right]$ and $r \in\left[1,\left(2^{*}\right)^{*}\right]$. Hence, the uniform bounds in Lemma 6.1] with respect to $\varepsilon_{1}$ give that, for each $\varepsilon_{2}$, there exist $u^{\varepsilon_{2}} \in W^{2,2}\left(\mathbb{T}^{d}\right)$ and $m^{\varepsilon_{2}} \in W^{2,2}\left(\mathbb{T}^{d}\right)$ with $\left(m^{\varepsilon_{2}}\right)^{\frac{\alpha+1}{2}} \in W^{1,2}\left(\mathbb{T}^{d}\right)$ such that, up to a not relabeled subsequence of $\left(m^{\varepsilon}, u^{\varepsilon}\right)_{\varepsilon_{1}}$,

$$
\begin{align*}
& u^{\varepsilon} \rightharpoonup u^{\varepsilon_{2}} \text { weakly in } W^{2,2}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+},  \tag{6.13}\\
& m^{\varepsilon} \rightharpoonup m^{\varepsilon_{2}} \text { weakly in } W^{2,2}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+},  \tag{6.14}\\
& \left(m^{\varepsilon}\right)^{\frac{\alpha+1}{2}} \rightharpoonup\left(m^{\varepsilon_{2}}\right)^{\frac{\alpha+1}{2}} \text { weakly in } W^{1,2}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+},  \tag{6.15}\\
& \sqrt{\varepsilon_{1}} u^{\varepsilon} \rightarrow 0 \text { in } W^{2 p, 2}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+},  \tag{6.16}\\
& \sqrt{\varepsilon_{1}} m^{\varepsilon} \rightarrow 0 \text { in } W^{2 p, 2}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+},  \tag{6.17}\\
& \left(m^{\varepsilon}\right)^{\frac{1}{2}} D u^{\varepsilon} \rightharpoonup\left(m^{\varepsilon_{2}}\right)^{\frac{1}{2}} D u^{\varepsilon_{2}} \text { weakly in } L^{2}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+},  \tag{6.18}\\
& \left(m^{\varepsilon}\right)^{\frac{1}{2}} D^{2} u^{\varepsilon} \rightharpoonup\left(m^{\varepsilon_{2}}\right)^{\frac{1}{2}} D^{2} u^{\varepsilon_{2}} \text { weakly in } L^{2}\left(\mathbb{T}^{d} ; \mathbb{M}_{d \times d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+},  \tag{6.19}\\
& \left(m^{\varepsilon}\right)^{\frac{\alpha+1}{2}} D u^{\varepsilon} \rightharpoonup\left(m^{\varepsilon_{2}}\right)^{\frac{\alpha+1}{2}} D u^{\varepsilon_{2}} \text { weakly in } W^{1,1}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+},  \tag{6.20}\\
& h\left(\boldsymbol{m}^{\varepsilon}\right) \rightarrow h\left(\boldsymbol{m}^{\varepsilon_{2}}\right) \text { in } L^{1}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+}, \tag{6.21}
\end{align*}
$$

where we also used the Rellich-Kondrachov theorem, the uniqueness of weak limits, and Assumption (h2). Observe that (6.20) holds weakly in $W^{1,1}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)$, and not only weakly- $\begin{gathered}\text { in }\end{gathered}$ $B V\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)$, by (6.13), (6.15), and the Rellich-Kondrachov theorem. Moreover, in view of (6.13), (6.14), and (6.21), we can assume that the subsequence also satisfies, for a.e. $x \in \mathbb{T}^{d}$,

$$
\begin{equation*}
D u^{\varepsilon}(x) \rightarrow D u^{\varepsilon_{2}}(x), m^{\varepsilon}(x) \rightarrow m^{\varepsilon_{2}}(x), \text { and } h\left(\boldsymbol{m}^{\varepsilon}\right)(x) \rightarrow h\left(\boldsymbol{m}^{\varepsilon_{2}}\right)(x) \text { as } \varepsilon_{1} \rightarrow 0^{+} . \tag{6.22}
\end{equation*}
$$

Note also that by (6.14), $\left(m^{\varepsilon}\right)^{\frac{1}{2}} \rightarrow\left(m^{\varepsilon_{2}}\right)^{\frac{1}{2}}$ in $L^{2}\left(\mathbb{T}^{d}\right)$ as $\varepsilon_{1} \rightarrow 0^{+}$, which, together with (6.18) and (6.19), yields

$$
\begin{align*}
& m^{\varepsilon} D u^{\varepsilon} \rightharpoonup m^{\varepsilon_{2}} D u^{\varepsilon_{2}} \text { weakly in } L^{1}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+}  \tag{6.23}\\
& m^{\varepsilon} D^{2} u^{\varepsilon} \rightharpoonup m^{\varepsilon_{2}} D^{2} u^{\varepsilon_{2}} \text { weakly in } L^{1}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+} .
\end{align*}
$$

Additionally, using the sequential lower semicontinuity of the norm in Lebesgue and Sobolev spaces with respect to the corresponding weak convergence, it follows from Lemma 6.1 and (6.13) -(6.20) that there is a positive constant, $C$, independent of $\varepsilon_{2}$ such that

$$
\begin{aligned}
& \left\|u^{\varepsilon_{2}}\right\|_{W^{1,2}\left(\mathbb{T}^{d}\right)}+\left\|\left(m^{\varepsilon_{2}}\right)^{\frac{\alpha+1}{2}}\right\|_{W^{1,2}\left(\mathbb{T}^{d}\right)}+\left\|\left(m^{\varepsilon_{2}}\right)^{\frac{1}{2}} D u^{\varepsilon_{2}}\right\|_{L^{2}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)}+\left\|\left(m^{\varepsilon_{2}}\right)^{\frac{1}{2}} D^{2} u^{\varepsilon_{2}}\right\|_{L^{2}\left(\mathbb{T}^{d} ; \mathbb{M}_{d \times d}\right)} \\
& \quad+\left\|\left(m^{\varepsilon_{2}}\right)^{\frac{\alpha+1}{2}} D u^{\varepsilon_{2}}\right\|_{W^{1,1}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)}+\left\|\sqrt{\varepsilon_{2}} u^{\varepsilon_{2}}\right\|_{W^{2,2}\left(\mathbb{T}^{d}\right)}+\left\|\sqrt{\varepsilon_{2}} m^{\varepsilon_{2}}\right\|_{W^{2,2}\left(\mathbb{T}^{d}\right)} \leqslant C .
\end{aligned}
$$

Thus, arguing as above, there exist $u \in W^{1,2}\left(\mathbb{T}^{d}\right) \cap L^{2^{*}}\left(\mathbb{T}^{d}\right)$ and $m \in L^{\frac{\alpha+1}{2} 2^{*}}\left(\mathbb{T}^{d}\right)$ with $m^{\frac{\alpha+1}{2}} \in$ $W^{1,2}\left(\mathbb{T}^{d}\right)$ such that, up to a not relabeled subsequence of $\left(m^{\varepsilon_{2}}, u^{\varepsilon_{2}}\right)_{\varepsilon_{2}}$,

$$
\begin{align*}
& u^{\varepsilon_{2}} \rightharpoonup u \text { weakly in } W^{1,2}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{2} \rightarrow 0^{+},  \tag{6.24}\\
& \left(m^{\varepsilon_{2}}\right)^{\frac{\alpha+1}{2}} \rightharpoonup m^{\frac{\alpha+1}{2}} \text { weakly in } W^{1,2}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{2} \rightarrow 0^{+},  \tag{6.25}\\
& \left(m^{\varepsilon_{2}}\right)^{\frac{1}{2}} D u^{\varepsilon_{2}} \rightharpoonup m^{\frac{1}{2}} D u \text { weakly in } L^{2}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right) \text { as } \varepsilon_{2} \rightarrow 0^{+}, \\
& \left(m^{\varepsilon_{2}}\right)^{\frac{\alpha+1}{2}} D u^{\varepsilon_{2}} \stackrel{\star}{ } m^{\frac{\alpha+1}{2}} D u \text { weakly-ᄎ in } B V\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right) \text { as } \varepsilon_{2} \rightarrow 0^{+},  \tag{6.26}\\
& m^{\varepsilon_{2}} D u^{\varepsilon_{2}} \rightharpoonup m D u \text { weakly in } L^{1}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right) \text { as } \varepsilon_{2} \rightarrow 0^{+},  \tag{6.27}\\
& \sqrt{\varepsilon_{2}} u^{\varepsilon_{2}} \rightarrow 0 \text { in } W^{1,2}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{2} \rightarrow 0^{+},  \tag{6.28}\\
& \sqrt{\varepsilon_{2}} m^{\varepsilon_{2}} \rightarrow 0 \text { in } W^{1,2}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{2} \rightarrow 0^{+},  \tag{6.29}\\
& \varepsilon_{2} \Delta m^{\varepsilon_{2}} \rightarrow 0 \text { in } L^{1}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{2} \rightarrow 0^{+},  \tag{6.30}\\
& h\left(\boldsymbol{m}^{\varepsilon_{2}}\right) \rightarrow h(\boldsymbol{m}) \text { in } L^{1}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{2} \rightarrow 0^{+} . \tag{6.31}
\end{align*}
$$

Moreover, using in addition the compact embedding $B V\left(\mathbb{T}^{d}\right) \hookrightarrow L^{s}\left(\mathbb{T}^{d}\right)$ for all $s \in\left[1,1^{*}\right)$, it follows from (6.24), (6.25), (6.26), (6.30), and (6.31) that, up to a further not relabeled subsequence of $\left(m^{\varepsilon_{2}}, u^{\varepsilon_{2}}\right)_{\varepsilon_{2}}$, we have for a.e. $x \in \mathbb{T}^{d}$,

$$
\begin{align*}
& u^{\varepsilon_{2}}(x) \rightarrow u(x), \quad m^{\varepsilon_{2}}(x) \rightarrow m(x), \quad\left(m^{\varepsilon_{2}}\right)^{\frac{\alpha+1}{2}}(x) D u^{\varepsilon_{2}}(x) \rightarrow m^{\frac{\alpha+1}{2}}(x) D u(x),  \tag{6.32}\\
& \varepsilon_{2} \Delta m^{\varepsilon_{2}}(x) \rightarrow 0, \quad h\left(\boldsymbol{m}^{\varepsilon_{2}}\right)(x) \rightarrow h(\boldsymbol{m})(x) \text { as } \varepsilon_{2} \rightarrow 0^{+}
\end{align*}
$$

Finally, because $\inf _{\mathbb{T}^{d}} m^{\varepsilon}>0$ and $\left(m^{\varepsilon}\right)_{\varepsilon_{1}}$ converges a.e. in $\mathbb{T}^{d}$ to $m^{\varepsilon_{2}}$ as $\varepsilon_{1} \rightarrow 0^{+}$, we conclude that $m^{\varepsilon_{2}} \geqslant 0$ a.e. in $\mathbb{T}^{d}$. In turn, the latter condition together with the second convergence in (6.32) yields $m \geqslant 0$ a.e. in $\mathbb{T}^{d}$. This concludes Step 1 .

Step 2. We claim that the pair $(m, u)$ with $m \geqslant 0$ a.e. in $\mathbb{T}^{d}$ determined in Step 1 is a weak solution to (6.2).

The argument is the same as the one in the proof of Theorem 1.1
Step 3. We prove that the pair $(m, u)$ with $m \geqslant 0$ a.e. in $\mathbb{T}^{d}$ determined in Step 1 satisfies $\int_{\mathbb{T}^{d}} m \mathrm{~d} x=1, \operatorname{div}(m D u)+\sigma^{2} \Delta m \in L^{\frac{\alpha+1}{2} 2^{*}}\left(\mathbb{T}^{d}\right)$, and (6.12).

Let $\varphi \in C^{\infty}\left(\mathbb{T}^{d}\right)$. Multiplying (6.4) by $\varphi$, integrating the resulting equality over $\mathbb{T}^{d}$, and using integration by parts, we obtain

$$
\begin{aligned}
\int_{\mathbb{T}^{d}} & \left(m^{\varepsilon}-1\right) \varphi \mathrm{d} x+\varepsilon_{1} \int_{\mathbb{T}^{d}} u^{\varepsilon} \varphi \mathrm{d} x+\varepsilon_{1} \int_{\mathbb{T}^{d}} \Delta^{p} u^{\varepsilon} \Delta^{p} \varphi \mathrm{~d} x+\varepsilon_{2} \int_{\mathbb{T}^{d}} u^{\varepsilon} \varphi \mathrm{d} x-\varepsilon_{2} \int_{\mathbb{T}^{d}} u^{\varepsilon} \Delta \varphi \mathrm{d} x \\
& =-\int_{\mathbb{T}^{d}} m^{\varepsilon} D u^{\varepsilon} \cdot D \varphi \mathrm{~d} x+\sigma^{2} \int_{\mathbb{T}^{d}} m^{\varepsilon} \Delta \varphi \mathrm{d} x
\end{aligned}
$$

Letting $\varepsilon_{1} \rightarrow 0^{+}$first and using (6.14), (6.16), (6.13), and (6.23), and then letting $\varepsilon_{2} \rightarrow 0^{+}$ and invoking (6.25), (6.28), and (6.27), we conclude that

$$
\begin{aligned}
\int_{\mathbb{T}^{d}}(m-1) \varphi \mathrm{d} x & =-\int_{\mathbb{T}^{d}} m D u \cdot D \varphi \mathrm{~d} x+\sigma^{2} \int_{\mathbb{T}^{d}} m \Delta \varphi \mathrm{~d} x \\
& =\left\langle\operatorname{div}(m D u)+\sigma^{2} \Delta m, \varphi\right\rangle_{\mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right), C^{\infty}\left(\mathbb{T}^{d}\right)} .
\end{aligned}
$$

Because $m-1 \in L^{\frac{\alpha+1}{2} 2^{*}}\left(\mathbb{T}^{d}\right)$ and $\varphi \in C^{\infty}\left(\mathbb{T}^{d}\right)$ is arbitrary, it follows that $\operatorname{div}(m D u)+\sigma^{2} \Delta m \in$ $L^{\frac{\alpha+1}{2}} 2^{*}\left(\mathbb{T}^{d}\right)$ and (6.12) holds a.e. in $\mathbb{T}^{d}$.

Finally, taking $\varphi=1$ in the equality above yields $\int_{\mathbb{T}^{d}} m \mathrm{~d} x=1$. This completes Step 3 .
Step 4. We prove that the pair $(m, u)$ with $m \geqslant 0$ a.e. in $\mathbb{T}^{d}$ determined in Step 1 satisfies (6.11).

Let $\varphi \in C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\varphi \geqslant 0$. Multiplying (6.3) by $\varphi$, integrating the resulting equality over $\mathbb{T}^{d}$, and using the integration by parts formula and the condition $\beta_{\varepsilon_{1}}(\cdot) \leqslant 0$, we get

$$
\begin{align*}
\int_{\mathbb{T}^{d}} & -u^{\varepsilon} \varphi \mathrm{d} x+\int_{\mathbb{T}^{d}} V\left(x, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right) \varphi \mathrm{d} x+\sigma^{2} \int_{\mathbb{T}^{d}} u^{\varepsilon} \Delta \varphi \mathrm{d} x+\varepsilon_{1} \int_{\mathbb{T}^{d}} m^{\varepsilon} \varphi \mathrm{d} x \\
& +\varepsilon_{1} \int_{\mathbb{T}^{d}} \Delta^{p} m^{\varepsilon} \Delta^{p} \varphi \mathrm{~d} x+\varepsilon_{2} \int_{\mathbb{T}^{d}} m^{\varepsilon} \varphi \mathrm{d} x-\varepsilon_{2} \int_{\mathbb{T}^{d}} m^{\varepsilon} \Delta \varphi \mathrm{d} x  \tag{6.33}\\
& =\int_{\mathbb{T}^{d}} \frac{\left|D u^{\varepsilon}\right|^{2}}{2} \varphi \mathrm{~d} x-\int_{\mathbb{T}^{d}} \beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right) \varphi \mathrm{d} x \geqslant \int_{\mathbb{T}^{d}} \frac{\left|D u^{\varepsilon}\right|^{2}}{2} \varphi \mathrm{~d} x .
\end{align*}
$$

First, we let $\varepsilon_{1} \rightarrow 0^{+}$and then $\varepsilon_{2} \rightarrow 0^{+}$. According to (6.13), (6.14), (6.17), (6.21), (6.22), (6.24), (6.25), (6.29), (6.31), (6.32), the sequential lower semicontinuity of the convex functional

$$
w \in L^{2}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right) \mapsto \int_{\mathbb{T}^{d}} \frac{|w(x)|^{2}}{2} \varphi(x) \mathrm{d} x \in \mathbb{R}_{0}^{+}
$$

with respect to the weak convergence in $L^{2}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)$, and the continuity of $V$ together with Fatou's lemma and Assumption ( $(\mathrm{H5c})$, we deduce that

$$
\int_{\mathbb{T}^{d}}-u \varphi \mathrm{~d} x+\int_{\mathbb{T}^{d}} V(x, m, h(\boldsymbol{m})) \varphi \mathrm{d} x+\sigma^{2} \int_{\mathbb{T}^{d}} u \Delta \varphi \mathrm{~d} x \geqslant \int_{\mathbb{T}^{d}} \frac{|D u|^{2}}{2} \varphi \mathrm{~d} x
$$

Equivalently,

$$
\left\langle-u-\frac{|D u|^{2}}{2}+V(x, m, h(\boldsymbol{m}))+\sigma^{2} \Delta u, \varphi\right\rangle_{\mathcal{D}^{\prime}\left(\mathbb{T}^{d}\right), C^{\infty}\left(\mathbb{T}^{d}\right)} \geqslant 0
$$

and so, (6.11) holds since $\varphi \in C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\varphi \geqslant 0$ is arbitrary.
Next, we address the deterministic case, $\sigma=0$, and improve the supersolution property from Proposition 5.5.

Corollary 6.3. Suppose that Assumptions (h1), (h2), (g1), and (H5) hold with $\sigma=0$. If $d \geqslant 8$, assume further that $\alpha>\frac{d-4}{2}$ in Assumption (H5). Then, the weak solution, $(m, u)$, of (6.2) given by Theorem 6.2 satisfies

$$
\left\{\begin{array}{l}
\left(-u-\frac{|D u|^{2}}{2}+V(x, m, h(\boldsymbol{m}))\right) m=0 \\
m-\operatorname{div}(m D u)-1=0
\end{array}\right.
$$

a.e. in $\mathbb{T}^{d}$.

Proof. Assume that $\sigma=0$. In view of Theorem 6.2 we are left to prove that

$$
\begin{equation*}
\left(-u-\frac{|D u|^{2}}{2}+V(x, m, h(\boldsymbol{m}))\right) m=0 \quad \text { a.e. in } \mathbb{T}^{d} . \tag{6.34}
\end{equation*}
$$

In what follows, either $d \leqslant 7$ and $\alpha>0$ or $d \geqslant 8$ and $\alpha>\frac{d-4}{2}$. Let $\varepsilon=\left(\varepsilon_{1}, \varepsilon_{2}\right) \in(0,1)^{2}$ and $\left(m^{\varepsilon}, u^{\varepsilon}\right) \in C^{\infty}\left(\mathbb{T}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\inf _{\mathbb{T}^{d}} m^{\varepsilon}>0$ be as in the proof of Theorem6.2, The arguments in the Step 4 of that proof show that by letting $\varepsilon_{1} \rightarrow 0^{+}$in (6.33), we have

$$
\int_{\mathbb{T}^{d}}\left(-u^{\varepsilon_{2}}-\frac{\left|D u^{\varepsilon_{2}}\right|^{2}}{2}+V\left(x, m^{\varepsilon_{2}}, h\left(\boldsymbol{m}^{\varepsilon_{2}}\right)\right)+\varepsilon_{2} m^{\varepsilon_{2}}-\varepsilon_{2} \Delta m^{\varepsilon_{2}}\right) \varphi \mathrm{d} x \geqslant 0
$$

for all $\varphi \in C^{\infty}\left(\mathbb{T}^{d}\right)$, with $\varphi \geqslant 0$. Hence, because $m^{\varepsilon_{2}} \geqslant 0$ a.e. in $\mathbb{T}^{d}$, we conclude that

$$
\begin{equation*}
\left(-u^{\varepsilon_{2}}-\frac{\left|D u^{\varepsilon_{2}}\right|^{2}}{2}+V\left(x, m^{\varepsilon_{2}}, h\left(\boldsymbol{m}^{\varepsilon_{2}}\right)\right)+\varepsilon_{2} m^{\varepsilon_{2}}-\varepsilon_{2} \Delta m^{\varepsilon_{2}}\right) m^{\varepsilon_{2}} \geqslant 0 \tag{6.35}
\end{equation*}
$$

a.e. in $\mathbb{T}^{d}$.

Next, we prove the converse inequality. Let $\varphi \in C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\varphi \geqslant 0$. Multiplying (6.3) by $m^{\varepsilon} \varphi$, integrating the resulting equality over $\mathbb{T}^{d}$, and using integration by parts, we get

$$
\begin{align*}
& -\int_{\mathbb{T}^{d}} u^{\varepsilon} m^{\varepsilon} \varphi \mathrm{d} x-\int_{\mathbb{T}^{d}} \frac{\left|D u^{\varepsilon}\right|^{2}}{2} m^{\varepsilon} \varphi \mathrm{d} x+\int_{\mathbb{T}^{d}} V\left(x, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right) m^{\varepsilon} \varphi \mathrm{d} x \\
& \quad+\varepsilon_{1} \int_{\mathbb{T}^{d}}\left(m^{\varepsilon}\right)^{2} \varphi \mathrm{~d} x+\varepsilon_{1} \int_{\mathbb{T}^{d}} \Delta^{p} m^{\varepsilon} \Delta^{p}\left(m^{\varepsilon} \varphi\right) \mathrm{d} x+\int_{\mathbb{T}^{d}} \beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right) m^{\varepsilon} \varphi \mathrm{d} x \\
& \quad+\varepsilon_{2} \int_{\mathbb{T}^{d}}\left(m^{\varepsilon}\right)^{2} \varphi \mathrm{~d} x-\varepsilon_{2} \int_{\mathbb{T}^{d}} \Delta m^{\varepsilon} m^{\varepsilon} \varphi \mathrm{d} x=0 \tag{6.36}
\end{align*}
$$

Using (6.13) and (6.14) together with the embeddings mentioned at the beginning of Step 1 of the previous proof, we have

$$
\begin{align*}
\lim _{\varepsilon_{1} \rightarrow 0^{+}} & \left(-\int_{\mathbb{T}^{d}} u^{\varepsilon} m^{\varepsilon} \varphi \mathrm{d} x+\varepsilon_{2} \int_{\mathbb{T}^{d}}\left(m^{\varepsilon}\right)^{2} \varphi \mathrm{~d} x-\varepsilon_{2} \int_{\mathbb{T}^{d}} \Delta m^{\varepsilon} m^{\varepsilon} \varphi \mathrm{d} x\right) \\
& =-\int_{\mathbb{T}^{d}} u^{\varepsilon_{2}} m^{\varepsilon_{2}} \varphi \mathrm{~d} x+\varepsilon_{2} \int_{\mathbb{T}^{d}}\left(m^{\varepsilon_{2}}\right)^{2} \varphi \mathrm{~d} x-\varepsilon_{2} \int_{\mathbb{T}^{d}} \Delta m^{\varepsilon_{2}} m^{\varepsilon_{2}} \varphi \mathrm{~d} x \tag{6.37}
\end{align*}
$$

From Assumptions H5(a)i , g1a, and H5(b)i , it follows that

$$
V\left(x, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right) \geqslant \frac{1}{\kappa_{2}}\left(m^{\varepsilon}\right)^{\alpha}-\kappa_{2}
$$

for some positive constant, $\kappa_{2}$, independent of $\varepsilon$. Thus, because $m^{\varepsilon} \varphi \geqslant 0$ a.e. in $\mathbb{T}^{d}$, Fatou's lemma, the continuity of $V$, and the convergences (6.15) and (6.22) imply that

$$
\begin{equation*}
\int_{\mathbb{T}^{d}} V\left(x, m^{\varepsilon_{2}}, h\left(\boldsymbol{m}^{\varepsilon_{2}}\right)\right) m^{\varepsilon_{2}} \varphi \mathrm{~d} x \leqslant \liminf _{\varepsilon_{1} \rightarrow 0^{+}} \int_{\mathbb{T}^{d}} V\left(x, m^{\varepsilon}, h\left(\boldsymbol{m}^{\varepsilon}\right)\right) m^{\varepsilon} \varphi \mathrm{d} x \tag{6.38}
\end{equation*}
$$

By definition, $\beta_{\varepsilon_{1}}(\cdot) \leqslant 0$ and $\beta_{\varepsilon_{1}}(s)=0$ if $s \geqslant \varepsilon_{1}$, and thus

$$
\left|\int_{\mathbb{T}^{d}} \beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right) m^{\varepsilon} \varphi \mathrm{d} x\right|=\left|\int_{\left\{x \in \mathbb{T}^{d} \cdot m^{\varepsilon}(x) \leqslant \varepsilon_{1}\right\}} \beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right) m^{\varepsilon} \varphi \mathrm{d} x\right| \leqslant \varepsilon_{1}\|\varphi\|_{L^{\infty}\left(\mathbb{T}^{d}\right)} C,
$$

where $C$ is the constant given by Lemma 6.1 Hence,

$$
\begin{equation*}
\lim _{\varepsilon_{1} \rightarrow 0^{+}}\left(\varepsilon_{1} \int_{\mathbb{T}^{d}}\left(m^{\varepsilon}\right)^{2} \varphi \mathrm{~d} x+\varepsilon_{1} \int_{\mathbb{T}^{d}} \Delta^{p} m^{\varepsilon} \Delta^{p}\left(m^{\varepsilon} \varphi\right) \mathrm{d} x+\int_{\mathbb{T}^{d}} \beta_{\varepsilon_{1}}\left(m^{\varepsilon}\right) m^{\varepsilon} \varphi \mathrm{d} x\right)=0 \tag{6.39}
\end{equation*}
$$

where we also used (6.17).
Next, we prove that

$$
\begin{equation*}
\lim _{\varepsilon_{1} \rightarrow 0^{+}} \int_{\mathbb{T}^{d}} \frac{\left|D u^{\varepsilon}\right|^{2}}{2} m^{\varepsilon} \varphi \mathrm{d} x=\int_{\mathbb{T}^{d}} \frac{\left|D u^{\varepsilon_{2}}\right|^{2}}{2} m^{\varepsilon_{2}} \varphi \mathrm{~d} x \tag{6.40}
\end{equation*}
$$

In view of the embeddings mentioned at the beginning of Step 1 of the previous proof, we have

$$
\begin{align*}
& m^{\varepsilon} \rightarrow m^{\varepsilon_{2}} \text { in } L^{p}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+} \text {and for all } p \in\left(1,\left(2^{*}\right)^{*}\right)  \tag{6.41}\\
& m^{\varepsilon} \rightarrow m^{\varepsilon_{2}} \text { in } L^{r}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+} \text {and for all } r \in\left(1, \frac{(\alpha+1) 2^{*}}{2}\right),  \tag{6.42}\\
& \left|D u^{\varepsilon}\right|^{2} \rightarrow\left|D u^{\varepsilon_{2}}\right|^{2} \text { in } L^{s}\left(\mathbb{T}^{d}\right) \text { as } \varepsilon_{1} \rightarrow 0^{+} \text {and for all } s \in\left(1, \frac{2^{*}}{2}\right) \tag{6.43}
\end{align*}
$$

If $d \leqslant 7$, then $\left(\frac{2^{*}}{2}\right)^{\prime}<\left(2^{*}\right)^{*}$. Accordingly, for some $p<\frac{2^{*}}{2}$ in (6.41) and some $s<\frac{2^{*}}{2}$ in (6.43), we get $\left|D u^{\varepsilon}\right|^{2} m^{\varepsilon} \rightarrow\left|D u^{\varepsilon_{2}}\right|^{2} m^{\varepsilon_{2}}$ strongly in $L^{q}\left(\mathbb{T}^{d}\right)$ with $\frac{1}{q}=\frac{1}{r}+\frac{1}{s}$ and $q>1$. Moreover, if $d \geqslant 8$ and $\alpha>\frac{d-4}{2}$, then there is $q>1$ such that $\frac{1}{q}=\frac{1}{r}+\frac{1}{s}$, where $r$ and $s$ are as in (6.42) and (6.43), respectively. Therefore, we get again $\left|D u^{\varepsilon}\right|^{2} m^{\varepsilon} \rightarrow\left|D u^{\varepsilon_{2}}\right|^{2} m^{\varepsilon_{2}}$ strongly in $L^{q}\left(\mathbb{T}^{d}\right)$.

Finally, by (6.36), (6.37), (6.38), (6.39), and (6.40), we have

$$
\int_{\mathbb{T}^{d}}\left(-u^{\varepsilon_{2}}-\frac{\left|D u^{\varepsilon_{2}}\right|^{2}}{2}+V\left(x, m^{\varepsilon_{2}}, h\left(\boldsymbol{m}^{\varepsilon_{2}}\right)\right)+\varepsilon_{2} m^{\varepsilon_{2}}-\varepsilon_{2} \Delta m^{\varepsilon_{2}}\right) m^{\varepsilon_{2}} \varphi \mathrm{~d} x \leqslant 0
$$

Because $\varphi \in C^{\infty}\left(\mathbb{T}^{d}\right)$ with $\varphi \geqslant 0$ is arbitrary, recalling (6.35), it follows that

$$
\begin{equation*}
\left(-u^{\varepsilon_{2}}-\frac{\left|D u^{\varepsilon_{2}}\right|^{2}}{2}+V\left(x, m^{\varepsilon_{2}}, h\left(\boldsymbol{m}^{\varepsilon_{2}}\right)\right)+\varepsilon_{2} m^{\varepsilon_{2}}-\varepsilon_{2} \Delta m^{\varepsilon_{2}}\right) m^{\varepsilon_{2}}=0 \quad \text { a.e. in } \mathbb{T}^{d} \tag{6.44}
\end{equation*}
$$

Let $A:=\left\{x \in \mathbb{T}^{d}:(6.32)\right.$ and $m(x) \geqslant 0$ hold $\}$. Then, $\mathcal{L}^{d}\left(\mathbb{T}^{d} \backslash A\right)=0$.
If $x \in A$ is such that $m(x)=0$, then clearly

$$
\begin{equation*}
\left(-u(x)-\frac{|D u(x)|^{2}}{2}+V(x, m(x), h(\boldsymbol{m})(x))\right) m(x)=0 . \tag{6.45}
\end{equation*}
$$

If $x \in A$ is such that $m(x)>0$, then $m^{\varepsilon_{2}}(x)>0$ for all sufficiently small $\varepsilon_{2}$; moreover,

$$
\begin{aligned}
\lim _{\varepsilon_{2} \rightarrow 0^{+}}\left|D u^{\varepsilon_{2}}(x)\right|^{2} m^{\varepsilon_{2}}(x) & =\lim _{\varepsilon_{2} \rightarrow 0^{+}}\left|D u^{\varepsilon_{2}}(x)\right|^{2}\left(m^{\varepsilon_{2}}\right)^{\alpha+1}(x)\left(m^{\varepsilon_{2}}\right)^{1-\alpha}(x) \\
& =|D u(x)|^{2} m(x)
\end{aligned}
$$

which, together with (6.32) and (6.44), proves that (6.45) holds for all such $x$. This completes the proof of (6.34).

## 7. Final remarks

The monotonicity method developed here is a powerful and flexible way to study MFGs. Several extensions can be considered using similar ideas, including the standard stationary MFG:

$$
\left\{\begin{array}{l}
H\left(x, D u, D^{2} u, m, h(\boldsymbol{m})\right)=\bar{H} \\
-\operatorname{div}\left(m D_{p} H\left(x, D u, D^{2} u, m, h(\boldsymbol{m})\right)\right)+\left(m D_{M_{i j}} H\left(x, D u, D^{2} u, m, h(\boldsymbol{m})\right)\right)_{x_{i} x_{j}}=0 .
\end{array}\right.
$$

In addition, there are several areas where future developments are likely. First, we foresee improvements in the regularity theory for weak solutions. Additional regularity is essential to prove the uniqueness of solutions. Second, the congestion problems examined here may enjoy further regularity properties; this topic was not explored in the present paper. Third, time-dependent MFGs are a natural application of our methods. Here, we need to develop a different regularization method because of the initial-terminal boundary conditions. Moreover, the regularity theory for these problems may differ substantially from the stationary case. Finally, our results may be of independent interest in the calculus of variations as certain MFGs are the Euler-Lagrange equation of integral functionals. These functionals are convex but, in many cases, non-coercive, and their study presents substantial challenges; see, for example, [29.

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