Phase Retrieval by Linear Algebra

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Abstract

The null vector method, based on a simple linear algebraic concept, is proposed as a solution to the phase retrieval problem. In the case with complex Gaussian random measurement matrices, a non-asymptotic error bound is derived, yielding an asymptotic regime of accurate approximation comparable to that for the spectral vector method.

1 Introduction

We consider the following phase retrieval problem: Let $A = [a_i]$ be a $n \times N$ random matrix with independently and identically distributed entries in N(0, 1) + iN(0, 1), i.e. circularly symmetric complex Gaussian random variables. Let $x_0 \in \mathbb{C}^n$ and $y = A^*x_0$. Suppose we are given A and b := |y| where |y| denote the vector such that $|y|(j) = |y(j)|, \forall j$. The aim of phase retrieval is to find x_0 .

Clearly this is a nonlinear inversion problem. Simple dimension count shows that, for the solution to be unique in general, the number of (nonnegative) data N needs to be at least twice the number n of unknown (complex) components. There are many approaches to phase retrieval, the most efficient and effective, especially when the problem size is large, being fixed point algorithms (see [3,4,6,7] and references therein) and non-convex optimization methods [1,2]. Phase retrieval has a wide range of applications, see [8] for a recent survey.

The following observation motivates our current approach: Let I be a subset of $\{1, \dots, N\}$ and I_c its complement such that $b(i) \leq b(j)$ for all $i \in I, j \in I_c$. In other words, $\{b(i) : i \in I\}$ are the "weaker" signals and $\{b(j) : j \in I_c\}$ the "stronger" signals. Let |I| be the cardinality of the set I. Since $b(i) = |a_i^* x_0|, i \in I$, are small, $\{a_i\}_{i \in I}$ is a set of sensing vectors nearly orthogonal to x_0 . Denote the sub-column matrices consisting of $\{a_i\}_{i \in I}$ and $\{a_j\}_{j \in I_c}$ by A_I

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and A_{I_c} , respectively. Define the null vector by the singular vector for the least singular value of A_I :

 $x_{\text{null}} := \arg\min\left\{ \|A_I^* x\|^2 : x \in C^n, \|x\| = \|x_0\| \right\}$

which can be computed by purely linear algebraic methods.

The goal of the paper is to establish a regime where x_{null} is an accurate approximation to x_0 .

2 Approximation theorem

Note that both x_{null} and the phase retrieval solution is at best uniquely defined up to a global phase factor. So we use the following error metric

$$\|x_0 x_0^* - x_{\text{null}} x_{\text{null}}^*\|^2 = 2\|x_0\|^4 - 2|x_0^* x_{\text{null}}|^2 \tag{1}$$

which has the advantage of being independent of the global phase factor.

The following theorem is our main result. **Theorem 2.1.** *Suppose*

$$\sigma := \frac{|I|}{N} < 1, \quad \nu = \frac{n}{|I|} < 1.$$
(2)

Then for any $\epsilon \in (0,1), \delta > 0$ and $t \in (0, \nu^{-1/2} - 1)$ the following error bound

$$\|x_0 x_0^* - x_{\text{null}} x_{\text{null}}^*\|^2 \leq \left(\left(\frac{2+t}{1-\epsilon}\right) \sigma + \epsilon \left(-2\ln(1-\sigma) + \delta\right) \right) \frac{2\|x_0\|^4}{\left(1 - (1+t)\sqrt{\nu}\right)^2}$$
(3)

holds with probability at least

$$1 - 2\exp\left(-N\delta^2 e^{-\delta}|1 - \sigma|^2/2\right) - \exp(-2\lfloor|I|\epsilon\rfloor^2/N) - Q \tag{4}$$

where Q has the asymptotic upper bound

$$2\exp\left\{-c\min\left[\frac{e^{2}t^{2}}{16}\left(\ln\sigma^{-1}\right)^{2}|I|^{2}/N,\ \frac{et}{4}|I|\ln\sigma^{-1}\right]\right\},\ \sigma\ll1,$$
(5)

with an absolute constant c.

Remark 2.2. To unpack the implications of Theorem 2.1, consider the following asymptotic: With ϵ and t fixed, let

$$n \gg 1$$
, $\sigma = \frac{|I|}{N} \ll 1$, $\frac{|I|^2}{N} \gg 1$, $\nu = \frac{n}{|I|} < 1$.

We have

$$\|x_0 x_0^* - x_{\text{null}} x_{\text{null}}^*\|^2 \le c_0 \sigma \|x_0\|^4 \tag{6}$$

with probability at least

$$1 - c_1 e^{-c_2 n} - c_3 \exp\left\{-c_4 \left(\ln \sigma^{-1}\right)^2 |I|^2 / N\right\}$$

for moderate constants c_0, c_1, c_2, c_3, c_4 .

The proof of Theorem 2.1 is given in the next section.

The spectral vector method [1, 2, 7] is another linear algebraic method and uses the leading singular vector x_{spec} of $B^* = \text{diag}[b]A^*$ to approximate x_0 where

$$x_{\text{spec}} := \arg \max \left\{ \|B^* x\|^2 : x \in \mathbb{C}^n, \|x\| = \|x_0\| \right\}.$$

The spectral vector method has a comparable performance guarantee to (6) which vanishes as $\sigma \to 0$, with probability close to 1 exponentially in n.

In practice, however, the null vector method significantly outperforms the spectral vector method in terms of accuracy and noise stability when b is contaminated with noise [4].

The drawback with both approaches is that the error metric vanishes only with infinitely many data, $N \to \infty$. For a finite data set, the null vector is best to be deployed in conjunction with a fast (locally) convergent fixed point algorithm such as alternating projection [4] or the Douglas-Rachford algorithm [3].

3 Proof of Theorem 2.1

The proof is based on the following two propositions. **Proposition 3.1.** There exists $x_{\perp} \in \mathbb{C}^n$ with $x_{\perp}^* x_0 = 0$ and $||x_{\perp}|| = ||x_0|| = 1$ such that

$$\frac{1}{4} \|x_0 x_0^* - x_{\text{null}} x_{\text{null}}^*\|^2 \le \frac{\|b_I\|^2}{\|A_I^* x_\bot\|^2}.$$
(7)

Proof. Since x_{null} is optimally phase-adjusted, we have

$$\beta := x_0^* x_{\text{null}} \ge 0 \tag{8}$$

and

$$x_0 = \beta x_{\text{null}} + \sqrt{1 - \beta^2} z \tag{9}$$

for some unit vector $z^*x_{\text{null}} = 0$. Then

$$x_{\perp} := -(1 - \beta^2)^{1/2} x_{\text{null}} + \beta z \tag{10}$$

is a unit vector satisfying $x_0^* x_{\perp} = 0$. Since x_{null} is a singular vector and z belongs in another singular subspace, we have

$$\begin{aligned} \|A_I^* x_0\|^2 &= \beta^2 \|A_I^* x_{\text{null}}\|^2 + (1 - \beta^2) \|A_I^* z\|^2, \\ \|A_I^* x_{\perp}\|^2 &= (1 - \beta^2) \|A_I^* x_{\text{null}}\|^2 + \beta^2 \|A_I^* z\|^2 \end{aligned}$$

from which it follows that

$$(2 - \beta^2) \|A_I^* x_0\|^2 - (1 - \beta^2) \|A_I^* x_\perp\|^2$$

$$= \|A_I^* x_{\text{null}}\|^2 + 2(1 - \beta^2)^2 \left(\|A_I^* z\|^2 - \|A_I^* x_{\text{null}}\|^2 \right) \ge 0.$$
(11)

By (11), (1) and $||b_I|| = ||A_I^* x_0||$, we also have

$$\frac{\|b_I\|^2}{\|A_I^* x_\perp\|^2} \ge \frac{1-\beta^2}{2-\beta^2} \ge \frac{1}{2}(1-\beta^2) = \frac{1}{4}\|x_0 x_0^* - x_{\text{null}} x_{\text{null}}^*\|^2.$$
(12)

Proposition 3.2. Let $A \in \mathbb{C}^{n \times N}$ be an *i.i.d.* complex standard Gaussian random matrix. Then for any $\epsilon > 0, \delta > 0, t > 0$

$$\|b_I\|^2 \le |I| \left(\left(\frac{2+t}{1-\epsilon}\right) \frac{|I|}{N} + \epsilon \left(-2\ln\left(1-\frac{|I|}{N}\right) + \delta\right) \right)$$

with probability at least

$$1 - 2\exp\left(-N\delta^{2}e^{-\delta}|1 - \sigma|^{2}/2\right) - 2\exp\left(-2\epsilon^{2}|1 - \sigma|^{2}\sigma^{2}N\right) - Q$$

where Q has the asymptotic upper bound

$$2\exp\left\{-c\min\left[\frac{e^2t^2}{16}\frac{|I|^2}{N}\left(\ln\sigma^{-1}\right)^2, \frac{et}{4}|I|\ln\sigma^{-1}\right]\right\}, \quad \sigma := \frac{|I|}{N} \ll 1.$$

The proof of Proposition 3.2 is given in the next section.

Now we turn to the proof of Theorem 2.1.

Without loss of the generality we may assume $||x_0|| = 1$. Otherwise, we replace x_0, x_{null} by $x_0/||x_0||$ and $x_{null}/||x_0||$, respectively. Let $Q = [Q_1 \ Q_2 \ \cdots \ Q_n]$ be a unitary transformation where $Q_1 = x_0$ or equivalently $x_0 = Qe_1$ where e_1 is the canonical vector with 1 as the first entry and zero elsewhere. Since unitary transformations do not affect the covariance structure of Gaussian random vectors, the matrix A^*Q is an i.i.d. complex standard Gaussian matrix.

Proposition 3.3. Let I be any set such that $b(i) \leq b(j)$ for all $i \in I$ and $j \in I_c = \{1, 2, ..., N\} \setminus I$. For any unitary matrix Q, let $A' \in C^{|I| \times (n-1)}$ be the sub-column matrix of A_I^*Q with its first column vector deleted. Then A' is an i.i.d. complex standard Gaussian random matrix.

Proof. First note that $A_I^*Q = (A^*Q)_I$, the row submatrix of A^*Q indexed by I. As noted already, A^*Q is an i.i.d. complex Gaussian matrix.

Since $x_0 = Qe_1$ and $b = |A^*Qe_1|$, I and I_c are entirely determined by the first column of A^*Q which is independent of the other columns of A^*Q . Consequently, the probability law of A' conditioned on the choice of I equals the probability law of A' for a fixed I. Therefore, A' is an i.i.d. complex standard Gaussian matrix.

Let $\{\nu_i\}_{i=1}^{n-1}$ be the singular values of A' in the ascending order. For ant $z \in \mathbb{C}^{n-1}$,

$$B' := A' \operatorname{diag}(z/|z|)$$

has the same set of singular values as A'. Again, we adopt the convention that z(j)/|z(j)| = 1when z(j) = 0. We have

$$||A'z|| = ||B'|z|||$$

and hence

$$||A'z|| = (||\Re(B')|z|||^2 + ||\Im(B')|z|||^2)^{1/2} \ge \sqrt{2} (||\Re(B')|z||| \wedge ||\Im(B')|z|||)$$

By the theory of Wishart matrices [5], the singular values $\{\nu_j^R\}_{j=1}^{n-1}, \{\nu_j^I\}_{j=1}^{n-1}$ (in the ascending order) of $\Re(B'), \Im(B')$ satisfy the probability bounds that for every t > 0 and $j = 1, \dots, n-1$

$$\mathbb{P}\left(\sqrt{|I|} - (1+t)\sqrt{n} \le \nu_j^R \le \sqrt{|I|} + (1+t)\sqrt{n}\right) \ge 1 - 2e^{-nt^2/2},\tag{13}$$

$$\mathbb{P}\left(\sqrt{|I|} - (1+t)\sqrt{n} \le \nu_j^I \le \sqrt{|I|} + (1+t)\sqrt{n}\right) \ge 1 - 2e^{-nt^2/2}.$$
 (14)

By Proposition 3.1 and (13)-(14), we have

$$\begin{aligned} \|x_0 x_0^* - x_{\text{null}} x_{\text{null}}^*\| &\leq \frac{\sqrt{2} \|b_I\|}{\|\Re(B') |y|\| \wedge \|\Im(B') |y|\|} \\ &\leq \sqrt{2} \|b_I\| (\nu_{n-1}^R \wedge \nu_{n-1}^I)^{-1} \\ &\leq \sqrt{2} \|b_I\| (\sqrt{|I|} - (1+t)\sqrt{n})^{-1} \end{aligned}$$

By Proposition 3.2, we obtain the desired bound (3). The success probability is at least the expression (13) minus $4e^{-nt^2/2}$ which equals the expression (4).

3.1 **Proof of Proposition 3.2**

By the Gaussian assumption, $b(i)^2 = |a_i^* x_0|^2$ has a chi-squared distribution with the probability density $e^{-z/2}/2$ on $z \in [0, \infty)$ and the cumulative distribution

$$F(\tau) := \int_0^\tau 2^{-1} \exp(-z/2) dz = 1 - \exp(-\tau/2).$$

Let

$$\tau_* = -2\ln(1 - |I|/N) \tag{15}$$

for which $F(\tau_*) = |I|/N$.

Define

$$\hat{I} := \{i : b(i)^2 \le \tau_*\} = \{i : F(b^2(i)) \le |I|/N\},\$$

and

$$\|\hat{b}\|^2 := \sum_{i \in \hat{I}} b(i)^2.$$

Let

$$\{\tau_1 \leq \tau_2 \leq \ldots \leq \tau_N\}$$

be the sorted sequence of $\{b(1)^2, \ldots, b(N)^2\}$ in magnitude. **Proposition 3.4. (i)** For any $\delta > 0$, we have

$$\tau_{|I|} \leq \tau_* + \delta \tag{16}$$

with probability at least

$$1 - \exp\left(-\frac{N}{2}\delta^2 e^{-\delta}|1 - |I|/N|^2\right)$$
(17)

(ii) For each $\epsilon > 0$, we have

$$|\hat{I}| \ge |I|(1-\epsilon) \tag{18}$$

or equivalently,

$$\tau_{\lfloor |I|(1-\epsilon)\rfloor} \le \tau_* \tag{19}$$

with probability at least

$$1 - 2\exp\left(-4\epsilon^2|1 - |I|/N|^2|I|^2/N\right)$$
(20)

Proof. (i) Since $F'(\tau) = \exp(-\tau/2)/2$,

$$|F(\tau + \epsilon) - F(\tau)| \ge \epsilon/2 \exp(-(\tau + \epsilon)/2).$$
(21)

For $\delta > 0$, let

$$\zeta := F(\tau_* + \delta) - F(\tau_*)$$

which by (21) satisfies

$$\zeta \ge \frac{\delta}{2} \exp(-\frac{1}{2}(\tau_* + \delta)). \tag{22}$$

Let $\{w_i : i = 1, ..., N\}$ be the i.i.d. indicator random variables

$$w_i = \chi_{\{b(i)^2 > \tau_* + \delta\}}$$

whose expectation is given by

$$\mathbb{E}[w_i] = 1 - F(\tau_* + \delta).$$

The Hoeffding inequality yields

$$P(\tau_{|I|} > \tau_* + \delta) = P\left(\sum_{i=1}^N w_i > N - |I|\right)$$

$$= P\left(N^{-1}\sum_{i=1}^N w_i - \mathbb{E}[w_i] > 1 - |I|/N - \mathbb{E}[w_i]\right)$$

$$= P\left(N^{-1}\sum_{i=1}^N w_i - \mathbb{E}[w_i] > \zeta\right)$$

$$\leq \exp(-2N\zeta^2).$$
(23)

Hence, for any fixed $\delta > 0$,

$$\tau_{|I|} \le \tau_* + \delta \tag{24}$$

holds with probability at least

$$1 - \exp(-2N\zeta^2) \geq 1 - \exp\left(-\frac{N\delta^2}{2}e^{-\tau_*-\delta}\right)$$
$$= 1 - \exp\left(-\frac{N\delta^2}{2}e^{-\delta}|1 - |I|/N|^2\right)$$

by (22).

(ii) Consider the following replacement

$$\begin{array}{ll} (a) & |I| \longrightarrow \lceil |I|(1-\epsilon) \rceil \\ (b) & \tau_* \longrightarrow F^{-1}(\lceil |I|(1-\epsilon) \rceil/N) \\ (c) & \delta \longrightarrow F^{-1}(|I|/N) - F^{-1}(\lceil |I|(1-\epsilon) \rceil/N) \\ (d) & \zeta \longrightarrow F^{-1}(\tau_* + \delta) - F^{-1}(\tau_*) = |I|/N - \lceil |I|(1-\epsilon) \rceil/N = \frac{||I|\epsilon|}{N} \end{array}$$

in the preceding argument. Then (23) becomes

$$\mathbb{P}\left(\tau_{\lceil |I|(1-\epsilon)\rceil} > F^{-1}(|I|/N)\right) \leq \exp(-2N\zeta^2) = \exp\left(-\frac{2\lfloor |I|\epsilon\rfloor^2}{N}\right).$$

That is,

$$\tau_{\lceil |I|(1-\epsilon)\rceil} \le \tau_*$$

holds with probability at least

$$1 - \exp(-2\lfloor |I|\epsilon \rfloor^2 / N).$$

Proposition 3.5. For each $\epsilon > 0$ and $\delta > 0$,

$$\frac{\|b_I\|^2}{|I|} \le \frac{\|\hat{b}\|^2}{|\hat{I}|} + \epsilon(\tau_* + \delta)$$
(25)

with probability at least

$$1 - 2\exp\left(-\frac{1}{2}\delta^2 e^{-\delta}|1 - |I|/N|^2N\right) - 2\exp\left(-2\epsilon^2|1 - |I|/N|^2\frac{|I|^2}{N}\right).$$
 (26)

Proof. Since $\{\tau_j\}$ is an increasing sequence, the function $T(m) = m^{-1} \sum_{i=1}^{m} \tau_i$ is also increasing. Consider the two alternatives either $|I| \ge |\hat{I}|$ or $|\hat{I}| \ge |I|$. For the latter,

$$||b_I||^2/|I| \le ||\hat{b}||^2/|\hat{I}|$$

due to the monotonicity of T.

For the former case $|I| \ge |\hat{I}|$, we have

$$T(|I|) = |I|^{-1} \sum_{i=1}^{|\hat{I}|} \tau_i + |I|^{-1} \sum_{i=|\hat{I}|+1}^{|I|} \tau_i$$

$$\leq T(|\hat{I}|) + |I|^{-1} (|I| - |\hat{I}|) \tau_{|I|}.$$

By Proposition 3.4 (ii) $|\hat{I}| \ge (1-\epsilon)|I|$ and hence

$$T(|I|) \leq T(|\hat{I}|) + |I|^{-1}(|I| - |I|(1 - \epsilon))\tau_{|I|} = T(|\hat{I}|) + \epsilon\tau_{|I|}$$

with probability at least given by (20).

By Proposition 3.4 (i), $\tau_{|I|} \leq \tau_* + \delta$ with probability at least given by (17).

Continuing the proof of Proposition 3.2, let us consider the i.i.d. centered, bounded random variables

$$Z_i := \frac{N^2}{|I|^2} \left[b(i)^2 \chi_{\tau_*} - \mathbb{E}[b(i)^2 \chi_{\tau_*}] \right]$$
(27)

where χ_{τ_*} is the characteristic function of the set $\{b(i)^2 \leq \tau_*\}$. Note that

$$\mathbb{E}(b(j)^{2}\chi_{\tau_{*}}) = \int_{0}^{\tau_{*}} 2^{-1}z \exp(-z/2)dz = 2 - (\tau_{*}+2)\exp(-\tau_{*}/2) \le 2|I|^{2}/N^{2} \quad (28)$$

and hence

$$-2 \le Z_i \le \sup\left\{\frac{N^2}{|I|^2}b(i)^2\chi_{\tau_*}\right\} = \frac{N^2}{|I|^2}\tau_*.$$
(29)

Next recall the Bernstein-inequality.

Proposition 3.6. [9] Let Z_1, \ldots, Z_N be *i.i.d.* centered sub-exponential random variables. Then for every $t \ge 0$, we have

$$\mathbb{P}\left\{N^{-1}|\sum_{i=1}^{N} Z_i| \ge t\right\} \le 2\exp\left\{-c\min(Nt^2/K^2, Nt/K)\right\},\tag{30}$$

where c is an absolute constant and

$$K = \sup_{p \ge 1} p^{-1} (\mathbf{E} |Z_j|^p)^{1/p}.$$

Remark 3.7. For K we have the following estimates

$$K \leq \frac{2N^2}{|I|^2} \sup_{p \geq 1} p^{-1} (\mathbb{E}|b(i)^2 \chi_{\tau_*}|^p)^{1/p}$$

$$\leq \frac{2N^2}{|I|^2} \tau_* \sup_{p \geq 1} p^{-1} (\mathbb{E}\chi_{\tau_*})^{1/p}$$

$$\leq \frac{2N^2}{|I|^2} \tau_* \sup_{p \geq 1} p^{-1} (1 - e^{-\tau_*/2})^{1/p}.$$
(31)

The maximum of the right hand side of (31) occurs at

$$p_* = -\ln(1 - e^{-\tau_*/2})$$

and hence

$$K \leq \frac{2N^2}{|I|^2} \frac{\tau_*}{p_*} (1 - e^{-\tau_*/2})^{1/p_*}.$$

We are interested in the regime

$$\tau_* \asymp 2|I|/N \ll 1$$

which implies

$$p_* \simeq -\ln\frac{\tau_*}{2} \simeq \ln\frac{N}{|I|}$$

and consequently

$$K \le \frac{4N}{e|I|} \left(\ln \frac{N}{|I|} \right)^{-1}, \quad \sigma = |I|/N \ll 1.$$
(32)

On the other hand, upon substituting the asymptotic bound (32) in the probability bound

$$Q = 2\exp\left\{-c\min(Nt^2/K^2, Nt/K)\right\}$$

of (30), we have

$$K \le 2 \exp\left\{-c \min\left[\frac{e^2 t^2}{16} \left(\ln \sigma^{-1}\right)^2 |I|^2 / N, \ \frac{et}{4} |I| \ln \sigma^{-1}\right]\right\}, \quad \sigma \ll 1.$$

The Bernstein inequality ensures that with high probability

$$\left|\frac{\|\hat{b}\|^2}{N} - \mathbb{E}(b^2(i)\chi_{\tau_*})\right| \le t \frac{|I|^2}{N^2}$$

By (18) and (28), we also have

$$\frac{\|\hat{b}\|^{2}}{|\hat{I}|} \leq \mathbb{E}(b(i)^{2}\chi_{\tau_{*}})\frac{N}{|\hat{I}|} + t\frac{|I|^{2}}{|\hat{I}|N}$$

$$\leq \left(\mathbb{E}(b(i)^{2}\chi_{\tau_{*}})\frac{N^{2}}{|I|^{2}} + t\right)\frac{|I|}{N}$$

$$\leq \frac{2+t}{1-\epsilon} \cdot \frac{|I|}{N}$$
(33)

By Prop. 3.5, we now have

$$\|b_I\|^2 \leq |I| \left(\frac{\|\hat{b}\|^2}{|\hat{I}|} + \epsilon \left(\tau_* + \delta\right)\right)$$

with probability at least given by (4), which together with (33) and (15) complete the proof of Proposition 3.2.

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