APPROXIMATE SOLUTIONS OF INVERSE PROBLEMS FOR NONLINEAR SPACE FRACTIONAL DIFFUSION EQUATIONS WITH RANDOMLY PERTURBED DATA

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ABSTRACT. This paper is concerned with backward problem for nonlinear space fractional diffusion with additive noise on the right-hand side and the final value. To regularize the instable solution, we develop some new regularized method for solving the problem. In the case of constant coefficients, we use the truncation methods. In the case of perturbed time dependent coefficients, we apply a new quasi-reversibility method. We also show the convergence rate between the regularized solution and the sought solution under some a priori assumption on the sought solution.

Keywords: Inverse problem for fractional heat equation, truncation method, approximate solutions, randomly perturbed source, randomly perturbed final value.

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1. Introduction

In this paper we focus on the problem of finding the initial function $u(x,0) = u_0(x)$ such that u(x,t), t>0 satisfies the following final value problem for the nonlinear equation with fractional Laplacian

$$\begin{cases}
 u_t + a(t)(-\Delta)^{\beta} u &= F(u) + g(x,t), & (x,t) \in \Omega \times (0,T), \\
 u_x(x,t) &= 0, & x \in \partial\Omega, \\
 u(x,T) &= u_T(x), & x \in \Omega,
\end{cases}$$
(1)

where $\beta > \frac{1}{2}$ is a given constant, see equation (52) for this restriction on β . The domain $\Omega = (0, \pi)$ is 1-D domain. The function F and g are called the source functions which will be defined later. a(t) is a given time dependent coefficient. The function u_T is called the final value data. $(-\Delta)^{\beta}$ is the fractional Laplacian that will be explained in Section 2. The space and time fractional diffusion has been studied recently in [6]. In this paper, we only consider the problem with fractional order of space variable defined by spectral theory. Our fractional Laplacian in this paper differs from the fractional Laplacian defined in [6].

The problem (1) with $\beta = a(t) = 1$ is the backward problem for classical parabolic equation. These problems are applied in fields such as the heat conduction theory [1], material science [16], hydrology [2, 15], groundwater contamination [17], digital remove blurred noiseless image [4] and also in many other practical applications of mathematical physics and engineering. It is well–known that the backward parabolic problem is severely ill–posed (see [11]). The solutions do not always exist, and in the case of existence, the solutions do not depend continuously on the given initial data. In fact, from small noise contaminated physical measurements, the corresponding solutions will have large errors. Therefore, some regularized methods are required to find approximate solution. If $\beta = 1$, the deterministic case for Problem (1) has been studied by [14, 21, 22].

The analysis of regularization methods for the stable solution of Problem (1) depends on the mathematical model for the noise term on the source function g and the final value data u_T . We suppose that the measurements are described by functions

$$g^{\text{obs}} = g + \text{"noise"}, \quad u_T^{\text{obs}} = u_T + \text{"noise"}.$$
 (2)

If the noise is considered as a deterministic quantity, it is natural to study the worst-case error. In the literature a number of efficient methods for the solution of (1) have been developed: see, for example, [3, 8], and the references therein.

If the errors are generated from uncontrollable sources such as wind, rain, humidity, etc, then the model is random. If the noise is modeled as a random quantity, the convergence of estimators $\tilde{u}(x,0)$ of u(x,0) should be studied in statistical terms. More details on ill-posedness of the problem (1) in the case of $F=0,\beta=1$ with random noise can be found in [13]. Methods for the deterministic cases cannot apply directly to this case. Our main purpose in the random noise case is finding suitable estimators $\tilde{u}(x,0)$ of u(x,0) and consider the expected square error $\mathbb{E}\|\tilde{u}(x,0)-u(x,0)\|$, also called the mean integrated square error (MISE).

There exist a considerable amount of literature on regularization methods for linear backward problem with random noise. In Cavalier [5], the author gave some theoretical examples about inverse problems with random noise. Mair and Ruymgaart [12] considered theoretical formulae for statistical inverse estimation in Hilbert spaces and applied the method to some examples. Recently, Hohage et al. [9] applied spectral cut-off (truncation method) and Tikhonov-type methods for solving linear statistical inverse problems including backward heat equation (See p. 2625, [9]). In the linear inhomogenous case of (1), i.e, $\beta = 1$ and F = 0, the Problem (1) has been recently studied in [13] in two space dimensions.

To the best of the authors' knowledge, the backward problem for nonlinear parabolic equation with random noise was not investigated in the literature. This is one of the motivations of our present paper. Next we discuss the difficulty of investigating the nonlinear problem. A well–known fact is the following: if F(u) = 0 then the problem (1) and (2) can be transformed into a linear operator with random case

$$u_T = Cu_0 + \text{"noise"}. \tag{3}$$

Where C is a linear bounded operator with an unbounded inverse. There are many well-known methods developed by Cavalier [5], Hohage et al [9], Siltanen [10], Trong et al [18] as above, for solving the latter linear model. However, when F depends on u, we can not transform Problem (1) into a linear one, this makes the nonlinear problem more difficult to study. Therefore, we have to develop some new methods to solve the nonlinear problem.

In this paper, using a similar random model given in [13], we consider the nonlinear problem as follows

$$\widetilde{u}_T(x_k) = u_T(x_k) + \sigma_k \epsilon_k, \quad \widetilde{g}_k(t) = g(x_k, t) + \vartheta \xi_k(t), \quad \text{for} \quad k = \overline{1, n},$$
(4)

where $x_k = \pi \frac{2k-1}{2n}$ and ϵ_k are unknown independent random errors. Moreover, $\epsilon_k \sim \mathcal{N}(0,1)$, and σ_k are unknown positive constants which are bounded by a positive constant V_{max} , i.e., $0 \leq \sigma_k < V_{max}$ for all $k = 1, \dots, n$. $\xi_k(t)$'s are Brownian motions. The noises $\epsilon_k, \xi_k(t)$ are mutually independent. A similar model with noise in equation (4) without the g part has been

recently considered by Tuan and Nane [19].

Next we give some details about our methods for the following two cases

The first case: a(t) = 1. First, we transform the problem (1) into a nonlinear integral equation, then we apply the Fourier truncation method (using the eigenfunctions $\cos(px)$, $p = 0, 1, 2 \cdots$ of the Laplacian in the interval $(0, \pi)$ with Neumann boundary conditions) associated with some techniques in nonparametric regression to establish a first regularized solution $\overline{U}_{M_n,n}(x,t)$ which satisfies (33). To obtain the estimate between $\overline{U}_{M_n,n}$ and u, we need some stronger assumptions on u, such as (27) and (29). The main result is Theorem 2.6. However, as pointed out in Remark 2.5, the assumptions (27) and (29) are difficult to come up in practice. Motivated by this, when g = 0 we develop a second regularized solution $\widehat{U}_{M_n,n}$ defined by (69) to obtain the estimate for $u \in C([0,T]; H^{\gamma}(\Omega))$ (see Remark 2.5 for more details). The main result for the second type of regularization is given in Theorem 2.8. It is important to realize that the second regularized solution is a modification of the first regularized solution. Our methods in this paper can be applied to solve many ill-posed problems of nonlinear PDEs such as Cauchy problem for nonlinear elliptic, nonlinear ultraparabolic, nonlinear strongly damped wave equations, and many others.

The second case: a(t) depend on t and is perturbed. Note that if the coefficient a in the main equation of (1) is not noisy then the Fourier truncation method in [21] can be applied to Problem (1). However, the difficulty occurs for (86) when the time dependent coefficient a(t) is noisy. Indeed, we assume that a(t) is noisy by observed random data $\overline{a}(t)$ which satisfy that

$$\overline{a}(t) = a(t) + \epsilon \overline{\xi}(t) \tag{5}$$

where $\overline{\xi}(t)$ is Brownian motion. If we have used a Fourier truncation solution for (1), then the regularized solution would contain some terms such as $\exp\left(p^{2\beta}\int_t^T\int_t^s\overline{a}(\tau)d\tau ds\right)$, which would lead to some complex computations. Hence, we don't follow the truncation method as in [21], instead we develop a new method to find a regularized solution. We will apply a new quasi-reversibility method for solving the problem. Further details of this method can be found in Tuan [21]. In this case our **main results are Theorems 3.2 and 3.7.**

In this paper, we only study the upper bound of the convergence rate. In a future work, we will study the minimax rate of convergence for finding the optimal rate. The problem of finding minimax rate is a very difficult and interesting problem.

2. Regularized solutions for backward problem for nonlinear fractional space diffusion

2.1. **Some Notation.** We first introduce notation, and then state the first set of our main results in this paper. We define fractional powers of the Neummann-Laplacian.

$$Af := -\Delta f. \tag{6}$$

Since A is a linear densely defined self-adjoint and positive definite elliptic operator on the connected bounded domain Ω with Neumann boundary condition, the eigenvalues of A satisfy

$$\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_p \le \cdots$$

with $\lambda_p = p^2 \to \infty$ as $p \to \infty$; see [7]. The corresponding eigenfunctions are denoted respectively by $\varphi_p(x) = \sqrt{\frac{2}{\pi}}\cos(px)$. Thus the eigenpairs (λ_p, ϕ_p) , p = 0, 1, 2, ..., satisfy

$$\begin{cases} A\varphi_p(x) = -\lambda_p \phi_p(x), & x \in \Omega \\ \partial_x \phi_p(x) = 0, & x \in \partial \Omega. \end{cases}$$

The functions φ_p are normalized so that $\{\phi_p\}_{p=0}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$. Defining

$$H^{\gamma}(\Omega) = \left\{ v \in L^{2}(\Omega) : \sum_{p=0}^{\infty} \lambda_{p}^{2\gamma} |\langle v, \phi_{p} \rangle|^{2} < +\infty \right\},\,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\Omega)$, then $H^{\gamma}(\Omega)$ is a Hilbert space equipped with norm

$$||v||_{H^{\gamma}(\Omega)} = \left(\sum_{p=1}^{\infty} \lambda_p^{2\gamma} |\langle v, \phi_p \rangle|^2\right)^{1/2}.$$

Next we define the **fractional Laplacian** operator using the spectral theory.

Definition 2.1. Let $f \in L^2(\Omega)$. For each $\beta > 0$, we define fractional Laplacian using the spectral theorem as follows

$$A^{\beta} f := (-\Delta)^{\beta} f = \sum_{p=1}^{\infty} p^{2\beta} \left\langle f, \phi_p \right\rangle \phi_p(x), \tag{7}$$

where $\phi_p(x) = \sqrt{\frac{2}{\pi}}\cos(px)$. More details on this fractional Laplacian can be found in [8].

In this section we assume that a(t) = 1 and $\beta > 1/2$.

2.2. The solution of the problem (1).

Lemma 2.2. If the problem (1) has solution u then it is given by

$$u(x,t) = \sum_{p=0}^{\infty} \left[e^{(T-t)p^{2\beta}} \left\langle u_T, \phi_p \right\rangle - \int_t^T e^{(s-t)p^{2\beta}} g_p(s) ds - \int_t^T e^{(s-t)p^{2\beta}} F_p(u)(s) ds \right] \phi_p(x)$$
 (8)

where
$$g_p(t) = \langle g(\cdot, t), \phi_p \rangle$$
 and $F_p(u)(t) = \langle F((u(\cdot, t))), \phi_p \rangle$

Proof. Suppose the Problem (1) has the solution u which given by Fourier series

$$u(x,t) = \sum_{p=1}^{\infty} u_p(t)\phi_p(x), \quad \text{where} \quad u_p(t) = \left\langle u(\cdot,t), \phi_p \right\rangle. \tag{9}$$

Multiplying both sides of the equation $u_t + (-\Delta)^{\beta} u = F(u(x,t)) + g(x,t)$ by $\phi_p(x)$ and integrating over Ω leads to

$$\frac{d}{dt}u_p(t) + p^{2\beta}u_p(t) = F_p(u(t)) + g_p(t).$$
(10)

Here we have used Definition (2.1). Multiplying both sides of (10) by $e^{p^{2\beta t}}$, and by taking the integral from t to T we get

$$\int_{t}^{T} \left(e^{sp^{2\beta}} u_{p}(s) \right)'(s) ds = \int_{t}^{T} e^{sp^{2\beta}} g_{p}(s) ds + \int_{t}^{T} e^{sp^{2\beta}} F_{p}(u(s)) ds \tag{11}$$

The latter equality can be transformed into

$$u_p(t) = e^{(T-t)p^{2\beta}} \left\langle u_T, \phi_p \right\rangle - \int_t^T e^{(s-t)p^{2\beta}} g_p(s) ds - \int_t^T e^{(s-t)p^{2\beta}} F_p(u)(s) ds \tag{12}$$

where we note that $u_p(T) = \langle u_T, \phi_p \rangle$. This completes the proof of Lemma.

First, we state following Lemmas that will be used in this paper

Lemma 2.3. [Lemma 2.4 in [19]] Let $p, n \in \mathbb{N}$ such that $0 \le p \le n-1$. Assume that u_T is piecewise C^1 on $[0, \pi]$. Then

$$\langle u_T, \phi_p \rangle = \begin{cases} \frac{1}{n} \sum_{k=1}^n u_T(x_k) - \widetilde{G}_{n0}, & p = 0, \\ \frac{\pi}{n} \sum_{k=1}^n u_T(x_k) \phi_p(x_k) - \widetilde{G}_{np}, & 1 \le p \le n - 1. \end{cases}$$
(13)

where

$$\widetilde{G}_{np} = \begin{cases}
\sqrt{\frac{2}{\pi}} \sum_{l=1}^{\infty} (-1)^{l} \langle u_{T}, \phi_{2ln} \rangle, & p = 0, \\
\sum_{l=1}^{\infty} (-1)^{l} [\langle u_{T}, \phi_{p+2ln} \rangle + \langle u_{T}, \phi_{-p+2ln} \rangle], & 1 \leq p \leq n-1.
\end{cases}$$
(14)

Applying Lemma 2.3 we obtain the next result.

Lemma 2.4. Let $p, n \in \mathbb{N}$ such that $0 \le p \le n-1$. Assume that $x \to g(x,t)$ is piecewise C^1 on $[0,\pi]$. Then

$$\langle g(\cdot,t),\phi_{p}\rangle = \begin{cases} \frac{1}{n}\sum_{k=1}^{n}g(x_{k},t) - \widetilde{H}_{n0}(t), & p = 0, \\ \frac{\pi}{n}\sum_{k=1}^{n}g(x_{k},t)\phi_{p}(x_{k}) - \widetilde{H}_{np}(t), & 1 \leq p \leq n - 1. \end{cases}$$
(15)

where

$$\widetilde{H}_{np}(t) = \begin{cases}
\sqrt{\frac{2}{\pi}} \sum_{l=1}^{\infty} (-1)^{l} \langle g(\cdot, t), \phi_{2ln} \rangle, & p = 0, \\
\sum_{l=1}^{\infty} (-1)^{l} \left[\langle g(\cdot, t), \phi_{p+2ln} \rangle + \langle g(\cdot, t), \phi_{-p+2ln} \rangle \right], & 1 \leq p \leq n-1.
\end{cases}$$
(16)

We use the following representation of u in the next lemma to find an estimator of u(x,t).

Lemma 2.5. Suppose that problem (1) has solution u, then u can be represented as follows

$$u(x,t) = \Phi_{M_{n},n}(u_{T})(x,t) - \widetilde{\Phi}_{M_{n},n}(g)(x,t)$$

$$- \sum_{p=0}^{M_{n}} e^{(T-t)p^{2\beta}} \widetilde{G}_{np} \phi_{p}(x) + \sum_{p=0}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} \widetilde{H}_{np}(s) ds \right] \phi_{p}(x)$$

$$- \sum_{p=0}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} F_{p}(u)(s) ds \right] \phi_{p}(x)$$

$$+ \sum_{p=M_{n}+1}^{\infty} \left[e^{(T-t)p^{2\beta}} u_{p}(T) - \int_{t}^{T} e^{(s-t)p^{2\beta}} g_{p}(s) ds - \int_{t}^{T} e^{(s-t)p^{2\beta}} F_{p}(u)(s) ds \right] \phi_{p}(x). \quad (17)$$

Here M_n is the parameter depending on n such that $0 < M_n < n$. The terms $\widetilde{H}, \widetilde{G}$ are defined in Lemma 2.4. $\Phi, \widetilde{\Phi}$ are defined for all $f \in L^2(\Omega)$ as follows

$$\mathbf{\Phi}_{M_n,n}(f)(x,t) = \frac{1}{n} \sum_{k=1}^n f(x_k,t) + \sum_{p=1}^{M_n} e^{(T-t)p^{2\beta}} \left[\frac{\pi}{n} \sum_{k=1}^n f(x_k,t) \phi_p(x_k) \right] \phi_p(x)$$
(18)

and

$$\widetilde{\Phi}_{M_n,n}(f)(x,t) = \frac{1}{n} \sum_{k=1}^n f(x_k,t) + \sum_{p=1}^{M_n} \left[\int_t^T e^{(s-t)p^{2\beta}} \left(\frac{\pi}{n} \sum_{k=1}^n f(x_k,t) \phi_p(x_k) \right) ds \right] \phi_p(x).$$
 (19)

Proof. By Lemma 2.2, we get

$$u(x,t) = \sum_{p=0}^{\infty} \left[e^{(T-t)p^{2\beta}} u_p(T) - \int_t^T e^{(s-t)p^{2\beta}} g_p(s) ds - \int_t^T e^{(s-t)p^{2\beta}} F_p(u)(s) ds \right] \phi_p(x)$$

$$= \sum_{p=0}^{M_n} \left[e^{(T-t)p^{2\beta}} u_p(T) - \int_t^T e^{(s-t)p^{2\beta}} g_p(s) ds - \int_t^T e^{(s-t)p^{2\beta}} F_p(u)(s) ds \right] \phi_p(x)$$

$$:= A_1$$

$$+ \sum_{p=M_n+1}^{\infty} \left[e^{(T-t)p^{2\beta}} u_p(T) - \int_t^T e^{(s-t)p^{2\beta}} g_p(s) ds - \int_t^T e^{(s-t)p^{2\beta}} F_p(u)(s) ds \right] \phi_p(x). \quad (20)$$

By using Lemma 2.3 and 2.4, we obtain

$$A_{1} = \frac{1}{n} \sum_{k=1}^{n} u_{T}(x_{k}) - \widetilde{G}_{n0} + \sum_{p=1}^{M_{n}} e^{(T-t)p^{2\beta}} \left[\frac{\pi}{n} \sum_{k=1}^{n} u_{T}(x_{k}) \phi_{p}(x_{k}) - \widetilde{G}_{np} \right] \phi_{p}(x)$$

$$- \int_{t}^{T} \left[\frac{1}{n} \sum_{k=1}^{n} g(x_{k}, s) - \widetilde{H}_{n0}(s) \right] ds - \sum_{p=1}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} \left(\frac{\pi}{n} \sum_{k=1}^{n} g(x_{k}, t) \phi_{p}(x_{k}) - \widetilde{H}_{np}(s) \right) ds \right] \phi_{p}(x)$$

$$- \sum_{p=0}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} F_{p}(u)(s) ds \right] \phi_{p}(x)$$

$$(21)$$

By a simple computation, the term A_1 is equal to

$$A_{1} = \Phi_{M_{n},n}(u_{T})(x,t) - \widetilde{\Phi}_{M_{n},n}(g)(x,t) - \sum_{p=0}^{M_{n}} e^{(T-t)p^{2\beta}} \widetilde{G}_{np} \phi_{p}(x) + \sum_{p=0}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} \widetilde{H}_{np}(s) ds \right] \phi_{p}(x) - \sum_{n=0}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} F_{p}(u)(s) ds \right] \phi_{p}(x).$$

$$(22)$$

Combining (20) and (22) gives the proof of the Lemma.

- 2.3. Fourier method and regularization. We make use of the following assumptions on the functions F, g
 - (i) $F \in L^{\infty}(\mathbb{R})$ and F is a Lipschitz function, i.e. there exists a positive constant K such that

$$|F(\xi_1) - F(\xi_2)| \le K|\xi_1 - \xi_2|, \quad \forall \xi_1, \xi_2 \in \mathbb{R}.$$
 (23)

(ii) There exists positive constant $\gamma > 1$ such that

$$\sup_{0 \le t \le T} \left[\sum_{p=0}^{\infty} p^{2\gamma} \left\langle g(\cdot, t), \phi_p \right\rangle^2 \right] \le \widetilde{E}_2. \tag{24}$$

(iii) The regularized parameter M_n satisfies

$$\lim_{n \to +\infty} \frac{(M_n + 1)e^{2TM_n^{2\beta}}}{n} \quad \text{bounded.}$$
 (25)

Theorem 2.6. Suppose $\beta > 1/2$ and a(t) = 1 in equation (1). We construct a regularized approximate solution of equation (1) denoted by $\overline{U}_{M_n,n}$ that is defined by the following nonlinear

integral equation

$$\overline{U}_{M_n,n}(x,t) = \Phi_{M_n,n}(\widetilde{u}_T)(x,t) - \widetilde{\Phi}_{M_n,n}(\widetilde{g})(x,t) - \sum_{p=0}^{M_n} \left[\int_t^T e^{(s-t)p^{2\beta}} F_p(\overline{U}_{M_n,n})(s) ds \right] \phi_p(x).$$
(26)

The terms $\Phi_{M_n,n}(\widetilde{u}_T)(x,t)$ and $\widetilde{\Phi}_{M_n,n}(\widetilde{g})(x,t)$ are defined above in equations (18) and (19) respectively. Assume that problem (1) has unique solution $u \in C([0,T];L^2(\Omega))$. If there exists \widetilde{P}_1 such that

$$\sup_{0 \le t \le T} \sum_{p=1}^{\infty} e^{2p^{2\beta}t} |\langle u(\cdot, t), \phi_p \rangle|^2 \le \widetilde{P}_1$$
 (27)

Then we have

$$\mathbf{E} \left\| \overline{U}_{M_n,n}(\cdot,t) - u(\cdot,t) \right\|_{L^2(\Omega)}^2 \le 6e^{-2M_n^{2\beta}t} \left[C_3 \frac{(M_n+1)e^{2TM_n^{2\beta}}}{n} + \widetilde{P}_1 \right] e^{6K(T-t)}. \tag{28}$$

If there exists $\alpha > 0$ and \widetilde{P}_2 such that

$$\sup_{0 \le t \le T} \sum_{p=1}^{\infty} p^{2\beta\alpha} e^{2p^{2\beta}t} | \langle u(\cdot, t), \phi_p \rangle |^2 \le \widetilde{P}_2$$
 (29)

Then we have

$$\mathbf{E} \| \overline{U}_{M_n,n}(\cdot,t) - u(\cdot,t) \|_{L^2(\Omega)}^2 \le 6e^{-2M_n^{2\beta}t} \left[C_3 \frac{(M_n+1)e^{2TM_n^{2\beta}}}{n} + M_n^{-2\beta\alpha} \widetilde{P}_2 \right] e^{6K(T-t)}.$$
 (30)

Remark 2.1. In the previous Theorem, with the estimate in (28), we could get the error estimate for t > 0 but the error estimate for t = 0 is not useful. Hence, we need to assume (30) to obtain the error estimate for t = 0. It is easy to see that for t = 0, the error is of order $\max\left(\frac{(M_n+1)e^{2TM_n^2\beta}}{n}, M_n^{-2\beta\alpha}\right)$.

Remark 2.2. Let us choose M_n as follows

$$e^{2TM_n^{2\beta}} = n^{\sigma}, \quad 0 < \sigma < 1. \tag{31}$$

Then we have

$$M_n := \left(\frac{\sigma}{2T}\log(n)\right)^{\frac{1}{2\beta}}.$$
 (32)

If (27) holds then the error $\mathbf{E} \| \overline{U}_{M_n,n}(\cdot,t) - u(\cdot,t) \|_{L^2(\Omega)}^2$ is of order $n^{-\frac{\sigma t}{T}}$.

If (29) holds then the error $\mathbf{E} \| \overline{U}_{M_n,n}(\cdot,t) - u(\cdot,t) \|_{L^2(\Omega)}^2$ is of order

$$n^{-\frac{\sigma t}{T}} \max \left(\frac{\left(\frac{\sigma}{2T} \log(n)\right)^{\frac{1}{2\beta}}}{n^{1-\sigma}}, \left(\frac{\sigma}{2T} \log(n)\right)^{-\alpha} \right).$$

Hence if (29) holds and t = 0 then the error $\mathbf{E} \| \overline{U}_{M_n,n}(\cdot,0) - u(\cdot,0) \|_{L^2(\Omega)}^2$ is of order

$$\max\left(\frac{\left(\frac{\sigma}{2T}\log(n)\right)^{\frac{1}{2\beta}}}{n^{1-\sigma}}, \left(\frac{\sigma}{2T}\log(n)\right)^{-\alpha}\right).$$

Proof of Theorem 2.6. We divide the proof into two parts.

Part 1. The existence and uniqueness of the solution to the nonlinear intergral equation (26). Let us put

$$\mathbb{G}\left(w(x,t)\right) = \Phi_{M_n,n}(\widetilde{u}_T)(x,t) - \widetilde{\Phi}_{M_n,n}(\widetilde{g})(x,t) - \sum_{p=0}^{M_n} \left[\int_t^T e^{(s-t)p^{2\beta}} F_p(w)(s) ds \right] \phi_p(x). \tag{33}$$

for $w \in C([0,T];L^2(\Omega))$. We claim that for every $v,w \in C([0,T];L^2(\Omega))$

$$\|\mathbb{G}^{m}(v) - \mathbb{G}^{m}(w)\| \le \sqrt{\frac{K^{2}Te^{2TM_{n}^{2\beta}C}}{m!}} |||v - w|||, \tag{34}$$

where |||.||| is the sup norm in $C([0,T];L^2(\Omega))$. For m=1, using Hölder's inequality we have

$$\|\mathbb{G}(v)(.,t) - \mathbb{G}(w)(.,t)\|_{L^{2}(\Omega)} = \frac{\pi}{2} \sum_{p=0}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} \left(F_{p}(v)(s) - F_{p}(w)(s) \right) ds \right]^{2}$$

$$\leq \frac{\pi}{2} \sum_{p=0}^{M_{n}} \left[\int_{t}^{T} e^{2(s-t)p^{2\beta}} ds \int_{t}^{T} \left(F_{p}(v)(s) - F_{p}(w)(s) \right)^{2} ds \right]$$

$$\leq \frac{\pi}{2} \sum_{p=0}^{M_{n}} \left[e^{2TM_{n}^{2\beta}} (T-t) \int_{t}^{T} \left(F_{p}(v)(s) - F_{p}(w)(s) \right)^{2} ds \right]$$

$$= e^{2TM_{n}^{2\beta}} (T-t) \sum_{p=0}^{M_{n}} \left[\int_{t}^{T} \left(F_{p}(v)(s) - F_{p}(w)(s) \right)^{2} ds \right]$$

$$\leq e^{2TM_{n}^{2\beta}} (T-t) \|F(v)(.,t) - F(w)(.,t)\|_{L^{2}(\Omega)}^{2}. \tag{35}$$

Using the fact that F is globally Lipschitz, we get

$$||F(v)(.,t) - F(w)(.,t)||_{L^{2}(\Omega)}^{2} \le K^{2}||v(.,t) - w(.,t)||_{L^{2}(\Omega)} \le CK^{2}|||v - w|||.$$
(36)

Combining (35) and (36), we conclude that (34) holds for m = 1. By a similar method as above, we can show that (34) holds for m = j for $j \in \mathbb{N}$. It is obvious that

$$\lim_{m \to +\infty} \sqrt{\frac{K^2 T e^{2T M_n^{2\beta} C}}{m!}} = 0. \tag{37}$$

It implies that there exists a positive integer number m_0 , such that \mathbb{G}^{m_0} is a contraction. It follows that the equation $\mathbb{G}^{m_0}v = v$ has unique solution $\overline{U}_{M_n,n} \in C([0,T];L^2(\Omega))$. We claim that $\mathbb{G}\left(\overline{U}_{M_n,n}\right) = \overline{U}_{M_n,n}$. In fact, we have

$$\mathbb{G}\left(\mathbb{G}^{m_0}\left(\overline{U}_{M_n,n}\right)\right) = \mathbb{G}\left(\overline{U}_{M_n,n}\right) \tag{38}$$

Hence

$$\mathbb{G}^{m_0}\left(\mathbb{G}\left(\overline{U}_{M_n,n}\right)\right) = \mathbb{G}\left(\overline{U}_{M_n,n}\right) \tag{39}$$

The latter equality leads to $\mathbb{G}\left(\overline{U}_{M_n,n}\right)$ is a fixed point of \mathbb{G}^{m_0} . By the uniqueness of the fixed point of \mathbb{G}^{m_0} , we can conclude that $\mathbb{G}\left(\overline{U}_{M_n,n}\right) = \overline{U}_{M_n,n}$. Part 1 is completely proved.

Part 2. The error estimate between the regularized solution $\overline{U}_{M_n,n}$ and the exact solution u.

From Lemma 2.3 we get

$$\overline{U}_{M_{n},n}(x,t) - u(x,t) = \underbrace{\Phi_{M_{n},n}(\widetilde{u}_{T})(x,t) - \Phi_{M_{n},n}(u_{T})(x,t)}_{B_{1,M,n}(x,t)} + \underbrace{\widetilde{\Phi}_{M_{n},n}(\widetilde{g})(x,t) - \widetilde{\Phi}_{M_{n},n}(g)(x,t)}_{B_{2,M,n}(x,t)} + \underbrace{\sum_{p=0}^{M_{n}} e^{(T-t)p^{2\beta}} \widetilde{G}_{np}\phi_{p}(x) - \sum_{p=0}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} \widetilde{H}_{np}(s) ds \right] \phi_{p}(x)}_{B_{4,M,n}(x,t)} - \underbrace{\sum_{p=0}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} \left(F_{p}(\overline{U}_{M_{n},n}) - F_{p}(u)(s) \right) ds \right] \phi_{p}(x)}_{B_{5,M,n}(x,t)} - \underbrace{\sum_{p=M_{n}+1}^{\infty} \left[e^{(T-t)p^{2\beta}} u_{p}(T) - \int_{t}^{T} e^{(s-t)p^{2\beta}} g_{p}(s) ds - \int_{t}^{T} e^{(s-t)p^{2\beta}} F_{p}(u)(s) ds \right] \phi_{p}(x)}_{B_{6,M_{n},n}(x,t)}}$$
(40)

This implies that

$$\|\overline{U}_{M_{n},n}(\cdot,t) - u(\cdot,t)\|_{L^{2}(\Omega)}^{2} \le 6\|B_{1,M,n}\|_{L^{2}(\Omega)}^{2} + 6\|B_{2,M,n}\|_{L^{2}(\Omega)}^{2} + 6\|B_{3,M,n}\|_{L^{2}(\Omega)}^{2} + 6\|B_{4,M,n}\|_{L^{2}(\Omega)}^{2} + 6\|B_{5,M,n}\|_{L^{2}(\Omega)}^{2} + 6\|B_{6,M,n}\|_{L^{2}(\Omega)}^{2}.$$

$$(41)$$

Step 1. Estimate $\mathbf{E} \|B_{1,M_n,n}\|_{L^2(\Omega)}^2$.

Using the fact that $\widetilde{u}_T(x_k) = u_T(x_k) + \sigma_k \epsilon_k$, we get

$$\Phi_{M_{n},n}(\widetilde{u}_{T})(x,t) - \Phi_{M_{n},n}(u_{T})(x,t)
= \frac{1}{n} \sum_{k=1}^{n} \left[\widetilde{u}_{T}(x_{k})(x_{k}) - u_{T}(x_{k}) \right]
+ \sum_{p=1}^{M_{n}} e^{(T-t)p^{2\beta}} \left[\frac{\pi}{n} \sum_{k=1}^{n} \left(u_{T}^{\epsilon}(x_{k}) - u_{T}(x_{k}) \right) \phi_{p}(x_{k}) \right] \phi_{p}(x)
= \frac{1}{n} \sum_{k=1}^{n} \sigma_{k} \epsilon_{k} + \sum_{p=1}^{M_{n}} e^{(T-t)p^{2\beta}} \left[\frac{\pi}{n} \sum_{k=1}^{n} \sigma_{k} \epsilon_{k} \phi_{p}(x_{k}) \right] \phi_{p}(x).$$
(42)

The Parseval's identity implies that

$$||B_{1,M_n,n}||_{L^2(\Omega)}^2 = \left\| \Phi_{M,n}(\widetilde{u}_T(\cdot,t) - \Phi_{M,n}(u_T)(\cdot,t)) \right\|^2$$

$$= \frac{1}{n^2} \Big[\sum_{k=1}^n \sigma_k \epsilon_k \Big]^2 + \sum_{n=1}^{M_n} e^{2(T-t)p^{2\beta}} \left[\frac{\pi}{n} \sum_{k=1}^n \sigma_k \epsilon_k \phi_p(x_k) \right]^2.$$
(43)

Since the noises ϵ_k are mutually independent, we obtain $\mathbf{E}(\epsilon_i \epsilon_k) = 0$. Hence

$$\mathbf{E}\|B_{1,M_{n},n}\|_{L^{2}(\Omega)}^{2} = \frac{1}{n^{2}} \sum_{k=1}^{n} \sigma_{k}^{2} \mathbf{E} \epsilon_{k}^{2} + \sum_{p=1}^{M_{n}} e^{2(T-t)p^{2\beta}} \frac{\pi^{2}}{n^{2}} \sum_{k=1}^{n} \sigma_{k}^{2} \mathbf{E} \epsilon_{k}^{2}$$

$$\leq \frac{V_{\max}^{2}}{n} + \frac{\pi^{2} V_{\max}^{2}}{n} \sum_{p=1}^{M_{n}} e^{2(T-t)p^{2\beta}} \leq \pi^{2} V_{\max}^{2} \frac{\sum_{p=0}^{M_{n}} e^{2(T-t)p^{2\beta}}}{n}. \tag{44}$$

Step 2. Estimate $\mathbf{E} \|B_{2,M_n,n}\|_{L^2(\Omega)}^2$

From equations (4), (18), (19), we deduce that

$$\widetilde{\Phi}_{M_n,n}(\widetilde{g})(\cdot,t) - \widetilde{\Phi}_{M_n,n}(g)(\cdot,t)
= \frac{1}{n} \sum_{k=1}^n \vartheta \xi_k(t) + \sum_{p=1}^{M_n} \left[\int_t^T e^{(s-t)p^{2\beta}} \left(\frac{\pi}{n} \sum_{k=1}^n \vartheta \xi_k(s) \phi_p(x_k) \right) ds \right] \phi_p(x).$$
(45)

The Parseval's identity implies that

$$\left\|\widetilde{\Phi}_{M_n,n}(\widetilde{g})(\cdot,t) - \widetilde{\Phi}_{M_n,n}(g)(\cdot,t)\right\|_{L^2(\Omega)}^2 = \frac{1}{n^2} \left[\sum_{k=1}^n \vartheta \xi_k(t)\right]^2 + \sum_{p=1}^{M_n} \left[\int_t^T e^{(s-t)p^{2\beta}} \left(\frac{\pi}{n} \sum_{k=1}^n \vartheta \xi_k(s) \phi_p(x_k)\right) ds\right]^2$$
(46)

From the properties of Brownian motion, we know that $\mathbf{E}[\xi_i(t)\xi_k(t)] = 0$ for $k \neq i$ and $\mathbf{E}\xi_k^2(t) = t$. By the Hölder's inequality, we obtain

$$\mathbf{E}\|B_{2,M_{n},n}\|_{L^{2}(\Omega)}^{2} \leq \frac{\vartheta^{2}t}{n} + \sum_{p=1}^{M_{n}} \left[\int_{t}^{T} e^{2(s-t)p^{2\beta}} ds \int_{t}^{T} \frac{\pi^{2}}{n^{2}} \sum_{k=1}^{n} \vartheta^{2} \mathbf{E} \xi_{k}^{2}(s) \phi_{p}^{2}(x_{k}) ds \right]$$

$$\leq \frac{\vartheta^{2}T}{n} + \vartheta^{2}T^{3} \frac{\sum_{p=1}^{M_{n}} e^{2(T-t)p^{2\beta}}}{n} \leq \vartheta^{2}(T+T^{3}) \frac{\sum_{p=0}^{M_{n}} e^{2(T-t)p^{2\beta}}}{n}.$$

$$(47)$$

Step 3. Estimate $||B_{3,M_n,n}||_{L^2(\Omega)}^2$.

First, we have the following inequality

$$|F_{p}(u(t))| \leq ||F(u(\cdot,t))||_{L^{2}(\Omega)} \leq ||F(0)||_{L^{2}(\Omega)} + K||u||_{L^{2}(\Omega)}$$

$$\leq ||F(0)||_{L^{2}(\Omega)} + \max(K,1) \sup_{0 \leq t \leq T} ||u(\cdot,t)||_{L^{2}(\Omega)}, \tag{48}$$

and $||u(.,0)||_{L^2(\Omega)} \leq E$. This implies that for all $p \in \mathbb{N}, p \geq 1$

$$\left| \left\langle u(\cdot, T), \phi_p \right\rangle \right| = \left| e^{-Tp^{2\beta}} u_p(0) + \int_0^T e^{(s-T)p^{2\beta}} F_p(u)(s) ds \right|$$

$$\leq e^{-Tp^{2\beta}} \|u(\cdot, 0)\|_{L^2(\Omega)} + \frac{1 - e^{-Tp^{2\beta}}}{p^{2\beta}} \|F(u)\|_{L^{\infty}} \leq \frac{E}{Tp^{2\beta}} + \frac{E}{p^{2\beta}} \leq \frac{\widetilde{E}_1}{p^{2\beta}}. \tag{49}$$

where $\widetilde{E}_1 = \frac{E}{T} + E$.

Now, we estimate the term \widetilde{G}_{np} for $p \geq 0$. From equations (16) and (49), we have the bound for \widetilde{G}_{n0}

$$\widetilde{G}_{n0} \le \sqrt{\frac{2}{\pi}} \sum_{l=1}^{\infty} |\langle u_T, \phi_{2ln} \rangle| \le \sqrt{\frac{2}{\pi}} \sum_{l=1}^{\infty} \frac{\widetilde{E}_1}{2^{2\beta} l^{2\beta} n^{2\beta}}.$$
(50)

and the bound for \widetilde{G}_{np}

$$\widetilde{G}_{np} \leq \sum_{l=1}^{\infty} \left| \left\langle u_T, \phi_{p+2ln} \right\rangle + \left\langle u_T, \phi_{-p+2ln} \right\rangle \right|$$

$$\leq \widetilde{E} \left[\sum_{l=1}^{\infty} \frac{1}{(p+2ln)^{2\beta}} + \sum_{l=1}^{\infty} \frac{1}{(-p+2ln)^{2\beta}} \right] \leq \frac{2\widetilde{E}}{4^{\beta}} \left(\sum_{l=1}^{\infty} \frac{1}{l^{2\beta}} \right) \frac{1}{n^{2\beta}}.$$
(51)

Since $\beta > \frac{1}{2}$ by assumption, we know that the series $\sum_{l=1}^{\infty} \frac{1}{l^{2\beta}}$ converges. Let us denote

$$C_1(\beta, \widetilde{E}_1) = 2\sqrt{\frac{2}{\pi}} \frac{\widetilde{E}}{4^{\beta}} \sum_{l=1}^{\infty} \frac{1}{l^{2\beta}}.$$
 (52)

Then combining (50) and (51) gives

$$\widetilde{G}_{np} \le \frac{C_1(\beta, \widetilde{E}_1)}{n^{2\beta}}, \quad \text{for all} \quad p \ge 0.$$
 (53)

This leads to the following estimate

$$||B_{3,M_n,n}||_{L^2(\Omega)}^2 = \sum_{p=0}^{M_n} e^{2(T-t)p^{2\beta}} \left| \widetilde{G}_{np} \right|^2 \le C_1^2(\beta, \widetilde{E}_1) \frac{\sum_{p=0}^{M_n} e^{2(T-t)p^{2\beta}}}{n^{4\beta}}.$$
 (54)

Step 4. Estimate $||B_{4,M_n,n}||_{L^2(\Omega)}^2$. Using the assumption (24), we have

$$\left\langle g(\cdot,t),\phi_{p}\right\rangle = p^{-\gamma}p^{\gamma} \langle g(\cdot,t),\phi_{p}\rangle \leq \frac{\widetilde{E}_{2}}{p^{\gamma}}$$
 (55)

for $\gamma > 1$. From equation (55), we estimate $\widetilde{H}_{n0}(t)$ as follows

$$\widetilde{H}_{n0}(t) \le \sqrt{\frac{2}{\pi}} \sum_{l=1}^{\infty} |\langle g(\cdot, t), \phi_{2ln} \rangle| \le \sqrt{\frac{2}{\pi}} \sum_{l=1}^{\infty} \frac{\widetilde{E}_2}{2^{\gamma} l^{\gamma} n^{\gamma}} \le \frac{C_2(\gamma, \widetilde{E}_2)}{n^{\gamma}}.$$
 (56)

For $p \ge 1$

$$\widetilde{H}_{np}(t) \leq \sum_{l=1}^{\infty} \left| \left\langle g\left(\cdot, t\right), \phi_{p+2ln} \right\rangle + \left\langle g\left(\cdot, t\right), \phi_{-p+2ln} \right\rangle \right| \\
\leq \widetilde{E}_{2} \left[\sum_{l=1}^{\infty} \frac{1}{(p+2ln)^{\gamma}} + \sum_{l=1}^{\infty} \frac{1}{(-p+2ln)^{\gamma}} \right] \leq \frac{2\widetilde{E}_{2}}{2^{\gamma}} \left(\sum_{l=1}^{\infty} \frac{1}{l^{\gamma}} \right) \frac{1}{n^{\gamma}} \leq \frac{C_{2}(\gamma, \widetilde{E}_{2})}{n^{\gamma}}.$$
(57)

where

$$C_2(\gamma, \widetilde{E}_2) = 2\sqrt{\frac{2}{\pi}} \frac{\widetilde{E}_2}{2^{\gamma}} \sum_{l=1}^{\infty} \frac{1}{l^{\gamma}}.$$

This leads to the following estimation

$$||B_{4,M_n,n}||_{L^2(\Omega)}^2 = \sum_{n=0}^{M_n} \left[\int_t^T e^{(s-t)p^{2\beta}} \widetilde{H}_{np}(s) ds \right]^2 \le T^2 C_2^2(\gamma, \widetilde{E}_2) \frac{\sum_{p=0}^{M_n} e^{2(T-t)p^{2\beta}}}{n^{2\gamma}}.$$
 (58)

Step 5. Estimate $\mathbf{E} \|B_{5,M_n,n}\|_{L^2(\Omega)}^2$. Using Hölder's inequality and Lipschitz property of F, we have

$$||B_{5,M_{n},n}||_{L^{2}(\Omega)}^{2} = \sum_{p=0}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} \left(F_{p}(\overline{U}_{M_{n},n}) - F_{p}(u)(s) \right) ds \right]^{2}$$

$$\leq \sum_{p=0}^{M_{n}} \int_{t}^{T} e^{2(s-t)p^{2\beta}} \left(F_{p}(\overline{U}_{M_{n},n}) - F_{p}(u)(s) \right)^{2} ds$$

$$\leq \int_{t}^{T} e^{2(s-t)M_{n}^{2\beta}} \left[\sum_{p=0}^{M_{n}} \left(F_{p}(\overline{U}_{M_{n},n}) - F_{p}(u)(s) \right)^{2} \right] ds$$

$$\leq K \int_{t}^{T} e^{2(s-t)M_{n}^{2\beta}} ||\overline{U}_{M_{n},n}(.,s) - u(.,s)||_{L^{2}(\Omega)}^{2} ds.$$
(59)

This implies that

$$\mathbf{E} \|B_{5,M_n,n}\|_{L^2(\Omega)}^2 \le K \int_t^T e^{2(s-t)M_n^{2\beta}} \mathbf{E} \|\overline{U}_{M_n,n}(.,s) - u(.,s)\|_{L^2(\Omega)}^2 ds. \tag{60}$$

Step 6. Estimate $||B_{6,M_n,n}||_{L^2(\Omega)}^2$.

• If (27) holds then we get

$$||B_{6,M_n,n}||_{L^2(\Omega)}^2 = \sum_{p=M_n+1}^{\infty} \left\langle u(\cdot,t), \phi_p \right\rangle^2 = \sum_{p=M_n+1}^{\infty} e^{-2p^{2\beta}t} e^{2p^{2\beta}t} \left\langle u(\cdot,t), \phi_p \right\rangle^2 \\ \leq e^{-2M_n^{2\beta}t} \widetilde{P}_1.$$
(61)

• If (29) holds then we get

$$||B_{6,M_{n},n}||_{L^{2}(\Omega)}^{2} = \sum_{p=M_{n}+1}^{\infty} \left\langle u(\cdot,t), \phi_{p} \right\rangle^{2}$$

$$= \sum_{p=M_{n}+1}^{\infty} p^{-2\beta\alpha} e^{-2p^{2\beta}t} e^{2p^{2\beta}t} p^{2\beta\alpha} \left\langle u(\cdot,t), \phi_{p} \right\rangle^{2} \leq M_{n}^{-2\beta\alpha} e^{-2M_{n}^{2\beta}t} \widetilde{P}_{2}.$$
 (62)

We divide proof of the theorem in two cases:

Case 1: When equation (27) holds.

Combining six steps above, we get

$$\begin{split} \mathbf{E} \| \overline{U}_{M_{n},n}(\cdot,t) - u(\cdot,t) \|_{L^{2}(\Omega)}^{2} \\ &\leq 6 \mathbf{E} \| B_{1,M_{n},n} \|^{2} + 6 \mathbf{E} \| B_{2,M_{n},n} \|^{2} + 6 \| B_{3,M_{n},n} \|^{2} \\ &+ 6 \| B_{4,M_{n},n} \|^{2} + 6 \mathbf{E} \| B_{5,M_{n},n} \|^{2} + 6 \| B_{6,M_{n},n} \|^{2} \\ &\leq 6 \left(\pi^{2} V_{\max}^{2} + \vartheta^{2}(T+T^{3}) \right) \frac{\sum_{p=0}^{M} e^{2(T-t)p^{2\beta}}}{n} \\ &+ 6 C_{1}^{2} (\beta, \widetilde{E}_{1}) \frac{\sum_{p=0}^{M_{n}} e^{2(T-t)p^{2\beta}}}{n^{4\beta}} + 6 T^{2} C_{2}^{2} (\gamma, \widetilde{E}_{2}) \frac{\sum_{p=0}^{M_{n}} e^{2(T-t)p^{2\beta}}}{n^{2\gamma}} \\ &+ 6 e^{-2M_{n}^{2\beta} t} \widetilde{P}_{1} + 6 K \int_{t}^{T} e^{2(s-t)M_{n}^{2\beta}} \mathbf{E} \| \overline{U}_{M_{n},n}(.,s) - u(.,s) \|^{2} ds. \end{split}$$

This leads to

$$\mathbf{E} \| \overline{U}_{M_{n},n}(\cdot,t) - u(\cdot,t) \|_{L^{2}(\Omega)}^{2} \\
\leq 6 \left[\left(\pi^{2} V_{\max}^{2} + \vartheta^{2} (T+T^{3}) \right) + C_{1}^{2} (\beta, \widetilde{E}_{1}) + T^{2} C_{2}^{2} (\gamma, \widetilde{E}_{2}) \right] \frac{(M_{n}+1) e^{2(T-t)M_{n}^{2\beta}}}{\min(n, n^{4\beta}, n^{2\gamma})} \\
+ 6 e^{-2M_{n}^{2\beta} t} \widetilde{P}_{1} + 6K \int_{t}^{T} e^{2(s-t)M^{2\beta}} \mathbf{E} \| \overline{U}_{M_{n},n}(.,s) - u(.,s) \|_{L^{2}(\Omega)}^{2} ds.$$

Multiplying the latter inequality with $e^{2M_n^{2\beta}t}$, we obtain

$$e^{2M_{n}^{2\beta}t}\mathbf{E}\|\overline{U}_{M_{n},n}(\cdot,t) - u(\cdot,t)\|_{L^{2}(\Omega)}^{2} \leq 6C_{3}\frac{(M+1)e^{2TM^{2\beta}}}{\min(n,n^{4\beta},n^{2\gamma})} + 6\widetilde{P}_{1}$$

$$+ 6K\int_{t}^{T}e^{2sM^{2\beta}}\mathbf{E}\|\overline{U}_{M_{n},n}(\cdot,s) - u(\cdot,s)\|_{L^{2}(\Omega)}^{2}ds.$$
 (63)

Since $6C_3 \frac{(M+1)e^{2TM^{2\beta}}}{\min(n,n^{4\beta},n^{2\gamma})}$ does not depend on t, using Gronwall's inequality, we obtain

$$e^{2M_n^{2\beta}t}\mathbf{E}\|\overline{U}_{M_n,n}(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)}^2 \le \left[6C_3 \frac{(M_n+1)e^{2TM_n^{2\beta}}}{n} + 6\widetilde{P}_1\right]e^{6K(T-t)}$$
(64)

Case 2: When equation (29) holds.

By similar method as in the previous case, we get

$$e^{2M_n^{2\beta}t}\mathbf{E}\|\overline{U}_{M_n,n}(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)}^2 \le \left[6C_3 \frac{(M_n+1)e^{2TM_n^{2\beta}}}{n} + 6M_n^{-2\beta\alpha}\widetilde{P}_2\right]e^{6K(T-t)}$$
 (65)

when (29) holds.

Remark 2.3. In a future work, we will study the random case for final value problem for the time and space fractional diffusion equation in the sense of Chen et al. [6].

Remark 2.4. In theorem 2.6 we assumed that the source function F is globally Lispchitz. In some applications of our model, the extension to locally Lipschitz source functions is required. Suppose that the source function $F: \mathbb{R} \to \mathbb{R}$ satisfies that

$$|F(u) - F(v)| \le K_F(Q) |u - v|,$$
 (66)

for each Q > 0 and for any u, v with $|u|, |v| \leq Q$, where

$$K_F(Q) := \sup \left\{ \left| \frac{F(u) - F(v)}{u - v} \right| : |u|, |v| \le Q, u \ne v \right\} < +\infty.$$

Suppose that $K_F(Q)$ is increasing and $\lim_{Q\to +\infty} K_F(Q) = +\infty$. In this case, as used by Tuan [20] we approximate F by \overline{F}_{Q_n} defined by

$$\overline{F}_{Q_n}(u(x,t)) = \begin{cases} F(Q_n), & u(x,t) > Q_n, \\ F(u(x,t)), & -Q_n \le u(x,t) \le Q_n, \\ F(-Q_n), & u(x,t) < -Q_n. \end{cases}$$

where the sequence $Q_n \to +\infty$ as $n \to +\infty$. Using equation (26), we introduce the following regularized solution

$$\overline{U}_{M_n,Q_n,n}(x,t) = \mathbf{\Phi}_{M_n,n}(\widetilde{u}_T)(x,t) - \widetilde{\Phi}_{M_n,n}(\widetilde{g})(x,t) - \sum_{p=0}^{M_n} \left[\int_t^T e^{(s-t)p^{2\beta}} \overline{F}_{Q_n,p}(\overline{U}_{M_n,Q_n,n})(s) ds \right] \phi_p(x).$$
(67)

Using a similar method as in the proof of Theorem 2.6, we can get the error estimate of u by $\overline{U}_{M_n,Q_n,n}$. We omit the details of the proof here.

Remark 2.5. In Theorem 2.6, to obtain the error estimate, we require the strong assumptions (27) and (29) about u. This is a limitation of Theorem 1. There are not many functions u that satisfies these conditions. Especially in practice, these conditions are more difficult to be satisfied and checked. To remove this limitation, we introduce a new regularization solution and introduce a new technique to estimate the error in Theorem 2.6. In fact, in the next theorem we only need a weaker assumption for u. We assume that $u \in C([0,T]; H^{\gamma}(0,\pi))$ for any $\gamma > 0$. This condition is more natural.

We have the following Lemma which gives a new representation of the solution when g = 0 in equation (1).

Lemma 2.7. Suppose that g = 0 and that the problem (1) has solution u then it is represented as follows

$$u(x,t) = \Phi_{M_n,n}(u_T)(x,t) - \sum_{p=0}^{M_n} e^{(T-t)p^{2\beta}} \widetilde{G}_{np} \phi_p(x)$$

$$- \sum_{p=0}^{M_n} \left[\int_t^T e^{(s-t)p^{2\beta}} F_p(u)(s) ds \right] \phi_p(x) + \sum_{p=M_n+1}^{\infty} \left[\int_0^t e^{(s-t)p^{2\beta}} F_p(u)(s) ds \right] \phi_p(x)$$

$$+ \sum_{p=M_n+1}^{\infty} e^{-tp^{2\beta}} u_p(0) \phi_p(x). \tag{68}$$

Proof. The proof is a simple adaptation of Lemma 2.2 and we omit it here.

Theorem 2.8. Let g = 0. Assume that 5KT < 1 (where K is the Lipschitz constant of F in equation (23) and the problem (1) has unique solution u such that $u \in C([0,T];H^{\gamma}(\Omega))$. We construct another regularized solution $\hat{U}_{M_n,n}$ which is defined by the following nonlinear integral

$$\widehat{U}_{M_{n},n}(x,t) = \Phi_{M_{n},n}(\widetilde{u}_{T})(x,t) - \sum_{p=0}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} F_{p}(\widehat{U}_{M_{n},n})(s) ds \right] \phi_{p}(x)
+ \sum_{p=M_{n}+1}^{\infty} \left[\int_{0}^{t} e^{(s-t)p^{2\beta}} F_{p}(\widehat{U}_{M_{n},n})(s) ds \right] \phi_{p}(x).$$
(69)

Moreover, we have the following estimate

$$\mathbf{E}\|\widehat{U}_{M_{n},n}(\cdot,t) - u(\cdot,t)\|_{L^{2}(\Omega)}^{2} \leq e^{-2M_{n}^{2\beta}t} \frac{\left(5\pi^{2}V_{\max}^{2} + C_{1}^{2}(\beta,\widetilde{E}_{1})\right) \frac{(M_{n}+1)e^{2TM_{n}^{2\beta}}}{n} + 5M_{n}^{-2\beta\gamma}\|u(0)\|_{H^{\gamma}(\Omega)}^{2}}{1 - 5kT}$$

$$(70)$$

Proof. Part A. The integral equation (69) has unique solution in $C([0,T];L^2(\Omega))$. Let us define on C([0,T];H) the following Bielecki norm

$$||f||_1 = \sup_{0 \le t \le T} e^{tM_n} ||f(t)||, \text{ for all } f \in C([0,T]; L^2(\Omega)).$$

It is easy to show that $\|.\|_1$ is a norm of $C([0,T];L^2(\Omega))$. For $w \in C([0,T];L^2(\Omega))$, we consider the following functional

$$\mathcal{J}(w)(t) = \Phi_{M_n,n}(\widetilde{u}_T)(x,t) - \sum_{p=0}^{M_n} \left[\int_t^T e^{(s-t)p^{2\beta}} F_p(w)(s) ds \right] \phi_p(x)
+ \sum_{p=M_n+1}^{\infty} \left[\int_0^t e^{(s-t)p^{2\beta}} F_p(w)(s) ds \right] \phi_p(x),$$
(71)

where $F_p(w)(s) = \langle F(s, w(s)), \phi_p \rangle$. We shall prove that, for every $w_1, w_2 \in C([0, T]; L^2(\Omega))$,

$$\|\mathcal{J}(w_1) - \mathcal{J}(w_2)\|_1 \le KT \|w_1 - w_2\|_1. \tag{72}$$

First, by using Hölder's inequality and Lipschitz condition of F, we have the following estimates for all $t \in [0, T]$

$$\sum_{p=0}^{M_n} \left(\int_t^T e^{(s-t)p^{2\beta}} \left[F_p(w_1)(s) - F_p(w_2)(s) \right] ds \right)^2 \\
\leq (T-t) \sum_{p=0}^{M_n} \int_t^T \left| e^{(s-t)p^{2\beta}} \left[F_p(w_1)(s) - F_p(w_2)(s) \right] \right|^2 ds \\
\leq (T-t) \sum_{p=0}^{M_n} \int_t^T e^{2(s-t)M_n^{2\beta}} \left| F_p(w_1)(s) - F_p(w_2)(s) \right|^2 ds \\
\leq K^2(T-t) \int_t^T e^{2(s-t)M_n^{2\beta}} \|w_1(s) - w_2(s)\|^2 ds \\
\leq e^{-2tM_n^{2\beta}} K^2(T-t)^2 \sup_{0 \leq s \leq T} e^{2sM_n^{2\beta}} \|w_1(s) - w_2(s)\|^2 \\
= e^{-2tM_n^{2\beta}} K^2(T-t)^2 \|w_1 - w_2\|_1^2, \tag{73}$$

and

$$\sum_{p=M_{n}+1}^{\infty} \left(\int_{0}^{t} e^{(s-t)M_{n}^{2\beta}} \left[F_{p}(w_{1})(s) - F_{p}(w_{2})(s) \right] ds \right)^{2}$$

$$\leq t \sum_{p=M_{n}+1}^{\infty} \int_{0}^{t} \left| e^{(s-t)M_{n}^{2\beta}} \left[F_{p}(w_{1})(s) - F_{p}(w_{2})(s) \right] \right|^{2} ds$$

$$\leq t \sum_{p=M_{n}+1}^{\infty} \int_{0}^{t} e^{2(s-t)M_{n}^{2\beta}} \left| F_{p}(w_{1})(s) - F_{p}(w_{2})(s) \right|^{2} ds$$

$$\leq K^{2}t \int_{0}^{t} e^{2(s-t)M_{n}^{2\beta}} \|w_{1}(s) - w_{2}(s)\|^{2} ds$$

$$\leq e^{-2tM_{n}^{2\beta}} K^{2}t^{2} \sup_{0 \leq s \leq T} e^{2sM_{n}^{2\beta}} \|w_{1}(s) - w_{2}(s)\|^{2}$$

$$= e^{-2tM_{n}^{2\beta}} K^{2}t^{2} \|w_{1} - w_{2}\|_{1}^{2}. \tag{74}$$

From the definition of \mathcal{J} in (71), we have

$$\mathcal{J}(w_1)(t) - \mathcal{J}(w_2)(t) = \sum_{p=0}^{M_n} \left(-\int_t^T e^{(s-t)M_n^{2\beta}} \left[F_p(w_1)(s) - F_p(w_2)(s) \right] ds \right) \phi_p(x)
+ \sum_{p=M_n+1}^{\infty} \left(\int_0^t e^{(s-t)M_n^{2\beta}} \left[F_p(w_1)(s) - F_p(w_2)(s) \right] ds \right) \phi_p(x).$$
(75)

Combining (73), (74), (75) and using the inequality $(a+b)^2 \leq (1+\theta)a^2 + (1+\frac{1}{\theta})b^2$ for any real numbers a, b and $\theta > 0$, we get the following estimate for all $t \in (0, T)$

$$\|\mathcal{J}(w_1)(.,t) - \mathcal{J}(w_2)(.,t)\|^2 \le e^{-2tM_n^{2\beta}} K^2 (1+\theta) t^2 \|w_1 - w_2\|_1^2 + e^{-2tM_n^{2\beta}} K^2 \left(1 + \frac{1}{\theta}\right) (T-t)^2 \|w_1 - w_2\|_1^2.$$
 (76)

By choosing $\theta = \frac{T-t}{t}$, we obtain

$$e^{2tM_n^{2\beta}} \|\mathcal{J}(w_1)(t) - \mathcal{J}(w_2)(t)\|^2 \le K^2 T^2 \|w_1 - w_2\|_1^2, \text{ for all } t \in (0, T).$$
 (77)

On other hand, letting t = T in (74), we deduce

$$e^{2TM_n^{2\beta}} \|\mathcal{J}(w_1)(T) - \mathcal{J}(w_2)(T)\|^2 \le K^2 T^2 \|w_1 - w_2\|_1^2. \tag{78}$$

By letting t = 0 in (73), we have

$$\|\mathcal{J}(w_1)(0) - \mathcal{J}(w_2)(0)\|^2 \le K^2 T^2 \|w_1 - w_2\|_1^2.$$
(79)

Combining (77), (78) and (79), we obtain

$$e^{tM_n^{2\beta}} \|\mathcal{J}(w_1)(t) - \mathcal{J}(w_2)(t)\| \le KT \|w_1 - w_2\|_1, \ 0 \le t \le T,$$

which leads to (72). Since KT < 1, we can conclude that \mathcal{J} is a contraction; by the Banach fixed point theorem, it follows that the equation $\mathcal{J}(w) = w$ has a unique solution $\widehat{U}_{M_n,n} \in C([0,T];L^2(\Omega))$.

Part B. Estimate $\mathbf{E} \| \widehat{U}_{M_n,n}(\cdot,t) - u(\cdot,t) \|_{L^2(\Omega)}^2$. We have

$$\widehat{U}_{M_{n},n}(x,t) - u(x,t) = \underbrace{\Phi_{M_{n},n}(\widetilde{u}_{T})(x,t) - \Phi_{M_{n},n}(u_{T})(x,t)}_{B_{1,M_{n},n}(x,t)} + \underbrace{\sum_{p=0}^{M_{n}} e^{(T-t)p^{2\beta}} \widetilde{G}_{np} \phi_{p}(x)}_{B_{3,M,n}(x,t)} - \underbrace{\sum_{p=0}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} \left(F_{p}(\widehat{U}_{M_{n},n}) - F_{p}(u)(s) \right) ds \right] \phi_{p}(x)}_{B_{7,M,n}(x,t)} + \underbrace{\sum_{p=M_{n}+1}^{\infty} \left[\int_{0}^{t} e^{(s-t)p^{2\beta}} \left(F_{p}(\widehat{U}_{M_{n},n}) - F_{p}(u)(s) \right) ds \right] \phi_{p}(x)}_{B_{8,M_{n},n}(x,t)} - \underbrace{\sum_{p=M_{n}+1}^{\infty} e^{-tp^{2\beta}} u_{p}(0) \phi_{p}(x)}_{B_{9,M_{n},n}(x,t)} \right]}_{B_{9,M_{n},n}(x,t)} \tag{80}$$

This implies that

$$\mathbf{E}\|\widehat{U}_{M_{n},n}(\cdot,t) - u(\cdot,t)\|_{L^{2}(\Omega)}^{2} \leq 5\mathbf{E}\|B_{1,M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2} + 5\|B_{3,M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2} + 5\mathbf{E}\|B_{7,M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2} + 5\mathbf{E}\|B_{8,M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2} + 5\|B_{9,M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2}.$$
(81)

Using Hölder's inequality and Lipschitz condition of F, we have

$$||B_{7,M_{n},n}(\cdot,t)||_{L^{2}(\Omega)}^{2} = \sum_{p=0}^{M_{n}} \left[\int_{t}^{T} e^{(s-t)p^{2\beta}} \left(F_{p}(\widehat{U}_{M_{n},n}) - F_{p}(u)(s) \right) ds \right]^{2}$$

$$\leq \sum_{p=0}^{M_{n}} \int_{t}^{T} e^{2(s-t)p^{2\beta}} \left(F_{p}(\widehat{U}_{M_{n},n}) - F_{p}(u)(s) \right)^{2} ds$$

$$\leq K \int_{t}^{T} e^{2(s-t)M_{n}^{2\beta}} ||\widehat{U}_{M_{n},n}(\cdot,s) - u(\cdot,s)||_{L^{2}(\Omega)}^{2} ds.$$
(82)

The term $\mathbf{E} \|B_{7,M_n,n}(\cdot,t)\|_{L^2(\Omega)}^2$ is bounded by

$$\mathbf{E}\|B_{7,M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2} \le K \int_{t}^{T} e^{2(s-t)M_{n}^{2\beta}} \mathbf{E}\|\widehat{U}_{M_{n},n}(\cdot,s) - u(\cdot,s)\|_{L^{2}(\Omega)}^{2} ds.$$
 (83)

By a similar way as above, we estimate $\mathbf{E} \|B_{8,M_n,n}(\cdot,t)\|_{L^2(\Omega)}^2$ as follows

$$\mathbf{E} \|B_{8,M_n,n}(\cdot,t)\|_{L^2(\Omega)}^2 \le K \int_0^t e^{2(s-t)M_n^{2\beta}} \mathbf{E} \|\widehat{U}_{M_n,n}(\cdot,s) - u(\cdot,s)\|_{L^2(\Omega)}^2 ds.$$
 (84)

Finally, we bound $||B_{9,M_n,n}(\cdot,t)||^2_{L^2(\Omega)}$ by

$$||B_{9,M_{n},n}(\cdot,t)||_{L^{2}(\Omega)}^{2} = \sum_{p=M_{n}+1}^{\infty} e^{-tp^{2\beta}} p^{-2\gamma} \left(p^{\gamma} u_{p}(0) \right)^{2}$$

$$\leq e^{-2tM_{n}^{2\beta}} M_{n}^{-2\beta\gamma} \sum_{p=M_{n}+1}^{\infty} \left(p^{\gamma} u_{p}(0) \right)^{2}$$

$$\leq M_{n}^{-2\beta\gamma} e^{-2tM_{n}^{2\beta}} ||u(0)||_{H^{\gamma}(\Omega)}^{2}.$$
(85)

We estimate the bounds for the other terms as in the proof of Theorem 2.6, hence we can conclude that

$$\mathbf{E}\|\widehat{U}_{M_{n},n}(\cdot,t) - u(\cdot,t)\|_{L^{2}(\Omega)}^{2} \leq \left(5\pi^{2}V_{\max}^{2} + C_{1}^{2}(\beta,\widetilde{E}_{1})\right) \frac{(M_{n}+1)e^{2(T-t)M_{n}^{2\beta}}}{n} + 5M_{n}^{-2\beta\gamma}e^{-2tM_{n}^{2\beta}}\|u(0)\|_{H^{\gamma}(\Omega)}^{2} + 5K\int_{0}^{T}e^{2(s-t)M_{n}^{2\beta}}\mathbf{E}\|\widehat{U}_{M_{n},n}(.,s) - u(.,s)\|_{L^{2}(\Omega)}^{2}ds,$$

where we combined (80), (81), (82),(83), (84), (85). Multiplying the latter inequality with $e^{2M_n^{2\beta}t}$, we obtain

$$e^{2M_n^{2\beta}t}\mathbf{E}\|\widehat{U}_{M_n,n}(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)}^2 \le \left(5\pi^2 V_{\max}^2 + C_1^2(\beta,\widetilde{E}_1)\right) \frac{(M_n + 1)e^{2TM_n^{2\beta}}}{n} + 5M_n^{-2\beta\gamma}\|u(0)\|_{H^{\gamma}(\Omega)}^2 + 5K\int_0^T e^{2sM_n^{2\beta}}\mathbf{E}\|\widehat{U}_{M_n,n}(\cdot,s) - u(\cdot,s)\|_{L^2(\Omega)}^2 ds.$$

Since $\widehat{U}_{M_n,n}$, $u \in C([0,T]; L^2(\Omega))$ we obtain that the function $e^{2M_n^{2\beta}t}\mathbf{E}\|\widehat{U}_{M_n,n}(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}^2$ is continuous on [0,T]. Therefore, there exists a positive

$$\widetilde{A} = \sup_{0 \le t \le T} e^{2M_n^{2\beta} t} \mathbf{E} \|\widehat{U}_{M_n,n}(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)}^2.$$

This implies that

$$\widetilde{A} \le \left(5\pi^2 V_{\max}^2 + C_1^2(\beta, \widetilde{E}_1)\right) \frac{(M_n + 1)e^{2TM_n^{2\beta}}}{n} + 5M_n^{-2\beta\gamma} \|u(\cdot, 0)\|_{H^{\gamma}(\Omega)}^2 + 5kT\widetilde{A}.$$

Hence

$$e^{2M_n^{2\beta}t}\mathbf{E}\|\widehat{U}_{M_n,n}(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)}^2$$

$$\leq \widetilde{A} \leq \frac{\left(5\pi^2 V_{\max}^2 + C_1^2(\beta,\widetilde{E}_1)\right) \frac{(M_n+1)e^{2TM_n^{2\beta}}}{n} + 5M_n^{-2\beta\gamma}\|u(\cdot,0)\|_{H^{\gamma}(\Omega)}^2}{1 - 5kT}.$$

3. Space fractional diffusion equation with randomly perturbed time dependent coefficients

3.1. **Problem setting and regularization method.** In this section, we consider the inverse problem for space fractional diffusion equation with perturbed time dependent coefficients

$$\begin{cases}
 u_t + a(t)(-\Delta)^{\beta} u &= F(u(x,t)) + g(x,t), & (x,t) \in \Omega \times (0,T), \\
 u_x(x,t) &= 0, & x \in \partial\Omega, \\
 u(x,T) &= u_T(x), & x \in \Omega,
\end{cases}$$
(86)

where $0 < a(t) < a_0$ for some positive number a_0 . Here, the source function $F : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function that satisfies: for each Q > 0 and for any u, v satisfying $|u|, |v| \le Q$, there holds

$$|F(u) - F(v)| \le K_F(Q) |u - v|,$$
 (87)

where

$$K(Q) := \sup \left\{ \left| \frac{F(u) - F(v)}{u - v} \right| : |u|, |v| \le Q, u \ne v \right\} < +\infty.$$

We note that the function $Q \to K(Q)$ is increasing and $\lim_{Q \to +\infty} K(Q) = +\infty$. Next, for the ease of the reader, we describe our regularized method and analysis.

First, we approximate u(x,T) and g(x,t) by

$$\overline{w}_{M_n,n}(x) = \frac{1}{n} \sum_{k=1}^n \widetilde{u}_T(x_k) + \sum_{p=1}^{M_n} \left[\frac{\pi}{n} \sum_{k=1}^n \widetilde{u}_T(x_k) \phi_p(x_k) \right] \phi_p(x). \tag{88}$$

and

$$\overline{g}_{M_n,n}(x,t) = \frac{1}{n} \sum_{k=1}^n \widetilde{g}(x_k,t) + \sum_{n=1}^{M_n} \left[\frac{\pi}{n} \sum_{k=1}^n \widetilde{g}(x_k,t) \phi_p(x_k) \right] \phi_p(x), \tag{89}$$

respectively. Second, we approximate F by \overline{F}_Q defined by

$$\overline{F}_Q\left(u(x,t)\right) = \begin{cases} F(Q), & u(x,t) > Q, \\ F(u(x,t)), & -Q \leq u(x,t) \leq Q, \\ F(-Q), & u(x,t) < -Q. \end{cases}$$

for all Q > 0. Since K is increasing function in $[0, +\infty)$, we choose a sequence $\{Q_n\}$ satisfying $Q_n \to +\infty$ as $n \to +\infty$. Using Lemma 1.1 of the paper [20], we also have the Lipschitz continuity of the function \overline{F}_Q from the following lemma

Lemma 3.1 ([20]). For $v_1.v_2 \in L^2(\Omega)$, we have

$$\|\overline{F}_{Q_n}(v_1) - \overline{F}_{Q_n}(v_2)\|_{L^2(\Omega)} \le 2K(Q_n)\|v_1 - v_2\|_{L^2(\Omega)}.$$
(90)

Next, since a(t) is noised by $\overline{a}(t)$, using the fractional Laplacian defined by the spectral theorem we have

$$a(t)(-\Delta)^{\beta} f = a(t) \sum_{n=0}^{\infty} p^{2\beta} < f, \phi_p > \phi_p(x), \ f \in L^2(\Omega).$$

We approximate the operator $a(t)(-\Delta)^{\beta}$ by regularized operator $\overline{a}(t)(-\Delta)^{\beta} - \overline{a}_0 \mathbf{R}_{n,\beta}$. By the observations and steps above, we present the following new regularized problem using the quasi-reversibility method.

Assume that $\overline{a}(t) < a_0$ for all $t \in [0,T]$. We study the following regularized problem with Neumann boundary condition

$$\begin{cases}
\frac{\partial W_{M_{n},n}}{\partial t} + \overline{a}(t)(-\Delta)^{\beta}W_{M_{n},n} - a_{0}\overline{\mathbf{R}}_{n,\beta}(W_{M_{n},n}) \\
= \overline{F}_{Q_{n}}(W_{M_{n},n}(x,t)) + \overline{g}_{M_{n},n}(x,t), \quad (x,t) \in \Omega \times (0,T), \\
\frac{\partial W_{M_{n},n}(x,t)}{\partial x} = 0, \quad x \in \partial\Omega, \\
W_{M_{n},n}(x,T) = \overline{w}_{M_{n},n}(x), \quad x \in \Omega,
\end{cases}$$
(91)

where $\overline{\mathbf{R}}_{n,\beta}$ is defined by

$$\overline{\mathbf{R}}_{n,\beta}(v) = \sum_{p>M_n a_0^{-\frac{1}{2\beta}}}^{\infty} p^{2\beta} \langle v, \phi_p \rangle_{L^2(\Omega)} \phi_p(x), \tag{92}$$

for any function $v \in L^2(\Omega)$.

In the remainder of this section we give two results on convergence rate of $W_{M_n,n}$ to u. The first result in the next section concerns the error estimate in $L^2(\Omega)$. The second result in subsection 3.3 concern the error estimate in the higher Sobolev space $H^{\beta}(\Omega)$.

3.2. Error estimate in $L^2(\Omega)$.

Theorem 3.2. Suppose that M_n satisfies

$$\lim_{n \to +\infty} \frac{(M_n + 1)e^{2TM_n^{2\beta}}}{n} \quad bounded, \tag{93}$$

and ϵ satisfies that

$$\lim_{\epsilon \to 0} e^{TM_n^{2\beta}} \epsilon \quad bounded. \tag{94}$$

Choose Q_n such that

$$\lim_{n \to +\infty} e^{2TK(Q_n)} e^{-2tM_n^{2\beta}} = 0, \ 0 < t \le T.$$
(95)

Then the problem (91) has unique solution $W_{M_n,n} \in C([0,T];L^2(\Omega))$. Furthermore, assume that $g,u \in L^{\infty}\left([0,T];\tilde{V}(\Omega)\right)$, where $\tilde{V}(\Omega)$ is defined in (117). we have

$$\mathbf{E} \|W_{M_n,n}(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)}^2 \le e^{(2K(Q_n)+4)T} e^{-2tM_n^{\beta}} \Phi(n,u,g,\delta,\epsilon), \tag{96}$$

where

$$\Phi(n, u, g, \delta, \epsilon) = \left(\pi^{2} V_{\max}^{2} + \left|\overline{C}(\delta, u)\right|^{2} + \pi^{2} T^{3} \vartheta^{2} + T \left|\overline{D}(\delta, g)\right|^{2}\right) \frac{(M_{n} + 1) e^{2T M_{n}^{2\beta}}}{n} + \epsilon^{2} e^{2T M_{n}^{2\beta}} T^{2} \|u\|_{L^{\infty}[0, T]; H^{2\beta}(\Omega))}^{2} + \left(\|u_{T}\|_{\tilde{V}(\Omega)}^{2} + T \|g\|_{L^{\infty}([0, T]; \tilde{V}(\Omega))}^{2} + T a_{0}^{2} \|u\|_{L^{\infty}([0, T]; \tilde{V}(\Omega))}^{2}\right).$$

Remark 3.1. Let us choose M_n as follows

$$e^{2TM_n^{2\beta}} = n^{\sigma}, \quad 0 < \sigma < 1. \tag{97}$$

Then we have

$$M_n := \left(\frac{\sigma}{2T} \log(n)\right)^{\frac{1}{2\beta}}.$$
 (98)

Choose $\epsilon < \overline{\theta} < n\epsilon$, and $0 < \sigma < \frac{\log(\frac{\overline{\theta}}{\epsilon})}{\log n}$. Then $n^{\sigma} \epsilon^2 < \overline{\theta}$. Moreover, it is easy to check that M_n satisfies the condition of Theorem 3.2. We can choose Q_n such that $e^{2K(Q_n)T}n^{-\frac{\sigma t}{T}}$ bounded as $n \to \infty$. In particular we can choose Q_n such that

$$K(Q_n) \le \frac{1}{2T} \log \left(\log(n) \right).$$
 (99)

We note that the term $\log \left(\log(n)\right) \to +\infty$ as $n \to \infty$. The choice of Q_n as in 99 is suitable since we recall that the function $Q \to K(Q)$ is increasing and $\lim_{Q \to +\infty} K(Q) = +\infty$.

Under the assumptions above we can deduce that the error $\mathbf{E} \|W_{M_n,n}(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)}^2$ is of order $\log(n)n^{-\frac{\sigma t}{T}}$ for $t \in (0,T]$.

Remark 3.2. The error in inequality 96 is not useful for t = 0. To get an approximation of u(x,0), we give another Lemma below.

Lemma 3.3. Assume that M_n satisfies

$$M_n^{\beta} > \frac{1}{T}\log(\frac{1}{T}). \tag{100}$$

Then there exists unique $t_n \in (0,T)$ such that

$$e^{-t_n M_n^{\beta}} = t_n. \tag{101}$$

Assume that u satisfies that $\frac{\partial u(x,t)}{\partial t} \in L^{\infty}(0,T;L^{2}(\Omega))$. Choose Q_{n} such that

$$\lim_{n \to +\infty} \frac{e^{2TK(Q_n)}}{M_n^{\beta}} = 0. \tag{102}$$

Then we have the following estimate

$$\mathbf{E} \|W_{M_{n},n}(\cdot,t_{n}) - u(\cdot,0)\|_{L^{2}(\Omega)}^{2} \leq 2\Phi(n,u,g,\delta,\epsilon)e^{(2K(Q_{n})+4)T} \frac{1}{M_{n}^{\beta}} + 2\frac{1}{M_{n}^{\beta}} \left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|_{L^{\infty}([0,T];L^{2}(\Omega))}^{2}.$$
(103)

Remark 3.3. Suppose that Q_n satisfies (102), then the error $\mathbf{E} \|W_{M_n,n}(\cdot,t_n) - u(\cdot,0)\|_{L^2(\Omega)}^2$ is of order $\frac{e^{2TK(Q_n)}}{M^{\beta}}$.

We give a proof of Lemma 3.3 by using the estimates in Theorem 3.2.

Proof of Lemma 3.3. First, consider the function $\varphi(z) = e^{-zM_n^{\beta}} - z$ for $z \in (0,T)$. Note that φ is decreasing function and $\varphi(0) = 1$. And

$$\varphi(T) = e^{-TM_n^{\beta}} - T < 0 \tag{104}$$

by the assumption (100). This implies that the equation $\varphi(z) = 0$ has unique solution $z_0 \in (0,T)$. Since z_0 depends on n, we can denote it by t_n . Using the inequality $e^m \ge m$ for m > 0, we deduce that

$$\frac{1}{t_n} = e^{t_n M_n^{\beta}} \ge t_n M_n^{\beta}.$$

Hence

$$t_n \le \sqrt{\frac{1}{M_n^{\beta}}}. (105)$$

Now, we will consider the error $\mathbf{E} \|W_{M_n,n}(\cdot,t_n) - u(\cdot,0)\|_{L^2(\Omega)}^2$. First, using the triangle inequality and the inequality $(a_0 + a_1)^2 \leq 2a_0^2 + 2a_1^2$ for any positive a_0, a_1 , we have

$$||W_{M_{n},n}(x,t_{n}) - u(x,0)||_{L^{2}(\Omega)}^{2} \leq \left(||W_{M_{n},n}(\cdot,t_{n}) - u(\cdot,t_{n})||_{L^{2}(\Omega)} + ||u(\cdot,t_{n}) - u(\cdot,0)||_{L^{2}(\Omega)}\right)^{2}$$

$$\leq 2||W_{M_{n},n}(\cdot,t_{n}) - u(\cdot,t_{n})||_{L^{2}(\Omega)}^{2} + 2||u(\cdot,t_{n}) - u(\cdot,0)||_{L^{2}(\Omega)}^{2}.$$
(106)

Since (96) holds for any t > 0, we obtain

$$\mathbf{E} \|W_{M_n,n}(\cdot,t_n) - u(\cdot,t_n)\|_{L^2(\Omega)}^2 \le e^{(2K(Q_n) + 4)T} e^{-2t_n M_n^{\beta}} \Phi(n, u, g, \delta, \epsilon).$$
 (107)

From the Newton-Leibniz formula, we get

$$\|u(\cdot,t_n) - u(\cdot,0)\|_{L^2(\Omega)}^2 = \left\| \int_0^{t_n} \frac{\partial u(\cdot,s)}{\partial s} ds \right\|_{L^2(\Omega)}^2 \le \left(\int_0^{t_n} \left\| \frac{\partial u(\cdot,s)}{\partial s} \right\|_{L^2(\Omega)} ds \right)^2$$

$$\le t_n^2 \left\| \frac{\partial u(\cdot,t)}{\partial t} \right\|_{L^\infty([0,T];L^2(\Omega))}^2. \tag{108}$$

Combining (106),(107),(108), we get

$$\mathbf{E} \|W_{M_{n},n}(\cdot,t_{n}) - u(\cdot,0)\|_{L^{2}(\Omega)}^{2} \leq 2e^{(2K(Q_{n})+4)T}e^{-2t_{n}M_{n}^{\beta}}\Phi(n,u,g,\delta,\epsilon) + 2t_{n}^{2} \left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|_{L^{\infty}([0,T];L^{2}(\Omega))}^{2}$$

$$\leq 2\Phi(n,u,g,\delta,\epsilon)e^{(2K(Q_{n})+4)T}\frac{1}{M_{n}^{\beta}} + 2\frac{1}{M_{n}^{\beta}} \left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|_{L^{\infty}([0,T];L^{2}(\Omega))}^{2}.$$
(109)

since we have used to $e^{-t_n M_n^{\beta}} = t_n \leq \sqrt{\frac{1}{M_n^{\beta}}}$.

To prove Theorem 3.2, first we need some Lemmas.

Lemma 3.4. The problem (91) has unique solution $W_{M_n,n} \in C([0,T];L^2(\Omega))$.

Proof. Let $b(t) = a_0 - a(t)$ and $\overline{b}(t) = a_0 - \overline{a}(t)$. The equation (91) can be transformed to the following equation

$$\frac{\partial W_{M_n,n}}{\partial t} - \overline{b}(t)(-\Delta)^{\beta}W_{M_n,n} + a_0\overline{\mathbf{R}}_{n,\beta}W_{M_n,n} = \overline{F}_{Q_n}(W_{M_n,n}(x,t)) + \overline{g}_{M_n,n}(x,t). \tag{110}$$

Set $\widehat{W}_{M_n,n}(x,t) = W_{M_n,n}(x,T-t)$ then we get

$$\frac{\partial \widehat{W}_{M_n,n}}{\partial t} + \overline{b}(t)(-\Delta)^{\beta}\widehat{W}_{M_n,n} = a_0 \overline{\mathbf{R}}_{n,\beta}\widehat{W}_{M_n,n} - \overline{F}_{Q_n}(\widehat{W}_{M_n,n}(x,t)) - \overline{g}_{M_n,n}(x,t), \tag{111}$$

and the initial condition

$$\widehat{W}_{M_n,n}(x,0) = \overline{w}_{M_n,n}(x). \tag{112}$$

For any function $v \in L^2(\Omega)$, set

$$\overline{H}(v) = a_0 \overline{\mathbf{R}}_{n,\beta} v - \overline{F}_{Q_n}(v(x,t)) - \overline{g}_{M_n,n}(x,t).$$

It is easy to check that \overline{H} is globally Lipschitz function. In fact, for $v_1, v_2 \in L^2(\Omega)$, we obtain using Lemma 3.1

$$\|\overline{H}(v_1) - \overline{H}(v_2)\|_{L^2(\Omega)} \le a_0 \|\overline{\mathbf{R}}_{n,\beta}v_1 - \overline{\mathbf{R}}_{n,\beta}v_2\|_{L^2(\Omega)} + \|\overline{F}_{Q_n}(v_1) - \overline{F}_{Q_n}(v_2)\|_{L^2(\Omega)}$$

$$\le M_n^{2\beta} \|v_1 - V_2\|_{L^2(\Omega)} + 2K(Q_n)\|v_1 - v_2\|_{L^2(\Omega)}.$$
(113)

By taking the inner product of (110) with $\phi_p(x)$, we get

$$\frac{d}{dt} \left\langle \widehat{W}_{M_n,n}(.,t), \phi_p \right\rangle + \overline{b}(t) p^{2\beta} \left\langle \widehat{W}_{M_n,n}(.,t), \phi_p \right\rangle = \left\langle \overline{H}(\widehat{W}_{M_n,n}(.,t)), \phi_p \right\rangle. \tag{114}$$

By multiplying both sides of the latter equality with $\exp\left(p^{2\beta}\int_0^t \overline{b}(s)ds\right)$, and then taking the integral from 0 and t, we transform the above differential equation into the following nonlinear integral equation

$$\widehat{W}_{M_n,n}(x,t) = \sum_{p=1}^{\infty} \exp\left(-p^{2\beta} \int_0^t \overline{b}(s)ds\right) \left[\left\langle \overline{w}_{M_n,n}, \phi_p \right\rangle\right] \phi_p(x)$$

$$+ \int_0^t \exp\left(-p^{2\beta} \int_s^t \overline{b}(\xi)d\xi\right) \left[\left\langle \overline{H}(\widehat{W}_{M_n,n}(.,s)), \phi_p \right\rangle ds\right] \phi_p(x).$$
(115)

Here $\widehat{W}_{M_n,n}$ is the mild solution of (111)-(112). The existence of mild solution of nonlinear integral equation (115) is proved similarly as in the proof of part 1 of Theorem 2.6.

Lemma 3.5. Recall the definition of $\overline{\mathbf{R}}_{n,\beta}$ in equation (92) and define $\overline{\mathbf{P}}_{n,\beta}$

$$\overline{\mathbf{P}}_{n,\beta}(v) = a_0 \sum_{p < M_n a_0^{-\frac{1}{2\beta}}} p^{2\beta} \langle v, \phi_p \rangle_{L^2(\Omega)} \phi_p(x), \tag{116}$$

for any function $v \in L^2(\Omega)$. Define the following space of functions

$$\widetilde{V}(\Omega) := \left\{ \theta \in L^2(\Omega) : \sum_{p=1}^{\infty} p^{4\beta} e^{2Ta_0^{\beta} p^{2\beta}} \left\langle \theta, \phi_p \right\rangle_{L^2(\Omega)}^2 < +\infty \right\}. \tag{117}$$

Then we have

$$\|\overline{\mathbf{P}}_{n,\beta}v\|_{L^2(\Omega)} \le M_n^{2\beta} \|v\|_{L^2(\Omega)}, \quad \text{for any} \quad v \in L^2(\Omega), \tag{118}$$

and

$$a_0 \|\overline{\mathbf{R}}_{n,\beta} v\|_{L^2(\Omega)} \le a_0 e^{-TM_n^{2\beta}} \|v\|_{\widetilde{V}(\Omega)} \quad \text{for any} \quad v \in \widetilde{V}(\Omega).$$
 (119)

Proof. We obtain

$$\|\overline{\mathbf{P}}_{n}v\|_{L^{2}(\Omega)}^{2} \leq a_{0}^{2} \sum_{p < M_{n} a_{0}^{-\frac{1}{2\beta}}}^{\infty} p^{4\beta} \langle v, \phi_{p} \rangle_{L^{2}(\Omega)}^{2}$$

$$\leq M_{n}^{4\beta} \sum_{p < M_{n} a_{0}^{-\frac{1}{2\beta}}}^{\infty} \langle v, \phi_{p} \rangle_{L^{2}(\Omega)}^{2}$$

$$\leq M_{n}^{4\beta} \|v\|_{L^{2}(\Omega)}^{2}. \tag{120}$$

and

$$a_{0}^{2} \|\overline{\mathbf{R}}_{n}v\|_{L^{2}(\Omega)}^{2} \leq a_{0}^{2} \sum_{p \geq M_{n} a_{0}^{-\frac{1}{2\beta}}}^{\infty} \exp\left(-2Ta_{0}p^{2\beta}\right) p^{4\beta} \exp\left(2Ta_{0}p^{2\beta}\right) \left\langle v, \phi_{p} \right\rangle_{L^{2}(\Omega)}^{2}$$

$$\leq a_{0}^{2} e^{-2TM_{n}^{2\beta}} \sum_{p=1}^{\infty} p^{4\beta} \exp\left(2Ta_{0}p^{2\beta}\right) \left\langle v, \phi_{p} \right\rangle_{L^{2}(\Omega)}^{2}$$

$$\leq a_{0}^{2} e^{-2TM_{n}^{2\beta}} \|v\|_{\widetilde{V}(\Omega)}^{2}. \tag{121}$$

Lemma 3.6. Assume that $u_T = u(.,T) \in \tilde{V}(\Omega)$ and $g \in L^{\infty}([0,T]; \tilde{V}(\Omega))$. Then the following estimates hold

$$\mathbf{E} \| \overline{w}_{M_n,n} - u_T \|_{L^2(\Omega)}^2 \le \left(\pi^2 V_{\text{max}}^2 + \left| \overline{C}(\delta, u) \right|^2 \right) \frac{M_n + 1}{n} + e^{-2TM_n^{2\beta}} \| u_T \|_{\tilde{V}(\Omega)}^2. \tag{122}$$

and

$$\mathbf{E} \|\overline{g}_{M_{n},n}(.,t) - g(.,t)\|_{L^{2}(\Omega)}^{2} \le \left(\pi^{2} T^{2} \vartheta^{2} + \left|\overline{D}(\delta,g)\right|^{2}\right) \frac{M_{n} + 1}{n} + e^{-2T M_{n}^{2\beta}} \|g\|_{L^{\infty}([0,T];\widetilde{V}(\Omega))},\tag{123}$$

where

$$\overline{C}(\delta, u) = \max\left(\frac{2}{4^{\delta}}, \frac{\sqrt{2}}{\sqrt{\pi}2^{\delta}}\right) \left(\sum_{l=1}^{\infty} \frac{1}{l^{\delta}}\right) \|u_T\|_{H^{\delta}(\Omega)},$$

and

$$\overline{D}(\delta, g) = \max\left(\frac{2}{4^{\delta}}, \frac{\sqrt{2}}{\sqrt{\pi}2^{\delta}}\right) \left(\sum_{l=1}^{\infty} \frac{1}{l^{\delta}}\right) \|g\|_{L^{\infty}([0,T]; H^{\delta}(\Omega))}.$$

for any $\delta > 1$.

Proof. In this proof, we will find the estimate between $u_T = u(.,T)$ and the approximate function $w_{M_n,n}$ defined in (88). By the formula of u_T in Lemma 2.3, we get

$$u_{T}(x) = \sum_{p=0}^{\infty} \left\langle u_{T}, \phi_{p} \right\rangle \phi_{p}(x)$$

$$= \frac{1}{n} \sum_{k=1}^{n} u_{T}(x_{k}) - \widetilde{G}_{n0} + \sum_{p=1}^{M_{n}} \left(\frac{\pi}{n} \sum_{k=1}^{n} u_{T}(x_{k}) \phi_{p}(x_{k}) - \widetilde{G}_{np} \right) \phi_{p}(x)$$

$$+ \sum_{n=M_{n}+1}^{\infty} \left\langle u_{T}, \phi_{p} \right\rangle \phi_{p}(x). \tag{124}$$

This together with (88) and the fact that $\tilde{u}_T(x_k) = u_T(x_k) + \sigma_k \epsilon_k$ gives

$$\|\overline{w}_{M_n,n} - u_T\|_{L^2(\Omega)}^2 \le \left[\frac{1}{n} \sum_{k=1}^n \sigma_k \epsilon_k - \widetilde{G}_{n0}\right]^2 + \sum_{p=1}^{M_n} \left[\frac{\pi}{n} \sum_{k=1}^n \sigma_k \epsilon_k \phi_p(x_k) - \widetilde{G}_{np}\right]^2 + \sum_{p=M_n+1}^{\infty} \left\langle u_T, \phi_p \right\rangle^2.$$

$$(125)$$

This implies that

$$\mathbf{E} \|\overline{w}_{M_{n},n} - u_{T}\|_{L^{2}(\Omega)}^{2} \leq \frac{1}{n^{2}} \sum_{k=1}^{n} \sigma_{k}^{2} \mathbf{E} \epsilon_{k}^{2} + \left| \widetilde{G}_{n0} \right|^{2} + \left[\frac{\pi^{2}}{n^{2}} \sum_{p=1}^{M_{n}} \sum_{k=1}^{n} \sigma_{k}^{2} \mathbf{E} \epsilon_{k}^{2} \phi_{p}^{2}(x_{k}) + \sum_{p=1}^{M_{n}} \left| \widetilde{G}_{np} \right|^{2} \right] + \sum_{p=M_{n}+1}^{\infty} \left\langle u_{T}, \phi_{p} \right\rangle^{2}.$$
(126)

Further, by the formula

$$||u_T||_{H^{\delta}(\Omega)}^2 = \sum_{n=0}^{\infty} p^{2\delta} \langle u_T, \phi_p \rangle^2$$
(127)

we obtain that for all $p \in \mathbb{N}, p \geq 1$

$$\left| \left\langle u_T, \phi_p \right\rangle \right| \le \frac{\|u_T\|_{H^{\delta}(\Omega)}}{p^{\delta}}.$$
 (128)

Now, we estimate the term \widetilde{G}_{np} for $p \geq 0$. By inequality (128), we have the bound for \widetilde{G}_{n0}

$$\widetilde{G}_{n0} \le \sqrt{\frac{2}{\pi}} \sum_{l=1}^{\infty} |\langle u_T, \phi_{2ln} \rangle| \le \sqrt{\frac{2}{\pi}} \sum_{l=1}^{\infty} \frac{\|u_T\|_{H^{\delta}(\Omega)}}{2^{\delta} l^{\delta} n^{\delta}}, \tag{129}$$

and the bound for \widetilde{G}_{np}

$$\widetilde{G}_{np} \leq \sum_{l=1}^{\infty} \left| \left\langle u_{T}, \phi_{p+2ln} \right\rangle + \left\langle u_{T}, \phi_{-p+2ln} \right\rangle \right|
\leq \|u_{T}\|_{H^{\delta}(\Omega)} \left[\sum_{l=1}^{\infty} \frac{1}{(p+2ln)^{\delta}} + \sum_{l=1}^{\infty} \frac{1}{(-p+2ln)^{\delta}} \right]
\leq \frac{2\|u(.,T)\|_{H^{\delta}(\Omega)}}{4^{\delta}} \left(\sum_{l=1}^{\infty} \frac{1}{l^{\delta}} \right) \frac{1}{n^{\delta}}.$$
(130)

Since $\delta > 1$, we know that the series $\sum_{l=1}^{\infty} \frac{1}{l^{\delta}}$ converges. Let us denote

$$\overline{C}(\delta, u) = \max\left(\frac{2}{4^{\delta}}, \frac{\sqrt{2}}{\sqrt{\pi}2^{\delta}}\right) \left(\sum_{l=1}^{\infty} \frac{1}{l^{\delta}}\right) \|u_T\|_{H^{\delta}(\Omega)}.$$

By (129) and (130), we get

$$\widetilde{G}_{np} \le \frac{\overline{C}(\delta, u)}{n^{\delta}}, \quad \text{for all} \quad p \ge 0.$$
 (131)

It follows from (126) that

$$\mathbf{E} \| \overline{w}_{M_{n},n} - u_{T} \|_{L^{2}(\Omega)}^{2} \\
\leq \frac{V_{\max}^{2}(\pi^{2}M_{n} + 1)}{n} + \frac{(M_{n} + 1)|\overline{C}(\delta, u)|^{2}}{n^{2\delta}} + \sum_{p=M_{n}+1}^{\infty} \left\langle u_{T}, \phi_{p} \right\rangle^{2} \\
\leq \frac{V_{\max}^{2}(\pi^{2}M_{n} + 1)}{n} + \frac{(M_{n} + 1)|\overline{C}(\delta, u)|^{2}}{n^{2\delta}} + e^{-2TM_{n}^{2}\beta} \sum_{p=M_{n}+1}^{\infty} e^{2Tp^{2\beta}} \left\langle u_{T}, \phi_{p} \right\rangle^{2} \\
\leq \left(\pi^{2}V_{\max}^{2} + |\overline{C}(\delta, u)|^{2}\right) \frac{M_{n} + 1}{n} + e^{-2TM_{n}^{2\beta}} \|u_{T}\|_{\tilde{V}(\Omega)}^{2}. \tag{132}$$

Assume that $g \in L^{\infty}([0,T]; H^{\delta}(\Omega))$ and let us define

$$\overline{D}(\delta, g) = \max\left(\frac{2}{4^{\delta}}, \frac{\sqrt{2}}{\sqrt{\pi}2^{\delta}}\right) \left(\sum_{l=1}^{\infty} \frac{1}{l^{\delta}}\right) \|g\|_{L^{\infty}([0,T]; H^{\delta}(\Omega))}.$$

In a similar way, we show that

$$\widetilde{H}_{np}(t) \le \frac{\overline{D}(\delta, g)}{n^{\delta}}, \quad \text{for all} \quad p \ge 0, t \in [0, T].$$
 (133)

and

$$\|\overline{g}_{M_{n},n}(.,t) - g(.,t)\|^{2} \leq \left[\frac{1}{n} \sum_{k=1}^{n} \vartheta \xi_{k}(t) - \widetilde{H}_{n0}(t)\right]^{2} + \sum_{p=1}^{M_{n}} \left[\frac{\pi}{n} \sum_{k=1}^{n} \vartheta \xi_{k}(t) \phi_{p}(x_{k}) - \widetilde{H}_{np}(t)\right]^{2} + \sum_{p=M_{n}+1}^{\infty} \left\langle g(\cdot,t), \phi_{p} \right\rangle^{2}.$$
(134)

From the properties of Brownian motion, we known that $\mathbf{E}[\xi_i(t)\xi_k(t)] = 0$ for $k \neq i$ and $\mathbf{E}\xi_k^2(t) = t$. By the Hölder inequality, we obtain

$$\mathbf{E}\|\overline{g}_{M_{n},n}(.,t) - g(.,t)\|^{2} \leq \frac{1}{n^{2}} \sum_{k=1}^{n} \vartheta^{2} \mathbf{E} \xi_{k}^{2}(t) + \sum_{p=1}^{M_{n}} \frac{\pi^{2}}{n^{2}} \sum_{k=1}^{n} \vartheta^{2} \mathbf{E} \xi_{k}^{2}(t) \phi_{p}^{2}(x_{k}) + \sum_{p=0}^{M_{n}} \left| \widetilde{H}_{np}(t) \right|^{2}$$

$$+ e^{-2TM_{n}^{\beta}} \sum_{p=M_{n}+1}^{\infty} e^{2Tp^{2\beta}} \left\langle g(\cdot,t), \phi_{p} \right\rangle^{2}$$

$$\leq \frac{T\vartheta^{2}(\pi^{2}M_{n}+1)}{n} + \frac{(M_{n}+1)\left|\overline{D}(\delta,g)\right|^{2}}{n^{2\delta}} + e^{-2TM_{n}^{2\beta}} \|g\|_{L^{\infty}([0,T];\widetilde{V}(\Omega))}^{2}$$

$$\leq \left(\pi^{2}T\vartheta^{2} + \left|\overline{D}(\delta,g)\right|^{2}\right) \frac{M_{n}+1}{n} + e^{-2TM_{n}^{2\beta}} \|g\|_{L^{\infty}([0,T];\widetilde{V}(\Omega))}^{2}.$$

$$(135)$$

Proof of Theorem 3.2. We now return the proof of Theorem. The main equation in (86) can be rewritten as follows

$$\frac{\partial u}{\partial t} + \overline{a}(t)(-\Delta)^{\beta}u = F(u(x,t)) + g(x,t) + \left(\overline{a}(t) - a(t)\right)(-\Delta)^{\beta}u. \tag{136}$$

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For $\rho_n > 0$, we put $\mathbf{Y}_{M_n,n}(x,t) = e^{\rho_n(t-T)} [W_{M_n,n}(x,t) - u(x,t)]$. Then, from the last two equalities, a simple computation gives

$$\frac{\partial \mathbf{Y}_{M_{n},n}}{\partial t} - \overline{b}(t)(-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} - \rho_{n} \mathbf{Y}_{M_{n},n}$$

$$= -e^{\rho_{n}(t-T)} a_{0} \overline{\mathbf{P}}_{n,\beta} \mathbf{Y}_{M_{n},n} + e^{\rho_{n}(t-T)} a_{0} \overline{\mathbf{R}}_{n,\beta} u - e^{\rho_{n}(t-T)} \left(\overline{a}(t) - a(t) \right) (-\Delta)^{\beta} u$$

$$+ e^{\rho_{n}(t-T)} \left[\overline{F}_{Q_{n}}(W_{M_{n},n}(x,t)) - F(u(x,t)) \right]$$

$$+ e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(x,t)) - g(x,t) \right], \quad (x,t) \in \Omega \times (0,T), \tag{137}$$

and

$$\frac{\partial \mathbf{Y}_{M_n,n}}{\partial x}|_{\partial\Omega} = 0, \ \mathbf{Y}_{M_n,n}(x,T) = \overline{w}_{M_n,n}(x) - u_T(x).$$

Here, we note that

$$\left\langle v, (-\Delta)^{\beta} v \right\rangle_{L^{2}(\Omega)} = \int_{\Omega} \left(\sum_{p=0}^{\infty} \langle v, \phi_{p} \rangle \phi_{p}(x) \right) \left(\sum_{p=0}^{\infty} p^{2\beta} \langle v, \phi_{p} \rangle \phi_{p}(x) \right) dx$$

$$= \sum_{p=0}^{\infty} p^{2\beta} \langle v, \phi_{p} \rangle^{2} = \|v\|_{H^{\beta}(\Omega)}^{2}. \tag{138}$$

By taking the inner product of the two sides of the latter equality with $\mathbf{Y}_{M_n,n}$ one deduces that

$$\frac{1}{2} \frac{d}{dt} \| \mathbf{Y}_{M_{n},n}(\cdot,t) \|_{L^{2}(\Omega)}^{2} - \overline{b}(t) \| \mathbf{Y}_{M_{n},n}(\cdot,t) \|_{H^{\beta}(\Omega)}^{2} - \rho_{n} \| \mathbf{Y}_{M_{n},n}(\cdot,t) \|_{L^{2}(\Omega)}^{2}$$

$$= \underbrace{\left\langle -e^{\rho_{n}(t-T)} a_{0} \overline{\mathbf{P}}_{n,\beta} \mathbf{Y}_{M_{n},n}, \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{1,n}} + \underbrace{\left\langle e^{\rho_{n}(t-T)} a_{0} \overline{\mathbf{R}}_{n,\beta} u, \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{2,n}}$$

$$+ \underbrace{\left\langle -e^{\rho_{n}(t-T)} \left(\overline{a}(t) - a(t) \right) (-\Delta)^{\beta} u, \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{3,n}}$$

$$+ \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{F}_{Q_{n}}(W_{M_{n},n}(\cdot,t)) - F(u(\cdot,t)) \right], \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{4,n}}$$

$$+ \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t) \right) - g(\cdot,t) \right], \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{5,n}}.$$
(139)

First, thanks to Lemma 3.5, we bound $\widetilde{\mathcal{J}}_{1,n}$ using the Cauchy-Schwartz inequality as follows

$$\left| \widetilde{\mathcal{J}}_{1,n} \right| \le \| \mathbf{P}_n \mathbf{Y}_{M_n,n} \|_{L^2(\Omega)} \| \mathbf{Y}_{M_n,n}(\cdot,t) \|_{L^2(\Omega)} \le M_n^{2\beta} \| \mathbf{Y}_{M_n,n}(\cdot,t) \|_{L^2(\Omega)}^2.$$
 (140)

Using Lemma 3.5 and Cauchy-Schwartz inequality, the term $\widetilde{\mathcal{J}}_{2,n}$ can be estimated by

$$\begin{aligned} \left| \widetilde{\mathcal{J}}_{2,n} \right| &\leq \frac{1}{2} e^{2\rho_n (t-T)} a_0^2 e^{-TM_n^{2\beta}} \|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^2 + \frac{1}{2} \|\mathbf{Y}_{M_n,n}(\cdot,t)\|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{2} a_0^2 e^{-2TM_n^{2\beta}} \|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^2 + \frac{1}{2} \|\mathbf{Y}_{M_n,n}(\cdot,t)\|_{L^2(\Omega)}^2, \end{aligned}$$
(141)

From the inequality $2\langle a_1, a_2 \rangle_{L^2(\Omega)} \le ||a_1||_{L^2(\Omega)}^2 + ||a_2||_{L^2(\Omega)}^2$ for any $a_i \in L^2(\Omega)$, (i = 1, 2), we infer

$$\begin{aligned} \left| \widetilde{\mathcal{J}}_{3,n} \right| &= \left| \left\langle -e^{\rho_n(t-T)} \left(\overline{a}(t) - a(t) \right) (-\Delta)^{\beta} u, \mathbf{Y}_{M_n,n} \right\rangle_{L^2(\Omega)} \right| \\ &\leq \frac{1}{2} e^{2\rho_n(t-T)} \left(\overline{a}(t) - a(t) \right)^2 \left\| (-\Delta)^{\beta} u \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \| \mathbf{Y}_{M_n,n}(\cdot,t) \|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{2} e^{2\rho_n(t-T)} \left(\overline{a}(t) - a(t) \right)^2 \| u \|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^2 + \frac{1}{2} \| \mathbf{Y}_{M_n,n}(\cdot,t) \|_{L^2(\Omega)}^2. \end{aligned}$$
(142)

Finally, since $\lim_{n\to+\infty}Q_n=+\infty$, for a sufficiently large n>0 such that $Q_n\geq \|u\|_{L^\infty([0,T];L^2(\Omega))}$. Moreover, we have $\overline{F}_{Q_n}(u(x,t))=F(u(x,t))$. Using the global Lipschitz property of \overline{F}_{Q_n} , one similarly has for $|\widetilde{\mathcal{J}}_{4,n}|$ the fact that

$$\begin{aligned} \left| \widetilde{\mathcal{J}}_{4,n} \right| &= \left| \left\langle e^{\rho_n(t-T)} \left[\overline{F}_{Q_n}(W_{M_n,n}(\cdot,t)) - F(u(\cdot,t)) \right], \mathbf{Y}_{M_n,n} \right\rangle_{L^2(\Omega)} \right| \\ &\leq \left\| e^{\rho_n(t-T)} \left[\overline{F}_{Q_n}(W_{M_n,n}(\cdot,t)) - \overline{F}_{Q_n}(u(x,t)) \right] \right\|_{L^2(\Omega)} \|\mathbf{Y}_{M_n,n}\|_{L^2(\Omega)} \\ &\leq 2K(Q_n) \|\mathbf{Y}_{M_n,n}(\cdot,t)\|_{L^2(\Omega)}^2. \end{aligned} \tag{143}$$

The term $|\widetilde{\mathcal{J}}_{5,n}|$ can be bounded by

$$\left| \widetilde{\mathcal{J}}_{5,n} \right| = \left| \left\langle e^{\rho_n(t-T)} \left[\overline{g}_{M_n,n}(\cdot,t) \right) - g(\cdot,t) \right], \mathbf{Y}_{M_n,n} \right\rangle_{L^2(\Omega)} \right|$$

$$\leq \frac{1}{2} e^{2\rho_n(t-T)} \left\| \overline{g}_{M_n,n}(\cdot,t) \right) - g(\cdot,t) \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \| \mathbf{Y}_{M_n,n} \|_{L^2(\Omega)}^2. \tag{144}$$

Combining (139), (140), (141), (142) and (144) gives

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{Y}_{M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2} - \rho_{n} \|\mathbf{Y}_{M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2} \\
\geq -M_{n}^{2\beta} \|\mathbf{Y}_{M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2} - \frac{1}{2} a_{0}^{2} e^{-2TM_{n}^{2\beta}} \|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2} \\
- \frac{1}{2} \|\mathbf{Y}_{M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2} - e^{2\rho_{n}(t-T)} \left(\overline{a}(t) - a(t)\right)^{2} \|u\|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^{2} \\
- \|\mathbf{Y}_{M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2} - 2K(Q_{n}) \|\mathbf{Y}_{M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2} \\
- \frac{1}{2} e^{2\rho_{n}(t-T)} \|\overline{g}_{M_{n},n}(\cdot,t)) - g(\cdot,t)\|_{L^{2}(\Omega)}^{2} - \frac{1}{2} \|\mathbf{Y}_{M_{n},n}\|_{L^{2}(\Omega)}^{2}. \tag{145}$$

By taking the integral from t to T and by a simple calculation yields

$$\|\mathbf{Y}_{M_{n},n}(\cdot,T)\|_{L^{2}(\Omega)}^{2} - \|\mathbf{Y}_{M_{n},n}(\cdot,t)\|_{L^{2}(\Omega)}^{2}$$

$$+ \int_{t}^{T} \left(a_{0}^{2}e^{-2TM_{n}^{2\beta}}\|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2} + \left(\overline{a}(s) - a(s)\right)^{2}\|u\|_{L^{\infty}[0,T];H^{2\beta}(\Omega))}^{2}\right) ds$$

$$+ \int_{t}^{T} e^{2\rho_{n}(s-T)} \|\overline{g}_{M_{n},n}(\cdot,s)) - g(\cdot,s)\|_{L^{2}(\Omega)}^{2} ds$$

$$\geq \int_{t}^{T} \left(2\rho_{n} - 2M_{n}^{2\beta} - 4K(Q_{n}) - 4\right) \|\mathbf{Y}_{M_{n},n}(\cdot,s)\|_{L^{2}(\Omega)}^{2} ds. \tag{146}$$

Let us choose $\rho_n = M_n^{2\beta}$. This leads to

$$\begin{split} e^{2\rho_{n}(t-T)} \|W_{M_{n},n}(\cdot,t) - u(\cdot,t)\|_{L^{2}(\Omega)}^{2} \\ &\leq \|\overline{w}_{M_{n},n}(\cdot) - u_{T}(\cdot)\|_{L^{2}(\Omega)}^{2} + Ta_{0}^{2}e^{-2TM_{n}^{\beta}} \|u\|_{L^{\infty}\left([0,T];\tilde{V}(\Omega)\right)}^{2} \\ &+ T \sup_{0 \leq t \leq T} \left\|\overline{g}_{M_{n},n}(\cdot,t)\right) - g(\cdot,t) \right\|_{L^{2}(\Omega)}^{2} + \|u\|_{L^{\infty}\left([0,T];H^{2\beta}(\Omega)\right)}^{2} \int_{0}^{T} \left(\overline{a}(s) - a(s)\right)^{2} ds \\ &+ \left(4K(Q_{n}) + 4\right) \int_{t}^{T} e^{2\rho_{n}(s-T)} \|W_{M_{n},n}(\cdot,s) - u(\cdot,s)\|_{L^{2}(\Omega)}^{2} ds. \end{split}$$

Hence, we obtain

$$e^{2\rho_{n}(t-T)}\mathbf{E}\|W_{M_{n},n}(\cdot,t) - u(\cdot,t)\|_{L^{2}(\Omega)}^{2}$$

$$\leq \mathbf{E}\|\overline{w}_{M_{n},n}(\cdot) - u_{T}(\cdot)\|_{L^{2}(\Omega)}^{2} + Ta_{0}^{2}e^{-2TM_{n}^{\beta}}\|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2}$$

$$+ \int_{t}^{T} e^{2\rho_{n}(s-T)}\mathbf{E}\|\overline{g}_{M_{n},n}(\cdot,s)) - g(\cdot,s)\|_{L^{2}(\Omega)}^{2} ds + \epsilon^{2}\|u\|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^{2} \int_{0}^{T} \mathbf{E}|\overline{\xi}(s)|^{2} ds$$

$$+ \left(4K(Q_{n}) + 4\right) \int_{t}^{T} e^{2\rho_{n}(s-T)}\mathbf{E}\|W_{M_{n},n}(\cdot,s) - u(\cdot,s)\|_{L^{2}(\Omega)}^{2} ds. \tag{147}$$

In the above we have used the fact that $\overline{a}(t) - a(t) = \epsilon \overline{\xi}(t)$ and $\mathbf{E} \left| \overline{\xi}(t) \right|^2 = t$. Using the second inequality of Lemma 3.6 and noting that $e^{2\rho_n(s-T)} \leq 1$ for $0 \leq s \leq T$, we have the following estimate

$$\int_{t}^{T} e^{2\rho_{n}(s-T)} \mathbf{E} \left\| \overline{g}_{M_{n},n}(\cdot,s) - g(\cdot,s) \right\|_{L^{2}(\Omega)}^{2} ds$$

$$\leq \int_{t}^{T} e^{2\rho_{n}(s-T)} \left[\left(\pi^{2} T^{2} \vartheta^{2} + \left| \overline{D}(\delta,g) \right|^{2} \right) \frac{M_{n}+1}{n} + e^{-2TM_{n}^{\beta}} \|g\|_{L^{\infty}([0,T];\widetilde{V}(\Omega))} \right] ds$$

$$\leq \int_{t}^{T} \left[\left(\pi^{2} T^{2} \vartheta^{2} + \left| \overline{D}(\delta,g) \right|^{2} \right) \frac{M_{n}+1}{n} + e^{-2TM_{n}^{\beta}} \|g\|_{L^{\infty}([0,T];\widetilde{V}(\Omega))} \right] ds$$

$$= \left[\left(\pi^{2} T^{2} \vartheta^{2} + \left| \overline{D}(\delta,g) \right|^{2} \right) \frac{M_{n}+1}{n} + e^{-2TM_{n}^{\beta}} \|g\|_{L^{\infty}([0,T];\widetilde{V}(\Omega))} \right] (T-t)$$

$$\leq \left(\pi^{2} T^{3} \vartheta^{2} + T |\overline{D}(\delta,g)|^{2} \right) \frac{M_{n}+1}{n} + T e^{-2TM_{n}^{\beta}} \|g\|_{L^{\infty}([0,T];\widetilde{V}(\Omega))}.$$
(148)

From the observations above and by using the first inequality in Lemma 3.6, we conclude that

$$e^{2\rho_{n}(t-T)}\mathbf{E}\|W_{M_{n},n}(\cdot,t) - u(\cdot,t)\|_{L^{2}(\Omega)}^{2}$$

$$\leq \left(\pi^{2}V_{\max}^{2} + \left|\overline{C}(\delta,u)\right|^{2} + \pi^{2}T^{3}\vartheta^{2} + T\left|\overline{D}(\delta,g)\right|^{2}\right)\frac{M_{n}+1}{n} + \epsilon^{2}T^{2}\|u\|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^{2}$$

$$+ e^{-2TM_{n}^{\beta}}\left(\|u_{T}\|_{\tilde{V}(\Omega)}^{2} + T\|g\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2} + Ta_{0}^{2}\|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2}\right)$$

$$+ \left(4K(Q_{n}) + 4\right)\int_{t}^{T} e^{2\rho_{n}(s-T)}\mathbf{E}\|W_{M_{n},n}(\cdot,s) - u(\cdot,s)\|_{L^{2}(\Omega)}^{2}ds. \tag{149}$$

Multiplying both sides by $e^{2TM_n^{2\beta}}$, we obtain

$$e^{2tM_n^{2\beta}}\mathbf{E}\|W_{M_n,n}(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}^2$$

$$\leq \left(\pi^{2}V_{\max}^{2} + \left|\overline{C}(\delta, u)\right|^{2} + \pi^{2}T^{3}\vartheta^{2} + T\left|\overline{D}(\delta, g)\right|^{2}\right) \frac{(M_{n} + 1)e^{2TM_{n}^{2\beta}}}{n} + \epsilon^{2}e^{2TM_{n}^{2\beta}}T^{2}\|u\|_{L^{\infty}([0, T]; H^{2\beta}(\Omega))}^{2} \\
+ \left(\|\mathbf{u}_{T}\|_{\tilde{V}(\Omega)}^{2} + T\|g\|_{L^{\infty}([0, T]; \tilde{V}(\Omega))}^{2} + Ta_{0}^{2}\|u\|_{L^{\infty}([0, T]; \tilde{V}(\Omega))}^{2}\right) \\
+ \left(4K(Q_{n}) + 4\right) \int_{t}^{T} e^{2sM_{n}^{\beta}} \mathbf{E}\|W_{M_{n}, n}(\cdot, s) - u(\cdot, s)\|_{L^{2}(\Omega)}^{2} ds.$$

Applying Gronwall's inequality, we get

$$e^{2tM_n^{2\beta}}\mathbf{E}\|W_{M_n,n}(\cdot,t)-u(\cdot,t)\|_{L^2(\Omega)}^2 \le e^{(4K(Q_n)+4)T}\Phi(n,u,g,\delta,\epsilon),$$

where

$$\Phi(n, u, g, \delta, \epsilon) = \left(\pi^{2} V_{\text{max}}^{2} + \left|\overline{C}(\delta, u)\right|^{2} + \pi^{2} T^{3} \vartheta^{2} + T \left|\overline{D}(\delta, g)\right|^{2}\right) \frac{(M_{n} + 1) e^{2T M_{n}^{2\beta}}}{n} + \epsilon^{2} e^{2T M_{n}^{2\beta}} T^{2} \|u\|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^{2} + \left(\|u_{T}\|_{\tilde{V}(\Omega)}^{2} + T \|g\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2} + T a_{0}^{2} \|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2}\right). \tag{150}$$

Hence

$$\mathbf{E} \|W_{M_n,n}(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)}^2 \le e^{(4K(Q_n)+4)T} e^{-2tM_n^{\beta}} \Phi(n,u,g,\delta,\epsilon).$$

3.3. Error estimate in $H^{\beta}(\Omega)$. In this subsection, we give error estimate between the regularized solution and the sought solution in higher Sobolev spaces.

Theorem 3.7. Suppose that M_n satisfies

$$\lim_{n \to +\infty} \frac{(M_n^{2\beta+1} + M_n^{2\beta})e^{2M_n^{2\beta}T}}{n} \quad bounded, \tag{151}$$

and ϵ satisfies the condition in equation (94). Let us choose Q_n such that

$$\lim_{n \to +\infty} e^{-2M_n^{2\beta} t} \exp\left(\frac{8}{b_0} K^2(Q_n) T\right) = 0, \ t \in (0, T].$$
 (152)

Assume that $g, u \in L^{\infty}\left([0,T]; \tilde{V}(\Omega)\right)$, where $\tilde{V}(\Omega)$ is defined in Lemma 3.5. Then we have

$$\mathbf{E}\|W_{M_n,n}(\cdot,t) - u(\cdot,t)\|_{H^{\beta}(\Omega)}^2 \le e^{-2M_n^{2\beta}t} \exp\left(\frac{8}{b_0}K^2(Q_n)(T-t)\right) \mathbf{\Pi}(n,u,g,\delta,\epsilon).$$
(153)

Where

$$\mathbf{\Pi}(n, u, g, \delta, \epsilon) = \left(\pi^{2} V_{\max}^{2} + \left|\overline{C}(\delta, u)\right|^{2} + \frac{8}{b_{0}} \pi^{2} T^{3} \vartheta^{2} + \frac{8}{b_{0}} T \left|\overline{D}(\delta, g)\right|^{2}\right) \frac{(M_{n}^{2\beta+1} + M_{n}^{2\beta}) e^{2M_{n}^{2\beta} T}}{n} + \frac{4e^{2M_{n}^{2\beta} T} \epsilon^{2} T^{2}}{b_{0}} \|u\|_{L^{\infty}[0, T]; H^{2\beta}(\Omega))}^{2} + \|u_{T}\|_{\tilde{V}(\Omega)}^{2} + \frac{8T}{b_{0}} \|g\|_{L^{\infty}([0, T]; \tilde{V}(\Omega))}^{2} + \frac{8T a_{0}^{2}}{b_{0}} \|u\|_{L^{\infty}([0, T]; \tilde{V}(\Omega))}^{2}$$
(154)

Remark 3.4. 1. It is easy to see that when t > 0 then the error $\mathbf{E} \|W_{M_n,n}(\cdot,t) - u(\cdot,t)\|_{H^{\beta}(\Omega)}^2$ is of order

$$e^{-2M_n^{2\beta}t} \exp\left(\frac{8}{b_0}K^2(Q_n)T\right). \tag{155}$$

One example for M_n and Q_n for Theorem 3.7 can be found in Remark 3.1. The estimate at t = 0 is showed by similar argument as in Lemma 3.3, so, we omit it here.

2. In Theorems 3.2 and 3.7 the upper bounds of the approximations are very complex. The reason is that the models in Section 3 are more complex than the ones in Section 2. Indeed, in section 3, the time dependent coefficient a(t) is noisy by random coefficients and the source term F is locally Lipschitz continuous. The estimates in this case are not simple.

Proof of Theorem 3.7. Assume that $\overline{b}(t) \geq b_0 > 0$. By taking the inner product of the two sides of equality (137) with $(-\Delta)^{\beta} \mathbf{Y}_{M_n,n}$ one deduces that

$$\frac{1}{2} \frac{d}{dt} \| \mathbf{Y}_{M_{n},n}(\cdot,t) \|_{H^{\beta}(\Omega)}^{2} - \overline{b}(t) \| (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \|_{L^{2}(\Omega)}^{2} - \rho_{n} \| \mathbf{Y}_{M_{n},n}(\cdot,t) \|_{H^{\beta}(\Omega)}^{2}$$

$$= \underbrace{\left\langle -e^{\rho_{n}(t-T)} a_{0} \overline{\mathbf{P}}_{n,\beta} \mathbf{Y}_{M_{n},n}, (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{6,n}} + \underbrace{\left\langle e^{\rho_{n}(t-T)} a_{0} \overline{\mathbf{R}}_{n,\beta} u, (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{8,n}} + \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{a}(t) - a(t) \right) (-\Delta)^{\beta} u, (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{9,n}} + \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{F}_{Q_{n}}(W_{M_{n},n}(\cdot,t)) - F(u(\cdot,t)) \right], -(-\Delta)^{\beta} \widehat{\mathbf{Y}}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{9,n}} + \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t)) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{10,n}} \cdot \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{10,n}} \cdot \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{10,n}} \cdot \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{10,n}} \cdot \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{10,n}} \cdot \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{10,n}} \cdot \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{10,n}} \cdot \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{10,n}} \cdot \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{10,n}} \cdot \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{10,n}} \cdot \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{10,n}} \cdot \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \right\rangle_{L^{2}(\Omega)}}_{=:\widetilde{\mathcal{J}}_{10,n}} \cdot \underbrace{\left\langle e^{\rho_{n}(t-T)} \left[\overline{$$

For $\widetilde{\mathcal{J}}_{6,n}$, we have

$$\begin{split} &|\widetilde{\mathcal{J}}_{6,n}| = \left| \left\langle -e^{\rho_n(t-T)} a_0 \overline{\mathbf{P}}_{n,\beta} \mathbf{Y}_{M_n,n}, (-\Delta)^{\beta} \mathbf{Y}_{M_n,n} \right\rangle_{L^2(\Omega)} \right| \\ &= e^{\rho_n(t-T)} a_0 \int_{\Omega} \left(\sum_{p < M_n a_0^{-\frac{1}{2\beta}}} p^{2\beta} < \mathbf{Y}_{M_n,n}, \phi_p > \phi_p(x) \right) \left(\sum_{p=0}^{\infty} p^{2\beta} < \mathbf{Y}_{M_n,n}, \phi_p > \phi_p(x) \right) dx \\ &= e^{\rho_n(t-T)} a_0 \sum_{p < M_n a_0^{-\frac{1}{2\beta}}} p^{4\beta} < \mathbf{Y}_{M_n,n}, \phi_p >^2 \\ &\leq M_n^{2\beta} \sum_{p < M_n a_0^{-\frac{1}{2\beta}}} p^{2\beta} < \mathbf{Y}_{M_n,n}, \phi_p >^2 \leq M_n^{2\beta} \|\mathbf{Y}_{M_n,n}\|_{H^{\beta}(\Omega)}^2. \end{split}$$

We bound $\widetilde{\mathcal{J}}_{7,n}$ by

$$\begin{split} \left| \widetilde{\mathcal{J}}_{7,n} \right| &\leq \frac{4}{b_0} e^{2\rho_n (t-T)} a_0^2 \left\| \overline{\mathbf{R}}_{n,\beta} \mathbf{Y}_{M_n,n} \right\|_{L^2(\Omega)}^2 + \frac{b_0}{4} \left\| (-\Delta)^{\beta} \mathbf{Y}_{M_n,n} \right\|_{L^2(\Omega)}^2 \\ &\leq \frac{4}{b_0} a_0^2 e^{-TM_n^{2\beta}} \left\| u \right\|_{L^{\infty} \left([0,T]; \widetilde{V}(\Omega) \right)}^2 + \frac{b_0}{4} \left\| (-\Delta)^{\beta} \mathbf{Y}_{M_n,n} \right\|_{L^2(\Omega)}^2. \end{split}$$

The term $\widetilde{\mathcal{J}}_{8,n}$ is bounded by

$$\begin{split} \left| \widetilde{\mathcal{J}}_{8,n} \right| &\leq \frac{4}{b_0} e^{2\rho_n(t-T)} \left(\overline{a}(t) - a(t) \right)^2 \left\| (-\Delta)^{\beta} u \right\|_{L^2(\Omega)}^2 + \frac{b_0}{4} \left\| (-\Delta)^{\beta} \mathbf{Y}_{M_n,n} \right\|_{L^2(\Omega)}^2 \\ &\leq \frac{4}{b_0} e^{2\rho_n(t-T)} \left(\overline{a}(t) - a(t) \right)^2 \|u\|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^2 + \frac{b_0}{4} \left\| (-\Delta)^{\beta} \mathbf{Y}_{M_n,n} \right\|_{L^2(\Omega)}^2. \end{split}$$

The term $\widetilde{\mathcal{J}}_{9,n}$ is estimated as follows

$$\begin{split} \left| \widetilde{\mathcal{J}}_{9,n} \right| &\leq \frac{4}{b_0} e^{2\rho_n(t-T)} \left\| \overline{F}_{Q_n}(W_{M_n,n}(\cdot,t)) - F\left(u(\cdot,t)\right) \right\|^2 + \frac{b_0}{4} \left\| (-\Delta)^{\beta} \mathbf{Y}_{M_n,n} \right\|_{L^2(\Omega)}^2 \\ &\leq \frac{4}{b_0} K^2(Q_n) \|\mathbf{Y}_{M_n,n}(\cdot,t)\|_{L^2(\Omega)}^2 + \frac{b_0}{4} \left\| (-\Delta)^{\beta} \mathbf{Y}_{M_n,n} \right\|_{L^2(\Omega)}^2 \\ &\leq \frac{4}{b_0} K^2(Q_n) \|\mathbf{Y}_{M_n,n}(\cdot,t)\|_{H^{\beta}(\Omega)}^2 + \frac{b_0}{4} \left\| (-\Delta)^{\beta} \mathbf{Y}_{M_n,n} \right\|_{L^2(\Omega)}^2. \end{split}$$

where in the latter inequality, we have noted that for all $v \in H^{\beta}(\Omega)$ then

$$||v||_{H^{\beta}(\Omega)}^{2} = \sum_{p=0}^{\infty} p^{2\beta} \langle v, \phi_{p} \rangle^{2} \ge \sum_{p=0}^{\infty} \langle v, \phi_{p} \rangle^{2} = ||v||_{L^{2}(\Omega)}^{2}.$$
 (156)

The term $|\widetilde{\mathcal{J}}_{10,n}|$ can be bounded by

$$\left| \widetilde{\mathcal{J}}_{10,n} \right| = \left| \left\langle e^{\rho_n(t-T)} \left[\overline{g}_{M_n,n}(\cdot,t) \right) - g(\cdot,t) \right], (-\Delta)^{\beta} \mathbf{Y}_{M_n,n} \right\rangle_{L^2(\Omega)} \right|$$

$$\leq \frac{4}{b_0} e^{2\rho_n(t-T)} \left\| \left| \overline{g}_{M_n,n}(\cdot,t) \right| - g(\cdot,t) \right\|_{L^2(\Omega)}^2 + \frac{b_0}{4} \left\| (-\Delta)^{\beta} \mathbf{Y}_{M_n,n} \right\|_{L^2(\Omega)}^2.$$

$$(157)$$

From the above observations, we obtain

$$\frac{1}{2} \frac{d}{dt} \| \mathbf{Y}_{M_{n},n}(\cdot,t) \|_{H^{\beta}(\Omega)}^{2} \\
\geq \overline{b}(t) \| (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \|_{L^{2}(\Omega)}^{2} + \rho_{n} \| \mathbf{Y}_{M_{n},n}(\cdot,t) \|_{H^{\beta}(\Omega)}^{2} - M_{n}^{2\beta} \| \mathbf{Y}_{M_{n},n} \|_{H^{\beta}(\Omega)}^{2} \\
- \frac{4a_{0}^{2}}{b_{0}} e^{-TM_{n}^{2\beta}} \| u \|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2} - \frac{b_{0}}{4} \| (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \|_{L^{2}(\Omega)}^{2} \\
- \frac{4}{b_{0}} e^{2\rho_{n}(t-T)} (\overline{a}(t) - a(t))^{2} \| u \|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^{2} - \frac{b_{0}}{4} \| (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \|_{L^{2}(\Omega)}^{2} \\
- \frac{4}{b_{0}} K^{2}(Q_{n}) \| \mathbf{Y}_{M_{n},n}(\cdot,t) \|_{H^{\beta}(\Omega)}^{2} - \frac{b_{0}}{4} \| (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \|_{L^{2}(\Omega)}^{2} \\
- \frac{4}{b_{0}} e^{2\rho_{n}(t-T)} \| \overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \|_{L^{2}(\Omega)}^{2} - \frac{b_{0}}{4} \| (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \|_{L^{2}(\Omega)}^{2} \\
= (\overline{b}(t) - b_{0}) \| (-\Delta)^{\beta} \mathbf{Y}_{M_{n},n} \|_{L^{2}(\Omega)}^{2} - \frac{4}{b_{0}} e^{2\rho_{n}(t-T)} (\overline{a}(t) - a(t))^{2} \| u \|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^{2} \\
- \frac{4}{b_{0}} e^{2\rho_{n}(t-T)} \| \overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t) \|_{L^{2}(\Omega)}^{2} - \frac{4a_{0}^{2}}{b_{0}} e^{-TM_{n}^{2\beta}} \| u \|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2} \\
+ \left(\rho_{n} - M_{n}^{2\beta} - \frac{4}{b_{0}} K^{2}(Q_{n}) \right) \| \mathbf{Y}_{M_{n},n}(\cdot,t) \|_{H^{\beta}(\Omega)}^{2}. \tag{158}$$

By the fact that $\overline{b}(t) \geq b_0$, we know that the term $(\overline{b}(t) - b_0) \| (-\Delta)^{\beta} \mathbf{Y}_{M_n,n} \|_{L^2(\Omega)}^2$ is non-negative. It follows from (158) that

$$\frac{d}{dt} \|\mathbf{Y}_{M_{n},n}(\cdot,t)\|_{H^{\beta}(\Omega)}^{2} + \frac{8a_{0}^{2}}{b_{0}} e^{-TM_{n}^{2\beta}} \|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2}
+ \frac{8}{b_{0}} e^{2\rho_{n}(t-T)} \left(\overline{a}(t) - a(t)\right)^{2} \|\mathbf{u}\|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^{2}
+ \frac{8}{b_{0}} e^{2\rho_{n}(t-T)} \|\overline{g}_{M_{n},n}(\cdot,t) - g(\cdot,t)\|_{L^{2}(\Omega)}^{2}
\geq 2 \left(\rho_{n} - M_{n}^{2\beta} - \frac{4}{b_{0}} K^{2}(Q_{n})\right) \|\mathbf{Y}_{M_{n},n}(\cdot,t)\|_{H^{\beta}(\Omega)}^{2}.$$
(159)

By taking the integral from t to T, we obtain that

$$\|\mathbf{Y}_{M_{n},n}(\cdot,T)\|_{H^{\beta}(\Omega)}^{2} - \|\mathbf{Y}_{M_{n},n}(\cdot,t)\|_{H^{\beta}(\Omega)}^{2} + \underbrace{\int_{t}^{T} \left(\frac{8a_{0}^{2}}{b_{0}}e^{-2TM_{n}^{2\beta}}\|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2} + \frac{8}{b_{0}}\left(\overline{a}(s) - a(s)\right)^{2}\|u\|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^{2}\right) ds}_{:=J_{11}} + \underbrace{\int_{t}^{T} e^{2\rho_{n}(s-T)}\frac{8}{b_{0}}\|\overline{g}_{M_{n},n}(\cdot,s)) - g(\cdot,s)\|_{L^{2}(\Omega)}^{2} ds}_{:=J_{12}} \\ \geq \int_{t}^{T} \left(2\rho_{n} - 2M_{n}^{2\beta} - \frac{8}{b_{0}}K^{2}(Q_{n})\right)\|\mathbf{Y}_{M_{n},n}(\cdot,s)\|_{H^{\beta}(\Omega)}^{2} ds.$$

$$(160)$$

Let us choose $\rho_n = M_n^{2\beta}$, we have that

$$\|\mathbf{Y}_{M_{n},n}(\cdot,t)\|_{H^{\beta}(\Omega)}^{2} \leq \frac{8}{b_{0}}K^{2}(Q_{n})\int_{t}^{T}\|\mathbf{Y}_{M_{n},n}(\cdot,s)\|_{H^{\beta}(\Omega)}^{2}ds + \|\mathbf{Y}_{M_{n},n}(\cdot,T)\|_{H^{\beta}(\Omega)}^{2} + J_{11} + J_{12}.$$
(161)

Next we give upper bounds for the terms J_{11} and J_{12} of (161). For J_{11} , by equation (5), we have

$$J_{11} \le \frac{8a_0^2}{b_0} (T - t) e^{-2TM_n^{2\beta}} \|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^2 + \frac{8}{b_0} \|u\|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^2 \int_t^T \epsilon^2 |\overline{\xi}(s)|^2 ds.$$
 (162)

Since $\mathbf{E}|\overline{\xi}(s)|^2 = s$, we have the following estimation

$$\mathbf{E}J_{11} \leq \frac{8Ta_{0}^{2}}{b_{0}}e^{-2TM_{n}^{2\beta}}\|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2} + \frac{8}{b_{0}}\|u\|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^{2} \int_{t}^{T} \epsilon^{2}\mathbf{E}|\overline{\xi}(s)|^{2}ds$$

$$\leq \frac{8Ta_{0}^{2}}{b_{0}}e^{-2TM_{n}^{2\beta}}\|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2} + \frac{4\epsilon^{2}T^{2}}{b_{0}}\|u\|_{L^{\infty}([0,T];H^{2\beta}(\Omega))}^{2}.$$
(163)

For J_{12} , by equation (123), we get

$$\mathbf{E}J_{12} \leq \frac{8}{b_0} \int_{t}^{T} \mathbf{E} \left\| \overline{g}_{M_{n},n}(\cdot,s) - g(\cdot,s) \right\|_{L^{2}(\Omega)}^{2} ds$$

$$\leq \frac{8}{b_0} \left[\left(\pi^{2} T^{2} \vartheta^{2} + \left| \overline{D}(\delta,g) \right|^{2} \right) \frac{M_{n} + 1}{n} + e^{-2TM_{n}^{2\beta}} \|g\|_{L^{\infty}([0,T];\widetilde{V}(\Omega))} \right] (T - t). \tag{164}$$

Now, we continue to estimate $\|\mathbf{Y}_{M_n,n}(\cdot,T)\|_{H^{\beta}(\Omega)}^2$ of (160). From (125), we obtain

$$\|\mathbf{Y}_{M_{n},n}(\cdot,T)\|_{H^{\beta}(\Omega)}^{2} = \|\overline{w}_{M_{n},n} - u_{T}\|_{H^{\beta}(\Omega)}^{2}$$

$$\leq \left[\frac{1}{n}\sum_{k=1}^{n}\sigma_{k}\epsilon_{k} - \widetilde{G}_{n0}\right]^{2} + \sum_{p=1}^{M_{n}}p^{2\beta}\left[\frac{\pi}{n}\sum_{k=1}^{n}\sigma_{k}\epsilon_{k}\phi_{p}(x_{k}) - \widetilde{G}_{np}\right]^{2}$$

$$+ \sum_{p=M_{n}+1}^{\infty}p^{2\beta}\left\langle u_{T},\phi_{p}\right\rangle^{2}$$

$$\leq \left[\frac{1}{n}\sum_{k=1}^{n}\sigma_{k}\epsilon_{k} - \widetilde{G}_{n0}\right]^{2} + M_{n}^{2\beta}\sum_{p=1}^{M_{n}}\left[\frac{\pi}{n}\sum_{k=1}^{n}\sigma_{k}\epsilon_{k}\phi_{p}(x_{k}) - \widetilde{G}_{np}\right]^{2}$$

$$+ \sum_{p=M_{n}+1}^{\infty}p^{2\beta}\left\langle u_{T},\phi_{p}\right\rangle^{2}. \tag{165}$$

Using the similar techniques as in the proof of Theorem 3.1, we get

$$\mathbf{E} \|\mathbf{Y}_{M_{n},n}(\cdot,T)\|_{H^{\beta}(\Omega)}^{2} \leq \left(\pi^{2} V_{\max}^{2} + \left|\overline{C}(\delta,u)\right|^{2}\right) \frac{M_{n}^{2\beta+1} + M_{n}^{2\beta}}{n}$$

$$+ e^{-2TM_{n}^{2}\beta} \sum_{p=M_{n}+1}^{\infty} p^{2\beta} e^{2Tp^{2\beta}} \left\langle u_{T}, \phi_{p} \right\rangle^{2}$$

$$\leq \left(\pi^{2} V_{\max}^{2} + \left|\overline{C}(\delta,u)\right|^{2}\right) \frac{M_{n}^{2\beta+1} + M_{n}^{2\beta}}{n} + e^{-2TM_{n}^{2\beta}} \|u_{T}\|_{\tilde{V}(\Omega)}^{2}.$$
 (166)

Combining equations (161), (163), (164), (166), we derive that

$$e^{2M_{n}^{2\beta}t}\mathbf{E}\|W_{M_{n},n}(\cdot,t)-u(\cdot,t)\|_{H^{\beta}(\Omega)}^{2}$$

$$\leq \left(\pi^{2}V_{\max}^{2}+\left|\overline{C}(\delta,u)\right|^{2}+\frac{8}{b_{0}}\pi^{2}T^{3}\vartheta^{2}+\frac{8}{b_{0}}T\left|\overline{D}(\delta,g)\right|^{2}\right)\frac{(M_{n}^{2\beta+1}+M_{n}^{2\beta})e^{2M_{n}^{2\beta}T}}{n}$$

$$+\frac{4e^{2M_{n}^{2\beta}T}\epsilon^{2}T^{2}}{b_{0}}\|u\|_{L^{\infty}[0,T];H^{2\beta}(\Omega))}^{2}$$

$$+\left(\|u_{T}\|_{\tilde{V}(\Omega)}^{2}+\frac{8T}{b_{0}}\|g\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2}+\frac{8Ta_{0}^{2}}{b_{0}}\|u\|_{L^{\infty}([0,T];\tilde{V}(\Omega))}^{2}\right)$$

$$+\frac{8}{b_{0}}K^{2}(Q_{n})\int_{t}^{T}e^{2M_{n}^{2\beta}s}\mathbf{E}\|W_{M_{n},n}(\cdot,s)-u(\cdot,s)\|_{H^{\beta}(\Omega)}^{2}ds.$$

Let us denote

$$\begin{split} & \Pi(n, u, g, \delta, \epsilon) \\ & = \left(\pi^2 V_{\text{max}}^2 + \left| \overline{C}(\delta, u) \right|^2 + \frac{8}{b_0} \pi^2 T^3 \vartheta^2 + \frac{8}{b_0} T \left| \overline{D}(\delta, g) \right|^2 \right) \frac{(M_n^{2\beta + 1} + M_n^{2\beta}) e^{2M_n^{2\beta} T}}{n} \\ & + \frac{4e^{2M_n^{2\beta} T} \epsilon^2 T^2}{b_0} \|u\|_{L^{\infty}[0, T]; H^{2\beta}(\Omega))}^2 + \|u_T\|_{\tilde{V}(\Omega)}^2 \\ & + \frac{8T}{b_0} \|g\|_{L^{\infty}([0, T]; \tilde{V}(\Omega))}^2 + \frac{8T a_0^2}{b_0} \|u\|_{L^{\infty}([0, T]; \tilde{V}(\Omega))}^2 \end{split}$$

then we have

$$\begin{split} e^{2M_n^{2\beta}t}\mathbf{E}\|W_{M_n,n}(\cdot,t) - u(\cdot,t)\|_{H^{\beta}(\Omega)}^2 &\leq \mathbf{\Pi}(n,u,g,\delta,\epsilon) \\ &+ \frac{8}{b_0}K^2(Q_n)\int_t^T e^{2M_n^{2\beta}s}\mathbf{E}\|W_{M_n,n}(\cdot,s) - u(\cdot,s)\|_{H^{\beta}(\Omega)}^2 ds. \end{split}$$

Using Gronwall's inequality, we obtain

$$e^{2M_n^{2\beta}t}\mathbf{E}\|W_{M_n,n}(\cdot,t)-u(\cdot,t)\|_{H^{\beta}(\Omega)}^2 \leq \exp\left(\frac{8}{b_0}K^2(Q_n)(T-t)\right)\mathbf{\Pi}(n,u,g,\delta,\epsilon).$$

This implies that

$$\mathbf{E} \|W_{M_n,n}(\cdot,t) - u(\cdot,t)\|_{H^{\beta}(\Omega)}^2 \le e^{-2M_n^{2\beta}t} \exp\left(\frac{8}{b_0} K^2(Q_n)(T-t)\right) \mathbf{\Pi}(n,u,g,\delta,\epsilon).$$

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