# Accelerated Methods for Non-Convex Optimization 

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#### Abstract

We present an accelerated gradient method for non-convex optimization problems with Lipschitz continuous first and second derivatives. The method requires time $O\left(\epsilon^{-7 / 4} \log (1 / \epsilon)\right)$ to find an $\epsilon$-stationary point, meaning a point $x$ such that $\|\nabla f(x)\| \leq \epsilon$. The method improves upon the $O\left(\epsilon^{-2}\right)$ complexity of gradient descent and provides the additional second-order guarantee that $\nabla^{2} f(x) \succeq-O\left(\epsilon^{1 / 2}\right) I$ for the computed $x$. Furthermore, our method is Hessian-free, i.e. it only requires gradient computations, and is therefore suitable for large scale applications.


## 1 Introduction

In this paper, we consider the optimization problem

$$
\begin{equation*}
\underset{x \in \mathbb{R}^{d}}{\operatorname{minimize}} f(x), \tag{1}
\end{equation*}
$$

where $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ has $L_{1}$-Lipschitz continuous gradient and $L_{2}$-Lipschitz continuous Hessian, but may be non-convex. Without further assumptions, finding a global minimum of this problem is computationally intractable: finding an $\epsilon$-suboptimal point for a $k$-times continuously differentiable function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ requires at least $\Omega\left((1 / \epsilon)^{d / k}\right)$ evaluations of the function and first $k$-derivatives, ignoring problem-dependent constants [28, §1.6]. Consequently, we aim for a weaker guarantee, looking for locally "optimal" points for problem (1); in particular, we seek stationary points, that is points $x$ with sufficiently small gradient:

$$
\begin{equation*}
\|\nabla f(x)\| \leq \epsilon \tag{2}
\end{equation*}
$$

The simplest method for obtaining a guarantee of the form (2) is gradient descent (GD). It is well-known [32] that if GD begins from a point $x_{1}$, then for any $\Delta_{f} \geq f\left(x_{1}\right)-\inf _{x} f(x)$ and $\epsilon>0$, it finds a point satisfying the bound (2) in $O\left(\Delta_{f} L_{1} \epsilon^{-2}\right)$ iterations. If one additionally assumes that the function $f$ is convex, substantially more is possible: GD then requires only $O\left(R L_{1} \epsilon^{-1}\right)$ iterations, where $R$ is an upper bound on the distance between $x_{1}$ and the set of minimizers of $f$. Moreover, in the same note [32], Nesterov also shows that acceleration and regularization techniques can reduce the iteration complexity to $\widetilde{O}\left(\sqrt{R L_{1}} \epsilon^{-1 / 2}\right) .{ }^{1}$

In the non-convex setting, it is possible to achieve better rates of convergence to stationary points assuming access to more than gradients, e.g. the full Hessian. Nesterov and Polyak [33] explore such possibilities with their work on the cubic-regularized Newton method, which they show computes an $\epsilon$-stationary point in $O\left(\Delta_{f} L_{2}^{0.5} \epsilon^{-3 / 2}\right)$ iterations (i.e. gradient and Hessian calculations). However, with a naive implementation, each such iteration requires explicit calculation of the Hessian $\nabla^{2} f(x)$ and the solution of multiple linear systems, with complexity $\widetilde{O}\left(d^{3}\right) .{ }^{2}$ More recently, Birgin et al.

[^0][8] extend cubic regularization to $p$ th order regularization, showing that iteration complexities of $O\left(\epsilon^{-(p+1) / p}\right)$ are possible given evaluations of the first $p$ derivatives of $f$. That is, there exist algorithms for which $\epsilon^{-(p+1) / p}$ calculations of the first $p$ derivatives of $f$ are sufficient to achieve the guarantee (2); naturally, these bounds ignore the computational cost of each iteration. More efficient rates are also known for various structured problems, such as finding KKT points for indefinite quadratic optimization problems [43] or local minima of $\ell_{p}$ "norms," $p \in(0,1)$, over linear constraints [16].

In this paper, we ask a natural question: using only gradient information, is it possible to improve on the $\epsilon^{-2}$ iteration complexity of gradient descent in terms of number of gradient calculations? We answer the question in the affirmative, providing an algorithm that requires at most

$$
\widetilde{O}\left(\Delta_{f} L_{1}^{\frac{1}{2}} L_{2}^{\frac{1}{4}} \epsilon^{-\frac{7}{4}}+\Delta_{f}^{1 / 2} L_{1}^{1 / 2} \epsilon^{-1}\right)
$$

gradient and Hessian-vector product evaluations to find an $x$ such that $\|\nabla f(x)\| \leq \epsilon$. For a summary of our results in relation to other work, see Table 1.

Another advantage of the cubic-regularized Newton method is that it provides a second-order guarantee of the form $\nabla^{2} f(x) \succeq-\sqrt{\epsilon} I$, thus giving a rate of convergence to points with zero gradient and positive semi-definite Hessian. Such second-order stationary points are finer approximations of local minima compared to first-order stationary points (with zero gradient). Our approach also provides this guarantee, and is therefore an example of a first-order method that converges to a second-order stationary point in time polynomial in the desired accuracy and with logarithmic dependence on the problem dimension. A notable consequence of this approach is that for strict saddle functions [24, 17]-with only non-degenerate stationary points-our approach converges linearly to local minimizers. We discuss this result in detail in Section 5.

Table 1 Runtime comparisons for finding a first-order stationary point (2)

|  | \# iterations | Hessian <br> free? | Gradient <br> Lipschitz? | Hessian <br> Lipschitz? | convex $f ?$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Gradient descent <br> (non-convex case) | $O\left(\Delta_{f} L_{1} \epsilon^{-2}\right)$ | Yes | Yes | No | No |
| Gradient descent <br> (convex case) [32] | $O\left(R L_{1} \epsilon^{-1}\right)$ | Yes | Yes | No | Yes |
| Proximal accelerated <br> gradient descent [32] | $\widetilde{O}\left(\left(R L_{1}\right)^{\frac{1}{2}} \epsilon^{-\frac{1}{2}}\right)$ | Yes | Yes | No | Yes |
| Cubic-regularized <br> Newton method [33] | $\widetilde{O}\left(\Delta_{f} L_{2}^{\frac{1}{2}} \epsilon^{-\frac{3}{2}}\right)$ | No | Yes | Yes | No |
| This paper <br> (Theorem 4.3) | $\widetilde{\boldsymbol{O}}\left(\boldsymbol{\Delta}_{f} \boldsymbol{L}_{1}^{\frac{1}{2}} \boldsymbol{L}_{2}^{\frac{1}{4}} \epsilon^{-\frac{7}{4}}\right)$ | Yes | Yes | Yes | No |

### 1.1 Related work and background

In the optimization and machine learning literature, there has been substantial recent work on the convergence properties of optimization methods for non-convex problems. One line of work investigates the types of local optima to which gradient-like methods converge, as well as convergence rates. In this vein, under certain reasonable assumptions (related to geometric properties of saddle points), Ge et al. [17] show that stochastic gradient descent (SGD) converges to second-order local optima (stationary points with positive semidefinite Hessian), while Lee et al. [24] show that GD generically converges to second-order local optima. Anandkumar and Ge [6] extend these ideas,
showing how to find a point that approximately satisfies the third-order necessary conditions for local optimality in polynomial time. While these papers used second-order smoothness assumptions to ensure convergence to stronger local minima than the simple stationary condition (2), they do not improve rates of convergence to stationarity.

A second line of work focuses on improving the slow convergence rates of SGD to stationary points (typically $O\left(\epsilon^{-4}\right)$ stochastic gradient evaluations are sufficient [18]) under appropriate structural conditions on $f$. One natural condition-common in the statistics and machine learning literature - is that $f$ is the sum of $n$ smooth non-convex functions. Indeed, the work of Reddi et al. [37] and Allen-Zhu and Hazan [3] achieves a rate of convergence $O\left(\epsilon^{-2}\right)$ for such problems without performing the $n$ gradient evaluations (one per function) that standard gradient descent requires in each iteration. These analyses extend variance-reduction techniques that apply to incremental convex optimization problems [22, 13]. Nonetheless, they do not improve on the $O\left(\epsilon^{-2}\right)$ iteration complexity of GD.

Additionally, a number of researchers apply accelerated gradient methods [31] to non-convex optimization problems, though we know no theoretical guarantees giving improved performance over standard gradient descent methods. Ghadimi and Lan [19] show how to modify Nesterov's accelerated gradient descent method so that it enjoys the same convergence guarantees as gradient descent on non-convex optimization problems, while maintaining the accelerated (optimal) firstorder convergence rates for convex problems. Li and Lin [25] develop an accelerated method for non-convex optimization and show empirically that on (non-convex) sparse logistic regression test problems their methods outperform other methods, including gradient descent.

While the subproblem that appears in the cubic-regularized Newton method is expensive to solve exactly, it is possible to consider methods in which such subproblems are solved only approximately by a low complexity Hessian-free procedure. A number of researchers investigate this approach, including Cartis et al. [12] and Bianconcini et al. [7]. These works exhibit strong empirical results, but their analyses do not improve on the $O\left(\epsilon^{-2}\right)$ evaluation complexity of gradient descent. Recently, Hazan and Koren [20] and Ho-Nguyen and Kılınc-Karzan [21] have shown how to solve the related quadratic non-convex trust-region problem using accelerated first-order methods; both these papers use accelerated eigenvector computations as a primitive, similar to our approach. It is therefore natural to ask whether acceleration can give faster convergence to stationary points of general non-convex functions, a question we answer in the affirmative.

Concurrently to and independently of this paper, ${ }^{3}$ Agarwal et al. [2] also answer this question affirmatively. They develop a method that uses fast approximate matrix inversion as a primitive to solve cubic-regularized Newton-type steps [33], and applying additional acceleration techniques they show how to find stationary points of general smooth non-convex objectives. Though the technical approach is somewhat different, their convergence rates to $\epsilon$-stationary points are identical to ours. They also specialize their technique to problems of the finite sum form $f=\frac{1}{n} \sum_{i=1}^{n} f_{i}$, showing that additional improvements in terms of $n$ are achievable.

### 1.2 Our approach

Our method is in the spirit of the techniques that underly accelerated gradient descent (AGD). While Nesterov's 1983 development of acceleration schemes may appear mysterious at first, there are multiple interpretations of AGD as the careful combination of different routines for function minimization. The estimate sequence ideas of Nesterov [31] and proximal point proofs [26, 40, 14] show how to view accelerated gradient descent as a trade-off between building function lower

[^1]bounds and directly making function progress. Bubeck et al. [10] develop an AGD algorithm with a geometric interpretation based on shrinking spheres, while the work of Allen-Zhu and Orecchia [5] shows that AGD may be viewed as a coupling between mirror descent [28] and gradient descent; this perspective highlights how to trade each method's advantages in different scenarios to achieve faster-accelerated-running time.

We follow a similar template of leveraging two competing techniques for making progress on computing a stationary point, but we deviate from standard analyses involving acceleration in our coupling of the algorithms. The first technique we apply is fairly well known. If the problem is locally non-convex, the Hessian must have a negative eigenvalue. In this case, under the assumption that the Hessian is Lipschitz continuous, moving in the direction of the corresponding eigenvector must make progress on the objective. Nesterov and Polyak [33] (and more broadly, the literature on cubic regularization) use this implicitly, while other researchers [17, 6] use this more explicitly to escape from saddle points.

The second technique is the crux of our approach. While $L_{1}$-Lipschitz continuity of $\nabla f$ ensures that the smallest eigenvalue of the Hessian is at least $-L_{1}$, we show that any stronger boundany deviation from this "worst possible" negative curvature - allows us to improve upon gradient descent. We show that if the smallest eigenvalue is at least $-\gamma$, which we call $-\gamma$-strong convexity, we can apply proximal point techniques [35, 32] and accelerated gradient descent to a carefully constructed regularized problem to obtain a faster running time. Our procedure proceeds by approximately minimizing a sequence of specially constructed such functions. This procedure is of independent interest since it can be applied in a standalone manner whenever the function is known to be globally almost convex.

By combining these procedures, we achieve our result. We run an accelerated (single) eigenvector routine - also known as principle components analysis (PCA)-to estimate the eigenvector corresponding to the smallest eigenvalue of the Hessian. Depending on the estimated eigenvalue we either move along the approximate eigenvector or apply accelerated gradient descent to a regularized sub-problem, where we carefully construct the regularization based on this smallest eigenvalue. Trading between these two cases gives our improved running time. We remark that an improvement over gradient descent is obtainable even if we use a simpler (non-accelerated) method for estimating eigenvectors, such as the power method. That said, an accelerated gradient descent subroutine for the regularized sub-problems we solve appears to be crucial to achieving faster convergence rates than gradient descent.

The remainder of the paper is structured as follows. Section 2 introduces the notation and existing results on which our approach is based. Section 3.1 introduces our method for accelerating gradient descent on "almost convex" functions, while Section 3.2 presents and explains our "negative curvature descent" subroutine. Section 4 joins the two building blocks to obtain our main result, while in Section 5, we show how our results give linear convergence to local minima for strict-saddle functions.

## 2 Notation and standard results

Here, we collect our (mostly standard) notation and a few basic results. Throughout this paper, norms $\|\cdot\|$ are the Euclidean norm; when applied to matrices $\|\cdot\|$ denotes the $\ell_{2}$-operator norm. All logarithms are base-e. For a symmetric matrix $A$, we let $\lambda_{\min }(A)$ and $\lambda_{\max }(A)$ denote its minimum and maximum eigenvalues, respectively. We also use the following definitions.

Definition 1 (Smoothness). A function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is $L_{1}$-smooth if its gradient is $L_{1}$-Lipschitz, that is, $\|\nabla f(x)-\nabla f(y)\| \leq L_{1}\|x-y\|$ for all $x, y$.

Definition 2 (Lipschitz Hessian). The Hessian of a twice differentiable function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is $L_{2}$-Lipschitz continuous if $\left\|\nabla^{2} f(x)-\nabla^{2} f(y)\right\| \leq L_{2}\|x-y\|$ for all $x, y$.

Definition 3 (Optimality gap). A function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ has optimality gap $\Delta_{f}>0$ at point $x$ if $f(x)-\inf _{y \in \mathbb{R}^{d}} f(y) \leq \Delta_{f}$.

We assume throughout without further mention that $f$ is $L_{1}$-smooth, has $L_{2}$-Lipschitz continuous Hessian, and has optimality gap $\Delta_{f}<\infty$ at the initial search point, generally denoted $z_{1}$.

The next definition is atypical, because we allow the strong convexity parameter $\sigma_{1}$ to be negative. Of course, if $\sigma_{1}<0$ the function may be non-convex, but we can use $\sigma_{1}$ to bound the extent to which the function is non-convex, similar to the ideas of lower $C^{2}$-functions in variational analysis [38]. As we show in Lemma 3.1 this "almost convexity" allows improvements in runtime over gradient descent.

Definition 4 (Generalized strong convexity and almost convexity). A function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is $\sigma_{1-}$ strongly convex if $\frac{\sigma_{1}}{2}\|y-x\|^{2} \leq f(y)-f(x)-\nabla f(x)^{T}(y-x)$ for some $\sigma_{1} \in \mathbb{R}$. For $\gamma=\max \left\{-\sigma_{1}, 0\right\}$, we call such functions $\gamma$-almost convex.

The next three results are standard but useful lemmas using the definitions above.
Lemma 2.1 (Nesterov [31], Theorem 2.1.5). Let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be L-smooth. Then for all $x, y \in \mathbb{R}^{d}$

$$
\left|f(y)-f(x)-\nabla f(x)^{T}(y-x)\right| \leq \frac{L_{1}}{2}\|y-x\|^{2}
$$

Lemma 2.2 (Nesterov and Polyak [33], Lemma 1). Let $f$ have $L_{2}$-Lipschitz Hessian. Then for all $x, y \in \mathbb{R}^{d}$

$$
\left\|\nabla f(y)-\nabla f(x)-\nabla^{2} f(x)(y-x)\right\| \leq \frac{L_{2}}{2}\|y-x\|^{2}
$$

and

$$
\left|f(y)-f(x)-\nabla f(x)^{T}(y-x)-\frac{1}{2}(y-x)^{T} \nabla^{2} f(x)(y-x)\right| \leq \frac{L_{2}}{6}\|y-x\|^{3}
$$

Lemma 2.3 (Boyd and Vandenberghe [9], Eqs. (9.9) and (9.14)). Let $f$ be $L_{1}$-smooth and $\mu$-strongly convex. Then for all $x$ the minimizer $x^{*}$ of $f$ satisfies

$$
2 \mu\left[f(x)-f\left(x^{*}\right)\right] \leq\|\nabla f(x)\|^{2} \leq 2 L_{1}\left[f(x)-f\left(x^{*}\right)\right] .
$$

Lemma 2.1 guarantees any $L_{1}$-smooth function is $\left(-L_{1}\right)$-strongly convex. A key idea in this paper is that if a function is ( $-\sigma_{1}$ )-strongly convex with $\sigma_{1} \geq 0$, standard convex proximal methods are still applicable, provided the regularization is sufficiently large. The following trivial lemma, stated for later reference, captures this idea.

Lemma 2.4. Suppose $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is $\left(-\sigma_{1}\right)$-strongly convex, where $\sigma_{1} \geq 0$. Then for any $x_{0} \in \mathbb{R}^{d}$ the function $g(x)=f(x)+\sigma_{1}\left\|x-x_{0}\right\|^{2}$ is $\left(\sigma_{1}\right)$-strongly convex.

Throughout this paper, we use a fully non-asymptotic big-O notation to be clear about the convergence rates of the algorithms we analyze and to avoid confusion involving relative values of problem-dependent constants (such as $d, L_{1}, L_{2}$ ).

Definition 5 (Big-O notation). Let $\mathcal{S}$ be a set and let $M_{1}, M_{2}: \mathcal{S} \rightarrow \mathbb{R}_{+}$. Then $M_{1}=O\left(M_{2}\right)$ if there exists $C \in \mathbb{R}_{+}$such that $M_{1}(s) \leq C \cdot M_{2}(s)$ for all $s \in \mathcal{S}$.

Throughout, we take $\mathcal{S} \subset[0, \infty)^{6}$ to be the set of tuples $\left(\epsilon, \delta, L_{1}, L_{2}, \Delta_{f}, d\right)$; sometimes we require the tuples to meet certain assumptions that we specify. The notation $\widetilde{O}(\cdot)$ hides logarithmic factors in problem parameters: we say that $M_{1}=\widetilde{O}\left(M_{2}\right)$ if $M_{1}=O\left(M_{2} \log \left(1+L_{1}+L_{2}+\Delta_{f}+d+1 / \delta+1 / \epsilon\right)\right)$.

Because we focus on gradient-based procedures, we measure the running time of our algorithms in terms of gradient operations, each of which we assume takes a (problem-dependent) amount of time $\mathrm{T}_{\text {grad }}$. The following assumption specifies this more precisely.

Assumption A. The following operations take $O\left(\mathrm{~T}_{\text {grad }}\right)$ time:

1. The evaluation $\nabla f(x)$ for a point $x \in \mathbb{R}^{d}$.
2. The evaluation of $\nabla^{2} f(x) v$ for some vector $v \in \mathbb{R}^{d}$ and point $x \in \mathbb{R}^{d}$. (See Remark 1 for justification of this assumption.)
3. Any arithmetic operation (addition, subtraction or multiplication) of two vectors of dimension at most d.

Based on Assumption A, we call an algorithm Hessian free if its basic operations take time at most $O\left(\mathrm{~T}_{\mathrm{grad}}\right)$.
Remark 1: By definition of the Hessian, we have that $\lim _{h \rightarrow 0} h^{-1}(\nabla f(x+h v)-\nabla f(x))=\nabla^{2} f(x) v$ for any $v \in \mathbb{R}^{d}$. Thus, a natural approximation to the product $\nabla^{2} f(x) v$ is to set

$$
p=\frac{\nabla f(x+h v)-\nabla f(x)}{h}
$$

for some small $h>0$. By Lemma 2.2, we immediately have

$$
\left\|p-\nabla^{2} f(x) v\right\| \leq h \frac{L_{2}\|v\|^{2}}{2}
$$

which allows sufficiently precise calculation by taking $h$ small. ${ }^{4}$
In a number of concrete cases, Hessians have structure that allows efficient computation of the product $v \mapsto \nabla^{2} f(x) v$. For example, in neural networks, one may compute $\nabla^{2} f(x) v$ using a back-propagation-like technique at the cost of at most two gradient evaluations [36, 39].

### 2.1 Building block 1: fast gradient methods

With the basic lemmas and definitions in place, we now recapitulate some of the classical development of accelerated methods. First, the following pseudo-code gives Nesterov's classical accelerated gradient descent method for strongly convex functions [31].

[^2]```
function Accelerated-Gradient-Descent \(\left(f, y_{1}, \epsilon, L_{1}, \sigma_{1}\right)\)
    Set \(\kappa=L_{1} / \sigma_{1}\) and \(z_{1}=y_{1}\)
    for \(j=1,2, \ldots\) do
        if \(\left\|\nabla f\left(y_{j}\right)\right\| \leq \epsilon\) then return \(y_{j}\)
        end if
\[
\begin{aligned}
& y_{j+1}=z_{j}-\frac{1}{L_{1}} \nabla f\left(z_{j}\right) \\
& z_{j+1}=\left(1+\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right) y_{j+1}-\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1} y_{j}
\end{aligned}
\]
end for
end function
```

The method Accelerated-Gradient-descent enjoys the following essentially standard guarantee when initialized at any iterate $z_{1}$ satisfying $f\left(z_{1}\right)-\inf _{x} f(x) \leq \Delta_{f}$.
Lemma 2.5. Let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be $\sigma_{1}>0$-strongly convex and $L_{1}$-smooth. Let $\epsilon>0$ and let $z_{j}$ denote the $j$ th iterate of Accelerated-gradient-Descent ( $f, z_{1}, \epsilon, L_{1}, \sigma_{1}$ ). If

$$
j \geq 1+\sqrt{\frac{L_{1}}{\sigma_{1}}} \log \left(\frac{4 L_{1}^{2} \Delta_{f}}{\sigma_{1} \epsilon^{2}}\right) \quad \text { then }\left\|\nabla f\left(z_{j}\right)\right\| \leq \epsilon .
$$

Proof. Let $z^{*}$ be the minimizer of $f$. If $\epsilon^{2} \geq 4 L_{1}^{2} \Delta_{f} / \sigma_{1}$, then

$$
\left\|\nabla f\left(z_{1}\right)\right\|^{2} \stackrel{(i)}{\leq} L_{1}^{2}\left\|z_{1}-z^{*}\right\|^{2} \stackrel{(i i)}{\leq} \frac{2 L_{1}^{2}}{\sigma_{1}}\left(f\left(z_{1}\right)-f\left(z^{*}\right)\right) \stackrel{(i i i)}{\leq} \frac{4 L_{1}^{2} \Delta_{f}}{\sigma_{1}} \leq \epsilon^{2},
$$

where inequality ( $i$ ) follows from smoothness of $f$ (Def. 1), inequality (ii) by the strong convexity of $f$ (Lemma 2.4), and inequality (iii) by the definition of $\Delta_{f}$. Thus the iteration ends at $j=1$.

For smaller $\epsilon$, we let $\kappa=L_{1} / \sigma_{1} \geq 1$ denote the condition number for the problem. Then Nesterov [31, Theorem 2.2.2] shows that for $k>1$

$$
f\left(z_{k}\right)-f\left(z^{*}\right) \leq L_{1}\left(1-\sqrt{\sigma_{1} / L_{1}}\right)^{k-1}\left\|z_{1}-z^{*}\right\|^{2} \leq 2 \kappa \exp \left(-(k-1) \kappa^{-\frac{1}{2}}\right) \Delta_{f} .
$$

Taking any $k \geq 1+\sqrt{\kappa} \log \frac{4 L_{1} \kappa \Delta_{f}}{\epsilon^{2}}$ yields

$$
f\left(z_{k}\right)-f\left(z^{*}\right) \leq \frac{\epsilon^{2}}{2 L_{1}} .
$$

Noting that $\|\nabla f(x)\|^{2} \leq 2 L_{1}\left(f(x)-f\left(z^{*}\right)\right)$ by Lemma 2.3, we obtain our result.

### 2.2 Building block 2: fast eigenvector computation

The final building block we use is accelerated approximate leading eigenvector computation. We consider two types of approximate eigenvectors. By a relative $\varepsilon$-approximate leading eigenvector of a positive semidefinite (PSD) matrix $H$, we mean a vector $v$ such that $\|v\|=1$ and $v^{T} H v \geq$ $(1-\varepsilon) \lambda_{\max }(H)$; similarly, an additive $\varepsilon$-approximate leading eigenvector of $H$ satisfies $\|v\|=1$ and
$v^{T} H v \geq \lambda_{\max }(H)-\varepsilon$. A number of methods compute such (approximate) leading eigenvectors, including the Lanczos method [23]. For concreteness, we state one lemma here, where in the lemma we let $\mathrm{T}_{\text {hess }}$ denote the larger of the times required to compute the matrix-vector product $H v$ or to add two vectors.
Lemma 2.6 (Accelerated top eigenvector computation). Let $H \in \mathbb{R}^{d \times d}$ be symmetric and PSD. There exists an algorithm that on input $\varepsilon, \delta \in(0,1)$ runs in $O\left(\mathrm{~T}_{\text {hess }} \log (d / \delta) \varepsilon^{-1 / 2}\right)$ time and, with probability at least $1-\delta$, returns a relative $\varepsilon$-approximate leading eigenvector $\widehat{v}$.

Notably, the Lanczos method [23, Theorem 3.2] achieves this complexity guarantee. While Lemma 2.6 relies on infinite precision arithmetic (the stability of the Lanczos method is an active area of research [34]), shift-and-invert preconditioning [15] also achieves the convergence guarantee to within poly-logarithmic factors in bounded precision arithmetic. This procedure reduces computing the top eigenvector of the matrix $H$ to solving a sequence of linear systems, and using fast gradient descent to solve the linear systems guarantees the running time in Lemma 2.6. (See Section 8 of [15] for the reduction and Theorem 4.1 of [4] for another statement of the result.) For simplicity -because we do not focus on such precision issues - we use Lemma 2.6 in the sequel.

For later use, we include a corollary of Lemma 2.6 in application to finding minimum eigenvectors of the Hessian $\nabla^{2} f(x)$ using matrix-vector multiplies. Recalling that $f$ is $L_{1}$-smooth, we know that the matrix $M:=L_{1} I-\nabla^{2} f(x)$ is PSD, and its eigenvalues are $\left\{L_{1} I-\lambda_{i}\right\}_{i=1}^{d} \subset\left[0,2 L_{1}\right]$, where $\lambda_{i}$ is the $i$ th eigenvalue of $\nabla^{2} f(x)$. The procedure referenced in Lemma 2.6 (Lanczos or another accelerated method) applied to the matrix $M$ thus, with probability at least $1-\delta$, provides a vector $\widehat{v}$ with $\|\widehat{v}\|=1$ such that

$$
L_{1}-\widehat{v}^{T} \nabla^{2} f(x) \widehat{v}=\widehat{v}^{T} M \widehat{v} \geq(1-\varepsilon) \lambda_{\max }(M) \geq(1-\varepsilon)\left(L_{1}-\lambda_{\min }\left(\nabla^{2} f(x)\right)\right)
$$

in time $O\left(\mathrm{~T}_{\text {grad }} \varepsilon^{-\frac{1}{2}} \log \frac{d}{\delta}\right)$. Rearranging this, we have

$$
\widehat{v}^{T} \nabla^{2} f(x) \widehat{v} \leq \varepsilon L_{1}+(1-\varepsilon) \lambda_{\min }\left(\nabla^{2} f(x)\right),
$$

and substituting $\epsilon /\left(2 L_{1}\right)$ for $\varepsilon$ yields the following summarizing corollary.
Corollary 2.7 (Finding the negative curvature). In the setting of the previous paragraph, there exists an algorithm that given $x \in \mathbb{R}^{d}$ computes, with probability at least $1-\delta$, an additive $\epsilon$ approximate smallest eigenvector $\widehat{v}$ of $\nabla^{2} f(x)$ in time $O\left(\mathrm{~T}_{\operatorname{grad}}\left(1+\log (d / \delta) \sqrt{L_{1} / \epsilon}\right)\right)$.

## 3 Two structured non-convex problems

With our preliminary results established, in this section we turn to two methods that form the core of our approach. Roughly, our overall algorithm will be to alternate between finding directions of negative curvature of $f$ and solving structured sub-problems that are nearly convex, meaning that the smallest eigenvalue of the Hessian has a lower bound $-\gamma, \gamma>0$, where $\gamma \ll L_{1}$. We turn to each of these pieces in turn.

### 3.1 Accelerated gradient descent for almost convex functions

The second main component of our general accelerated method is a procedure for finding stationary points of smooth non-convex functions that are not too non-convex. By not too non-convex, we mean $\gamma$-almost convexity, as in Def. 4, that is, that

$$
f(y) \geq f(x)+\nabla f(x)^{T}(y-x)-\frac{\gamma}{2}\|x-y\|^{2} \quad \text { for all } x, y,
$$

where $\gamma \geq 0$. The next procedure applies to such almost convex functions, and builds off of a suggestion of Nesterov [32] to use regularization coupled with accelerated gradient descent to improve convergence guarantees for finding a stationary point of $f$. The idea, as per Lemma 2.4 , is to add a regularizing term of the form $\gamma\left\|z-z_{0}\right\|^{2}$ to make the $\gamma$-almost convex function $f$ become $\gamma$-strongly convex. As we describe in the sequel, we solve a sequence $j=1,2, \ldots$ of such proximal sub-problems

$$
\begin{equation*}
\underset{z}{\operatorname{minimize}} g_{j}(z):=f(z)+\gamma\left\|z-z_{j}\right\|^{2} \tag{3}
\end{equation*}
$$

quickly using accelerated gradient descent. Whenever $\gamma \ll L_{1}$, the regularized model $g_{j}$ of $f$ has better fidelity to $f$ than the model $f(z)+\frac{L_{1}}{2}\left\|z-z_{j}\right\|^{2}$ (which is essentially what gradient descent attempts to minimize), allowing us to make greater progress in finding stationary points of $f$. We now present the Almost-Convex-AGD procedure.

```
function Almost-Convex-AGD \(\left(f, z_{1}, \epsilon, \gamma, L_{1}\right)\)
    for \(j=1,2, \ldots\) do
        if \(\left\|\nabla f\left(z_{j}\right)\right\| \leq \epsilon\) then return \(z_{j}\)
        end if
        Let \(g_{j}(z)=f(z)+\gamma\left\|z-z_{j}\right\|^{2}\) as in model (3).
        \(\epsilon^{\prime}=\epsilon \sqrt{\gamma /\left(50\left(L_{1}+2 \gamma\right)\right)}\)
        \(z_{j+1} \leftarrow \operatorname{AcCELERATED-GRADIENT-DESCENT}\left(g_{j}, z_{j}, \epsilon^{\prime}, L_{1}, \gamma\right)\)
    end for
end function
```

Recalling the definition $\Delta_{f} \geq f\left(z_{1}\right)-\inf _{x} f(x)$, we have the following convergence guarantee.
Lemma 3.1. Let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be be $\min \left\{\sigma_{1}, 0\right\}$-almost convex and $L_{1}$-smooth. Let $\gamma \geq \sigma_{1}$ and let $0<\gamma \leq L_{1}$. Then Almost-CONVEX-AGD $\left(f, z_{1}, \epsilon, \gamma, L_{1}\right)$ returns a vector $z$ such that $\|\nabla f(z)\| \leq \epsilon$ and

$$
\begin{equation*}
f\left(z_{1}\right)-f(z) \geq \min \left\{\gamma\left\|z-z_{1}\right\|^{2}, \frac{\epsilon}{\sqrt{10}}\left\|z-z_{1}\right\|\right\} \tag{4}
\end{equation*}
$$

in time

$$
\begin{equation*}
O\left(\mathrm{~T}_{\operatorname{grad}}\left(\sqrt{\frac{L_{1}}{\gamma}}+\frac{\sqrt{\gamma L_{1}}}{\epsilon^{2}}\left(f\left(z_{1}\right)-f(z)\right)\right) \log \left(2+\frac{L_{1}^{3} \Delta_{f}}{\gamma^{2} \epsilon^{2}}\right)\right) \tag{5}
\end{equation*}
$$

Before providing the proof, we remark that the runtime guarantee $(5)$ is an improvement over the convergence guarantees of standard gradient descent—which scale as $O\left(\mathrm{~T}_{\text {grad }} \Delta_{f} L_{1} \epsilon^{-2}\right)$ —whenever $\gamma \ll L_{1}$.
Proof. Because $f$ is $-\sigma_{1}$-strongly convex and $\gamma \geq \sigma_{1}$, Lemma 2.4 guarantees that $g_{j}$ is $\gamma$-strongly convex. This strong convexity also guarantees that $g_{j}$ has a unique minimizer, which we denote $z_{j}^{*}$.

Let $j_{*}$ be the time at which the routine terminates (we set $j_{*}=\infty$ if this does not occur; our analysis addresses this case). Let $j \in\left[1, j_{*}\right) \cap \mathbb{N}$ be arbitrary. We have by Line 7 and Lemma 2.5 (recall that $g_{j}$ is convex and $L_{1}+2 \gamma$ smooth) that $\left\|\nabla g_{j}\left(z_{j+1}\right)\right\|^{2} \leq \frac{\epsilon^{2} \gamma}{L_{1}+2 \gamma}$. Moreover, because $j<j_{*}$, we have $\left\|\nabla g_{j}\left(z_{j}\right)\right\|=\left\|\nabla f\left(z_{j}\right)\right\| \geq \epsilon$ by our termination criterion and definition (3) of $g_{j}$. Consequently, $\left\|\nabla g_{j}\left(z_{j+1}\right)\right\|^{2} \leq \frac{\gamma}{L_{1}+2 \gamma}\left\|\nabla g_{j}\left(z_{j}\right)\right\|^{2}$, and applying Lemma 2.3 to the $\left(L_{1}+2 \gamma\right)$-smooth and $\gamma$-strongly convex function $g_{j}$ yields that

$$
g_{j}\left(z_{j+1}\right)-g_{j}\left(z_{j}^{*}\right) \leq \frac{1}{2 \gamma}\left\|\nabla g_{j}\left(z_{j+1}\right)\right\|^{2} \leq \frac{1}{2\left(L_{1}+2 \gamma\right)}\left\|\nabla g_{j}\left(z_{j}\right)\right\|^{2} \leq g_{j}\left(z_{j}\right)-g_{j}\left(z_{j}^{*}\right)
$$

Thus we have $g_{j}\left(z_{j+1}\right) \leq g_{j}\left(z_{j}\right)$ and

$$
f\left(z_{j+1}\right)=g_{j}\left(z_{j+1}\right)-\gamma\left\|z_{j+1}-z_{j}\right\|^{2} \leq g_{j}\left(z_{j}\right)-\gamma\left\|z_{j+1}-z_{j}\right\|^{2}=f\left(z_{j}\right)-\gamma\left\|z_{j+1}-z_{j}\right\|^{2} .
$$

Inducting on the index $j$, we have

$$
\begin{equation*}
-\Delta_{f} \leq f\left(z_{j_{*}}\right)-f\left(z_{1}\right) \leq-\gamma \sum_{j=1}^{j_{*}-1}\left\|z_{j+1}-z_{j}\right\|^{2} \tag{6}
\end{equation*}
$$

Equation (6) shows that to bound the number of iterations of the algorithm it suffices to lower bound the differences $\left\|z_{j+1}-z_{j}\right\|$. Using the condition $\left\|\nabla g_{j}\left(z_{j+1}\right)\right\| \leq \epsilon \sqrt{\frac{\gamma}{50\left(L_{1}+2 \gamma\right)}} \leq \frac{1}{10} \epsilon$, we have

$$
\left\|z_{j+1}-z_{j}\right\|=\frac{1}{2 \gamma}\left\|\nabla f\left(z_{j+1}\right)-\nabla g_{j}\left(z_{j+1}\right)\right\| \geq \frac{1}{2 \gamma}\left(\left\|\nabla f\left(z_{j+1}\right)\right\|-\frac{\epsilon}{10}\right),
$$

where the inequality is a consequence of the triangle inequality. By our termination criterion, we know that if $j+1<j_{*}$ then $\left\|\nabla f\left(z_{j+1}\right)\right\| \geq \epsilon$ and therefore $\left\|z_{j+1}-z_{j}\right\| \geq \frac{9 \epsilon}{20 \gamma} \geq \frac{\epsilon}{\gamma \sqrt{5}}$. Substituting this bound into (6) yields

$$
-\Delta_{f} \leq f\left(z_{j_{*}}\right)-f\left(z_{1}\right) \leq-\gamma \sum_{j=1}^{j_{*}-2}\left\|z_{j+1}-z_{j}\right\|^{2} \leq-\left(j_{*}-1\right) \cdot \frac{\epsilon^{2}}{5 \gamma}
$$

and therefore

$$
\begin{equation*}
j_{*} \leq 1+\frac{5 \gamma}{\epsilon^{2}}\left[f\left(z_{1}\right)-f\left(z_{j_{*}}\right)\right] \leq 1+\frac{5 \gamma}{\epsilon^{2}} \Delta_{f} . \tag{7}
\end{equation*}
$$

Note that the method calls Accelerated-Gradient-Descent (Line 7) with accuracy parameter $\epsilon^{\prime}=\epsilon \sqrt{\gamma /\left(50\left(L_{1}+2 \gamma\right)\right)}$; using $\gamma \leq L_{1}$ we may apply Lemma 2.5 to bound the running time of each call by

$$
O\left(\mathrm{~T}_{\text {grad }}\left(1+\sqrt{\frac{L_{1}+2 \gamma}{\gamma}} \log \frac{4\left(L_{1}+2 \gamma\right)^{2} \Delta_{f}}{\gamma\left(\epsilon^{\prime}\right)^{2}}\right)\right)=O\left(\mathrm{~T}_{\text {grad }} \sqrt{\frac{L_{1}}{\gamma}} \log \left(2+\frac{L_{1}^{3} \Delta_{f}}{\gamma^{2} \epsilon^{2}}\right)\right) .
$$

The method Almost-convex-AGD performs at most $j^{*}$ iterations (Eq. (7)), and combining the preceding display with this iteration bound yields the running time (5).

All that remains is to prove the progress bound (4). By application of the triangle inequality and Jensen's inequality, we have

$$
\left\|z_{j^{*}}-z_{1}\right\|^{2} \leq\left(\sum_{j=1}^{j^{*}-1}\left\|z_{j+1}-z_{j}\right\|\right)^{2} \leq j^{*} \cdot \sum_{j=1}^{j^{*}-1}\left\|z_{j+1}-z_{j}\right\|^{2}
$$

Combing this bound with the earlier progress guarantee (6) yields $f\left(z_{1}\right)-f\left(z_{j^{*}}\right) \geq \frac{\gamma}{j^{*}}\left\|z_{j}^{*}-z_{1}\right\|^{2}$, and since by $(7)$ either $j^{*} \leq 1$ or $j^{*} \leq 10 \frac{\gamma}{\epsilon^{2}}\left[f\left(z_{1}\right)-f\left(z_{j^{*}}\right)\right]$ the result follows.

### 3.2 Exploiting negative curvature

Our first sub-routine either declares the problem locally "almost convex" or finds a direction of $f$ that has negative curvature, meaning a direction $v$ such that $v^{T} \nabla^{2} f(x) v<0$. The idea to make progress on $f$ by moving in directions of descent on the Hessian is of course well-known, and relies on the fact that if at a point $z$ the function $f$ is "very" non-convex, i.e. $\lambda_{\min }\left(\nabla^{2} f(z)\right) \leq-\alpha / 2$ for some $\alpha>0$, then we can reduce the objective significantly (by a constant fraction of $L_{2}^{-2} \alpha^{3}$ at least) by taking a step in a direction of negative curvature. Conversely, if $\lambda_{\min }\left(\nabla^{2} f(z)\right) \geq-\alpha / 2$, the function $f$ is "almost convex" in a neighborhood of $z$, suggesting that gradient-like methods on $f$ directly should be effective. With this in mind, we present the routine Negative-curvature-descent, which, given a function $f$, initial point $z_{1}$, and a few additional tolerance parameters, returns a vector $z$ decreasing $f$ substantially by moving in Hessian-based directions.

```
function Negative-curvature-descent \(\left(z_{1}, f, L_{2}, \alpha, \Delta_{f}, \delta\right)\)
    Set \(\delta^{\prime}=\delta /\left(1+12 L_{2}^{2} \Delta_{f} / \alpha^{3}\right)\)
    for \(j=1,2, \ldots\) do
        Find a vector \(v_{j}\) such that \(\left\|v_{j}\right\|=1\) and, with probability at least \(1-\delta^{\prime}\),
            \(\lambda_{\min }\left(\nabla^{2} f\left(z_{j}\right)\right) \geq v_{j}^{T} \nabla^{2} f\left(z_{j}\right) v_{j}-\alpha / 2\)
            using a leading eigenvector computation \(\triangleright\) see Corollary 2.7
            if \(v_{j}^{T} \nabla^{2} f\left(z_{j}\right) v_{j} \leq-\alpha / 2\) then \(\triangleright\) Make at least \(\alpha^{3} / 12 L_{2}^{2}\) progress
                                    \(z_{j+1} \leftarrow z_{j}-\frac{2\left|v_{j}^{T} \nabla^{2} f\left(z_{j}\right) v_{j}\right|}{L_{2}} \operatorname{sign}\left(v_{j}^{T} \nabla f\left(z_{j}\right)\right) v_{j}\)
            else
                                    \(\triangleright\) w.h.p., \(\lambda_{\min }\left(\nabla^{2} f\left(z_{j}\right)\right) \geq-\alpha\)
            return \(z_{j}\)
        end if
    end for
end function
```

We provide a formal guarantee for the method Negative-Curvature-descent in the following lemma. Before stating the lemma, we recall that $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ has $L_{2}$-Lipschitz Hessian and that $\Delta_{f} \geq f\left(z_{1}\right)-\inf _{x} f(x)$.

Lemma 3.2. Let the function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be $L_{1}$-smooth and have $L_{2}$-Lipschitz continuous Hessian, $\alpha>0,0<\delta<1$ and $z_{1} \in \mathbb{R}^{d}$. If we call Negative-curvature-descent $\left(z_{1}, f, L_{2}, \alpha, \Delta_{f}, \delta\right)$ then the algorithm terminates at iteration $j$ for some

$$
\begin{equation*}
j \leq 1+\frac{12 L_{2}^{2}\left(f\left(z_{1}\right)-f\left(z_{j}\right)\right)}{\alpha^{3}} \leq 1+\frac{12 L_{2}^{2} \Delta_{f}}{\alpha^{3}}, \tag{8}
\end{equation*}
$$

and with probability at least $1-\delta$

$$
\begin{equation*}
\lambda_{\min }\left(\nabla^{2} f\left(z_{j}\right)\right) \geq-\alpha \tag{9}
\end{equation*}
$$

Furthermore, each iteration requires time at most

$$
\begin{equation*}
O\left(\mathrm{~T}_{\mathrm{grad}}\left[1+\sqrt{\frac{L_{1}}{\alpha}} \log \left(\frac{d}{\delta}\left(1+12 \frac{L_{2}^{2} \Delta_{f}}{\alpha^{3}}\right)\right)\right]\right) . \tag{10}
\end{equation*}
$$

Proof. Assume that the method has not terminated at iteration $k$. Let

$$
\eta_{k}=\frac{2\left|v_{k}^{T} \nabla^{2} f\left(z_{k}\right) v_{k}\right|}{L_{2}} \operatorname{sign}\left(v_{k}^{T} \nabla f\left(z_{k}\right)\right)
$$

denote the step size used at iteration $k$, so that $z_{k+1}=z_{k}-\eta_{k} v_{k}$ as in Line 5. By the $L_{2}$-Lipschitz continuity of the Hessian, we have

$$
\left|f\left(z_{k}-\eta_{k} v_{k}\right)-f\left(z_{k}\right)+\eta_{k} v_{k}^{T} \nabla f\left(z_{k}\right)-\frac{1}{2} \eta_{k}^{2} v_{k}^{T} \nabla^{2} f\left(z_{k}\right) v_{k}\right| \leq \frac{L_{2}}{6}\left\|\eta_{k} v_{k}\right\|^{3} .
$$

Noting that $\eta_{k} v_{k}^{T} \nabla f\left(z_{k}\right) \geq 0$ by construction, we rearrange the preceding inequality to obtain

$$
f\left(z_{k+1}\right)-f\left(z_{k}\right) \leq \frac{\eta_{k}^{2}}{2}\left(\frac{L_{2}}{3}\left|\eta_{k}\right|+v_{k}^{T} \nabla^{2} f\left(z_{k}\right) v_{k}\right)=-\frac{2\left|v_{k}^{T} \nabla^{2} f\left(z_{k}\right) v_{k}\right|^{3}}{3 L_{2}^{2}} \stackrel{(i)}{\leq}-\frac{\alpha^{3}}{12 L_{2}^{2}},
$$

where inequality $(i)$ uses that $\left|v_{k}^{T} \nabla^{2} f\left(z_{k}\right) v_{k}\right|>\alpha / 2$, as the stopping criterion has not been met. Telescoping the above equation for $k=1,2, \ldots, j-1$, we conclude that at the final iteration

$$
\Delta_{f} \geq f\left(z_{1}\right)-f\left(z_{j}\right) \geq \frac{\alpha^{3}}{12 L_{2}^{2}}(j-1)
$$

which gives the bound (8).
We turn to inequality (9). Recall the definition of $\delta^{\prime}=\frac{\delta}{1+12 L_{2}^{2} \Delta_{f} / \alpha^{3}}$, which certainly satisfies $\delta^{\prime} \leq \delta / j$ if $j$ is the final iteration of the algorithm (as the bound (8) is deterministic). Now, at the last iteration, we have by definition of the final iterate that $v_{j}^{T} \nabla^{2} f\left(z_{j}\right) v_{j} \geq-\frac{\alpha}{2}$, and thus, if $v_{j}$ is an additive $\alpha / 2$-approximate smallest eigenvector, we have $\lambda_{\min }\left(\nabla^{2} f\left(z_{j}\right)\right) \geq v_{j}^{T} \nabla^{2} f\left(z_{j}\right) v_{j}-\alpha$. Applying a union bound, the probability that the approximate eigenvector method fails to return an $\alpha / 2$-approximate eigenvector in any iteration is bounded by $\delta^{\prime} j \leq \delta$, giving the result.

Finally, equation (10) is immediate by Corollary 2.7 .

## 4 An accelerated gradient method for non-convex optimization

Now that we have provided the two subroutines Negative-curvature-descent and Almost-CONVEX-AGD, which (respectively) find directions of negative curvature and solve nearly convex problems, we combine them carefully to provide an accelerated gradient method for smooth nonconvex optimization. The idea behind our Accelerated-non-convex-method is as follows. At the beginning of each iteration $k$ we use Negative-curvature-descent to make progress until we reach a point $\widehat{x}_{k}$ where the function is almost convex (Def. 4) in a neighborhood of the current iterate. For a parameter $\alpha \geq 0$, we define the convex penalty

$$
\begin{equation*}
\rho_{\alpha}(x):=L_{1}\left[\|x\|-\frac{\alpha}{L_{2}}\right]_{+}^{2}, \tag{11}
\end{equation*}
$$

where $[t]_{+}=\max \{t, 0\}$. We then modify the function $f(x)$ by adding the penalty $\rho_{\alpha}$ and defining

$$
f_{k}(x)=f(x)+\rho_{\alpha}\left(x-\widehat{x}_{k}\right) .
$$

The function $f_{k}(x)$ is globally almost convex, as we show in Lemma 4.1 to come, so that the method Almost-convex-AGD applied to the function $f_{k}(x)$ quickly reduces the objective $f$.

We trade between curvature minimization and accelerated gradient using the parameter $\alpha$ in the definition (11) of $\rho$, which governs acceptable levels of non-convexity. By carefully choosing $\alpha$, the combined method has convergence rate $\widetilde{O}\left(\epsilon^{-7 / 4}\right)$, which we we prove in Theorem 4.3 .

```
Algorithm 1 Acceleration of smooth non-linear optimization
    function ACCELERATED-NON-CONVEX-METHOD \(\left(x_{1}, f, \epsilon, L_{1}, L_{2}, \alpha, \Delta_{f}, \delta\right)\)
        Set \(K:=\left\lceil 1+\Delta_{f}\left(12 L_{2}^{2} / \alpha^{3}+\sqrt{10} L_{2} /(\alpha \epsilon)\right)\right\rceil\) and \(\delta^{\prime \prime}:=\frac{\delta}{K}\)
        for \(k=1,2, \ldots\) do
            if \(\alpha<L_{1}\) then
                \(\widehat{x}_{k} \leftarrow \operatorname{NegATIVE-CURVATURE-DESCENT}\left(x_{k}, f, L_{2}, \alpha, \Delta_{f}, \delta^{\prime \prime}\right)\)
            else
                \(\widehat{x}_{k} \leftarrow x_{k}\)
            end if
            if \(\left\|\nabla f\left(\widehat{x}_{k}\right)\right\| \leq \epsilon\) then
                return \(\widehat{x}_{k} \quad \triangleright\) guarantees w.h.p., \(\lambda_{\min }\left(\nabla^{2} f\left(\widehat{x}_{k}\right)\right) \geq-2 \alpha\)
            end if
            Set \(f_{k}(x)=f(x)+L_{1}\left(\left[\left\|x-\widehat{x}_{k}\right\|-\alpha / L_{2}\right]_{+}\right)^{2}\)
            \(x_{k+1} \leftarrow\) Almost-CONVEX-AGD \(\left(f_{k}, \widehat{x}_{k}, \epsilon / 2,3 \alpha, 5 L_{1}\right)\)
        end for
    end function
```


### 4.1 Preliminaries: convexity and iteration bounds

Before coming to the theorem giving a formal guarantee for ACCELERATED-NON-CONVEX-METHOD, we provide two technical lemmas showing that the internal subroutines are well-behaved. The first lemma confirms that the regularization technique (11) transforms a locally almost convex function into a globally almost convex function (Def. 4), so we can efficiently apply ALmost-CONVEX-AGD to it.

Lemma 4.1. Let $f$ be $L_{1}$-smooth and have $L_{2}$-Lipschitz continuous Hessian. Let $x_{0} \in \mathbb{R}^{d}$ be such that $\nabla^{2} f\left(x_{0}\right) \succeq-\alpha I$ for some $\alpha \geq 0$. The function $f_{\alpha}(x):=f(x)+\rho_{\alpha}\left(x-x_{0}\right)$ is $3 \alpha$-almost convex and $5 L_{1}$-smooth.

Proof. It is clear that $\rho=\rho_{\alpha}$ is convex, as it is an increasing convex function of a positive argument [9, Chapter 3.2]. We claim that $\rho$ is $4 L_{1}$-smooth. Indeed, the gradient

$$
\nabla \rho(x)=2 L_{1} \frac{x}{\|x\|}\left[\|x\|-\frac{\alpha}{L_{2}}\right]_{+}
$$

is continuous by inspection and differentiable except at $\|x\|=\frac{\alpha}{L_{2}}$. For $\|x\|<\alpha / L_{2}$, we have $\nabla^{2} \rho(x)=0$, and for $\|x\|>\alpha / L_{2}$ we have

$$
\begin{equation*}
\nabla^{2} \rho(x)=2 L_{1}\left(I+\frac{\alpha}{L_{2}}\left(\frac{x x^{T}}{\|x\|^{3}}-\frac{I}{\|x\|}\right)\right) \tag{12}
\end{equation*}
$$

which satisfies $0 \preceq \nabla^{2} \rho(x) \preceq 4 L_{1} I$ for all $x$. As $\nabla \rho(x)$ is continuous, we conclude that $\rho$ is $4 L_{1}$-smooth. The $L_{1}$-smoothness of $f$ then implies that the sum $f(x)+\rho\left(x-x_{0}\right)$ is $5 L_{1}$ smooth.

To argue almost convexity of $f+\rho$, we show that $\nabla^{2} f(x)+\nabla^{2} \rho\left(x-x_{0}\right) \succeq-3 \alpha I$ almost everywhere, which is equivalent to Definition 4 when the gradient is continuous. For $\left\|x-x_{0}\right\|<$
$2 \alpha / L_{2}$, we have by Lipschitz continuity of $\nabla^{2} f$ that

$$
\nabla^{2} f(x) \succeq \nabla^{2} f\left(x_{0}\right)-L_{2}\left\|x-x_{0}\right\| I \succeq-3 \alpha I,
$$

which implies the result because $\rho$ is convex. For $\left\|x-x_{0}\right\|>2 \alpha / L_{2}$, inspection of the Hessian (12) shows that $\nabla^{2} \rho\left(x-x_{0}\right) \succeq L_{1} I$. Since $\nabla^{2} f(x) \succeq-L_{1} I$ almost everywhere by the $L_{1}$-smoothness of $f$, we conclude that $\nabla^{2} f(x)+\nabla^{2} \rho\left(x-x_{0}\right) \succeq 0$ whenever $\nabla^{2} f(x)$ exists.

The next lemma provides a high probability guarantee on the correctness and number of iterations of Accelerated-non-convex-method. (There is randomness in the eigenvector computation subroutine invoked within Negative-curvature-descent.) As always, we let $\Delta_{f} \geq$ $f\left(x_{1}\right)-\inf _{x} f(x)$.
Lemma 4.2. Let $f$ be $L_{1}$-smooth with $L_{2}$-Lipschitz continuous Hessian, $\epsilon>0, \delta \in(0,1)$, and $\alpha \in$ $\left[0, L_{1}\right]$. Then with probability at least $1-\delta$, the method Accelerated-non-Convex-method $\left(x_{1}\right.$, $\left.f, \epsilon, L_{1}, L_{2}, \alpha, \Delta_{f}, \delta\right)$ terminates after $t$ iterations with $\left\|\nabla f\left(\widehat{x}_{t}\right)\right\| \leq \epsilon$, where $t$ satisfies

$$
t \leq \begin{cases}2+\Delta_{f}\left(\frac{12 L_{2}^{2}}{\alpha^{3}}+\frac{\sqrt{10} L_{2}}{\alpha \epsilon}\right) & \text { if } \alpha<L_{1}  \tag{13}\\ 2+\Delta_{f} \frac{16 L_{1}}{3 \epsilon^{2}} & \text { if } \alpha=L_{1}\end{cases}
$$

Further, $\lambda_{\min }\left(\nabla^{2} f\left(\widehat{x}_{k}\right)\right) \geq-2 \alpha$ for all $k \leq t$.
Proof. Before beginning the proof proper, we provide a quick bound on the size of the difference between iterates $\widehat{x}_{k}$ and $\widehat{x}_{k-1}$, which will imply progress in function values across iterations of Alg. 1. In each iteration that the convergence criterion $\left\|\nabla f\left(\widehat{x}_{k}\right)\right\| \leq \epsilon$ is not met- that is, whenever $\left\|\nabla f\left(\widehat{x}_{k}\right)\right\|>\epsilon$-we have that

$$
\epsilon \leq\left\|\nabla f\left(\widehat{x}_{k}\right)\right\| \stackrel{(i)}{\leq}\left\|\nabla f_{k-1}\left(\widehat{x}_{k}\right)\right\|+\left\|\nabla \rho\left(\widehat{x}_{k}-\widehat{x}_{k-1}\right)\right\| \stackrel{(i i)}{\leq} \frac{\epsilon}{2}+2 L_{1}\left[\left\|\widehat{x}_{k}-\widehat{x}_{k-1}\right\|-\frac{\alpha}{L_{2}}\right]_{+}
$$

In inequality $(i)$ we used the triangle inequality and definition of $f_{k-1}=f+\rho\left(\cdot-x_{k-1}\right)$ and inequality ( $i i$ ) used that the call to Almost-convex-AGD returns $\widehat{x}_{k}$ with $\left\|\nabla f_{k-1}\left(x_{k}\right)\right\| \leq \epsilon / 2$. Rearranging yields

$$
\begin{equation*}
\frac{\epsilon}{4 L_{1}} \leq\left[\left\|\widehat{x}_{k}-\widehat{x}_{k-1}\right\|-\frac{\alpha}{L_{2}}\right]_{+}=\left\|\widehat{x}_{k}-\widehat{x}_{k-1}\right\|-\frac{\alpha}{L_{2}} \tag{14}
\end{equation*}
$$

where the equality is implied because $\epsilon>0$.
Now we consider two cases, the first the simpler case that $\alpha=L_{1}$ is large enough that we never search for negative curvature, and the second that $\alpha<L_{1}$ so that we find directions of negative curvature in the method.

Case 1: large $\alpha$ In this case, we have that $\alpha=L_{1}$, so that $x_{k}=\widehat{x}_{k}$ for all iterations $k$ (Line 7 of the algorithm). Assume that at iteration $k$ that the algorithm has not terminated, so $\left\|\nabla f\left(\widehat{x}_{k}\right)\right\| \geq \epsilon$. Then inequality (14) gives $\frac{\epsilon}{4 L_{1}}<\left\|\widehat{x}_{k}-\widehat{x}_{k-1}\right\|$. By Lemma 4.1 we know that $f_{k}$ is $3 L_{1}$-almost convex (Def. 4) and $5 L_{1}$-smooth; therefore we may apply Lemma 3.1 with $\gamma=3 \alpha=3 L_{1}$ to lower bound the progress of the call to Almost-convex-AGD in Line 13 of Alg. 1 to obtain

$$
\begin{align*}
f\left(\widehat{x}_{k-1}\right)-f\left(\widehat{x}_{k}\right) & \geq \min \left\{3 L_{1}\left\|\widehat{x}_{k-1}-\widehat{x}_{k}\right\|^{2}, \frac{\epsilon}{\sqrt{10}}\left\|\widehat{x}_{k-1}-\widehat{x}_{k}\right\|\right\} \\
& \geq \min \left\{3 \frac{\epsilon^{2}}{16 L_{1}}, \frac{\epsilon^{2}}{4 \sqrt{10} L_{1}}\right\} \geq \frac{\epsilon^{2}}{16 L_{1}} . \tag{15}
\end{align*}
$$

Telescoping this display, we have for any iteration $s$ at which the algorithm has not terminated that

$$
\Delta_{f} \geq \sum_{k=2}^{s} f\left(\widehat{x}_{k-1}\right)-f\left(\widehat{x}_{k}\right) \geq(s-1) \frac{3 \epsilon^{2}}{16 L_{1}}
$$

which yields the second case of the bound (13). The inequality $\nabla^{2} f\left(\widehat{x}_{j}\right) \succeq-2 \alpha I$ holds trivially because $f$ is $L_{1}$-smooth.

Case 2: small $\alpha$ In this case, we assume that $\alpha<L_{1}$. Let $K=\left\lceil 1+\Delta_{f}\left(\frac{12 L_{2}^{2}}{\alpha^{3}}+\frac{\sqrt{10} L_{2}}{\alpha \epsilon}\right)\right\rceil$ and $\delta^{\prime \prime}=\frac{\delta}{K}$ as in line 2 of Alg. 1. By Lemma 3.2 and a union bound, with probability at least $1-\delta$, for all $k \leq K$ the matrix inequality $\nabla^{2} f\left(\widehat{x}_{k}\right) \succeq-2 \alpha I$ holds, so that we perform our subsequent analysis (for $k \leq K$ ) conditional on this event without appealing to any randomness.

Equation (14) implies that at iteration $1<k \leq K$ exactly one of following three cases is true:
(i) The termination criterion $\left\|\nabla f\left(\widehat{x}_{k}\right)\right\| \leq \epsilon$ holds and Alg. 1 terminates.
(ii) Negative-curvature-Descent (Line 5) constructs $\widehat{x}_{k} \neq x_{k}$, and (i) fails.
(iii) Neither (i) nor (ii) holds, and $\left\|\widehat{x}_{k}-\widehat{x}_{k-1}\right\| \geq \alpha / L_{2}$.

We claim that in case (ii) or (iii), we have

$$
\begin{equation*}
f\left(\widehat{x}_{k-1}\right)-f\left(\widehat{x}_{k}\right) \geq \min \left\{\frac{\alpha \epsilon}{L_{2} \sqrt{10}}, \frac{\alpha^{3}}{12 L_{2}^{2}}\right\} . \tag{16}
\end{equation*}
$$

Deferring the proof of claim (16), we note that it immediately gives a quick proof of the result. Assume, in order to obtain a contradiction that after $K$ iterations the algorithm has not terminated it follows that:

$$
\Delta_{f} \geq f\left(\widehat{x}_{1}\right)-f\left(\widehat{x}_{K}\right)=\sum_{k=2}^{K} f\left(\widehat{x}_{k}\right)-f\left(\widehat{x}_{k+1}\right) \geq(K-1) \min \left\{\frac{\alpha \epsilon}{L_{2} \sqrt{10}}, \frac{\alpha^{3}}{12 L_{2}^{2}}\right\}
$$

Substituting for $K=\left\lceil 1+\Delta_{f}\left(\frac{12 L_{2}^{2}}{\alpha^{3}}+\frac{\sqrt{10} L_{2}}{\alpha \epsilon}\right)\right\rceil$ as in line 2 yields a contradiction and therefore the algorithm terminates after at most $K$ iterations which is the first case of the bound (13).

Let us now prove the claim (16). First, assume case (ii). Then Negative-curvature-descent requires at least two iterations, so Lemma 3.2 implies

$$
\frac{12 L_{2}^{2}\left(f\left(x_{k}\right)-f\left(\widehat{x}_{k}\right)\right)}{\alpha^{3}} \geq 1
$$

Combining this with the fact that $f\left(x_{k}\right) \leq f\left(\widehat{x}_{k-1}\right)$ by the progress bound (4) in Lemma 3.1 (Almost-convex-AGD decreases function values), we have

$$
f\left(\widehat{x}_{k-1}\right)-f\left(\widehat{x}_{k}\right) \geq f\left(x_{k}\right)-f\left(\widehat{x}_{k}\right) \geq \frac{\alpha^{3}}{12 L_{2}^{2}} .
$$

Let us now consider the case that (iii) holds. By Lemma 4.1 we know that the constructed function $f_{k}$ is $3 \alpha$-almost convex (Def. 4) and $5 L_{1}$-smooth, therefore we may apply Lemma 3.1 with $\gamma=3 \alpha$ to lower bound the progress of the entire inner loop of Alg. 1 by

$$
f\left(\widehat{x}_{k-1}\right)-f\left(\widehat{x}_{k}\right) \geq \min \left\{\gamma\left\|\widehat{x}_{k-1}-x_{k}\right\|^{2}, \frac{\epsilon}{\sqrt{10}}\left\|\widehat{x}_{k-1}-x_{k}\right\|\right\} \geq \min \left\{\frac{3 \alpha^{3}}{L_{2}^{2}}, \frac{\alpha \epsilon}{L_{2} \sqrt{10}}\right\}
$$

as desired.

### 4.2 Main result

With Lemmas 4.1 and 4.2 in hand, we may finally present our main result.
Theorem 4.3. Let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be $L_{1}$-smooth and have $L_{2}$-Lipschitz continuous Hessian. Let

$$
\alpha=\min \left\{L_{1}, \max \left\{\epsilon^{2} \Delta_{f}^{-1}, \epsilon^{1 / 2} L_{2}^{1 / 2}\right\}\right\}
$$

and $\delta \in(0,1)$. Then with probability at least $1-\delta$, Accelerated-non-Convex-method $\left(x_{1}, f\right.$, $\left.\epsilon, L_{1}, L_{2}, \alpha, \Delta_{f}, \delta\right)$ returns a point $x$ that satisfies

$$
\|\nabla f(x)\| \leq \epsilon \quad \text { and } \quad \lambda_{\min }\left(\nabla^{2} f(x)\right) \geq-2 \epsilon^{1 / 2} L_{2}^{1 / 2}
$$

in time

$$
O\left(\mathrm{~T}_{\mathrm{grad}}\left(\Delta_{f} L_{1}^{1 / 2} L_{2}^{1 / 4} \epsilon^{-7 / 4}+\Delta_{f}^{1 / 2} L_{1}^{1 / 2} \epsilon^{-1}+1\right) \log \tau\right),
$$

where $\tau=1+1 / \epsilon+1 / \delta+d+L_{1}+L_{2}+\Delta_{f}$.
Proof. We split this proof into two cases: (I) small alpha when $\alpha<L_{1}$ and hence $\alpha=\frac{\epsilon^{2}}{\Delta_{f}}$ or $\sqrt{L_{2} \epsilon}$, this is the non-trivial case requiring solution to a reasonably small accuracy; and (II) when $\alpha=L_{1}$, when the algorithm is roughly equivalent to gradient descent (and $\epsilon$ is large enough that we do not require substantial accuracy).

Case I: Small $\alpha$ We proceed in two steps. First, we bound the number of eigenvector calculations that Negative-curvature-descent performs by providing a progress guarantee for each of them using Lemma 3.2 and arguing that making too much progress is impossible. After this, we perform a similar calculation for the total number of gradient calculations throughout calls to Almost-CONVEX-AGD, this time applying Lemma 3.1.

We begin by bounding the number of eigenvector calculations. When $\alpha<L_{1}$, its definition implies $\epsilon \leq \min \left\{L_{1}^{2} / L_{2}, \Delta_{f}^{1 / 2} L_{1}^{1 / 2}\right\}$. Let $j_{k}^{*}$ be the total number of times the method Negative-curvature-descent invokes the eigenvector computation subroutine (Line 4 of Negative-curvature-descent) during iteration $k$ of the method Accelerated-non-convex-method, let $k^{*}$ denote the total number of iterations of Accelerated-non-convex-method, and define $q:=\sum_{k=1}^{k^{*}} j_{k}^{*}$ as the total number of eigenvector computations. Then by telescoping the bound (8) in Lemma 3.2 and using that $f\left(x_{k}\right) \leq f\left(\widehat{x}_{k-1}\right)$ by the progress bound (4) in Lemma 3.1 (Almost-CONVEX-AGD decreases function values), we have

$$
\sum_{k=1}^{k^{*}}\left(j_{k}^{*}-1\right) \leq \sum_{k=1}^{k^{*}} \frac{12 L_{2}^{2}}{\alpha^{3}}\left(f\left(x_{k}\right)-f\left(\widehat{x}_{k}\right)\right) \leq \sum_{k=1}^{k^{*}} \frac{12 L_{2}^{2}}{\alpha^{3}}\left(f\left(\widehat{x}_{k-1}\right)-f\left(\widehat{x}_{k}\right)\right) \leq \frac{12 \Delta_{f} L_{2}^{2}}{\alpha^{3}} .
$$

Substituting the bound on $k^{*}$ that Lemma 4.2 supplies, we see that with probability at least $1-\delta$,

$$
\begin{equation*}
q \leq \frac{12 \Delta_{f} L_{2}^{2}}{\alpha^{3}}+k^{*} \leq 1+\Delta_{f}\left(\frac{24 L_{2}^{2}}{\alpha^{3}}+\frac{\sqrt{10} L_{2}}{\alpha \epsilon}\right) \stackrel{(i)}{\leq} 1+28 \Delta_{f} L_{2}^{1 / 2} \epsilon^{-3 / 2} \tag{17}
\end{equation*}
$$

where inequality ( $i$ ) follows by our construction that $\alpha \geq \epsilon^{1 / 2} L_{2}^{1 / 2}$. Inequality (17) thus provides a bound on the total number of fast eigenvector calculations we require.

We use the bound (17) to bound the total cost of calls to Negative-curvature-Descent. The tolerated failure probability $\delta^{\prime \prime}$ defined in line 2 satisfies

$$
\frac{1}{\delta^{\prime \prime}}=\frac{1+\Delta_{f}\left(12 L_{2}^{2} / \alpha^{3}+\sqrt{10} L_{2} /(\alpha \epsilon)\right)}{\delta} \leq \frac{1+16 \Delta_{f} L_{2}^{1 / 2} \epsilon^{-3 / 2}}{\delta}
$$

so that $\log \frac{1}{\delta^{\prime \prime}}=O(\log \tau)$. By Lemma 3.2, Eq. (10), the cost of each iteration during Negative-CURVATURE-DESCENT is, using $\max \left\{\epsilon^{2} \Delta_{f}^{-1}, \sqrt{\epsilon L_{2}}\right\}=\alpha<L_{1}$, at most

$$
O\left(\mathrm{~T}_{\mathrm{grad}}\left[1+\sqrt{\frac{L_{1}}{\alpha}} \log \left(\frac{d}{\delta^{\prime \prime}}\left(1+\frac{12 L_{2}^{2} \Delta_{f}}{\alpha^{3}}\right)\right)\right]\right)=O\left(\mathrm{~T}_{\mathrm{grad}} \frac{L_{1}^{\frac{1}{2}}}{\left(L_{2} \epsilon\right)^{\frac{1}{4}} \vee\left(\epsilon \Delta_{f}^{\frac{1}{2}}\right)} \log \tau\right) .
$$

Multiplying this time complexity by $q$ as bounded in expression (17) gives that the total cost of the calls to Negative-curvature-descent is

$$
\begin{equation*}
O\left(\mathrm{~T}_{\mathrm{grad}}\left(\Delta_{f} L_{1}^{1 / 2} L_{2}^{1 / 4} \epsilon^{-7 / 4}+L_{1}^{1 / 2} \Delta_{f}^{1 / 2} \epsilon^{-1}\right) \log \tau\right) \tag{18}
\end{equation*}
$$

We now compute the total cost of calling Almost-convex-AGD. Using the time bound (5) of Lemma 3.1, the cost of calling Almost-convex-AGD in iteration $k$ with almost convexity parameter $\gamma=3 \alpha$ is bounded by the sum of

$$
\begin{equation*}
O\left(\mathrm{~T}_{\mathrm{grad}} \sqrt{\frac{L_{1}}{\gamma}} \log \tau\right) \text { and } O\left(\mathrm{~T}_{\mathrm{grad}} \frac{\sqrt{\gamma L_{1}}}{\epsilon^{2}}\left[f_{k}\left(x_{k}\right)-f_{k}\left(x_{k+1}\right)\right] \log \tau\right) \tag{19}
\end{equation*}
$$

We separately bound the total computational cost of each of the terms (19).
Using the bound $k^{*} \leq 1+\Delta_{f} \frac{16 L_{2}^{1 / 2}}{\epsilon^{3 / 2}}$ as in expression (17) for the total number of iterations of Alg. 1, we see that the first of the time bounds (19) yields identical total cost to the eigenvector computations (18), because $\gamma^{-\frac{1}{2}}=O\left(\alpha^{-\frac{1}{2}}\right)=O\left(1 / \sqrt[4]{\epsilon L_{2}}\right)$. Thus we consider the second term in expression (19). Using the fact that $\left[f_{k}\left(x_{k}\right)-f_{k}\left(x_{k+1}\right)\right] \leq\left[f\left(x_{k}\right)-f\left(x_{k+1}\right)\right]$ by definition of $x_{k+1}$ and the method Almost-convex-AGD, we telescope to find

$$
\sum_{k=1}^{k^{*}}\left[f_{k}\left(x_{k}\right)-f_{k}\left(x_{k+1}\right)\right] \leq \sum_{k=1}^{k^{*}}\left[f\left(x_{k}\right)-f\left(x_{k+1}\right)\right] \leq \Delta_{f}
$$

Noting that by assumption that the almost convexity parameter $\gamma=3 \alpha$, we have $\sqrt{\gamma / 3}=\sqrt{\alpha} \leq$ $L_{2}^{1 / 4} \epsilon^{1 / 4}+\epsilon \Delta_{f}^{-1 / 2}$, telescoping the second term of the bound (19) on the cost of Almost-convexAGD immediately gives the total computational cost bound

$$
O\left(\mathrm{~T}_{\operatorname{grad}} \frac{\Delta_{f} L_{1}^{1 / 2}}{\epsilon^{2}}\left(L_{2}^{1 / 4} \epsilon^{1 / 4}+\frac{\epsilon}{\sqrt{\Delta_{f}}}\right) \log \tau\right)
$$

over all calls of Almost-convex-AGD. This is evidently our desired result that the total computational cost when $\alpha<L_{1}$ is (18).

Case II: Large $\alpha$ When $\alpha=L_{1}$, the algorithm becomes roughly equivalent to gradient descent, because Negative-curvature-descent is not required, so that we need only bound the total computational cost of calls to Almost-convex-AGD. The bound (19) on the computational
effort of each such call again applies, and noting that $L_{1}=\alpha=3 \gamma$ in this case, we replace the bounds (19) with the two terms

$$
O\left(\mathrm{~T}_{\text {grad }} \log \tau\right) \text { and } O\left(\mathrm{~T}_{\text {grad }} \frac{L_{1}}{\epsilon^{2}}\left[f\left(x_{k}\right)-f\left(x_{k+1}\right)\right] \log \tau\right) .
$$

As in Case I, we may telescope the second time bound to obtain total computational effort $O\left(\mathrm{~T}_{\text {grad }}\left(1+\Delta_{f} \frac{L_{1}}{\epsilon^{2}}\right) \log \tau\right)$, while applying the iteration bound (13) of Lemma 4.2 to the first term similarly yields the bound $O\left(\mathrm{~T}_{\text {grad }}\left(1+\Delta_{f} \frac{L_{1}}{\epsilon^{2}}\right) \log \tau\right)$ on the total computational cost. To conclude the proof we observe that $\alpha=L_{1}$ implies $L_{1} \leq \max \left\{\left(\epsilon L_{2}\right)^{1 / 2}, \epsilon^{2} / \Delta_{f}\right\}$ and that $L_{1} \leq$ $\max \left\{\left(\epsilon L_{2}\right)^{1 / 4} L_{1}^{1 / 2}, \epsilon^{2} / \Delta_{f}\right\}$. Therefore,

$$
O\left(\mathrm{~T}_{\mathrm{grad}}\left(1+\Delta_{f} \frac{L_{1}}{\epsilon^{2}}\right) \log \tau\right)=O\left(\mathrm{~T}_{\mathrm{grad}}\left(1+\frac{\Delta_{f} L_{1}^{1 / 2} L_{2}^{1 / 4}}{\epsilon^{7 / 4}}\right) \log \tau\right)
$$

which gives our desired total time.

We provide a bit of discussion to help explicate this result. Much of the complication in the statement of Theorem 4.3 is a consequence of our desire for generality in possible parameter values. In common settings in which points reasonably close to stationarity are desired-when the accuracy $\epsilon$ is small enough-we may simplify the theorem substantially, as the following corollary demonstrates.

Corollary 4.4. Let the conditions of Theorem 4.3 hold, and in addition assume that $\epsilon \leq \sqrt[3]{\Delta_{f}^{2} / L_{2}}$. Then the total computational cost of Alg. 1 is at most

$$
\widetilde{O}\left(\mathrm{~T}_{\mathrm{grad}} \Delta_{f} \frac{L_{1}^{1 / 2} L_{2}^{1 / 4}}{\epsilon^{7 / 4}}\right)
$$

To elucidate the relative importance of acceleration in the approximate eigenvector or gradient descent computation in Accelerated-non-convex-method, we may also consider replacing them with (respectively) the power method (rather than the Lanczos method) or standard gradient descent. We first consider the accelerated (approximate) eigenvector routine. With probability at least $1-\delta$, the power method finds an $\alpha$-additive approximate maximum or minimum eigenvector of the matrix $\nabla^{2} f(x) \in \mathbb{R}^{d \times d}$, with operator norm bounded as $\left\|\nabla^{2} f(x)\right\| \leq L_{1}$, in time $O\left(\frac{L_{1}}{\alpha} \log \frac{d}{\delta}\right)$ (compare this with Corollary 2.7). In this case, substituting $\alpha \asymp \epsilon^{4 / 9}$, rather than $\alpha \asymp \epsilon^{1 / 2}$ in Theorem 4.3, and mimicking the preceding proof yields total time complexity of order $\epsilon^{-16 / 9} \ll \epsilon^{-2}$, ignoring all other problem-dependent constants. That is, non-accelerated eigenvector routines can still yield faster than $\epsilon^{-2}$ rates of convergence.

Conversely, it appears that accelerated gradient descent is more central to our approach. Indeed, the term involving $\sqrt{\gamma L_{1}}$ in the bound (13) is important, as it allows us to carefully trade "almost" convexity $\gamma$ with accuracy $\epsilon$ to achieve fast rates of convergence. Replacing the accelerated gradient descent method with gradient descent in Almost-convex-AGD eliminates the possibility for such optimal trading. Of course, our procedure would still produce output with the second order guarantee $\nabla^{2} f(x) \succeq-2 \epsilon^{1 / 2} L_{2}^{1 / 2} I_{d \times d}$.

## 5 Accelerated (linear) convergence to local minimizers of strictsaddle functions

In this section, we show how to apply Accelerated-non-convex-method and Theorem 4.3 to find local minimizers for generic non-pathological smooth optimization problems with linear rates of convergence. Of course, it is in general NP-hard to even check if a point is a local minimizer of a smooth nonconvex optimization problem [27, 30], so we require a few additional assumptions in this case. In general, second-order stationary points need not be local minima; consequently, we consider strict-saddle functions, which are functions such that all eigenvalues of the Hessian are non-zero at all critical points, so that second-order stationary points are indeed local minima. Such structural assumptions have been important in recent work on first-order methods for general smooth minimization [24, 17, 41], and in a sense "random" functions generally satisfy these conditions (cf. the discussion of Morse functions in the book [1]). To make our discussion formal, consider the following quantitative definition.

Definition 6. A twice differentiable function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is $\left(\varepsilon, \sigma_{-}, \sigma_{+}\right)$-strict-saddle if for any point $x$ such that $\|\nabla f(x)\| \leq \varepsilon$, $\lambda_{\min }\left(\nabla^{2} f(x)\right) \in\left(-\infty, \sigma_{-}\right] \cup\left[\sigma_{+}, \infty\right)$.

Some definitions of strict-saddle include a radius $R$ bounding the distance between any point $x$ satisfying $\|\nabla f(x)\| \leq \varepsilon$ and $\lambda_{\min }\left(\nabla^{2} f(x)\right) \geq \sigma_{+}$and a local minimizer $x^{+}$of $f$, and they assume that $f$ is $\sigma_{+}$-strongly convex in a ball of radius $2 R$ around any local minimizer. Our assumption on the Lipschitz continuity of $\nabla^{2} f$ obviates the need for such conditions, allowing the following simplified definition.

Definition 7. Let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ have $L_{2}$-Lipschitz continuous Hessian. We call $f \sigma_{1}$-strict-saddle if it is $\left(\sigma_{1}^{2} / L_{2}, \sigma_{1}, \sigma_{1}\right)$-strict-saddle.

With this definition in mind, we present Algorithm 2, which leverages Algorithm 1 to obtain linear convergence (in the desired accuracy $\epsilon$ ) to a local minimizer of strict-saddle functions. The algorithm proceeds in two phases, first finding a region of strong convexity, and in the second phase solving a regularized version of resulting locally convex problem in this region. That the first phase of Alg. 2 terminates in a neighborhood of a local optimum of $f$, where $f$ is convex in this neighborhood, is an immediate consequence of the strict-saddle property coupled with the gradient and Hessian bounds of Theorem 4.3. We can then apply (accelerated) gradient descent to quickly find the local optimum, which we describe rigorously in the following theorem.

Theorem 5.1. Let $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be $L_{1}$-smooth, have $L_{2}$-Lipschitz continuous Hessian, and be $\sigma_{1}$-strict-saddle. Let $\epsilon \geq 0$ and $\delta \in(0,1)$. With probability at least $1-\delta$, Accelerated-strict-$\operatorname{SADDLE}-\operatorname{METHOD}\left(x_{1}, f, \epsilon, L_{1}, L_{2}, \sigma_{1}, \Delta_{f}, \delta\right)$ returns a point $x$ that satisfies $\|\nabla f(x)\| \leq \epsilon$ in time

$$
O\left(\mathrm{~T}_{\text {grad }}\left[\sqrt{\frac{L_{1}}{\sigma_{1}}} \log \left(\tau^{\prime}+\frac{1}{\epsilon}\right)+\frac{L_{1}^{1 / 2} L_{2}^{2} \Delta_{f}}{\sigma_{1}^{7 / 2}} \log \tau^{\prime}\right]\right),
$$

where $\tau^{\prime}=1+L_{1} / \sigma_{1}+1 / \delta+d+L_{2}+\Delta_{f}$. When $\epsilon \leq \frac{\sigma_{1}^{2}}{16 L_{2}}$, with the same probability there exists a local minimizer $x_{+}^{\star}$ of $f$ such that

$$
\begin{equation*}
\left\|x-x_{+}^{\star}\right\| \leq \frac{2 \epsilon}{\sigma_{1}} \text { and } f(x)-f\left(x_{+}^{\star}\right) \leq \frac{2 L_{1} \epsilon^{2}}{\sigma_{1}^{2}} . \tag{20}
\end{equation*}
$$

```
Algorithm 2 Acceleration of smooth strict-saddle optimization
    function Accelerated-Strict-Saddle-Method \(\left(x_{1}, f, \epsilon, L_{1}, L_{2}, \sigma_{1}, \Delta_{f}, \delta\right)\)
    Phase one
        Set \(\varepsilon=\max \left\{\epsilon, \frac{\sigma_{1}^{2}}{16 L_{2}}\right\}\)
        Set \(\alpha=\min \left\{L_{1}, \max \left\{\varepsilon^{2} \Delta_{f}^{-1}, \varepsilon^{1 / 2} L_{2}^{1 / 2}\right\}\right\} \quad \triangleright\) as in Theorem 4.3
        \(x_{+} \leftarrow\) Accelerated-non-convex-method \(\left(x_{1}, f, \varepsilon, L_{1}, L_{2}, \alpha, \Delta_{f}, \delta\right)\)
    Phase two:
        if \(\epsilon<\varepsilon\) then \(\quad \triangleright\) non-trivial case
            Set \(f_{+}(x)=f(x)+L_{1}\left[\left\|x-x_{+}\right\|-\frac{\sigma_{1}}{4 L_{2}}\right]_{+}^{2}\)
            return Accelerated-Gradient-descent \(\left(f_{+}, x_{+}, \epsilon, 5 L_{1}, \sigma_{1} / 2\right)\)
        else
            return \(x_{+}\)
        end if
    end function
```

Proof. The result in the low accuracy regime in which $\epsilon>\frac{\sigma_{1}^{2}}{16 L_{2}}$ is immediate by Theorem 4.3, and we therefore focus on the case that $\epsilon \leq \frac{\sigma_{1}^{2}}{16 L_{2}}$. We perform our analysis conditional on the event, which holds with probability at least $1-\delta$, that the guarantees of Theorem 4.3 hold. That is, that $x_{+}$generated in Line 4 satisfies

$$
\begin{equation*}
\left\|\nabla f\left(x_{+}\right)\right\| \leq \frac{\sigma_{1}^{2}}{16 L_{2}} \text { and } \nabla^{2} f\left(x_{+}\right) \succeq-\frac{\sigma_{1}}{4} I, \tag{21}
\end{equation*}
$$

and that it is computed in time

$$
\begin{aligned}
T_{1} & =O\left(\mathrm{~T}_{\text {grad }}\left[\left(\frac{L_{1}^{1 / 2} L_{2}^{2} \Delta_{f}}{\sigma_{1}^{7 / 2}}+\frac{L_{1}^{1 / 2} L_{2} \Delta_{f}^{1 / 2}}{\sigma_{1}^{2}}+1\right) \log \tau^{\prime}\right]\right) \\
& \stackrel{(i)}{=} O\left(\mathrm{~T}_{\operatorname{grad}} \sqrt{\frac{L_{1}}{\sigma_{1}}}\left[\frac{L_{2}^{2} \Delta_{f}}{\sigma_{1}^{3}}+1\right] \log \tau^{\prime}\right),
\end{aligned}
$$

where $\tau^{\prime}$ is as in the theorem statement. Equality $(i)$ is a consequence of the inequalities $1 \leq$ $\sqrt{L_{1} / \sigma_{1}}$ and $1+a+a^{2}=O\left(1+a^{2}\right)$ for $a \geq 0$.

In conjunction with Definition 7, the bounds (21) imply that $\nabla^{2} f\left(x_{+}\right) \succeq \sigma_{1} I$. Recalling Lemma 4.1, and the bound (12) from its proof, a trivial calculation involving the Lipschitz continuity of $\nabla^{2} f$ shows that $f_{+}(x)=f(x)+L_{1}\left[\left\|x-x_{+}\right\|-\sigma_{1} / 4 L_{2}\right]_{+}^{2}$ is $\sigma_{1} / 2$-strongly convex. Additionally, we have immediately that $f_{+}$is $5 L_{1}$-smooth.

Let $x_{+}^{\star}$ be the unique global minimizer of $f_{+}$. By the strong convexity of $f_{+}$, we may bound the distance between $x_{+}$and $x_{+}^{\star}$ (recall Lemma 2.3) by

$$
\left\|x_{+}-x_{+}^{\star}\right\| \leq \frac{2\left\|\nabla f_{+}\left(x_{+}\right)\right\|}{\sigma_{1}}=\frac{2\left\|\nabla f\left(x_{+}\right)\right\|}{\sigma_{1}} \leq \frac{\sigma_{1}}{8 L_{2}},
$$

where final inequality is immediate from the gradient bound (21). By construction, $f_{+}=f$ on the ball $\left\{x:\left\|x-x_{+}\right\| \leq \sigma_{1} / 4 L_{2}\right\}$, and as $x_{+}^{\star}$ belongs to the interior of this ball, it is a local minimizer of $f$.

Let $x$ be the point produced by the call to Accelerated-gradient-descent. By Lemma 2.5, $x$ satisfies $\left\|\nabla f_{+}(x)\right\| \leq \epsilon$ and is computed in time

$$
T_{2}:=\mathrm{T}_{\mathrm{grad}}+\mathrm{T}_{\mathrm{grad}} \sqrt{\frac{10 L_{1}}{\sigma_{1}}} \log \left(\frac{200 L_{1}^{2} \Delta_{f}}{\sigma_{1} \epsilon^{2}}\right)=O\left(\mathrm{~T}_{\text {grad }} \sqrt{\frac{L_{1}}{\sigma_{1}}} \log \left(\tau^{\prime}+\frac{1}{\epsilon}\right)\right) .
$$

The strong convexity of $f_{+}$once more (Lemma 2.3) implies

$$
\left\|x-x_{+}^{\star}\right\| \leq \frac{\left\|\nabla f_{+}(x)\right\|}{\sigma_{1} / 2} \leq \frac{2 \epsilon}{\sigma_{1}} \leq \frac{\sigma_{1}}{8 L_{2}},
$$

which gives the distance bound in expression (20). Combining $\left\|x-x_{+}^{\star}\right\| \leq \frac{\sigma_{1}}{8 L_{2}}$ and $\left\|x_{+}-x_{+}^{\star}\right\| \leq$ $\frac{\sigma_{1}}{8 L_{2}}$, we have that $\left\|x-x_{+}\right\| \leq \frac{\sigma_{1}}{4 L_{2}}$, and therefore $f(x)=f_{+}(x)$ and $\|\nabla f(x)\|=\left\|\nabla f_{+}(x)\right\| \leq \epsilon$. The functional bound (20) then follows from the $L_{1}$-smoothness of $f$ and that $\nabla f\left(x_{+}^{\star}\right)=0$, as

$$
f(x)-f\left(x_{+}^{\star}\right) \leq \nabla f\left(x_{+}^{\star}\right)^{T}\left(x-x_{+}^{\star}\right)+\frac{L_{1}}{2}\left\|x-x_{+}^{\star}\right\|^{2}=\frac{L_{1}}{2}\left\|x-x_{+}^{\star}\right\|^{2} .
$$

The running time guarantee follows by summing $T_{1}$ and $T_{2}$ above.

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[^0]:    ${ }^{1}$ The notation $\widetilde{O}$ hides logarithmic factors. See Definition 5 .
    ${ }^{2}$ Technically, $\widetilde{O}\left(d^{\omega}\right)$ where $\omega<2.373$ is the matrix multiplication constant [42].

[^1]:    ${ }^{3} \mathrm{~A}$ preprint of the current paper appears on the arXiv [11].

[^2]:    ${ }^{4}$ We assume infinite precision arithmetic in this paper: see discussion in Section 2.2.

