The asymptotic behavior of solutions to the repulsive n-body problem

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Abstract

The n-body problem with a purely repulsive Coulomb interaction is considered. It is shown that for large times t the distance between any two particles grows linearly in t. The trajectory of each particle is asymptotically a straight line with a fixed velocity which is different for different particles.

1 Introduction

Consider n point-charges with spatial coordinates $x_i(t) \in \mathbb{R}^3$, i = 1, ..., n, depending on time $t \in \mathbb{R}$. The charges are all of the same sign, and for simplicity all the particles have the same charge and the same mass which is set equal to unity; the results below remain true without this assumption, as long as all the charges have the same sign. The particles interact via a repulsive Coulomb force so that the evolution of the system is governed by the usual n-body equations, but with the repulsive sign, written here in first order form:

$$\dot{x}_i = v_i, \qquad \dot{v}_i = \sum_{j=1,\dots,n, j \neq i, \frac{x_i - x_j}{|x_i - x_j|^3}, \qquad i = 1,\dots, n.$$
 (1)

The energy of the system, which is recalled in the next section, is conserved and positive definite. Hence the v_i remain bounded and $x_i \neq x_j$ for $i \neq j$ as long as the solution exists, and hence every solution exists globally in time.

It is natural to ask how the system behaves for $t \to \infty$ (and for $t \to -\infty$), and one expects that due to the repulsive interaction the system spreads out in space. Indeed, in [6] it is conjectured that the distance between any two particles goes to infinity for $t \to \infty$.

We shall prove that there are two constants $c_1, c_2 > 0$ which depend only on the initial data such that for t sufficiently large,

$$c_1 t \le |x_i(t) - x_j(t)| \le c_2 t, \ i \ne j.$$
 (2)

This will follow from a relation between the potential energy of the system and its kinetic energy relative to its asymptotic configuration, cf. the theorem in the next section. This relation further implies that there exist parameters $x_i^*, v_i^* \in \mathbb{R}^3$, $i = 1, \ldots, n$, such that

$$x_i(t) = x_i^* + t \, v_i^* + \mathcal{O}(\ln t), \ t \to \infty, \tag{3}$$

and $v_i^* \neq v_j^*$ for $i \neq j$.

Our analysis is based on arguments which are completely analogous to those used in [5] for the plasma-physics case of the Vlasov-Poisson system, which can be thought of as the continuum, mean-field limit of the system (1) as $n \to \infty$, see also [9, 10]. We are not aware that these arguments, which are simple enough, have previously been used in the present context. The repulsive n-body problem, which can also be viewed as a special case of the so-called charged n-body problem with the repulsive Coulomb interaction dominating the Newtonian gravitational attraction, and which is certainly less important and less intriguing than the classical, gravitational n-body problem, appears in various places in the literature. In addition to [6] we mention [1, 2, 3, 4, 7, 8].

2 Results and proofs

Throughout this section we consider a solution to the system (1) with initial data and $n \in \mathbb{N}$ arbitrary but fixed. The kinetic and potential energies of the solution are defined as

$$E_{\text{kin}}(t) := \frac{1}{2} \sum_{1 \le i \le n} |v_i(t)|^2, \qquad E_{\text{pot}}(t) := \frac{1}{2} \sum_{1 \le i \ne j \le n} \frac{1}{|x_i(t) - x_j(t)|},$$

and the total energy is conserved,

$$E(t) := E_{kin}(t) + E_{pot}(t) = E(0).$$

This fact is well known and easily checked. It implies that as long as the solution exists, the velocities of the particles remain bounded and their relative distances remain bounded away from zero,

$$|v_i(t)| \le \sqrt{2E(0)}, \ |x_i(t) - x_j(t)| \ge \frac{1}{\sqrt{2E(0)}}, \ 1 \le i \ne j \le n,$$
 (4)

so in particular the solution exists globally in t, which is well known as well. In addition to the standard energy quantities we define

$$E_{\rm kin}^{\rm rel}(t) := \frac{1}{2} \sum_{1 \le i \le n} \left| v_i(t) - \frac{x_i(t)}{t} \right|^2.$$
 (5)

We refer to this quantity as the *relative kinetic energy*; notice that according to (3) the limit of $x_i(t)/t$ for $t \to \infty$ is the asymptotic velocity of the *i*th particle. We obtain the following identities.

Theorem. (a) For all $t \in \mathbb{R}$,

$$\frac{d}{dt} \left[t^2 E_{\rm kin}^{\rm rel}(t) + t^2 E_{\rm pot}(t) \right] = t E_{\rm pot}(t).$$

(b) For all t > 1,

$$E_{\rm kin}^{\rm rel}(t) + E_{\rm pot}(t) = \frac{C}{t} - \frac{1}{t} \int_{1}^{t} E_{\rm kin}^{\rm rel}(s) \, ds$$

with C > 0 independent of t.

Proof. For the proof we recall the quantity

$$I(t) := \frac{1}{2} \sum_{1 \le i \le n} |x_i(t)|^2,$$

which up to the factor 1/2 is the moment of inertia and has the well known property that

$$\begin{split} \frac{d^2}{dt^2} I(t) &= \frac{d}{dt} \sum_{1 \le i \le n} x_i(t) \cdot v_i(t) \\ &= 2 E_{\text{kin}}(t) + \sum_{1 \le i \ne j \le n} x_i(t) \cdot \frac{x_i(t) - x_j(t)}{|x_i(t) - x_j(t)|^3} \\ &= 2 E_{\text{kin}}(t) + E_{\text{pot}}(t) = 2 E(t) - E_{\text{pot}}(t). \end{split}$$

Using this and conservation of energy the assertion in part (a) follows:

$$\frac{d}{dt} \left[t^2 E_{\text{kin}}^{\text{rel}}(t) + t^2 E_{\text{pot}}(t) \right] = \frac{d}{dt} \left[t^2 E(t) + I(t) - t \frac{d}{dt} I(t) \right]
= 2t E(t) - t (2E(t) - E_{\text{pot}}(t)) = t E_{\text{pot}}(t).$$

If we abbreviate $g(t) = t^2 E_{\text{kin}}^{\text{rel}}(t)$ and $h(t) = g(t) + t^2 E_{\text{pot}}(t)$, we can for $t \neq 0$ rewrite the identity in (a) in the form

$$\frac{d}{dt}h(t) = \frac{1}{t}h(t) - \frac{1}{t}g(t).$$

We read this as a linear, inhomogeneous ODE for the function h which we solve by the variation of constants formula, taking t=1 as initial time; any other initial time $t_0>0$ would do just as well:

$$h(t) = t h(1) - t \int_{1}^{t} \frac{g(s)}{s^{2}} ds.$$

If we recall what g and h stand for and divide by t^2 this proves part (b). \square We exploit this theorem as follows.

Corollary. (a) For all $t \ge 1$, $E_{pot}(t) \le C/t$, and Eqn. (2) holds.

- (b) For i = 1, ..., n, limiting velocities $v_i^* = \lim_{t \to \infty} v_i(t)$ exist, and for all $t \ge 1$, $|v_i(t) v_i^*| \le C/t$. If $i \ne j$, then $v_i^* \ne v_j^*$.
- (c) In addition, there exist points $x_i^* \in \mathbb{R}^3$, i = 1, ..., n, such that Eqn. (3) holds.

Proof. The estimate for $E_{\text{pot}}(t)$ follows directly from part (b) of the theorem. It implies the lower bound in (2), while the upper bound is obvious from the bound for the velocities in (4).

The lower bound on $|x_i(t) - x_j(t)|$ for $i \neq j$ and Eqn. (1) imply that $|\dot{v}_i(t)| \leq C/t^2$. Hence for all $t' \geq t \geq 1$,

$$|v_i(t') - v_i(t)| \le \frac{C}{t}. (6)$$

The bound for the velocities in (4) implies that for any sequence $t_k \to \infty$ the sequence $v_i(t_k)$ has a convergent subsequence, but due to (6) its limit does not depend on the subsequence or the chosen sequence so that $v_i(t)$ has a limit. Taking $t' \to \infty$ in (6) yields the error estimate in part (b). Clearly,

$$x_i(t) = x_i(1) + (t-1)v_i^* + \int_1^t (v_i(s) - v_i^*) ds,$$

and since by the error estimate in (b) the integral is $O(\ln t)$, Eqn. (3) is established as well. If we express $|x_i(t) - x_j(t)|$ using this formula and observe that this difference grows linearly in t if $i \neq j$, we can conclude that $v_i^* \neq v_j^*$ for $i \neq j$, and the proof is complete.

- Concluding Remarks. (a) The linear growth rate in (2) is sharp, since it gives both a lower and an upper bound for $|x_i(t) x_j(t)|$. It is conceivable that the error terms in part (b) of the corollary can be improved, but probably not by the rather innocent estimates used here.
 - (b) Part (b) of the theorem implies that $E_{\rm kin}^{\rm rel}(t)$ decays at least like 1/t, but this rate is not sharp, since by the same identity,

$$\int_{1}^{\infty} E_{\rm kin}^{\rm rel}(s) \, ds < \infty;$$

if not, the right hand side of the identity would become negative for large t, which is not possible. If we substitute the asymptotics from part (b) of the corollary we see that for large t, $E_{\rm kin}^{\rm rel}(t) \leq C \ln^2 t/t^2$, which is consistent with the convergence of the above integral.

(c) It is far from clear whether the asymptotic behavior which was obtained in [5] by completely analogous methods for the Vlasov-Poisson system in the plasma physics case is optimal. The author conjectures that it is not, and the wish to understand this issue lead him to the *n*-body considerations which he reports here.

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