OPTIMAL ESTIMATION VIA NONANTICIPATIVE RATE DISTORTION FUNCTION AND APPLICATIONS TO TIME-VARYING GAUSS-MARKOV PROCESSES*

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Abstract. In this paper, we develop finite-time horizon causal filters using the nonanticipative rate distortion theory. We apply the developed theory to design optimal filters for time-varying multidimensional Gauss-Markov processes, subject to a mean square error fidelity constraint. We show that such filters are equivalent to the design of an optimal {encoder, channel, decoder}, which ensures that the error satisfies a fidelity constraint. Moreover, we derive a universal lower bound on the mean square error of any estimator of time-varying multidimensional Gauss-Markov processes in terms of conditional mutual information. Unlike classical Kalman filters, the filter developed is characterized by a reverse-waterfilling algorithm, which ensures that the fidelity constraint is satisfied. The theoretical results are demonstrated via illustrative examples.

Key words. Causal filters, nonanticipative rate distortion function, mean square error distortion, reverse-water filling, universal lower bound.

1. Introduction. Motivated by real-time control applications, of communication system design, Gorbunov and Pinsker in [2] introduced the so-called nonanticipatory ϵ -entropy of general processes, (see [2, Introduction I]). The nonanticipative ϵ -entropy is equivalent to Shannon's Rate Distortion Function (RDF) [3,4] with an additional causality constraint on the optimal reproduction or estimator.

Along the same lines, for a two-sample Gaussian process, Bucy in [5] derived a causal estimator using the Distortion Rate Function¹ (DRF) subject to a causality constraint. Galdos and Gustafson in [7] applied the classical RDF to design reduced order estimators. Tatikonda, in his Ph.D. thesis [8], introduced the so-called sequential RDF, which is a variant of the nonanticipatory ϵ -entropy and related this to the Optimal Performance Theoretically Attainable (OPTA) by causal codes, as defined by Neuhoff and Gilbert in [9]. Moreover, in [8], the author computed the sequential RDF of a scalar-valued Gaussian process described by discrete recursion driven by an Independent and Identically Distributed (IID) Gaussian noise process, subject to a Mean Square Error (MSE) fidelity constraint. In addition, the author of [8] illustrated by construction, how to communicate the Gaussian process, optimally over a memoryless Additive Gaussian Noise (AGN) channel subject to a power constraint, that is, by designing the {encoder, decoder} so that the AGN channel operates at its capacity and the sequential RDF is achieved. In [10] the authors showed that if the Gaussian process is unstable then sequential RDF is bounded below by the sum of logarithms of the absolute values of the unstable eigenvalues, and that a necessary condition for asymptotic stability of a linear control system over a limited-

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¹The DRF is the dual of the RDF (see [6]).

rate communication channel is "the capacity of the channel, noiseless or noisy, is larger than the sum of logarithms of the absolute values of the unstable eigenvalues of the open loop control system". Similar conditions are derived by many authors via alternative methods [11–13].

In [14] the authors re-visited the relation between information theory and filtering theory, by introducing the so-called Nonanticipative RDF (NRDF), and derived existence of optimal solutions. Moreover, under the assumption that the solution to the NRDF is time-invariant, the form of the optimal reproduction distribution is derived. This expression is applied to derive a sub-optimal causal filter for time-invariant multidimensional partially observed Gaussian processes described by discrete-time recursions. For fully observed Gaussian processes the solution given in [14] is optimal and generalizes the solution given in [10] to multidimensional Gaussian processes with MSE distortion instead of per letter distortion. Recently, Stavrou *et al.* in [15] showed that nonanticipative ϵ -entropy, sequential RDF, and NRDF are equivalent notions. The optimal reproduction distribution which minimizes directed information from one process to another process subject to average distortion constraint is given in [16].

The NRDF has been used in many other communication-related problems. For example, Derpich and Østergaard in [17] applied the nonanticipatory ϵ -entropy of the scalar Gaussian process subject to a MSE fidelity constraint, to derive several bounds on the OPTA by causal and zero-delay codes. Kourtellaris *et al.* in [18] illustrated the simplicity of jointly designing an {encoder, channel, decoder} operating optimally in real-time, for a Binary Symmetric Markov process subject to a Hamming distance distortion function, which is communicated over a finite state channel with unit memory on past channel outputs (with some symmetry) subject to a transmission cost constraint. The NRDF is also applied in control-related problems using zero-delay communication constraints. For example, Tanaka *et al.* in [19] investigated a time-varying multidimensional fully observed Gauss-Markov process with letter-by-letter distortion motivated by the utility of such communication model in real-time communications for control. In addition, in [19] the authors apply semidefinite programming to find, numerically, optimal solutions to the sequential RDF (or NRDF) of time-varying fully observed Gauss-Markov sources.

1.1. Problem Statement. In this paper we investigate the following estimation problem: given an arbitrary random process, we wish to design an optimal communication system so that at the output of this system the estimated process satisfies an end-to-end average fidelity or distortion criterion.

This problem is equivalent to the design of an optimal {encoder, decoder}, which communicates the arbitrary process and reconstructs it at the output of the decoder. Formally, the problem can be cast as follows:

PROBLEM 1. (Information-based estimation) Given

- (a) an arbitrary random process $\{X_t : t = 0, ..., n\}$ taking values in complete separable metric spaces $\{X_t : t = 0, ..., n\}$, with conditional distribution $\{P_{X_t|X^{t-1}}(dx_t|x^{t-1}) : t = 0, ..., n\}, x^{t-1} \triangleq \{x_0, x_1, ..., x_{t-1}\};$
- (b) a distortion function or fidelity of reproducing x_t by $y_t \in \mathcal{Y}_t \subseteq \mathcal{X}_t, t = 0, 1, \ldots, n$, defined by a real-valued measurable function $d_{0,n}(\cdot, \cdot)$

(1.1)
$$d_{0,n}(x^n, y^n) \triangleq \sum_{t=0}^n \rho_t(T^t x^n, T^t y^n) \in [0, \infty],$$

where $T^t x^n \subseteq \{x_0, x_1, \ldots, x_t\}, T^t y^n \subseteq \{y_0, y_1, \ldots, y_t\}$ is either fixed or nonincreasing with time² for $t = 0, 1, \ldots, n$,

we wish to determine an optimal probabilistic {encoder, channel, decoder} which communicates $\{X_t : t = 0, ..., n\}$ and reconstructs it at the output of the decoder or estimator, while it satisfies the end-to-end average fidelity given by

(1.2)
$$\frac{1}{n+1}\mathbb{E}\left\{d_{0,n}(X^n,Y^n)\right\} \le D, \quad \forall D \in [0,\infty).$$

The above definition of estimation problem ensures fidelity (1.2) is satisfied, hence it is fundamentally different from standard approaches of estimation theory, such as, MSE estimation. In general, to achieve such fidelity, for any $D \in [D_{\min}, \infty]$, we know from Shannon's information theory [3], that we need to design the actual observation process or sensor from which the estimator is constructed. This is equivalent the construction of the {encoder, channel, decoder}, as shown in Fig. 1.1. This point

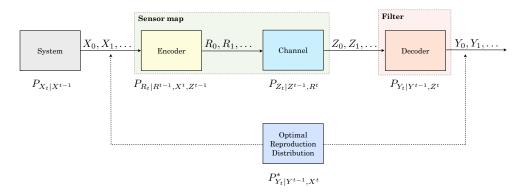


Fig. 1.1: Block diagram of Problem 1 with probabilistic {encoder, channel, decoder}.

of view was recognized by Gorbunov and Pinsker [2], and Bucy [5] several years ago. Our main objective is to address Problem 1 using information-theoretic measures. The natural information-theoretic measure to addresse Problem 1 is the NRDF; this is justified by the equivalence of NRDF and nonanticipatory ϵ -entropy.

In the next section, we describe the contributions and the fundamental differences between information-based estimation via NRDF and Bayesian estimation theory.

1.2. Relation between Bayesian Estimation and Estimation using NRDF. In Bayesian filtering [20, 21], one is given a model that generates the unobserved process $\{X_t : t = 0, ..., n\}$, via its conditional distribution $\{P_{X_t|X^{t-1}}(dx_t|x^{t-1}) : t = 0, ..., n\}$, or via discrete-time recursive dynamics, and a model that generates observed data obtained from sensors $\{Z_t : t = 0, ..., n\}$, via its conditional distribution $\{P_{Z_t|Z^{t-1},X^t}(dz_t|z^{t-1},x^t) : t = 0, ..., n\}$, while an estimate of the unobserved process $\{X_t : t = 0, ..., n\}$, denoted by $\{\hat{X}_t : t = 0, ..., n\}$, is constructed causally, based on the observed data $\{Z_t : t = 0, ..., n\}$. Thus, in Bayesian filtering theory, both models which generate the unobserved and observed processes,

²For example $\rho_t(T^t x^n, T^t y^n) = \rho(x_t, y_t), t = 0, \dots, n$, where $\rho(\cdot, \cdot)$ is a distance metric.

 $\{X_t: t=0,\ldots,n\}$ and $\{Z_t: t=0,\ldots,n\}$, respectively, are given *á priori*, while the estimator $\{\hat{X}_t: t=0,\ldots,n\}$ is a nonanticipative functional of the past information $Z^{t-1}, t=0,\ldots,n$, often computed recursively, like Kalman filter. Fig. 1.2 illustrates the block diagram of the Bayesian filtering problem.

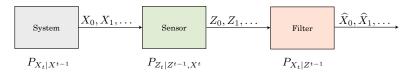


Fig. 1.2: Bayesian Filtering Problem.

On the other hand, in information-based estimation, defined in Problem 1, one is given the process $\{X_t : t = 0, ..., n\}$ and a fidelity criterion, and the objective is to determine the optimal nonanticipative reproduction conditional distribution $\{P_{Y_t|Y^{t-1},X^t}^*(dy_t|y^{t-1},x^t) : t = 0,...,n\}$ corresponding to NRDF, denoted hereinafter by $R_{0,n}^{na}(D)$, and to realize this distribution by an {encoder, channel, decoder} so that the end-to-end distortion (1.2) is met.

As a result, in Problem 1, the observation model is constructed by the cascade of the {encoder, channel} and the filter is the decoder, which satisfies the end-to-end fidelity (1.2).

1.3. Contributions. The main contributions of this paper are the following: (R1) We give a *closed form expression* for the optimal nonanticipative reproduction conditional distribution, $\{P_{Y_t|Y^{t-1},X^t}^*: t = 0, ..., n\}$, which achieves the infimum of the Finite-Time Horizon (FTH) NRDF³. Then, we identify some of its properties, which are necessary for the design of the optimal {encoder, decoder} pair. (R2) We apply our framework to a *time-varying multidimensional fully observed*

Gauss-Markov process $\{X_t : t = 0, ..., n\}$ with MSE distortion, and we show the following:

- (1) The parametric expression of $R_{0,n}^{na}(D)$ is characterized by a time-space reversewaterfilling;
- (2) At each time *n* the value $R_{0,n}^{na}(D)$ is achieved by an optimal {encoder, channel, decoder}, where the channel is a Multiple Input Multiple Output (MIMO) Additive Gaussian Noise (AGN) channel, the encoder operates at the capacity of the AGN channel, and (1.2) holds with equality.
- (3) At each time n, we give the universal lower bound on the MSE of any causal estimator of the Gauss-Markov process.

Contribution (**R1**) generalizes [14, 15], in that we remove the assumption that the optimal reproduction distribution $\{P_{Y_t|Y^{t-1},X^t}^*: t = 0,\ldots,n\}$ is time-invariant, the source is Markov, and distortion is single-letter. This leads to recursive computation of the optimal nonstationary distribution $\{P_{Y_t|Y^{t-1},X^t}^*: t = 0,\ldots,n\}$, backwards in time; i.e., starting at time t = n and going backwards to time t = 0. Contribution (**R2**) demonstrates that for time-varying multidimensional fully-observed Gauss-Markov processes, the parametric expression of the NRDF, $R_{0,n}^{na}(D)$, is characterized by a time-space reverse-waterfilling. To solve the time-space reverse-waterfilling, we propose an iterative algorithm which computes numerically the value of $R_{0,n}^{na}(D)$, and we present examples to illustrate the effectiveness of the algorithm. The Markovian

³In the sequel, when we refer to FTH NRDF we just say NRDF.

property of the optimal reproduction distribution, implies that the optimal distribution is $\{P_{Y_t|Y^{t-1},X_t}^*: t = 0,\ldots,n\}$. This is realized by an {encoder, channel, decoder}, with probability of estimation error decaying exponentially, under certain conditions. The universal lower bound on the MSE of any estimator generalizes the well-known bound of a Gaussian RV given in [22]. The new recursive estimator is finite-dimensional, and ensures the fidelity constraint is met. The time-space reverse-waterfilling implies that given a distortion level, the optimal state estimation is chosen based on an optimal threshold policy, in time and space (dimension). This is the fundamental difference from the well-known Kalman filter equations.

The rest of the paper is structured as follows. In Section 2, we provide the notation used throughout the paper. In Section 3, we introduce NRDF for general processes. In Section 4, we describe the form of the optimal nonstationary (time-varying) reproduction distribution of the NRDF. In Section 5, we concentrate on evaluating the NRDF for time-varying multidimensional Gaussian processes with memory, present examples in the context of realizable filtering theory, and we derive a universal lower bound to the mean square error of any estimator of Gaussian processes based on NRDF. We draw conclusions and discuss future directions in Section 6.

2. Notation. We let $\mathbb{R} = (-\infty, \infty), \mathbb{Z} = \{\dots, -1, 0, 1, \dots\}, \mathbb{N} = \{1, 2, \dots\}, \mathbb$ $\mathbb{N}_0 = \{0, 1, \ldots\}, \mathbb{N}_0^n \triangleq \{0, 1, \ldots, n\}. \mathbb{E}\{\cdot\}$ represents the expectation of its argument. $\sigma\{\cdot\}$ represents the σ -algebra of events generated by its argument. For a non-square matrix $A \in \mathbb{R}^{n \times m}$, we denote its transpose by A^{T} . For a square matrix $A \in \mathbb{R}^{n \times n}$, we denote by diag $\{A\}$ the matrix having A_{ii} , $i = 1, \ldots, n$, on its diagonal and zero elsewhere. We denote the source alphabet spaces by the measurable space $\{(\mathcal{X}_n, \mathcal{B}(\mathcal{X}_n)) : n \in \mathbb{Z}\},$ where $\mathcal{X}_n, n \in \mathbb{Z}$ are complete separable metric spaces or Polish spaces, and $\mathcal{B}(\mathcal{X}_n)$ are Borel σ -algebras of subsets of \mathcal{X}_n . We denote points in $\mathcal{X}^{\mathbb{Z}} \triangleq \times_{n \in \mathbb{Z}} \mathcal{X}_n$ by $x_{-\infty}^{\infty} \triangleq \{\dots, x_{-1}, x_0, x_1, \dots\} \in \mathcal{X}^{\mathbb{Z}}$, and their restrictions to finite coordinates for any $(m, n) \in \mathbb{N}_0$ by $x_m^n \triangleq \{x_m, \dots, x_0, x_1, \dots, x_n\} \in \mathcal{X}_m^n$, $n \ge m$. We denote by $\mathcal{B}(\mathcal{X}^{\mathbb{Z}}) \triangleq \otimes_{t \in \mathbb{Z}} \mathcal{B}(\mathcal{X}_t)$ the σ -algebra on $\mathcal{X}^{\mathbb{Z}}$ generated by cylinder sets $\{\mathbf{x} = (\dots, x_{-1}, x_0, x_1, \dots) \in \mathcal{X}^{\mathbb{Z}} : x_j \in A_j, \ j \in \mathbb{Z}\}, A_j \in \mathcal{B}(\mathcal{X}_j), j \in \mathbb{Z}. \text{ Thus, } \mathcal{B}(\mathcal{X}_m^n) \text{ denote the } \sigma\text{-algebras of cylinder sets in } \mathcal{X}_m^n, \text{ with bases over } A_j \in \mathcal{B}(\mathcal{X}_j), j \in \{m, m+1\}\}$ $1, \ldots, n$, $m \le n$, $(m, n) \in \mathbb{Z}$. For a Random Variable (RV) $X : (\Omega, \mathcal{F}) \longmapsto (\mathcal{X}, \mathcal{B}(\mathcal{X}))$ we denote the distribution induced by X on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ by $\mathbf{P}_X(dx) \equiv \mathbf{P}(dx)$. We denote the set of such probability distributions by $\mathcal{M}(\mathcal{X})$. We denote the conditional distribution of RV Y given X = x (i.e., fixed) by $\mathbf{P}_{Y|X}(dy|X = x) \equiv \mathbf{P}_{Y|X}(dy|x)$. Such conditional distributions are equivalently described by stochastic kernels or transition functions [23] $\mathbf{K}(\cdot|\cdot)$ on $\mathcal{B}(\mathcal{Y}) \times \mathcal{X}$, mapping \mathcal{X} into $\mathcal{M}(\mathcal{Y})$ (space of distributions), i.e., $x \in \mathcal{X} \longmapsto \mathbf{K}(\cdot|x) \in \mathcal{M}(\mathcal{Y})$, and such that for every $A \in \mathcal{B}(\mathcal{Y})$, the function $\mathbf{K}(A|\cdot)$ is $\mathcal{B}(\mathcal{X})$ -measurable. We denote the set of such stochastic kernels by $\mathcal{Q}(\mathcal{Y}|\mathcal{X})$.

3. NRDF on General Alphabets. In this section, we introduce the definition of NRDF for general processes taking values in Polish spaces (complete separable metric spaces), that include finite, countable, and continuous alphabet spaces. Source Distribution. The process $\{X_0, X_1, \ldots\}$ is described by the collection of conditional probability distributions $\{\mathbf{P}_{X_n|X^{n-1}}(\cdot|x^{n-1}) : x^{n-1} \in \mathcal{X}^{n-1}, n \in \mathbb{N}_0\}$. For each $n \in \mathbb{N}_0$, we let $\mathbf{P}_{X_n|X^{n-1}}(\cdot|\cdot) \equiv P_n(\cdot|\cdot) \in \mathcal{Q}_n(\mathcal{X}_n|\mathcal{X}^{n-1})$, and for n = 0, we

(3.1)
$$P_{0,n}(A_{0,n}) \triangleq \int_{A_0} P_0(dx_0) \dots \int_{A_n} P_n(dx_n | x^{n-1}), \quad A_t \in \mathcal{B}(\mathcal{X}_t), \ A_{0,n} = \times_{t=0}^n A_t.$$

set $\mathbf{P}_{X_0|X^{-1}} = P_0(dx_0)$. We define the probability distribution on \mathcal{X}^n by

Thus, for each $n \in \mathbb{N}_0$, $P_{0,n}(\cdot) \in \mathcal{M}(\mathcal{X}^n)$.

Reproduction Distribution. The reproduction process $\{\ldots, Y_2, Y_1, Y_0, Y_1, \ldots\} \equiv \{Y^{-1}, Y_0, Y_1, \ldots,\}$ is described by the collection of conditional distributions $\{\mathbf{P}_{Y_n|Y^{n-1},X^n}(\cdot|y^{n-1},x^n): (y^{n-1},x^n) \in \mathcal{Y}^{n-1} \times \mathcal{X}^n, n \in \mathbb{N}_0\}$, i.e., $y^n \equiv (y^{-1}, y_0^n), x^n \equiv x_0^n$. For each $n \in \mathbb{N}_0$, we let $\mathbf{P}_{Y_n|Y^{n-1},X^n}(\cdot|\cdot,\cdot) \equiv Q_n(\cdot|\cdot,\cdot) \in \mathcal{Q}_n(\mathcal{Y}_n|\mathcal{Y}^{n-1} \times \mathcal{X}^n)$, and for n = 0, $\mathbf{P}_{Y_0|Y^{-1},X_0} = Q_0(dy_0|y^{-1},x_0)$. The RV Y^{-1} is the initial data with fixed distribution $\mathbf{P}_{Y^{-1}}(dy^{-1}) = \mu(dy^{-1})$. We define the family of conditional probability distributions on \mathcal{Y}_0^n parametrized by $(y^{-1},x^n) \in \mathcal{Y}^{-1} \times \mathcal{X}^n$ by

$$\vec{Q}_{0,n}(B_{0,n}|y^{-1},x^n) \triangleq \int_{B_0} Q_0(dy_0|y^{-1},x_0)\dots$$
(3.2)
$$\int_{B_n} Q_n(dy_n|y^{n-1},x^n), \quad B_t \in \mathcal{B}(\mathcal{Y}_t), \ B_{0,n} = \times_{t=0}^n B_t.$$

Thus, for each $n \in \mathbb{N}_0$, $\overrightarrow{Q}_{0,n}(\cdot|y^{-1}, x^n) \in \mathcal{M}(\mathcal{Y}_0^n)$, $(y^{-1}, x^n) \in \mathcal{Y}^{-1} \times \mathcal{X}^n$. Given a $P_{0,n}(\cdot) \in \mathcal{M}(\mathcal{X}^n)$ a $\overrightarrow{Q}_{0,n}(\cdot|y^{-1}, x^n) \in \mathcal{M}(\mathcal{Y}_0^n)$, and a fixed distribution $\mu(dy^{-1})$, we define the following distributions. The joint distribution on $\mathcal{X}^n \times \mathcal{Y}_0^n$ given $Y^{-1} = y^{-1}$ is defined by

(3.3)

$$\mathbf{P}^{\overrightarrow{Q}}(A_{0,n} \times B_{0,n} | y^{-1}) \triangleq (P_{0,n} \otimes \overrightarrow{Q}_{0,n}) \left(\times_{t=0}^{n} (A_{t} \times B_{t}) | y^{-1} \right) \\
= \int_{A_{0}} P_{0}(dx_{0}) \int_{B_{0}} Q_{0}(dy_{0} | y^{-1}, x_{0}) \dots \\
\int_{A_{n}} P_{n}(dx_{n} | x^{n-1}) \int_{B_{n}} Q_{n}(dy_{n} | y^{n-1}, x^{n}).$$

The marginal distribution on \mathcal{Y}_0^n given $Y^{-1} = y^{-1}$ is defined by

$$\Pi_{0,n}^{\overrightarrow{Q}}(B_{0,n}|y^{-1}) \triangleq \int_{B_{0,n}} \int_{\mathcal{X}^n} (P_{0,n} \otimes \overrightarrow{Q}_{0,n}) (dx^n, dy_0^n | y^{-1}) \\ = \int_{B_{0,n}} \Pi_0^{\overrightarrow{Q}} (dy_0 | y^{-1}) \dots \Pi_n^{\overrightarrow{Q}} (dy_n | y^{n-1}).$$

The product probability distribution $\overrightarrow{\Pi}_{0,n}^{\overrightarrow{Q}}(\cdot|y^{-1}) : \mathcal{B}(\mathcal{X}^n) \otimes \mathcal{B}(\mathcal{Y}_0^n) \longmapsto [0,1]$ conditioned on $Y^{-1} = y^{-1}$, is defined by

$$\vec{\Pi}_{0,n}^{\vec{Q}} \left(A_{0,n} \times B_{0,n} | y^{-1} \right) \triangleq \left(P_{0,n} \times \Pi_{0,n}^{\vec{Q}} \right) \left(\times_{t=0}^{n} (A_t \times B_t) | y^{-1} \right) \\ = \int_{A_0} P_0(dx_0) \int_{B_0} \Pi_0^{\vec{Q}}(dy_0 | y^{-1}) \dots \int_{A_n} P_n(dx_n | x^{n-1}) \int_{B_n} \Pi_n^{\vec{Q}}(dy_n | y^{n-1}).$$

We define the relative entropy between the joint distribution $\mathbf{P}^{\overrightarrow{Q}}(dx^n, dy_0^n | y^{-1})$ and the product distribution $\overrightarrow{\Pi}_{0,n}^{\overrightarrow{Q}}(dx^n, dy_0^n | y^{-1})$, averaged over the initial distribution $\mu(dy^{-1})$, as follows:

$$\mathbb{D}(P_{0,n} \otimes \overrightarrow{Q}_{0,n} || \overrightarrow{\Pi}_{0,n}^{\overrightarrow{Q}}) = \int_{\mathcal{X}^n \times \mathcal{Y}^n} \log \left(\frac{P_{0,n}(\cdot) \otimes \overrightarrow{Q}_{0,n}(\cdot | y^{-1}, x^n)}{P_{0,n}(\cdot) \otimes \Pi_{0,n}^{\overrightarrow{Q}}(\cdot | y^{-1})} (x^n, y_0^n) \right)$$

$$(3.4) \qquad \qquad P_{0,n}(dx^n) \otimes \overrightarrow{Q}_{0,n}(dy_0^n | y^{-1}, x^n) \otimes \mu(dy^{-1})$$

$$\stackrel{(a)}{=} \int_{\mathcal{X}^n \times \mathcal{Y}^n} \log \left(\frac{\overrightarrow{Q}_{0,n}(\cdot | y^{-1}, x^n)}{\Pi_{0,n}^{\overrightarrow{Q}}(\cdot | y^{-1})} (y_0^n) \right)$$

$$(3.5) \qquad \qquad P_{0,n}(dx^n) \otimes \overrightarrow{Q}_{0,n}(dy_0^n | y^{-1}, x^n) \otimes \mu(dy^{-1})$$

$$(3.6) \qquad \equiv \mathbb{I}_{0,n}(P_{0,n}, \overrightarrow{Q}_{0,n})$$

where (a) is due to the chain rule of relative entropy (see [24]). In (3.6) the notation $\mathbb{I}_{0,n}(\cdot,\cdot)$ indicates the functional dependence on $\{P_{0,n}, \overrightarrow{Q}_{0,n}\}$ (the dependence on $\mu(dy^{-1})$ is omitted). By [24, Theorem 5], the set of distributions $\overrightarrow{Q}_{0,n}(\cdot|y^{-1},x^n) \in \mathcal{M}(\mathcal{Y}_0^n)$ is convex, and by [24, Theorem 6], $\mathbb{I}_{0,n}(P_{0,n},\cdot)$ is a convex functional of $\overrightarrow{Q}_{0,n}(\cdot|y^{-1},x^n) \in \mathcal{M}(\mathcal{Y}_0^n)$.

Given the distortion function of reproducing x_t by $y_t, t = 0, 1, ..., n$, defined by (1.1), the fidelity constraint set is defined as follows.

$$\vec{\mathcal{Q}}_{0,n}(D) \triangleq \left\{ \vec{Q}_{0,n}(\cdot|y^{-1},x^n) \in \mathcal{M}(\mathcal{Y}_0^n) : \frac{1}{n+1} \mathbf{E}_{\mu}^{\vec{\mathcal{Q}}} \left\{ d_{0,n}(X^n,Y^n) \right\} \le D \right\}, \ D \ge 0$$

where $\mathbf{E}_{\mu}^{\overrightarrow{Q}}\{\cdot\}$ indicates that the joint distribution is induced by $\{P_{0,n}(dx^n), \overrightarrow{Q}_{0,n}(dy_0^n|y^{-1}, x^n), \mu(dy^{-1})\}$ defined by (3.3). Clearly, $\overrightarrow{\mathcal{Q}}_{0,n}(D)$ is a convex set.

DEFINITION 3.1. (NRDF)

The NRDF is defined by

(3.7)
$$R_{0,n}^{na}(D) \triangleq \inf_{\vec{Q}_{0,n}(dy_0^n | y^{-1}, x^n) \in \vec{\mathcal{Q}}_{0,n}(D)} \mathbb{I}_{0,n}(P_{0,n}, \vec{Q}_{0,n}).$$

By the above discussion the NRDF is a convex optimization problem. Sufficient conditions for existence of an optimal solution to the convex optimization problem (3.7) are given in [24, Theorem III.13].

For completeness, in the next remark we give the connection of the NRDF to the classical Shannon RDF [4] and nonanticipatory ϵ -entropy [2].

REMARK 1. (RDF and nonanticipatory ϵ -entropy)

Consider the distribution $P_{0,n}(\cdot) \in \mathcal{M}(\mathcal{X}^n)$ and the conditional distribution $Q_{0,n}^{nc}(dy_0^n | y^{-1}, x^n) \in \mathcal{M}(\mathcal{Y}_0^n), (y^{-1}, x^n) \in \mathcal{Y}^{-1} \times \mathcal{X}^n$, which is a non-causal distribution, because by Bayes' rule $Q_{0,n}^{nc}(dy_0^n | y^{-1}, x^n) = \bigotimes_{t=0}^n Q_t^{nc}(dy_t | y^{t-1}, x^n)$. The conditional distribution on \mathcal{Y}_0^n given $Y^{-1} = y^{-1}$, and the joint distribution on $\mathcal{X}^n \times \mathcal{Y}_0^n$ are induced as follows.

(3.8)
$$\Pi_{0,n}^{Q^{\mathrm{nc}}}(dy_0^n|y^{-1}) = \int_{\mathcal{X}^n} Q_{0,n}^{\mathrm{nc}}(dy_0^n|y^{-1},x^n) \otimes P_{0,n}(dx^n),$$

(3.9)
$$\mathbf{P}^{Q^{\mathrm{nc}}}(dx^n, dy_0^n | y^{-1}) = P_{0,n}(dx^n) \otimes Q_{0,n}^{\mathrm{nc}}(dy_0^n | y^{-1}, x^n).$$

Define the fidelity constraint

(3.10)
$$\mathcal{Q}_{0,n}(D) \triangleq \left\{ Q_{0,n}^{\mathrm{nc}}(dy_0^n | y^{-1}, x^n) \in \mathcal{M}(\mathcal{Y}_0^n) : \frac{1}{n+1} \mathbf{E}_{\mu}^{Q^{\mathrm{nc}}} \left\{ d_{0,n}(X^n, Y^n) \right\} \le D \right\}, \ D \ge 0.$$

The classical RDF [4] is defined by

(3.11)
$$R_{0,n}(D) \triangleq \inf_{\substack{Q_{0,n}^{nc}(dy_0^n | y^{-1}, x^n) \in \mathcal{Q}_{0,n}(D)}} I(X^n; Y_0^n | Y^{-1}),$$

where $I(X^n; Y_0^n | Y^{-1})$ is the conditional mutual information given by

$$I(X^{n}; Y_{0}^{n} | Y^{-1}) = \int_{\mathcal{X}^{n} \times \mathcal{Y}^{n}} \log \left(\frac{Q_{0,n}^{\mathrm{nc}}(\cdot | y^{-1}, x^{n})}{\Pi_{0,n}^{Q^{\mathrm{nc}}}(\cdot | y^{-1})}(y_{0}^{n}) \right)$$

(3.12)
$$P_{0,n}(dx^n) \otimes Q_{0,n}^{\mathrm{nc}}(dy_0^n | y^{-1}, x^n) \otimes \mu(dy^{-1})$$

(3.13)
$$\equiv \mathbb{I}_{0,n}(P_{0,n}, Q_{0,n}^{\mathrm{nc}}).$$

Unfortunately, classical RDF does not give causal estimators, because the optimal reproduction distribution in (3.11) is $\{Q_t^{nc}(dy_t|y^{t-1},x^n):t\in\mathbb{N}_0^n\}$; hence, in general, it is non-causal with respect to $\{X_0, \ldots, X_n\}$. This let Gorbunov and Pinsker in [2] to define the notion of nonanticipatory ϵ -entropy, as follows

(3.14)
$$R_{0,n}^{\varepsilon}(D) \triangleq \inf_{\substack{\mathcal{Q}_{0,n}(D): Q_{0,t}^{\mathrm{nc}}(dy_{0}^{t}|y^{-1},x^{n}) = Q_{0,t}^{GP}(dy_{0}^{t}|y^{-1},x^{t})}}_{t=0,\dots,n} I(X^{n}; Y_{0}^{n}|Y^{-1}).$$

We note that conditional independence $Q_{0,t}^{nc}(dy_0^t|y^{-1},x^n) = Q_{0,t}^{GP}(dy_0^t|y^{-1},x^t), t =$ $0, \ldots, n$ is a causality restriction of the reproduction distribution in (3.11).

The equivalence of the nonanticipatory ϵ -entropy, $R_{0,n}^{\varepsilon}(D)$, and NRDF, $R_{0,n}^{na}(D)$, is a direct consequence of the following equivalent characterization of conditional independence statements shown in [15].

MC3: $P_t(dx_{t+1}|x^t, y^t) = P_t(dx_{t+1}|x^t)$, for each t = 0, 1, ..., n-1, $\forall n \in \mathbb{N}_0$; **MC4:** $Q_{0,t}^{nc}(dy_0^t|y^{-1}, x^t, x_{t+1}^n) = \overrightarrow{Q}_{0,t}(dy_0^t|y^{-1}, x^t)$, for each t = 0, 1, ..., n-1, $\forall n \in \mathbb{N}_0$;

In view of the above statements, the NRDF defined by (3.7) is equivalent to the nonanticipatory ϵ -entropy defined by (3.14), that is, $R_{0,n}^{na}(D) = R_{0,n}^{\varepsilon}(D)$.

4. Optimal Nonstationary Reproduction Distribution. In this section, we describe the form of the optimal nonstationary (time-varying) reproduction distribution that achieves the infimum in (3.7).

First, we state the following properties regarding the convexity and continuity of the NRDF, $R_{0,n}^{na}(D)$, that are necessary for the development of our results.

1) $R_{0,n}^{na}(D)$ is a convex, non-increasing function of $D \in [0,\infty)$.

2) If $\hat{R}_{0,n}^{na}(D) < \infty$, then $R_{0,n}^{na}(\cdot)$ is continuous on $D \in [0,\infty)$. Note that 1) is similar to the one derived in [15, Lemma IV.4]. Also, for 2) recall that a bounded and convex function is continuous. Since $R_{0,n}^{na}(D)$ is non-increasing, it is bounded outside the neighbourhood of D = 0 and it is also continuous on $(0, \infty)$. In other words, if $R_{0,n}^{na}(D) < \infty$ then $R_{0,n}^{na}(D)$ is bounded and hence continuous on $[0, \infty)$. Moreover, since $R_{0,n}^{na}(D)$ is convex and non-increasing then its inverse function, $D_{0,n}(R^{na})$, exists and it is convex, non-increasing function of $R^{na} \in [0, \infty)$. $D_{0,n}(R^{na})$ is called FTH Nonanticipative Distortion Rate Function (NDRF) and is given by

(4.1)
$$D_{0,n}(R^{na}) = \inf_{\frac{1}{n+1} \mathbb{I}_{0,n}(P_{0,n}, \vec{Q}_{0,n}) \le R^{na}} \mathbf{E}_{\mu}^{\vec{Q}} \left\{ d_{0,n}(X^{n}, Y^{n}) \right\}$$

The NRDF defined by (3.7) is a convex optimization problem, and thus, if there exists an interior point in the set $\overrightarrow{Q}_{0,n}(D)$, it can be reformulated using Lagrange duality theorem [25, Theorem 1, pp. 224-225] as an unconstrained problem as follows.

$$R_{0,n}^{na}(D) = \sup_{s \le 0} \inf_{\vec{Q}_{0,n}(\cdot|y^{-1},x^{n}) \in \mathcal{M}(\mathcal{Y}_{0}^{n})} \left\{ \mathbb{I}_{0,n}(P_{0,n},\vec{Q}_{0,n}) - s \frac{1}{n+1} \mathbf{E}_{\mu}^{\vec{Q}} \left\{ d_{0,n}(X^{n},Y^{n}) \right\} \right\}.$$

Next, we state Theorem 4.1, which is used in the subsequent analysis to compute the NRDF, $R_{0,n}^{na}(D)$, of time-varying multidimensional Gauss-Markov processes.

THEOREM 4.1. (Optimal nonstationary reproduction distributions) Suppose there exists a $\overrightarrow{Q}_{0,n}^*(\cdot|y^{-1},x^n) \in \overrightarrow{Q}_{0,n}(D)$, which solves (3.7), and that $\mathbb{I}_{0,n}(P_{0,n}, \overrightarrow{Q}_{0,n})$ is Gâteaux differentiable in every direction of $\{Q_t(\cdot|y^{t-1},x^t) : t \in \mathbb{N}_0^n\}$ for a fixed $P_{0,n}(\cdot) \in \mathcal{M}(\mathcal{X}^n)$ and $\mu(dy^{-1}) \in \mathcal{M}(\mathcal{Y}^{-1})$. Then, the following hold: (1) The optimal reproduction distributions denoted by $\{Q_t^*(\cdot|y^{t-1},x^t) \in \mathcal{M}(\mathcal{Y}_t) : t \in \mathbb{N}_0^n\}$ are given by the following recursive equations backwards in time. For t = n:

(4.3)
$$Q_n^*(dy_n|y^{n-1},x^n) = \frac{e^{s\rho_n(T^nx^n,T^ny^n)}\Pi_n^{\overrightarrow{Q}^*}(dy_n|y^{n-1})}{\int_{\mathcal{Y}_n} e^{s\rho_n(T^nx^n,T^ny^n)}\Pi_n^{\overrightarrow{Q}^*}(dy_n|y^{n-1})}$$

For $t = n - 1, n - 2, \dots, 0$:

(4.4)
$$Q_t^*(dy_t|y^{t-1}, x^t) = \frac{e^{s\rho_t(T^tx^n, T^ty^n) - g_{t,n}(x^t, y^t)} \prod_t^{\overline{Q}^*}(dy_t|y^{t-1})}{\int_{\mathcal{Y}_t} e^{s\rho_t(T^tx^n, T^ty^n) - g_{t,n}(x^t, y^t)} \prod_t^{\overline{Q}^*}(dy_t|y^{t-1})}$$

where s < 0, $\Pi_t^{\overrightarrow{Q}^*}(\cdot|y^{t-1}) \in \mathcal{M}(\mathcal{Y}_t)$ and $g_{t,n}(x^t, y^t)$ is given by (4.5)

$$g_{t,n}(x^{t}, y^{t}) = -\int_{\mathcal{X}_{t+1}} P_{t+1}(dx_{t+1}|x^{t}) \log\left(\int_{\mathcal{Y}_{t+1}} e^{s\rho_{t+1}(T^{t+1}x^{n}, T^{t+1}y^{n}) - g_{t+1,n}(x^{t+1}, y^{t+1})} \prod_{t+1}^{\overrightarrow{Q}^{*}}(dy_{t+1}|y^{t})\right),$$

$$g_{n,n}(x^{n}, y^{n}) = 0.$$

(2) The NRDF is given by

(3) If $R_{0,n}^{na}(D) > 0$ then s < 0, and

$$(4.7) \quad \frac{1}{n+1} \sum_{t=0}^{n} \int_{\mathcal{X}^{t} \times \mathcal{Y}^{t}} \rho_{t}(T^{t}x^{n}, T^{t}y^{n})(P_{0,t} \otimes \overrightarrow{Q}_{0,t}^{*})(dx^{t}, dy_{0}^{t}|y^{-1}) \otimes \mu(dy^{-1}) = D.$$

Proof. The sequence of minimizations over $\{Q_t(\cdot|y^{t-1}, x^t) : t \in \mathbb{N}_0^n\}$ in (4.2) is a nested optimization problem. Hence, we can introduce the dynamic programming recursive equations. Then, we carry out the infimum starting at the last stage over $Q_n(\cdot|y^{n-1}, x^n) \in \mathcal{M}(\mathcal{Y}_n)$ and sequentially move backwards in time to determine $Q_n^*(\cdot|y^{n-1}, x^n), Q_{n-1}^*(\cdot|y^{n-2}, x^{n-1}), \ldots, Q_0^*(\cdot|y^{-1}, x_0)$. The procedure is straightforward and we omit it due to space limitations. \Box

We note that Theorem 4.1 is fundamentally different from [14, Theorem IV.4]. In the latter, it is assumed that all elements $\{Q_t(dy_t|y^{t-1}, x^t) : t \in \mathbb{N}_0^n\}$ are identical.

From the above theorem, for a given distribution $P_{0,n}(\cdot) \in \mathcal{M}(\mathcal{X}^n)$, we can identify the dependence of the optimal nonstationary reproduction distribution on past and present symbols of the information process $\{X_t : t \in \mathbb{N}_0^n\}$, but not its dependence on past reproduction symbols. In what follows, we give certain properties of the information structure of the optimal nonstationary reproduction distribution that achieves the infimum in (3.7).

Information structure of the optimal nonstationary reproduction distribution.

(1) The dependence of $Q_n^*(dy_n|y^{n-1}, x^n)$ on $x^n \in \mathcal{X}^n$ is determined by the dependence of $\rho_n(T^nx^n, T^ny^n)$ on $x^n \in \mathcal{X}^n$ as follows:

(1.1) If $\rho_t(T^tx^n, T^iy^n) = \bar{\rho}(x_t, y^t), t = 0, \dots, n$, then $Q_n^*(dy_n|y^{n-1}, x^n) = Q_n^*(dy_n|y^{n-1}, x_n)$, while for $t = n - 1, n - 2, \dots, 0$, the dependence of $Q_t^*(dy_t|y^{t-1}, x^t)$ on $x^t \in \mathcal{X}^t$ is determined from the dependence of $g_{t,n}(x^t, y^t)$ on $x^t \in \mathcal{X}^t$.

determined from the dependence of $g_{t,n}(x^t, y^t)$ on $x^t \in \mathcal{X}^t$. (1.2) If $P_t(dx_t|x^{t-1}) = P_t(dx_t|x^{t-1}_{t-1-L})$, where L is a non-negative finite integer, and $\rho_t(T^tx^n, T^ty^n) = \bar{\rho}(x^t_{t-N}, y_t)$, where N is a non-negative finite integer, then $Q_t^*(dy_t|y^{t-1}, x^t) = Q_t^*(dy_t|y^{t-1}, x^t_{t-J})$, where $J = \max\{N, L\}$.

(2) If $g_{t,n}(x^t, y^t) = \hat{g}_{t,n}(x^t, y^{t-1}), \ \forall t \in \mathbb{N}_0^{n-1}$ then the optimal reproduction distribution (4.4) reduces to

$$Q_t^*(dy_t|y^{t-1}, x^t) = \frac{e^{s\rho_t(T^tx^n, T^ty^n)} \prod_t^{\vec{Q}^*}(dy_t|y^{t-1})}{\int_{\mathcal{Y}_t} e^{s\rho_t(T^tx^n, T^ty^n)} \prod_t^{\vec{Q}^*}(dy_t|y^{t-1})}$$

To further understand the dependence of the optimal nonstationary reproduction distributions (4.3), (4.4) on past reproductions, we state an alternative characterization of the nonstationary solution of $R_{0,n}^{na}(D)$, as a maximization over a certain class of functions. We use this additional characterization to derive lower bounds on $R_{0,n}^{na}(D)$, which are achievable.

THEOREM 4.2. (Characterization of solution of NRDF) An alternative characterization of NRDF is

$$R_{0,n}^{na}(D) = \sup_{s \le 0} \sup_{\{\lambda_t \in \Psi_s^t: t \in \mathbb{N}_0^n\}} \left\{ sD - \frac{1}{n+1} \sum_{t=0}^n \int_{\mathcal{X}^t \times \mathcal{Y}^{t-1}} \int_{\mathcal{Y}_t} g_{t,n}(x^t, y^t) Q_t^*(dy_t | y^{t-1}, x^t) + \log\left(\lambda_t(x^t, y^{t-1})\right) P_t(dx_t | x^{t-1}) \otimes (P_{0,t-1} \otimes \overrightarrow{Q}_{0,t-1}^*) (dx^{t-1}, dy_0^{t-1} | y^{-1}) \otimes \mu(dy^{-1}) \right\}$$

where

(4.9)

$$\Psi_{s}^{t} \triangleq \left\{ \lambda_{t}(x^{t}, y^{t-1}) \geq 0: \int_{\mathcal{X}^{t-1}} \left(\int_{\mathcal{X}_{t}} e^{s\rho_{t}(T^{t}x^{n}, T^{t}y^{n}) - g_{t,n}(x^{t}, y^{t})} \lambda_{t}(x^{t}, y^{t-1}) P_{t}(dx_{t} | x^{t-1}) \right) \\ \otimes \mathbf{P}^{\overrightarrow{Q}^{*}}(dx^{t-1} | y^{t-1}) \leq 1 \right\}$$

and $g_{n,n}(x^n, y^n) = 0$, and for $t \in \mathbb{N}_0^{n-1}$,

$$g_{t,n}(x^t, y^t) = -\int_{\mathcal{X}_{t+1}} P_{t+1}(dx_{t+1}|x^t) \log\left(\lambda_{t+1}(x^{t+1}, y^t)\right)^{-1}$$

For $s \in (-\infty, 0]$ a necessary and sufficient condition for $\{\lambda_t(\cdot, \cdot) : t = 0, ..., n\}$ to achieve the supremum of (4.8) is the existence of a probability distribution $\Pi_t^{\overrightarrow{Q}^*}(\cdot|y^{t-1}) \in \mathcal{M}(\mathcal{Y}_t)$ such that

$$\lambda_t(x^t, y^{t-1}) = \left(\int_{\mathcal{Y}_t} e^{s\rho_t(T^t x^n, T^t y^n) - g_{t,n}(x^t, y^t)} \Pi_t^{\overrightarrow{Q}^*}(dy_t | y^{t-1}) \right)^{-1}, \ t \in \mathbb{N}_0^n.$$

Proof. See Appendix A. \Box

Theorem 4.2 is crucial in the computation of $R_{0,n}^{na}(D)$ for any given source (with memory), simply because apart from Gaussian or memoryless sources, to solve a rate distortion problem explicitly, one needs to identify the dependence of the optimal reproduction distribution on past reproduction symbols, Y^{t-1} , and in general to find the information structure of the optimal reproduction distribution. In the next section, we use the previous theorems to derive $R_{0,n}^{na}(D)$ for the Gaussian source.

5. NRDF of Time-Varying Multidimensional Gauss-Markov Processes. In this section, we apply Theorem 4.1 and Theorem 4.2 from Section 4 to time-varying multidimensional Gauss-Markov processes in state-space form, and we obtain the following results:

(1) the analytical expression of the optimal nonstationary reproduction distribution that achieves the infimum of the NRDF and the analytical expression of the NRDF subject to a square error distortion;

(2) a realization of the optimal nonstationary reproduction distribution in the sense of Fig. 5.1 that allows us to obtain the optimal filter;

(3) a universal lower bound on the MSE of any causal estimator of Gaussian processes.

The analytical expression of the NRDF is found by developing a time-space algorithm, which is a generalization of the standard reverse-waterfilling algorithm derived in [6, Section 10.3.3] for independent Gaussian RV. Toward this, illustrative examples that verify our theory are presented.

The time-varying multidimensional Gauss-Markov processes defined as follows.

DEFINITION 5.1. (Time-varying multidimensional Gauss-Markov process) The source process is modeled as a time-varying p-dimensional Gauss-Markov process defined by

(5.1)
$$X_{t+1} = A_t X_t + B_t W_t, \ X_0 = x_0, \ t \in \mathbb{N}_0^{n-1},$$

where $A_t \in \mathbb{R}^{p \times p}, B_t \in \mathbb{R}^{p \times k}, t \in \mathbb{N}_0^{n-1}$. We assume **(G1)** $X_0 \in \mathbb{R}^p$ is Gaussian $N(0; \Sigma_{X_0})$;

(G2) $\{W_t : t \in \mathbb{N}_0^n\}$ is a k-dimensional IID Gaussian $N(0; I_k)$ sequence, independent of X_0 ;

(G3) The distortion function is defined by $d_{0,n}(x^n, y^n) \triangleq \sum_{t=0}^n \rho_t(T^t x^n, T^t y^n) = \sum_{t=0}^n ||x_t - y_t||_2^2.$

Information Structure. By Theorem 4.1 and the Markovian property of (5.1), the optimal nonstationary reproduction distribution given by (4.3)-(4.4) is Markov with respect to $\{X_0, \ldots, X_n\}$, that is, $\{Q_t^*(dy_t|y^{t-1}, x^t) \equiv Q_t^*(dy_t|y^{t-1}, x_t) : t \in \mathbb{N}_0^n\}$ (see the comments below Theorem 4.1 on information structures of the optimal reproduction distribution). Since $\{X_t : t \in \mathbb{N}_0^n\}$ is Markov and the distortion function is squared error, then by [10] the optimal reproduction process $\{Y_t^* : t \in \mathbb{N}_0^n\}$ is Gaussian, and the joint process $\{(X_t, Y_t) : t \in \mathbb{N}_0^n\}$ is also Gaussian. In what follows, we also show the Gaussianity of the structure of the optimal reproduction distribution $\{Q_t^*(dy_t|y^{t-1}, x_t) : t \in \mathbb{N}_0^n\}$.

Starting from stage n and going backwards, we can show that $\{Q_t^* (dy_t | y^{t-1}, x_t) : t \in \mathbb{N}_0^n\}$ are conditional Gaussian distributions.

Stage n. Since the exponential term $||y_n - x_n||_2^2$ in the Right-Hand Side (RHS) of (4.3) is quadratic in (x_n, y_n) , and $\{X_t : t \in \mathbb{N}_0^n\}$ is Gaussian, then it follows that a Gaussian distribution $Q_n(\cdot|y^{n-1}, x_n)$, for a fixed realization of (y^{n-1}, x_n) , and a Gaussian distribution $\Pi_n^{\overrightarrow{Q}}(\cdot|y^{n-1})$ satisfy both the left and right sides of (4.3). This implies that $Q_n^*(\cdot|y^{n-1}, x_n)$ and $\Pi_n^{\overrightarrow{Q}^*}(\cdot|y^{n-1})$ are both Gaussian for fixed (y^{n-1}, x_n) and y^{n-1} , with conditional means which are linear in (y^{n-1}, x_n) and y^{n-1} , respectively, and conditional covariances which are independent of (y^{n-1}, x_n) and y^{n-1} , respectively.

Stages $t \in \{n-1, n-2, \ldots, 1, 0\}$. By (4.4), evaluated at t = n-1, then $g_{n-1,n}(x_{n-1}, y^{n-1})$ will include terms of quadratic form in x_{n-1} and y^{n-1} . Repeating this argument recursively, it can be verified that at any time $t \in \mathbb{N}_0^{n-1}$, the optimal reproduction distribution $Q_t^*(\cdot|y^{t-1}, x_t)$ is conditionally Gaussian with conditional means linear with respect to (x_t, y^{t-1}) , and conditional covariances independent of $(x_t, y^{t-1}), t \in \mathbb{N}_0^{n-1}$.

By induction, we then deduce that the optimal reproduction distributions are conditionally Gaussian, and they are realized using a general equation of the form

(5.2)
$$Y_t = \bar{A}_t X_t + \bar{B}_t Y^{t-1} + V_t^c, \ t \in \mathbb{N}_0^n,$$

where $\bar{A}_t \in \mathbb{R}^{p \times p}$, $\bar{B}_t \in \mathbb{R}^{p \times tp}$, and $\{V_t^c : t \in \mathbb{N}_0^n\}$ is an independent sequence of Gaussian vectors $\{N(0; Q_t) : t \in \mathbb{N}_0^n\}$.

Next, we simplify the computation by introducing the following preprocessing at the encoder and decoder associated with channel (5.2) (as shown in Fig. 5.1). *Preprocessing at Encoder.* Introduce (i) the estimation error $\{K_t : t \in \mathbb{N}_0^n\}$ of

 $\{X_t : t \in \mathbb{N}_0^n\}$ based on $\{Y_0, \ldots, Y_{t-1}\}$, and (ii) its covariance $\{\Pi_{t|t-1} : t \in \mathbb{N}_0^n\}$, defined by

(5.3)
$$K_t \triangleq X_t - \widehat{X}_{t|t-1}, \ \widehat{X}_{t|t-1} \triangleq \mathbb{E}\left\{X_t | \sigma\{Y^{t-1}\}\right\}, \ \Pi_{t|t-1} \triangleq \mathbb{E}\{K_t K_t^{\mathrm{T}}\}, \ t \in \mathbb{N}_0^n,$$

where $\sigma\{Y^{t-1}\}$ is the σ -algebra (observable events) generated by the sequence $\{Y^{t-1}\}$. The covariance is diagonalized by introducing a unitary transformation $\{E_t : t \in \mathbb{N}_0^n\}$ such that

(5.4)
$$E_t \Pi_{t|t-1} E_t^{\mathsf{T}} = \Lambda_t$$
, where $\Lambda_t \triangleq \operatorname{diag}\{\lambda_{t,1}, \dots, \lambda_{t,p}\}, t \in \mathbb{N}_0^n$.

To facilitate the computation, we introduce the scaling process $\{\Gamma_t : t \in \mathbb{N}_0^n\}$, where $\Gamma_t \triangleq E_t K_t, t \in \mathbb{N}_0^n$, has independent Gaussian components but all of the components are correlated.

Preprocessing at Decoder. Analogously, we introduce the error process $\{\tilde{K}_t : t \in \mathbb{N}_0^n\}$ and the scaling process $\{\tilde{\Gamma} : t \in \mathbb{N}_0^n\}$ defined by

(5.5)
$$\widetilde{K}_t \triangleq Y_t - \widehat{X}_{t|t-1}, \text{ and } \widetilde{\Gamma}_t \triangleq \Phi_t Z_t, \ Z_t \triangleq (\Theta_t E_t K_t + V_t^c), \ t \in \mathbb{N}_0^n.$$

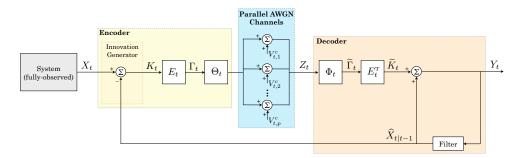


Fig. 5.1: Realization of the optimal nonstationary reproduction distribution of multidimensional Gaussian process.

The square error fidelity criterion $d_{0,n}(\cdot, \cdot)$ is not affected by the above processing of $\{(X_t, Y_t) : t \in \mathbb{N}_0^n\}$, since the preprocessing at both the encoder and decoder does not affect the form of the squared error distortion function, that is,

(5.6)
$$d_{0,n}(X^n, Y^n) = d_{0,n}(K^n, \tilde{K}^n) = \frac{1}{n+1} \sum_{t=0}^n ||\tilde{K}_t - K_t||_2^2$$
$$= \frac{1}{n+1} \sum_{t=0}^n ||\tilde{\Gamma}_t - \Gamma_t||_2^2 = d_{0,n}(\Gamma^n, \tilde{\Gamma}^n).$$

Using basic properties of conditional entropy, it can be shown that the following expressions are equivalent.

$$R_{0,n}^{na}(D) = R_{0,n}^{na,K^{n},\tilde{K}^{n}}(D) \triangleq \inf_{\{Q_{t}: t=0,\dots,n\}: \mathbb{E}\left\{d_{0,n}(K^{n},\tilde{K}^{n})\right\} \leq D} \sum_{t=0}^{n} I(K_{t};\tilde{K}_{t}|\tilde{K}^{t-1})$$

(5.7)
$$= R_{0,n}^{na,\Gamma^{n},\tilde{\Gamma}^{n}}(D) \triangleq \inf_{\{Q_{t}: t=0,\dots,n\}: \mathbb{E}\left\{d_{0,n}(\Gamma^{n},\tilde{\Gamma}^{n})\right\} \leq D} \sum_{t=0}^{n} I(\Gamma_{t};\tilde{\Gamma}_{t}|\tilde{\Gamma}^{t-1}).$$

Next, we derive the main theorem which gives the closed form expression of the NRDF for multidimensional Gaussian process (5.1) by considering the feedback realization scheme shown in Fig. 5.1, where $\{V_c^t : t \in \mathbb{N}_0^n\}$ is Gaussian $\{N(0; Q_t) : t \in \mathbb{N}_0^n\}$, and $\{\Theta_t, \Phi_t : t \in \mathbb{N}_0^n\}$ are the matching matrices to be determined.

THEOREM 5.2. $(R_{0,n}^{na}(D) \text{ of time-varying multidimensional Gauss-Markov process})$

(1) The NRDF, $R_{0,n}^{na}(D)$, of the Gauss-Markov process (5.1), is given by

(5.8)
$$R_{0,n}^{na}(D) = \frac{1}{2} \frac{1}{n+1} \sum_{t=0}^{n} \sum_{i=1}^{p} \log\left(\frac{\lambda_{t,i}}{\delta_{t,i}}\right), \ \delta_{t,i} \le \lambda_{t,i}, \ t \in \mathbb{N}_{0}^{n}, \ i = 1, \dots, p,$$

(5.9)
$$\equiv \frac{1}{2} \frac{1}{n+1} \sum_{t=0} \sum_{i=1}^{t} \log \left\{ \max \left(1, \frac{\lambda_{t,i}}{\delta_{t,i}} \right) \right\},$$

where $\Lambda_t = E_t \Pi_{t|t-1} E_t^T$,

(5.10)
$$\Pi_{t|t-1} \triangleq \mathbb{E}\left\{ \left(X_t - \mathbb{E}\left\{ X_t | \sigma\{Y^{t-1}\} \right\} \right) \left(X_t - \mathbb{E}\left\{ X_t | \sigma\{Y^{t-1}\} \right\} \right)^T \right\}$$

(5.11)
$$\delta_{t,i} \triangleq \begin{cases} \xi, & \text{if } \xi \leq \lambda_{t,i} \\ \lambda_{t,i}, & \text{if } \xi > \lambda_{t,i} \end{cases}, \ \forall t, i$$

and ξ is chosen such that

(5.12)
$$\frac{1}{n+1} \sum_{t=0}^{n} \sum_{i=1}^{p} \delta_{t,i} = D.$$

(2) The error $X_t - \mathbb{E}\{X_t | \sigma\{Y^{t-1}\}\}$ is Gaussian $N(0; \Pi_{t|t-1}), \hat{X}_{t|t-1} \triangleq \mathbb{E}\{X_t | \sigma\{Y^{t-1}\}\},$ and $\Pi_{t|t-1}$ are given by the Kalman filter equations

(5.13)
$$\widehat{X}_{t+1|t} = A_t \widehat{X}_{t|t-1} + A_t \Pi_{t|t-1} (E_t^T H_t E_t)^T M_t^{-1} \left(Y_t - \widehat{X}_{t|t-1} \right),$$
$$\Pi_{t+1|t} = A_t \Pi_{t|t-1} A_t^T - A_t \Pi_{t|t-1} (E_t^T H_t E_t)^T M_t^{-1} (E_t^T H_t E_t) \Pi_{t|t-1} A_t^T$$

(5.14)
$$+ B_t B_t^{T} = A_t E_t^{T} \Delta_t E_t A_t^{T} + B_t B_t^{T}, \quad \Pi_{0|-1} = \bar{\Pi}_{0|-1}, \quad t \in \mathbb{N}_0^n,$$

(5.15)
$$M_t = E_t^{'} H_t E_t \Pi_{t|t-1} (E_t^{'} H_t E_t)^{'} + E_t^{'} \Phi_t Q_t \Phi_t^{'} E_t = E_t^{'} H_t \Lambda_t E_t$$

where

(5.16)
$$\eta_{t,i} = 1 - \frac{\delta_{t,i}}{\lambda_{t,i}}, \ i = 1, \dots, p, \ H_t \triangleq \operatorname{diag}\{\eta_{t,1}, \dots, \eta_{t,p}\},$$
$$\Delta_t = \operatorname{diag}\{\delta_{t,1}, \dots, \delta_{t,p}\}, \ \Phi_t \triangleq \sqrt{H_t \Delta_t Q_t^{-1}}, \ t \in \mathbb{N}_0^n.$$

(3) The realization of the optimal time-varying (nonstationary) reproduction distribution illustrated in Fig. 5.1 is given by

(5.17)
$$Y_{t} = E_{t}^{^{T}} H_{t} E_{t} \left(X_{t} - \widehat{X}_{t|t-1} \right) + E_{t}^{^{T}} \Phi_{t} V_{t}^{c} + \widehat{X}_{t|t-1}$$
$$= \widehat{X}_{t|t-1} + E_{t}^{^{T}} \Phi_{t} Z_{t}, \quad Z_{t} = \Theta_{t} E_{t} \left(X_{t} - \widehat{X}_{t|t-1} \right) + V_{t}^{c}, \quad \Theta_{t} = \Phi_{t}^{-1} H_{t}.$$

(4) The filter estimate satisfies

(5.18)
$$\widehat{X}_{t|t-1} = A_{t-1}Y_{t-1}, \ \widehat{X}_{0|-1} = \mathbb{E}\{X_0|\sigma\{Y^{-1}\}\}, \ t \in \mathbb{N}_0^n$$
(5.19)
$$\widehat{X}_{t|t} = Y_t$$

and the optimal reproduction process is

(5.20)
$$Y_t = A_{t-1}Y_{t-1} + E_t^{^T} \Phi_t Z_t, \ Z_t = \Theta_t E_t \left(X_t - A_{t-1}Y_{t-1} \right) + V_t^c.$$

- (5) The processes $\{Y_t : t \in \mathbb{N}_0^n\}$ and $\{\tilde{K}_t : t \in \mathbb{N}_0^n\}$ generate the same information, *i.e.*, $\sigma\{Y^t\} = \sigma\{\tilde{K}^t\}, t \in \mathbb{N}_0^n$.
 - *Proof.* See Appendix B. \Box

We make the following observations regarding Theorem 5.2. REMARK 2.

(1) The main features of Theorem 5.2 are the following:

First, by (5.20) the information structure of the optimal reproduction for the specific Gaussian source with memory given by (5.1) is Markov, i.e.,

(5.21)
$$Q_t^*(dy_t|y^{t-1}, x^t) \equiv Q_t^*(dy_t|y_{t-1}, x_t)$$

Hence, the output process $\{Y_t : t \in \mathbb{N}_0^n\}$ is first order Markov.

Second, the time-space reverse-waterfilling property (5.8)-(5.11), states that if the reproduction error $\delta_{t,i}$ is above the eigenvalue $\lambda_{t,i}$ of the error covariance $\Pi_{t|t-1}$, then the time-space component $X_{t,i}^{4}$ is not reconstructed by $Y_{t,i}^{5}$, for $t \in \mathbb{N}_{0}^{n}$, i = 1, ..., p. The behavior of $\delta_{t,i}$ is described by the reversewaterfilling expression (5.11), and the level ξ depends on D, i.e., the overall fidelity of the error.

(2) For each t + 1, $\hat{X}_{t+1|t} = A_t Y_t$, given by (5.13), is the estimator of X_{t+1} based on Y^t . In addition, the time-space reverse-waterfilling is part of the estimation algorithm. This is a variant of the Kalman filter.

The following remark, is a direct consequence of Theorem 5.2, and illustrates the connection between $R_{0,n}^{na}(D)$ and $D_{0,n}(R^{na})$ given by (4.1).

Remark 3.

From Theorem 5.2 the NRDF of the Gaussian process (5.1) is given by

(5.22)
$$R_{0,n}^{na}(D) = \frac{1}{2} \frac{1}{n+1} \sum_{t=0}^{n} \sum_{i=1}^{p} \log \left\{ \max\left(1, \frac{\lambda_{t,i}}{\delta_{t,i}}\right) \right\} \stackrel{(a)}{\equiv} \frac{1}{n+1} \sum_{t=0}^{n} \sum_{i=1}^{p} R_{t,i}^{na}(\delta_{t,i})$$

where (a) follows if we let

(5.23)
$$R_{t,i}^{na}(\delta_{t,i}) \triangleq \frac{1}{2} \log \left\{ \max\left(1, \frac{\lambda_{t,i}}{\delta_{t,i}}\right) \right\}, \ t \in \mathbb{N}_0^n, \ i = 1, \dots, p.$$

By (5.23) we obtain

(5.24)
$$\delta_{t,i} = \lambda_{t,i} e^{-2R_{t,i}^{na}}, \ t \in \mathbb{N}_0^n, \ i = 1, \dots, p.$$

Utilizing (5.12), we obtain

(5.25)
$$D = \frac{1}{n+1} \sum_{t=0}^{n} \delta_t = \frac{1}{n+1} \sum_{t=0}^{n} \sum_{i=1}^{p} \delta_{t,i}, \ \delta_t \triangleq \sum_{i=1}^{p} \delta_{t,i}.$$

Substituting (5.24) into (5.25) we obtain

(5.26)
$$D_{0,n}(R^{na}) = \frac{1}{n+1} \sum_{t=0}^{n} \delta_t = \frac{1}{n+1} \sum_{t=0}^{n} \sum_{i=1}^{p} \lambda_{t,i} e^{-2R_{t,i}^{na}}.$$

 $^{{}^{4}}X_{t,i}$ is the time-space component of the vector process $\{X_t: t \in \mathbb{N}_0^n\}$.

 $^{{}^{5}}Y_{t,i}$ is the time-space component of the vector process $\{Y_t: t \in \mathbb{N}_0^n\}$.

Next, we utilize the closed form expressions of the NRDF and FTH NDRF evaluated for time-varying multidimensional Gauss-Markov process to derive a lower bound on the MSE given in terms of conditional mutual information $I(X^n; Y_0^n | Y^{-1})$.

THEOREM 5.3. (Universal lower bound on mean square error) Let $\{X_t : t \in \mathbb{N}_0^n\}$ be the multidimensional Gauss-Markov process given by (5.1) and let $\{\tilde{Y}_t : t \in \mathbb{N}_0^n\}$ be any estimator (not necessarily Gaussian) of $\{X_t : t \in \mathbb{N}_0^n\}$. The mean square error is bounded below by

(5.27)
$$\frac{1}{n+1}\sum_{t=0}^{n} \mathbb{E}\left\{||X_t - \widetilde{Y}_t||_2^2\right\} \ge \frac{1}{n+1}\sum_{t=0}^{n}\sum_{i=1}^{p}\lambda_{t,i}e^{-2I(X_{t,i};\widetilde{Y}_{t,i}|\widetilde{Y}_{t-1,i})}$$

Proof. Let $D = \frac{1}{n+1} \sum_{t=0}^{n} \mathbb{E}\left\{ ||X_t - \widetilde{Y}_t||_2^2 \right\}$ where

$$\mathbb{E}\left\{||X_t - \widetilde{Y}_t||_2^2\right\} = \sum_{i=1}^p \delta_{t,i} \text{ with } D \in [0,\infty).$$

Since, in general, $R_{t,i}^{na} \leq I(X_{t,i}; \widetilde{Y}_{t,i} | \widetilde{Y}_{t-1,i}), t \in \mathbb{N}_0^n, i = 1, \ldots, p$, then by (5.26), we obtain

(5.28)
$$\frac{1}{n+1} \sum_{t=0}^{n} \mathbb{E}\left\{ ||X_t - \widetilde{Y}_t||_2^2 \right\} = D_{0,n}(R^{na}) = \frac{1}{n+1} \sum_{t=0}^{n} \sum_{i=1}^{p} \lambda_{t,i} e^{-2R_{t,i}^{na}} \\ \ge \frac{1}{n+1} \sum_{t=0}^{n} \sum_{i=1}^{p} \lambda_{t,i} e^{-2I(X_{t,i};\widetilde{Y}_{t,i}|\widetilde{Y}_{t-1,i})},$$

which is the desired result. This completes the proof. \square

Notice that from Remark 2, (2), if we substitute $\tilde{Y}_t = \hat{X}_{t|t-1} = A_{t-1}Y_{t-1}$ in Theorem 5.3, then we have the lower bound (5.27).

In the next remark, we relate degenerated versions of the lower bound given by (5.27) to existing results in the literature.

REMARK 4. (Relations to existing results)

(a) [26, Theorem 5.8.1], [27] Let $X = (X_1, \ldots, X_p)$ be a p-dimensional Gaussian vector with distribution $X \sim N(0; \Gamma_X)$ and $Y = (Y_1, \ldots, Y_p)$ be its reproduction vector. Then, for any D > 0,

(5.29)
$$R(D) \triangleq \inf_{Q(dy|x):\mathbb{E}||X-Y||_2^2 \le D} I(X;Y) = \frac{1}{2} \sum_{i=1}^p \log\left\{ \max\left(1,\frac{\lambda_i}{\xi}\right) \right\}$$

where {λ_i: i = 1,..., p} are the eigenvalues of Γ_X and ξ > 0 is a constant uniquely determined by ∑_{i=1}^p min{λ_i, ξ} = D. Note that the solution of classical RDF in (5.29) is based on reverse-waterfilling method (see [26, Lemma 5.8.2]). The above results are also obtained from Theorem 5.2, if we assume model (5.1) generates an IID sequence {X_t : t ∈ N₀ⁿ} (by setting A_t = 0, B_t = I). In such case, Â_{t|t-1} = EX_t = 0 and Π_{t|t-1} = EX_tX_t^T = Γ_X.
(b) Assume X ~ N(0; σ_X²). By [26, Theorem 1.8.7] the following holds.

$$R(D) = \min_{\substack{Q(dy|x): \ \mathbb{E}||X-Y||_2^2 \le D}} I(X;Y) = \frac{1}{2} \log \left\{ \max\left(1, \frac{\sigma_X^2}{D}\right) \right\}, \ D \ge 0,$$
$$D(R) = \min_{\substack{Q(dy|x): \ \mathbb{E}||X-Y||_2^2 \le D}} \mathbb{E} \left\{ ||X-Y||_2^2 \right\} = \sigma_X^2 e^{-2R}.$$

$$U = \min_{\substack{Q(dy|x): \ I(X;Y) \le R}} \mathbb{E}\left\{ ||X - I||_2 \right\} = 0$$

The realization scheme to achieve the classical RDF or the DRF is the following.

(5.30)
$$Y = \left(1 - \frac{D}{\sigma_X^2}\right) X + V^c, \ V^c \sim N\left(0; D(1 - \frac{D}{\sigma_X^2})\right).$$

Note that (5.30) is a degenerated version of (5.17) assuming the model of (5.1) generates IID sequence $\{X_t : t \in \mathbb{N}_0^n\}$ as in (a), and the connection to Theorem 5.2 is established by setting $E_t = 1$, $H_t = 1 - \frac{D}{\sigma_X^2}$, $\widehat{X}_{t|t-1} = 0$, $\Phi_t = H_t D$ and $V_t^c \sim N(0; 1)$.

(c) (Lower bound on MSE [26, 1.8.8], [22]) Given a Gaussian RV $X \sim N(0; \sigma_X^2)$, then for any real valued RV \tilde{Y} (not necessarily Gaussian) the MSE is bounded below by

(5.31)
$$\mathbb{E}||X - \widetilde{Y}||_2^2 \ge \sigma_X^2 e^{-2I(X;\overline{Y})}$$

The RDF of the Gaussian RV $X \sim N(0; \sigma_X^2)$ and the lower bound in (5.31), are utilized in [22, 26] to derive optimal coding and decoding schemes for transmitting a Gaussian message $\theta \sim N(0; \sigma_{\theta}^2)$ over an AWGN channel with feedback, $Y_t = X_t(\theta, Y^{t-1}) + V_t^c$, $t \in \mathbb{N}_0^n$, where $\{V_t^c : t \in \mathbb{N}_0^n\}$ is IID Gaussian process. Although we do not pursue such problems in this paper, we note that Theorems 5.2 and 5.3 are necessary in order to derive optimal coding schemes for additive Gaussian channels with memory (including additive Gaussian memoryless channels).

5.1. Examples. In this section, we numerically compute the NRDF of *time-varying* Gauss-Markov process, using Theorem 5.2. For these examples, the utility of the reverse-waterfilling algorithm is necessary even when the process elements are scalar (i.e., p = 1). For process elements in higher dimensions (i.e., $p \ge 2$), the complexity of the problem increases, since the reverse-waterfilling algorithm must be solved both in time and space units. We overcome this obstacle by proposing an iterative algorithmic technique that allocates information of the Gaussian process and distortion levels optimally.

REMARK 5. (Relations to existing results)

The examples presented here deal with the time-space aspects of the reverse-waterfilling algorithm. This is fundamentally different from [15, Section IV.C] where it is assumed that the optimal reproduction distributions $\{Q_t^*(dy_t|y^{t-1}, x_t) = Q^*(dy_t|y^{t-1}, x_t) : t \in \mathbb{N}_0\}$ are time-invariant (identical).

EXAMPLE 1. Consider the following two-dimensional Gauss-Markov process

(5.32)
$$\begin{bmatrix} X_{t+1,1} \\ X_{t+1,2} \end{bmatrix} = \underbrace{\begin{bmatrix} -\alpha_t & 1 \\ -\beta_t & 0 \end{bmatrix}}_{A_t} \begin{bmatrix} X_{t,1} \\ X_{t,2} \end{bmatrix} + \underbrace{\begin{bmatrix} \sigma_{W_{t,1}} & 0 \\ 0 & \sigma_{W_{t,2}} \end{bmatrix}}_{B_t} \begin{bmatrix} W_{t,1} \\ W_{t,2} \end{bmatrix} t = 0, 1, 2, \ i = 1, 2,$$

where $W_{t,i} \sim N(0;1)$, $\sigma_{W_{t,i}}W_{t,i} \sim N(0;\sigma_{W_{t,i}}^2)$ and $\{A_t, B_t\}$ are time-varying matrices. This example corresponds to (5.1) for p = k = n = 2. For this example, we

Algorithm 1 Rate distortion allocation	ı algorithm: 'I	'he vector case
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Initialize:

The number of time-steps n; the number of channels p the distortion level D; the error tolerance ϵ ; the initial covariance matrix $\overline{\Pi}_{0|-1}$ of the error process K_0 , the state-space matrices A_t and B_t of the time-varying multidimensional Gauss-Markov process X_t given by (5.1).

Set $\xi = D$; flag = 0. while flag = 0 do Compute $\delta_{t,i} \forall t, i$ as follows: for t = 0 : n do Perform Singular Value Decomposition: $[E_t, \Lambda_t] = \text{SVD}(\Pi_{t|t-1})$ Δ_t is computed according to (5.11). Use $A_t B_t$ and Δ_t to compute $\Pi_{t+1|t}$ according to (5.14). end for if $|\frac{1}{n+1}\sum_{t=0}^n \sum_{i=1}^p \delta_{t,i} - D| \le \epsilon$ then flag $\leftarrow 1$ else Re-adjust ξ as follows: $\xi \leftarrow \xi + \beta(D - \frac{1}{n+1}\sum_{t=0}^n \sum_{i=1}^p \delta_{t,i})$, where $\beta \in (0, 1]$ is a proportionality gain and affects the rate of convergence. end if end while

choose the distortion level D = 3 and consider the following matrices $\{A_t, B_t\}$:

$$A_{0} = \begin{bmatrix} -0.5 & 1\\ -0.4 & 0 \end{bmatrix}, B_{0} = \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}$$
$$A_{1} = \begin{bmatrix} -0.4 & 1\\ -0.5 & 0 \end{bmatrix}, B_{1} = \begin{bmatrix} 0.9 & 0\\ 0 & 1.4 \end{bmatrix}$$
$$A_{2} = \begin{bmatrix} -0.9 & 1\\ -0.5 & 0 \end{bmatrix}, B_{2} = \begin{bmatrix} 1.2 & 0\\ 0 & 1.3 \end{bmatrix}$$

The initial covariance matrix of the error process K_t is

$$\bar{\Pi}_{0|-1} = \begin{bmatrix} 0.6 & 0.2\\ 0.2 & 0.4 \end{bmatrix}.$$

Recall that the covariance matrix of the error process K_t given by (5.14) is simplified to

(5.33) $\Pi_{t+1|t} = A_t E_t^{^{T}} \{ \operatorname{diag} \{ \delta_{t,1}, \delta_{t,2} \} \} E_t A_t^{^{T}} + B_t B_t^{^{T}}, \ t = 0, 1, 2, \ \Pi_{0|-1} = \bar{\Pi}_{0|-1}$

and $\delta_{t,i}$ given by (5.11) becomes

(5.34)
$$\delta_{t,i} = \min\{\lambda_{t,1}, \xi\}, \ t = 0, 1, 2, \ i = 1, 2.$$

Now let us implement Algorithm 1 for error tolerance $\epsilon = 10^{-3}$. We choose an initial $\xi = \xi_0$ to start our iterations. A good starting point is $\xi_0 = D$. For $\overline{\Pi}_{0|-1}$ we perform

Singular Value Decomposition (SVD) and we obtain the unitary matrix

$$E_0 = \begin{bmatrix} -0.8507 & -0.5257\\ -0.5257 & 0.8507 \end{bmatrix}$$

and the eigenvalues in a diagonal matrix that correspond to the levels of the noise $\lambda_{0,1}$ and $\lambda_{0,2}$, i.e.,

$$\Lambda_0 = \begin{bmatrix} 0.7236 & 0\\ 0 & 0.2764 \end{bmatrix}$$

For $\xi = \xi_0 = D = 3$ and $(\lambda_{0,1}, \lambda_{0,2}) = (0.72, 0.28)$ we compute Δ_0 using (5.34). Hence,

$$\Delta_0 = \Lambda_0 = \begin{bmatrix} 0.7236 & 0\\ 0 & 0.2764 \end{bmatrix}.$$

Using A_0 , B_0 , Δ_0 and E_0 we compute $\Pi_{1|0}$ using (5.33)

$$\Pi_{1|0} = \begin{bmatrix} 1.3500 & 0.0400\\ 0.0400 & 1.0960 \end{bmatrix}$$

and the procedure of (a) computing the SVD of $\Pi_{1|0}$, (b) computing Δ_1 is repeated as it is done for $\Pi_{1|0}$. Similarly, the procedure is repeated for all t = 0, 1, ..., n. At the end, for the given ξ we check if $|\frac{1}{n+1}\sum_{t=0}^{n}\sum_{i=1}^{p}\delta_{t,i} - D| \leq \epsilon$. If it does, we stop the iterations and the last ξ is the level we want. If not, we update ξ as $\xi \leftarrow \xi + \beta (D - \frac{1}{n+1}\sum_{t=0}^{n}\sum_{i=1}^{p}\delta_{t,i})$ and we repeat the procedure for all t again. For this example, the final reverse-waterfilling is found in 9 iterations and it is shown in Figure 5.2.

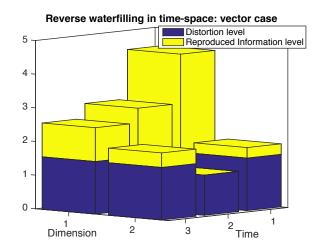


Fig. 5.2: Reverse-waterfilling in time-space for n=2 time units and p=2 space units.

By (5.8) we compute the NRDF:

$$R_{0,2}^{na}(D) = \frac{1}{2} \frac{1}{2+1} \sum_{t=0}^{2} \sum_{i=1}^{2} \log\left(\frac{\lambda_{t,i}}{\delta_{t,i}}\right) = 0.6330 \ bits/source \ symbol$$

In the next corollary, we degrade the results derived in Theorem 5.2 to the case of *time-varying scalar Gauss-Markov process*. This corollary emphasizes on the fact that even in its simplest form, i.e., when p = 1, the computation of FTH NRDF for time-varying Gauss-Markov processes can only be evaluated numerically by utilizing algorithmic methods. Note that in the sequel, when we refer to the scalar Gaussian process, for simplicity we will not make use of the dimension subscript, that is, $\lambda_{t,1} \equiv \lambda_t, \delta_{t,1} \equiv \delta_t, \eta_{t,1} \equiv \eta_t, q_{t,1} \equiv q_t$ etc.

COROLLARY 5.4. $(R_{0,n}^{na}(D) \text{ of time-varying scalar Gauss-Markov process})$ This corresponds to (5.1) by setting p = k = 1, $A_t = \alpha_t$, $B_t = \sigma_{W_t}$, i.e., $\sigma_{W_t}W_t \sim N(0; \sigma_{W_t}^2)$ giving

(5.35)
$$X_{t+1} = \alpha_t X_t + \sigma_{W_t} W_t, \ W_t \sim N(0;1), \ X_0 \sim N(0;\sigma_{X_0}^2), \ t = 0, 1, \dots, n$$

where $\{\alpha_t, \sigma_{W_t} : t = 0, 1, \dots, n\}$ are time varying. Then $\sigma_{X_t}^2 \triangleq \operatorname{Var}(X_t)$, satisfies $\sigma_{X_{t+1}}^2 = \alpha_t^2 \sigma_{X_t}^2 + \sigma_{W_t}^2$, $\sigma_{X_0}^2 = \sigma_0^2$, $t \in \mathbb{N}_0^n$. In this case, by Theorem 5.2, and (5.8) we obtain

(5.36)
$$R_{0,n}^{na}(D) = \frac{1}{2} \frac{1}{n+1} \sum_{t=0}^{n} \log\left(\frac{\lambda_t}{\delta_t}\right)$$

where

(5.37)
$$\delta_t \triangleq \begin{cases} \xi & \text{if } \xi \leq \lambda_t \\ \lambda_t & \text{if } \xi > \lambda_t \end{cases}, \ \forall t$$

with ξ fixed such that $\frac{1}{n+1}\sum_{t=0}^{n} \delta_t = D, \delta_t = \min_t \{\lambda_t, \xi\}$ and $\Pi_{t|t-1} = \Lambda_t = \lambda_t$, (i.e., $E_t = 1$), $H_t = \eta_t = 1 - \frac{\delta_t}{\lambda_t}$, $t = 0, \ldots, n$. By (5.15), we obtain

(5.38)
$$M_t = \lambda_t H_t^2 + H_t \delta_t = H_t \left(\lambda_t H_t + \delta_t\right) = H_t \left(\lambda_t \left(1 - \frac{\delta_t}{\lambda_t}\right) + \delta_t\right) = \lambda_t H_t.$$

Also, by (5.14), we obtain

$$\lambda_{t+1} = \alpha_t^2 \lambda_t - \alpha_t^2 \lambda_t^2 H_t^2 M^{-1} + \sigma_{W_t}^2 \stackrel{(a)}{=} \alpha_t^2 \lambda_t - \alpha_t^2 \lambda_t^2 H_t^2 H_t^{-1} \lambda_t^{-1} + \sigma_{W_t}^2$$
(5.39)
$$= \alpha_t^2 \lambda_t - \alpha_t^2 \lambda_t^2 H_t^2 + \sigma_{W_t}^2 = \alpha_t^2 \lambda_t - \alpha_t^2 \lambda_t^2 H_t^2 H_t^{-1} \lambda_t^{-1} + \sigma_{W_t}^2 = \alpha_t^2 \lambda_t - \alpha_t^2 \lambda_t^2 H_t^2 H_t^{-1} \lambda_t^{-1} + \sigma_{W_t}^2 = \alpha_t^2 \lambda_t - \alpha_t^2 \lambda_t^2 H_t^2 H_t^{-1} \lambda_t^{-1} + \sigma_{W_t}^2 = \alpha_t^2 \lambda_t - \alpha_t^2 \lambda_t^2 H_t^2 H_t^{-1} \lambda_t^{-1} + \sigma_{W_t}^2 = \alpha_t^2 \lambda_t - \alpha_t^2 \lambda_t^2 H_t^2 H_t^{-1} \lambda_t^{-1} + \sigma_{W_t}^2 = \alpha_t^2 \lambda_t - \alpha_t^2 \lambda_t^2 H_t^2 H_t^{-1} \lambda_t^{-1} + \sigma_{W_t}^2 = \alpha_t^2 \lambda_t - \alpha_t^2 \lambda_t^2 H_t^2 H_t^2 H_t^{-1} \lambda_t^{-1} + \sigma_{W_t}^2 = \alpha_t^2 \lambda_t - \alpha_t^2 \lambda_t^2 H_t^2 H_t^2 H_t^{-1} \lambda_t^{-1} + \sigma_{W_t}^2 = \alpha_t^2 \lambda_t - \alpha_t^2 \lambda_t^2 H_t^2 H_t^2 H_t^2 + \alpha_t^2 + \alpha_t^2 \lambda_t^2 H_t^2 H_t^2 + \alpha_t^2 + \alpha_$$

$$=\alpha_t^2\lambda_t - \alpha_t^2\lambda_t H_t + \sigma_{W_t}^2 = \alpha_t^2\lambda_t - \alpha_t^2\lambda_t \left(1 - \frac{\partial_t}{\lambda_t}\right) + \sigma_{W_t}^2 = \alpha_t^2\delta_t + \sigma_{W_t}^2, \ \bar{\lambda}_0 = \sigma_{X_0}^2$$

where (a) follows from (5.38).

Similarly to Algorithm 1, we structure Algorithm 2 for rate distortion allocation.

EXAMPLE 2. For this example, we choose the distortion level D = 2 and use the following $\{a_t^2, \sigma_{W_t}^2\}$:

$$(a_0^2, \sigma_{W_0}^2) = (1, 1), \quad (a_1^2, \sigma_{W_1}^2) = (0.2, 1.3), \quad (a_2^2, \sigma_{W_2}^2) = (1.8, 0.7).$$

The initial variance is $\sigma_{X_0} = 1$. Hence, $\overline{\lambda}_0 = \sigma_{X_0} = 1$.

Now let us implement Algorithm 2 for error tolerance $\epsilon = 10^{-3}$. We choose an initial $\xi = \xi_0$ to start our iterations. A good starting point is $\xi_0 = D$. Using (5.37), $\delta_0 = \min\{1, 2\} = 1$. Then, using (5.39), $\lambda_1 = \alpha_0^2 \delta_0 + \sigma_{W_0}^2$ and thus δ_1 is computed. Similarly, the procedure is repeated for all t = 0, 1, ..., n. At the end, for the given ξ

Algorithm 2 Rate distortion a	allocation	algorithm:	The scalar of	case
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Initialize:

The number of time-steps n; the distortion level D; the error tolerance ϵ ; the initial variance $\bar{\lambda}_0 = \sigma_{X_0}^2$ of the initial state X_0 , the values a_t and $\sigma_{W_t}^2$ of the time-varying scalar Gauss-Markov process X_t given by (5.35).

Set $\xi = D$; flag = 0.

while flag = 0 do Compute $\delta_t \forall t$ as follows: for t = 0 : n do δ_t is computed according to (5.37). Use a_t and $\sigma_{W_t}^2$ to compute λ_{t+1} according to (5.39). end for if $|\frac{1}{n+1}\sum_{t=0}^n \delta_t - D| \le \epsilon$ then flag $\leftarrow 1$ else Re-adjust ξ as follows: $\xi \leftarrow \xi + \beta(D - \frac{1}{n+1}\sum_{t=0}^n \delta_t)$, where $\beta \in (0, 1]$ is a proportionality gain and affects the rate of convergence. end if end while

we check if $|\frac{1}{n+1}\sum_{t=0}^{n} \delta_t - D| \leq \epsilon$. If it does, we stop the iterations and the last ξ is the level we want. If not, we update ξ as $\xi \leftarrow \xi + \beta(D - \frac{1}{n+1}\sum_{t=0}^{n} \delta_t)$ and we repeat the procedure for all t again.

For this example, the final reverse-waterfilling is found after 15 iterations and it is shown in Figure 5.2.

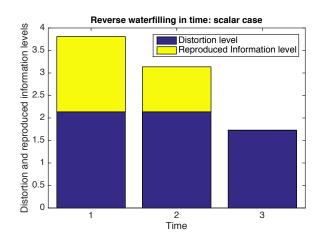


Fig. 5.3: Reverse-waterfilling in time for n=2 time units.

By (5.36) we compute the NRDF:

$$R_{0,2}^{na}(D) = \frac{1}{2} \frac{1}{2+1} \sum_{t=0}^{2} \log\left(\frac{\lambda_t}{\delta_t}\right) = 0.2314 \text{ bits/source symbol}$$

5.2. Realization of (5.20). In this section, we exemplify the relation between information-based estimation via NRDF and the fact that the latter can also be seen as a realization of an {encoder, channel, decoder} processing information optimally with zero-delay. For simplicity we consider scalar process (p = 1). Note that this concept is precisely the one described in Fig. 1.1.

EXAMPLE 3. (Realization of (5.20) for scalar processes) Let X_t be the scalar time-varying Gauss-Markov process defined by (5.35), and recall that $\lambda_t = \alpha_{t-1}^2 \delta_{t-1} + \sigma_{W_{t-1}}^2$ and $R^{na}(D) = \frac{1}{2} \sum_{t=0}^n \log\left(\frac{\delta_{t,i}}{\lambda_{t,i}}\right)$ (see (5.39) and (5.36), respectively).

Using (5.20) for p = 1, we obtain the following expression:

(5.40)
$$Y_t = \Phi_t Z_t + \alpha_{t-1} Y_{t-1}, \ Z_t = \Theta_t \left(X_t - \alpha_{t-1} Y_{t-1} \right) + V_t^c,$$

where

(5.41)
$$\Phi_t \triangleq \sqrt{\frac{H_t \delta_t}{q_t}} \quad and \quad \Theta_t \triangleq \sqrt{\frac{H_t q_t}{\delta_t}}.$$

Note that H_t, δ_t are defined in Corollary 5.4 and q_t is the variance of the scalar noise process $V_t^c \sim N(0; q_t)$.

Next, we consider the FTH information capacity of a memoryless AWGN channel with or without feedback with Gaussian noise process given as follows

(5.42)
$$C_{0,n}(P) = \frac{1}{2} \frac{1}{n+1} \sum_{t=0}^{n} \log\left(1 + \frac{P_t}{q_t}\right),$$

where P_t is the power level allocated at each time.

Suppose that this channel is used once per source symbol, that is, the coding rate between the source symbols and the channel symbols is 1 [28, Definition 2.1]. For the realization in (5.40), the smallest achievable distortion is obtained by setting (5.36)=(5.42) that yields

(5.43)
$$\delta_t^{\min} = \frac{\lambda_t q_t}{q_t + P_t} = \frac{\left(\alpha_{t-1}^2 \delta_{t-1}^{\min} + \sigma_{W_{t-1}}^2\right) q_t}{q_t + P_t}, \ t \in \mathbb{N}_1^n,$$

where

(5.44)
$$D_{\min} = \frac{1}{n+1} \sum_{t=0}^{n} \delta_t^{\min}.$$

Evaluating $\lambda_t = \alpha_{t-1}^2 \delta_{t-1} + \sigma_{W_{t-1}}^2$ at δ_t^{\min} the following feedback encoder operates at FTH information capacity.

(5.45)
$$Z_t = \sqrt{\frac{P_t}{\lambda_t}} K_t + q_t = \sqrt{\frac{P_t}{\alpha_{t-1}^2 \delta_{t-1}^{\min} + \sigma_{W_{t-1}}^2}} K_t + q_t,$$
$$K_t = X_t - \mathbb{E}\{X_t | \sigma\{Y^{t-1}\}\} = X_t - \alpha_{t-1}Y_{t-1},$$

where Z_t is the observation process containing the data.

In addition, the decoder (or the filter) is given by the realization in (5.40).

By (5.41), the scaling factor Φ_t which guarantees the minimum end-to-end error is

(5.46)
$$\Phi_t = \sqrt{\frac{\alpha_{t-1}^2 \delta_{t-1}^{\min} + \sigma_{W_{t-1}}^2}{P_t}} \frac{P_t}{q_t + P_t}$$

Substituting (5.46) into (5.40) we obtain

(5.47)
$$Y_t = \alpha_{t-1}Y_{t-1} + \sqrt{\frac{\alpha_{t-1}^2 \delta_{t-1}^{\min} + \sigma_{W_{t-1}}^2}{P_t}} \frac{P_t}{q_t + P_t} Z_t$$

Finally, the average end-to-end distortion at each time instant is computed by evaluating the expectation

$$D_t = \mathbb{E}\{|X_t - Y_t|^2\} = \frac{(\alpha_{t-1}^2 \delta_{t-1}^{\min} + \sigma_{W_{t-1}}^2)q_t^2 + (\alpha_{t-1}^2 \delta_{t-1}^{\min} + \sigma_{W_{t-1}}^2)q_t P_t}{(P_t + q_t)^2}$$
$$= \frac{(\alpha_{t-1}^2 \delta_{t-1}^{\min} + \sigma_{W_{t-1}}^2)q_t}{q_t + P_t} = \delta_t^{\min}.$$

The realization of (5.40) with an {encoder, channel, decoder} operating with zerodelay is illustrated in Fig. 5.4.

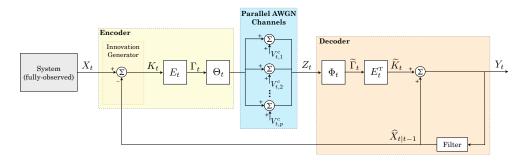


Fig. 5.4: Realization of the optimal reproduction of $R_{0,n}^{na}(D)$ given by (5.40). The scalings Θ_t and Φ_t are given by (5.45) and (5.46) respectively.

6. Conclusions and Future Directions. In this paper, we derived informationbased causal filters via nonanticipative rate distortion theory in finite-time horizon. We exemplified our theoretical framework to time-varying multidimensional Gauss-Markov process subject to a MSE fidelity, and we demonstrated that obtaining such filters is equivalent to the design of an optimal {encoder, channel, decoder}, which ensures that the error fidelity is met. Unlike classical Kalman filters, the new information-based causal filter is characterized by a reverse-waterfilling algorithm. Moreover, we established a universal lower bound on the MSE of any estimator of a Gaussian random process.

The results derived in this paper makes pave the way to generalizing the proposed framework to Gaussian sources governed by partially observed Gauss-Markov

processes. Part of ongoing research focuses on how filtering with fidelity criteria affects stability and performance of control systems.

Appendix A. Proof of Theorem 4.2. Let $s \leq 0, \lambda_n \in \Psi_s^n$ and $\overrightarrow{Q}_{0,n}^*(\cdot|y^{-1},x^n) \in \overrightarrow{\mathcal{Q}}_{0,n}(D)$ be given. Then, using the fact that

$$\frac{1}{n+1}\sum_{t=0}^n \int_{\mathcal{X}^t \times \mathcal{Y}^t} \rho_t(T^t x^n, T^t y^n)(P_{0,t} \otimes \overrightarrow{Q}_{0,t}) \left(dx^t, dy_0^t | y^{-1}\right) \mu(dy^{-1}) \le D$$

we obtain

$$\begin{split} &\frac{1}{n+1}\mathbb{I}_{0,n}(P_{0,n},\overrightarrow{Q}_{0,n}) - sD + \frac{1}{n+1}\sum_{t=0}^{n}\int_{\mathcal{X}^{t}\times\mathcal{Y}^{t}}g_{t,n}(x^{t},y^{t})(P_{0,t}\otimes\overrightarrow{Q}_{0,t})\left(dx^{t},dy_{0}^{t}|y^{-1}\right)\otimes\mu(dy^{-1}) \\ &- \frac{1}{n+1}\sum_{t=0}^{n}\int_{\mathcal{X}^{t}\times\mathcal{Y}^{t-1}}\log\left(\lambda_{t}(x^{t},y^{t-1})\right)\left(P_{0,t}\otimes\overrightarrow{Q}_{0,t}\right)\left(dx^{t},dy_{0}^{t}|y^{-1}\right)\otimes\mu(dy^{-1}) \\ &\geq \frac{1}{n+1}\sum_{t=0}^{n}\int_{\mathcal{X}^{t}\times\mathcal{Y}^{t}}\log\left(\frac{Q_{t}^{*}(dy_{t}|y^{t-1},x^{t})}{\Pi_{t}^{\overrightarrow{Q}^{*}}(dy_{t}|y^{t-1})}\right)\left(P_{0,t}\otimes\overrightarrow{Q}_{0,t}\right)\left(dx^{t},dy_{0}^{t}|y^{-1}\right)\otimes\mu(dy^{-1}) \\ &- s\frac{1}{n+1}\sum_{t=0}^{n}\int_{\mathcal{X}^{t}\times\mathcal{Y}^{t}}\rho_{t}\left(T^{t}x^{n},T^{t}y^{n}\right)(P_{0,t}\otimes\overrightarrow{Q}_{0,t})\left(dx^{t},dy_{0}^{t}|y^{-1}\right)\otimes\mu(dy^{-1}) \\ &+ \frac{1}{n+1}\sum_{t=0}^{n}\int_{\mathcal{X}^{t}\times\mathcal{Y}^{t}}g_{t,n}(x^{t},y^{t})(P_{0,t}\otimes\overrightarrow{Q}_{0,t})\left(dx^{t},dy_{0}^{t}|y^{-1}\right)\otimes\mu(dy^{-1}) \\ &- \frac{1}{n+1}\sum_{t=0}^{n}\int_{\mathcal{X}^{t}\times\mathcal{Y}^{t}}\log\left(\lambda_{t}(x^{t},y^{t-1})\right)\left(P_{0,t}\otimes\overrightarrow{Q}_{0,t}\right)\left(dx^{t},dy_{0}^{t}|y^{-1}\right)\otimes\mu(dy^{-1}) \\ &= \frac{1}{n+1}\sum_{t=0}^{n}\int_{\mathcal{X}^{t-1}\times\mathcal{Y}^{t-1}}\left\{\int_{\mathcal{X}_{t}\times\mathcal{Y}^{t}}\log\left(\frac{Q_{t}^{*}(dy_{t}|y^{t-1},x^{t})e^{-s\rho_{t}(T^{t}x^{n},T^{t}y^{n})+g_{t,n}(x^{t},y^{t})}{\Pi_{t}^{\overrightarrow{Q}^{*}}(dy_{t}|y^{t-1})\lambda_{t}(x^{t},y^{t-1})}\right) \\ &Q_{t}^{*}(dy_{t}|y^{t-1},x^{t})\otimes P_{t}(dx_{t}|x^{t-1})\right\}\left(P_{0,t-1}\otimes\overrightarrow{Q}_{0,t-1}\right)\left(dx^{t-1},dy_{0}^{t-1}|y^{-1})\lambda_{t}(x^{t},y^{t-1})\right) \\ &Q_{t}^{*}(dy_{t}|y^{t-1},x^{t})\otimes P_{t}(dx_{t}|x^{t-1})\right\}\left(P_{0,t-1}\otimes\overrightarrow{Q}_{0,t-1}\right)\left(dx^{t-1},dy_{0}^{t-1}|y^{-1})\lambda_{t}(x^{t},y^{t-1})\right) \\ &Q_{t}^{*}(dy_{t}|y^{t-1},x^{t})\otimes P_{t}(dx_{t}|x^{t-1})\right\}\left(P_{0,t-1}\otimes\overrightarrow{Q}_{0,t-1}\right)\left(dx^{t-1},dy_{0}^{t-1}|y^{-1})\lambda_{t}(x^{t},y^{t-1})\right) \\ &Q_{t}^{*}(dy_{t}|y^{t-1},x^{t})\otimes P_{t}(dx_{t}|x^{t-1})\right\}\left(P_{0,t-1}\otimes\overrightarrow{Q}_{0,t-1}\right)\left(dx^{t-1},dy_{0}^{t-1}|y^{-1})\right)\right)\right)\right\} \\ &= \frac{1}{n+1}\sum_{t=0}^{n}\left\{1-\int_{\mathcal{Y}^{t}}\Pi_{0}^{\overrightarrow{Q}^{*}}(dy_{0}^{t}|y^{-1})\otimes\mu(dy^{-1})\right. \\ &\left(\int_{\mathcal{X}^{t-1}}\int_{\mathcal{X}^{t}}e^{s\rho_{t}(T^{t}x^{n},T^{t}y^{n})-g_{t,n}(x^{t},y^{t})\lambda_{t}(x^{t},y^{t-1})P_{t}(dx_{t}|x^{t-1})\otimes\mathbf{P}_{t}^{\overrightarrow{Q}^{*}}(dx^{t-1}|y^{t-1})\right)\right)\right\}$$

where (a) follows from the inequality $\log x \ge 1 - \frac{1}{x}$, x > 0, and (b) follows from (4.9).

Hence, we obtain

$$\begin{split} &R_{0,n}^{na}(D) \stackrel{(c)}{\geq} \\ &\sup_{s \le 0} \sup_{\lambda \in \Psi_s} \left\{ sD - \frac{1}{n+1} \sum_{t=0}^n \int_{\mathcal{X}^t \times \mathcal{Y}^t} g_{t,n}(x^t, y^t) (P_{0,t} \otimes \overrightarrow{Q}_{0,t}) \left(dx^t, dy_0^t | y^{-1} \right) \otimes \mu(dy^{-1}) \right. \\ &+ \frac{1}{n+1} \sum_{t=0}^n \int_{\mathcal{X}^t \times \mathcal{Y}^{t-1}} \log \left(\lambda_t(x^t, y^{t-1}) \right) P_t(dx_t | x^{t-1}) (P_{0,t-1} \otimes \overrightarrow{Q}_{0,t-1}) \left(dx^{t-1}, dy_0^{t-1} | y^{-1} \right) \otimes \mu(dy^{-1}) \right\} \end{split}$$

However, equality in (c) holds if

$$\lambda_t(x^t, y^{t-1}) \triangleq \left(\int_{\mathcal{Y}_t} e^{s\rho_t(T^t x^n, T^t y^n) - g_{t,n}(x^t, y^t)} \Pi_t^{\overrightarrow{Q}^*}(dy_t | y^{t-1}) \right)^{-1}, \ \forall t \in \mathbb{N}_0^n$$

This completes the proof.

Appendix B. Proof of Theorem 5.2.

(1) The derivation is based on the fact that the feedback realization scheme of Fig. 5.1 is generally an upper bound on the NRDF, $R_{0,n}^{na}(D)$, of the Gaussian process, and this realization gives (5.8). The achievability of this upper bound is established by evaluating the lower bound in (5.8) which is done recursively moving backward in time, utilizing the expression we obtained in Theorem 4.2.

Upper Bound. First, consider the realization of Fig. 5.1. Define $\{H_t : t \in \mathbb{N}_0^n\}$ as in (5.16). By Fig. 5.1, we obtain

(B.1)

$$\tilde{K}_{t} = E_{t}^{^{\mathrm{T}}} H_{t} E_{t} \left(X_{t} - \mathbb{E} \left\{ X_{t} | \sigma \{ Y^{t-1} \} \right\} \right) + E_{t}^{^{\mathrm{T}}} \Phi_{t} V_{t}^{c} = E_{t}^{^{\mathrm{T}}} H_{t} E_{t} K_{t} + E_{t}^{^{\mathrm{T}}} \Phi_{t} V_{t}^{c}, \ t \in \mathbb{N}_{0}^{n}$$

where $\{V_t^c : t \in \mathbb{N}_0^n\}$ is a zero mean independent Gaussian process with covariance $\operatorname{Cov}(V_t^c) = Q_t = \operatorname{diag}\{q_{t,1}, \ldots, q_{t,p}\}$, and $\{\Phi_t : t \in \mathbb{N}_0^n\}$ is to be determined. Next, we show that by letting $\Phi_t = \sqrt{H_t \Delta_t Q_t^{-1}}$, and $\Delta_t \triangleq \operatorname{diag}\{\delta_{t,1}, \ldots, \delta_{t,p}\}$, then $\Pi_{t|t-1} = \mathbb{E}\left\{K_t K_t^{\mathrm{T}}\right\}$, and also $\frac{1}{n+1}\mathbb{E}\left\{\sum_{t=0}^n ||X_t - Y_t||_2^2\right\} = \frac{1}{n+1}\mathbb{E}\left\{\sum_{t=0}^n ||K_t - \tilde{K}_t||_2^2\right\} = D$. Clearly, by (5.3), (5.5), and (B.1), we obtain

$$\frac{1}{n+1} \sum_{t=0}^{n} \mathbb{E} \left\{ \left(X_{t} - Y_{t} \right)^{^{\mathrm{T}}} \left(X_{t} - Y_{t} \right) \right\} = \frac{1}{n+1} \sum_{t=0}^{n} \operatorname{trace} \left(\mathbb{E} \left\{ \left(K_{t} - \tilde{K}_{t} \right) \left(K_{t} - \tilde{K}_{t} \right)^{^{\mathrm{T}}} \right\} \right) \\
= \frac{1}{n+1} \sum_{t=0}^{n} \operatorname{trace} \mathbb{E} \left\{ \left(K_{t} - E_{t}^{^{\mathrm{T}}} H_{t} E_{t} K_{t} - E_{t}^{^{\mathrm{T}}} \Phi_{t} V_{t}^{c} \right) \left(K_{t} - E_{t}^{^{\mathrm{T}}} H_{t} E_{t} K_{t} - E_{t}^{^{\mathrm{T}}} \Phi_{t} V_{t}^{c} \right)^{^{\mathrm{T}}} \right\} \\
= \frac{1}{n+1} \sum_{t=0}^{n} \operatorname{trace} \left\{ E_{t}^{^{\mathrm{T}}} \left(\left(I - H_{t} \right) \operatorname{diag}(\lambda_{t,1}, \dots, \lambda_{t,p}) \left(I - H_{t} \right)^{^{\mathrm{T}}} + \left(\Phi_{t} Q_{t} \Phi_{t}^{^{\mathrm{T}}} \right) \right) E_{t} \right\} \\
\stackrel{(a)}{=} \frac{1}{n+1} \sum_{t=0}^{n} \operatorname{trace} \left\{ \operatorname{diag}(\delta_{t,1}, \dots, \delta_{t,p}) \right\} = D,$$

where (a) holds by setting Φ_t as in (5.16). By (5.7), the NRDF can be written as

follows:

where (b) follows from the fact that conditioning reduces entropy (see also [15, Lemma V.1, Remark V.2]), (c) follows again from the fact that $\tilde{K}_t = E_t^{\mathrm{T}} H_t E_t K_t + E_t^{\mathrm{T}} \Phi_t V_t^c$ is a memoryless Gaussian channel, and (d) follows from the orthogonality of K_t and V_t^c . Actually, by [15, Lemma V.1, Remark V.2], it can be shown that the inequalities (b), (c), (d) are equalities.

Next, we compute the entropies appearing in (B.3) from the covariances of the corresponding processes. The covariance of the Gaussian zero mean term $E_t^{\mathsf{T}} \Phi_t V_t^c$, $t \in \mathbb{N}_0^n$, is given by

The covariance of $\tilde{K}_t, t \in \mathbb{N}_0^n$, is given by

$$\mathbb{E}\left\{\tilde{K}_{t}\tilde{K}_{t}^{^{\mathrm{T}}}\right\} = \mathbb{E}\left\{\left(E_{t}^{^{\mathrm{T}}}H_{t}E_{t}K_{t} + E_{t}^{^{\mathrm{T}}}\Phi_{t}V_{t}^{c}\right)\left(E_{t}^{^{\mathrm{T}}}H_{t}E_{t}K_{t} + E_{t}^{^{\mathrm{T}}}\Phi_{t}V_{t}^{c}\right)^{^{\mathrm{T}}}\right\}$$
$$= E_{t}^{^{\mathrm{T}}}\left(\operatorname{diag}\{\eta_{t,1}^{2}\lambda_{t,1},\ldots,\eta_{t,p}^{2}\lambda_{t,p}\} + \operatorname{diag}\{\eta_{t,1}\delta_{t,1},\ldots,\eta_{t,p}\delta_{t,p}\}\right)E_{t}$$
$$(B.5) \qquad = E_{t}^{^{\mathrm{T}}}\operatorname{diag}\{\lambda_{t,1} - \delta_{t,1},\ldots,\lambda_{t,p} - \delta_{t,p}\}E_{t}, \ t \in \mathbb{N}_{0}^{n}.$$

Using (B.5) we obtain the first term of (B.3) as follows⁶

(B.6)
$$\sum_{t=0}^{n} H(\tilde{K}_{t}) = \frac{1}{2} \sum_{t=0}^{n} \sum_{i=1}^{p} \log \left\{ (2\pi e) \left(\lambda_{t,i} - \delta_{t,i} \right)^{+} \right\}.$$

Also, by (B.4), we obtain the second term in (B.3) as follows.

(B.7)
$$\sum_{t=0}^{n} H(E_t^{\mathrm{T}} \Phi_t V_t^c) = \frac{1}{2} \sum_{t=0}^{n} \sum_{i=1}^{p} \log \left\{ (2\pi e) \left(\eta_{t,i} \delta_{t,i} \right) \right\}.$$

This problem can be cast into the following convex optimization problem

(B.8)
$$\min_{\frac{1}{n+1}\sum_{t=0}^{n}\sum_{i=1}^{p}\delta_{\infty,i}=D}\frac{1}{n+1}\sum_{t=0}^{n}\sum_{i=1}^{p}\max\left\{0,\frac{1}{2}\log\left(\frac{\lambda_{t,i}}{\delta_{t,i}}\right)\right\}$$

⁶Note that $(\cdot)^+ \triangleq \max\{0, \cdot\}.$

Since this is a convex optimization problem, we use Lagrange multipliers to construct the following augmented functional

(B.9)
$$J(D) = \frac{1}{2} \sum_{t=0}^{n} \sum_{i=1}^{p} \log\left(\frac{\lambda_{t,i}}{\delta_{t,i}}\right) - s \frac{1}{n+1} \sum_{t=0}^{n} \sum_{i=1}^{p} \delta_{t,i}, \quad s \le 0.$$

Differentiating with respect to $\delta_{t,i}$ and setting equal to zero, we obtain

(B.10)
$$\frac{\partial J}{\partial \delta_{t,i}} = -\frac{1}{2\delta_{t,i}} - s = 0 \Longrightarrow s = -\frac{1}{2\delta_{t,i}} \quad \text{or} \quad \delta_{t,i} = \hat{\xi}, \quad \hat{\xi} \ge 0.$$

Evidently, the optimal information allocation to the various descriptions results in an equal distortion for the components of the time-invariant multidimensional Gauss-Markov process. This is feasible if the constant $\hat{\xi}$ in (B.10) is less than $\lambda_{t,i} \forall t, i$. As the total distortion level increases, the constant $\hat{\xi}$ also increases until it exceeds $\lambda_{t,i}$ for some t, i. If we increase the total distortion, we must use the Karush-Kuhn-Tucker (KKT) conditions [29] to find the minimum in the convex optimization problem (B.8). By applying KKT conditions we obtain

(B.11)
$$\frac{\partial J}{\partial \delta_{t,i}} = -\frac{1}{2\delta_{t,i}} - s, \quad s \le 0$$

where s is chosen so that

(B.12)
$$\frac{\partial J}{\partial \delta_{t,i}} = \begin{cases} 0 & \text{if } \delta_{t,i} \le \lambda_{t,i} \\ \le 0 & \text{if } \delta_{t,i} > \lambda_{t,i} \end{cases}$$

It is easy to verify that the solution of KKT conditions yields

(B.13)
$$\delta_{t,i} \triangleq \begin{cases} \xi & \text{if } \xi \leq \lambda_{t,i} \\ \lambda_{t,i} & \text{if } \xi > \lambda_{t,i} \end{cases}, \ \forall t, t \in \mathbb{R}$$

where ξ is chosen such that $\frac{1}{n+1} \sum_{t=0}^{n} \sum_{i=1}^{p} \delta_{t,i} = D$ and $\delta_{t,i} = \min\{\xi, \lambda_{t,i}\}$. Using (B.6) and (B.7) in (B.3) we have the following upper bound

$$\begin{aligned} R_{0,n}^{na,K^{n},\tilde{K}^{n}}(D) &\leq \frac{1}{n+1} \sum_{t=0}^{n} I(K_{t};\tilde{K}_{t}|,\tilde{K}^{t-1}) \\ &\leq \frac{1}{2} \frac{1}{n+1} \sum_{t=0}^{n} \sum_{i=1}^{p} \log\left\{\frac{(\lambda_{t,i}-\delta_{t,i})^{+}}{\eta_{t,i}\delta_{t,i}}\right\} = \frac{1}{2} \frac{1}{n+1} \sum_{t=0}^{n} \sum_{i=1}^{p} \log\left(\frac{\lambda_{t,i}}{\delta_{t,i}}\right), \end{aligned}$$

where $\delta_{t,i} = \min\{\xi, \lambda_{t,i}\}, t \in \mathbb{N}_0^n, i = 1, \dots, p$, and $\frac{1}{n+1} \sum_{t=0}^n \sum_{i=1}^p \delta_{t,i} = D$. Note that if $\delta_{t,i} = \lambda_{t,i}$, for $t \in \mathbb{N}_0^n$ and $i = 1, \dots, p$, then no data are estimated.

Lower Bound. Here, we apply Theorem 4.2 recursively, to obtain a lower bound for the NRDF, $R_{0,n}^{na}(D) = R_{0,n}^{na,K^n,\tilde{K}^n}(D)$, which is precisely (5.8). Let $\bar{p}(\cdot|,\cdot)$ and $\bar{p}(\cdot)$ denote the conditional and unconditional densities, respectively.

Let $\bar{p}(\cdot|,\cdot)$ and $\bar{p}(\cdot)$ denote the conditional and unconditional densities, respectively. Using the property of $\{\lambda_t(\cdot,\cdot,): t = 0,\ldots,n\}$ corresponding to the fact that $\lambda_t(k^t, \tilde{k}^{t-1}) \equiv \lambda_t(k_t, \tilde{k}^{t-1}), t = 0,\ldots,n$ and by Theorem 4.2, an alternative expression for the NRDF, $R_{0,n}^{na,K^n,\tilde{K}^n}(D)$ is the following. (B 15)

$$(\mathbf{D},\mathbf{I}\mathbf{J})$$

$$R_{0,n}^{na,\mathbf{A}}(D) = \sup_{s \le 0} \sup_{\{\lambda_t(k_t, \tilde{k}^{t-1}) \in \Psi_s^t: t \in \mathbb{N}_0^n\}} \{\text{term-}(0) + \dots + \text{term-}(n-1) + \text{term-}(n)\}$$

where

:

$$\begin{aligned} \text{term-}(0) &\equiv -\frac{1}{n+1} \int_{\mathcal{K}_0} \Big(\int_{\tilde{\mathcal{K}}_0} g_{0,n}(\tilde{k}_0) \bar{p}(\tilde{k}_0 | k_0) d\tilde{k}_0 \Big) \bar{p}(k_0 |) dk_0 + \frac{1}{n+1} \int_{\mathcal{K}_0} \log \Big(\lambda_0(k_0) \Big) \bar{p}(k_0) dk_0 \\ \text{term-}(1) &\equiv -\frac{1}{n+1} \int_{\mathcal{K}_1 \times \tilde{\mathcal{K}}_0} \Big(\int_{\tilde{\mathcal{K}}_1} g_{1,n}(\tilde{k}^1) \bar{p}(\tilde{k}_1 | \tilde{k}_0, k_1) d\tilde{k}_1 \Big) \bar{p}(k_1, \tilde{k}_0) dk_1 d\tilde{k}_0 \\ &+ \frac{1}{n+1} \int_{\mathcal{K}_1 \times \tilde{\mathcal{K}}_0} \log \Big(\lambda_0(k_1, \tilde{k}_0) \Big) \bar{p}(k_1, \tilde{k}_0) dk_1 d\tilde{k}_0 \end{aligned}$$

$$\operatorname{term-(n-2)} \equiv -\frac{1}{n+1} \int_{\mathcal{K}_{n-2} \times \tilde{\mathcal{K}}^{n-3}} \left(\int_{\tilde{\mathcal{K}}_{n-2}} g_{n-2,n}(\tilde{k}^{n-2}) \bar{p}(\tilde{k}_{n-2} | \tilde{k}^{n-3}, k_{n-2}) d\tilde{k}_{n-2} \right) \bar{p}(k_{n-2}, \tilde{k}^{n-3}) dk_{n-2} d\tilde{k}^{n-3} dk_{n-2} d\tilde{k}^{n-3} + \frac{1}{n+1} \int_{\mathcal{K}_{n-2} \times \tilde{\mathcal{K}}^{n-3}} \log \left(\lambda_{n-2}(k_{n-2}, \tilde{k}^{n-3}) \right) \bar{p}(k_{n-2}, \tilde{k}^{n-3}) dk_{n-2} d\tilde{k}^{n-3} dk_{n-2} d\tilde{k}^{n-3} dk_{n-1} d\tilde{k}^{n-2} d\tilde{k}^{n-3} dk_{n-1} d\tilde{k}^{n-2} d\tilde{k}^{n-2}$$

term-(n)
$$\equiv sD + \frac{1}{n+1} \int_{\mathcal{K}_n \times \tilde{\mathcal{K}}^{n-1}} \log\left(\lambda_n(k_n, \tilde{k}^{n-1})\right) \bar{p}(k_n, \tilde{k}^{n-1}) dk_n d\tilde{k}^{n-1}$$

and

(B.16)

$$\Psi_{s}^{t} \triangleq \Big\{\lambda_{t}(k_{t}, \tilde{k}^{t-1}) \ge 0: \int_{\mathcal{K}_{t}} e^{s||k_{t} - \tilde{k}_{t}||_{2}^{2} - g_{t,n}(\tilde{k}^{t})} \lambda_{t}(k_{t}, \tilde{k}^{t-1}) \bar{p}(k_{t}|\tilde{k}^{t-1}) dk_{t} \le 1 \Big\}, \ t \in \mathbb{N}_{0}^{n},$$

(B.17)
$$g_{t,n}(\tilde{k}^n) = 0,$$

(B.17)
$$g_{t,n}(\tilde{k}^t) = -\int_{\mathcal{K}_{t+1}} \log\left(\lambda_{t+1}(k_{t+1}, \tilde{k}^t)\right)^{-1} \bar{p}(k_{t+1}|, \tilde{k}^t) dk_{t+1}, \ t \in \mathbb{N}_0^{n-1}.$$

Clearly, if $g_{t,n}(\tilde{k}^t) = \bar{g}_{t,n}(\tilde{k}^{t-1})$, i.e., it is independent of (\tilde{k}_t) , for $t \in \mathbb{N}_0^{n-1}$, then by Theorem 4.1, the RHS terms in (B.15) involving $g_{t,n}(\cdot, \cdot)$, $t \in \mathbb{N}_0^{n-1}$, will not appear (because the optimal reproduction distribution will not involve such terms). Since $g_{n,n}(\cdot, \cdot) = 0$, by (B.16), (B.17), $\lambda_n(k_n, \tilde{k}^{n-1})$ determines $g_{n-1,n}(\cdot, \cdot), \lambda_{n-1}(\cdot, \cdot)$ determines $g_{n-2,n}(\cdot, \cdot)$ and so on, and the RHS of (B.15) involves supremum over $\{\lambda_t(\cdot, \cdot): t \in \mathbb{N}_0^n\}$, then any choice of $\{\lambda_t(\cdot, \cdot): t \in \mathbb{N}_0^n\}$ gives a lower bound. The main idea, implemented below, uses the property of distortion function, and the source distribution, to show that $\{\lambda_t(\cdot, \cdot): t \in \mathbb{N}_0^n\}$ can be chosen so that $g_{t,n}(\tilde{k}^t) = \bar{g}_{t,n}(\tilde{k}^{t-1}), t \in \mathbb{N}_0^{n-1}$, giving a lower bound which is achievable, and that the optimal reproduction distribution is of the form

$$\bar{p}(\tilde{k}_t|, \tilde{k}^{t-1}, k_t) = \frac{e^{s||k_t - \tilde{k}_t||_2^2} \bar{p}(\tilde{k}_t|\tilde{k}^{t-1})}{\int_{\tilde{\mathcal{K}}_t} e^{s||k_t - \tilde{k}_t||_2^2} \bar{p}(\tilde{k}_t|\tilde{k}^{t-1})}.$$

Step t = **n.** The set Ψ_s^n is defined as follows:

(B.18)
$$\Psi_s^n \triangleq \Big\{ \lambda_n(k_n, \tilde{k}^{n-1}) \ge 0 : \int_{\mathcal{K}_n} e^{s||k_n - \tilde{k}_n||_2^2} \lambda_n(k_n, \tilde{k}^{n-1}) \bar{p}(k_n|\tilde{k}^{n-1}) dk_n \le 1 \Big\},$$

where $\bar{p}(k_n|\tilde{k}^{n-1})$ denotes the conditional density of k_n given (\tilde{k}^{n-1}) . Take $\lambda_n(k_n, \tilde{k}^{n-1}) \in \Psi^n_s$ such that

(B.19)
$$\lambda_n(k_n, \tilde{k}^{n-1}) = \frac{\alpha_n}{\bar{p}(k_n | \tilde{k}^{n-1})}$$

for some α_n not depending on k_n , and substitute (B.19) into the integral inequality in (B.18) to obtain

$$\alpha_n \int_{\mathcal{K}_n} e^{s||k_n - \tilde{k}_n||_2^2} dk_n \le 1.$$

By change of variable of integration then

(B.20)
$$\alpha_n \int_{-\infty}^{\infty} e^{s||z_n||_2^2} dz_n = \alpha_n \sqrt{\left(-\frac{\pi}{s}\right)^p} = \alpha_n \left(-\frac{\pi}{s}\right)^{\frac{p}{2}} \le 1,$$

where "s" is the non-positive Lagrange multiplier.

Moreover, α_n is chosen so that the inequality of (B.20) holds with equality, giving

(B.21)
$$\alpha_n = \frac{1}{\int e^{s||z_n||_2^2} dz_n} = \left(-\frac{s}{\pi}\right)^{\frac{p}{2}}, \ \lambda_n(k_n, \tilde{k}^{n-1}) = \frac{(-\frac{s}{\pi})^{p/2}}{\bar{p}_n(k_n|\tilde{k}^{n-1})}.$$

Substituting (B.21) into the term-(n) of (B.15) gives

$$\underline{\operatorname{term-}(\mathbf{n})} = sD + \frac{1}{n+1}\log\alpha_n - \frac{1}{n+1}\int_{\mathcal{K}_n\times\tilde{\mathcal{K}}^{n-1}}\log\left(\bar{p}(k_n|\tilde{k}^{n-1})\right)\bar{p}(k_n,\tilde{k}^{n-1})dk_nd\tilde{k}^{n-1}$$
(B.22)

$$= sD + \frac{1}{n+1} \log\left(-\frac{s}{\pi}\right)^{\frac{p}{2}} + \frac{1}{n+1} H(K_n | \tilde{K}^{n-1}).$$

The choice of $\lambda_n(\cdot, \cdot)$ given by (B.21) determines $g_{n-1,n}(\cdot)$ given by

$$g_{n-1,n}(\tilde{k}^{n-1}) = -\int_{\mathcal{K}_n} \bar{p}(dk_n | \tilde{k}^{n-1}) \log\left(\lambda_n(k_n, \tilde{k}^{n-1})\right)^{-1}$$

$$\stackrel{(a)}{=} -\int_{\mathcal{K}_n} \bar{p}(dk_n | \tilde{k}^{n-1}) \log\left(\frac{\bar{p}(k_n | \tilde{k}^{n-1})}{\alpha_n}\right)$$

$$= \log \alpha_n + H(K_n | \tilde{K}^{n-1} = \tilde{k}^{n-1}), \ \alpha_n = \left(-\frac{s}{\pi}\right)^{\frac{p}{2}}$$

$$\stackrel{(b)}{\leq} \log \alpha_n + H(K_n | \tilde{K}^{n-2} = \tilde{k}^{n-2})$$

$$= \log\left(-\frac{s}{\pi}\right)^{\frac{p}{2}} + H(K_n | \tilde{K}^{n-2} = \tilde{k}^{n-2}) \equiv \bar{g}_{n-1,n}(\tilde{k}^{n-2})$$

where (a) follows from the fact that $\left(\lambda_n(k_n, \tilde{k}^{n-1})\right)^{-1} = \frac{\bar{p}(k_n|\tilde{k}^{n-1})}{\alpha_n}$, and (b) from the fact that conditioning reduces entropy.

When the upper bound in (B.23) is substituted into the second expression of term-(n-1) of (B.15) involving $g_{n-1,n}(\cdot)$, it gives

$$-\frac{1}{n+1}\int_{\mathcal{K}_{n-1}\times\tilde{\mathcal{K}}^{n-2}} \left(\int_{\tilde{\mathcal{K}}_{n-1}} g_{n-1,n}(\tilde{k}^{n-1})\bar{p}(\tilde{k}_{n-1}|\tilde{k}^{n-2},k_{n-1})d\tilde{k}_{n-1}\right)\bar{p}(k_{n-1},\tilde{k}^{n-2})dk_{n-1}d\tilde{k}^{n-2}$$

$$\geq -\frac{1}{n+1}\int_{\mathcal{K}_{n-1}\times\tilde{\mathcal{K}}^{n-2}} \left(\int_{\tilde{\mathcal{K}}_{n-1}} \bar{g}_{n-1,n}(\tilde{k}^{n-2})\bar{p}(\tilde{k}_{n-1}|\tilde{k}^{n-2},k_{n-1})d\tilde{k}_{n-1}\right)\bar{p}(k_{n-1},\tilde{k}^{n-2})dk_{n-1}d\tilde{k}^{n-2}$$

Step t = **n** - **1.** The set Ψ_s^{n-1} is defined as follows (using $g_{n-1,n}(\tilde{k}^{n-1}) \equiv \bar{g}_{n-1,n}(\tilde{k}^{n-2})$ given by (B.23) obtained in step t = n)

$$\begin{split} \Psi_s^{n-1} &\triangleq \Big\{ \lambda_{n-1}(k_{n-1}, \tilde{k}^{n-2}) \ge 0 : \\ \text{(B.24)} &\int_{\mathcal{K}_{n-1}} e^{s||k_{n-1} - \tilde{k}_{n-1}||_2^2 - \bar{g}_{n-1,n}(\tilde{k}^{n-2})} \lambda_{n-1}(k_{n-1}, \tilde{k}^{n-2}) \bar{p}(k_{n-1}|\tilde{k}^{n-2}) dk_{n-1} \le 1 \Big\}. \end{split}$$

Take $\lambda_{n-1}(k_{n-1}, \tilde{k}^{n-2}) \in \Psi_s^{n-1}$ such that

(B.25)
$$\lambda_{n-1}(k_{n-1}, \tilde{k}^{n-2}) = \frac{\alpha_{n-1}(k^{n-2})}{\bar{p}(k_{n-1}|\tilde{k}^{n-2})}$$

for some $\alpha_{n-1}(\tilde{k}^{n-2})$ not depending on k_{n-1} , and substitute (B.25) into the integral inequality in (B.24) to obtain

$$\alpha_{n-1}(\tilde{k}^{n-2})e^{-\bar{g}_{n-1,n}(\tilde{k}^{n-2})}\int_{\mathcal{K}_{n-1}}e^{s||k_{n-1}-\tilde{k}_{n-1}||_{2}^{2}}dk_{n-1}\leq 1.$$

By change of variable of integration then

$$\alpha_{n-1}(\tilde{k}^{n-2})e^{-\bar{g}_{n-1,n}(\tilde{k}^{n-2})}\int_{-\infty}^{\infty}e^{s||z_{n-1}||_{2}^{2}}dz_{n-1} = \alpha_{n-1}(\tilde{k}^{n-2})e^{-\bar{g}_{n-1,n}(\tilde{k}^{n-2})}\left(-\frac{\pi}{s}\right)^{\frac{p}{2}} \le 1.$$

Hence,

(B.26)
$$\alpha_{n-1}(\tilde{k}^{n-2}) \left(-\frac{\pi}{s}\right)^{\frac{p}{2}} \le e^{\bar{g}_{n-1,n}(\tilde{k}^{n-2})}.$$

Moreover, $\alpha_{n-1}(\cdot)$ is chosen so that the inequality in (B.26) holds with equality, giving

(B.27)
$$\alpha_{n-1}(\tilde{k}^{n-2}) = \frac{e^{\bar{g}_{n-1,n}(\tilde{k}^{n-2})}}{\left(-\frac{\pi}{s}\right)^{\frac{p}{2}}} = e^{\log\alpha_n + H(K_n|\tilde{K}^{n-2} = \tilde{k}^{n-2})} \left(-\frac{s}{\pi}\right)^{\frac{p}{2}} = e^{(1-\frac{s}{2})^p} e^{H(K_n|\tilde{K}^{n-2} = \tilde{k}^{n-2})},$$

where (c) holds due to (B.23). Therefore, (B.25) is given by

(B.28)
$$\lambda_{n-1}(k_{n-1},\tilde{k}^{n-1}) = \frac{\left(-\frac{s}{\pi}\right)^p e^{H(K_n|\tilde{K}^{n-2}=\tilde{k}^{n-2})}}{\bar{p}(k_{n-1}|\tilde{k}^{n-2})}$$

Substituting (B.28) into the term-(n-1) of (B.15) gives

$$\underbrace{\operatorname{term-(n-1)}}_{[k]} \stackrel{(d)}{=} - \frac{1}{n+1} \int_{\mathcal{K}_{n-1} \times \tilde{\mathcal{K}}^{n-2}} \left(\int_{\tilde{\mathcal{K}}_{n-1}} \bar{g}_{n-1,n}(\tilde{k}^{n-2}) \bar{p}(\tilde{k}_{n-1} | \tilde{k}^{n-2}, k_{n-1}) d\tilde{k}_{n-1} \right) \\ \times \bar{p}(k_{n-1}, \tilde{k}^{n-2} |) dk_{n-1} d\tilde{k}^{n-2} \\ (B.29) \qquad + \frac{1}{n+1} \int_{\mathcal{K}_{n-1} \times \tilde{\mathcal{K}}^{n-2}} \log \left(\lambda_{n-1}(k_{n-1}, \tilde{k}^{n-2}) \right) \bar{p}(k_{n-1}, \tilde{k}^{n-2}) dk_{n-1} d\tilde{k}^{n-2} \\ \stackrel{(e)}{=} -\frac{1}{n+1} \log \left(-\frac{s}{\pi} \right)^{\frac{p}{2}} - \frac{1}{n+1} H(K_n | \tilde{K}^{n-2}) \\ + \frac{1}{n+1} \int_{\mathcal{K}_{n-1} \times \tilde{\mathcal{K}}^{n-2}} \log \left(\frac{\alpha_{n-1}(\tilde{k}^{n-2})}{\bar{p}(k_{n-1} | \tilde{k}^{n-2})} \right) \bar{p}(k_{n-1}, \tilde{k}^{n-2}) dk_{n-1} d\tilde{k}^{n-2} \\ = -\frac{1}{n+1} \log \left(-\frac{s}{\pi} \right)^{\frac{p}{2}} - \frac{1}{n+1} H(K_n | \tilde{K}^{n-2}) + \frac{1}{n+1} \log \left(-\frac{s}{\pi} \right)^p \\ + \frac{1}{n+1} H(K_n | \tilde{K}^{n-2}) + \frac{1}{n+1} H(K_{n-1} | \tilde{K}^{n-2}) \\ (B.30) \qquad = \frac{1}{n+1} \log \left(-\frac{s}{\pi} \right)^{\frac{p}{2}} + \frac{1}{n+1} H(K_{n-1} | \tilde{K}^{n-2}),$$

where (d) follows from the fact that $g_{n-1,n}(\tilde{k}^{n-1}) \leq \bar{g}_{n-1,n}(\tilde{k}^{n-2})$ (see (B.23)) and (e) follows by substituting (B.23) and (B.25) into the second and third expression of (B.29), respectively.

The choice of $\lambda_{n-1}(\cdot, \cdot)$ (given by (B.28)) determines $g_{n-2,n}(\cdot)$ given by

$$g_{n-2,n}(\tilde{k}^{n-2}) = -\int_{\mathcal{K}_{n-1}} \bar{p}(k_{n-1}|, \tilde{k}^{n-2}) \log\left(\lambda_{n-1}(k_{n-1}, \tilde{k}^{n-2})\right)^{-1}$$

$$\stackrel{(f)}{=} -\int_{\mathcal{K}_{n-1}} \bar{p}(k_{n-1}|\tilde{k}^{n-2}) \log\left(\frac{\bar{p}(k_{n-1}|\tilde{k}^{n-2})}{\alpha_{n-1}(\tilde{k}^{n-2})}\right), \ \alpha_{n-1}(\tilde{k}^{n-2}) = \left(-\frac{s}{\pi}\right)^{p} e^{H(K_{n}|\tilde{K}^{n-2}=\tilde{k}^{n-2})}$$

$$= \log\left(\alpha_{n-1}(\tilde{k}^{n-2})\right) - \int_{\mathcal{K}_{n-1}} \bar{p}(k_{n-1}|\tilde{k}^{n-2}) \log\left(\bar{p}(k_{n-1}|\tilde{k}^{n-2})\right)$$

$$= \log\left(-\frac{s}{\pi}\right)^{p} + H(K_{n}|\tilde{K}^{n-2}=\tilde{k}^{n-2}) + H(K_{n-1}|\tilde{K}^{n-2}=\tilde{k}^{n-2})$$

$$\stackrel{(g)}{\leq} \log\left(-\frac{s}{\pi}\right)^{p} + H(K_{n}|, \tilde{K}^{n-3}=\tilde{k}^{n-3}) + H(K_{n-1}|\tilde{K}^{n-3}=\tilde{k}^{n-3})$$

$$\equiv \bar{g}_{n-2,n}(\tilde{k}^{n-3}),$$

where (f) follows from the fact that $\left(\lambda_{n-1}(k_{n-1},\tilde{k}^{n-2})\right)^{-1} = \frac{\bar{p}(k_{n-1}|\tilde{k}^{n-2})}{\alpha_{n-1}(\tilde{k}^{n-2})}$, and (g) follows from the fact that conditioning reduces entropy.

When the upper bound in (B.31) is substituted into the second expression of term-(n-2) of (B.15) involving $g_{n-2,n}(\cdot)$, it gives

$$-\frac{1}{n+1}\int_{\mathcal{K}_{n-2}\times\tilde{\mathcal{K}}^{n-3}}\left(\int_{\tilde{\mathcal{K}}_{n-2}}g_{n-2,n}(\tilde{k}^{n-2})\bar{p}(\tilde{k}_{n-2}|\tilde{k}^{n-3},k_{n-2})d\tilde{k}_{n-2}\right)\bar{p}(k_{n-2},\tilde{k}^{n-3})dk_{n-2}d\tilde{k}^{n-3}$$

$$\geq -\frac{1}{n+1}\int_{\mathcal{K}_{n-2}\times\tilde{\mathcal{K}}^{n-3}}\left(\int_{\tilde{\mathcal{K}}_{n-2}}\bar{g}_{n-2,n}(\tilde{k}^{n-3})\bar{p}(\tilde{k}_{n-2}|\tilde{k}^{n-3},k_{n-2})d\tilde{k}_{n-2}\right)\bar{p}(k_{n-2},\tilde{k}^{n-3})dk_{n-2}d\tilde{k}^{n-3}$$

Step t = **n** - **2.** The set Ψ_s^{n-2} is defined as follows (using $g_{n-2,n}(\tilde{k}^{n-2}) \equiv \bar{g}_{n-2,n}(\tilde{k}^{n-3})$ given by (B.31) obtained in step t = n - 1).

(B.32)

$$\Psi_{s}^{n-2} \triangleq \Big\{ \lambda_{n-2}(k_{n-2}, \tilde{k}^{n-3}) \ge 0 : \\ \int_{\mathcal{K}_{n-2}} e^{s||k_{n-2} - \tilde{k}_{n-2}||_{2}^{2} - \bar{g}_{n-2,n}(\tilde{k}^{n-3})} \lambda_{n-2}(k_{n-2}, \tilde{k}^{n-3}) \bar{p}(k_{n-2}|\tilde{k}^{n-3}) dk_{n-2} \le 1 \Big\}.$$

Take $\lambda_{n-2}(k_{n-2}, \tilde{k}^{n-3}) \in \Psi_s^{n-2}$ such that

(B.33)
$$\lambda_{n-2}(k_{n-2}, \tilde{k}^{n-3}) = \frac{\alpha_{n-2}(\tilde{k}^{n-3})}{\bar{p}(k_{n-2}|\tilde{k}^{n-3})}$$

for some $\alpha_{n-2}(\tilde{k}^{n-3})$ not depending on k_{n-2} , and substitute (B.33) into the integral inequality in (B.32) to obtain

$$\alpha_{n-2}(\tilde{k}^{n-3})e^{-\bar{g}_{n-2,n}(\tilde{k}^{n-3})}\int_{\mathcal{K}_{n-2}}e^{s||k_{n-2}-\tilde{k}_{n-2}||_{2}^{2}}dk_{n-2}\leq 1.$$

By change of variable of integration then

$$\alpha_{n-2}(\tilde{k}^{n-3})e^{-\bar{g}_{n-2,n}(\tilde{k}^{n-3})}\int_{-\infty}^{\infty}e^{s||z_{n-2}||_{2}^{2}}dz_{n-2} = \alpha_{n-2}(\tilde{k}^{n-3})e^{-\bar{g}_{n-2,n}(\tilde{k}^{n-3})}\left(-\frac{\pi}{s}\right)^{\frac{p}{2}} \le 1.$$

Hence,

(B.34)
$$\alpha_{n-2}(\tilde{k}^{n-3})\left(-\frac{\pi}{s}\right)^{\frac{p}{2}} \le e^{\bar{g}_{n-2,n}(\tilde{k}^{n-3})}.$$

Moreover, $\alpha_{n-2}(\cdot)$ is chosen so that the inequality in (B.34) holds with equality, giving

$$\alpha_{n-2}(\tilde{k}^{n-3}) = \frac{e^{\bar{g}_{n-2,n}(\tilde{k}^{n-3})}}{\int e^{s||z_{n-2}||_2^2} dz_{n-2}} = e^{\log\alpha_{n-1}(\tilde{k}^{n-2}) + H(K_{n-1}|\tilde{K}^{n-3} = \tilde{k}^{n-3})} \left(-\frac{s}{\pi}\right)^{\frac{p}{2}}$$
(B.35)
$$= \left\{ \left(-\frac{s}{\pi}\right)^{\frac{p}{2}} \right\}^3 e^{H(K_n|\tilde{K}^{n-3} = \tilde{k}^{n-3}) + H(K_{n-1}|\tilde{K}^{n-3} = \tilde{k}^{n-3})}.$$

Therefore, (B.33) is given by

(B.36)
$$\lambda_{n-2}(k_{n-2}, \tilde{k}^{n-3}) = \frac{\left\{\left(-\frac{s}{\pi}\right)^{\frac{p}{2}}\right\}^3 e^{H(K_n|\tilde{K}^{n-3}=\tilde{k}^{n-3})+H(K_{n-1}|\tilde{K}^{n-3}=\tilde{k}^{n-3})}{\bar{p}(k_{n-2}|\tilde{k}^{n-3})}.$$

Substituting (B.36) into term-(n-2) of (B.15) gives

$$\frac{Term - (n-2):}{\geq} \int_{\mathcal{K}_{n-2} \times \tilde{\mathcal{K}}^{n-3}} \left(\int_{\tilde{\mathcal{K}}_{n-2}} \bar{g}_{n-2,n}(\tilde{k}^{n-3}) \bar{p}(\tilde{k}_{n-2} | \tilde{k}^{n-3}, k_{n-2}) d\tilde{k}_{n-2} \right) \\
\bar{p}(k_{n-2}, \tilde{k}^{n-3}) dk_{n-2} d\tilde{k}^{n-3} \\
(B.37) + \int_{\mathcal{K}_{n-2} \times \tilde{\mathcal{K}}^{n-3}} \log \left(\lambda_{n-2}(k_{n-2}, \tilde{k}^{n-3}) \right) \bar{p}(k_{n-2}, \tilde{k}^{n-3}) dk_{n-2} d\tilde{k}^{n-3} \\
\stackrel{(i)}{=} -\frac{1}{n+1} \log \left\{ \left(-\frac{s}{\pi} \right)^{\frac{p}{2}} \right\}^2 - H(K_n | \tilde{K}^{n-3}) - H(K_{n-1} | \tilde{K}^{n-3}) \\
+ \frac{1}{n+1} \int_{\mathcal{K}_{n-2} \times \tilde{\mathcal{K}}^{n-3}} \log \left(\alpha_{n-2}(\tilde{k}^{n-3}) \right) \bar{p}(k_{n-2}, \tilde{k}^{n-3}) dk_{n-2} d\tilde{k}^{n-3}$$

$$(B.38) - \frac{1}{n+1} \int_{\mathcal{K}_{n-2} \times \tilde{\mathcal{K}}^{n-3}} \log\left(\bar{p}(k_{n-2}|\tilde{k}^{n-3})\right) \bar{p}(k_{n-2}, \tilde{k}^{n-3}) dk_{n-2} d\tilde{k}^{n-3}$$

$$= -\frac{1}{n+1} \log\left\{\left(-\frac{s}{\pi}\right)^{\frac{p}{2}}\right\}^2 - \frac{1}{n+1} H(K_n|\tilde{K}^{n-3})$$

$$-\frac{1}{n+1} H(K_{n-1}|\tilde{K}^{n-3}) + \frac{1}{n+1} \log\left\{\left(-\frac{s}{\pi}\right)^{\frac{p}{2}}\right\}^3$$

$$+ H(K_n|\tilde{K}^{n-3}) + \frac{1}{n+1} H(K_{n-1}|\tilde{K}^{n-3}) + \frac{1}{n+1} H(K_{n-2}|\tilde{K}^{n-3})$$

$$= \frac{1}{n+1} \log\left(-\frac{s}{\pi}\right)^{\frac{p}{2}} + \frac{1}{n+1} H(K_{n-2}|\tilde{K}^{n-3}),$$

where (h) follows from the fact that $g_{n-2,n}(\tilde{k}^{n-2}) \leq \bar{g}_{n-2,n}(\tilde{k}^{n-3})$ (see (B.31)), and (i) follows by substituting (B.31) and (B.33) into the second and third expression of (B.37), respectively.

By applying induction, we obtain the following lower bound for the NRDF.

where (j) follows from the fact that

$$H(K_t | \tilde{K}^{t-1}) = H(X_t - \mathbb{E} \{ X_t | \sigma \{ K^{t-1} \} \} | \tilde{K}^{t-1})$$

= $H(X_t | \tilde{K}^{t-1}) = H(X_t) = \frac{1}{2} \sum_{t=0}^n \log 2\pi e |\Lambda_t|.$

Next, we show how to find the Lagrangian multiplier "s" so that the lower bound (B.39) equals $\frac{1}{2} \sum_{t=0}^{n} \sum_{i=1}^{p} \log\left(\frac{\lambda_{t,i}}{\delta_{t,i}}\right)$. To this end, we need to ensure existence of some s < 0 such that the following identity holds.

$$sD + \frac{1}{2}\frac{1}{n+1}\sum_{n=0}^{n}\sum_{i=1}^{p}\log\left(-\frac{s}{\pi}\right) + \frac{1}{2}\frac{1}{n+1}\sum_{t=0}^{n}\log 2\pi e|\Lambda_{t}| = \frac{1}{2}\frac{1}{n+1}\sum_{t=0}^{n}\sum_{i=1}^{p}\log\left(\frac{\lambda_{t,i}}{\delta_{t,i}}\right).$$

After some algebra, the previous expression can be simplified into the following expression.

$$\frac{1}{2}\log e^{2s\frac{1}{(n+1)}\sum_{t=0}^{n}\sum_{i=1}^{p}\delta_{t,i}} + \frac{1}{2}\frac{1}{n+1}\sum_{t=0}^{n}\sum_{i=1}^{p}\log\left(-\frac{s}{\pi}\right) = \frac{1}{2}\frac{1}{n+1}\sum_{t=0}^{n}\sum_{i=1}^{p}\log\frac{1}{2\pi e\delta_{t,i}}.$$

In turn, from the equation above we obtain

$$\frac{1}{2}\frac{1}{n+1}\sum_{t=0}^{n}\sum_{i=1}^{p}\log e^{2s\delta_{t,i}}\left(-\frac{s}{\pi}\right) = \frac{1}{2}\frac{1}{n+1}\sum_{t=0}^{n}\sum_{i=1}^{p}\log\frac{1}{2\pi e\delta_{t,i}} \Longrightarrow \delta_{t,i} = -\frac{1}{2s},$$

where $\delta_{t,i} = \{\xi, \lambda_{t,i}\}$. Now, if $\delta_{t,i} = \xi$ then $\delta_{t,i} = -\frac{1}{2s}$ and the NRDF is bounded below by the following expression

$$R_{0,n}^{na,K^n,\tilde{K}^n}(D) \ge \frac{1}{2} \frac{1}{n+1} \sum_{t=0}^n \sum_{i=1}^p \log\left(\frac{\lambda_{t,i}}{\delta_{t,i}}\right), \ \frac{1}{n+1} \sum_{t=0}^n \sum_{i=1}^p \delta_{t,i} = D.$$

(2) The estimation error $\hat{X}_{t|t-1}$ is given by the modified Kalman filter equations (5.13)-(5.15) (see [20, Theorem 1.1, pp. 158]). Note that (5.15) is computed as follows.

(B.40)
$$M_{t} = E_{t}^{^{\mathrm{T}}} H_{t} E_{t} \Pi_{t|t-1} (E_{t}^{^{\mathrm{T}}} H_{t} E_{t})^{^{\mathrm{T}}} + E_{t}^{^{\mathrm{T}}} \Phi_{t} Q_{t} \Phi_{t}^{^{\mathrm{T}}} E_{t}$$
$$\stackrel{(a)}{=} E_{t}^{^{\mathrm{T}}} H_{t} \Lambda_{t} H_{t} E_{t} + E_{t}^{^{\mathrm{T}}} H_{t} \Delta_{t} E_{t} = E_{t}^{^{\mathrm{T}}} H_{t} \Lambda_{t} E_{t},$$

where (a) follows if by setting $\Phi_t = \sqrt{H_t \Delta_t Q_t^{-1}}$. By substituting (5.15) into (5.14) we obtain

(B.41)
$$\Pi_{t+1|t} = A_t E_t^{\mathsf{T}} \Delta_t E_t A_t^{\mathsf{T}} + B_t B_t^{\mathsf{T}}.$$

(3) Next, we determine the realization of the optimal reproduction distribution. Recall that $\Pi_{t|t-1}$ is given by (5.10). Therefore, to determine $\Pi_{t|t-1}$, we need the equation of the error $e_t \triangleq X_t - \hat{X}_{t|t-1}$, hence the equation of the least-squares filter of X_t given all the previous outputs Y^{t-1} , namely $\hat{X}_{t|t-1}$. From Fig. 5.1, we deduce that $Y_t = \tilde{K}_t + \hat{X}_{t|t-1}$, where $\{\hat{X}_{t|t-1} : t \in \mathbb{N}_0\}$ is obtained from the modified Kalman filter $\hat{X}_{t|t-1}$. Thus, we obtain (5.17).

(4) By substituting (B.40) in (5.13) we obtain the updated version of $\widehat{X}_{t|t-1}$ as follows.

$$(B.42) \\ \widehat{X}_{t+1|t} = A_t \widehat{X}_{t|t-1} + A_t \Pi_{t|t-1} (E_t^{^{\mathrm{T}}} H_t E_t)^{^{\mathrm{T}}} M_t^{-1} \left(Y_t - \widehat{X}_{t|t-1} \right) \\ = A_t \widehat{X}_{t|t-1} + A_t \Pi_{t|t-1} E_t^{^{\mathrm{T}}} H_t E_t E_t^{^{\mathrm{T}}} \operatorname{diag} \{ \frac{1}{\lambda_{t,1}}, \dots, \frac{1}{\lambda_{t,p}} \} H_t^{-1} E_t \left(Y_t - \widehat{X}_{t|t-1} \right) \\ = A_t \widehat{X}_{t|t-1} + A_t E_t^{^{\mathrm{T}}} \Lambda_t E_t E_t^{^{\mathrm{T}}} H_t \operatorname{diag} \{ \frac{1}{\lambda_{t,1}}, \dots, \frac{1}{\lambda_{t,p}} \} H_t^{-1} E_t \left(Y_t - \widehat{X}_{t|t-1} \right) = A_t Y_t.$$

Using (B.42), we obtain (5.18) and since $\widehat{X}_{t+1|t} = A_t \widehat{X}_{t|t}$ we also obtain (5.19). Finally, by substituting (5.18) in (5.17) we obtain (5.20).

(5) To show the last stage of our theorem, we note that $\mathcal{V}_t \triangleq Y_t - \mathbb{E}\left\{Y_t | \sigma\{Y^{t-1}\}\right\}$ is the innovation process of (5.17), and that $\mathcal{V}_t \triangleq Y_t - \widehat{X}_{t|t-1} \equiv \widetilde{K}_t$. Moreover, since $\widetilde{K}_t = E_t^{\mathrm{T}} \Phi_t Z_t$ and $\{E_t^{\mathrm{T}} \Phi_t : t \in \mathbb{N}_0^n\}$ are invertible, then the statement holds. This completes the proof.

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