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EDGE ELEMENT METHOD FOR OPTIMAL CONTROL OF STATIONARY MAXWELL SYSTEM WITH GAUSS LAW*

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Abstract. A novel edge element method is proposed for the optimal control of the stationary Maxwell system with a nonvanishing charge density. The proposed approach does not involve the usual saddle-point formulation and features a positive definite structure in the associated equality constraints, for which optimal preconditioners are available in combination with conjugate gradient iteration. Our main results include error estimates and strong convergence for both the optimal edge element solution and the associated discrete Gauss laws. In particular, our analysis helps improve significantly the convergence rate established by Ciarlet, Wu, and Zou [*SIAM J. Numer. Anal.*, 52 (2014), pp. 779–807] for the edge element method for the stationary Maxwell system. Numerical experiments are presented to verify the theoretical results.

Key words. Maxwell equations, Gauss law, edge elements, optimality system, error estimates

AMS subject classifications. 78A30, 78M50, 78M10

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1. Introduction.

This work shall examine an edge element approximation and the analysis of the following optimal control problem:

$$\text{(P)} \qquad \min \frac{1}{2}\int_\Omega \epsilon|\boldsymbol{E}-\boldsymbol{E}_d|^2\,dx + \frac{\kappa}{2}\int_\Omega \epsilon|\boldsymbol{u}|^2\,dx,$$

subject to the stationary Maxwell equations with a nonvanishing charge density

$$(1) \qquad \begin{cases} \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}) = \epsilon\boldsymbol{u} & \text{in } \Omega,\\ \operatorname{div}(\epsilon\boldsymbol{E}) = \rho & \text{in } \Omega,\\ \boldsymbol{E}\times\boldsymbol{n} = 0 & \text{on } \Gamma,\end{cases}$$

and to the Gauss law for the applied current source

$$(2) \qquad \operatorname{div}(\epsilon\boldsymbol{u}) = 0 \quad \text{in } \Omega.$$

The precise mathematical assumptions on the data involved in (P) will be specified in section 2.

Several mathematical and numerical studies on electromagnetic optimal control problems can be found in the literature. However, they were mainly focused on the cases where the stationary system (1) was replaced by the corresponding time-dependent system [2, 15, 20, 21, 29] or the curl-curl-elliptic system [11, 26, 27]; namely,

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either a time derivative term $\partial \boldsymbol{E}/\partial t$ or a zero-order term is added in the first equation of (1). In these cases, the divergence law, i.e., the second equation in (1), can be automatically ensured at the discrete level from the first equation of (1) when edge element methods are used for discretization. Thus, only two symmetric and positive definite systems (corresponding to (1) and its adjoint system) need to be solved in the discrete optimality conditions. However, the situation will be much trickier and more difficult when the stationary system (1) is considered instead of the corresponding time-dependent system or the curl-curl-elliptic system. Not much has been studied for this stationary optimal control problem except the recent work [28], where the optimal control (P), constrained with the stationary state system (1) for $\rho \equiv 0$, $\epsilon \equiv 1$, and a nonlinear magnetic permeability $\mu = \mu(x, |\mathbf{curl}\, \boldsymbol{E}|)$, was investigated both mathematically and numerically. In this case, as most approximations do for the stationary state system (1), a Lagrange multiplier ∇p is included in the left-hand side of the first equation so that (1) becomes a saddle-point system. A mixed finite element method was proposed in [28] for this saddle-point system using the lowest-order edge elements of Nédélec's first family and the continuous piecewise linear elements to approximate $\boldsymbol{E}$ and p, respectively. The error estimates of the proposed finite element method were also established. These mathematical and numerical results obtained in [28] are naturally valid for the stationary optimal control problem (P) with the linear state system (1). We note that two (resp., linearized) indefinite saddle-point systems (corresponding to (1) and its adjoint system) need to be solved at each iteration (resp., each inner iteration) when an iterative method is applied for solving the discretized version of (P) (resp., (P) constrained with (1) with a nonlinear magnetic permeability $\mu = \mu(x, |\mathbf{curl}\, \boldsymbol{E}|)$). This is an essential difference between the minimization (P) constrained with the stationary state system (1) and its time-dependent version or the curl-curl-elliptic version. It is much more difficult to solve the resulting indefinite saddle-point systems than the similar symmetric and positive definite systems, for which efficient and nearly optimal preconditioners are available such as the multigrid and Hiptmair–Xu preconditioners [8, 10] or the overlapping and nonoverlapping domain decomposition preconditioners [13, 22]. One of the most popular methods for solving such discrete indefinite saddle-point systems is the preconditioned inexact Uzawa iterative methods, but they converge in a reasonable rate only when two efficient preconditioners are available for the curl-curl system and the corresponding Schur complement system (see [12, 13] and the references therein). But this is usually quite difficult to realize in most applications.

There is another fundamental issue that needs our full attention when we solve the optimal control problem (P) numerically. We see that both the continuous optimal solutions $\boldsymbol{E}$ and $\boldsymbol{u}$ satisfy the Gauss law (see (1) and (2)). It is physically and mathematically important whether the finite element methods used could guarantee the global strong convergence of the Gauss law for the discrete optimal solutions. This is still an open question for all existing finite element approximations of optimal control problems governed by both stationary and nonstationary Maxwell systems.

This work is mainly motivated by two numerical challenges we have discussed above. In order to treat these two numerical difficulties, a novel edge element method was proposed recently for solving the stationary Maxwell equations (1) in [7] (for the case $\rho = 0$) and in [5] (for the general charge density). In contrast to most existing edge element schemes (see, e.g., [4, 6, 9, 18]), the new method does not involve any saddle-point structure. Instead, it requires only the resolution of a symmetric and positive definite system, which can be solved by efficient and nearly optimal preconditioners, including [8, 10, 13, 22]. More importantly, the new edge element method ensures the optimal convergence rate [5, 7] and strong convergence of the Gauss law in some proper norm [5]. It is natural to ask whether the edge element method [5, 7] with all its advantages can be extended and transferred to the optimal control problem (P). This is exactly the main objective of the current work.

On the basis of our earlier work in [5, 28], we therefore aim at developing an efficient finite element method for the optimal control problem (P) without a saddle-point structure so that the strong convergence of the Gauss law can be ensured for the discrete optimal solutions. We now describe our basic strategy to realize this aim. In order to drop both Gauss laws for the state $\boldsymbol{E}$ and the control $\boldsymbol{u}$ in (1)–(2), we introduce two additional terms $\gamma\epsilon\boldsymbol{E}$ and $\gamma\epsilon\nabla\chi$ with a parameter $\gamma > 0$ in (1), where $\chi \in H_0^1(\Omega)$ is the unique solution of the variational equation:

$$(\epsilon\nabla\chi, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = -(\rho, \psi)_{L^2(\Omega)} \quad \forall\psi \in H_0^1(\Omega). \tag{3}$$

This leads to the following family of optimal control problems that we consider:

$$(\mathrm{P}^\gamma) \qquad \begin{cases} \min \dfrac{1}{2}\displaystyle\int_\Omega \epsilon|\boldsymbol{E} - \boldsymbol{E}_d|^2\,dx + \dfrac{\kappa}{2}\displaystyle\int_\Omega \epsilon|\boldsymbol{u}|^2\,dx, \\ \text{s.t.} \quad \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}) + \gamma\epsilon\boldsymbol{E} = \epsilon(\boldsymbol{u} + \gamma\nabla\chi) \qquad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad \boldsymbol{E}\times\boldsymbol{n} = 0 \qquad\qquad\qquad\qquad \text{on } \Gamma. \end{cases}$$

Hereafter, we discretize the state $\boldsymbol{E}$ and the control $\boldsymbol{u}$ in (P^γ) by the lowest-order edge elements of Nédélec's first family and consider γ as a function depending on the mesh size. Based on this concept, we propose the following finite element approximation:

$$(\mathrm{P}_h) \quad \begin{cases} \displaystyle\min_{\boldsymbol{E}_h,\boldsymbol{u}_h\in\boldsymbol{\mathcal{ND}}_h} \dfrac{1}{2}\int_\Omega \epsilon|\boldsymbol{E}_h - \boldsymbol{E}_d|^2\,dx + \dfrac{\kappa}{2}\int_\Omega \epsilon|\boldsymbol{u}_h|^2\,dx, \\ \text{s.t.} \ (\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_h, \mathbf{curl}\,\boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon\boldsymbol{E}_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \\ \qquad\qquad = (\epsilon(\boldsymbol{u}_h + \gamma(h)\nabla\chi_h), \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{ND}}_h, \end{cases}$$

where $\boldsymbol{\mathcal{ND}}_h$ denotes the space of lowest-order edge elements of Nédélec's first family [19] with vanishing tangential traces. Furthermore, χ_h is an appropriate continuous piecewise linear approximation of χ. The precise mathematical formulation for (P_h) will be presented in section 4.

The proposed finite element approach (P_h) turns out to be very efficient, and there are three main reasons for this, as we shall demonstrate later, where the second and third ones present two important novel features in numerical solutions of the optimal control problem (P). First, the method ensures strong convergence of (P_h) towards (P) with optimal convergence rate (Theorem 4.10). Second, it guarantees strong convergence of all Gauss laws involved, including the discrete optimal control, the discrete optimal state, and the discrete adjoint state (Theorem 4.7). More importantly, the equality constraint in (P_h) features a positive definite structure; i.e., no saddle-point structure appears in (P_h). This makes the resulting numerical method much more favorable than the existing mixed finite element methods, especially when the state Maxwell system (1) involves a nonlinear magnetic permeability $\mu = \mu(x, |\mathbf{curl}\,\boldsymbol{E}|)$ as considered in [28], where two linearized indefinite saddle-point systems need to be solved at each inner iteration when an iterative algorithm is applied for the optimal control problem. In addition, there is another novel feature in our new formulation and method, which will be seen clearly in our subsequent numerical analysis: The use of weighting coefficients ϵ and $\gamma\epsilon$, respectively, in the objective functionals and the state equations for (P^γ) and (P_h) is crucial for the optimal convergence of the resulting finite element method. This seems to be the first indication of the essential impact of the coefficients in the mathematical and numerical studies of electromagnetic optimal control problems governed by Maxwell systems.

Our strategy to prove error estimates for (P_h) is based on the use of the solution operator of a discrete mixed variational problem (see section 4.2) in combination with various optimal control and finite element techniques. Here, for the proposed finite element method (P_h), we are able to prove the convergence rate of $\gamma(h)+h^s$ for some exponent $s \in (0.5, 1]$, depending on the regularity of the optimal solution to (P). In particular, this result (see Corollary 4.12) significantly improves the recently obtained convergence rate of $\sqrt{\gamma(h)}+h^s$ for the edge element approximation of the stationary Maxwell system (1) with a nonvanishing charge density in [5].

We remark that one drawback of our proposed method lies in the stronger assumption on the desired electric field $\boldsymbol{E}_d$. We may notice that the mathematical and numerical analysis of the optimal control system (P) requires only $\boldsymbol{E}_d \in \boldsymbol{L}^2(\Omega)$ (see, e.g., [28]). But we need the additional assumption $\operatorname{div}(\epsilon \boldsymbol{E}_d) = \rho \in L^2(\Omega)$ for our analyses in this work. This condition appears to be reasonable from the physical point of view, as it is in agreement with the Gauss law about electricity (see Remark 2.3). Nonetheless, the condition may not hold if noisy data are allowed in the desired electric field $\boldsymbol{E}_d$.

The rest of this paper is organized as follows. In next section, we introduce our notation and general assumptions for (P), including some preliminary results. Section 3 is devoted to the mathematical analysis for (P) and (P^γ), including the strong convergence of (P^γ) towards (P) with a reasonable convergence rate. In section 4, we analyze the finite element approximation (P_h). Our main results include the strong convergence of the finite element solution with optimal convergence rate and the strong convergence of the Gauss law in the discrete optimal state, the discrete optimal adjoint state, and the optimal discrete control.

2. Preliminaries. We start by introducing our notation and general assumptions for (P). Throughout this work, unless it is specified explicitly, we shall use c to denote a generic positive constant, which is independent of the mesh size, the triangulation, and the quantities/fields of interest. For a given Hilbert space V, we use the notation $\|\cdot\|_V$ and $(\cdot,\cdot)_V$ for a standard norm and a standard inner product in V. The Euclidean norm in $\mathbb{R}^3$ is denoted by $|\cdot|$. Furthermore, if V is continuously embedded in another normed function space Y, we write $V \hookrightarrow Y$. We use a bold typeface to indicate a three-dimensional vector-valued function or a Hilbert space of three-dimensional vector-valued functions. In our analysis, we mainly use the following Hilbert spaces:

$$
\begin{aligned}
\boldsymbol{H}(\mathrm{div}) &= \big\{\boldsymbol{q} \in \boldsymbol{L}^2(\Omega) \bigm| \operatorname{div} \boldsymbol{q} \in L^2(\Omega)\big\},\\
\boldsymbol{H}_0(\mathrm{div}) &= \big\{\boldsymbol{q} \in \boldsymbol{H}(\mathrm{div}) \bigm| \boldsymbol{q}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\big\},\\
\boldsymbol{H}(\mathrm{div}{=}0) &= \big\{\boldsymbol{q} \in \boldsymbol{H}(\mathrm{div}) \bigm| \operatorname{div} \boldsymbol{q} = 0 \text{ in } \Omega\big\},\\
\boldsymbol{H}(\mathbf{curl}) &= \big\{\boldsymbol{q} \in \boldsymbol{L}^2(\Omega) \bigm| \mathbf{curl}\, \boldsymbol{q} \in \boldsymbol{L}^2(\Omega)\big\},\\
\boldsymbol{H}_0(\mathbf{curl}) &= \big\{\boldsymbol{q} \in \boldsymbol{H}(\mathbf{curl}) \bigm| \boldsymbol{q}\times\boldsymbol{n} = 0 \text{ on } \Gamma\big\},
\end{aligned}
$$

where the div- and **curl**-operators as well as the tangential and normal traces are understood in the sense of distributions. The state space associated with (P) is given by the Hilbert space

$$\boldsymbol{V} := \{\boldsymbol{q} \in \boldsymbol{H}_0(\mathbf{curl}) \mid \epsilon\boldsymbol{q} \in \boldsymbol{H}(\mathrm{div})\},$$

endowed with the inner product

$$(\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{V}} := (\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{H}(\mathbf{curl})} + (\mathrm{div}\,(\epsilon\boldsymbol{v}), \mathrm{div}\,(\epsilon\boldsymbol{w}))_{L^2(\Omega)} \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$$

and the norm $\|\cdot\|_{\boldsymbol{V}} = (\cdot,\cdot)_{\boldsymbol{V}}^{1/2}$. Furthermore, the control space associated with (P) is given by the Hilbert space

$$\boldsymbol{U} := \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) \mid \epsilon\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div}{=}0)\},$$

endowed with the inner product $(\cdot,\cdot)_{\boldsymbol{U}} = (\cdot,\cdot)_{\boldsymbol{L}^2(\Omega)}$ and the norm $\|\cdot\|_{\boldsymbol{U}} = (\cdot,\cdot)_{\boldsymbol{L}^2(\Omega)}^{1/2}$.

Remark 2.1. It follows from the definition that

$$\boldsymbol{U} = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) \mid \epsilon\boldsymbol{u} \in \boldsymbol{H}(\mathrm{div}{=}0)\} = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) \mid (\epsilon\boldsymbol{u}, \nabla\phi)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \phi \in H_0^1(\Omega)\}.$$

Therefore, as proposed in [28], an ϵ-divergence-free control $\boldsymbol{u} \in \boldsymbol{U}$ can be realized by including the variational equality

$$(\epsilon\boldsymbol{u}, \nabla\phi)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \phi \in H_0^1(\Omega)$$

as an explicit control constraint of (P) in place of (2). But this control constraint is naturally eliminated in (P$^\gamma$); see Remark 3.3.

ASSUMPTION 2.2 (general assumptions for (P)). *We assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with a connected Lipschitz boundary Γ. The electric permittivity $\epsilon : \Omega \to \mathbb{R}$ and the magnetic permeability $\mu : \Omega \to \mathbb{R}$ are of class $L^\infty(\Omega)$ and satisfy*

$$0 < \underline{\mu} \le \mu(x) \le \overline{\mu} \quad \textit{a.e. in } \Omega \quad \textit{and} \quad 0 < \underline{\epsilon} \le \epsilon(x) \le \overline{\epsilon} \quad \textit{a.e. in } \Omega \tag{4}$$

for some positive real constants $\underline{\mu} < \overline{\mu}$ and $\underline{\epsilon} < \overline{\epsilon}$. Moreover, $\kappa > 0$ denotes the control cost constant, and the desired electric field $\boldsymbol{E}_d \in \boldsymbol{L}^2(\Omega)$ satisfies the Gauss law:

$$\mathrm{div}\,(\epsilon\boldsymbol{E}_d) = \rho \quad \textit{in } \Omega \iff (\epsilon\boldsymbol{E}_d, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = -(\rho, \psi)_{L^2(\Omega)} \quad \forall \psi \in H_0^1(\Omega), \tag{5}$$

where $\rho \in L^2(\Omega)$ is the charge density.

Remark 2.3. In this work, Ω represents a large holdall domain that may contain different materials including conductors and inductors. We refer the reader to [24] for low-frequency electromagnetic optimal control problems with multiply connected conductors.

We note that (5) arises from the Gauss law about electricity. As $\boldsymbol{E}_d$ is the desired electric field, $\boldsymbol{D}_d := \epsilon\boldsymbol{E}_d$ is then the desired electric displacement field. According to the Gauss law about electricity, the divergence of the electric displacement field yields the free electric charge density, namely (5).

We notice that, since the boundary Γ is connected, there exists a constant $\hat{c} > 0$, depending only on Ω, such that

$$\|\boldsymbol{E}\|_{\boldsymbol{L}^2(\Omega)} \le \hat{c}\left(\|\mathbf{curl}\,\boldsymbol{E}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,(\epsilon\boldsymbol{E})\|_{L^2(\Omega)}\right) \quad \forall \boldsymbol{E} \in \boldsymbol{V}. \tag{6}$$

The inequality (6) follows from a classical indirect argument by using the compactness of the embedding $\boldsymbol{V} \hookrightarrow \boldsymbol{L}^2(\Omega)$ [25] and the fact that

$$\{\boldsymbol{y} \in \boldsymbol{V} \mid \mathbf{curl}\,\boldsymbol{y} = 0, \mathrm{div}\,(\epsilon\boldsymbol{y}) = 0\} = \{0\},$$

which holds due to the connectedness of Γ (see, e.g., [1]). Also, the Ladyzhenskaya–Babuška–Brezzi (LBB) condition

$$\sup_{0\neq \boldsymbol{E}\in \boldsymbol{H}_0(\mathbf{curl})} \frac{|(\epsilon \boldsymbol{E},\nabla\psi)_{\boldsymbol{L}^2(\Omega)}|}{\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl})}} \geq \frac{(\epsilon\nabla\psi,\nabla\psi)_{\boldsymbol{L}^2(\Omega)}}{\|\nabla\psi\|_{\boldsymbol{H}(\mathbf{curl})}} \geq c\|\psi\|_{H_0^1(\Omega)} \quad \forall \psi \in H_0^1(\Omega) \tag{7}$$

is satisfied with a constant $c>0$ depending only on $\underline{\epsilon}$ and Ω. In fact, since $\nabla H_0^1(\Omega) \subset \boldsymbol{H}_0(\mathbf{curl})$ and $\mathbf{curl}\,\nabla \equiv 0$, we may insert $\boldsymbol{E}=\nabla\psi$ in (7) to get the LBB condition.

3. Mathematical analysis. We consider a mixed variational formulation for the stationary Maxwell equations (1): For a given $\boldsymbol{u}\in\boldsymbol{U}$, find $\boldsymbol{E}\in\boldsymbol{V}$ such that

$$\begin{cases} (\mu^{-1}\mathbf{curl}\,\boldsymbol{E},\mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\epsilon\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} & \forall \boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl}),\\ (\epsilon\boldsymbol{E},\nabla\psi)_{\boldsymbol{L}^2(\Omega)} = -(\rho,\psi)_{L^2(\Omega)} & \forall\psi\in H_0^1(\Omega). \end{cases} \tag{8}$$

It is standard to verify that, for every $\boldsymbol{u}\in\boldsymbol{U}$, the mixed variational formulation (8) admits a unique solution $\boldsymbol{E}\in\boldsymbol{V}$. This follows from a well-known theory for mixed variational problems (see [3]) together with the Poincaré–Friedrichs-type inequality (6) and the LBB condition (7). Next, we introduce the solution operator associated with (8) as

$$\boldsymbol{G}:\boldsymbol{U}\to\boldsymbol{V}, \quad \boldsymbol{u}\mapsto\boldsymbol{E},$$

that assigns to every control $\boldsymbol{u}\in\boldsymbol{U}$ the unique solution $\boldsymbol{E}\in\boldsymbol{V}$ of the mixed variational formulation (8). The solution operator $\boldsymbol{G}:\boldsymbol{U}\to\boldsymbol{V}$ is bounded and affine linear such that it is infinitely Fréchet differentiable. Its Fréchet derivative at $\boldsymbol{z}\in\boldsymbol{U}$ in the direction $\boldsymbol{u}\in\boldsymbol{U}$ is given by $\boldsymbol{G}'(\boldsymbol{z})\boldsymbol{u}=\boldsymbol{E}_z$, where $\boldsymbol{E}_z\in\boldsymbol{V}$ is the solution of the following mixed variational equations:

$$\begin{cases} (\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_z,\mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\epsilon\boldsymbol{u},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} & \forall \boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl}),\\ (\epsilon\boldsymbol{E}_z,\nabla\psi)_{\boldsymbol{L}^2(\Omega)} = 0 & \forall\psi\in H_0^1(\Omega). \end{cases} \tag{9}$$

Employing the solution operator, we may reformulate the optimal control problem (P) as a minimization problem in Hilbert spaces:

$$\min_{\boldsymbol{u}\in\boldsymbol{U}} f(\boldsymbol{u}) := \frac{1}{2}\int_\Omega \epsilon|\boldsymbol{G}(\boldsymbol{u})-\boldsymbol{E}_d|^2\,dx + \frac{\kappa}{2}\int_\Omega \epsilon|\boldsymbol{u}|^2\,dx. \tag{P}$$

By classical arguments (see [16, 23]), the minimization problem (P) admits a unique solution $\overline{\boldsymbol{u}}\in\boldsymbol{U}$, and its necessary and sufficient optimality condition is given by

$$f'(\overline{\boldsymbol{u}})\boldsymbol{u}=0 \quad \forall \boldsymbol{u}\in\boldsymbol{U}. \tag{10}$$

THEOREM 3.1. *A control $\overline{\boldsymbol{u}}\in\boldsymbol{U}$ with the associated electric field $\overline{\boldsymbol{E}}\in\boldsymbol{V}$ is the* (*unique*) *optimal solution of* (P) *if and only if there exists a unique $\overline{\boldsymbol{p}}\in\boldsymbol{V}$ such that the triple* $(\overline{\boldsymbol{u}},\overline{\boldsymbol{E}},\overline{\boldsymbol{p}})$ *satisfies*

$$\begin{cases} (\mu^{-1}\mathbf{curl}\,\overline{\boldsymbol{E}},\mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\epsilon\overline{\boldsymbol{u}},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} & \forall \boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl}),\\ (\epsilon\overline{\boldsymbol{E}},\nabla\psi)_{\boldsymbol{L}^2(\Omega)} = -(\rho,\psi)_{L^2(\Omega)} & \forall\psi\in H_0^1(\Omega), \end{cases} \tag{11a}$$

$$\begin{cases} (\mu^{-1}\mathbf{curl}\,\overline{\boldsymbol{p}},\mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\overline{\boldsymbol{E}}-\boldsymbol{E}_d),\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} & \forall \boldsymbol{v}\in \boldsymbol{H}_0(\mathbf{curl}),\\ (\epsilon\overline{\boldsymbol{p}},\nabla\psi)_{\boldsymbol{L}^2(\Omega)} = 0 & \forall\psi\in H_0^1(\Omega), \end{cases} \tag{11b}$$

$$\overline{\boldsymbol{u}} = -\kappa^{-1}\overline{\boldsymbol{p}}. \tag{11c}$$

Proof. The existence of a unique solution $\overline{\boldsymbol{p}} \in \boldsymbol{V}$ of the mixed variational problem (11b) follows from [3] along with (5), (6), and (7). Inserting $\boldsymbol{v} = \boldsymbol{G}'(\overline{\boldsymbol{u}})\boldsymbol{u}$ with $\boldsymbol{u} \in \boldsymbol{U}$ in (11b) yields

$$(12) \qquad (\mu^{-1}\mathbf{curl}\,\overline{\boldsymbol{p}}, \mathbf{curl}\,(\boldsymbol{G}'(\overline{\boldsymbol{u}})\boldsymbol{u}))_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\overline{\boldsymbol{E}} - \boldsymbol{E}_d), \boldsymbol{G}'(\overline{\boldsymbol{u}})\boldsymbol{u})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{u} \in \boldsymbol{U}.$$

We also know that $\boldsymbol{G}'(\overline{\boldsymbol{u}})\boldsymbol{u}$ satisfies (9), with $\boldsymbol{z} = \overline{\boldsymbol{u}}$ and $\boldsymbol{E}_z = \boldsymbol{G}'(\overline{\boldsymbol{u}})\boldsymbol{u}$, and hence inserting $\boldsymbol{v} = \overline{\boldsymbol{p}}$ in (9) gives

$$(13) \qquad (\mu^{-1}\mathbf{curl}\,(\boldsymbol{G}'(\overline{\boldsymbol{u}})\boldsymbol{u}), \mathbf{curl}\,\overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} = (\epsilon\boldsymbol{u}, \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{u} \in \boldsymbol{U}.$$

From (11b), (12), and (13), we come to the conclusion that

$$\begin{aligned} f'(\overline{\boldsymbol{u}})\boldsymbol{u} &= (\epsilon(\overline{\boldsymbol{E}} - \boldsymbol{E}_d), \boldsymbol{G}'(\overline{\boldsymbol{u}})\boldsymbol{u})_{\boldsymbol{L}^2(\Omega)} + \kappa(\epsilon\overline{\boldsymbol{u}}, \boldsymbol{u})_{\boldsymbol{L}^2(\Omega)} \\ &= (\epsilon(\overline{\boldsymbol{p}} + \kappa\overline{\boldsymbol{u}}), \boldsymbol{u})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{u} \in \boldsymbol{U}. \end{aligned}$$

Thus, the necessary and sufficient optimality condition (10) is nothing but

$$(14) \qquad (\epsilon(\overline{\boldsymbol{p}} + \kappa\overline{\boldsymbol{u}}), \boldsymbol{u})_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \boldsymbol{u} \in \boldsymbol{U}.$$

Now, the second variational equality in (11b) implies $\epsilon\overline{\boldsymbol{p}} \in \boldsymbol{H}(\text{div=0})$. This regularity property implies that $\overline{\boldsymbol{p}} + \kappa\overline{\boldsymbol{u}} \in \boldsymbol{U}$. Then we can insert $\boldsymbol{u} = \overline{\boldsymbol{p}} + \kappa\overline{\boldsymbol{u}}$ in (14) to obtain

$$(\epsilon(\overline{\boldsymbol{p}} + \kappa\overline{\boldsymbol{u}}), \overline{\boldsymbol{p}} + \kappa\overline{\boldsymbol{u}})_{\boldsymbol{L}^2(\Omega)} = 0 \qquad \Longleftrightarrow \qquad \overline{\boldsymbol{u}} = -\kappa^{-1}\overline{\boldsymbol{p}}.$$

This completes the proof. ∎

In what follows, we shall denote by $\overline{\boldsymbol{u}} \in \boldsymbol{U}$ the unique optimal solution of (P) with the corresponding optimal electric field $\overline{\boldsymbol{E}} \in \boldsymbol{V}$ and the adjoint state $\overline{\boldsymbol{p}} \in \boldsymbol{V} \cap \boldsymbol{U}$ satisfying (11). Thanks to (11c), we can see that the optimal control enjoys the regularity property

$$(15) \qquad \overline{\boldsymbol{u}} \in \boldsymbol{V} \cap \boldsymbol{U}.$$

3.1. Sensitivity analysis of (P$^\gamma$). This section is devoted to the sensitivity analysis of (P$^\gamma$), namely, to establish an error estimate depending on the parameter γ. First, we note that the variational formulation for the associated state equation in (P$^\gamma$) is given by

$$(16) \quad (\mu^{-1}\mathbf{curl}\,\boldsymbol{E}^\gamma, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + \gamma(\epsilon\boldsymbol{E}^\gamma, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\boldsymbol{u} + \gamma\nabla\chi), \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \; \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}).$$

By the Lax–Milgram lemma, the variational equality (16) admits for every $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ a unique solution $\boldsymbol{E}^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$. We denote the corresponding solution operator by

$$\boldsymbol{G}^\gamma : \boldsymbol{L}^2(\Omega) \to \boldsymbol{H}_0(\mathbf{curl}), \quad \boldsymbol{u} \mapsto \boldsymbol{E}^\gamma.$$

Some elementary properties of this operator are listed below for later use.

LEMMA 3.2. *The solution operator $\boldsymbol{G}^\gamma : \boldsymbol{L}^2(\Omega) \to \boldsymbol{H}_0(\mathbf{curl})$ satisfies $\boldsymbol{G}^\gamma(0) = \nabla\chi$ and $\operatorname{div}(\epsilon\boldsymbol{G}^\gamma(\boldsymbol{u})) = \rho$ for all $\boldsymbol{u} \in \boldsymbol{U}$ and all $\gamma > 0$.*

Proof. Let $\gamma > 0$. Since $\mathbf{curl}\,\nabla \equiv 0$, we can easily see that

$$(\mu^{-1}\mathbf{curl}\,(\nabla\chi), \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + \gamma(\epsilon\nabla\chi, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\epsilon(0 + \gamma\nabla\chi), \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \; \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}),$$

which implies $\boldsymbol{G}^\gamma(0) = \nabla\chi$ by the definition of $\boldsymbol{G}^\gamma$. Now, for any $\boldsymbol{u} \in \boldsymbol{U}$, we insert $\boldsymbol{v} = \nabla\psi$ with $\psi \in H_0^1(\Omega)$ in (16) and use (3) to see that $\boldsymbol{E}^\gamma := \boldsymbol{G}^\gamma(\boldsymbol{u})$ satisfies

$$(17) \quad \gamma(\epsilon\boldsymbol{E}^\gamma, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\boldsymbol{u} + \gamma\nabla\chi), \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = -\gamma(\rho, \psi)_{L^2(\Omega)} \quad \forall \psi \in H_0^1(\Omega). \quad ∎$$

Similarly to (P), we reformulate (P^γ) as a minimization problem in Hilbert spaces:

$$(\mathrm{P}_\gamma) \qquad \min_{\boldsymbol{u}\in \boldsymbol{L}^2(\Omega)} f^\gamma(\boldsymbol{u}) := \frac{1}{2}\int_\Omega \epsilon|\boldsymbol{G}^\gamma(\boldsymbol{u}) - \boldsymbol{E}_d|^2\,dx + \frac{\kappa}{2}\int_\Omega \epsilon|\boldsymbol{u}|^2\,dx.$$

Remark 3.3. We emphasize that the formulation (P^γ) removes the original divergence constraint on the control $\boldsymbol{u}$ as the control space of (P^γ) is now given by $\boldsymbol{L}^2(\Omega)$ instead of $\boldsymbol{U}$ as in (P). Nonetheless, we will see later that the optimal control of (P_γ) belongs to $\boldsymbol{U}$. Similarly to (P), (P^γ) admits a unique optimal solution, with its necessary and sufficient optimality conditions described as in the following theorem, whose proof is basically analogous to that of Theorem 3.1.

THEOREM 3.4. *Let $\gamma > 0$. A control $\overline{\boldsymbol{u}}^\gamma \in \boldsymbol{L}^2(\Omega)$ with the associated electric field $\overline{\boldsymbol{E}}^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$ is the* (*unique*) *optimal solution of* (P^γ) *if and only if there exists a unique $\overline{\boldsymbol{p}}^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$ such that the triple $(\overline{\boldsymbol{u}}^\gamma, \overline{\boldsymbol{E}}^\gamma, \overline{\boldsymbol{p}}^\gamma)$ satisfies*

$$\text{(18a)} \qquad (\mu^{-1}\mathbf{curl}\,\overline{\boldsymbol{E}}^\gamma, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + \gamma(\epsilon\overline{\boldsymbol{E}}^\gamma, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\overline{\boldsymbol{u}}^\gamma + \gamma\nabla\chi), \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}),$$

$$\text{(18b)} \qquad (\mu^{-1}\mathbf{curl}\,\overline{\boldsymbol{p}}^\gamma, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + \gamma(\epsilon\overline{\boldsymbol{p}}^\gamma, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\overline{\boldsymbol{E}}^\gamma - \boldsymbol{E}_d), \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}),$$

$$\text{(18c)} \qquad \overline{\boldsymbol{u}}^\gamma = -\kappa^{-1}\overline{\boldsymbol{p}}^\gamma.$$

An important consequence of the optimality system for (P^γ) is the following structural property for the optimal triple $(\overline{\boldsymbol{u}}^\gamma, \overline{\boldsymbol{E}}^\gamma, \overline{\boldsymbol{p}}^\gamma)$ of (P^γ).

PROPOSITION 3.5. *For every $\gamma > 0$, let $(\overline{\boldsymbol{u}}^\gamma, \overline{\boldsymbol{E}}^\gamma, \overline{\boldsymbol{p}}^\gamma) \in \boldsymbol{L}^2(\Omega) \times \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{H}_0(\mathbf{curl})$ be the optimal triple of* (P^γ) *satisfying* (18). *Then it holds that*

$$\text{(19)} \qquad \overline{\boldsymbol{u}}^\gamma \in \boldsymbol{V}\cap\boldsymbol{U}, \quad \overline{\boldsymbol{E}}^\gamma \in \boldsymbol{V}, \quad \operatorname{div}(\epsilon\overline{\boldsymbol{E}}^\gamma) = \rho, \quad \overline{\boldsymbol{p}}^\gamma \in \boldsymbol{V}\cap\boldsymbol{U}.$$

Proof. For a fixed $\gamma > 0$, inserting $\boldsymbol{v} = \nabla\psi$ with $\psi \in H_0^1(\Omega)$ in (18a) yields

$$\text{(20)} \qquad \begin{aligned} \gamma(\epsilon\overline{\boldsymbol{E}}^\gamma, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} &= (\epsilon(\overline{\boldsymbol{u}}^\gamma + \gamma\nabla\chi), \nabla\psi)_{\boldsymbol{L}^2(\Omega)} \\ &= (\epsilon(-\kappa^{-1}\overline{\boldsymbol{p}}^\gamma + \gamma\nabla\chi), \nabla\psi)_{\boldsymbol{L}^2(\Omega)} \\ &= -\kappa^{-1}(\epsilon\overline{\boldsymbol{p}}^\gamma, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} - \gamma(\rho, \psi)_{L^2(\Omega)} \quad \forall \psi \in H_0^1(\Omega), \end{aligned}$$

where we have used (18c) and (3). Analogously, setting $\boldsymbol{v} = \nabla\psi$ with $\psi \in H_0^1(\Omega)$ in (18b) implies that

$$\text{(21)} \qquad \begin{aligned} \gamma(\epsilon\overline{\boldsymbol{p}}^\gamma, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} &= (\epsilon(\overline{\boldsymbol{E}}^\gamma - \boldsymbol{E}_d), \nabla\psi)_{\boldsymbol{L}^2(\Omega)} \\ &\underbrace{=}_{(5)} (\epsilon\overline{\boldsymbol{E}}^\gamma, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} + (\rho, \psi)_{L^2(\Omega)} \quad \forall \psi \in H_0^1(\Omega). \end{aligned}$$

From (20) and (21), it follows that

$$(\gamma^2 + \kappa^{-1})(\epsilon\overline{\boldsymbol{p}}^\gamma, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi \in H_0^1(\Omega).$$

In other words, $\operatorname{div}(\epsilon\overline{\boldsymbol{p}}^\gamma) = 0$, so $\overline{\boldsymbol{p}}^\gamma \in \boldsymbol{V}\cap\boldsymbol{U}$. Then it follows from (18c) that $\overline{\boldsymbol{u}}^\gamma \in \boldsymbol{V}\cap\boldsymbol{U}$. Consequently, in view of Lemma 3.2, we obtain that

$$\overline{\boldsymbol{E}}^\gamma \in \boldsymbol{V} \quad \text{and} \quad \operatorname{div}(\epsilon\overline{\boldsymbol{E}}^\gamma) = \rho.$$

This completes the proof. ∎

In what follows, for every $\gamma > 0$, let $(\overline{\boldsymbol{u}}^\gamma, \overline{\boldsymbol{E}}^\gamma, \overline{\boldsymbol{p}}^\gamma) \in (\boldsymbol{V} \cap \boldsymbol{U}) \times \boldsymbol{V} \times (\boldsymbol{V} \cap \boldsymbol{U})$ denote the optimal triple of (P^γ) satisfying (18).

THEOREM 3.6. *There exists a constant $c > 0$, independent of γ, such that*

$$\|\overline{\boldsymbol{u}}^\gamma - \overline{\boldsymbol{u}}\|_{\boldsymbol{H}(\mathbf{curl})} + \|\overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}}\|_{\boldsymbol{H}(\mathbf{curl})} + \|\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}}\|_{\boldsymbol{H}(\mathbf{curl})} \le c\gamma \quad \forall\, \gamma > 0\,.$$

Proof. As $\overline{\boldsymbol{u}}^\gamma$ is the optimal solution of (P^γ), it follows that

$$f^\gamma(\overline{\boldsymbol{u}}^\gamma) \le f^\gamma(0) \underbrace{=}_{Lemma\, 3.2} \frac{1}{2}\int_\Omega \epsilon|\nabla\chi - \boldsymbol{E}_d|^2\, dx \quad \forall\gamma > 0.$$

This implies the existence of a constant $c > 0$, independent of $\gamma > 0$, such that

$$\|\overline{\boldsymbol{E}}^\gamma\|^2_{\boldsymbol{L}^2(\Omega)} + \|\overline{\boldsymbol{u}}^\gamma\|^2_{\boldsymbol{L}^2(\Omega)} \le c \quad \forall\gamma > 0. \tag{22}$$

Then (22), along with (18c), implies that $\{\overline{\boldsymbol{u}}^\gamma\}_{\gamma>0}, \{\overline{\boldsymbol{E}}^\gamma\}_{\gamma>0}$, and $\{\overline{\boldsymbol{p}}^\gamma\}_{\gamma>0}$ are all bounded in $\boldsymbol{L}^2(\Omega)$.

Setting $\boldsymbol{v} = \overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}}$ in (18a) and (11a), respectively, then subtracting the resulting equalities, we infer that

$$\begin{aligned} (\mu^{-1}\mathbf{curl}\,(\overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}}), \mathbf{curl}\,(\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}}))_{\boldsymbol{L}^2(\Omega)} + \gamma(\epsilon\overline{\boldsymbol{E}}^\gamma, \overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} \\ = (\epsilon(\overline{\boldsymbol{u}}^\gamma + \gamma\nabla\chi - \overline{\boldsymbol{u}}), \overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} \\ \underbrace{=}_{(11\mathrm{c})-(18\mathrm{c})} \quad -\kappa^{-1}\|\epsilon^{1/2}\,(\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})\,\|^2_{\boldsymbol{L}^2(\Omega)} + \gamma(\epsilon\nabla\chi, \overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{23}$$

Similarly, setting $\boldsymbol{v} = \overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}}$ in (18b) and (11b), respectively, then subtracting the resulting equations yields that

$$\begin{aligned} (\mu^{-1}\mathbf{curl}\,(\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}}), \mathbf{curl}\,(\overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}}))_{\boldsymbol{L}^2(\Omega)} + \gamma(\epsilon\overline{\boldsymbol{p}}^\gamma, \overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} \\ = \|\epsilon^{1/2}(\overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}})\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall\gamma > 0. \end{aligned} \tag{24}$$

In view of (23) and (24), we obtain

$$\begin{aligned} &\|\epsilon^{1/2}(\overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}})\|^2_{\boldsymbol{L}^2(\Omega)} + \kappa^{-1}\|\epsilon^{1/2}\,(\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})\,\|^2_{\boldsymbol{L}^2(\Omega)} \\ &= \gamma\big((\epsilon\overline{\boldsymbol{p}}^\gamma, \overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} + (\epsilon(\nabla\chi - \overline{\boldsymbol{E}}^\gamma), \overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)}\big) \\ &\le \gamma\big(\|\epsilon^{1/2}\overline{\boldsymbol{p}}^\gamma\|_{\boldsymbol{L}^2(\Omega)} + \|\epsilon^{1/2}(\nabla\chi - \overline{\boldsymbol{E}}^\gamma)\|_{\boldsymbol{L}^2(\Omega)}\big)\big(\|\epsilon^{1/2}(\overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}})\|_{\boldsymbol{L}^2(\Omega)} + \|\epsilon^{1/2}\,(\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})\,\|_{\boldsymbol{L}^2(\Omega)}\big). \end{aligned}$$

From the above estimate and the boundedness of $\{\overline{\boldsymbol{E}}^\gamma\}_{\gamma>0}$ and $\{\overline{\boldsymbol{p}}^\gamma\}_{\gamma>0}$ in $\boldsymbol{L}^2(\Omega)$, it follows that

$$\|\overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega)} + \|\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}}\|_{\boldsymbol{L}^2(\Omega)} \le c\gamma \quad \forall\gamma > 0.$$

Then making use of (11c) and (33c), we have

$$\|\overline{\boldsymbol{u}}^\gamma - \overline{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)} + \|\overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega)} + \|\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}}\|_{\boldsymbol{L}^2(\Omega)} \le c\gamma \quad \forall\gamma > 0. \tag{25}$$

Now, inserting $\boldsymbol{v} = \overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}}$ in (18a) and (11a), respectively, then subtracting the resulting equations, we obtain that

$$\begin{aligned} &\|\mu^{-1/2}\mathbf{curl}\,(\overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}})\|^2_{\boldsymbol{L}^2(\Omega)} \\ &= \gamma(\epsilon(\nabla\chi - \overline{\boldsymbol{E}}^\gamma), \overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} + (\epsilon(\overline{\boldsymbol{u}}^\gamma - \overline{\boldsymbol{u}}), \overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} \\ &\le \gamma\overline{\epsilon}\|\nabla\chi - \overline{\boldsymbol{E}}^\gamma\|_{\boldsymbol{L}^2(\Omega)}\|\overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega)} + \overline{\epsilon}\|\overline{\boldsymbol{u}}^\gamma - \overline{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega)}\|\overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall\gamma > 0. \end{aligned} \tag{26}$$

Analogously, we insert $\boldsymbol{v} = \overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}}$ in (18b) and (11b) to obtain

$$(\mu^{-1}\mathbf{curl}\,\overline{\boldsymbol{p}}^\gamma, \mathbf{curl}\,\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} + \gamma(\epsilon\overline{\boldsymbol{p}}^\gamma, \overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\overline{\boldsymbol{E}}^\gamma - \boldsymbol{E}_d), \overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)}$$

and

$$(\mu^{-1}\mathbf{curl}\,\overline{\boldsymbol{p}}, \mathbf{curl}\,\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\overline{\boldsymbol{E}} - \boldsymbol{E}_d), \overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)}.$$

Then, subtracting these two identities yields

$$\begin{aligned} &\|\mu^{-1/2}\mathbf{curl}\,(\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})\|^2_{\boldsymbol{L}^2(\Omega)} \\ &= -\gamma(\epsilon\overline{\boldsymbol{p}}^\gamma, \overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} + (\epsilon(\overline{\boldsymbol{E}}^\gamma - \overline{\boldsymbol{E}}), \overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} \\ &\le \gamma\bar{\epsilon}\|\overline{\boldsymbol{p}}^\gamma\|_{\boldsymbol{L}^2(\Omega)}\|\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}}^\gamma\|_{\boldsymbol{L}^2(\Omega)} + \bar{\epsilon}\|\boldsymbol{E}^\gamma - \overline{\boldsymbol{E}}^\gamma\|_{\boldsymbol{L}^2(\Omega)}\|\overline{\boldsymbol{p}}^\gamma - \overline{\boldsymbol{p}}^\gamma\|_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{27}$$

Now the desired estimate in Theorem 3.6 is a direct consequence of (25)–(27). ☐

4. Finite element method. This section is devoted to the analysis of the finite element approximation (P_h) we proposed in the introduction. From now on, the domain $\Omega \subset \mathbb{R}^3$ is additionally assumed to be Lipschitz polyhedral. We consider a family $\{\mathcal{T}_h\}_{h>0}$ of triangulations of Ω consisting of tetrahedral elements T such that

$$\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} T.$$

For every element $T \in \mathcal{T}_h$, we denote by h_T the diameter of T, by ρ_T the diameter of the largest ball contained in T, and by h the maximal diameter of all elements, i.e., $h := \max\{h_T \mid T \in \mathcal{T}_h\}$. We assume $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform, i.e., there exist two positive constants ϱ and ϑ such that

$$\frac{h_T}{\rho_T} \le \varrho \quad \text{and} \quad \frac{h}{h_T} \le \vartheta \quad \forall\, T \in \mathcal{T}_h,\ \forall\, h > 0.$$

Let us denote the space of lowest-order edge elements of Nédélec's first family [19] with vanishing tangential traces and the space of continuous piecewise linear elements with vanishing traces by

$$\begin{aligned} \boldsymbol{\mathcal{ND}}_h &:= \big\{\boldsymbol{E}_h \in \boldsymbol{H}_0(\mathbf{curl}) \;\big|\; \boldsymbol{E}_{h|T} = \boldsymbol{a}_T + \boldsymbol{b}_T \times x \quad \text{with } \boldsymbol{a}_T, \boldsymbol{b}_T \in \mathbb{R}^3 \quad \forall T \in \mathcal{T}_h\big\}, \\ \Theta_h &:= \big\{\phi_h \in H^1_0(\Omega) \;\big|\; \phi_{h|T} = \boldsymbol{a}_T \cdot x + b_T \quad \text{with } \boldsymbol{a}_T \in \mathbb{R}^3,\, b_T \in \mathbb{R} \quad \forall T \in \mathcal{T}_h\big\}. \end{aligned}$$

By the well-known discrete de Rham diagram (cf. [18, p. 150]), we know that

$$\nabla\Theta_h \subset \boldsymbol{\mathcal{ND}}_h. \tag{28}$$

In what follows, we consider the parameter γ as a function of the mesh size of the discretization; i.e., $\gamma = \gamma(h)$. This function is supposed to be bounded; i.e., there exists a constant $c > 0$, independent of $h > 0$, such that

$$0 < \gamma(h) \le c \quad \forall h > 0. \tag{29}$$

Now we introduce the finite element solution $\chi_h \in \Theta_h$ of (3):

$$(\epsilon\nabla\chi_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = -(\rho, \psi_h)_{\boldsymbol{L}^2(\Omega)} \quad \forall\psi_h \in \Theta_h. \tag{30}$$

Then we shall consider the following finite element approximation of (16).

For every given $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$, find $\boldsymbol{E}_h \in \boldsymbol{\mathcal{ND}}_h$ such that, for all $\boldsymbol{v}_h \in \boldsymbol{\mathcal{ND}}_h$, it holds that

$$(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_h, \mathbf{curl}\,\boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon \boldsymbol{E}_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\boldsymbol{u} + \gamma(h)\nabla\chi_h), \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)}. \tag{31}$$

We denote the (discrete) solution operator associated with (31) by

$$\boldsymbol{G}_h : \boldsymbol{L}^2(\Omega) \to \boldsymbol{\mathcal{ND}}_h, \quad \boldsymbol{u} \mapsto \boldsymbol{E}_h,$$

that assigns to every $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ the unique solution $\boldsymbol{E}_h \in \boldsymbol{\mathcal{ND}}_h$ of (31). For later use, we introduce the subspace $\boldsymbol{X}_h^{(\epsilon)}$ of $\boldsymbol{\mathcal{ND}}_h$ consisting of all discrete ϵ-divergence-free edge element functions:

$$\boldsymbol{X}_h^{(\epsilon)} := \left\{ \boldsymbol{u}_h \in \boldsymbol{\mathcal{ND}}_h \;\middle|\; (\epsilon \boldsymbol{u}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi_h \in \Theta_h \right\}. \tag{32}$$

Then, making use of this subspace, we can drive a discrete counterpart of Lemma 3.2.

LEMMA 4.1. *For every $h > 0$, the operator $\boldsymbol{G}_h : \boldsymbol{L}^2(\Omega) \to \boldsymbol{\mathcal{ND}}_h$ satisfies $\boldsymbol{G}_h(0) = \nabla\chi_h$ and $(\epsilon \boldsymbol{G}_h(\boldsymbol{u}_h), \nabla\phi_h)_{\boldsymbol{L}^2(\Omega)} = -(\rho, \phi_h)_{L^2(\Omega)}$ for all $\boldsymbol{u}_h \in \boldsymbol{X}_h^{(\epsilon)}$ and $\phi_h \in \Theta_h$.*

Proof. For $h > 0$, we can easily see by using $\mathbf{curl}\,\nabla \equiv 0$ that

$$\begin{aligned}(\mu^{-1}\mathbf{curl}\,(\nabla\chi_h), \mathbf{curl}\,\boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon\nabla\chi_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} &= \gamma(h)(\epsilon\nabla\chi_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \\ &= (\epsilon(0 + \gamma(h)\nabla\chi_h), \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{ND}}_h,\end{aligned}$$

which implies $\boldsymbol{G}_h(0) = \nabla\chi_h$ by using the definition of $\boldsymbol{G}_h$ and the fact that $\nabla\Theta_h \subset \boldsymbol{\mathcal{ND}}_h$. Now, inserting $\boldsymbol{v}_h = \nabla\psi_h$ with $\psi_h \in \Theta_h$ in (31), we see that $\boldsymbol{E}_h := \boldsymbol{G}_h(\boldsymbol{u}_h)$ for every $\boldsymbol{u}_h \in \boldsymbol{X}_h^{(\epsilon)}$ satisfies

$$\gamma(h)(\epsilon \boldsymbol{E}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\boldsymbol{u}_h + \gamma(h)\nabla\chi_h), \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = -\gamma(h)(\rho, \psi_h)_{L^2(\Omega)} \quad \forall \psi_h \in \Theta_h,$$

where the last equality holds due to $\boldsymbol{u}_h \in \boldsymbol{X}_h^{(\epsilon)}$ and (30). ∎

Now, by introducing the objective functional

$$f_h : \boldsymbol{L}^2(\Omega) \to \mathbb{R}, \quad f_h(\boldsymbol{u}) := \frac{1}{2}\int_\Omega \epsilon|\boldsymbol{G}_h(\boldsymbol{u}) - \boldsymbol{E}_d|^2\,dx + \frac{\kappa}{2}\int_\Omega \epsilon|\boldsymbol{u}|^2\,dx,$$

we propose the finite element approximation for (P^γ) as follows:

$$\min_{\boldsymbol{u}_h \in \boldsymbol{\mathcal{ND}}_h} f_h(\boldsymbol{u}_h). \tag{P_h}$$

4.1. Convergence analysis for (P_h). For the convergence and error estimates of the finite element approximation (P_h), we first present its necessary and sufficient optimality condition, whose proof is analogous to that of Theorem 3.1.

THEOREM 4.2. *Let $h > 0$. A function $\overline{\boldsymbol{u}}_h \in \boldsymbol{\mathcal{ND}}_h$ is the (unique) optimal solution of (P_h) if and only if there exists a unique $\overline{\boldsymbol{p}}_n \in \boldsymbol{\mathcal{ND}}_h$ such that the following holds for all $\boldsymbol{v}_h \in \boldsymbol{\mathcal{ND}}_h$:*

$$(\mu^{-1}\mathbf{curl}\,\overline{\boldsymbol{E}}_h, \mathbf{curl}\,\boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon\overline{\boldsymbol{E}}_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\overline{\boldsymbol{u}}_h + \gamma(h)\nabla\chi_h), \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)}, \tag{33a}$$

$$(\mu^{-1}\mathbf{curl}\,\overline{\boldsymbol{p}}_h, \mathbf{curl}\,\boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon\overline{\boldsymbol{p}}_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\overline{\boldsymbol{E}}_h - \boldsymbol{E}_d), \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)}, \tag{33b}$$

$$\overline{\boldsymbol{u}}_h = -\kappa^{-1}\overline{\boldsymbol{p}}_h. \tag{33c}$$

Based on the optimality system (33) and Lemma 4.1, we obtain a discrete counterpart of Proposition 3.5. This result is essential to our convergence analysis.

PROPOSITION 4.3. *For every $h > 0$, let $\boldsymbol{X}_h^{(\epsilon)}$ be the space as defined in* (32), *and let $\overline{\boldsymbol{u}}_h, \overline{\boldsymbol{E}}_h, \overline{\boldsymbol{p}}_h \in \boldsymbol{\mathcal{ND}}_h$ be the optimal triple of* (P_h) *satisfying* (33). *Then it holds that*

$$\overline{\boldsymbol{u}}_h, \overline{\boldsymbol{p}}_h \in \boldsymbol{X}_h^{(\epsilon)} \quad \text{and} \quad (\epsilon\overline{\boldsymbol{E}}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = -(\rho, \psi_h)_{L^2(\Omega)} \quad \forall \psi_h \in \Theta_h, \forall h > 0. \tag{34}$$

Proof. Thanks to (28), we may insert $\boldsymbol{v}_h = \nabla\psi_h$ with $\psi_h \in \Theta_h$ in (33a) and use (33c) and (30) to obtain

$$\begin{aligned}\gamma(h)(\epsilon\overline{\boldsymbol{E}}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} &= (\epsilon(\overline{\boldsymbol{u}}_h + \gamma(h)\nabla\chi_h), \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} \\ &= -\kappa^{-1}(\epsilon\overline{\boldsymbol{p}}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon\nabla\chi_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} \\ &= -\kappa^{-1}(\epsilon\overline{\boldsymbol{p}}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} - \gamma(h)(\rho, \psi_h)_{L^2(\Omega)} \quad \forall \psi_h \in \Theta_h.\end{aligned} \tag{35}$$

Similarly, inserting $\boldsymbol{v}_h = \nabla\psi_h$ with $\psi_h \in \Theta_h$ in (33b) yields

$$\begin{aligned}\gamma(h)(\epsilon\overline{\boldsymbol{p}}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} &= (\epsilon(\overline{\boldsymbol{E}}_h - \boldsymbol{E}_d), \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} \\ &= (\epsilon\overline{\boldsymbol{E}}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} + (\rho, \psi_h)_{L^2(\Omega)} \quad \forall \psi_h \in \Theta_h,\end{aligned} \tag{36}$$

where we have used (5). Then we infer from (35) and (36) that

$$\begin{aligned}\gamma(h)^2(\epsilon\overline{\boldsymbol{p}}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} &= \gamma(h)\left((\epsilon\overline{\boldsymbol{E}}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} + (\rho, \psi_h)_{L^2(\Omega)}\right) \\ &= -\kappa^{-1}(\epsilon\overline{\boldsymbol{p}}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} \quad \forall \psi_h \in \Theta_h,\end{aligned}$$

from which it follows that $(\gamma(h)^2 + \kappa^{-1})(\epsilon\overline{\boldsymbol{p}}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = 0$ for all $\psi_h \in \Theta_h$. Thus, we come to the desired conclusion that

$$\begin{aligned}\overline{\boldsymbol{p}}_h \in \boldsymbol{X}_h^{(\epsilon)} &\underbrace{\Longrightarrow}_{(33c)} \quad \overline{\boldsymbol{u}}_h \in \boldsymbol{X}_h^{(\epsilon)} \\ &\underbrace{\Longrightarrow}_{Lemma\,4.1} \quad (\epsilon\overline{\boldsymbol{E}}_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = -(\rho, \psi_h)_{L^2(\Omega)} \ \forall \psi_h \in \Theta_h.\end{aligned}$$

∎

The upcoming lemma states the discrete compactness property for $\boldsymbol{X}_h^{(\epsilon)}$. The discrete compactness property for Nédélec's edge elements in the case $\epsilon \equiv 1$ goes back to Kikuchi [14].

LEMMA 4.4. *Let $\{\boldsymbol{z}_h\}_{h>0}$ be a uniformly bounded sequence in $\boldsymbol{H}_0(\mathbf{curl})$ satisfying $\boldsymbol{z}_h \in \boldsymbol{X}_h^{(\epsilon)}$ for all $h > 0$. Then, there exists a subsequence $\{\boldsymbol{z}_{h_n}\}_{n=1}^{\infty} \subset \{\boldsymbol{z}_h\}_{h>0}$ with $h_n \to 0$ as $n \to \infty$ such that*

$$\begin{aligned}\boldsymbol{z}_{h_n} \to \boldsymbol{z} \quad &\textit{strongly in } \boldsymbol{L}^2(\Omega) \textit{ as } n \to \infty, \\ \mathbf{curl}\, \boldsymbol{z}_{h_n} \rightharpoonup \mathbf{curl}\, \boldsymbol{z} \quad &\textit{weakly in } \boldsymbol{L}^2(\Omega) \textit{ as } n \to \infty\end{aligned}$$

for some $\boldsymbol{z} \in \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{U}$, i.e., $\operatorname{div}(\epsilon\boldsymbol{z}) = 0$ *in* Ω.

Proof. The assertion is well known (see, e.g., [18]). We provide the proof only for the convenience of the reader. In view of the discrete Helmholtz decomposition, for every $h > 0$, there exists a unique pair $(\boldsymbol{y}_h^1, \theta_h^1) \in \boldsymbol{X}_h^{(1)} \times \Theta_h$ such that

$$\boldsymbol{z}_h = \boldsymbol{y}_h^1 + \nabla\theta_h^1. \tag{37}$$

Due to the uniform boundedness $\{\boldsymbol{z}_h\}_{h>0} \subset \boldsymbol{H}_0(\mathbf{curl})$, the sequence $\{\boldsymbol{y}_h^1\}_{h>0}$ is uniformly bounded in $\boldsymbol{H}_0(\mathbf{curl})$. Thus, employing the discrete compactness property [14] for $\epsilon \equiv 1$, we find a subsequence $\{\boldsymbol{y}_{h_n}^1\}_{n=1}^\infty \subset \{\boldsymbol{y}_h^1\}_{h>0}$ with $h_n \to 0$ as $n \to \infty$ such that

$$\begin{aligned} \boldsymbol{y}_{h_n}^1 &\to \boldsymbol{y}^1 \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \text{ as } n \to \infty, \\ \mathbf{curl}\, \boldsymbol{y}_{h_n}^1 &\rightharpoonup \mathbf{curl}\, \boldsymbol{y}^1 \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \text{ as } n \to \infty \end{aligned} \tag{38}$$

for some $\boldsymbol{y}^1 \in \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}{=}0)$. Now, making use of the classical Helmholtz decomposition, there exists a unique pair $(\boldsymbol{y}^\epsilon, \theta^\epsilon) \in \boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{U} \times H_0^1(\Omega)$ such that

$$\boldsymbol{y}^1 = \boldsymbol{y}^\epsilon + \nabla\theta^\epsilon. \tag{39}$$

We now show that $\boldsymbol{z}_{h_n} \to \boldsymbol{y}^\epsilon$ strongly in $\boldsymbol{L}^2(\Omega)$ as $n \to \infty$. Since $\boldsymbol{y}^\epsilon \in \boldsymbol{U}$ and $\boldsymbol{z}_{h_n} \in \boldsymbol{X}_{h_n}^{(\epsilon)}$ holds for all $n \in \mathbb{N}$,

$$(\epsilon(\boldsymbol{z}_{h_n} - \boldsymbol{y}^\epsilon), \nabla\phi_{h_n})_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \phi_{h_n} \in \Theta_{h_n},\ \forall n \in \mathbb{N}. \tag{40}$$

From (37), (39), and (40), we obtain that

$$(\epsilon(\boldsymbol{z}_{h_n} - \boldsymbol{y}^\epsilon), \boldsymbol{z}_{h_n} - \boldsymbol{y}^\epsilon)_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\boldsymbol{z}_{h_n} - \boldsymbol{y}^\epsilon), \boldsymbol{y}_{h_n}^1 - \boldsymbol{y}^1 + \nabla\theta^\epsilon - \nabla\phi_{h_n})_{\boldsymbol{L}^2(\Omega)}\ \forall \phi_{h_n} \in \Theta_{h_n},\ \forall n \in \mathbb{N},$$

and so

$$\underline{\epsilon}\|\boldsymbol{z}_{h_n} - \boldsymbol{y}^\epsilon\|_{\boldsymbol{L}^2(\Omega)} \le \overline{\epsilon}\|\boldsymbol{y}_{h_n}^1 - \boldsymbol{y}^1\|_{\boldsymbol{L}^2(\Omega)} + \overline{\epsilon}\|\nabla\theta^\epsilon - \nabla\phi_{h_n}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \phi_{h_n} \in \Theta_{h_n},\ \forall n \in \mathbb{N}.$$

Now employing (38) and the fact that $\{\Theta_{h_n}\}_{n=1}^\infty$ is dense in $H_0^1(\Omega)$, the above inequality implies the strong convergence $\boldsymbol{z}_{h_n} \to \boldsymbol{y}^\epsilon$ in $\boldsymbol{L}^2(\Omega)$ as $n \to \infty$. ◻

In what follows, for every $h > 0$, we shall denote by $(\overline{\boldsymbol{u}}_h, \overline{\boldsymbol{E}}_h, \overline{\boldsymbol{p}}_h) \in \boldsymbol{X}_h^{(\epsilon)} \times \boldsymbol{\mathcal{ND}}_h \times \boldsymbol{X}_h^{(\epsilon)}$ the optimal triple of (P$_h$) satisfying (33). Let us now prove the strong convergence of $(\overline{\boldsymbol{u}}_h, \overline{\boldsymbol{E}}_h, \overline{\boldsymbol{p}}_h)$ to $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{E}}, \overline{\boldsymbol{p}})$ as $h \to 0$ in the following theorem.

THEOREM 4.5. *Suppose that* $\lim_{h\to 0} \gamma(h) = 0$. *Then,*

$$\lim_{h\to 0} \|\overline{\boldsymbol{u}}_h - \overline{\boldsymbol{u}}\|_{\boldsymbol{H}(\mathbf{curl})} = \lim_{h\to 0} \|\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}\|_{\boldsymbol{H}(\mathbf{curl})} = \lim_{h\to 0} \|\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}}\|_{\boldsymbol{H}(\mathbf{curl})} = 0.$$

Proof. For every $h > 0$, the fact that $\overline{\boldsymbol{u}}_h$ is the unique solution of (P$_h$) yields

$$f_h(\overline{\boldsymbol{u}}_h) \le f_h(0) \underbrace{=}_{\textit{Lemma } 4.1} \frac{1}{2}\|\epsilon^{1/2}(\nabla\chi_h - \boldsymbol{E}_d)\|_{\boldsymbol{L}^2(\Omega)}^2 \quad \forall h > 0.$$

Therefore, in view of (30), there exists a constant $c > 0$, independent of h, such that

$$\|\overline{\boldsymbol{E}}_h\|_{\boldsymbol{L}^2(\Omega)} + \|\overline{\boldsymbol{u}}_h\|_{\boldsymbol{L}^2(\Omega)} \le c \quad \forall h > 0 \quad \underbrace{\Longrightarrow}_{(33c)} \quad \|\overline{\boldsymbol{p}}_h\|_{\boldsymbol{L}^2(\Omega)} \le c \quad \forall h > 0. \tag{41}$$

Now, setting $\boldsymbol{v}_h = \overline{\boldsymbol{E}}_h$ in (33a) and $\boldsymbol{v}_h = \overline{\boldsymbol{p}}_h$ in (33b) and then employing (41) and (29), we obtain

$$\|\mathbf{curl}\, \overline{\boldsymbol{E}}_h\|_{\boldsymbol{L}^2(\Omega)} \le c \quad \text{and} \quad \|\mathbf{curl}\, \overline{\boldsymbol{p}}_h\|_{\boldsymbol{L}^2(\Omega)} \le c \quad \forall h > 0. \tag{42}$$

From (41) and (42), we conclude that the sequences $\{\overline{\boldsymbol{u}}_h\}_{h>0}, \{\overline{\boldsymbol{E}}_h\}_{h>0}, and \{\overline{\boldsymbol{p}}_h\}_{h>0}$ are all uniformly bounded in $\boldsymbol{H}_0(\mathbf{curl})$. Therefore, there exists a subsequence $\{(\overline{\boldsymbol{u}}_{h_n}, \overline{\boldsymbol{E}}_{h_n}, \overline{\boldsymbol{p}}_{h_n})\}_{n=1}^{\infty} \subset \{(\overline{\boldsymbol{u}}_h, \overline{\boldsymbol{E}}_h, \overline{\boldsymbol{p}}_h)\}_{h>0}$ with $h_n \to 0$ as $n \to \infty$ such that

$$\left\{ \begin{array}{lll} \overline{\boldsymbol{u}}_{h_n} & \rightharpoonup & \widetilde{\boldsymbol{u}} \quad \text{weakly in } \boldsymbol{H}_0(\mathbf{curl}) \text{ as } n \to \infty, \\ \overline{\boldsymbol{E}}_{h_n} & \rightharpoonup & \widetilde{\boldsymbol{E}} \quad \text{weakly in } \boldsymbol{H}_0(\mathbf{curl}) \text{ as } n \to \infty, \\ \overline{\boldsymbol{p}}_{h_n} & \rightharpoonup & \widetilde{\boldsymbol{p}} \quad \text{weakly in } \boldsymbol{H}_0(\mathbf{curl}) \text{ as } n \to \infty \end{array} \right. \tag{43}$$

for some $\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{p}} \in \boldsymbol{H}_0(\mathbf{curl})$. From (33c), we know $\boldsymbol{u}_{h_n} = -\kappa^{-1}\boldsymbol{p}_{h_n}$ for all $n \in \mathbb{N}$. Thus, (43) implies that

$$\widetilde{\boldsymbol{u}} = -\kappa^{-1}\widetilde{\boldsymbol{p}}. \tag{44}$$

Now, we denote by $\mathcal{I}_h : \mathcal{C}_0^\infty(\Omega) \to \Theta_h$ the nodal interpolation operator corresponding to the finite element space Θ_h. By virtue of Proposition 4.3, it holds for every $\psi \in \mathcal{C}_0^\infty(\Omega)$ that

$$\left\{ \begin{array}{l} (\epsilon \boldsymbol{u}_{h_n}, \nabla \mathcal{I}_{h_n}\psi)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall n \in \mathbb{N}, \\ (\epsilon \boldsymbol{p}_{h_n}, \nabla \mathcal{I}_{h_n}\psi)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall n \in \mathbb{N}, \\ (\epsilon \boldsymbol{E}_{h_n}, \nabla \mathcal{I}_{h_n}\psi)_{\boldsymbol{L}^2(\Omega)} = -(\rho, \mathcal{I}_h\psi)_{L^2(\Omega)} \quad \forall n \in \mathbb{N}. \end{array} \right. \tag{45}$$

Then, passing to the limit $n \to \infty$ in (45), we obtain from (43) that

$$(\epsilon\widetilde{\boldsymbol{u}}, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = (\epsilon\widetilde{\boldsymbol{p}}, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \text{and} \quad (\epsilon\widetilde{\boldsymbol{E}}, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = -(\rho, \psi)_{L^2(\Omega)} \ \forall \psi \in \mathcal{C}_0^\infty(\Omega).$$

Consequently, since $\mathcal{C}_0^\infty(\Omega) \subset H_0^1(\Omega)$ is dense, we come to the conclusion that

$$\left\{ \begin{array}{l} (\epsilon\widetilde{\boldsymbol{u}}, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi \in H_0^1(\Omega), \\ (\epsilon\widetilde{\boldsymbol{p}}, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi \in H_0^1(\Omega), \\ (\epsilon\widetilde{\boldsymbol{E}}, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} = -(\rho, \psi)_{L^2(\Omega)} \quad \forall \psi \in H_0^1(\Omega). \end{array} \right. \tag{46}$$

Next, let $\boldsymbol{\mathcal{N}}_h : \boldsymbol{\mathcal{C}}_{\mathbf{0}}^{\boldsymbol{\infty}}(\Omega) \to \boldsymbol{\mathcal{ND}}_h$ denote the curl-conforming Nédélec interpolation operator corresponding to the finite element space $\boldsymbol{\mathcal{ND}}_h$. According to (33a), we have for every $\boldsymbol{v} \in \boldsymbol{\mathcal{C}}_{\mathbf{0}}^{\boldsymbol{\infty}}(\Omega)$ that

$$\begin{aligned} &(\mu^{-1}\mathbf{curl}\,\overline{\boldsymbol{E}}_{h_n}, \mathbf{curl}\,\boldsymbol{\mathcal{N}}_{h_n}\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + \gamma(h_n)(\epsilon\overline{\boldsymbol{E}}_{h_n}, \boldsymbol{\mathcal{N}}_{h_n}\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \\ &\qquad\qquad = (\epsilon(\overline{\boldsymbol{u}}_{h_n} + \gamma(h_n)\nabla\chi_{h_n}), \boldsymbol{\mathcal{N}}_{h_n}\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \quad \forall n \in \mathbb{N}. \end{aligned} \tag{47}$$

Passing to the limit $n \to \infty$ in (47), we obtain from (43) and $\lim_{n\to\infty}\gamma(h_n) = 0$ that

$$(\mu^{-1}\mathbf{curl}\,\widetilde{\boldsymbol{E}}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\epsilon\widetilde{\boldsymbol{u}}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{C}}_{\mathbf{0}}^{\boldsymbol{\infty}}(\Omega).$$

Therefore, as $\boldsymbol{\mathcal{C}}_{\mathbf{0}}^{\boldsymbol{\infty}}(\Omega) \subset \boldsymbol{H}_0(\mathbf{curl})$ is dense, it follows that

$$(\mu^{-1}\mathbf{curl}\,\widetilde{\boldsymbol{E}}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\epsilon\widetilde{\boldsymbol{u}}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}). \tag{48}$$

Analogously, we deduce from (33b), (43), and $\lim_{n\to\infty}\gamma(h_n) = 0$ that

$$(\mu^{-1}\mathbf{curl}\,\widetilde{\boldsymbol{p}}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\widetilde{\boldsymbol{E}} - \boldsymbol{E}_d), \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}). \tag{49}$$

We can see from (44), (46), and (48)–(49) that the weak limit $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{p}})$ satisfies the necessary and sufficient optimality condition for (P), and consequently

$$(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{p}}) = (\overline{\boldsymbol{u}}, \overline{\boldsymbol{E}}, \overline{\boldsymbol{p}}).$$

In particular, the weak limit is independent of the subsequence $\{(\overline{\boldsymbol{u}}_{h_n}, \overline{\boldsymbol{E}}_{h_n}, \overline{\boldsymbol{p}}_{h_n}\}_{n=1}^{\infty}$, and consequently (43) holds for the whole sequence, i.e.,

$$\left\{\begin{array}{lll} \overline{\boldsymbol{u}}_h & \rightharpoonup & \overline{\boldsymbol{u}} \quad \text{weakly in } \boldsymbol{H}_0(\mathbf{curl}) \text{ as } h \to 0, \\ \overline{\boldsymbol{E}}_h & \rightharpoonup & \overline{\boldsymbol{E}} \quad \text{weakly in } \boldsymbol{H}_0(\mathbf{curl}) \text{ as } h \to 0, \\ \overline{\boldsymbol{p}}_h & \rightharpoonup & \overline{\boldsymbol{p}} \quad \text{weakly in } \boldsymbol{H}_0(\mathbf{curl}) \text{ as } h \to 0. \end{array}\right. \tag{50}$$

Now, making use of Lemma 4.4, we obtain from Proposition 4.3 and (50) that

$$\left\{\begin{array}{lll} \overline{\boldsymbol{u}}_h & \to & \overline{\boldsymbol{u}} \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \text{ as } h \to 0, \\ \overline{\boldsymbol{p}}_h & \to & \overline{\boldsymbol{p}} \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \text{ as } h \to 0. \end{array}\right. \tag{51}$$

Setting $\boldsymbol{v}_h = \boldsymbol{v} = \overline{\boldsymbol{p}}_h$ in (33b) and (11b) yields

$$(\mu^{-1}\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}}), \mathbf{curl}\,\overline{\boldsymbol{p}}_h)_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon\overline{\boldsymbol{p}}_h, \overline{\boldsymbol{p}}_h)_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}), \overline{\boldsymbol{p}}_h)_{\boldsymbol{L}^2(\Omega)},$$

from which it follows that

$$\begin{aligned}\|\mu^{-1/2}\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}})\|^2_{\boldsymbol{L}^2(\Omega)} = -\gamma(h)(\epsilon\overline{\boldsymbol{p}}_h, \overline{\boldsymbol{p}}_h)_{\boldsymbol{L}^2(\Omega)} + (\epsilon(\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}), \overline{\boldsymbol{p}}_h)_{\boldsymbol{L}^2(\Omega)} \\ - (\mu^{-1}\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}}), \mathbf{curl}\,\overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Then, passing to the limit $h \to 0$, (50), (51), and $\lim_{h\to 0}\gamma(h) = 0$ imply

$$\lim_{h\to 0}\|\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}})\|_{\boldsymbol{L}^2(\Omega)} = 0.$$

Together with (51), this strong convergence yields

$$\lim_{h\to 0}\|\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}}\|_{\boldsymbol{H}(\mathbf{curl})} = 0 \quad \underbrace{\Longrightarrow}_{(33c)} \quad \lim_{h\to 0}\|\overline{\boldsymbol{u}}_h - \overline{\boldsymbol{u}}\|_{\boldsymbol{H}(\mathbf{curl})} = 0. \tag{52}$$

It remains now to prove the strong convergence of $\{\overline{\boldsymbol{E}}_h\}_{h>0}$ in $\boldsymbol{H}_0(\mathbf{curl})$. First, we verify the strong convergence in $\boldsymbol{L}^2(\Omega)$ by inserting $\boldsymbol{v}_h = \boldsymbol{v} = \overline{\boldsymbol{E}}_h$ in (33b) and (11b):

$$(\mu^{-1}\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}}), \mathbf{curl}\,\overline{\boldsymbol{E}}_h)_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon\overline{\boldsymbol{p}}_h, \overline{\boldsymbol{E}}_h)_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}), \overline{\boldsymbol{E}}_h)_{\boldsymbol{L}^2(\Omega)},$$

from which it follows that

$$\begin{aligned}\|\epsilon^{1/2}(\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}})\|^2_{\boldsymbol{L}^2(\Omega)} =&(\mu^{-1}\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}}), \mathbf{curl}\,\overline{\boldsymbol{E}}_h)_{\boldsymbol{L}^2(\Omega)} \\ &+ \gamma(h)(\epsilon\overline{\boldsymbol{p}}_h, \overline{\boldsymbol{E}}_h)_{\boldsymbol{L}^2(\Omega)} - (\epsilon(\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}), \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)}.\end{aligned} \tag{53}$$

Then, passing to the limit $h \to 0$ in (53), we obtain from (50), (52), and $\lim_{h\to 0}\gamma(h) = 0$ that

$$\lim_{h\to 0}\|\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega)} = 0. \tag{54}$$

Similarly, by setting $\boldsymbol{v}_h = \boldsymbol{v} = \overline{\boldsymbol{E}}_h$ in (33a) and (11a), we deduce from (50), (52), and $\lim_{h\to 0}\gamma(h) = 0$ that

$$\lim_{h\to 0}\|\mathbf{curl}\,(\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}})\|_{\boldsymbol{L}^2(\Omega)} = 0. \tag{55}$$

From (54)–(55), we come to the conclusion that $\lim_{h\to 0}\|\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}\|_{\boldsymbol{H}(\mathbf{curl})} = 0$. ∎

4.2. Error estimates. We first show that the newly proposed finite element approximation (P_h) ensures the global strong convergence of all three Gauss laws for the discrete optimal control, state, and adjoint state $\overline{\boldsymbol{u}}_h$, $\overline{\boldsymbol{E}}_h$, and $\overline{\boldsymbol{p}}_h$, which satisfy the optimality system (33). We recall that, using the fact that Ω is Lipschitz polyhedral, there is a constant $\delta \in (0.5, 1]$ such that [1]

$$\boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div}) \hookrightarrow \boldsymbol{H}^\delta(\Omega) \quad \text{and} \quad \boldsymbol{H}(\mathbf{curl}) \cap \boldsymbol{H}_0(\mathrm{div}) \hookrightarrow \boldsymbol{H}^\delta(\Omega). \tag{56}$$

The results in the following lemma were verified in [5, Lemma 3.9].

LEMMA 4.6. *Suppose that $\epsilon \in W^{1,\infty}(\Omega)$ and $s \in (0.5, 1]$. Then, there exists a constant $c > 0$, independent of h and $\boldsymbol{z}_h$, such that*

$$\|\mathrm{div}\,(\epsilon \boldsymbol{z}_h)\|_{H^{-s}(\Omega)} \le c h^{s+\delta-1} \|\mathbf{curl}\, \boldsymbol{z}_h\|_{\boldsymbol{L}^2(\Omega)}$$

for all $h > 0$ and all $\boldsymbol{z}_h \in \boldsymbol{X}_h^{(\epsilon)}$. Moreover, the solution $\chi_h \in \Theta_h$ of (30) satisfies

$$\|\mathrm{div}\,(\epsilon \nabla \chi_h) - \rho\|_{H^{-s}(\Omega)} \le c h^{s+\delta-1} \|\rho\|_{H^{\delta-1}(\Omega)} \quad \forall h > 0.$$

THEOREM 4.7. *Suppose that $\epsilon \in W^{1,\infty}(\Omega)$, and $s \in (0.5, 1]$. Then, there exists a positive constant c, independent of h, $\overline{\boldsymbol{u}}_h$, $\overline{\boldsymbol{E}}_h$, and $\overline{\boldsymbol{p}}_h$, such that for all $h > 0$,*

$$\|\mathrm{div}(\epsilon \overline{\boldsymbol{u}}_h)\|_{H^{-s}(\Omega)} + \|\mathrm{div}(\epsilon \overline{\boldsymbol{p}}_h)\|_{H^{-s}(\Omega)} + \|\mathrm{div}(\epsilon \overline{\boldsymbol{E}}_h) - \rho\|_{H^{-s}(\Omega)} \le c h^{s+\delta-1}.$$

Proof. From Proposition 4.3, we know that $\overline{\boldsymbol{u}}_h, \overline{\boldsymbol{p}}_h \in \boldsymbol{X}_h^{(\epsilon)}$ for all $h > 0$. Therefore, Lemma 4.6 together with the uniform boundedness of $\{\overline{\boldsymbol{u}}_h\}_{h>0}$ and $\{\overline{\boldsymbol{p}}_h\}_{h>0}$ in $\boldsymbol{H}_0(\mathbf{curl})$ (see Theorem 4.5) implies

$$\|\mathrm{div}\,(\epsilon \overline{\boldsymbol{u}}_h)\|_{H^{-s}(\Omega)} + \|\mathrm{div}\,(\epsilon \overline{\boldsymbol{p}}_h)\|_{H^{-s}(\Omega)} \le c h^{s+\delta-1} \quad \forall h > 0. \tag{57}$$

Making use again of Proposition 4.3 along with (30), we have that

$$(\epsilon \overline{\boldsymbol{E}}_h, \nabla \psi_h)_{\boldsymbol{L}^2(\Omega)} = -(\rho, \psi_h)_{L^2(\Omega)} = (\epsilon \nabla \chi_h, \nabla \psi_h)_{\boldsymbol{L}^2(\Omega)} \quad \forall \psi \in \Theta_h,$$

from which it follows that

$$(\epsilon(\overline{\boldsymbol{E}}_h - \nabla \chi_h), \nabla \psi_h)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi \in \Theta_h \quad \Longrightarrow \quad \overline{\boldsymbol{E}}_h - \nabla \chi_h \in \boldsymbol{X}_h^{(\epsilon)} \quad \forall h > 0.$$

Then using Lemma 4.6 we can derive

$$\begin{aligned} \|\mathrm{div}\,(\epsilon \overline{\boldsymbol{E}}_h) - \rho\|_{H^{-s}(\Omega)} &\le \|\mathrm{div}\,(\epsilon(\overline{\boldsymbol{E}}_h - \nabla \chi_h))\|_{H^{-s}(\Omega)} + \|\mathrm{div}\,(\epsilon \nabla \chi_h) - \rho\|_{H^{-s}(\Omega)} \\ &\le c h^{s+\delta-1} (\|\mathbf{curl}\, \overline{\boldsymbol{E}}_h\|_{\boldsymbol{L}^2(\Omega)} + \|\rho\|_{H^{\delta-1}(\Omega)}) \quad \forall h > 0. \end{aligned} \tag{58}$$

Therefore, since $\{\overline{\boldsymbol{E}}_h\}_{h>0}$ is uniformly bounded in $\boldsymbol{H}_0(\mathbf{curl})$, the desired assertion follows from (57)–(58). ∎

As our main goal, we will derive next the error estimates for the optimal control, state, and adjoint state of the proposed edge element method (P_h). To do so, we introduce the following discrete mixed variational problem.

For a given $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl})$, find the solution $\boldsymbol{E}_h = \Phi_h(\boldsymbol{E}) \in \boldsymbol{\mathcal{ND}}_h$ to

$$\left\{ \begin{aligned} (\mu^{-1} \mathbf{curl}\, \boldsymbol{E}_h, \mathbf{curl}\, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} &= (\mu^{-1} \mathbf{curl}\, \boldsymbol{E}, \mathbf{curl}\, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} && \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{ND}}_h, \\ (\epsilon \boldsymbol{E}_h, \nabla \psi_h)_{\boldsymbol{L}^2(\Omega)} &= (\epsilon \boldsymbol{E}, \nabla \psi_h) && \forall \psi_h \in \Theta_h. \end{aligned} \right. \tag{59}$$

It is standard to verify that, for every $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl})$, the mixed discrete variational problem (59) admits a unique solution $\Phi_h \boldsymbol{E} := \boldsymbol{E}_h \in \boldsymbol{\mathcal{ND}}_h$ for all $h > 0$, satisfying

$$\|\Phi_h \boldsymbol{E} - \boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl})} \leq c \left(\inf_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{ND}}_h} \|\boldsymbol{v}_h - \boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl})} \right) \quad \forall \boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl}). \tag{60}$$

This follows again from a well-known theory for mixed variational problems (see, e.g., [18, Theorem 2.45]) by utilizing (6)–(7), the discrete Poincaré–Friedrichs-type inequality [9, Theorem 4.7],

$$\|\boldsymbol{E}_h\|_{\boldsymbol{L}^2(\Omega)} \leq c\|\mathbf{curl}\, \boldsymbol{E}_h\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{E}_h \in \boldsymbol{X}_h^{(\epsilon)},\ \forall h > 0, \tag{61}$$

and the discrete LBB condition,

$$\sup_{0 \neq \boldsymbol{E}_h \in \boldsymbol{\mathcal{ND}}_h} \frac{|(\epsilon \boldsymbol{E}_h, \nabla \psi_h)_{\boldsymbol{L}^2(\Omega)}|}{\|\boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl})}} \geq \frac{(\epsilon \nabla \psi_h, \nabla \psi_h)_{\boldsymbol{L}^2(\Omega)}}{\|\nabla \psi_h\|_{\boldsymbol{H}(\mathbf{curl})}} \geq c\|\psi_h\|_{H_0^1(\Omega)} \quad \forall \psi_h \in \Theta_h, \tag{62}$$

with a constant $c > 0$ depending only on $\underline{\epsilon}$ and Ω. Notice that (62) holds due to the inclusion $\nabla \Theta_h \subset \boldsymbol{\mathcal{ND}}_h$. Now, making use of the operator $\Phi_h : \boldsymbol{H}_0(\mathbf{curl}) \to \boldsymbol{\mathcal{ND}}_h$, we obtain the following important identity for our subsequent analysis.

LEMMA 4.8. *It holds for all $h > 0$ that*

$$\begin{aligned} &\|\mu^{-1/2}\mathbf{curl}\,(\overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})\|^2_{\boldsymbol{L}^2(\Omega)} + \kappa^{-1}\|\mu^{-1/2}\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \Phi_h \overline{\boldsymbol{p}})\|^2_{\boldsymbol{L}^2(\Omega)} \\ &= \gamma(h)\left[(\epsilon(\nabla\chi_h - \overline{\boldsymbol{E}}_h), \overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} + \kappa^{-1}(\epsilon \overline{\boldsymbol{p}}_h, \Phi_h \overline{\boldsymbol{p}} - \overline{\boldsymbol{p}}_h)_{\boldsymbol{L}^2(\Omega)}\right] \\ &+ \kappa^{-1}(\epsilon(\overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}}), \overline{\boldsymbol{p}} - \Phi_h \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} + \kappa^{-1}(\epsilon(\Phi_h \overline{\boldsymbol{E}} - \overline{\boldsymbol{E}}), \overline{\boldsymbol{p}}_h - \Phi_h \overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

Proof. In view of the state equations (18a) and (33a), we have that

$$\begin{aligned} &(\mu^{-1}\mathbf{curl}\,(\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}), \mathbf{curl}\, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon \overline{\boldsymbol{E}}_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \\ &\quad = (\epsilon(\overline{\boldsymbol{u}}_h - \overline{\boldsymbol{u}} + \gamma(h)\nabla\chi_h), \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{ND}}_h. \end{aligned} \tag{63}$$

Making use of the operator Φ_h and setting $\boldsymbol{v}_h = \overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}}$ in (63), we obtain

$$\begin{aligned} &\|\mu^{-1/2}\mathbf{curl}\,(\overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})\|^2_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon \overline{\boldsymbol{E}}_h, \overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} \\ &= (\epsilon(\overline{\boldsymbol{u}}_h - \overline{\boldsymbol{u}} + \gamma(h)\nabla\chi_h), \overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} \\ &\underbrace{=}_{(18c)\&(33c)} -\kappa^{-1}(\epsilon(\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}}), \overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon\nabla\chi_h, \overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} \\ &= -\kappa^{-1}(\epsilon(\overline{\boldsymbol{p}}_h - \Phi_h \overline{\boldsymbol{p}}), \overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} + \kappa^{-1}(\epsilon(\overline{\boldsymbol{p}} - \Phi_h \overline{\boldsymbol{p}}), \overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} \\ &\quad + \gamma(h)(\epsilon\nabla\chi_h, \overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)}, \end{aligned}$$

which implies

$$\begin{aligned} &\|\mu^{-1/2}\mathbf{curl}\,(\overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})\|^2_{\boldsymbol{L}^2(\Omega)} + \kappa^{-1}(\epsilon(\overline{\boldsymbol{p}}_h - \Phi_h \overline{\boldsymbol{p}}), \overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} \\ &= \gamma(h)(\epsilon(\nabla\chi_h - \overline{\boldsymbol{E}}_h), \overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} + \kappa^{-1}(\epsilon(\overline{\boldsymbol{p}} - \Phi_h \overline{\boldsymbol{p}}), \overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{64}$$

Similarly, we deduce from the adjoint equations (18b) and (33b) that

$$\begin{aligned} &(\mu^{-1}\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}}), \mathbf{curl}\, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon \overline{\boldsymbol{p}}_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \\ &\quad = (\epsilon(\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}), \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{ND}}_h. \end{aligned} \tag{65}$$

Then, making use of the operator Φ_h and setting $\boldsymbol{v}_h = \boldsymbol{p}_h - \Phi_h \overline{\boldsymbol{p}}$ in (65), we derive

$$\begin{aligned}\|\mu^{-1/2}\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})\|^2_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon\overline{\boldsymbol{p}}_h, \overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)}\\ = (\epsilon(\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}), \overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)}\\ = (\epsilon(\overline{\boldsymbol{E}}_h - \Phi_h\overline{\boldsymbol{E}}), \overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} + (\epsilon(\Phi_h\overline{\boldsymbol{E}} - \overline{\boldsymbol{E}}), \overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)},\end{aligned}$$

which implies

$$\begin{aligned}(\epsilon(\overline{\boldsymbol{E}}_h - \Phi_h\overline{\boldsymbol{E}}), \overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} = \|\mu^{-1/2}\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})\|^2_{\boldsymbol{L}^2(\Omega)}\\ + \gamma(h)(\epsilon\overline{\boldsymbol{p}}_h, \overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)} - (\epsilon(\Phi_h\overline{\boldsymbol{E}} - \overline{\boldsymbol{E}}), \overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)}.\end{aligned} \tag{66}$$

Applying (66) to (64), we come to the desired identity:

$$\begin{aligned}\|\mu^{-1/2}\mathbf{curl}\,(\overline{\boldsymbol{E}}_h - \Phi_h\overline{\boldsymbol{E}})\|^2_{\boldsymbol{L}^2(\Omega)} + \kappa^{-1}\|\mu^{-1/2}\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})\|^2_{\boldsymbol{L}^2(\Omega)}\\ = \gamma(h)(\epsilon(\nabla\chi_h - \overline{\boldsymbol{E}}_h), \overline{\boldsymbol{E}}_h - \Phi_h\overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)} + \kappa^{-1}(\epsilon(\overline{\boldsymbol{p}} - \Phi_h\overline{\boldsymbol{p}}), \overline{\boldsymbol{E}}_h - \Phi_h\overline{\boldsymbol{E}})_{\boldsymbol{L}^2(\Omega)}\\ + \kappa^{-1}\gamma(h)(\epsilon\overline{\boldsymbol{p}}_h, \Phi_h\overline{\boldsymbol{p}} - \overline{\boldsymbol{p}}_h)_{\boldsymbol{L}^2(\Omega)} + \kappa^{-1}(\epsilon(\Phi_h\overline{\boldsymbol{E}} - \overline{\boldsymbol{E}}), \overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})_{\boldsymbol{L}^2(\Omega)}.\end{aligned} \tag{67}$$

∎

We now recall a classical error estimate for the curl-conforming Nédélec interpolant $\boldsymbol{\mathcal{N}_h}$ in the space $\boldsymbol{H}^s(\mathbf{curl}) := \{\boldsymbol{E} \in \boldsymbol{H}^s(\Omega) \mid \mathbf{curl}\,\boldsymbol{E} \in \boldsymbol{H}^s(\Omega)\}$ [6].

LEMMA 4.9. *For $s \in (1/2, 1]$, there exists a constant $c > 0$, independent of h and $\boldsymbol{E}$, such that for all $h > 0$,*

$$\|\boldsymbol{E} - \boldsymbol{\mathcal{N}_h}\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl})} \le ch^s\|\boldsymbol{E}\|_{\boldsymbol{H}^s(\mathbf{curl})} \quad \forall \boldsymbol{E} \in \boldsymbol{H}^s(\mathbf{curl}). \tag{68}$$

We are now ready to establish our main result.

THEOREM 4.10. *Suppose that $\overline{\boldsymbol{E}}, \overline{\boldsymbol{p}} \in \boldsymbol{H}^s(\mathbf{curl})$ for some $s \in (0.5, 1]$. Then, there exists a constant $c > 0$, independent of h, $\overline{\boldsymbol{u}}_h$, $\overline{\boldsymbol{E}}_h$, and $\overline{\boldsymbol{p}}_h$, such that*

$$\|\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}\|_{\boldsymbol{H}(\mathbf{curl})} + \|\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}}\|_{\boldsymbol{H}(\mathbf{curl})} + \|\overline{\boldsymbol{u}}_h - \overline{\boldsymbol{u}}\|_{\boldsymbol{H}(\mathbf{curl})} \le c(\gamma(h) + h^s)$$

for all $h > 0$.

Proof. In view of the regularity assumption $\overline{\boldsymbol{E}}, \overline{\boldsymbol{p}} \in \boldsymbol{H}^s(\mathbf{curl})$ with $s \in (0.5, 1]$ along with (60) and (68), there is a constant $c > 0$, independent of h, such that

$$\|\Phi_h\overline{\boldsymbol{E}} - \overline{\boldsymbol{E}}\|_{\boldsymbol{H}(\mathbf{curl})} + \|\Phi_h\overline{\boldsymbol{p}} - \overline{\boldsymbol{p}}\|_{\boldsymbol{H}(\mathbf{curl})} \le ch^s \quad \forall h > 0. \tag{69}$$

On the other hand, according to Lemma 4.8, we have the following estimate:

$$\begin{aligned}\underline{\mu}^{-1}\|\mathbf{curl}\,(\overline{\boldsymbol{E}}_h - \Phi_h\overline{\boldsymbol{E}})\|^2_{\boldsymbol{L}^2(\Omega)} + \kappa^{-1}\underline{\mu}^{-1}\|\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})\|^2_{\boldsymbol{L}^2(\Omega)}\\ \le \Big(\gamma(h)\big(\|\epsilon(\nabla\chi_h - \overline{\boldsymbol{E}}_h)\|_{\boldsymbol{L}^2(\Omega)} + \kappa^{-1}\|\epsilon\overline{\boldsymbol{p}}_h\|_{\boldsymbol{L}^2(\Omega)}\big) + \kappa^{-1}\|\epsilon(\overline{\boldsymbol{p}} - \Phi_h\overline{\boldsymbol{p}})\|_{\boldsymbol{L}^2(\Omega)}\\ + \kappa^{-1}\|\epsilon(\overline{\boldsymbol{E}} - \Phi_h\overline{\boldsymbol{E}})\|_{\boldsymbol{L}^2(\Omega)}\Big)\Big(\|\overline{\boldsymbol{E}}_h - \Phi_h\overline{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega)} + \|\overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}}\|_{\boldsymbol{L}^2(\Omega)}\Big) \quad \forall h > 0.\end{aligned}$$

Then applying (69) to the above estimate yields

$$\begin{aligned}(\|\mathbf{curl}\,(\overline{\boldsymbol{E}}_h - \Phi_h\overline{\boldsymbol{E}})\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}})\|_{\boldsymbol{L}^2(\Omega)})^2\\ \le c(\gamma(h) + h^s)(\|\overline{\boldsymbol{E}}_h - \Phi_h\overline{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega)} + \|\overline{\boldsymbol{p}}_h - \Phi_h\overline{\boldsymbol{p}}\|_{\boldsymbol{L}^2(\Omega)}) \quad \forall h > 0.\end{aligned} \tag{70}$$

Using the definition of Φ_h and Proposition 4.3, we have for every $h > 0$ that

$$(\epsilon(\overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}}), \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}), \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi_h \in \Theta_h,$$
$$(\epsilon(\overline{\boldsymbol{p}}_h - \Phi_h \overline{\boldsymbol{p}}), \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = (\epsilon(\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}}), \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \psi_h \in \Theta_h.$$

In other words, it holds that

$$\overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}} \in \boldsymbol{X}_h^{(\epsilon)} \quad \text{and} \quad \overline{\boldsymbol{p}}_h - \Phi_h \overline{\boldsymbol{p}} \in \boldsymbol{X}_h^{(\epsilon)} \quad \forall h > 0,$$

so it follows from the discrete Poincaré–Friedrichs-type inequality (61) that

$$\begin{aligned} \|\overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega)} + \|\overline{\boldsymbol{p}}_h - \Phi_h \overline{\boldsymbol{p}}\|_{\boldsymbol{L}^2(\Omega)} \le c\big(&\|\mathbf{curl}\,(\overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})\|_{\boldsymbol{L}^2(\Omega)} \\ &+ \|\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \Phi_h \overline{\boldsymbol{p}})\|_{\boldsymbol{L}^2(\Omega)}\big) \quad \forall h > 0. \end{aligned} \tag{71}$$

Applying (71) to (70), we deduce that

$$\|\mathbf{curl}\,(\overline{\boldsymbol{E}}_h - \Phi_h \overline{\boldsymbol{E}})\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,(\overline{\boldsymbol{p}}_h - \Phi_h \overline{\boldsymbol{p}})\|_{\boldsymbol{L}^2(\Omega)} \le c(\gamma(h) + h^s)\ \forall h > 0. \tag{72}$$

Combining the estimates (71)–(72) along with (69), we finally obtain the estimate

$$\|\overline{\boldsymbol{E}}_h - \overline{\boldsymbol{E}}\|_{\boldsymbol{H}(\mathbf{curl})} + \|\overline{\boldsymbol{p}}_h - \overline{\boldsymbol{p}}\|_{\boldsymbol{H}(\mathbf{curl})} \le c(\gamma(h) + h^s) \quad \forall h > 0. \tag{73}$$

Now the desired estimate follows from this estimate above and the optimality conditions (11c) and (33c). ∎

Remark 4.11. We can easily observe that Theorem 4.10 ensures the optimal convergence rate for our proposed finite element optimal control method (P_h) if we take $\gamma = O(h)$. Note that our analysis can help improve the error estimate in [5]. In fact, by making use of the operator Φ_h, we are able to significantly improve the convergence rate of $\sqrt{\gamma(h)} + h^s$ achieved in [5] for the edge element approximation of the stationary Maxwell system (1) with a nonvanishing charge density. Our improved result is provided in the following corollary, whose proof is analogous to that of Theorem 4.10.

COROLLARY 4.12. *Let* $\mathbf{f} \in \boldsymbol{H}(\mathrm{div}{=}0)$ *and* $\boldsymbol{z} \in \boldsymbol{H}_0(\mathbf{curl})$ *denote the unique solution of*

$$\left\{ \begin{aligned} (\mu^{-1}\mathbf{curl}\,\boldsymbol{z}, \mathbf{curl}\,\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} &= (\mathbf{f}, \boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} && \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ (\epsilon \boldsymbol{z}, \nabla\psi)_{\boldsymbol{L}^2(\Omega)} &= -(\rho, \psi)_{L^2(\Omega)} && \forall \psi \in H_0^1(\Omega). \end{aligned} \right. \tag{74}$$

Furthermore, for every $h > 0$*, let* $\boldsymbol{z}_h \in \boldsymbol{\mathcal{ND}}_h$ *denote the unique solution of*

$$\begin{aligned} (\mu^{-1}\mathbf{curl}\,\boldsymbol{z}_h, \mathbf{curl}\,\boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} + \gamma(h)(\epsilon \boldsymbol{z}_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} &= (\mathbf{f} + \gamma(h)\epsilon\nabla\chi_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \\ &\qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{ND}}_h, \end{aligned} \tag{75}$$

where $\chi_h \in \Theta_h$ *is the solution of*

$$(\epsilon\nabla\chi_h, \nabla\psi_h)_{\boldsymbol{L}^2(\Omega)} = -(\rho, \psi_h)_{\boldsymbol{L}^2(\Omega)} \quad \forall \psi_h \in \Theta_h. \tag{76}$$

Then, if $\boldsymbol{z} \in \boldsymbol{H}^s(\mathbf{curl})$ *for some* $s \in (0.5, 1]$*, there exists a constant* $c > 0$*, independent of* h*,* $\boldsymbol{z}$*, and* $\boldsymbol{z}_h$*, such that*

$$\|\boldsymbol{z} - \boldsymbol{z}_h\|_{\boldsymbol{H}(\mathbf{curl})} \le c(\gamma(h) + h^s \|\boldsymbol{z}\|_{\boldsymbol{H}^s(\mathbf{curl})}) \quad \forall h > 0.$$

Proof. Thanks to the regularity assumption $\boldsymbol{z} \in \boldsymbol{H}^s(\mathbf{curl})$, we obtain from (60) and (68) that

$$\|\boldsymbol{z} - \Phi_h \boldsymbol{z}\|_{\boldsymbol{H}(\mathbf{curl})} \leq c h^s \|\boldsymbol{z}\|_{\boldsymbol{H}^s(\mathbf{curl})} \quad \forall h > 0. \tag{77}$$

Now, making use of the operator Φ_h (see (59) for its definition), we infer that

$$\begin{aligned}
(\mu^{-1}\mathbf{curl}\,(\boldsymbol{z}_h - \Phi_h \boldsymbol{z}), \mathbf{curl}\,\boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} &= (\mu^{-1}\mathbf{curl}\,(\boldsymbol{z}_h - \boldsymbol{z}), \mathbf{curl}\,\boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \\
\underbrace{=}_{(75)\&(74)} -\gamma(h)(\epsilon \boldsymbol{z}_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} &+ (\mathbf{f} + \epsilon\gamma(h)\nabla\chi_h, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} - (\mathbf{f}, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \\
&= \gamma(h)(\epsilon(\nabla\chi_h - \boldsymbol{z}_h), \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{ND}}_h.
\end{aligned}$$

Thus, inserting $\boldsymbol{v}_h = \boldsymbol{z}_h - \Phi_h \boldsymbol{z} \in \boldsymbol{\mathcal{ND}}_h$, we obtain that

$$\begin{aligned}
\|\mu^{-1/2}\mathbf{curl}\,(\boldsymbol{z}_h - \Phi_h z)\|^2_{\boldsymbol{L}^2(\Omega)} &= \gamma(h)(\epsilon(\nabla\chi_h - \boldsymbol{z}_h), \boldsymbol{z}_h - \Phi_h \boldsymbol{z})_{\boldsymbol{L}^2(\Omega)} \\
&\leq \gamma(h)\|\epsilon(\nabla\chi_h - \boldsymbol{z}_h)\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{z}_h - \Phi_h \boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega)} \\
&\leq c\gamma(h)\|\boldsymbol{z}_h - \Phi_h \boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall h > 0.
\end{aligned} \tag{78}$$

On the other hand, by the definition of Φ_h (see (59)) and (74)–(76) we infer that

$$\begin{aligned}
(\epsilon(\boldsymbol{z}_h - \Phi_h \boldsymbol{z}), \nabla\psi_h) &= (\epsilon(\boldsymbol{z}_h - \boldsymbol{z}), \nabla\psi_h) = (\epsilon \boldsymbol{z}_h, \nabla\psi_h) + (\rho, \psi_h) \\
&= (\epsilon\nabla\chi_h, \nabla\psi_h) + (\rho, \psi_h) = 0 \quad \forall \psi_h \in \Theta_h, \;\; \forall h > 0.
\end{aligned}$$

Consequently, we have $\boldsymbol{z}_h - \Phi_h \boldsymbol{z} \in \boldsymbol{X}_h^{(\epsilon)}$ for all $h > 0$, so we may apply the discrete Poincaré–Friedrichs-type inequality (61) to (78) to deduce that

$$\overline{\mu}^{-1}\|\mathbf{curl}\,(\boldsymbol{z}_h - \Phi_h \boldsymbol{z})\|_{\boldsymbol{L}^2(\Omega)} \leq c\gamma(h) \quad \forall h > 0.$$

Then, this inequality together with (61) implies

$$\|\boldsymbol{z}_h - \Phi_h \boldsymbol{z}\|_{\boldsymbol{H}(\mathbf{curl})} \leq c\gamma(h) \quad \forall h > 0. \tag{79}$$

Finally, we obtain from (77) and (79) that

$$\begin{aligned}
\|\boldsymbol{z}_h - \boldsymbol{z}\|_{\boldsymbol{H}(\mathbf{curl})} &\leq \|\boldsymbol{z}_h - \Phi_h \boldsymbol{z}\|_{\boldsymbol{H}(\mathbf{curl})} + \|\boldsymbol{z} - \Phi_h \boldsymbol{z}\|_{\boldsymbol{H}(\mathbf{curl})} \\
&\leq c(\gamma(h) + h^s\|\boldsymbol{z}\|_{\boldsymbol{H}^s(\mathbf{curl})}) \quad \forall h > 0.
\end{aligned}$$

This completes the proof. ∎

Remark 4.13. We know from Theorem 4.10 (cf. Corollary 4.12) that the choice

$$\gamma(h) = h$$

yields the desired optimal error estimate of order $O(h^s)$, where the index $s \in (0.5, 1]$ is determined by the regularity of the true solution. We recall that the results in [5] require the choice $\gamma(h) = h^2$ for the same optimal estimate $O(h^s)$. However, if h is sufficiently small, then the choice $\gamma(h) = h^2$ is much smaller than $\gamma(h) = h$ and increases the numerical effort considerably because the conditioning of the edge element system (31) is much worse. For this reason, we suggest choosing $\gamma(h) = h$ for the numerical solution of (P_h) to achieve the optimal error estimate with a reduced computational effort.

5. Numerical experiments. We present two numerical examples serving as a numerical illustration of Theorems 4.5 and 4.10.

5.1. Example 1 with a smooth optimal solution. As the first example, we consider the model optimal control problem (P) that has an analytical and smooth optimal solution, with the computational domain $\Omega = (0,1)^3$, the parameters $\mu = \epsilon = 1$, $\rho = 0$, and $\kappa = 1$, and the desired state $\boldsymbol{E}_d$ given by

$$\boldsymbol{E}_d(x) = (4\pi^4 + 1)\begin{pmatrix}\sin(\pi x_2)\sin(\pi x_3)\\ 0\\ 0\end{pmatrix}.$$

Then by straightforward computations we can verify that the three functions

$$\overline{\boldsymbol{E}}(x) = \begin{pmatrix}\sin(\pi x_2)\sin(\pi x_3)\\ 0\\ 0\end{pmatrix}, \quad \overline{\boldsymbol{u}}(x) = 2\pi^2\begin{pmatrix}\sin(\pi x_2)\sin(\pi x_3)\\ 0\\ 0\end{pmatrix},$$

$$\overline{\boldsymbol{p}}(x) = -2\pi^2\begin{pmatrix}\sin(\pi x_2)\sin(\pi x_3)\\ 0\\ 0\end{pmatrix}$$

satisfy the sufficient and necessary optimality system (11). Thus, the optimal solution of (P) is given by $\overline{\boldsymbol{u}}$. For all the examples in this section, we have solved the finite element approximation (P_h) using the open source software FEniCS [17]. The computational domain Ω was triangulated with a regular mesh of mesh size h, and the optimality system (33) was solved by MUMPS (MUltifrontal Massively Parallel sparse direct Solver). As pointed out in Remarks 4.11 and 4.13, in order to guarantee the optimal convergence rate in the finite element solution, we choose $\gamma(h) = h$. Furthermore, we employ the following quantity to compute the approximate order of convergence:

$$\mathrm{EOC} = \frac{\log\|\overline{\boldsymbol{u}}_{h_1} - \overline{\boldsymbol{u}}\|_{\boldsymbol{H}(\mathbf{curl})} - \log\|\overline{\boldsymbol{u}}_{h_2} - \overline{\boldsymbol{u}}\|_{\boldsymbol{H}(\mathbf{curl})}}{\log h_1 - \log h_2}$$

for two consecutive mesh sizes h_1 and h_2. Table 1 displays the $\boldsymbol{H}(\mathbf{curl})$-norm error between the analytical solution $\overline{\boldsymbol{u}}$ and the finite element solution $\overline{\boldsymbol{u}}_h$ for different mesh sizes. As we can see from the table, the finite element solution $\overline{\boldsymbol{u}}_h$ converges to the analytical solution $\overline{\boldsymbol{u}}$ as h decreases. Moreover, by Theorem 4.10 we know a convergence rate of $s = 1$ should be obtained due to the nice regularity properties $\overline{\boldsymbol{u}}, \overline{\boldsymbol{E}}, \overline{\boldsymbol{p}} \in \boldsymbol{H}^1(\mathbf{curl})$ for this example. This theoretical prediction is confirmed by our numerical results, as we see EOC approximates $s \approx 1$.

5.2. Example 2 with a nonsmooth optimal solution. In this example, we choose the nonconvex polyhedral computational domain

$$\Omega = \big\{(0,1/4)\times(0,1/2)\times(0,1)\big\} \setminus \big\{[1/8,1/4]\times[1/8,1/2]\times[0,1]\big\} \tag{80}$$

and the parameters $\mu = \epsilon = 1$, $\rho = 0$, and $\kappa = 1$. For convenience, we now include an additional shift control in our objective functional:

$$\min_{\boldsymbol{u}\in\boldsymbol{U}} \frac{1}{2}\int_\Omega |\boldsymbol{G}(\boldsymbol{u}) - \boldsymbol{E}_d|^2\,dx + \frac{1}{2}\int_\Omega |\boldsymbol{u} - \boldsymbol{u}_d|^2\,dx. \tag{P}$$

Table 1
Convergence history.

$h/\sqrt{2}$	$\|\overline{\boldsymbol{u}}_h - \overline{\boldsymbol{u}}\|_{\boldsymbol{H}(\mathbf{curl})}$	EOC
2^{-2}	13.5704	–
2^{-3}	6.94488	0.96644
2^{-4}	3.48798	0.99356
2^{-5}	1.74486	0.99929

Here, the desired state and the shift control are set to be

$$\boldsymbol{E}_d = \boldsymbol{G}(\boldsymbol{u}_d) \quad \text{and} \quad \boldsymbol{u}_d = \boldsymbol{G}(\mathbf{f}),$$

with $\mathbf{f} = 10^3(1,1,1)^T$. We note that, since $\mu \equiv \epsilon \equiv 1$ and $\rho \equiv 0$, the desired state and the shift control enjoy the regularity property $\boldsymbol{E}_d, \boldsymbol{u}_d \in \boldsymbol{H}^\delta(\mathbf{curl})$, with δ as in (56). As Ω is a nonconvex Lipschitz polyhedron, this exponent is strictly less than one, $\delta \in (0.5, 1)$. We also point out that the analytical solutions for $\boldsymbol{E}_d = \boldsymbol{G}(\boldsymbol{u}_d)$ and $\boldsymbol{u}_d = \boldsymbol{G}(\mathbf{f})$ are unknown. For our numerical experiment, we approximate them by their finite element approximations with a very fine mesh size $h = 2^{-8}\sqrt{2}$.

By our specific construction, the optimal solution of (P) is exactly given by $\overline{\boldsymbol{u}} = \boldsymbol{u}_d$, and all our results in this work can be naturally extended to (P) in the presence of the shift control $\boldsymbol{u}_d$. Table 2 displays the $\boldsymbol{H}(\mathbf{curl})$-norm error between the exact solution $\overline{\boldsymbol{u}}$ and our finite element solution $\overline{\boldsymbol{u}}_h$ with $\gamma(h) = h$. As the mesh size h decreases, we observe that the optimal solution approaches the exact one. See Figure 1 for the computed optimal electric field with different meshes. Furthermore, by Theorem 4.10 we know we can only expect a convergence rate $\delta \in (0.5, 1)$, as the computational domain (80) features a nonconvex structure in this example. This theoretical prediction is also reasonably confirmed by our numerical results with $\delta \approx 0.7$.

Table 2
Convergence history.

$h/\sqrt{2}$	$\|\overline{\boldsymbol{u}}_h - \overline{\boldsymbol{u}}\|_{\boldsymbol{H}(\mathbf{curl})}$	EOC
2^{-4}	0.153028	-
2^{-5}	0.118692	0.366573
2^{-6}	0.073895	0.683666

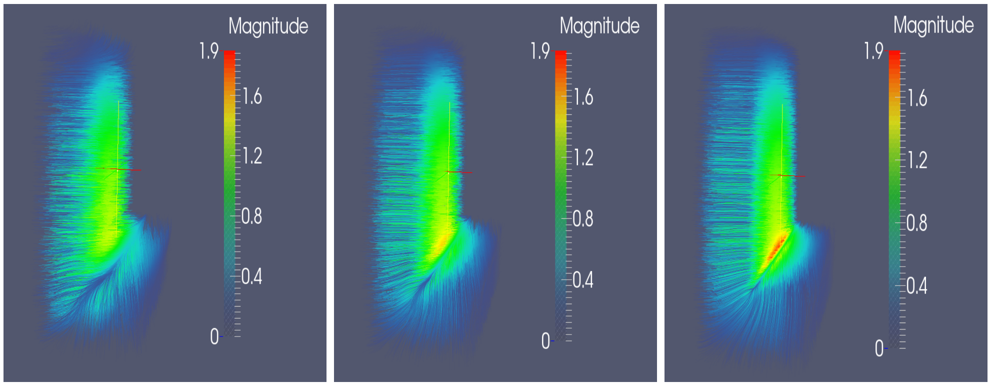

FIG. 1. *Computed optimal electric field* $\overline{\boldsymbol{E}}_h$ *with mesh size* $h = \sqrt{2}\,2^{-k}$ *for* $k = 4$ *(left plot),* $k = 5$ *(middle plot),* $k = 6$ *(right plot).*

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