NON-UNIQUE LIFTING OF INTEGER VARIABLES IN MINIMAL INEQUALITIES*

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Abstract. We explore the lifting question in the context of cut-generating functions. Most of the prior literature on this question focuses on cut-generating functions that have the unique lifting property. We develop a general theory for understanding the lifting question for cut-generating functions that do not necessarily have the unique lifting property.

Key words. cutting plane theory, cut-generating functions, lattice-free sets

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1. Introduction. Let $S \subseteq \mathbb{R}^n \setminus \{0\}$ be a closed set and consider the model

(1.1)
$$X_S(R,P) := \left\{ (s,y) \in \mathbb{R}^k_+ \times \mathbb{Z}^\ell_+ : Rs + Py \in S \right\},$$

where $k, \ell \in \mathbb{Z}_+$, $R \in \mathbb{R}^{n \times k}$, and $P \in \mathbb{R}^{n \times \ell}$. We allow k = 0 or $\ell = 0$, but not both. The assumption that S is closed and $0 \notin S$ implies that $(0,0) \notin \operatorname{conv}(X_S(R,P))$ [13, Lemma 2.1]. We search for valid inequalities that separate (0,0) from $X_S(R,P)$.

A cut-generating (function) pair (ψ, π) for S is a pair of functions $\psi, \pi \colon \mathbb{R}^n \to \mathbb{R}$ such that for every $k, \ell \in \mathbb{Z}_+, R = (r^1, \ldots, r^k) \in \mathbb{R}^{n \times k}$, and $P = (p^1, \ldots, p^\ell) \in \mathbb{R}^{n \times \ell}$, the inequality

(1.2)
$$\sum_{i=1}^{k} \psi(r^{i}) s_{i} + \sum_{j=1}^{\ell} \pi(p^{j}) y_{j} \ge 1$$

is satisfied by all points $(s, y) \in \operatorname{conv}(X_S(R, P))$. Note that $(0, 0) \in \mathbb{R}^k \times \mathbb{Z}^\ell$ does not satisfy (1.2), so the inequality separates (0, 0) from $\operatorname{conv}(X_S(R, P))$. Sometimes we refer to cut-generating pairs as valid cut-generating pairs or valid pairs to emphasize that they give valid inequalities of the form (1.2); inequality (1.2) is known as a cutting plane or a cut. The literature studying model (1.1) and cut-generating pairs is extensive. We refer the reader to the surveys [30, 14, 4, 8, 9] and Chapter 6 of [16], and the references within, for an overview of the field.

There is a natural partial order on the set of valid pairs, namely $(\psi', \pi') \leq (\psi, \pi)$ if and only if $\psi' \leq \psi$ and $\pi' \leq \pi$. Since each point $(s, y) \in X_S(R, P)$ is nonnegative, the relation $(\psi', \pi') \leq (\psi, \pi)$ indicates that all cuts obtained from (ψ, π) are implied by those obtained from (ψ', π') . The minimal elements under this partial order are called *minimal valid pairs*.

The connection between S-free sets and cut-generating functions has been instrumental in making cut-generating functions a computational tool for mixed-integer optimization. A set $B \subseteq \mathbb{R}^n$ is called a *convex* 0-*neighborhood* if B is convex and $0 \in int(B)$. If B is a convex 0-neighborhood and $S \cap int(B) = \emptyset$, then B is called

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an S-free convex 0-neighborhood. If there does not exist a strict superset of B that is also an S-free convex 0-neighborhood, then B is called a maximal S-free convex 0-neighborhood. A sublinear¹ function $\gamma : \mathbb{R}^n \to \mathbb{R}$ is called a representation of B if $B = \{r \in \mathbb{R}^n : \gamma(r) \leq 1\}$. A convex 0-neighborhood may have several representations, with the classic gauge function being one such representation. Representations of closed convex 0-neighborhoods was the main topic of study in [7, 12], where it was established that there always exists a smallest representation γ^* for a convex 0-neighborhood B, i.e., $\gamma^* \leq \gamma$ for all representations γ of B.

The following recipe provides one way of creating a cut-generating pair:

- 1. Fix a maximal S-free convex 0-neighborhood B.
- 2. Let γ^* be the smallest representation of *B*.
- 3. The pair $(\psi, \pi) = (\gamma^*, \gamma^*)$ is a cut-generating pair.

Unfortunately, this recipe falls short of creating a minimal cut-generating pair because the pair $(\psi, \pi) = (\gamma^*, \gamma^*)$ is only "partially minimal". Indeed, one can show that for any other cut-generating pair $(\psi', \pi') \leq (\psi, \pi)$, one must have $\psi' = \psi$. However, there may exist another function $\pi' \leq \pi$ such that (ψ, π') is also a valid pair. This motivates the following definition. Let *B* be a maximal *S*-free convex 0-neighborhood and let ψ be the smallest representation of *B*. Then $\pi : \mathbb{R}^n \to \mathbb{R}$ is a *lifting* of ψ if (ψ, π) is a valid cut-generating pair. Note that ψ is a lifting of itself. The set of all liftings of ψ is partially ordered by pointwise dominance, so one can define *minimal liftings*.

The lifting approach to create cut-generating pairs is useful because for some structured sets S, the smallest representations of maximal S-free convex 0- neighborhoods have nice, easy-to-compute "formulas". Moreover, for some classes of maximal S-free convex 0-neighborhoods, nice "formulas" exist for minimal liftings of the smallest representation. For a survey of these ideas, see [4] and Section 6.3.4 in [16].

We say that a function $\psi : \mathbb{R}^n \to \mathbb{R}$ is a valid function for S if (ψ, ψ) is a valid cutgenerating pair for S. The recipe above depends on the observation that the smallest representation of any S-free convex 0-neighborhood (not necessarily maximal) is a valid function for S [4, Theorem 4.12]. However, not all valid functions of S are representations of S-free convex 0-neighborhoods. The notion of a lifting of ψ can be easily extended to any valid function ψ for $S: \pi$ is a lifting of ψ if (ψ, π) forms a cut-generating pair for S. Under pointwise dominance, minimal elements of the set of liftings of a valid function ψ for S will be called minimal liftings of ψ .

1.1. Unique minimal liftings. Let $B \subseteq \mathbb{R}^n$ be a maximal S-free convex 0neighborhood. A central notion in the study of minimal liftings of the smallest representation ψ of B is the extended lifting region R(B) defined to be

(1.3) $R(B) := \{ r \in \mathbb{R}^n \colon \pi_1(r) = \pi_2(r) \text{ for all minimal liftings } \pi_1, \pi_2 \text{ of } \psi \}.$

If $R(B) = \mathbb{R}^n$, then ψ has a unique minimal lifting. Moreover, nice "formulas" for this unique lifting can be derived in terms of ψ ; see Section 6 of the survey [4]. A large class of maximal S-free convex 0-neighborhoods with this unique lifting property has been identified and studied in many recent papers on minimal liftings [1, 2, 6, 10, 24]. However, the same literature shows that there are many choices for B that satisfy $R(B) \subsetneq \mathbb{R}^n$. The purpose of this manuscript is to describe minimal valid pairs that arise from such maximal S-free convex 0-neighborhoods, that is, from maximal S-free convex 0-neighborhoods without the unique lifting property.

 $^{^1\}mathrm{A}$ function is sublinear if it is convex and subadditive.

Let $p^* \in \mathbb{R}^n$ and assume that $p^* \notin R(B)$. This means that there exist two minimal liftings of ψ that disagree on p^* . When considering a model $X_S(R, P)$ in which p^* is a column of P, one would like to develop cuts that have a small coefficient $\pi(p^*)$. To this end, it is of interest to examine the smallest possible value that any minimal lifting of ψ can achieve at p^* , which is denoted by

(1.4)
$$V_{\psi}(p^*) := \inf\{\pi(p^*) : \pi \text{ minimal lifting of } \psi\}.$$

We aim to find a minimal lifting in the collection

(1.5) $\mathcal{L}_{\psi,p^*} := \{ \pi : \mathbb{R}^n \to \mathbb{R} : \pi \text{ is a minimal lifting of } \psi \text{ and } \pi(p^*) = V_{\psi}(p^*) \}.$

For the setting when n = 2 and $S = \mathbb{Z}^2$, Dey and Wolsey [24] studied $V_{\psi}(p^*)$ and showed that \mathcal{L}_{ψ,p^*} is nonempty. In general, \mathcal{L}_{ψ,p^*} is nonempty, and we show this in Proposition 18 in Appendix A.

By definition of $V_{\psi}(p^*)$ and the extended lifting region, all $\pi \in \mathcal{L}_{\psi,p^*}$ agree on $\{p^*\} \cup R(B)$. Are there more values on which these liftings agree? Analogous to the extended lifting region, we define the *fixing region* \mathcal{F}_{ψ,p^*} corresponding to p^* to be the set of points on which all minimal liftings in \mathcal{L}_{ψ,p^*} agree, that is

(1.6)
$$\mathcal{F}_{\psi,p^*} := \{ p \in \mathbb{R}^n : \pi_1(p) = \pi_2(p) \text{ for all } \pi_1, \pi_2 \in \mathcal{L}_{\psi,p^*} \}.$$

If $\mathcal{F}_{\psi,p^*} = \mathbb{R}^n$, then there exists a unique lifting in \mathcal{L}_{ψ,p^*} . In other words, after finding the optimal lifting coefficient $V_{\psi}(p^*)$ for p^* , the lifting coefficients for all other vectors are uniquely determined for all minimal liftings that assign $V_{\psi}(p^*)$ to the vector p^* . If there exists a p^* such that $\mathcal{F}_{\psi,p^*} = \mathbb{R}^n$, then we say that ψ and the underlying set B are one point fixable.

Using the fixing region, the recipe provided above can be modified to create minimal cut-generating pairs.

- 1. Fix a maximal S-free convex 0-neighborhood B that is one point fixable.
- 2. Let ψ be the smallest representation of B.
- 3. Find $p^* \in \mathbb{R}^n$ such that $\mathcal{F}_{\psi,p^*} = \mathbb{R}^n$.
- 4. Then $\mathcal{L}_{\psi,p^*} = \{\pi\}$ and the pair (ψ, π) is a minimal cut-generating pair.

In this paper, we study the structure of the fixing region and one-point fixability. What is a good description of the fixing region? How does the fixing region depend on p^* ? How much does the fixing region cover? We explore questions such as these. Our work is motivated by Section 7 of [24], which initiated the study of this problem.

1.2. Statement of results. To state our results, we need the set

$$W_S := \{ w \in \mathbb{R}^n : s + \lambda w \in S , \forall s \in S, \forall \lambda \in \mathbb{Z} \}.$$

The importance of W_S is that any minimal lifting π of a valid function ψ satisfies $\pi(r+w) = \pi(r)$ for all $r \in \mathbb{R}^n$ and $w \in W_S$ (see Proposition 1).

Let $B \subseteq \mathbb{R}^n$ be a maximal S-free convex 0-neighborhood and let ψ be the corresponding smallest representation.

- 1. Let $p^* \in \mathbb{R}^n$. In Theorem 9, we use the structure of B to identify a nonempty set $\mathcal{X}(B, p^*) \subseteq \mathbb{R}^n$ such that $R(B) \subsetneq \mathcal{X}(B, p^*) + W_S \subseteq \mathcal{F}_{\psi, p^*}$. It is not known if this inner approximation of the fixing region is always equal to \mathcal{F}_{ψ, p^*} .
- 2. In Proposition 15, we use the inner approximation in Theorem 9 to show that certain Type 3 triangles are one point fixable. As a corollary, in Proposition 17 we

show that Type 3 triangles resulting from the so-called mixing set are one point fixable. This also follows from [24, Theorem 5]; we use different, more geometric techniques. See [23, 27] for more on the mixing set.

- 3. Theorem 14 says if our inner approximation $\mathcal{X}(B, p^*) + W_S$ of \mathcal{F}_{ψ,p^*} equals \mathbb{R}^n (implying that *B* is one point fixable), then the (S+t)-free convex 0-neighborhood B+t is one point fixable for any $t \in \mathbb{R}^n$ such that B+t is a 0-neighborhood. In other words, one point fixability is preserved under translations. If an *S*-free 0-neighborhood is used to derive cuts around a basic feasible solution of a mixedinteger linear program, then, by using this translation invariance, these cuts can be transformed to cuts around a different basic feasible solution. A more detailed discussion of this point is provided in [10] and [4]. Theorem 14 is in Subsection 3.4.
- 4. In Section 2, we develop a theory of partial cut-generating pairs, which are cutgenerating pairs that are only defined on subsets of \mathbb{R}^n . Partial cut-generating pairs, which were first developed in this paper, have been subsequently used in [5] to prove structural results about the infinite models in integer programming. One way to think of the results in Section 2 is that they are analogous to classic "lifting" results like Hahn-Banach theorems in analysis [19], and "lifting" valid inequalities from faces of a polytope to the full polytope (see, e.g., Section 7.2 in [16]).

2. Partial cut-generating functions. We denote the columns of a matrix A by col(A). For a set X and any $d \in \mathbb{N}$, X^d will denote the d-wise Cartesian product of X with itself. Let $\mathcal{R}, \mathcal{P} \subseteq \mathbb{R}^n$, $\psi : \mathcal{R} \to \mathbb{R}$, and $\pi : \mathcal{P} \to \mathbb{R}$. We define (ψ, π) to be a valid pair for $(S, \mathcal{R}, \mathcal{P})$ if for every $k, \ell \in \mathbb{Z}_+, R \in \mathcal{R}^k$, and $P \in \mathcal{P}^\ell$, the inequality

(2.1)
$$\sum_{r \in \operatorname{col}(R)} \psi(r) s_r + \sum_{p \in \operatorname{col}(P)} \pi(p) y_p \ge 1$$

is satisfied by all points $(s, y) \in X_S(R, P)$. Here, s_r denotes the continuous variable associated with $r \in \operatorname{col}(R)$ and y_p denotes the integer variable associated with $p \in \operatorname{col}(P)$. The concepts of a valid function $\psi : \mathcal{R} \to \mathbb{R}$ for (S, \mathcal{R}) and a minimal valid pair for $(S, \mathcal{R}, \mathcal{P})$ are defined analogously to the case $\mathcal{R} = \mathbb{R}^n$ and $\mathcal{P} = \mathbb{R}^n$. For $\mathcal{P} \subseteq \mathbb{R}^n$, we say $\pi : \mathcal{P} \to \mathbb{R}$ is a *lifting of a valid function* ψ for (S, \mathcal{R}) , if (ψ, π) is a valid pair for $(S, \mathcal{R}, \mathcal{P})$. The concept of a minimal lifting of ψ is analogously defined. When \mathcal{R} and \mathcal{P} are strict subsets of \mathbb{R}^n , we refer to ψ as a *partial cut-generating* function and (ψ, π) as a partial cut-generating pair.

Using this terminology, valid pairs for S defined in Section 1 become valid pairs for $(S, \mathbb{R}^n, \mathbb{R}^n)$ and valid functions for S become valid functions for (S, \mathbb{R}^n) . In the remaining text, we will be careful about explicitly stating \mathcal{R} and \mathcal{P} whenever we speak about valid functions or pairs.

Minimal cut-generating pairs for $(S, \mathbb{R}^n, \mathbb{R}^n)$ satisfy certain structural properties. The next proposition shows that some of these results also hold for partial cut-generating pairs, and setting $\mathcal{R} = \mathcal{P} = \mathbb{R}^n$ recovers the setting of cut-generating pairs. Similarly to the translation set W_S for classic cut-generating pairs, define

$$W_{S}^{+} := \{ w \in \mathbb{R}^{n} : s + \lambda w \in S , \forall s \in S, \forall \lambda \in \mathbb{Z}_{+} \}$$

for partial cut-generating pairs. Note that $W_S = W_S^+ \cap (-W_S^+)$.

PROPOSITION 1. Let $S \subseteq \mathbb{R}^n \setminus \{0\}$ be a closed set. Let $\mathcal{R}, \mathcal{P} \subseteq \mathbb{R}^n$ and $\psi : \mathcal{R} \to \mathbb{R}$ be a valid function for (S, \mathcal{R}) .

(a) For any minimal lifting π of ψ , $\pi(p) \leq \pi(p+w)$ for all $p \in \mathcal{P}$ and $w \in W_S^+$ such that $p+w \in \mathcal{P}$. So, $\pi(p) = \pi(p+w)$ for all $p \in \mathcal{P}$ and $w \in W_S$ such that $p+w \in \mathcal{P}$.

(b) Define $\psi^* : \mathcal{R} \to \mathbb{R}$ to be

$$\psi^*(r) := \inf\{\psi(r+w) : w \in W_S^+ \text{ such that } r+w \in \mathcal{R}\}.$$

Then (ψ, ψ^*) is a valid partial cut-generating pair for $(S, \mathcal{R}, \mathcal{R})$. (c) If $\mathcal{R} = \mathcal{P}$, then every minimal lifting π of ψ satisfies $\pi \leq \psi^*$.

Proof. Let $\mathcal{K} \subseteq \mathbb{R}^n$ and take $\sigma : \mathcal{K} \to \mathbb{R}$ to be a (not necessarily minimal) lifting of ψ . Thus, (ψ, σ) is a valid pair for $(S, \mathcal{R}, \mathcal{K})$. Define $\sigma^* : \mathcal{K} \to \mathbb{R}$ to be

$$\sigma^*(p) := \inf_{w \in W_S^+} \bigg\{ \sigma(p+w) : \ p+w \in \mathcal{K} \bigg\}.$$

First, we show that (ψ, σ^*) is a valid pair for $(S, \mathcal{R}, \mathcal{K})$.

Let $k, \ell \in \mathbb{Z}_+$, $R \in \mathcal{R}^k$, $P \in \mathcal{K}^\ell$, and $(s, y) \in X_S(R, P)$. Let $W \in \mathbb{R}^{n \times \ell}$ be any matrix with $\operatorname{col}(W) \subseteq W_S^+$ such that $P+W \in \mathcal{K}^\ell$. Let $(\bar{s}, \bar{y}) \in \mathbb{R}_+^k \times \mathbb{Z}_+^\ell$ be constructed as follows: $\bar{s}_r = s_r$ for each $r \in \operatorname{col}(R)$ and $\bar{y}_{p+w} = y_p$ for each $p + w \in \operatorname{col}(P + W)$. Since $\bar{w} \in W_S^+$ by definition of W, it follows that $R\bar{s} + (P+W)\bar{y} = Rs + Py + \bar{w} \in S$. Thus, since (ψ, σ) is a valid pair for $(S, \mathcal{R}, \mathcal{K})$,

$$\sum_{r \in \operatorname{col}(R)} \psi(r)\overline{s}_r + \sum_{p+w \in \operatorname{col}(P+W)} \sigma(p+w)\overline{y}_{p+w} \ge 1.$$

The above holds for all matrices $W \in \mathbb{R}^{n \times \ell}$ whose columns are in W_S^+ and $P + W \in \mathcal{K}^{\ell}$. Taking an infimum over all such W gives

$$\sum_{r \in \operatorname{col}(R)} \psi(r) s_r + \sum_{p \in \operatorname{col}(P)} \sigma^*(p) y_p$$

$$= \sum_{r \in \operatorname{col}(R)} \psi(r) s_r + \inf_W \left\{ \sum_{p+w \in \operatorname{col}(P+W)} \sigma(p+w) y_p \right\}$$

$$= \inf_W \left\{ \sum_{r \in \operatorname{col}(R)} \psi(r) s_r + \sum_{p+w \in \operatorname{col}(P+W)} \sigma(p+w) \overline{y}_{p+w} \right\}$$

$$\geq 1.$$

Thus, (ψ, σ^*) is also a valid pair for $(S, \mathcal{R}, \mathcal{K})$. Setting $\sigma = \psi$ and $\mathcal{K} = \mathcal{R}$ gives (b).

Let π be a minimal lifting of ψ . Set $\sigma = \pi$ and $\mathcal{K} = \mathcal{P}$. Since $\pi^* \leq \pi$ and π is minimal, we obtain $\pi^* = \pi$. Hence, $\pi(p) = \pi^*(p) \leq \pi(p+w)$ for all $p \in \mathcal{P}$ and $w \in W_S^+$ such that $p + w \in \mathcal{P}$. This proves (a).

Finally, assume that $\mathcal{P} = \mathcal{R}$. Since π is a minimal lifting of ψ , $\pi(r) \leq \psi(r)$ for all $r \in \mathcal{R}$. By $(a), \pi(p) \leq \pi(p+w) \leq \psi(p+w)$ for all $p \in \mathcal{P}$ and $w \in W_S^+$ such that $p+w \in \mathcal{P} = \mathcal{R}$. Taking an infimum over all such $w \in W_S^+$, we obtain (c).

Theorem 2 follows from standard calculations involving cut-generating functions, so the proof is omitted.

THEOREM 2. Let (ψ, π) be a minimal valid pair for $(S, \mathcal{R}, \mathcal{P})$. Then ψ and π are both subadditive over \mathcal{R} and \mathcal{P} , respectively, i.e., $\psi(r + r') \leq \psi(r) + \psi(r')$ for all $r, r' \in \mathcal{R}$ such that $r + r' \in \mathcal{R}$, and $\pi(p + p') \leq \pi(p) + \pi(p')$ for all $p, p' \in \mathcal{P}$ such that $p + p' \in \mathcal{P}$. Also, ψ is positively homogeneous over \mathcal{R} , i.e., for all $r \in \mathcal{R}$ and $\lambda > 0$ such that $\lambda r \in \mathcal{R}$, we have $\psi(\lambda r) = \lambda \psi(r)$. Given $\mathcal{R} \subseteq \mathcal{R}' \subseteq \mathbb{R}^n$, $\mathcal{P} \subseteq \mathcal{P}' \subseteq \mathbb{R}^n$, and a valid pair (ψ, π) for $(S, \mathcal{R}, \mathcal{P})$, a natural question is that of extension: do there always exist functions ψ', π' such that (ψ', π') is valid for $(S, \mathcal{R}', \mathcal{P}')$ and ψ', π' are extensions of ψ, π , i.e., they coincide on \mathcal{R} and \mathcal{P} respectively? The answer to the question is 'no', in general. Indeed, choosing $\mathcal{R} = \emptyset$ and $\mathcal{P} = \mathbb{R}^n$, we obtain Gomory and Johnson's pure integer model, where the discontinuous valid functions π cannot be appended to any ψ to give a valid pair for the full mixed-integer model (see [22]). On the positive side, the next result gives a sufficient condition for when partial cut-generating pairs can be extended.

For a set $X \subseteq \mathbb{R}^n$, we use cone(X) to denote the convex cone generated by X.

THEOREM 3. Let $\mathcal{R} \subseteq \mathcal{R}' \subseteq \mathbb{R}^n, \mathcal{P} \subseteq \mathcal{P}' \subseteq \mathbb{R}^n$ and (ψ, π) be a valid pair for $(S, \mathcal{R}, \mathcal{P})$. If $\mathcal{R}', \mathcal{P}' \subseteq \operatorname{cone}(\mathcal{R})$, then there exist functions $\psi' : \mathcal{R}' \to \mathbb{R}, \pi' : \mathcal{P}' \to \mathbb{R}$ such that (ψ', π') is a minimal valid pair for $(S, \mathcal{R}', \mathcal{P}')$ and $(\psi', \pi') \leq (\psi, \pi)$ on $\mathcal{R} \times \mathcal{P}$.

Proof. For $r' \in \mathcal{R}'$, define

$$\nu_{\psi}(r') := \inf \left\{ \sum_{r \in \mathcal{R}} \psi(r)h(r) : \begin{array}{c} r' = \sum_{r \in \mathcal{R}} rh(r) \text{ and} \\ h : \mathcal{R} \to \mathbb{R}_+ \text{ has finite support} \end{array} \right\}.$$

Similarly, for $p' \in \mathcal{P}'$ define

$$\nu_{\pi}(p') := \inf \left\{ \sum_{r \in \mathcal{R}} \psi(r)h(r) + \sum_{p \in \mathcal{P}} \pi(p)g(p) : \begin{array}{l} p' = \sum_{r \in \mathcal{R}} rh(r) + \sum_{p \in \mathcal{P}} pg(p), \\ h : \mathcal{R} \to \mathbb{R}_+ \text{ has finite support and} \\ g : \mathcal{P} \to \mathbb{Z}_+ \text{ has finite support} \end{array} \right\}.$$

Since $\mathcal{R}', \mathcal{P}' \subseteq \operatorname{cone}(\mathcal{R})$, the infima defining $\nu_{\psi}(r')$ and $\nu_{\pi}(p')$ are over nonempty sets. Thus, $\nu_{\psi}(r') \in [-\infty, \infty)$ for all $r' \in \mathcal{R}'$ and $\nu_{\pi}(p') \in [-\infty, \infty)$ for all $p' \in \mathcal{P}'$.

Define the functions $\tilde{\psi} : \mathcal{R}' \to \mathbb{R}$ and $\tilde{\pi} : \mathcal{P}' \to \mathbb{R}$ to be

$$\tilde{\psi}(r') := \begin{cases} \nu_{\psi}(r') & \text{if } \nu_{\psi}(r') > -\infty \\ \psi(r') & \text{if } \nu_{\psi}(r') = -\infty \text{ and } r' \in \mathcal{R} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{\pi}(p') := \begin{cases} \nu_{\pi}(p') & \text{if } \nu_{\pi}(p') > -\infty \\ \pi(p') & \text{if } \nu_{\pi}(p') = -\infty \text{ and } p' \in \mathcal{P} \\ 0 & \text{otherwise.} \end{cases}$$

Let $r \in \mathcal{R}$ and define $h : \mathcal{R} \to \mathbb{R}_+$ to be h(r) = 1 and h(r') = 0 for all $r' \in \mathcal{R} \setminus \{r\}$. If $\nu_{\psi}(r) = -\infty$, then $\tilde{\psi}(r) \leq \psi(r)$. If $\nu_{\psi}(r) = -\infty$, then

$$\tilde{\psi}(r) = \nu_{\psi}(r) \le \sum_{r \in \mathcal{R}} \psi(r)h(r) = \psi(r).$$

Hence, $\tilde{\psi}(r) \leq \psi(r)$ for every $r \in \mathcal{R}$. Similarly, $\tilde{\pi}(p) \leq \pi(p)$ for every $p \in \mathcal{P}$. Therefore, $(\tilde{\psi}, \tilde{\pi}) \leq (\psi, \pi)$ on $\mathcal{R} \times \mathcal{P}$. Zorn's Lemma implies that any valid pair is pointwise larger than some minimal valid pair (see, for example, Proposition A.1. in [10]), so it is sufficient to show that $(\tilde{\psi}, \tilde{\pi})$ is valid for $(S, \mathcal{R}', \mathcal{P}')$. Let R' and P' be matrices with columns in \mathcal{R}' and \mathcal{P}' , respectively. Consider $(s', y') \in X_S(R', P')$ and let $\varepsilon > 0$. Let $r' \in \operatorname{col}(R') \subseteq \mathcal{R}' \subseteq \operatorname{cone}(\mathcal{R})$. By the definition of ψ , there exists a function $h_{r'} : \mathcal{R} \to \mathbb{R}_+$ with finite support such that

$$r' = \sum_{r \in \mathcal{R}} r h_{r'}(r)$$
 and $\tilde{\psi}(r') > \left(\sum_{r \in \mathcal{R}} \psi(r) h_{r'}(r)\right) - \varepsilon.$

Similarly, for each $p' \in \operatorname{col}(P')$, there exist functions $h_{p'} : \mathcal{R} \to \mathbb{R}_+$ and $g_{p'} : \mathcal{P} \to \mathbb{Z}_+$, both with finite support, such that

$$p' = \sum_{r \in \mathcal{R}} rh_{p'}(r) + \sum_{p \in \mathcal{P}} pg_{p'}(p) \quad \text{and} \quad \tilde{\pi}(r) > \left(\sum_{r \in \mathcal{R}} \psi(r)h_{p'}(r) + \sum_{p \in \mathcal{P}} \pi(p)g_{p'}(p)\right) - \varepsilon.$$

Define the matrix $R \in \mathbb{R}^{n \times |\operatorname{col}(R)|}$ to have columns

$$\operatorname{col}(R) := \bigcup_{r' \in R'} \operatorname{support}(h_{r'}) \cup \bigcup_{p' \in P'} \operatorname{support}(g_{p'}),$$

and the matrix $P \in \mathbb{R}^{n \times |\mathrm{col}(P)|}$ to have columns

$$\operatorname{col}(P) := \bigcup_{r' \in R'} \operatorname{support}(g_{p'}).$$

Define $(\tilde{s}, \tilde{y}) \in \mathbb{R}^{|\operatorname{col}(R)|}_+ \times \mathbb{Z}^{|\operatorname{col}(P)|}_+$ component-wise to be

$$\tilde{s}_r := \sum_{r' \in R'} h_{r'}(r) s'_{r'} + \sum_{p' \in P'} h_{p'}(r) y'_{p'} \quad \forall \ r \in col(R), \text{ and}$$
$$\tilde{y}_p := \sum_{p' \in P'} g_{p'}(p) y'_{p'} \qquad \forall \ p \in col(P).$$

Using the fact that $(s', y') \in X_S(R', P')$ and the definitions of \tilde{s} and \tilde{y} , it follows that $R\tilde{s} + P\tilde{y} = R's' + P'y' \in S$. This implies that $(\tilde{s}, \tilde{y}) \in X_S(R, P)$. Set $M := \sum_{r' \in R'} s'_{r'} + \sum_{p' \in P'} y'_{p'}$. The value M is a constant because s' and y' are fixed. Since (ψ, π) is valid for $(S, \mathcal{R}, \mathcal{P})$, we see that

$$\begin{split} &\sum_{r'\in\mathcal{R}'}\tilde{\psi}(r')s'_{r'} + \sum_{p'\in\mathcal{P}'}\tilde{\pi}(p')y'_{p'} \\ &\geq \sum_{r'\in\mathcal{R}'}\left[\sum_{r\in\mathcal{R}}\psi(r)h_{r'}(r) - \varepsilon\right]s'_{r'} + \sum_{p'\in\mathcal{P}'}\left[\sum_{r\in\mathcal{R}}\psi(r)h_{p'}(r) + \sum_{p\in\mathcal{P}}\pi(p)g_{p'}(p) - \varepsilon\right]y'_{p} \\ &= \sum_{\substack{r\in\mathcal{R}\\r'\in\mathcal{R}'}}\psi(r)h_{r'}(r)s'_{r'} + \sum_{\substack{r\in\mathcal{R}\\p'\in\mathcal{P}'}}\psi(r)h_{p'}(r)y'_{p'} + \sum_{\substack{p\in\mathcal{P}\\p'\in\mathcal{P}'}}\pi(p)g_{p'}(p)y'_{p'} - \varepsilon M \\ &= \sum_{r\in\mathcal{R}}\psi(r)\tilde{s}_r + \sum_{p\in\mathcal{P}}\pi(p)\tilde{y}_p - \varepsilon M \\ \geq 1 - \varepsilon M. \end{split}$$

Letting $\varepsilon \to 0$ yields

$$\sum_{r' \in \mathcal{R}'} \tilde{\psi}(r') s'_{r'} + \sum_{p' \in \mathcal{P}'} \tilde{\pi}(p') y'_{p'} \ge 1.$$

Hence, $(\tilde{\psi}, \tilde{\pi})$ is a valid pair for $(S, \mathcal{R}', \mathcal{P}')$.

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3. The fixing region for truncated affine lattices. We now examine the fixing region \mathcal{F}_{ψ,p^*} for a valid function ψ and different choices of p^* . For the rest of the paper, we assume that $S = (b + \mathbb{Z}^n) \cap P$, where $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$ and $P \subseteq \mathbb{R}^n$ is a rational polyhedron. These S were called *polyhedrally-truncated affine lattices* in [10].

Let $p^* \in \mathbb{R}^n$ and recall \mathcal{L}_{ψ,p^*} from (1.5). One way of finding a minimal lifting of ψ is to find a function $\pi \in \mathcal{L}_{\psi,p^*}$. Proposition 18 in Appendix A shows that \mathcal{L}_{ψ,p^*} is nonempty. An important ingredient for finding $\pi \in \mathcal{L}_{\psi,p^*}$ is the value $V_{\psi}(p^*)$ from (1.4). In [24], Dey and Wolsey gave the following algebraic formula for $V_{\psi}(p^*)$:

(3.1)
$$V_{\psi}(p^*) = \sup_{w \in \mathbb{R}^n, N \in \mathbb{N}} \left\{ \frac{1 - \psi(w)}{N} : w + Np^* \in S \right\}.$$

A more geometric description of $V_{\psi}(p^*)$ was given in [2]. Let $B \subseteq \mathbb{R}^n$ be a maximal S-free convex 0-neighborhood. Because S is a truncated affine lattice, B is a polyhedron of the form

$$(3.2) B = \{ r \in \mathbb{R}^n : a^i \cdot r \le 1 \ \forall \ i \in I \},$$

where I is a finite set indexing the facets of B [3, 29]. Also, the smallest representation of B is

(3.3)
$$\psi_B(r) = \max_{i \in I} a^i \cdot r$$

If B is any S-free 0-neighborhood of the form (3.2), even if it is not maximal, then (3.3) gives a valid function for (S, \mathbb{R}^n) . This fact will be used later.

For $\lambda > 0$, define $Pyr(B, \lambda, p^*)$ to be the pyramid in $\mathbb{R}^n \times \mathbb{R}_+$ with $\frac{1}{\lambda}(p^*, 1)$ as the apex and $B \times \{0\}$ as the base, i.e.

(3.4)
$$\operatorname{Pyr}(B,\lambda,p^*) := \{(r,r_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+ : a^i \cdot r + (\lambda - a^i \cdot p^*)r_{n+1} \le 1 \ \forall \ i \in I\}.$$

The following was shown in [2, Theorem 11].

PROPOSITION 4. Let $B \subseteq \mathbb{R}^n$ be a maximal S-free convex 0-neighborhood and let $\psi := \psi_B : \mathbb{R}^n \to \mathbb{R}$ be obtained from B using (3.3). If $p^* \in \mathbb{R}^n$, then

$$V_{\psi}(p^*) = \inf \left\{ \lambda > 0 : \operatorname{Pyr}(B, \lambda, p^*) \text{ is } (S \times \mathbb{Z}) \text{-} free \right\}.$$

In [2], the authors studied a variant of $Pyr(B, \lambda, p^*)$ in which r_{n+1} was not constrained to be nonnegative. Their characterization of $V_{\psi}(p^*)$ in Proposition 4 is simply given by $Pyr(B, \lambda, p^*)$ is $(S \times \mathbb{Z}_+)$ -free. However, their proof also holds for the current definition of $Pyr(B, \lambda, p^*)$ and Proposition 4.

3.1. A geometric perspective on \mathcal{L}_{ψ,p^*} . The main tool for our geometric approach to understanding \mathcal{L}_{ψ,p^*} is the polyhedron $\operatorname{Pyr}(B, V_{\psi}(p^*), p^*)$ from (3.4).

Let $B \subseteq \mathbb{R}^n$ be a maximal S-free convex 0-neighborhood of the form (3.2), $\psi := \psi_B : \mathbb{R}^n \to \mathbb{R}$ be the valid function for (S, \mathbb{R}^n) obtained from B using (3.3), and $p^* \in \mathbb{R}^n$. A point $(\bar{x}, \bar{x}_{n+1}) \in S \times \mathbb{Z}_+$ with $\bar{x}_{n+1} \ge 1$ such that $\operatorname{Pyr}(B, V_{\psi}(p^*), p^*)$ contains (\bar{x}, \bar{x}_{n+1}) is called a *blocking point* for $\operatorname{Pyr}(B, V_{\psi}(p^*), p^*)$.

It was shown in [2, Theorem 11] that there is at least one blocking point for $Pyr(B, V_{\psi}(p^*), p^*)$ for every $p^* \in \mathbb{R}^n$. Lemma 5 relates the algebraic formula (3.1) for $V_{\psi}(p^*)$ and the important geometric notion of a blocking point for $Pyr(B, V_{\psi}(p^*), p^*)$. Since blocking points always exist, Lemma 5 implies that the supremum in (3.1) is actually a maximum and the infimum in Proposition 4 is actually a minimum.

LEMMA 5. Let $B \subseteq \mathbb{R}^n$ be a maximal S-free convex 0-neighborhood of the form (3.2). Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be the valid function for (S, \mathbb{R}^n) obtained from B using (3.3). If $(\bar{x}, \bar{x}_{n+1}) \in S \times \mathbb{Z}_+$ is a blocking point of $\operatorname{Pyr}(B, V_{\psi}(p^*), p^*)$, then

$$(\bar{x} - \bar{x}_{n+1}p^*, \bar{x}_{n+1}) \in \underset{w \in \mathbb{R}^n, N \in \mathbb{N}}{\arg \max} \left\{ \frac{1 - \psi(w)}{N} : w + Np^* \in S \right\}.$$

Conversely, if $(w, N) \in \mathbb{R}^n \times \mathbb{N}$ is a maximizer of (3.1), then $(w + Np^*, N)$ is a blocking point of $Pyr(B, V_{\psi}(p^*), p^*)$.

Proof. By (3.4), (\bar{x}, \bar{x}_{n+1}) is a blocking point of $Pyr(B, V_{\psi}(p^*), p^*)$ if and only if

$$a^{i} \cdot \bar{x} + (V_{\psi}(p^{*}) - a^{i} \cdot p^{*}) \bar{x}_{n+1} \le 1 \quad \forall \ i \in I,$$

and there exists some $i^* \in I$ such that $a^{i^*} \cdot \bar{x} + (V_{\psi}(p^*) - a^{i^*} \cdot p^*) \bar{x}_{n+1} = 1$. So, (\bar{x}, \bar{x}_{n+1}) is a blocking point of $Pyr(B, V_{\psi}(p^*), p^*)$ if and only if $\bar{x}_{n+1}V_{\psi}(p^*) + \max_{i \in I} \{a^i \cdot (\bar{x} - \bar{x}_{n+1}p^*)\} = 1$. By (3.1), the latter condition holds if and only if

$$V_{\psi}(p^*) = \frac{1 - \psi(\bar{x} - \bar{x}_{n+1}p^*)}{\bar{x}_{n+1}} = \sup_{w \in \mathbb{R}^n, N \in \mathbb{N}} \left\{ \frac{1 - \psi(w)}{N} : w + Np^* \in S \right\}.$$

This completes the proof.

3.2. A universal upper bound. In order to determine what vectors are in \mathcal{F}_{ψ,p^*} , we first show an upper bound on the value of minimal liftings of ψ and then show that this upper bound is tight. Theorem 7 gives an upper bound using the function $\psi_{[p^*,B]}^* : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$(3.5) \quad \psi^*_{[p^*,B]}((r,r_{n+1})) := \inf\{\psi_{\operatorname{Pyr}(B,V_{\psi}(p^*),p^*)}((r,r_{n+1}) + (w,z)) : (w,z) \in W^+_{S \times \mathbb{Z}_+}\}.$$

In (3.5), $\operatorname{Pyr}(B, V_{\psi}(p^*), p^*)$ is the set from (3.4), and $\psi_{\operatorname{Pyr}(B, V_{\psi}(p^*), p^*)}$ is obtained from (3.3) using $\operatorname{Pyr}(B, V_{\psi}(p^*), p^*)$ written as $\operatorname{Pyr}(B, V_{\psi}(p^*), p^*) = \{(r, r_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+ : \overline{a}^i \cdot r \leq 1 \forall i \in I\}$. We caution the reader that the formula (3.3) was introduced for *B* that contain 0 in the interior. However, the formula is a well-defined one, even if 0 lies on the boundary, as is the case for $\operatorname{Pyr}(B, V_{\psi}(p^*), p^*)$. While there is an interpretation of $\psi_{\operatorname{Pyr}(B, V_{\psi}(p^*), p^*)}$ as a cut-generating function for $(S \times \mathbb{Z}_+, \mathbb{R}^n \times \mathbb{R}_+)$, it is not important in what follows. What is important is Theorem 7, which shows that the restriction of $\psi_{[p^*,B]}^*$ to $\mathbb{R}^n \times \{0\}$ is a universal upper bound for all minimal liftings $\pi \in \mathcal{L}_{\psi,p^*}$. We view (3.5) as a formula via (3.3) applied to $\operatorname{Pyr}(B, V_{\psi}(p^*), p^*)$. The following technical lemma will be useful for establishing this upper bound.

LEMMA 6. Let B be a convex 0-neighborhood of the form (3.2). Let $p^* \in \mathbb{R}^n$ and $\lambda > 0$. For $(\bar{r}, \bar{r}_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+$ and $\mu \ge 0$, define $r' := (\bar{r}, \bar{r}_{n+1}) - \mu(p^*, 1)$. Then $\psi_{\operatorname{Pyr}(B,\lambda,p^*)}((\bar{r}, \bar{r}_{n+1})) = \psi_{\operatorname{Pyr}(B,\lambda,p^*)}(r') + \mu\psi_{\operatorname{Pyr}(B,\lambda,p^*)}((p^*, 1))$.

Proof. First, we show

$$\arg \max_{i \in I} \{ a^{i} \cdot \bar{r} + (\lambda - a^{i} \cdot p^{*}) \bar{r}_{n+1} \} = \arg \max_{i \in I} \{ a^{i} \cdot (\bar{r} - \bar{r}_{n+1} p^{*}) \}$$

=
$$\arg \max_{i \in I} \{ (a^{i}, (\lambda - a^{i} \cdot p^{*})) \cdot r' \}.$$

The first and second terms are equal since $\lambda \bar{r}_{n+1}$ is a constant, while the first and the third terms are equal because, for every $i \in I$

$$a^{i} \cdot \bar{r} + (\lambda - a^{i} \cdot p^{*}) \bar{r}_{n+1} = a^{i} \cdot (\bar{r} - \mu p^{*}) + (\lambda - a^{i} \cdot p^{*}) (\bar{r}_{n+1} - \mu) + \lambda \mu = (a^{i}, (\lambda - a^{i} \cdot p^{*})) \cdot r'$$

For
$$i^* \in \arg \max_{i \in I} \{ a^i \cdot \bar{r} + (\lambda - a^i \cdot p^*) \bar{r}_{n+1} \},$$

 $\psi_{\operatorname{Pyr}(B,\lambda,p^*)}((\bar{r},\bar{r}_{n+1})) = a^{i^*} \cdot \bar{r} + (\lambda - a^{i^*} \cdot p^*) \bar{r}_{n+1}$
 $= (a^{i^*}, (\lambda - a^{i^*} \cdot p^*)) \cdot r' + (a^{i^*}, (\lambda - a^{i^*} \cdot p^*)) \cdot \mu(p^*, 1)$
 $= \psi_{\operatorname{Pyr}(B,\lambda,p^*)}(r') + \mu \psi_{\operatorname{Pyr}(B,\lambda,p^*)}((p^*, 1)).$

The last equation holds because $(a^{i^*}, (\lambda - a^{i^*} \cdot p^*)) \cdot (p^*, 1) = \lambda = \psi_{\text{Pyr}(B,\lambda,p^*)}((p^*, 1)).$

THEOREM 7. Let B be a maximal S-free 0-neighborhood of the form (3.2) and ψ be the valid function for (S, \mathbb{R}^n) obtained from B using (3.3). Let $p^* \in \mathbb{R}^n$ and consider $\psi^*_{[p^*,B]}$ defined in (3.5). For $\pi \in \mathcal{L}_{\psi,p^*}$ and $p \in \mathbb{R}^n$, $\pi(p) \leq \psi^*_{[p^*,B]}((p,0))$.

Proof. To reduce notation in this proof, set $\Delta := \operatorname{Pyr}(B, V_{\psi}(p^*), p^*)$. Let $\pi \in \mathcal{L}_{\psi, p^*}$. Define $\mathcal{R} := (\mathbb{R}^n \times \{0\}) \cup \{(p^*, 1)\}$ and $\mathcal{P} := \mathbb{R}^n \times \{0\}$. Using (ψ, π) , which is a valid pair for $(S, \mathbb{R}^n, \mathbb{R}^n)$, we will create functions $\hat{\psi} : \mathbb{R}^{n+1} \to \mathbb{R}$ and $\hat{\pi} : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $(\hat{\psi}, \hat{\pi})$ is a valid pair for $(S \times \mathbb{Z}_+, \mathcal{R}, \mathcal{P})$. Since $\mathbb{R}^n \times \mathbb{R}_+ \subseteq \operatorname{cone}(\mathcal{R})$, we will be able to apply Theorem 3 to obtain a minimal valid pair (ψ', π') for $(S, \mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}_+)$ that equals (ψ, π) on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies $(\psi', \pi') \leq (\psi_{\Delta}, \psi^*_{[p^*, B]})$ when restricted to $(\mathbb{R}^n \times \{0\}) \times (\mathbb{R}^n \times \{0\})$.

Define $\hat{\psi} : \mathcal{R} \to \mathbb{R}$ by $\hat{\psi}((r,0)) = \psi(r)$ for all $r \in \mathbb{R}^n$ and $\hat{\psi}((p^*,1)) = V_{\psi}(p^*)$. Define $\hat{\pi} : \mathcal{P} \to \mathbb{R}$ by $\hat{\pi}((p,0)) = \pi(p)$ for all $p \in \mathbb{R}^n$.

CLAIM 8. $(\hat{\psi}, \hat{\pi})$ is valid for $(S \times \mathbb{Z}_+, \mathcal{R}, \mathcal{P})$.

Proof of Claim. Consider matrices $R \in \mathbb{R}^{(n+1)\times k}$ and $P \in \mathbb{R}^{(n+1)\times \ell}$ with columns in \mathcal{R} and \mathcal{P} , respectively. Let $(\bar{s}, \bar{y}) \in X_{S \times \mathbb{Z}_+}(R, P)$. Using two cases, we show that $(\hat{\psi}, \hat{\pi})$ and (\bar{s}, \bar{y}) satisfy (1.2). First, assume that $(p^*, 1) \notin \operatorname{col}(R)$ or $\bar{s}_{(p^*, 1)} = 0$. Since (ψ, π) is valid for $(S, \mathbb{R}^n, \mathbb{R}^n)$, it follows that

$$\sum_{r\in\operatorname{col}(R)} \hat{\psi}(r)\bar{s}_r + \sum_{p\in\operatorname{col}(P)} \hat{\pi}(p)\bar{y}_p = \sum_{(r',0)\in\operatorname{col}(R)} \psi(r')\bar{s}_r + \sum_{(p',0)\in\operatorname{col}(P)} \pi(p)\bar{y}_p \ge 1.$$

Now, assume that $(p^*, 1) \in \operatorname{col}(R)$ and $\bar{s}_{(p^*, 1)} \neq 0$. Since $R\bar{s} + P\bar{y} \in S \times \mathbb{Z}_+$ and $\mathcal{P} \subseteq \mathbb{R}^n \times \{0\}$, we have $\bar{s}_{(p^*, 1)} \in \mathbb{Z}_+$. Define $\tilde{R} \in \mathbb{R}^{n \times (k-1)}$ by its columns $\operatorname{col}(\tilde{R}) := \{r \in \mathbb{R}^n : (r, 0) \in \operatorname{col}(R)\}$, that is, the columns of \tilde{R} are the columns of $R \setminus \{(p^*, 1)\}$ projected to the first n coordinates. Similarly, define $\tilde{P} \in \mathbb{R}^{n \times (\ell+1)}$ by the columns $\operatorname{col}(\tilde{P}) := \{p \in \mathbb{R}^n : (p, 0) \in \operatorname{col}(P)\} \cup \{p^*\}$. Consider the pair $(\tilde{s}, \tilde{y}) \in \mathbb{R}^{k-1} \times \mathbb{R}^{\ell+1}$ defined by $\tilde{s}_r = \bar{s}_{(r,0)}$ for each $r \in \operatorname{col}(\tilde{R})$

Consider the pair $(\hat{s}, \hat{y}) \in \mathbb{R}^{k-1} \times \mathbb{R}^{\ell+1}$ defined by $\hat{s}_r = \bar{s}_{(r,0)}$ for each $r \in \operatorname{col}(R)$ and $\tilde{y}_p = \bar{y}_{(p,0)}$ for each $p \in \operatorname{col}(\tilde{P}) \setminus \{p^*\}$ and $\tilde{y}_{p^*} = \bar{y}_{(p^*,0)} + \bar{s}_{(p^*,1)}$. Since $\bar{s}_{(p^*,1)} \in \mathbb{Z}_+$ and $R\bar{s} + P\bar{y} \in S \times \mathbb{Z}_+$, it follows that $\tilde{R}\tilde{s} + \tilde{P}\tilde{y} \in S$. Thus, $(\tilde{s}, \tilde{y}) \in X_S(\tilde{R}, \tilde{P})$. By rearranging terms, we see that

$$\begin{split} &\sum_{r \in \operatorname{col}(R)} \hat{\psi}(r) \bar{s}_r + \sum_{p \in \operatorname{col}(P)} \hat{\pi}(p) \bar{y}_p \\ &= \sum_{r \in \operatorname{col}(R) \setminus \{(p^*, 1)\}} \hat{\psi}((r, 0)) \bar{s}_r + \hat{\psi}((p^*, 1)) \bar{s}_{(p^*, 1)} + \sum_{p \in \operatorname{col}(P)} \hat{\pi}(p) \bar{y}_p \\ &= \sum_{r \in \operatorname{col}(R) \setminus \{(p^*, 1)\}} \hat{\psi}((r, 0)) \bar{s}_r + V_{\psi}(p^*) \bar{s}_{(p^*, 1)} + \sum_{p \in \operatorname{col}(P)} \hat{\pi}(p) \bar{y}_p \\ &= \sum_{r \in \operatorname{col}(\tilde{R})} \psi(r) \tilde{s}_r + \sum_{p \in \operatorname{col}(\tilde{P})} \pi(p) \tilde{y}_p \\ \geq 1, \end{split}$$

 \diamond

where the last inequality holds since (ψ, π) is a valid pair for $(S, \mathbb{R}^n, \mathbb{R}^n)$.

By Theorem 3, there exist functions $\psi' : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ and $\pi' : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ such that (ψ', π') is a minimal valid pair for $(S \times \mathbb{Z}_+, \mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}_+)$ and $(\psi', \pi') \leq (\hat{\psi}, \hat{\pi})$ on $\mathcal{R} \times \mathcal{P}$. Thus, by construction of $(\hat{\psi}, \hat{\pi})$, we have $\psi'((r, 0)) \leq \psi(r)$ and $\pi'((p, 0)) \leq \pi(p)$ for all $r, p \in \mathbb{R}^n$. Since ψ is a minimal valid function for (S, \mathbb{R}^n) , we also have that $\psi'((r, 0)) = \psi(r)$ for all $r \in \mathbb{R}^n$. Similarly, since π is a minimal lifting of $\psi, \pi'((p, 0)) = \pi(p)$ for all $p \in \mathbb{R}^n$. By definition of $V_{\psi}(p^*)$, this implies that

(3.6)
$$\psi'((p^*,1)) = \hat{\psi}((p^*,1)) = V_{\psi}(p^*)$$

We now show $\psi'((r, r_{n+1})) \leq \psi_{\Delta}((r, r_{n+1}))$ for $(r, r_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$. Note that $(r, r_{n+1}) = (r, 0) + r_{n+1}(p^*, 1)$. By Lemma 6,

(3.7)
$$\psi_{\Delta}((r,0)) + r_{n+1}\psi_{\Delta}((p^*,1)) = \psi_{\Delta}((r,r_{n+1})).$$

Note that $\psi'((r,0)) \le \hat{\psi}((r,0)) = \psi(r) = \psi_{\Delta}((r,0))$. Thus,

 π

$$\begin{aligned}
\psi'((r, r_{n+1})) &\leq \psi'((r, 0)) + r_{n+1}\psi'((p^*, 1)) & \text{by Theorem 2} \\
&\leq \hat{\psi}((r, 0)) + r_{n+1}\hat{\psi}((p^*, 1)) & \text{by (3.6)} \\
\end{aligned}$$
(3.8)
$$= \psi_{\Delta}((r, 0)) + r_{n+1}\psi_{\Delta}((p^*, 1)) & \\
&= \psi_{\Delta}((r, r_{n+1})). & \text{by (3.7)}
\end{aligned}$$

Let $p \in \mathbb{R}^n$. By Proposition 1, (3.8), and $\pi'((p,0)) = \pi(p)$, we obtain

$$\begin{aligned} (p) &= \pi'((p,0)) \\ &\leq \inf\{\psi'((p,0) + (w,z)) : (w,z) \in W^+_{S \times \mathbb{Z}_+}\} \\ &\leq \inf\{\psi_{\Delta}((p,0) + (w,z)) : (w,z) \in W^+_{S \times \mathbb{Z}_+}\} \\ &= \psi^*_{[p^*,B]}((p,0)). \end{aligned}$$

3.3. Towards a description of the fixing region. In this subsection, let B be a maximal S-free convex 0-neighborhood of the form (3.2), let $\psi := \psi_B$ be the valid function for (S, \mathbb{R}^n) obtained from B using (3.3), and let $p^* \in \mathbb{R}^n$. In this subsection, we define a collection of polyhedra (given by explicit inequalities) whose union $\mathcal{X}(B, p^*)$ will be shown to be a subset of \mathcal{F}_{ψ, p^*} . The results in this subsection consider the pyramid $Pyr(B, \lambda, p^*)$ only for the value $\lambda = V_{\psi}(p^*)$. So, in order to reduce notation, we frequently use the notation

$$\Delta := \operatorname{Pyr}(B, V_{\psi}(p^*), p^*)$$

Let $\tilde{B} \subseteq \mathbb{R}^d$ be an S-free 0-neighborhood that takes one of the following forms: either $\tilde{B} = \{r \in \mathbb{R}^d : a^i \cdot r \leq 1 \forall i \in I\}$ or \tilde{B} is a pyramid of the form (3.4), which we write as $\tilde{B} = \{r \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : a^i \cdot r \leq 1 \forall i \in I\}$. For $x \in \mathbb{R}^d$, the spindle corresponding to x is defined to be

(3.9)
$$R_{\tilde{B}}(x) := \{ r \in \mathbb{R}^d : (a^i - a^k) \cdot r \le 0 \text{ and } (a^i - a^k) \cdot (x - r) \le 0 \ \forall i \in I \},\$$

where $\psi_{\tilde{B}}$ is defined according to $(3.3)^2$ and $k \in I$ is the index such that $\psi_{\tilde{B}}(x) = a^k \cdot x$. Spindles were originally used in [2, 24].

²We remind the reader that formula (3.3) is well-defined for any choice of B containing 0.

Theorem 9 provides a geometric inner approximation of the fixing region \mathcal{F}_{ψ,p^*} . The inner approximation is given by the set

(3.10)
$$\mathcal{X}(B,p^*) := \left(\bigcup_{(\bar{x},\bar{x}_{n+1})\in\Delta\cap(S\times\mathbb{Z}_+)} \left(\bigcup_{i=0}^{\bar{x}_{n+1}} R_B(\bar{x}-\bar{x}_{n+1}p^*) + ip^* \right) \right)$$

THEOREM 9. The set $\mathcal{X}(B, p^*)$ satisfies $\mathcal{X}(B, p^*) + W_S \subseteq \mathcal{F}_{\psi, p^*}$. Also, if $\pi \in \mathcal{L}_{\psi, p^*}$, $q \in \mathcal{X}(B, p^*)$ and $w \in W_S$, then

$$\pi(q+w) = \pi(q) = \psi^*_{[p^*,B]}((q,0)),$$

where $\psi^*_{[p^*,B]}$ is the function defined in (3.5).

We require some tools to prove Theorem 9. For $q \in \mathbb{R}^n$, consider lifting q after p^* has been lifted, that is, consider the smallest value that a minimal lifting of ψ can take at q after the lifting is restricted to take value $V_{\psi}(p^*)$ at p^* . To this end, define

(3.11)
$$V_{\psi}(q; p^*) := \inf \left\{ \pi(q) : \pi \in \mathcal{L}_{\psi, p^*} \right\}.$$

Proposition 4 states that $V_{\psi}(p^*)$ can be computed by constructing the pyramid $\Delta \subseteq \mathbb{R}^n \times \mathbb{R}_+$. Thus, because $V_{\psi}(q; p^*)$ is defined after $V_{\psi}(p^*)$ is fixed, Δ should affect $V_{\psi}(q; p^*)$. This leads us to examine points $(q, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}_+$. For $\lambda > 0, \bar{q} \in \mathbb{R}_+$, and $i \in I$, where I is the index set defining B in (3.2), consider the inequality

$$(3.12) \quad a^{i} \cdot r + (V_{\psi}(p^{*}) - a^{i} \cdot p^{*})r_{n+1} + (\lambda - a^{i} \cdot q - (V_{\psi}(p^{*}) - a^{i} \cdot p^{*})\bar{q})r_{n+2} \le 1.$$

We can apply the pyramid operator Pyr defined in (3.4) using Δ as a base to obtain the iterated pyramid (3.13)

$$\operatorname{Pyr}(\Delta,\lambda,(q,\bar{q})) = \left\{ (r,r_{n+1},r_{n+2}) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ : \begin{array}{c} r_{n+1} - \bar{q}r_{n+2} \ge 0 \text{ and} \\ (3.12) \text{ holds } \forall i \in I \end{array} \right\}.$$

Geometrically, the iterated pyramid in (3.13) is the pyramid (assuming that it is bounded) in \mathbb{R}^{n+2} with base $\Delta \times \{0\}$ and apex $\frac{1}{\lambda}(q,\bar{q},1)$. In this new pyramid, the inequality $r_{n+1} - \bar{q}r_{n+2} \ge 0$ is the result of lifting the inequality $r_{n+1} \ge 0$ defining Δ .

The first result that we need to prove Theorem 9 is the following generalization of a result about spindles in [2, 24].

PROPOSITION 10. Let $(x, x_{n+1}) \in \Delta \cap (S \times \mathbb{Z}_+)$. If $(q, \bar{q}) \in R_{\Delta}((x, x_{n+1})) \cap (\mathbb{R}^n \times \mathbb{R}_+)$, then

$$\psi_{\Delta}((q,\bar{q})) = \inf\{\lambda > 0 : \operatorname{Pyr}(\Delta,\lambda,(q,\bar{q})) \text{ is } (S \times \mathbb{Z} \times \mathbb{Z}) \text{-free}\},\$$

where ψ_{Δ} is defined from Δ using (3.3).

The proof of Proposition 10 is technical and provided in Appendix B.

The next result shows that $V_{\psi}(q; p^*)$ can be computed by constructing a pyramid in \mathbb{R}^{n+2} with base $\Delta \times \{0\}$ using (3.13). So, in order to sequentially lift variables to find a $\pi \in \mathcal{L}_{\psi,p^*}$, we can repeatedly apply the pyramid operator Pyr using the set from the previous lifted variable as a new base.

PROPOSITION 11. Let $q \in \mathbb{R}^n$. The value $V_{\psi}(q; p^*)$ satisfies

$$(3.14) V_{\psi}(q; p^*) = \inf \left\{ \lambda > 0 : \operatorname{Pyr}(\Delta, \lambda, (q, 0)) \text{ is } (S \times \mathbb{Z} \times \mathbb{Z}) \text{-free} \right\}.$$

The proof of Proposition 11 is in Appendix C and is similar to that of Proposition 4.

Proposition 11 characterizes $V_{\psi}(q; p^*)$ using a pyramid that depends on (q, 0). Proposition 12 states that $V_{\psi}(q; p^*)$ can be determined using pyramids that depend on certain translations of (q, 0) while holding p^* fixed.

PROPOSITION 12. If $q \in \mathbb{R}^n$ and $\bar{q} \in \mathbb{Z}_+$, then

$$V_{\psi}(q; p^*) = \inf \{ \lambda > 0 : \operatorname{Pyr}(\Delta, \lambda, (q, 0)) \text{ is } (S \times \mathbb{Z} \times \mathbb{Z}) \text{-free } \} \\ = \inf \{ \lambda > 0 : \operatorname{Pyr}(\Delta, \lambda, (q, \bar{q})) \text{ is } (S \times \mathbb{Z} \times \mathbb{Z}) \text{-free } \}.$$

Proof. The first equation follows from Proposition 11. Define the linear transformation $U: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ by

$$U(y, y_{n+1}, y_{n+2}) = (y, y_{n+1} + y_{n+2}\bar{q}, y_{n+2})$$

Note that U is invertible and $U^{-1}(y, y_{n+1}, y_{n+2}) = (y, y_{n+1} - y_{n+2}\bar{q}, y_{n+2})$. Since $\bar{q} \in \mathbb{Z}$, the map U is unimodular. Both U and U^{-1} map $S \times \mathbb{Z} \times \mathbb{Z}$ onto itself, and therefore, they map $(S \times \mathbb{Z} \times \mathbb{Z})$ -free sets to $(S \times \mathbb{Z} \times \mathbb{Z})$ -free sets.

Let $\lambda > 0$ and $(r, r_{n+1}, r_{n+2}) \in Pyr(\Delta, \lambda, (q, 0))$. For each $i \in I$, (3.13) implies

$$a^{i} \cdot r + (V_{\psi}(p^{*}) - a^{i} \cdot p^{*})r_{n+1} + (\lambda - a^{i} \cdot q)r_{n+2} \le 1$$

Thus,

$$(3.15) \ a^{i} \cdot r + (V_{\psi}(p^{*}) - a^{i} \cdot p^{*})(r_{n+1} + r_{n+2}\bar{q}) + (\lambda - a^{i} \cdot q - (V_{\psi}(p^{*}) - a^{i} \cdot p^{*})\bar{q})r_{n+2} \le 1.$$

Equation (3.15) is equivalent to $U(r, r_{n+1}, r_{n+2}) = (r, r_{n+1} + r_{n+2}\bar{q}, r_{n+2})$ satisfying (3.12) for each $i \in I$. Also, $r_{n+1} \ge 0$ because $(r, r_{n+1}, r_{n+2}) \in Pyr(\Delta, \lambda, (q, 0))$. Thus, $(r_{n+1}-r_{n+2}\bar{q})+r_{n+2}\bar{q}=r_{n+1}\ge 0$ and $U(r, r_{n+1}, r_{n+2})$ satisfies every inequality defining $Pyr(\Delta, \lambda, (q, \bar{q}))$ in (3.13). So, $U(r, r_{n+1}, r_{n+2}) \in Pyr(\Delta, \lambda, (q, \bar{q}))$ and

$$U \operatorname{Pyr}(\Delta, \lambda, (q, 0)) \subseteq \operatorname{Pyr}(\Delta, \lambda, (q, \bar{q})).$$

It remains to show $U^{-1} \operatorname{Pyr}(\Delta, \lambda, (q, \bar{q})) \subseteq \operatorname{Pyr}(\Delta, \lambda, (q, 0))$. If $(r, r_{n+1}, r_{n+2}) \in \operatorname{Pyr}(\Delta, \lambda, (q, \bar{q}))$, then the (n+1)-st component of $U^{-1}(r, r_{n+1}, r_{n+2})$ is $r_{n+1} - r_{n+2}\bar{q}$. Because (r, r_{n+1}, r_{n+2}) satisfies the inequalities (3.13), we have $r_{n+1} - r_{n+2}\bar{q} \ge 0$. Using arguments from the first part of this proof, we have $U^{-1} \operatorname{Pyr}(\Delta, \lambda, (q, \bar{q})) \subseteq \operatorname{Pyr}(\Delta, \lambda, (q, 0))$. So, $U \operatorname{Pyr}(\Delta, \lambda, (q, 0)) = \operatorname{Pyr}(\Delta, \lambda, (q, \bar{q}))$.

Since U and U^{-1} preserve $(S \times \mathbb{Z} \times \mathbb{Z})$ -free sets, the previous argument implies that if $Pyr(\Delta, \lambda, (q, 0))$ is $(S \times \mathbb{Z} \times \mathbb{Z})$ -free, then $Pyr(\Delta, \lambda, (q, \bar{q}))$ is $(S \times \mathbb{Z} \times \mathbb{Z})$ -free, and vice versa. This gives the desired result.

For $t \in \mathbb{R}$, define $H_t := \mathbb{R}^n \times \{t\}$. The next proposition shows that translating $H_0 \cap R_\Delta(\bar{x}, \bar{x}_{n+1})$ by tp^* is equal to projecting $H_t \cap R_\Delta(\bar{x}, \bar{x}_{n+1})$ onto the first n coordinates. The proof of Proposition 13 is given in Appendix D.

PROPOSITION 13. If $(\bar{x}, \bar{x}_{n+1}) \in (S \times \mathbb{Z}_+) \cap \Delta$ is a blocking point of Δ and $t \in \mathbb{R}$, then $H \subseteq B$ $(\bar{z}, \bar{z}, -) = (H \subseteq B, (\bar{z}, \bar{z}, -)) + t(x^*, 1)$

$$H_t \cap R_{\Delta}(\bar{x}, \bar{x}_{n+1}) = (H_0 \cap R_{\Delta}(\bar{x}, \bar{x}_{n+1})) + t(p^*, 1)$$

= $(R_B(\bar{x} - \bar{x}_{n+1}p^*) \times \{0\}) + t(p^*, 1)$.

We can now prove Theorem 9.

Proof of Theorem 9. Recall that $\Delta = \operatorname{Pyr}(B, V_{\psi}(p^*), p^*)$. Let $(\bar{x}, \bar{x}_{n+1}) \in \Delta \cap$ $(S \times \mathbb{Z}_+)$. If $\bar{x}_{n+1} = 0$, then $\bar{x} \in B \cap S$. In this case, $\bigcup_{i=0}^{\bar{x}_{n+1}} R_B(\bar{x} - \bar{x}_{n+1}p^*) + ip^* =$ $R_B(\bar{x})$. It is well-known (see, for example, [24, 23, 15, 2]) that $R_B(\bar{x}) + W_S \subseteq R(B)$, where R(B) is the extended lifting region (1.3) and $R_B(\bar{x})$ is the spindle corresponding to \bar{x} given in (3.9). Thus, by the definition of R(B), we obtain

$$\left(\bigcup_{i=0}^{\bar{x}_{n+1}} R_B(\bar{x} - \bar{x}_{n+1}p^*) + ip^*\right) + W_S = R_B(\bar{x}) + W_S \subseteq \mathcal{F}_{\psi,p^*}.$$

It is left to show $\bigcup_{i=0}^{\bar{x}_{n+1}} R_B(\bar{x} - \bar{x}_{n+1}p^*) + ip^* \subseteq \mathcal{F}_{\psi,p^*}$ when $x_{n+1} \ge 1$, i.e., when (\bar{x}, \bar{x}_{n+1}) is a blocking point of Δ . Let $\psi_{[p^*,B]}^* : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ be from (3.5), $\pi \in \mathcal{L}_{\psi,p^*}$, and $q \in R_B(\bar{x} - \bar{x}_{n+1}p^*) + ip^*$ for some $i \in \{0, \ldots, \bar{x}_{n+1}\}$. Note that

$$\begin{aligned}
V_{\psi}(q;p^{*}) &\leq \pi(q) & \text{by the definition of } V_{\psi}(q;p^{*}) \\
&\leq \psi^{*}_{[p,B^{*}]}((q,0)) & \text{by Theorem 7} \\
&= \inf_{\substack{(w,z)\in W^{+}_{S\times\mathbb{Z}_{+}}}} \psi_{\Delta}((q,0) + (w,z)) \\
&\leq \psi_{\Delta}((q,i)).
\end{aligned}$$

By Proposition 13, $(q, i) \in R_{\Delta}(\bar{x}, \bar{x}_{n+1}) \cap H_i$, so by Proposition 10 with $\bar{q} = i$,

$$\psi_{\Delta}((q,i)) = \inf\{\lambda > 0 : \operatorname{Pyr}(\Delta,\lambda,(q,i)) \text{ is } (S \times \mathbb{Z} \times \mathbb{Z}) \text{-free}\}.$$

By Proposition 12,

$$\psi_{\Delta}((q,i)) = V_{\psi}(q;p^*).$$

Thus, $V_{\psi}(q; p^*) = \pi(q) = \psi^*_{[p^*,B]}((q,0))$. Note that π was chosen arbitrarily in \mathcal{L}_{ψ,p^*} . Hence, every function in \mathcal{L}_{ψ,p^*} agrees on q. By definition of \mathcal{F}_{ψ,p^*} and Proposition 1 (a), it follows that

$$\left(\bigcup_{i=0}^{\bar{x}_{n+1}} R_B(\bar{x} - \bar{x}_{n+1}p^*) + ip^*\right) + W_S \subseteq \mathcal{F}_{\psi,p^*}.$$

3.4. Translation invariance of fixing region.

THEOREM 14. Let B be a maximal S-free convex 0-neighborhood and let $t \in \mathbb{R}^n$ such that $0 \in int(B+t)$. Thus, B+t is a maximal (S+t)-free convex 0-neighborhood. For $p^* \in \mathbb{R}^n$ and $\hat{p} := p^* + V_{\psi}(p^*)t \in \mathbb{R}^n$,

$$\mathcal{X}(B, p^*) + W_S = \mathbb{R}^n$$
 if and only if $\mathcal{X}(B+t, \hat{p}) + W_{S+t} = \mathbb{R}^n$.

Theorem 14 states that if a given maximal S-free convex 0-neighborhood B is one point fixable, then any translation B + t such that B + t is (S + t)-free is also one point fixable. The proof of Theorem 14 is technical in nature and is similar to that of Theorem 3.1 in [10]. For this reason, we provide the proof in Appendix E.

4. Application: Fixing Regions of Type 3 triangles. In this section, we find minimal liftings for Type 3 triangles. Type 3 triangles, which are defined precisely below, are maximal S-free convex 0-neighborhoods in \mathbb{R}^2 that contains exactly three points of S, one in the relative interior of each facet. In Subsection 4.1, we identify conditions that guarantee that a Type 3 triangle is one point fixable. In Subsection 4.2, we show that a family of Type 3 triangles coming from the extensively studied mixing set problem satisfies this sufficient condition.

In this section, let $S = \mathbb{Z}^2 + b$ for $b = (b_1, b_2) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$. Without loss of generality, we assume that $-1 \leq b_1, b_2 \leq 0$, and, by relabeling the coordinates, we assume that $-1 \leq b_2 \leq b_1 \leq 0$. Thus, the origin (0,0) is contained in the interior of the triangle $\operatorname{conv}\{\bar{s}^1, \bar{s}^2, \bar{s}^3\}$, where $\bar{s}^1 := (1 + b_1, 1 + b_2), \bar{s}^2 := (b_1, 1 + b_2)$, and $\bar{s}^3 := (b_1, b_2)$.

For $\gamma_1, \gamma_2, \gamma_3 > 0$ with $\gamma_2, \gamma_3 < 1$, define the vectors

$$q^{1} = q^{1}(\gamma_{1}) := \left(\frac{1}{(1,\gamma_{1}) \cdot (b_{1}+1,b_{2}+1)}, \frac{\gamma_{1}}{(1,\gamma_{1}) \cdot (b_{1}+1,b_{2}+1)}\right),$$

$$(4.1) \qquad q^{2} = q^{2}(\gamma_{2}) := \left(\frac{-1}{(-1,\gamma_{2}) \cdot (b_{1},b_{2}+1)}, \frac{\gamma_{2}}{(-1,\gamma_{2}) \cdot (b_{1},b_{2}+1)}\right),$$

$$q^{3} = q^{3}(\gamma_{3}) := \left(\frac{\gamma_{3}}{(\gamma_{3},-1) \cdot (b_{1},b_{2})}, \frac{-1}{(\gamma_{3},-1) \cdot (b_{1},b_{2})}\right),$$

and the triangle

(4.2)
$$T(\gamma_1, \gamma_2, \gamma_3) := \{ (x_1, x_2) \in \mathbb{R}^2 : q^i \cdot (x_1, x_2) \le 1 \ \forall \ i \in \{1, 2, 3\} \}.$$

Each triangle in the collection $\{T(\gamma_1, \gamma_2, \gamma_3) : \gamma_1, \gamma_2, \gamma_3 > 0 \text{ and } \gamma_2, \gamma_3 < 1\}$ is a maximal *S*-free convex 0-neighborhood in \mathbb{R}^2 such that each facet contains one of the points $\overline{s}^1, \overline{s}^2$ and \overline{s}^3 from *S* in their relative interior. In the literature, these triangles are referred to as *Type 3 triangles*. See [24] and the references therein for more on Type 3 triangles and the classification of *S*-free sets in \mathbb{R}^2 .

4.1. Sufficient condition for Type 3 triangles to be one point fixable. Let $T := T(\gamma_1, \gamma_2, \gamma_3)$ be a Type 3 triangle. Using (4.1), define the pyramid

$$P := \{ (x_1, x_2, x_3) \in \mathbb{R}^2 \times \mathbb{R}_+ : q^2(\gamma_2) \cdot (x_1, x_2) \le 1, (4.3) \qquad q^1(\gamma_1) \cdot (x_1, x_2) + (1 - \frac{(1, \gamma_1) \cdot (b_1 + 1, b_2 + 2)}{(1, \gamma_1) \cdot (b_1 + 1, b_2 + 1)}) x_3 \le 1, q^3(\gamma_3) \cdot (x_1, x_2) + (\frac{1}{2} - \frac{(\gamma_3, -1) \cdot (1 + b_1, 2 + b_2)}{2(\gamma_3, -1) \cdot (b_1, b_2)}) x_3 \le 1 \}.$$

Observe $T \times \{0\} = P \cap \{(x_1, x_2, x_3) : x_3 = 0\}$. Also, P contains the $S \times \mathbb{Z}$ points $(s^1, z^1) := (1 + b_1, 1 + b_2, 0), (s^2, z^2) := (b_1, 1 + b_2, 0), (s^3, z^3) := (b_1, b_2, 0), (s^4, z^4) := (1 + b_1, 2 + b_2, 1), (s^5, z^5) := (b_1, 1 + b_2, 1), and <math>(s^6, z^6) := (1 + b_1, 1 + b_2, 2)$, and P has three facets, F_1, F_2 , and F_3 , containing $\{(s^1, z^1), (s^4, z^4)\}, \{(s^2, z^2), (s^5, z^5)\}, and \{(s^3, z^3), (s^6, z^6)\}, respectively. In order to apply Theorem 9 to <math>T$, we need to show $P = Pyr(T, V_{\psi}(p^*), p^*)$ for some $p^* \in \mathbb{R}^2$. In the next result, we give sufficient conditions on $(\gamma_1, \gamma_2, \gamma_3)$ for such a p^* to exist, i.e., we give sufficient conditions for T to be one point fixable. Note that $W_S = \mathbb{Z}^2$ because $S = \mathbb{Z}^2 + b$.

PROPOSITION 15. Let $T := T(\gamma_1, \gamma_2, \gamma_3)$ be a Type 3 triangle and P be of the form (4.3). Let ψ be the valid function for (S, \mathbb{R}^2) obtained from T using (3.3).

- (i) P is a pyramid whose apex $a^* = (a_1^*, a_2^*, a_3^*)$ satisfies $a_3^* > 0$ if and only if $\gamma_2(2 \gamma_3 + 2\gamma_1\gamma_3) \gamma_1\gamma_3 > 0$.
- (ii) If P is $(S \times \mathbb{Z})$ -free, then $\mathcal{X}(T, p^*) + W_S = \mathbb{R}^n$ for $p^* = \frac{1}{a_3^*}(a_1^*, a_2^*)$. Thus, \mathcal{L}_{ψ, p^*} contains a unique function.

Proof. The apex of P is $a^* = (a_1^*, a_2^*, a_3^*)$, where

$$a_{1}^{*} = b_{1} + \frac{\gamma_{2}(2 + 2\gamma_{1} - \gamma_{3})}{\gamma_{2}(2 - \gamma_{3} + 2\gamma_{1}\gamma_{3}) - \gamma_{1}\gamma_{3}},$$

$$a_{2}^{*} = b_{2} + \frac{\gamma_{1}(2 - \gamma_{3} + 2\gamma_{2}\gamma_{3}) - (1 + \gamma_{2})(-2 + \gamma_{3})}{\gamma_{2}(2 - \gamma_{3} + 2\gamma_{1}\gamma_{3}) - \gamma_{1}\gamma_{3}},$$

$$a_{3}^{*} = \frac{2(1 + \gamma_{1} + \gamma_{2} - \gamma_{2}\gamma_{3})}{\gamma_{2}(2 - \gamma_{3} + 2\gamma_{1}\gamma_{3}) - \gamma_{1}\gamma_{3}}.$$

In order for P to be a pyramid with apex a^* satisfying $a_3^* > 0$, it is enough to show that $2(1 + \gamma_1 + \gamma_2 - \gamma_2\gamma_3) > 0$ and $\gamma_2(2 - \gamma_3 + 2\gamma_1\gamma_3) - \gamma_1\gamma_3 > 0$. The first inequality holds since $\gamma_3 < 1$ and the second holds by hypothesis. Hence, (i) holds.

By Proposition 18, \mathcal{L}_{ψ,p^*} is nonempty. By Theorem 9, in order to see that \mathcal{L}_{ψ,p^*} contains a unique function, it is sufficient to show that $\mathcal{X}(T,p^*) + W_S = \mathcal{X}(T,p^*) + \mathbb{Z}^2 = \mathbb{R}^2$. We draw inspiration from [24]. The crucial observation is that $P = \text{Pyr}(T, V_{\psi}(p^*), p^*)$ for the choice of p^* in the hypothesis.

Figure 8 in [24] labels the vertices of the spindles $R_T(s^1)$, $R_T(s^2)$ and $R_T(s^3)$ for T (recall (3.9) for the definition of a spindle). For completeness, we reproduce the labels in Figure 1 with the values v^i and δ_i defined below. The vertices of T are

$$v^{1} = \left(b_{1} + \frac{1+\gamma_{1}}{1+\gamma_{1}\gamma_{3}}, b_{2} + \frac{\gamma_{3}+\gamma_{1}\gamma_{3}}{1+\gamma_{1}\gamma_{3}}\right),$$

$$v^{2} = \left(b_{1} + \frac{\gamma_{2}}{\gamma_{1}+\gamma_{2}}, b_{2} + \frac{1+\gamma_{1}+\gamma_{2}}{\gamma_{1}+\gamma_{2}}\right), \text{ and}$$

$$v^{3} = \left(b_{1} + \frac{-\gamma_{2}}{1-\gamma_{2}\gamma_{3}}, b_{2} + \frac{-\gamma_{2}\gamma_{3}}{1-\gamma_{2}\gamma_{3}}\right).$$

The values of δ_i for $i \in \{1, 2, 3\}$ are

$$\delta_1 = \frac{1 + \gamma_1 \gamma_3}{1 + \gamma_1 + \gamma_2 - \gamma_2 \gamma_3}, \quad \delta_2 = \frac{\gamma_1 + \gamma_2}{1 + \gamma_1 + \gamma_2 - \gamma_2 \gamma_3}, \quad \delta_3 = \frac{1 + \gamma_1 - \gamma_2 \gamma_3 - \gamma_1 \gamma_2 \gamma_3}{1 + \gamma_1 + \gamma_2 - \gamma_2 \gamma_3}.$$

The δ_i 's are convex coefficients satisfying $s^i = \delta_i v^i + (1 - \delta_i) v^{i+1}$ for i = 1, 2, 3, where $v^4 = v^1$. One observes that $\delta_i \in [0, 1]$ holds because $\gamma_i > 0$ and $\gamma_2, \gamma_3 < 1$.

Define the region $K := \operatorname{conv}\{c^2, k, j, i, g, e^1\}$ (see Figure 1). The literature [24, 15], [2, Theorem 4] shows that $\mathbb{R}^2 \setminus (K + \mathbb{Z}^2)$ is contained in R(T), which is contained in $\mathcal{X}(T, p^*) + \mathbb{Z}^2$. Hence, if we can show that $K \subseteq \mathcal{X}(T, p^*) + \mathbb{Z}^2$, then $K + \mathbb{Z}^2 \subseteq \mathcal{X}(T, p^*) + \mathbb{Z}^2$ implying that $\mathbb{R}^2 = \mathcal{X}(T, p^*) + \mathbb{Z}^2$, thus completing the proof.

To this end, write K as $K = \bigcup_{i=1}^{5} K_i$, where

$$\begin{split} K_1 &= \operatorname{conv}\{l, e^1, g, u\}, \quad K_2 &= \operatorname{conv}\{u, m, i, g\}, \\ K_3 &= \operatorname{conv}\{m, j, k, p^*\}, \quad K_4 &= \operatorname{conv}\{c^2, k, p^*, l\}, \quad \text{and} \ K_5 &= \operatorname{conv}\{l, p^*, m, u\}. \end{split}$$

CLAIM 16. $K_1 \subseteq R_T(s^4 - p^*), K_2 \subseteq R_T(s^5 - p^*) + (1, 1), K_3 \subseteq R_T(s^4 - p^*) + p^*, K_4 \subseteq R_T(s^5 - p^*) + p^*, and K_5 \subseteq R_T(s^6 - 2p^*) + p^*.$

The proof of Claim 16 is technical and appears in Appendix F. By Theorem 9, $R_T(s^4-p^*), R_T(s^4-p^*)+p^*, R_T(s^5-p^*)+(1,1), R_T(s^5-p^*)+p^*, \text{ and } R_T(s^6-2p^*)+p^*$ are contained in $\mathcal{X}(T,p^*) + \mathbb{Z}^2$.

4.2. Type 3 triangles from the mixing set. Proposition 15 assumes the pyramid P is $(S \times \mathbb{Z})$ -free. This is satisfied by *mixing set* Type 3 triangles [23, 27].



FIG. 1. The spindles of T given in [24]. $K := \operatorname{conv}\{c^2, k, j, i, g, e^1\}$ is shaded, and o is the origin.

The mixing set is considered a fundamental set in mixed-integer programming theory. The facet-defining inequalities of this set are called "mixing" inequalities as they are supposed to "mix" the well-known mixed-integer rounding (MIR) inequalities. The mixing set appears as a relaxation of several important problems [27] such as production planning, capacitated facility location, and capacitated network design. Recently, inequalities closely related to mixing inequalities have had a huge impact in solving stochastic integer programs [28]. Mixing inequalities can be used for general mixed-integer linear programs, and there are several studies of its properties [21, 20]. Several generalizations of the mixing set have been studied as well [31, 18, 17].

If a Type 3 triangle satisfies $b \in int(conv \{(0, -1), (0, -1/2), (-1, -1)\})$, then we say that it is a *mixing set Type 3 triangle*. With this additional constraint on b, the mixing set Type 3 triangles are defined by b satisfying $-1 < b_2 < b_1 < 0$ and $b_1 - 2b_2 > 1$. Define $\delta_b = -b_1^2 - b_2^2 + b_1b_2 - b_2$ and observe that

$$\delta_b := b_1(b_2 - b_1) - b_2(1 + b_2) > 0.$$

Consider the Type 3 triangle $T(b) := T(\frac{b_2-b_1}{b_1}, \frac{b_1-b_2}{1+b_1}, \frac{b_1}{b_1-b_2-1})$ defined by

$$T(b) = \{ (x_1, x_2) \in \mathbb{R}^2 : (\frac{-b_1}{\delta_b}) x_1 + (\frac{b_1 - b_2}{\delta_b}) x_2 \le 1, (\frac{-b_1 - 1}{\delta_b}) x_1 + (\frac{b_1 - b_2}{\delta_b}) x_2 \le 1, (\frac{-b_1}{\delta_b}) x_1 + (\frac{b_1 - b_2 - 1}{\delta_b}) x_2 \le 1 \}.$$

By construction, $T(b) \cap S = \{(b_1, b_2), (b_1, 1 + b_2), (1 + b_1, 1 + b_2)\}$. Note that the constraints on b imply that $\gamma_1, \gamma_2, \gamma_3 > 0$ and $\gamma_2, \gamma_3 < 1$, as required. Substituting these values of $\gamma_1, \gamma_2, \gamma_3$ into (4.3), we obtain the pyramid

(4.4)

$$P(b) := \{ (x_1, x_2, x_3) \in \mathbb{R}^2 \times \mathbb{R}_+ : (\frac{-b_1}{\delta_b}) x_1 + (\frac{b_1 - b_2}{\delta_b}) x_2 - (\frac{b_1 - b_2}{\delta_b}) x_3 \leq 1, \\ (\frac{-b_1 - 1}{\delta_b}) x_1 + (\frac{b_1 - b_2}{\delta_b}) x_1 + (\frac{b_1 - b_2}{\delta_b}) x_2 \leq 1, \\ (\frac{-b_1}{\delta_b}) x_1 + (\frac{b_1 - b_2 - 1}{\delta_b}) x_2 + (\frac{2 - b_1 + 2b_2}{\delta_b}) x_3 \leq 1 \}.$$

We verify the two conditions in Proposition 15 to conclude that there exists a $p^* \in \mathbb{R}^2$ satisfying one point fixability for mixing set triangles. The condition $\gamma_2(2 - p^*)$

 $\gamma_3 + 2\gamma_1\gamma_3) - \gamma_1\gamma_3 > 0$ can be checked using $\gamma_1 = \frac{b_2 - b_1}{b_1}, \gamma_2 = \frac{b_1 - b_2}{1 + b_1}, \gamma_3 = \frac{b_1}{b_1 - b_2 - 1},$ and the constraints $-1 < b_2 < b_1 < 0$. Next, we verify $int(P(b)) \cap (S \times \mathbb{Z}) = \emptyset$.

PROPOSITION 17. P(b) is $(S \times \mathbb{Z})$ -free if T(b) is a mixing set Type 3 triangle.

Proof. For $t \in \mathbb{Z}_+$, define $H_t := \mathbb{R}^2 \times \{t\}$. Since $P(b) \cap H_0 = T(b) \times \{0\}$ is S-free, we only need to show relint $(P(b) \cap H_t) \cap (S \times \{t\}) = \emptyset$ for $t \ge 1$.

For $t \geq 1$, define the split sets

$$\begin{split} C_1 &:= \{ (x_1, x_2, t) \in \mathbb{R}^3 : \ t \le x_2 \le t + 1 \} + (b_1, b_2, 0), \\ C_2 &:= \{ (x_1, x_2, t) \in \mathbb{R}^3 : \ 0 \le -2x_1 + x_2 \le 1 \} + (b_1, b_2, 0), \text{ and} \\ C_3 &:= \left\{ (x_1, x_2, t) \in \mathbb{R}^3 : \ \frac{t}{2} \le -x_1 + x_2 \le \frac{t}{2} + \frac{1}{2} \right\} + (b_1, b_2, 0). \end{split}$$

The splits C_1, C_2 and C_3 have no points from $S \times \{t\}$ in their relative interior. So, if we show relint $(P(b) \cap H_t) \subseteq \operatorname{relint}(C_1) \cup \operatorname{relint}(C_2) \cup \operatorname{relint}(C_3)$, then P(b) will be $(S \times \mathbb{Z})$ -free, completing the proof. To this end, assume $(x_1^*, x_2^*, t) \in \operatorname{relint}(P(b) \cap H_t) \setminus (\operatorname{relint}(C_1) \cup \operatorname{relint}(C_2))$. This implies that (x_1^*, x_2^*, t) does not strictly satisfy some inequality defining C_1 and some inequality defining C_2 . This leads to four cases. Case 1. Suppose $x_2^* - b_2 \leq t$ and $-2(x_1^* - b_1) + (x_2^* - b_2) \leq 0$. Observe that

$$\begin{split} & (\frac{-b_1}{\delta_b})x_1^* + (\frac{b_1 - b_2 - 1}{\delta_b})x_2^* + (\frac{2 - b_1 + 2b_2}{2\delta_b}) \\ & \geq \quad (\frac{-b_1}{\delta_b})(\frac{2b_1 + x_2^* - b_2}{2}) + (\frac{b_1 - b_2 - 1}{\delta_b})x_2^* + (\frac{2 - b_1 + 2b_2}{2\delta_b}) \\ & = \quad (\frac{b_1 - 2b_2 - 2}{2\delta_b})x_2^* + (\frac{2 - b_1 + 2b_2}{2\delta_b})t + (\frac{-2b_1^2 + b_1b_2}{2\delta_b}) \\ & \geq \quad (\frac{b_1 - 2b_2 - 2}{2\delta_b})(t + b_2) + (\frac{2 - b_1 + 2b_2}{2\delta_b})t + (\frac{-2b_1^2 + b_1b_2}{2\delta_b}) = 1. \end{split}$$

The first inequality follows from the assumption $-2(x_1^*-b_1)+(x_2^*-b_2) \leq 0$, and the second from the assumption $x_2^*-b_2 \leq t$. This contradicts $(x_1^*, x_2^*, t) \in \operatorname{relint}(P(b) \cap H_t)$ because the third inequality defining P(b) is violated.

Case 2. Suppose $x_2^* - b_2 \le t$ and $-2(x_1^* - b_1) + (x_2^* - b_2) \ge 1$. We claim $(x_1^*, x_2^*, t) \in \text{relint}(C_3)$. It is sufficient to show $\frac{t}{2} < -(x_1^* - b_1) + (x_2^* - b_2) < \frac{t}{2} + \frac{1}{2}$. Because $(x_1^*, x_2^*, t) \in \text{relint}(P(b) \cap H_t)$, the third inequality defining P(b) bounds x_2^* :

$$x_2^* > \frac{-b_1}{1+b_2-b_1}x_1^* + \frac{t}{2} + \frac{1+b_2}{2(1+b_2-b_1)}t + \frac{-\delta_b}{1+b_2-b_1}$$

Using this, we see that

$$\begin{aligned} &-(x_1^*-b_1)+(x_2^*-b_2) \\ > &-(x_1^*-b_1)+(\frac{-b_1}{1+b_2-b_1}x_1^*+\frac{t}{2}+\frac{1+b_2}{2(1+b_2-b_1)}t+\frac{-\delta_b}{1+b_2-b_1})-b_2 \\ = & \frac{t}{2}+(\frac{-1-b_2}{1+b_2-b_1})x_1^*+(\frac{1+b_2}{2(1+b_2-b_1)})t+(\frac{b_1+b_1b_2}{1+b_2-b_1}) \\ \geq & \frac{t}{2}+(\frac{-1-b_2}{1+b_2-b_1})(\frac{x_2^*-b_2-1+2b_1}{2})+(\frac{1+b_2}{2(1+b_2-b_1)})t+(\frac{b_1+b_1b_2}{1+b_2-b_1}) \\ = & \frac{t}{2}+(\frac{-1-b_2}{2(1+b_2-b_1)})x_2^*+(\frac{1+b_2}{2(1+b_2-b_1)})t+(\frac{2b_2+b_2^*+1}{2(1+b_2-b_1)}) \\ \geq & \frac{t}{2}+(\frac{-1-b_2}{2(1+b_2-b_1)})(t+b_2)+(\frac{1+b_2}{2(1+b_2-b_1)})t+(\frac{2b_2+b_2^*+1}{2(1+b_2-b_1)}) \\ = & \frac{t}{2}+\frac{1+b_2}{2(1+b_2-b_1)}>\frac{t}{2}. \end{aligned}$$

The second inequality follows from $-2(x_1^* - b_1) + (x_2^* - b_2) \ge 1$ and $\frac{-1-b_2}{-b_1+b_2+1} < 0$, the third follows from $x_2^* \le t + b_2$, and the fourth follows from $\frac{1+b_2}{2(1+b_2-b_1)} > 0$.

Since $(x_1^*, x_2^*, t) \in \operatorname{relint}(P(b) \cap H_t)$, the second inequality defining P(b) implies

$$\begin{array}{rcl} -(x_1^*-b_1)+(x_2^*-b_2) &<& -x_1^*+b_1+\left(\frac{\delta_b}{b_1-b_2}+\frac{1+b_1}{b_1-b_2}x_1^*\right)-b_2\\ &=& \left(\frac{1+b_2}{b_1-b_2}\right)x_1^*+\frac{-b_2-b_1b_2}{b_1-b_2}\\ &\leq& \left(\frac{1+b_2}{b_1-b_2}\right)\left(\frac{2b_1+x_2^*-b_2-1}{2}\right)+\frac{-b_2-b_1b_2}{b_1-b_2}\\ &=& \left(\frac{1+b_2}{2(b_1-b_2)}\right)x_2^*+\left(\frac{2b_1-4b_2-b_2^2-1}{2(b_1-b_2)}\right)\\ &\leq& \left(\frac{1+b_2}{2(b_1-b_2)}\right)(t+b_2)+\left(\frac{2b_1-4b_2-b_2^2-1}{2(b_1-b_2)}\right)\\ &=& \frac{t}{2}+\left(\frac{1-b_1+2b_2}{b_1-b_2}\right)\frac{t}{2}+\left(\frac{2b_1-3b_2-1}{2(b_1-b_2)}\right)=\frac{t}{2}+\frac{1}{2}. \end{array}$$

The second inequality follows since $-2(x_1^*-b_1)+(x_2^*-b_2) \ge 1$ and $\frac{1+b_2}{-b_1+b_2+1} > 0$, the third follows since $x_2^* \le t+b_2$, and the fourth follows since $t \ge 1$ and $1 < b_1 - 2b_2$. *Case 3.* Suppose $x_2^* - b_2 \ge t+1$ and $-2(x_1^*-b_1)+(x_2^*-b_2) \le 0$. Observe that

$$\begin{aligned} & \left(\frac{-b_1}{\delta_b}\right) x_1^* + \left(\frac{b_1 - b_2}{\delta_b}\right) x_2^* - \left(\frac{b_1 - b_2}{\delta_b}\right) t \\ \geq & \left(\frac{-b_1}{\delta_b}\right) \left(\frac{2b_1 + x_2^* - b_2}{2}\right) + \left(\frac{b_1 - b_2}{\delta_b}\right) x_2^* - \left(\frac{b_1 - b_2}{\delta_b}\right) t \\ = & \left(\frac{b_1 - 2b_2}{2\delta_b}\right) x_2^* - \left(\frac{b_1 - b_2}{\delta_b}\right) t + \left(\frac{-2b_1^2 + b_1b_2}{2\delta_b}\right) \\ \geq & \left(\frac{b_1 - 2b_2}{2\delta_b}\right) (t + 1 + b_2) - \left(\frac{b_1 - b_2}{\delta_b}\right) t + \left(\frac{-2b_1^2 + b_1b_2}{2\delta_b}\right) \\ = & \left(\frac{-b_1}{2\delta_b}\right) t + \left(\frac{b_1}{2\delta_b}\right) + 1 \\ \geq & \left(\frac{-b_1}{2\delta_b}\right) + \left(\frac{b_1}{2\delta_b}\right) + 1 = 1. \end{aligned}$$

The first inequality follows since $\frac{-b_1}{\delta_b} > 0$ and $-2(x_1^* - b_1) + (x_2^* - b_2) \ge 0$, the second inequality follows since $b_1 - 2b_2 > 1$ and $x_2^* \ge t + 1 + b_2$, and the third inequality follows since $t \ge 1$. This contradicts that $(x_1^*, x_2^*, t) \in \operatorname{relint}(P(b) \cap H_t)$ because the first inequality defining P(b) is violated.

Case 4. Suppose $x_2^* - b_2 \ge t + 1$ and $-2(x_1^* - b_1) + (x_2^* - b_2) \ge 1$. Observe

$$\begin{aligned} (\frac{-b_1-1}{\delta_b})x_1^* + (\frac{b_1-b_2}{\delta_b})x_2^* &\geq (\frac{-b_1-1}{\delta_b})(\frac{x_2^*-1+2b_1-b_2}{2}) + (\frac{b_1-b_2}{\delta_b})x_2^* \\ &= (\frac{b_1-2b_2-1}{2\delta_b})x_2^* + (\frac{-b_1-1}{\delta_b})(\frac{2b_1-b_2-1}{2}) \\ &\geq (\frac{b_1-2b_2-1}{2\delta_b})(2+b_2) + (\frac{-b_1-1}{\delta_b})(\frac{2b_1-b_2-1}{2}) \\ &= 1 + \frac{b_1-2b_2-1}{2\delta_b} > 1. \end{aligned}$$

The first inequality comes from $\frac{-b_1-1}{\delta_b} < 0$ and $-2(x_1^*-b_1) + (x_2^*-b_2) \ge 1$. The second inequality comes from the fact that $b_1 - 2b_2 > 1$ and $\delta_b > 0$ so the term $\frac{b_1-2b_2-1}{2\delta_b}$ is positive; furthermore, $x_2^* \ge t + 1 + b_2 \ge 2 + b_2 > 0$ since $t \ge 1$ and $-1 < b_2$. The last inequality follows because $\delta_b > 0$ and $b_1 - 2b_2 > 1$. This contradicts $(x_1^*, x_2^*, t) \in \operatorname{relint}(P(b) \cap H_t)$ as the second inequality defining P(b) is violated. \Box

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Appendix A. Nonemptiness of \mathcal{L}_{ψ,p^*} .

PROPOSITION 18. \mathcal{L}_{ψ,p^*} is nonempty.

Proof. Define

$$\phi(p) := \inf_{w \in \mathbb{R}^n, N \in \mathbb{N}} \bigg\{ \psi(w) + NV_{\psi}(p^*) : w + Np^* \in p + W_S \bigg\}.$$

It was shown in [24] that ϕ is a lifting of ψ and $\phi(p^*) = V_{\psi}(p^*)$. The proof in [24] considers \mathbb{R}^2 and $S = \mathbb{Z}^2$, but the proof holds for \mathbb{R}^n and general $S \subseteq \mathbb{R}^n$. By Zorn's Lemma, there is a minimal lifting π of ψ such that $\pi \leq \phi$. By (1.4), $V_{\psi}(p^*) \leq \pi(p^*) \leq \phi(p^*) = V_{\psi}(p^*)$. Thus, $\pi \in \mathcal{L}_{\psi,p^*}$ by (1.5).

Appendix B. Proof of Proposition 10.

Proof of Proposition 10. Recall $\Delta := Pyr(B, V_{\psi}(p^*), p^*)$. Define $\bar{\lambda} := \psi_{\Delta}((q, \bar{q}))$. Since ψ_{Δ} takes the form (3.3), there exists some $k \in I$ such that

(B.1)
$$\bar{\lambda} = a^k \cdot q + (V_{\psi}(p^*) - a^k \cdot p^*)\bar{q} \ge a^i \cdot q + (V_{\psi}(p^*) - a^i \cdot p^*)\bar{q} \quad \forall i \in I.$$

As $(q,\bar{q}) \in R_{\Delta}(x,x_{n+1})$, it follows that $\psi_{\Delta}((x,x_{n+1})) = a^k \cdot x + (V_{\psi}(p^*) - a^k \cdot p^*)x_{n+1}$.

We claim $(x, x_{n+1}, 1) \in Pyr(\Delta, \overline{\lambda}, (q, \overline{q}))$. It suffices to check that $(x, x_{n+1}, 1)$ satisfies the inequalities from (3.12) hold. Let $i \in I$ and consider the corresponding equation given in (3.12). By (B.1) and $(x, x_{n+1}) \in \Delta$,

$$a^{i} \cdot x + (V_{\psi}(p^{*}) - a^{i} \cdot p^{*})x_{n+1} + (\bar{\lambda} - a^{i} \cdot q - (V_{\psi}(p^{*}) - a^{i} \cdot p^{*})\bar{q})$$

$$\leq a^{i} \cdot x + (V_{\psi}(p^{*}) - a^{i} \cdot p^{*})x_{n+1} \leq 1.$$

Thus, $(x, x_{n+1}, 1) \in Pyr(\Delta, \overline{\lambda}, (q, \overline{q}))$. The latter inequality is an equation when i = k, which implies that $(x, x_{n+1}, 1) \in int(Pyr(\Delta, \overline{\lambda} - \varepsilon, (q, \overline{q})) \text{ for } \varepsilon > 0 \text{ such that } \overline{\lambda} - \varepsilon > 0$. To complete the proof, it is enough to show $Pyr(\Delta, \overline{\lambda}, (q, \overline{q}))$ is $(S \times \mathbb{Z} \times \mathbb{Z})$ -free.

Let $(s, z_{n+1}, z_{n+2}) \in S \times \mathbb{Z} \times \mathbb{Z}$; we want $(s, z_{n+1}, z_{n+2}) \notin \operatorname{int}(\operatorname{Pyr}(\Delta, \bar{\lambda}, (q, \bar{q})))$. By definition of $\operatorname{Pyr}(\Delta, \bar{\lambda}, (q, \bar{q}))$, we assume $z_{n+2} \ge 0$. If $z_{n+1} < 0$, then $z_{n+1} - \bar{q}z_{n+2} < 0$ because $\bar{q}, z_{n+2} \ge 0$. So, the inequality $r_{n+1} - \bar{q}r_{n+2} \ge 0$ in (3.13) separates (s, z_{n+1}, z_{n+2}) from $\operatorname{int}(\operatorname{Pyr}(\Delta, \bar{\lambda}, (q, \bar{q})))$.

Assume $z_{n+1} \ge 0$. Since $(s, z_{n+1}) \in S \times \mathbb{Z}_+$ and Δ is $(S \times \mathbb{Z})$ -free, there exists an $i \in I$ such that $a^i \cdot s + (V_{\psi}(p^*) - a^i \cdot p^*)z_{n+1} \ge 1$. Thus,

$$a^{i} \cdot s + (V_{\psi}(p^{*}) - a^{i} \cdot p^{*})z_{n+1} + (\bar{\lambda} - a^{i} \cdot q - (V_{\psi}(p^{*}) - a^{i} \cdot p^{*})\bar{q})z_{n+2}$$

$$\geq 1 + (\bar{\lambda} - a^{i} \cdot q - (V_{\psi}(p^{*}) - a^{i} \cdot p^{*})\bar{q})z_{n+2}$$

$$\geq 1,$$

where the last inequality follows from (B.1). This completes the proof.

Appendix C. Proof of Proposition 11.

Proof of Proposition 11. Recall $\Delta := Pyr(B, V_{\psi}(p^*), p^*)$. Consider the model

(C.1)
$$\left\{ (s, y_{p^*}, y_q) \in \mathbb{R}^n_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+ : \sum_{r \in \mathbb{R}^n} rs_r + p^* y_{p^*} + qy_q \in S \right\}.$$

Note that $(s, y_{p^*}, y_q) \in (C.1)$ if and only if $(s, y_{p^*}, y_q) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ and

(C.2)
$$\sum_{r \in \mathbb{R}^n} (r, 0, 0) s_r + (p^*, 1, 0) y_{p^*} + (q, 0, 1) y_q \in S \times \mathbb{Z} \times \mathbb{Z}.$$

CLAIM 19. Let $\lambda > 0$. If the inequality

(C.3)
$$\sum_{r \in \mathbb{R}^n} \psi(r) s_r + V_{\psi}(p^*) y_{p^*} + \lambda y_q \ge 1$$

is valid for (C.1), then $Pyr(\Delta, \lambda, (q, 0))$ is $(S \times \mathbb{Z} \times \mathbb{Z})$ -free.

Proof of Claim. Let $(\overline{x}, \overline{x}_{n+1}, \overline{x}_{n+2}) \in S \times \mathbb{Z} \times \mathbb{Z}$. If $\overline{x}_{n+1} < 0$ or $\overline{x}_{n+2} < 0$, then by the definition of $\operatorname{Pyr}(\Delta, \lambda, (q, 0))$, $(\overline{x}, \overline{x}_{n+1}, \overline{x}_{n+2}) \notin \operatorname{Pyr}(\Delta, \lambda, (q, 0))$. So assume $(\overline{x}, \overline{x}_{n+1}, \overline{x}_{n+2}) \in S \times \mathbb{Z}_+ \times \mathbb{Z}_+$. Let $\overline{r} = \overline{x} - \overline{x}_{n+1}p^* + \overline{x}_{n+2}q$, $\overline{z}_1 = \overline{x}_{n+1}$, $\overline{z}_2 = \overline{x}_{n+2}$ and $\overline{s}_r = 1$ if $r = \overline{r}$ and $\overline{s}_r = 0$ otherwise. Note that

$$\sum_{r \in \mathbb{R}^n} r\overline{s}_r + p^*\overline{z}_1 + q\overline{z}_2 = \overline{x} \in S.$$

Since (C.3) is valid for (C.1), it follows that

$$1 \leq \sum_{r \in \mathbb{R}^n} \psi(r)\overline{s}_r + V_{\psi}(p^*)\overline{z}_1 + \lambda \overline{z}_2$$

= $\psi(\overline{r}) + V_{\psi}(p^*)\overline{x}_{n+1} + \lambda \overline{x}_{n+2}$
= $\max_{i \in I} \{a_i \cdot (\overline{x} - \overline{x}_{n+1}p^* - \overline{x}_{n+2}q) + V_{\psi}(p^*)\overline{x}_{n+1} + \lambda \overline{x}_{n+2}\}$
= $\max_{i \in I} \{a_i \cdot \overline{x} + (V_{\psi}(p^*) - a_i \cdot p^*)\overline{x}_{n+1} + (\lambda - a_i \cdot q)\overline{x}_{n+2}\}.$

 \diamond

 \diamond

Hence, $Pyr(\Delta, \lambda, (q, 0))$ is $(S \times \mathbb{Z} \times \mathbb{Z})$ -free.

The converse of the Claim 19 is also true.

CLAIM 20. If $\lambda > 0$ and $Pyr(\Delta, \lambda, (q, 0))$ is $(S \times \mathbb{Z} \times \mathbb{Z})$ -free, then (C.3) is valid for (C.1).

Proof of Claim. Consider $\Psi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\Psi(r, r_{n+1}, r_{n+2}) := \max_{i \in I} \left\{ a^i \cdot r + (V_{\psi}(p^*) - a^i \cdot p^*)r_{n+1} + (\lambda - a^i \cdot q)r_{n+2} \right\}.$$

Let $(s, y_{p^*}, y_q) \in (C.1)$. From the observation above, $(s, y_{p^*}, y_q) \in (C.2)$. Note that $\Psi(r, 0, 0) = \psi(r), \Psi(p^*, 1, 0) = V_{\psi}(p^*)$, and $\Psi(q, 0, 1) = \lambda$. It follows that

$$\sum_{r \in \mathbb{R}^n} \psi(r) s_r + V_{\psi}(p^*) y_{p^*} + \lambda y_q$$

=
$$\sum_{r \in \mathbb{R}^n} \Psi(r, 0, 0) s_r + \Psi(p^*, 1, 0) y_{p^*} + \Psi(q, 0, 1) y_q \ge 1.$$

Hence, (C.3) is valid for (C.1).

By Theorem 3 with $\mathcal{R} = \mathbb{R}^n$ and $\mathcal{P} = \{p_1^*, p_2^*\}, V_{\psi}(p_2^*; p_1^*)$ is the infimum of $\lambda > 0$ such that (C.3) is valid for (C.1). The result now follows from Claims 19 and 20.

Appendix D. Proof of Proposition 13.

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Proof of Proposition 13. From (3.4) and (3.9), we have

$$R_{\Delta}(\bar{x}, \bar{x}_{n+1}) = \left\{ (r, r_{n+1}) : (a^i - a^k) \cdot r + r_{n+1}((a^k - a^i) \cdot p^*) \le 0 \text{ and} \\ (a^i - a^k) \cdot (\bar{x} - r) + (\bar{x}_{n+1} - r_{n+1})((a^k - a^i) \cdot p^*) \le 0 \quad \forall i \in I \end{array} \right\}$$

where $k \in I$ is such that $\psi(x) = a^k \cdot x$. Therefore,

$$H_t \cap R_\Delta(\bar{x}, \bar{x}_{n+1})$$

$$= \left\{ \begin{array}{ll} (r,t): & (a^{i}-a^{k})\cdot r+t((a^{k}-a^{i})\cdot p^{*}) \leq 0 \text{ and} \\ (a^{i}-a^{k})\cdot (\bar{x}-r)+(\bar{x}_{n+1}-t)((a^{k}-a^{i})\cdot p^{*}) \leq 0 \quad \forall \ i \in I \end{array} \right\}$$
$$= \left\{ \begin{array}{ll} (\tilde{r}+tp^{*},t): & (a^{i}-a^{k})\cdot \tilde{r} \leq 0 \text{ and} \\ (a^{i}-a^{k})\cdot (\bar{x}-\tilde{r})+\bar{x}_{n+1}((a^{k}-a^{i})\cdot p^{*}) \leq 0 \quad \forall \ i \in I \end{array} \right\}$$
$$= (H_{0} \cap R_{\Delta}(\bar{x},\bar{x}_{n+1}))+t(p^{*},1).$$

A similar calculation shows $H_0 \cap R_\Delta(\bar{x}, \bar{x}_{n+1}) = R_B(\bar{x} - \bar{x}_{n+1}p^*) \times \{0\}.$

Appendix E. Proof of Theorem 14.

The first lemma required for Theorem 14 is an extension of the so-called 'Collision Lemma' (Lemma 3.2 in [10]). In this appendix, set $\Delta := Pyr(B, V_{\psi}(p^*), p^*)$.

PROPOSITION 21. Let B be a maximal S-free convex 0-neighborhood in \mathbb{R}^n of the form (3.2). Let $p^* \in \mathbb{R}^n$ and $(\overline{x}, \overline{x}_{n+1}), (\overline{y}, \overline{y}_{n+1}) \in \Delta \cap (S \times \mathbb{Z})$, and $i_x, i_y \in I$ satisfy $(a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*) \cdot (\overline{x}, \overline{x}_{n+1}) = (a^{i_y}, V_{\psi}(p^*) - a^{i_y} \cdot p^*) \cdot (\overline{y}, \overline{y}_{n+1}) = 1$. Let $k_x, k_y \in \mathbb{Z}$ be such that $0 \leq k_x \leq \overline{x}_{n+1}$ and $0 \leq k_y \leq \overline{y}_{n+1}$, and let $(x, k_x) \in R_{\Delta}(\overline{x}, \overline{x}_{n+1})$ and $(y, k_y) \in R_{\Delta}(\overline{y}, \overline{y}_{n+1})$. If $x - y \in W_S$, then

$$(a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*) \cdot (x, k_x) = (a^{i_y}, V_{\psi}(p^*) - a^{i_y} \cdot p^*) \cdot (y, k_y)$$

If $(x, k_x) \in \text{int}(R_{\Delta}(\overline{x}, \overline{x}_{n+1}))$ and $(y, k_y) \in \text{int}(R_{\Delta}(\overline{y}, \overline{y}_{n+1}))$, then $(a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*) = (a^{i_y}, V_{\psi}(p^*) - a^{i_y} \cdot p^*)$.

Proof. Let $(x, k_x) \in R_{\Delta}(\overline{x}, \overline{x}_{n+1})$ and $(y, k_y) \in R_{\Delta}(\overline{y}, \overline{y}_{n+1})$. Assume to the contrary that $(a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*) \cdot (x, k_x) < (a^{i_y}, V_{\psi}(p^*) - a^{i_y} \cdot p^*) \cdot (y, k_y)$ and consider $(\overline{y}, \overline{y}_{n+1}) + (x - y, k_x - k_y)$ (if the inequality is reversed then consider $(\overline{x}, \overline{x}_{n+1}) + (y - x, k_y - k_x)$ instead). Since $x - y \in W_S$ and $k_y \leq \overline{y}_{n+1}$, it follows that $(z, z_{n+1}) := (\overline{y}, \overline{y}_{n+1}) + (x - y, k_x - k_y) = (\overline{y} + (x - y), (\overline{y}_{n+1} - k_y) + k_x) \in S \times \mathbb{Z}$. We claim that $(z, z_{n+1}) \in int(\Delta)$, contradicting that Δ is $(S \times \mathbb{Z})$ -free. We will show this using the half-space definition of Δ from (3.4).

Take $i \in I$ and define $\alpha_i := (a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*)$. If $i = i^x$, then

$$\begin{aligned} \alpha_{i^x} \cdot (z, z_{n+1}) &\leq 1 - \alpha_{i^x}(y, k_y) + \alpha_{i^x} \cdot (x, k_x) & \text{since } (\overline{y}, \overline{y}_{n+1}) \in S \times \mathbb{Z}_+ \\ &< 1 - \alpha_{i^y}(y, k_y) + \alpha_{i^x} \cdot (x, k_x) & \text{since } (y, k_y) \in R_\Delta(\overline{y}, \overline{y}_{n+1}) \\ &\leq 1 & \text{since } a_{i^x} \cdot (x, k_x) < a_{i^y} \cdot (y, k_y) \end{aligned}$$

If $i = i^y$, then

$$\begin{aligned} \alpha_{i_y} \cdot (z, z_{n+1}) &= 1 - \alpha_{i_y}(y, k_y) + \alpha_{i_y} \cdot (x, k_x) & \text{ since } (\overline{y}, \overline{y}_{n+1}) \in S \times \mathbb{Z}_+ \\ &< 1 - \alpha_{i_x}(x, k_x) + \alpha_{i_x} \cdot (x, k_x) & \text{ since } a_{i^x} \cdot (x, k_x) < a_{i^y} \cdot (y, k_y) \\ &= 1. \end{aligned}$$

If $i \in I \setminus \{i^x, i^y\}$, then

$$\begin{aligned} \alpha_i \cdot (z, z_{n+1}) &\leq 1 + \alpha_i \cdot (x, k_x) - \alpha_i \cdot (y, k_y) & \text{since } (\overline{y}, \overline{y}_{n+1}) \in S \times \mathbb{Z}_+ \\ &\leq 1 + \alpha_i \cdot (x, k_x) - \alpha_{i^y} \cdot (y, k_y) & \text{since } (y, k_y) \in R_\Delta(\overline{y}, \overline{y}_{n+1}) \\ &< 1 + \alpha_i \cdot (x, k_x) - \alpha_{i^x} \cdot (x, k_x) & \text{since } a_{i^x} \cdot (x, k_x) < a_{i^y} \cdot (y, k_y) \\ &< 1 & \text{since } (x, k_x) \in R_\Delta(\overline{x}, \overline{x}_{n+1}). \end{aligned}$$

Hence, $(z, z_{n+1}) \in int(\Delta)$ gives a contradiction.

Assume $(x, k_x) \in \operatorname{int} (R_{\Delta}(\overline{x}, \overline{x}_{n+1}))$ and $(y, k_y) \in \operatorname{int} (R_{\Delta}(\overline{y}, \overline{y}_{n+1}))$. Assume to the contrary that $\alpha_{i^x} \neq \alpha_{i^y}$. Since $\alpha_{i^x} \neq \alpha_{i^y}$ and $(y, k_y) \in \operatorname{int} (R_{\Delta}(\overline{y}, \overline{y}_{n+1}))$, we have $\alpha_{i^x} \cdot (y, k_y) < \alpha_{i^y} \cdot (y, k_y)$ and

(E.1)
$$\alpha_{i_x} \cdot (\overline{y} - y, \overline{y}_{n+1} - k_y) < \alpha_{i_x} \cdot (\overline{y} - y, \overline{y}_{n+1} - k_y).$$

From the previous argument that $\alpha_{i^x}(x,k_x) = \alpha_{i^y}(y,k_y)$. Let $i \in I$. If $i = i^x$, then

$$\begin{aligned} \alpha_{i^{x}} \cdot (z, z_{n+1}) &= \alpha_{i^{x}} \cdot (\overline{y} - y, \overline{y}_{n+1} - k_{y}) + \alpha_{i^{x}} \cdot (x, k_{x}) \\ &< \alpha_{i^{y}} \cdot (\overline{y} - y, \overline{y}_{n+1} - k_{y}) + \alpha_{i^{x}} \cdot (x, k_{x}) \quad \text{from (E.1)} \\ &= 1 - \alpha_{i^{y}}(y, k_{y}) + \alpha_{i^{x}} \cdot (x, k_{x}) \quad \text{since } (\overline{y}, \overline{y}_{n+1}) \in S \times \mathbb{Z}_{+} \\ &= 1. \end{aligned}$$

If $i = i^y$, then

$$\begin{aligned} \alpha_{i^{y}} \cdot (z, z_{n+1}) &= 1 - \alpha_{i^{y}}(y, k_{y}) + \alpha_{i^{y}} \cdot (x, k_{x}) & \text{since } (\overline{y}, \overline{y}_{n+1}) \in S \times \mathbb{Z}_{+} \\ &= 1 - \alpha_{i^{x}}(x, k_{x}) + \alpha_{i^{y}} \cdot (x, k_{x}) \\ &< 1 & \text{since } (x, k_{x}) \in R_{\Delta}(\overline{x}, \overline{x}_{n+1}). \end{aligned}$$

If $i \in I \setminus \{i^x, i^y\}$, then

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$$\begin{aligned} &\alpha_i \cdot (z, z_{n+1}) \\ &= \alpha_i \cdot (\overline{y} - y, \overline{y}_{n+1} - k_y) + \alpha_i \cdot (x, k_x) \\ &< \alpha_{i^y} \cdot (\overline{y} - y, \overline{y}_{n+1} - k_y) + \alpha_i \cdot (x, k_x) \qquad \text{since } (y, k_y) \in R_\Delta(\overline{y}, \overline{y}_{n+1}) \\ &< 1 - \alpha_{i^y}(y, k_y) + \alpha_{i^x} \cdot (x, k_x) \qquad \text{since } (x, k_x) \in R_\Delta(\overline{x}, \overline{x}_{n+1}) \\ &= 1. \end{aligned}$$

This shows $(z, z_{n+1}) \in int(\Delta)$, which is a contradiction.

LEMMA 22. [Theorem 9.4 in [26]] Let $P_{\omega} \subseteq \mathbb{R}^n, \omega \in \Omega$ be a (possibly infinite) family of polyhedra such that any bounded set intersects only finitely many polyhedra, and $\bigcup_{\omega \in \Omega} P_{\omega} = \mathbb{R}^n$. Suppose there is a family of functions $A_{\omega} : P_{\omega} \to \mathbb{R}^n, \omega \in \Omega$ such that A_{ω} is continuous over P_{ω} for each $\omega \in \Omega$, and for every pair $\omega_1, \omega_2 \in \Omega$, $A_{\omega_1}(x) = A_{\omega_2}(x)$ for all $x \in P_{\omega_1} \cap P_{\omega_2}$. Then there is a unique, continuous map $A : \mathbb{R}^n \to \mathbb{R}^n$ that equals A_{ω} when restricted to P_{ω} for each $\omega \in \Omega$.

Proof. This follows from a direct application of Theorem 9.4 in Chapter III of [26] by noting that polyhedra are closed sets.

PROPOSITION 23. Let B be a maximal S-free convex 0-neighborhood in \mathbb{R}^n such that $\operatorname{int}(B \cap \operatorname{conv}(S)) \neq \emptyset$. Then any bounded set $U \subseteq \mathbb{R}^n$ intersects a finite number of polyhedra from $\mathcal{X}(B, p^*) + W_S$.

Proof. Recall that B is a full-dimensional set, so, by construction, Δ is fulldimensional. Also, $\operatorname{int}(\operatorname{conv}(S) \cap B) \neq \emptyset$ and $\operatorname{int}(\operatorname{conv}(S \times \mathbb{Z}) \cap \Delta) \neq \emptyset$. Set $\tilde{U} := U \times [0,1] \subseteq \mathbb{R}^{n+1}$. $\tilde{U} \subseteq \mathbb{R}^{n+1}$ is bounded, and by Theorem 2.7 in [10], \tilde{U} intersects finitely many polyhedra from $R(\Delta) + W_{S \times \mathbb{Z}_+} = R(\Delta) + W_S \times \{0\}$. Say for $i = 1, \ldots, k$, \tilde{U} intersects $\tilde{P}_i + (w_i, 0), (w_i, 0) \in W_S \times \{0\}$ and \tilde{P}_i is a polyhedron in $R(\Delta)$.

For $t \in \mathbb{Z}$, Proposition 13 states that the projection of $H_t \cap (\tilde{P}_i + (w_i, 0))$ onto \mathbb{R}^n is $\operatorname{proj}_{\mathbb{R}^n}(H_0 \cap \tilde{P}_i) + tp^* + w_i$, where $\operatorname{proj}_{\mathbb{R}^n}(\cdot)$ denotes the projection onto the first *n* coordinates. By definition of $\mathcal{X}(B, p^*) + W_S$, all polyhedra in $\mathcal{X}(b, p^*) + W_S$ are of the form $\operatorname{proj}_{\mathbb{R}^n}(H_0 \cap \tilde{P}_i) + tp^* + w_i$, where $t \leq x_{n+1}$ for some blocking point $(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ corresponding to Δ . Since \tilde{U} is bounded, $H_t \cap \tilde{U} \cap (\tilde{P}_i + (w_i, 0)) \neq \emptyset$ for only a finite number of integral *t*, for each $i = 1, \ldots, k$. Hence, *U* only intersects a finite number of polyhedra from $\mathcal{X}(B, p^*)$.

For $\varepsilon > 0$ and $x \in \mathbb{R}^d$, define $D(x; \varepsilon) := \{ y \in \mathbb{R}^d : ||x - y|| < \varepsilon \}.$

PROPOSITION 24. Let B be a maximal S-free convex 0-neighborhood in \mathbb{R}^n . For $p^* \in \mathbb{R}^n$, the set $\mathcal{X}(B, p^*) + W_S$ is closed.

Proof. Let $x \notin \mathcal{X}(B, p^*) + W_S$ and consider D(x, 1). From Proposition 23, D(x, 1)intersects only finite many polyhedra P_1, \ldots, P_k from $\mathcal{X}(B, p^*) + W_S$. Each P_i is closed, so the finite union $\bigcup_{i=1}^k P_i$ is too. Since $x \notin \bigcup_{i=1}^k P_i$, there exists $\varepsilon > 0$ such that $D(x; \varepsilon) \subseteq D(x; 1)$ does not intersect P_i for $i = 1, \ldots, k$. So, $D(x; \varepsilon) \cap (\mathcal{X}(B, p^*) + W_S) = \emptyset$. This implies $\mathbb{R}^n \setminus (\mathcal{X}(B, p^*) + W_S)$ is open, so $\mathcal{X}(B, p^*) + W_S$ is closed. \square

Let t be as in Theorem 14. For $i \in I$, set $a_t^i := \frac{a^i}{1+a^i \cdot t}$. Observe

$$B + t = \{ r \in \mathbb{R}^n : a_t^i \cdot r \le 1 \ \forall i \in I \},\$$

and

$$\Delta + (t,0) = \left\{ (r,r_{n+1}) \in \mathbb{R}^{n+1} : a_t^i \cdot r + \left(\frac{V_{\psi}(p^*) - a^i \cdot p^*}{1 + a^i \cdot t} \right) r_{n+1} \le 1 \ \forall i \in I \right\}.$$

The apex of $\Delta + (t, 0)$ is $\frac{1}{V_{\psi}(p^*)}(p^* + V_{\psi}(p^*)t, 1)$. Define

(E.2)
$$\hat{p} := p^* + V_{\psi}(p^*)t.$$

For each $k \in \mathbb{Z}, k \ge 0$, and $i \in I$ define $T_i^k : \mathbb{R}^n \to \mathbb{R}^n$ to be

$$T_i^k(x) := x + (a^i, V_{\psi}(p^*) - a^i p^*) \cdot (x, k)t.$$

The next result follows from a direct calculation.

PROPOSITION 25. The function T_i^k is invertible with the inverse defined by

$$(T_i^k)^{-1}(x) = x - \left(a_t^i, \frac{V_{\psi}(p^*) - a^i \cdot p^*}{1 + a^i \cdot t}\right) \cdot (x, k)t.$$

LEMMA 26. Let $(\overline{x}, \overline{x}_{n+1}), (\overline{y}, \overline{y}_{n+1}) \in \Delta \cap (S \times \mathbb{Z})$ and $i_x, i_y \in I$ be such that $(a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*) \cdot (\overline{x}, \overline{x}_{n+1}) = (a^{i_y}, V_{\psi}(p^*) - a^{i_y} \cdot p^*) \cdot (\overline{y}, \overline{y}_{n+1}) = 1$. Assume $(z, k_x) \in R_{\Delta}(\overline{x}, \overline{x}_{n+1}) + (w_x, 0)$ and $(z, k_y) \in R_{\Delta}(\overline{y}, \overline{y}_{n+1}) + (w_y, 0)$, where $w_x, w_y \in W_S$, $k_i \in \mathbb{Z}_+$, $k_x \leq \overline{x}_{n+1}$, and $k_y \leq \overline{y}_{n+1}$. Then $T_{i_x}^{k_x}(z - w_x, k_x) + w_x = T_{i_y}^{k_y}(z - w_y, k_y) + w_y$.

Proof. A direct calculation shows

$$\begin{aligned} T_{i_x}^{k_x}(z - w_x, k_x) + w_x \\ &= (z - w_x) + (a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*) \cdot (z, k_x)t + w_x & \text{by definition,} \\ &= z + (a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*) \cdot (z, k_x)t \\ &= z + (a^{i_y}, V_{\psi}(p^*) - a^{i_y} \cdot p^*) \cdot (z, k_y)t & \text{by Proposition 21,} \\ &= (z - w_y) + (a^{i_y}, V_{\psi}(p^*) - a^{i_y} \cdot p^*) \cdot (z, k_y)t + w_y & \text{by definition,} \\ &= T_{i_y}^{k_y}(z - w_y, k_y) + w_y \end{aligned}$$

PROPOSITION 27. Let $(\overline{x}, \overline{x}_{n+1}) \in \Delta$. Consider $R_B(\overline{x} - \overline{x}_{n+1}p^*) + kp^*$ for $k \in \mathbb{Z}_+, k \leq x_{n+1}$. If $i_x \in I$ satisfies $(a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*) \cdot (\overline{x}, \overline{x}_{n+1}) = 1$, then

$$T_{i_x}^k \left(R_B(\overline{x} - \overline{x}_{n+1}p^*) + kp^* \right) = R_{B+t}(\overline{x} + t - \overline{x}_{n+1}\hat{p}) + k\hat{p},$$

where \hat{p} is defined in (E.2).

Proof. Let $y \in R_B(\overline{x} - \overline{x}_{n+1}p^*) + kp^*$. Note that $(y,k) \in R_{\Delta}((\overline{x},\overline{x}_{n+1}))$ by Proposition 13. Also, $T_{i_x}^k(y) \in R_{B+t}(\overline{x} + t - \overline{x}_{n+1}\hat{p}) + k\hat{p}$ if and only if $(T_{i_x}^k(y), k) \in R_{\Delta+(t,0)}((\overline{x} + t, \overline{x}_{n+1}))$. We will show this latter sufficient condition.

We first show that for $i \in I$, if $[(a^i, V_{\psi}(p^*) - a^i \cdot p^*) - (a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*)] \cdot (y, k) \leq 0$, then $(a^i_t, \frac{V_{\psi}(p^*) - a^i \cdot p^*}{1 + a^i \cdot t}) \cdot (T^k_{i_x}(y), k) \leq a^{i_x} \cdot y + k \left(V_{\psi}(p^*) - a^{i_x} \cdot p^*\right)$ with equality for $i = i_x$. Observe that

$$\begin{split} & \left(a_{t}^{i}, \frac{V_{\psi}(p^{*}) - a^{i} \cdot p^{*}}{1 + a^{i} \cdot t}\right) \cdot (T_{i_{x}}^{k}(y), k) \\ &= \left(\frac{a^{i}}{1 + a^{i} \cdot t}, \frac{V_{\psi}(p^{*}) - a^{i} \cdot p^{*}}{1 + a^{i} \cdot t}\right) \cdot \left(y + (a^{i_{x}} \cdot y + (V_{\psi}(p^{*}) - a^{i_{x}} \cdot p^{*})k)t, k\right) \\ &= \frac{a^{i} \cdot y + k\left(V_{\psi}(p^{*}) - a^{i} \cdot p^{*}\right) + (a^{i_{x}} \cdot y)(a^{i} \cdot t) + (a^{i} \cdot t)k\left(V_{\psi}(p^{*}) - a^{i_{x}} \cdot p^{*}\right)}{1 + a^{i} \cdot t} \\ &\leq \frac{a^{i_{x}} \cdot y + k\left(V_{\psi}(p^{*}) - a^{i_{x}} \cdot p^{*}\right) + (a^{i_{x}} \cdot y)(a^{i} \cdot t) + (a^{i} \cdot t)k\left(V_{\psi}(p^{*}) + a^{i_{x}} \cdot p^{*}\right)}{1 + a^{i} \cdot t} \\ &= \frac{(1 + a^{i} \cdot t)\left(a^{i_{x}} \cdot y + k\left(V_{\psi}(p^{*}) - a^{i_{x}} \cdot p^{*}\right)\right)}{1 + a^{i} \cdot t} \\ &= a^{i_{x}} \cdot y + k\left(V_{\psi}(p^{*}) - a^{i_{x}} \cdot p^{*}\right), \end{split}$$

where the inequality holds because $[(a^i, V_{\psi}(p^*) - a^i \cdot p^*) - (a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*)] \cdot (y, k) \leq 0$. Equality holds if $i = i_x$.

Similarly, for $i \in I$ such that $[(a^i, V_{\psi}(p^*) - a^i \cdot p^*) - (a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*)] \cdot (y, k) \leq 0$, it follows that $(a^i_t, \frac{V_{\psi}(p^*) - a^i \cdot p^*}{1 + a^i \cdot t}) \cdot (\overline{x} + t - T^k_{i_x}(y), \overline{x}_{n+1} - k) \leq 1 - (a^{i_x} \cdot y + (V_{\psi}(p^*) - a^{i_x} \cdot p^*))k$ with equality for $i = i_x$.

Since $(y,k) \in R_{\Delta}((\overline{x},\overline{x}_{n+1}))$, it follows that $[(a^i, V_{\psi}(p^*) - a^i \cdot p^*) - (a^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*)] \cdot (y,k) \leq 0$ for each $i \in I$. Applying the arguments to each $i \in I$, with equality for $i = i_x$, we see that

$$\left[(a_t^i, \frac{V_{\psi}(p^*) - a^i \cdot p^*}{1 + a^i \cdot t}) - (a_t^{i_x}, \frac{V_{\psi}(p^*) - a^{i_x} \cdot p^*}{1 + a^{i_x} \cdot t})\right] \cdot (T_{i_x}^k(y), k) \le 0,$$

and

$$\left[(a_t^i, \frac{V_{\psi}(p^*) - a^i \cdot p^*}{1 + a^i \cdot t}) - (a_t^{i_x}, \frac{V_{\psi}(p^*) - a^{i_x} \cdot p^*}{1 + a^{i_x} \cdot t}) \right] \cdot (\overline{x}_{n+1} + t - T_{i_x}^k(y), \overline{x}_{n+1} - k) \le 0.$$

Hence, $(T_{i_x}^k(y), k) \in R_{\Delta+(t,0)}((\overline{x}+t, \overline{x}_{n+1}))$, so

$$T_{i_x}^k \left(R_B(\overline{x} - \overline{x}_{n+1}) + kp^* \right) \subseteq R_{B+t}(\overline{x} + t - \overline{x}_{n+1}\hat{p}) + k\hat{p}.$$

Using similar reasoning applied to $(T_{i_r}^k)^{-1}$, we get the reverse inclusion.

Proof of Theorem 14. Recall \hat{p} in (E.2). We show if $\mathcal{X}(B, p^*) + W_S = \mathbb{R}^n$, then $\mathcal{X}(B+t, \hat{p}) + W_{S+t} = \mathbb{R}^n$. The converse is proved by switching the roles of (B, p^*) and $(B+t, \hat{p})$.

A direct calculation shows that $W_S = W_{S+t}$ (see Proposition 2.1 in [10]). If B is a half-space, then the lifting region is equal to \mathbb{R}^n . The extended lifting region is contained in $\mathcal{X}(B, p^*) + W_S$, so $\mathcal{X}(B, p^*) + W_S = \mathcal{X}(B+t, \hat{p}) + W_{S+t} = \mathbb{R}^n$. Thus, assume that B is not a half-space.

Define the map $A : \mathbb{R}^n \to \mathbb{R}^n$ by

$$A(y) := T_{i_x}^k(y-u) + u, \quad \text{if } y \in R_B(w(z)) + kp^* + u_z$$

where $z = (\overline{x}, \overline{x}_{n+1})$ is a blocking point of Δ , $k \in \{0, \ldots, \overline{x}_{n+1}\}, u \in W_S$, and $(a_t^{i_x}, V_{\psi}(p^*) - a^{i_x} \cdot p^*) \cdot (\overline{x}, \overline{x}_{n+1}) = 1$. Since $\mathcal{X}(B, p^*) + W_S = \mathbb{R}^n$, each y is in some $R_B(\overline{x} - \overline{x}_{n+1}p^*) + kp^* + u$. A is well defined from Lemma 26.

By assumption, $\mathbb{R}^n = \mathcal{X}(B, p^*) + W_S$. Using Proposition 27, we have

$$\begin{aligned} A(\mathbb{R}^{n}) &= A\left(\mathcal{X}(B,p^{*}) + W_{S}\right) \\ &= A\left(\bigcup_{(\bar{x},\bar{x}_{n+1})\in\Delta\cap(S\times\mathbb{Z}_{+}), u\in W_{S}} \left(\bigcup_{i=0}^{\overline{x}_{n+1}} (R_{B}(\bar{x}-\bar{x}_{n+1}p^{*})+ip^{*}+u)\right)\right) \\ &= \bigcup_{(\bar{x},\bar{x}_{n+1})\in\Delta\cap(S\times\mathbb{Z}_{+}), u\in W_{S}} \left(\bigcup_{i=0}^{\overline{x}_{n+1}} A(R_{B}(\bar{x}-\bar{x}_{n+1}p^{*})+ip^{*}+u)\right) \\ &= \bigcup_{(\bar{x},\bar{x}_{n+1})\in\Delta\cap(S\times\mathbb{Z}_{+}), u\in W_{S}} \left(\bigcup_{i=0}^{\overline{x}_{n+1}} R_{B+t}(\overline{x}+t-\overline{x}_{n+1}\hat{p})+i\hat{p}+u\right) \\ &= \left(\bigcup_{(\bar{x},\bar{x}_{n+1})\in\Delta\cap(S\times\mathbb{Z}_{+})} \left(\bigcup_{i=0}^{\overline{x}_{n+1}} R_{B+t}(\overline{x}+t-\overline{x}_{n+1}\hat{p})+i\hat{p}\right)\right) + W_{S+t} \\ &= \mathcal{X}(B+m,\hat{p}) + W_{S+t}. \end{aligned}$$

So, A maps the translated fixing region to the translated fixing region.

Suppose $A(y_1) = A(y_2)$ for some $y_1, y_2 \in \mathbb{R}^n$. Let $\alpha := A(y_1) = A(y_2)$. By definition, for j = 1, 2, there exists a blocking point $(\overline{x}^j, \overline{x}^j_{n+1}) \in S \times \mathbb{Z}_+, k_j \in \mathbb{Z}_+$ with $k_j \leq \overline{x}^j_{n+1}$, and $w_j \in W_S$ such that $y_j \in R_B(\overline{x}^j - \overline{x}^j_{n+1}p^*) + k_jp^* + w_j$. Moreover

$$\alpha = A(y_1) = T_{i_{x_1}}^{k_1}(y_1 - w_1) + w_1 = T_{i_{x_2}}^{k_2}(y_2 - w_2) + w_2 = A(y_2).$$

By Proposition 27, $\alpha \in R_{B+t}(\overline{x}^j + t - \overline{x}_{n+1}^j \hat{p}) + k_j \hat{p} + w_j$, for $j \in \{1, 2\}$. So, $(\alpha, k_j) \in R_{\Delta+(t,0)}((\overline{x}^j + t, \overline{x}_{n+1}^j)) + (w_j, 0)$, for $j \in \{1, 2\}$. Lemma 26 applied to $(T_{i_{x_1}}^{k_1})^{-1}$ and

 $(T_{i_{x_2}}^{k_2})^{-1}$ shows

$$(T_{i_{x_1}}^{k_1})^{-1} \left(T_{i_{x_1}}^{k_1} (y_1 - w_1) + w_1 - w_1 \right) + w_1 = (T_{i_{x_2}}^{k_2})^{-1} \left(T_{i_{x_2}}^{k_2} (y_2 - w_2) + w_2 - w_2 \right) + w_2.$$

Applying the definition of $\left(T_{i_{x_j}}^{k_j}\right)^{-1}$ for j = 1, 2, we see $y_1 = y_2$. Hence, A is injective.

By Lemma 22 and Proposition 23, A is continuous. The Invariance of Domain Theorem (see [11, 25]) states that A is an open map. So, the translated fixing region is open because A maps \mathbb{R}^n to the translated fixing region. By Proposition 24, the translated fixing region is also closed. Because the translated fixing region is nonempty, this implies that it must be \mathbb{R}^n . Thus, B + t is one point fixable.

Appendix F. Case Analysis for K_i from Claim 16.

Proof of Claim 16. To prove this claim, we first construct the half-space definition of the spindles $R_T(s^4 - p^*)$, $R_T(s^5 - p^*)$, and $R_T(s^6 - 2p^*)$. Consider the vectors q^1, q^2 , and q^3 that define T, see (4.1). Since $(s^4, z^4) = (s^4, 1) \in P$ is contained in the same facet as $(s^1, 0)$ (see the discussion following (4.3)), we see that

(F.1)
$$R_T(s^4 - p^*) = \{x \in \mathbb{R}^2 : (q^i - q^1) \cdot x \le 0, (q^i - q^1) \cdot (s^4 - p^* - x) \le 0 \ \forall \ i \in \{2, 3\}\}.$$

Similarly, because $(s^5, 1)$ and $(s^6, 2)$ share a facet with $(s^2, 0)$ and $(s^3, 0)$, respectively,

(F.2)

$$R_T(s^5 - p^*) = \{x \in \mathbb{R}^2 : (q^i - q^2) \cdot x \le 0, (q^i - q^2) \cdot (s^5 - p^* - x) \le 0 \ \forall i \in \{1, 3\}\}$$

$$R_T(s^6 - 2p^*) = \{x \in \mathbb{R}^2 : (q^i - q^3) \cdot x \le 0, (q^i - q^3) \cdot (s^6 - 2p^* - x) \le 0 \ \forall i \in \{1, 2\}\}.$$

Consider the collection of points $K_1 = \operatorname{conv}\{l, e^1, g, u\}$. In order to prove $K_1 \subseteq R_T(s^4 - p^*)$, is is enough to show that $\{l, e^1, g, u\} \subseteq R_T(s^4 - p^*)$. Consider the point $l \in \{l, e^1, g, u\}$. Using the values in Figure 1 along with (F.1) and the definition $s^4 = (1 + b_1, 2 + b_2)$, it is straight forward, yet tedious, to show that the four values $(q^i - q^1) \cdot l, (q^i - q^1) \cdot (s^4 - p^* - l), i \in \{2, 3\}$ are all contained in

$$Q := \left\{ \begin{array}{c} 0, \ \frac{-1}{(1,\gamma_1)\cdot(1+b_1,b+b_2)}, \ \frac{-\gamma_1}{(1,\gamma_1)\cdot(1+b_1,b+b_2)}, \ \frac{-1+\gamma_2}{(-1,\gamma_2)\cdot(b_1,b_2)}, \\ \frac{-2+\gamma_3}{(-2\gamma_3,-2)\cdot(b_1,b_2)}, \ \frac{\gamma_3}{(-2\gamma_3,2)\cdot(b_1,b_2)}, \ \frac{-1+\gamma_3}{(\gamma_3,-1)\cdot(b_1,b_2)}, \\ \frac{-1-\gamma_1}{(1,\gamma_1)\cdot(1+b_1,1+b_2)}, \ \frac{-b_1(1+\gamma_1\gamma_3)-(1+\gamma_1)}{(1+b_1+\gamma_1(1+b_2))(-b_2+b_1\gamma_3)}, \ \frac{b_1(-1+\gamma_3)+b_2(-1+\gamma_2)+\gamma_2}{(b_1-(1+b_2)\gamma_2)(-b_2+b_1\gamma_3)}, \\ \frac{-(b_1+1)+\gamma_1b_2-(\gamma_1+2\gamma_1^2\gamma_3)}{(1+b_1+\gamma_1(1+b_2))(-b_2+b_1\gamma_3)}, \ \frac{\gamma_2+b_1(-1+\gamma_2\gamma_3)}{(b_1-(1+b_2)\gamma_2)(-b_2+b_1\gamma_3)}, \end{array} \right\}.$$

Because $\gamma_1, \gamma_2, \gamma_3 > 0$, $\gamma_2, \gamma_3 < 1$, and $-1 \leq b_2 \leq b_1 \leq 0$, a direct calculation shows that every value in Q is nonpositive. Hence, from (F.1), $l \in R_T(s^4 - p^*)$. Similar arguments show that when the four inner products defining (F.1) are evaluated at any point in $\{l, e^1, g, u\}$, the result is in Q. Hence $\{l, e^1, g, u\} \subseteq R_T(s^4 - p^*)$.

The inclusions $K_2 \subseteq R_T(s^5 - p^*) + (1, 1)$, $K_3 \subseteq R_T(s^4 - p^*) + p^*$, $K_4 \subseteq R_T(s^5 - p^*) + p^*$, and $K_5 \subseteq R_T(s^6 - 2p^*) + p^*$ use similar proofs. So, we only prove $K_2 \subseteq R_T(s^5 - p^*) + (1, 1)$. For this, it is enough to show that $\{u - (1, 1), m - (1, 1), i - (1, 1), g - (1, 1)\} \subseteq R_T(s^5 - p^*)$. However, substituting these four values in for x in (F.2) yields values in Q. Hence, $\{u - (1, 1), m - (1, 1), i - (1, 1), g - (1, 1)\} \subseteq R_T(s^5 - p^*)$.