# Correction of high-order BDF convolution quadrature for fractional evolution equations 

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#### Abstract

We develop proper correction formulas at the starting $k-1$ steps to restore the desired $k^{\text {th }}$-order convergence rate of the $k$-step BDF convolution quadrature for discretizing evolution equations involving a fractional-order derivative in time. The desired $k^{\text {th }}$-order convergence rate can be achieved even if the source term is not compatible with the initial data, which is allowed to be nonsmooth. We provide complete error estimates for the subdiffusion case $\alpha \in(0,1)$, and sketch the proof for the diffusion-wave case $\alpha \in(1,2)$. Extensive numerical examples are provided to illustrate the effectiveness of the proposed scheme. Keywords: fractional evolution equation, convolution quadrature, initial correction, backward difference formula, nonsmooth, incompatible data, error estimates


## 1 Introduction

We are interested in the convolution quadrature (CQ) generated by high-order backward difference formulas (BDFs) for solving the fractional-order evolution equation (with $0<\alpha<1$ )

$$
\left\{\begin{array}{l}
\partial_{t}^{\alpha}(u(t)-v)-A u(t)=f(t), \quad 0<t<T  \tag{1.1}\\
u(0)=v
\end{array}\right.
$$

where $f$ is a given function, and $\partial_{t}^{\alpha} u$ denotes the left-sided Riemann-Liouville fractional time derivative of order $\alpha$, defined by (cf. [17])

$$
\begin{equation*}
\partial_{t}^{\alpha} u(t):=\frac{1}{\Gamma(1-\alpha)} \frac{d}{d t} \int_{0}^{t}(t-s)^{-\alpha} u(s) d s \tag{1.2}
\end{equation*}
$$

where $\Gamma(z):=\int_{0}^{\infty} s^{z-1} e^{-s} d s$ is the Gamma function. Under the initial condition $u(0)=v$, the RiemannLiouville fractional derivative $\partial_{t}^{\alpha}(u-v)$ in the model 1.1 is identical with the usual Caputo fractional derivative [17, pp. 91].

In the model (1.1), the operator $A$ denotes either the Laplacian $\Delta$ on a polyhedral domain $\Omega \subset$ $\mathbb{R}^{d}(d=1,2,3)$ with a homogenous Dirichlet boundary condition, or its discrete approximation $\Delta_{h}$ by Galerkin finite element method. Thus the operator $A$ satisfies the following resolvent estimate (cf. [1, Example 3.7.5 and Theorem 3.7.11] and [34])

$$
\begin{equation*}
\left\|(z-A)^{-1}\right\|_{L^{2}(\Omega) \rightarrow L^{2}(\Omega)} \leq c_{\phi} z^{-1}, \quad \forall z \in \Sigma_{\phi} \tag{1.3}
\end{equation*}
$$

[^0]for all $\phi \in(\pi / 2, \pi)$, where $\Sigma_{\theta}:=\{z \in \mathbb{C} \backslash\{0\}:|\arg z|<\theta\}$ is a sector of the complex plane $\mathbb{C}$. The model (1.1) covers a broad range of applications related to anomalous diffusion discovered in the past two decades, e.g., conformational dynamics of protein molecules, contaminant transport in complex geological formations and relaxation in polymer systems; see [32].

There has been much recent interest in developing high-order schemes for problem 1.1), especially spectral methods [18, 39, 38, 2, 3] and discontinuous Galerkin [7, 25, 26, 27. In this work, we develop robust high-order schemes based on CQs generated by high-order BDFs. The CQ developed by Lubich [20, 21, 22] provides a flexible framework for constructing high-order methods to discretize the fractional derivative $\partial_{t}^{\alpha} u$. By its very construction, it inherits the stability properties of linear multistep methods, which greatly facilitates the analysis of the resulting numerical scheme, in a way often strikingly opposed to standard quadrature formulas [22, pp. 504]. Hence, it has been widely applied to discretize the model (1.1) and its variants, especially the CQ generated by BDF1 and BDF2 (with BDFk denoting BDF of order $k$ ). In the literature, the CQ generated by BDF1 is commonly known as the Grünwald-Letnikov formula.

By assuming that the solution is sufficiently smooth, which is equivalent to assuming smoothness of the initial data $v$ and imposing certain compatibility conditions on the source term $f$ at $t=0$, the stability and convergence of the numerical solutions of fractional evolution equations have been investigated in [6, 10, 35, 37, 40]. In general, if the source term $f$ is not compatible with the given initial data, the solution $u$ of the model 1.1 will exhibit weak singularity at $t=0$, which will deteriorate the convergence rate of the numerical solutions. This has been widely recognized in fractional ODEs [8, 9] and PDEs [5, 15, 31. In particular, direct implementation of the CQ generated by high-order BDFs for discretizing the fractional evolution equations generally only yields first-order accuracy. To restore the theoretical rate $O\left(\tau^{k}\right)$ of $\mathrm{BDF} k$, two different strategies have been proposed.

For fractional ODEs, one idea is to use starting weights [20] to correct the CQ in discretizing the fractional time derivative, cf. 2.1 below:

$$
\bar{\partial}_{\tau}^{\alpha} \varphi^{n}=\frac{1}{\tau^{\alpha}} \sum_{j=0}^{n} b_{n-j} \varphi^{j}+\sum_{j=0}^{M} w_{n, j} \varphi^{j}
$$

The starting term $\sum_{j=0}^{M} w_{n, j} u_{j}$ is to capture all leading singularities so as to recover a uniform $O\left(\tau^{k}\right)$ rate of the scheme, where $M \in \mathbb{N}$ and the weights $w_{n, j}$ generally depend on both $\alpha$ and $k$. This approach works well for fractional evolution ODEs, however, the extension of this approach to fractional evolution PDEs relies on expanding the solution into power series of $t$, which requires imposing certain compatibility conditions on the source term.

The second idea is to split the source term $f$ into $f(t)=f(0)+(f(t)-f(0))$ and to approximate $f(0)$ by $\bar{\partial}_{\tau} \partial_{t}^{-1} f(0)$, with a similar treatment of the initial data $v$. This leads to a corrected BDF2 at the first step and restores the $O\left(\tau^{2}\right)$ accuracy for any fixed $t_{n}>0$. The idea was first introduced in [23] for solving a variant of formulation (1.1) in the diffusion-wave case and then systematically developed in [5] for BDF2, and was recently extended to the model 1.1 in [15] for both subdiffusion and diffusion-wave cases. Higher-order extension of this approach is possible, but is still not available in the literature.

The goal of this work is to develop robust high-order BDFs for fractional evolution equations along the second strategy [5, 15]. Instead of extending this strategy to each high-order BDF method, separately, we develop a systematic strategy for correcting initial steps for high-order BDFs, based on a few simple criteria, cf. (2.13) and (2.14) for the model (1.1). These criteria emerge naturally from solution representations, and are purely algebraic in nature and straightforward to construct. The explicit correction coefficients will be given for BDFs up to order 6 . For $\mathrm{BDF} k$, the correction is only needed at the starting $k-1$ steps and thus the resulting scheme is easy to implement.

We develop proper corrections for high-order BDFs for both subdiffusion, i.e., $\alpha \in(0,1)$, and diffusion wave, i.e., $\alpha \in(1,2)$. It is noteworthy that for $\alpha \in(1,2)$, high-order BDFs can be either unconditionally or conditionally stable, depending on the fractional order $\alpha$, and in the latter case, an explicit CFL condition on the time step size $\tau$ is given. Theoretically, the corrected BDF $k$ achieves the $k^{\text {th }}$-order accuracy at any fixed time $t=t_{n}$ (when $t_{n}$ is bounded from below), and the error bound depends only
on data regularity, without assuming any compatibility conditions on the source term or extra regularity on the solution (cf. Theorems 2.2 and 3.2 . These results are supported by the numerical experiments in Section 4

The rest of the paper is organized as follows. In Section 2 we develop the correction for the subdiffusion case, including the motivations of the algebraic criteria for choosing the correction coefficients. The extension of the approach to the diffusion wave case is given in Section 3 Numerical results are presented in Section 4 to illustrate the efficiency and robustness of the corrected schemes. Appendix A gives an alternative interpretation of our correction method in terms of Lubich's convolution quadrature for operator-valued convolution integrals. Some lengthy proofs are given in Appendices B, C and D. Throughout, the notation $c$ denotes a generic positive constant, whose value may differ at each occurrence, but it is always independent of the time step size $\tau$ and the solution $u$.

## 2 BDFs for Subdiffusion and its Correction

Let $\left\{t_{n}=n \tau\right\}_{n=0}^{N}$ be a uniform partition of the interval $[0, T]$, with a time step size $\tau=T / N$. The CQ generated by $\mathrm{BDF} k, k=1, \ldots, 6$, approximates the fractional derivative $\partial_{t}^{\alpha} \varphi\left(t_{n}\right)$ by

$$
\begin{equation*}
\bar{\partial}_{\tau}^{\alpha} \varphi^{n}:=\frac{1}{\tau^{\alpha}} \sum_{j=0}^{n} b_{j} \varphi^{n-j} \tag{2.1}
\end{equation*}
$$

with $\varphi^{n}=\phi\left(t_{n}\right)$, where the weights $\left\{b_{j}\right\}_{j=0}^{\infty}$ are the coefficients in the power series expansion

$$
\begin{equation*}
\delta_{\tau}(\zeta)^{\alpha}=\frac{1}{\tau^{\alpha}} \sum_{j=0}^{\infty} b_{j} \zeta^{j} \quad \text { with } \quad \delta_{\tau}(\zeta):=\frac{1}{\tau} \sum_{j=1}^{k} \frac{1}{j}(1-\zeta)^{j} \tag{2.2}
\end{equation*}
$$

Below we often write $\delta(\zeta)=\delta_{1}(\zeta)$. The coefficients $b_{j}$ can be computed efficiently by the fast Fourier transform [29, 33] or recursion [36]. Correspondingly, the BDF for solving (1.1) seeks approximations $U^{n}, n=1, \ldots, N$, to the exact solution $u\left(t_{n}\right)$ by

$$
\begin{equation*}
\bar{\partial}_{\tau}^{\alpha}(U-v)^{n}-A U^{n}=f\left(t_{n}\right) \tag{2.3}
\end{equation*}
$$

If the solution $u$ is smooth and has sufficiently many vanishing derivatives at 0 , then $U^{n}$ converges at a rate of $O\left(\tau^{k}\right)$ [20, 22]. However, it generally only exhibits a first-order accuracy when solving fractional evolution equations, due to the weak solution singularity at 0 , even if the initial data $v$ and source term $f$ are smooth 30. This has been observed numerically [5, 15]. For $\alpha=1, \mathrm{BDF} k$ is known to be $A\left(\vartheta_{k}\right)$ stable with angle $\vartheta_{k}=90^{\circ}, 90^{\circ}, 86.03^{\circ}, 73.35^{\circ}, 51.84^{\circ}, 17.84^{\circ}$ for $k=1,2,3,4,5,6$, respectively [12, pp. 251].

To restore the $k^{\text {th }}$-order accuracy, we correct $\mathrm{BDF} k$ at the starting $k-1$ steps by (as usual, the summation disappears if the upper index is smaller than the lower one)

$$
\begin{array}{ll}
\bar{\partial}_{\tau}^{\alpha}(U-v)^{n}-A U^{n}=a_{n}^{(k)}(A v+f(0))+f\left(t_{n}\right)+\sum_{\ell=1}^{k-2} b_{\ell, n}^{(k)} \tau^{\ell} \partial_{t}^{\ell} f(0), & 1 \leq n \leq k-1  \tag{2.4}\\
\bar{\partial}_{\tau}^{\alpha}(U-v)^{n}-A U^{n}=f\left(t_{n}\right), & k \leq n \leq N
\end{array}
$$

where $a_{n}^{(k)}$ and $b_{\ell, n}^{(k)}$ are coefficients to be determined below. They are constructed so as to improve the accuracy of the overall scheme to $O\left(\tau^{k}\right)$ for a general initial data $v \in D(A)$ and a possibly incompatible right-hand side $f$. The only difference between (2.4) and the standard scheme (2.3) lies in the correction terms at the starting $k-1$ steps. Hence, the proposed scheme (2.4) is easy to implement.

Remark 2.1. In the scheme (2.4), the derivative $\partial_{t}^{\ell} f(0)$ may be replaced by its $(k-\ell-1)$-order finite difference approximation $f^{(\ell)}$, without sacrificing its accuracy.

Remark 2.2. The correction in (2.4) is minimal in the sense that there is no other correction scheme which modifies only the $k-1$ starting steps while retaining the $O\left(\tau^{k}\right)$ convergence. This does not rule out corrections with more starting steps. We give an interesting correction closely related to (2.4) in Appendix A.

### 2.1 Derivation of the correction criteria

Now we derive the criteria for choosing the coefficients $a_{j}^{(k)}$ and $b_{\ell, j}^{(k)}$, cf. (2.13) and (2.14), using Laplace transform and its discrete analogue, the generating function [23, 34]. We denote by ${ }^{\wedge}$ taking Laplace transform, and for a given sequence $\left(f^{n}\right)_{n=0}^{\infty}$, denote by $\tilde{f}(\zeta)$ the generating function, which is defined by $\widetilde{f}(\zeta):=\sum_{n=0}^{\infty} f^{n} \zeta^{n}$. First we split the right hand side $f$ into

$$
\begin{equation*}
f(t)=f(0)+\sum_{\ell=1}^{k-2} \frac{t^{\ell}}{\ell!} \partial_{t}^{\ell} f(0)+R_{k} \tag{2.5}
\end{equation*}
$$

and $R_{k}$ is the corresponding local truncation error, given by

$$
\begin{equation*}
R_{k}=f(t)-f(0)-\sum_{\ell=1}^{k-2} \frac{t^{\ell}}{\ell!} \partial_{t}^{\ell} f(0)=\frac{t^{k-1}}{(k-1)!} \partial_{t}^{k-1} f(0)+\frac{t^{k-1}}{(k-1)!} * \partial_{t}^{k} f \tag{2.6}
\end{equation*}
$$

where $*$ denotes Laplace convolution. Thus the function $w(t):=u(t)-v$ satisfies

$$
\begin{equation*}
\partial_{t}^{\alpha} w-A w=A v+f(0)+\sum_{\ell=1}^{k-2} \frac{t^{\ell}}{\ell!} \partial_{t}^{\ell} f(0)+R_{k} \tag{2.7}
\end{equation*}
$$

with $w(0)=0$. Since $w(0)=0$, the identity $\widehat{\partial_{t}^{\alpha} w}(z)=z^{\alpha} \widehat{w}(z)$ holds [17, Remark 2.8, pp. 84], and thus by Laplace transform, we obtain

$$
z^{\alpha} \widehat{w}(z)-A \widehat{w}(z)=z^{-1}(A v+f(0))+\sum_{\ell=1}^{k-2} \frac{1}{z^{\ell+1}} \partial_{t}^{\ell} f(0)+\widehat{R}_{k}(z)
$$

By inverse Laplace transform, the function $w(t)$ can be readily represented by

$$
\begin{equation*}
w(t)=\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}} e^{z t} K(z)(A v+f(0)) d z+\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}} e^{z t} z K(z)\left(\sum_{\ell=1}^{k-2} \frac{1}{z^{\ell+1}} \partial_{t}^{\ell} f(0)+\widehat{R}_{k}(z)\right) d z \tag{2.8}
\end{equation*}
$$

with the kernel function

$$
\begin{equation*}
K(z)=z^{-1}\left(z^{\alpha}-A\right)^{-1} \tag{2.9}
\end{equation*}
$$

In the representation 2.8, the contour $\Gamma_{\theta, \delta}$ is defined by

$$
\Gamma_{\theta, \delta}=\{z \in \mathbb{C}:|z|=\delta,|\arg z| \leq \theta\} \cup\left\{z \in \mathbb{C}: z=\rho e^{ \pm \mathrm{i} \theta}, \rho \geq \delta\right\}
$$

oriented with an increasing imaginary part. Throughout, we choose the angle $\theta$ such that $\pi / 2<\theta<$ $\min (\pi, \pi / \alpha)$ and hence $z^{\alpha} \in \Sigma_{\theta^{\prime}}$ with $\theta^{\prime}=\alpha \theta<\pi$ for all $z \in \Sigma_{\theta}$. By the resolvent estimate (1.3), there exists a constant $c$ which depends only on $\theta$ and $\alpha$ such that

$$
\begin{equation*}
\left\|\left(z^{\alpha}-A\right)^{-1}\right\| \leq c z^{-\alpha} \quad \text { and } \quad\|K(z)\| \leq c|z|^{-1-\alpha}, \quad \forall z \in \Sigma_{\theta} \tag{2.10}
\end{equation*}
$$

Next, we give a representation of the discrete solution $W^{n}:=U^{n}-v$, which follows from lengthy but simple computations, cf. Appendix B.

Theorem 2.1. The discrete solution $W^{n}:=U^{n}-v$ is represented by

$$
\begin{align*}
W^{n} & =\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau}} e^{z t_{n}} \mu\left(e^{-z \tau}\right) K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right)(A v+f(0)) d z \\
& +\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau}} e^{z t_{n}} \delta_{\tau}\left(e^{-z \tau}\right) K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right) \sum_{\ell=1}^{k-2}\left(\frac{\gamma_{\ell}\left(e^{-z \tau}\right)}{\ell!}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} e^{-z t_{j}}\right) \tau^{\ell+1} \partial_{t}^{\ell} f(0) d z  \tag{2.11}\\
& +\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau}} e^{z t_{n}} \delta_{\tau}\left(e^{-z \tau}\right) K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right) \tau \widetilde{R}_{k}\left(e^{-z \tau}\right) d z
\end{align*}
$$

with the contour $\Gamma_{\theta, \delta}^{\tau}:=\left\{z \in \Gamma_{\theta, \delta}:|\Im(z)| \leq \pi / \tau\right\}$ (oriented with an increasing imaginary part), where the functions $\mu(\zeta)$ and $\gamma_{\ell}(\zeta)$ are respectively defined by

$$
\begin{equation*}
\mu(\zeta)=\delta(\zeta)\left(\frac{\zeta}{1-\zeta}+\sum_{j=1}^{k-1} a_{j}^{(k)} \zeta^{j}\right) \quad \text { and } \quad \gamma_{\ell}(\zeta)=\left(\zeta \frac{d}{d \zeta}\right)^{\ell} \frac{1}{1-\zeta} \tag{2.12}
\end{equation*}
$$

By comparing the kernel functions in (2.8) and (2.11), we deduce that in order to have $O\left(\tau^{k}\right)$ accuracy, the following three conditions should be satisfied for $z \in \Gamma_{\theta, \delta}^{\tau}$ :

$$
\begin{aligned}
& \left|\delta_{\tau}\left(e^{-z \tau}\right)-z\right| \leq c|z|^{k+1} \tau^{k}, \quad\left|\mu\left(e^{-z \tau}\right)-1\right| \leq c|z|^{k} \tau^{k} \\
& \left|\left(\frac{\gamma_{\ell}\left(e^{-z \tau}\right)}{\ell!}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} e^{-z t_{j}}\right) \tau^{\ell+1}-\frac{1}{z^{\ell+1}}\right| \leq c|z|^{k-\ell-1} \tau^{k}
\end{aligned}
$$

Note that for $\mathrm{BDF} k$, the estimate $\left|\delta_{\tau}\left(e^{-z \tau}\right)-z\right| \leq c|z|^{k+1} \tau^{k}$ holds automatically (cf. Lemma B. 1 in Appendix B). It suffices to impose the following algebraic criteria (changing $e^{-z \tau}$ to $\zeta$ and $z \tau$ to $1-\zeta$ ): for $\operatorname{BDF} k$, choose the coefficients $\left\{a_{j}^{(k)}\right\}_{j=1}^{k-1}$ and $\left\{b_{\ell, j}^{(k)}\right\}_{j=1}^{k-1}$ such that

$$
\begin{align*}
&|\mu(\zeta)-1| \leq c|1-\zeta|^{k}  \tag{2.13}\\
&\left|\frac{\gamma_{\ell}(\zeta)}{\ell!}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} \zeta^{j}-\frac{1}{\delta(\zeta)^{\ell+1}}\right| \leq c|1-\zeta|^{k-\ell-1}, \quad \ell=1, \ldots, k-2 \tag{2.14}
\end{align*}
$$

where the functions $\mu(\zeta)$ and $\gamma_{\ell}(\zeta)$ are defined in (2.12). It can be verified that for $\mathrm{BDF} k, k=3, \ldots, 6$, the leading singularities on the left hand side of 2.14 do cancel out, and thus the criterion can be satisfied.

### 2.2 Computation of the coefficients $a_{j}^{(k)}$ and $b_{\ell, j}^{(k)}$

First we compute the coefficients $a_{j}^{(k)}$. To this end, we rewrite $\sum_{j=1}^{k-1} a_{j}^{(k)} \zeta^{j}$ as

$$
\begin{equation*}
\sum_{j=1}^{k-1} a_{j}^{(k)} \zeta^{j}=\zeta \sum_{j=0}^{k-2} c_{j}(1-\zeta)^{j} \tag{2.15}
\end{equation*}
$$

Consequently, by writing $\zeta=1-(1-\zeta)$, expanding the summation and collecting terms, we obtain (with the convention $c_{-2}=c_{-1}=0$ )

$$
\begin{aligned}
\mu(\zeta) & =\sum_{j=1}^{k} \frac{1}{j}(1-\zeta)^{j}\left(\frac{\zeta}{1-\zeta}+\zeta \sum_{j=0}^{k-2} c_{j}(1-\zeta)^{j}\right) \\
& =\sum_{j=0}^{k-1} \frac{1}{j+1}(1-\zeta)^{j}\left(1-(1-\zeta)-\sum_{j=0}^{k} c_{j-2}(1-\zeta)^{j}+\sum_{j=0}^{k-1} c_{j-1}(1-\zeta)^{j}\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{j=0}^{k-1} \frac{1}{j+1}(1-\zeta)^{j} \sum_{j=1}^{k} \frac{1}{j}(1-\zeta)^{j}-\sum_{j=2}^{k-1}\left(\sum_{\ell=0}^{j} \frac{1}{j-\ell+1} c_{\ell-2}\right)(1-\zeta)^{j} \\
& +\sum_{j=1}^{k-1}\left(\sum_{\ell=0}^{j} \frac{1}{j-\ell+1} c_{\ell-1}\right)(1-\zeta)^{j}+O\left((1-\zeta)^{k}\right) \\
= & 1+\sum_{j=1}^{k-1}\left(\frac{1}{j+1}-\frac{1}{j}-\sum_{\ell=0}^{j} \frac{1}{j-\ell+1} c_{\ell-2}+\sum_{\ell=0}^{j} \frac{1}{j-\ell+1} c_{\ell-1}\right)(1-\zeta)^{j}+O\left((1-\zeta)^{k}\right) \\
= & 1+\sum_{j=1}^{k-1}\left(-\frac{1}{j(j+1)}-\sum_{\ell=1}^{j-1} \frac{1}{j-\ell} c_{\ell-1}+\sum_{\ell=0}^{j-1} \frac{1}{j-\ell} c_{\ell}\right)(1-\zeta)^{j}+O\left((1-\zeta)^{k}\right) .
\end{aligned}
$$

Thus by choosing $c_{\ell}, \ell=0, \ldots, k-2$, such that

$$
\begin{equation*}
\sum_{\ell=0}^{j-1} \frac{1}{j-\ell} c_{\ell}=\frac{1}{j(j+1)}+\sum_{\ell=1}^{j-1} \frac{1}{j-\ell} c_{\ell-1}, \quad j=1, \ldots, k-1 \tag{2.16}
\end{equation*}
$$

Criterion (2.13) follows. The coefficients $a_{j}^{(k)}$ can be computed recursively from 2.16 and 2.15), and are given in Table 1. It is worth noting that the result for $k=2$ recovers exactly the correction in [15], and thus our algebraic construction generalizes the approach in [15].

Table 1: The coefficients $a_{j}^{(k)}$ computed by 2.15

| order of BDF | $a_{1}^{(k)}$ | $a_{2}^{(k)}$ | $a_{3}^{(k)}$ | $a_{4}^{(k)}$ | $a_{5}^{(k)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $k=2$ | $\frac{1}{2}$ |  |  |  |  |
| $k=3$ | $\frac{11}{12}$ | $-\frac{5}{12}$ |  |  |  |
| $k=4$ | $\frac{31}{24}$ | $-\frac{7}{6}$ | $\frac{3}{8}$ |  |  |
| $k=5$ | $\frac{1181}{720}$ | $-\frac{177}{80}$ | $\frac{341}{240}$ | $-\frac{251}{720}$ |  |
| $k=6$ | $\frac{2837}{1440}$ | $-\frac{2543}{720}$ | $\frac{17}{5}$ | $-\frac{1201}{720}$ | $\frac{95}{288}$ |

Next we compute the coefficients $b_{\ell, j}^{(k)}$. First we expand $\frac{\gamma_{\ell}(\zeta)}{\ell!}-\frac{1}{\delta(\zeta)^{\ell+1}}$ in $1-\zeta$ as

$$
\begin{equation*}
\frac{\gamma_{\ell}(\zeta)}{\ell!}-\frac{1}{\delta(\zeta)^{\ell+1}}=\sum_{j=0}^{k-\ell-2} g_{\ell, j}^{(k)}(1-\zeta)^{j}+O\left(|1-\zeta|^{k-\ell-1}\right) \tag{2.17}
\end{equation*}
$$

and then choose the coefficients $b_{\ell, j}^{(k)}, j=1, \ldots, k-1$ to satisfy 2.14 . To this end, we rewrite $\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} \zeta^{j}$ into the following form:

$$
\begin{equation*}
\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} \zeta^{j}=\zeta \sum_{j=0}^{k-2} d_{\ell, j}^{(k)}(1-\zeta)^{j}=\sum_{j=0}^{k-2} d_{\ell, j}^{(k)}(1-\zeta)^{j}-\sum_{j=1}^{k-1} d_{\ell, j-1}^{(k)}(1-\zeta)^{j} \tag{2.18}
\end{equation*}
$$

Then it suffices to choose

$$
\begin{array}{ll}
d_{\ell, 0}^{(k)}=-g_{\ell, 0}^{(k)}, & \\
d_{\ell, j}^{(k)}=d_{\ell, j-1}^{(k)}-g_{\ell, j}^{(k)} & \text { for } j=1, \ldots, k-\ell-2 \\
d_{\ell, j}^{(k)}=0 & \text { for } j=k-\ell-1, \ldots, k-2
\end{array}
$$

Now the coefficients $b_{\ell, j}^{(k)}$ can be computed recursively using 2.17, 2.19 and 2.18, and the results are given in Table 2. Note that for $k=4$ and 6 , the coefficients $b_{k-2, j}, j=1,2 \ldots, k-1$ vanish identically.

Table 2: The coefficients $b_{\ell, j}^{(k)}$.

| order of BDF |  | $b_{\ell, 1}^{(k)}$ | $b_{\ell, 2}^{(k)}$ | $b_{\ell, 3}^{(k)}$ | $b_{\ell, 4}^{(k)}$ | $b_{\ell, 5}^{(k)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k=3$ | $\ell=1$ | $\frac{1}{12}$ | 0 |  |  |  |
| $k=4$ | $\ell=1$ | $\frac{1}{6}$ | $-\frac{1}{12}$ | 0 |  |  |
|  | $\ell=2$ | 0 | 0 | 0 |  |  |
| $k=5$ | $\ell=1$ | $\frac{59}{240}$ | $-\frac{29}{120}$ | $\frac{19}{240}$ | 0 |  |
|  | $\ell=2$ | $\frac{1}{240}$ | $-\frac{1}{240}$ | 0 | 0 |  |
|  | $\ell=3$ | $\frac{1}{720}$ | 0 | 0 | 0 |  |
| $k=6$ | $\ell=1$ | $\frac{77}{240}$ | $-\frac{7}{15}$ | $\frac{73}{240}$ | $-\frac{3}{40}$ | 0 |
|  | $\ell=2$ | $\frac{1}{96}$ | $-\frac{1}{60}$ | $\frac{1}{160}$ | 0 | 0 |
|  | $\ell=3$ | $-\frac{1}{360}$ | $\frac{1}{720}$ | 0 | 0 | 0 |
|  | $\ell=4$ | 0 | 0 | 0 | 0 | 0 |

### 2.3 Error estimates

Last we state the error estimate for 2.4 . The proof relies on the splitting $u\left(t_{n}\right)-U^{n}=w\left(t_{n}\right)-W^{n}$ and the representations $(2.8)$ and 2.11$)$, and then bounding each term using 2.10 . The details can be found in Appendix C.

Theorem 2.2. Let Criteria 2.13 and 2.14 hold. Then for the solution $U^{n}$ to the corrected scheme (2.4), the following error estimate holds for any $t_{n}>0$

$$
\begin{aligned}
\left\|U^{n}-u\left(t_{n}\right)\right\|_{L^{2}(\Omega)} \leq & c \tau^{k}\left(t_{n}^{\alpha-k}\|f(0)+A v\|_{L^{2}(\Omega)}+\sum_{\ell=1}^{k-1} t_{n}^{\alpha+\ell-k}\left\|\partial_{t}^{\ell} f(0)\right\|_{L^{2}(\Omega)}\right. \\
& \left.+\int_{0}^{t_{n}}\left(t_{n}-s\right)^{\alpha-1}\left\|\partial_{s}^{k} f(s)\right\|_{L^{2}(\Omega)} d s\right)
\end{aligned}
$$

Remark 2.3. Theorem 2.2 implies that for any fixed $t_{n}>0$, the convergence rate is $O\left(\tau^{k}\right)$ for $B D F k$. In order to have a uniform rate $O\left(\tau^{k}\right)$, the following compatibility conditions are needed:

$$
f(0)+A v=0, \quad \text { and } \quad \partial_{t}^{(\ell)} f(0)=0, \quad \ell=1, \ldots, k-1,
$$

concurring with known results on convolution quadrature [22]. In the absence of these conditions, the error estimate deteriorates as $t \rightarrow 0$, which is consistent with the corresponding regularity theory: the solution (and its derivatives) exhibits weak singularity at $t=0$ [30].

Remark 2.4. The error estimate in Theorem 2.2 requires $A v \in L^{2}(\Omega)$, i.e., the initial data $v$ is reasonably smooth. Upon minor modifications of the proof in Appendix $C$, one can derive a similar error estimate for $v \in L^{2}(\Omega)$ :

$$
\left\|U^{n}-u\left(t_{n}\right)\right\|_{L^{2}(\Omega)} \leq c \tau^{k}\left(t_{n}^{-k}\|v\|_{L^{2}(\Omega)}+\sum_{\ell=0}^{k-1} t_{n}^{\alpha+\ell-k}\left\|\partial_{t}^{\ell} f(0)\right\|_{L^{2}(\Omega)}+\int_{0}^{t_{n}}\left(t_{n}-s\right)^{\alpha-1}\left\|\partial_{s}^{k} f(s)\right\|_{L^{2}(\Omega)} d s\right) .
$$

## 3 Corrected BDF for diffusion-wave problem

Now we extend the strategy in Section 2 to the diffusion-wave problem, i.e., $1<\alpha<2$ :

$$
\partial_{t}^{\alpha}(u(t)-v-t b)-A u(t)=f(t)
$$

with the initial conditions $u(0)=v$ and $u^{\prime}(0)=b$, where

$$
\begin{equation*}
\partial_{t}^{\alpha} u(t):=\frac{1}{\Gamma(2-\alpha)} \frac{d^{2}}{d t^{2}} \int_{0}^{t}(t-s)^{1-\alpha} u(s) d s \tag{3.1}
\end{equation*}
$$

The main differences from the subdiffusion case lie in the extra initial condition $b$ and better temporal smoothing property [15]. A straightforward implementation of $\mathrm{BDF} k$ can fail to yield the $O\left(\tau^{k}\right)$ rate, as the subdiffusion case, and further requires unnecessarily high regularity on $f$. We shall develop a corrected scheme to take care of both issues. First, in order to fully exploit the extra smoothing, we rewrite the source term $f$ as $f=\partial_{t} g$ with $g=\partial_{t}^{-1} f$. Then the diffusion-wave equation can be rewritten as

$$
\begin{equation*}
\partial_{t}^{\alpha}(u-v-t b)-A u=\partial_{t} g \tag{3.2}
\end{equation*}
$$

Next we correct the starting $k-1$ steps, and seek approximations $U^{n}, n=1, \ldots, N$, by

$$
\begin{array}{ll}
\bar{\partial}_{\tau}^{\alpha}(U-v-t b)^{n}-A U^{n}=a_{n}^{(k)} A v+c_{n}^{(k)} \tau A b+\bar{\partial}_{\tau} g^{n}+\sum_{\ell=1}^{k-2} b_{\ell, n}^{(k)} \tau^{\ell-1} \partial_{t}^{\ell-1} f(0), & 1 \leq n \leq k-1  \tag{3.3}\\
\bar{\partial}_{\tau}^{\alpha}(U-v-t b)^{n}-A U^{n}=\bar{\partial}_{\tau} g^{n} &
\end{array}
$$

The scheme involves $\bar{\partial}_{\tau} g^{n}$, instead of $f^{n}$, which enables one to relax the regularity requirement on $f$. The correction terms are to ensure the desired $O\left(\tau^{k}\right)$ rate.

Now we derive the criterion for choosing the coefficients in 3.3) using Laplace transform and generating function. First, since $g(0)=0, g(t)$ can be split into

$$
\begin{equation*}
g(t)=\sum_{\ell=1}^{k-2} \frac{t^{\ell}}{\ell!} \partial_{t}^{\ell} g(0)+R_{k}=\sum_{\ell=1}^{k-2} \frac{t^{\ell}}{\ell!} \partial_{t}^{\ell-1} f(0)+R_{k} \tag{3.4}
\end{equation*}
$$

where $R_{k}$ is the local truncation error $R_{k}=\frac{t^{k-1}}{(k-1)!} \partial_{t}^{k-1} g(0)+\frac{t^{k-1}}{(k-1)!} * \partial_{t}^{k} g(t)$. With the splitting 3.4), the function $w=u-v-t b$ satisfies

$$
\partial_{t}^{\alpha} w-A w=A v+t A b+\sum_{\ell=1}^{k-2} \partial_{t} \frac{t^{\ell}}{\ell!} \partial_{t}^{\ell-1} f(0)+\partial_{t} R_{k}
$$

Then by Laplace transform, we derive a representation of the continuous solution $w(t)$ :

$$
\begin{equation*}
w(t)=\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}} e^{z t} K(z)\left(A v+z^{-1} A b\right) d z+\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta} \delta} e^{z t} z K(z)\left(\sum_{\ell=1}^{k-2} \frac{1}{z^{\ell}} \partial_{t}^{\ell-1} f(0)+z \widehat{R}_{k}(z)\right) d z \tag{3.5}
\end{equation*}
$$

where the angle $\theta \in(\pi / 2, \pi)$ is sufficiently close to $\pi / 2^{\delta}$ such that $\alpha^{\ell} \xi^{1}<\pi$, and $\delta$ is small.
Since $\mathrm{BDF} k$ is $A\left(\vartheta_{k}\right)$-stable, the scheme (3.3) is unconditionally stable for any $\alpha<\alpha^{*}(k):=\pi /\left(\pi-\vartheta_{k}\right)$. The critical value $\alpha^{*}(k)$ is $1.91,1.68,1.40$ and 1.11 for $k=3, \ldots, 6$. In contrast, for $\alpha \geq \alpha^{*}(k)$, it is only conditionally stable. Note that for any $\alpha \in(1,2)$, the curve $\delta\left(e^{-\mathrm{i} \theta}\right)^{\alpha}$ is not tangent to the real axis at the origin (i.e., $\theta$ close to zero). This naturally gives rise to the following condition.

Condition 3.1. Let $r(A)$ be the numerical radius of $A$, and the following condition holds: (i) The fractional order $\alpha<\alpha^{*}(k)$ or (ii) The fractional order $\alpha \geq \alpha^{*}(k)$ and $\tau^{\alpha} r(A) \leq c(\alpha, k)-\gamma$ for some $\gamma>0$, where the constant $c(\alpha, k)$ is given by the intersection point of $\left\{\delta(\zeta)^{\alpha}:|\zeta|=1\right\}$ with the negative real axis (closest to the origin).

Remark 3.1. Condition 3.1(ii) specifies the CFL condition on the time step size $\tau$ (so it holds only if $r(A)<\infty)$. The CFL constant $c(\alpha, k)$ is not available in closed form, but can be determined numerically; see Fig. 1 for the values.


Figure 1: The CFL constant $c(\alpha, k)$ for $\operatorname{BDF} k, k=3,4,5,6$, at different $\alpha$ values.
It is interesting to observe the qualitative differences of BDFs of different order. For example, the CFL constant $c(\alpha, 6)$ of BDF6 does not approach zero even for $\alpha$ tends to 2 ; and there is an interval of $\alpha$ values for which the CFL constant $c(\alpha, 4)$ for BDF4 is larger than $c(\alpha, 3)$ for BDF3, i.e., BDF4 is less stringent in time step size.

The next result gives the representation of the solution $W^{n}=U^{n}-v-t_{n} b$, which follows from simple yet lengthy computations, cf. Appendix D.
Theorem 3.1. Under Condition 3.1, the discrete solution $W^{n}:=U^{n}-v-t_{n} b$ is given by

$$
\begin{align*}
W^{n}= & \frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau}} e^{z t_{n}} \mu\left(e^{-z \tau}\right) K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right) A v d z \\
& +\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau}} e^{z t_{n}} K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right) \delta_{\tau}\left(e^{-z \tau}\right)\left(\gamma_{1}\left(e^{-z \tau}\right)+\sum_{j=1}^{k-1} c_{j}^{(k)} e^{-z t_{j}}\right) \tau^{2} A b d z \\
& +\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau}} e^{z t_{n}} \delta_{\tau}\left(e^{-z \tau}\right) K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right) \sum_{\ell=1}^{k-2}\left(\delta\left(e^{-z \tau}\right) \frac{\gamma_{\ell}\left(e^{-z \tau}\right)}{\ell!}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} e^{-z t_{j}}\right) \tau^{\ell} \partial_{t}^{\ell-1} f(0) d z \\
& +\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau},}^{z t_{n}} e_{\tau}\left(e^{-z \tau}\right)^{2} K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right) \tau \widetilde{R}_{k}\left(e^{-z \tau}\right) d z \tag{3.6}
\end{align*}
$$

with the contour $\Gamma_{\theta, \delta}^{\tau}:=\left\{z \in \Gamma_{\theta, \delta}:|\Im(z)| \leq \pi / \tau\right\}$ (oriented with an increasing imaginary part), for some $\theta$ sufficiently close to $\pi / 2$, where $\mu(\zeta)$ and $\gamma_{\ell}(\zeta)$ are defined in (2.12).

Proceeding like before, from the solution representations (3.5) and (3.6), we deduce the following algebraic criteria for choosing the coefficients $a_{j}^{(k)}, c_{j}^{(k)}$ and $b_{\ell, n}^{(k)}$ :

$$
\begin{align*}
&|\mu(\zeta)-1| \leq c|1-\zeta|^{k},  \tag{3.7}\\
&\left|\gamma_{1}(\zeta)+\sum_{j=1}^{k-1} c_{j}^{(k)} \zeta^{j}-\frac{1}{\delta(\zeta)^{2}}\right| \leq c|1-\zeta|^{k-2},  \tag{3.8}\\
&\left|\delta(\zeta) \frac{\gamma_{\ell}(\zeta)}{\ell!}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} \zeta^{j}-\frac{1}{\delta(\zeta)^{\ell}}\right| \leq c|1-\zeta|^{k-\ell}, \quad \ell=1,2, \ldots, k-2, \tag{3.9}
\end{align*}
$$

where the functions $\mu(\zeta)$ and $\gamma_{\ell}(\zeta)$ are defined in 2.12.

By comparing Criterion (3.7) with 2.13), and respectively Criterion (3.8 with 2.14 , the coefficients $a_{j}^{(k)}$ are identical with that for $\alpha \in(0,1)$, and respectively $c_{j}^{(k)}$ with $b_{1, j}^{(k)}$ for $\alpha \in(0,1)$. However, due to the presence of the extra factor $\delta(\zeta)$, the coefficients $b_{\ell, j}^{(k)}$ are different from that of the case $0<\alpha<1$, and have to be determined. The procedure for computing $b_{\ell, j}^{(k)}$ is similar to that in Section 2.2, and the results are given in Table 3 .

Table 3: The coefficients $b_{\ell, j}^{(k)}$ according to Criterion (3.9).

| order of BDF |  | $b_{\ell, 1}^{(k)}$ | $b_{\ell, 2}^{(k)}$ | $b_{\ell, 3}^{(k)}$ | $b_{\ell, 4}^{(k)}$ | $b_{\ell, 5}^{(k)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k=3$ | $\ell=1$ | $\frac{1}{12}$ | $-\frac{1}{12}$ |  |  |  |
| $k=4$ | $\ell=1$ | $\frac{5}{24}$ | $-\frac{1}{3}$ | $\frac{1}{8}$ |  |  |
|  | $\ell=2$ | 0 | 0 | 0 |  |  |
| $k=5$ | $\ell=1$ | $\frac{257}{720}$ | $-\frac{187}{240}$ | $\frac{137}{240}$ | $-\frac{107}{240}$ |  |
|  | $\ell=2$ | $\frac{1}{240}$ | $-\frac{1}{120}$ | $\frac{1}{240}$ | 0 |  |
|  | $\ell=3$ | $-\frac{1}{720}$ | $\frac{1}{720}$ | 0 | 0 |  |
| $k=6$ | $\ell=1$ | $\frac{749}{1440}$ | $-\frac{1031}{720}$ | $\frac{31}{20}$ | $-\frac{577}{720}$ | $\frac{47}{288}$ |
|  | $\ell=2$ | $\frac{1}{80}$ | $-\frac{1}{30}$ | $\frac{7}{240}$ | $-\frac{1}{120}$ | 0 |
|  | $\ell=3$ | $-\frac{1}{288}$ | $\frac{1}{180}$ | $-\frac{1}{480}$ | 0 | 0 |
|  | $\ell=4$ | 0 | 0 | 0 | 0 | 0 |

Last, we state the error estimate for the approximation $U^{n}$. The proof is similar to that of Theorem 3.2. but with $g=\partial_{t}^{-1} f$ in place of $f$. It is briefly sketched in Appendix E.

Theorem 3.2. Let Criteria (3.7, (3.8) and 3.9 hold, and Condition 3.1 be fulfilled. Then for the solution $U^{n}$ to (3.3), the following error estimate holds for any $t_{n}>0$

$$
\begin{aligned}
\left\|U^{n}-u\left(t_{n}\right)\right\|_{L^{2}(\Omega)} \leq c \tau^{k} & \left(t_{n}^{\alpha-k}\|f(0)+A v\|_{L^{2}(\Omega)}+t_{n}^{\alpha+1-k}\|A b\|_{L^{2}(\Omega)}\right. \\
& \left.+\sum_{\ell=1}^{k-2} t_{n}^{\alpha+\ell-k}\left\|\partial_{t}^{\ell} f(0)\right\|_{L^{2}(\Omega)}+\int_{0}^{t_{n}}\left(t_{n}-s\right)^{\alpha-2}\left\|\partial_{s}^{k-1} f(s)\right\|_{L^{2}(\Omega)} d s\right)
\end{aligned}
$$

Remark 3.2. Theorem 3.2 only requires $(k-1)^{\text {th }}$ order derivative of $f$ in time, instead of $k^{\text {th }}$ order derivative of $f$ as in Theorem 2.2. Thus it indeed relaxes the regularity condition.

## 4 Numerical experiments and discussions

Now we present numerical results to show the efficiency and accuracy of the schemes (2.4) and (3.3) in one-spatial dimension, on the unit interval $\Omega=(0,1)$. In space, it is discretized with the piecewise linear Galerkin finite element method [13]: we divide $\Omega$ into $M$ equally spaced subintervals with a mesh size $h=1 / M$. Since the convergence behavior of the spatial discretization is well understood, we focus on the temporal convergence. In the computation, we fix the time step size $\tau$ at $\tau=t / N$, where $t$ is the time of interest. We measure the accuracy by the normalized errors $e^{N}=\left\|u\left(t_{N}\right)-U^{N}\right\|_{L^{2}(\Omega)} /\left\|u\left(t_{N}\right)\right\|_{L^{2}(\Omega)}$, where the reference solution $u\left(t_{N}\right)$ is computed using a much finer mesh. All the computations are carried out in MATLAB R2015a on a personal laptop, and further, in order to observe error beyond double precision, we employ the Multiprecision Computing Toolbox ${ }^{1}$ for MATLAB.

[^1]
### 4.1 Numerical results for subdiffusion

In the subdiffusion case, we consider the following two examples:
(a) $v=x(1-x) \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$ and $f \equiv 0 ;$
(b) $v \equiv 0$ and $f(x, t)=\cos (t)\left(1+\chi_{(0,1 / 2)}(x)\right)$.

The numerical results for case (a) by the corrected scheme 2.4 are presented in Table 4 where the numbers in the bracket denote the theoretical rate predicted by Theorem 2.2 It converges steadily at an $O\left(\tau^{k}\right)$ rate for all BDFs, which agrees well with the theory, showing clearly its robustness. Surprisingly, the asymptotic convergence of BDF6 kicks in only at a relatively small time step size, at $N=50$, which contrasts sharply with other BDF schemes. Thus in the preasymptotic regime, BDF5 is preferred over BDF6. To further illustrate Theorem 2.2 , in Fig. 2, we plot the numerical solution by BDF5 and its error profile. The solution decays first rapidly and then slowly, resulting in an initial layer. This layer shows clearly the limited temporal regularity of the solution at 0 and as a result, the approximation error near 0 is predominant, partly confirming the prefactor $t_{n}^{\alpha-k}$ in Theorem 2.2.

Table 4: The $L^{2}$-norm error $e^{N}$ for case (a) at $t_{N}=1$, by the corrected scheme (2.4) with $h=1 / 100$.

| $\alpha$ | $k \backslash N$ | 50 | 100 | 200 | 400 | 800 | rate |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.25 | 2 | $5.66 \mathrm{e}-5$ | $1.39 \mathrm{e}-5$ | $3.46 \mathrm{e}-6$ | $8.64 \mathrm{e}-7$ | $2.16 \mathrm{e}-7$ | $\approx 2.00$ (2.00) |
|  | 3 | $2.29 \mathrm{e}-6$ | $2.76 \mathrm{e}-7$ | $3.39 \mathrm{e}-8$ | $4.20 \mathrm{e}-9$ | $5.23 \mathrm{e}-10$ | $\approx 3.01$ (3.00) |
|  | 4 | $1.42 \mathrm{e}-7$ | 8.33e-9 | $5.04 \mathrm{e}-10$ | $3.10 \mathrm{e}-11$ | $1.91 \mathrm{e}-12$ | $\approx 4.02$ (4.00) |
|  | 5 | $1.26 \mathrm{e}-8$ | $3.41 \mathrm{e}-10$ | $1.01 \mathrm{e}-11$ | $3.07 \mathrm{e}-13$ | $9.45 \mathrm{e}-15$ | $\approx 5.03$ (5.00) |
|  | 6 | $1.09 \mathrm{e}-5$ | $1.60 \mathrm{e}-9$ | $2.55 \mathrm{e}-13$ | $3.82 \mathrm{e}-15$ | 5.83e-17 | $\approx 6.04$ (6.00) |
| 0.5 | 2 | $1.74 \mathrm{e}-4$ | $4.30 \mathrm{e}-5$ | $1.07 \mathrm{e}-5$ | $2.65 \mathrm{e}-6$ | $6.62 \mathrm{e}-7$ | $\approx 2.00$ (2.00) |
|  | 3 | 7.73e-6 | $9.29 \mathrm{e}-7$ | $1.14 \mathrm{e}-7$ | $1.41 \mathrm{e}-8$ | $1.76 \mathrm{e}-9$ | $\approx 3.01$ (3.00) |
|  | 4 | $5.12 \mathrm{e}-7$ | $2.98 \mathrm{e}-8$ | $1.80 \mathrm{e}-9$ | $1.10 \mathrm{e}-10$ | $6.83 \mathrm{e}-12$ | $\approx 4.02$ (4.00) |
|  | 5 | $4.75 \mathrm{e}-8$ | $1.27 \mathrm{e}-9$ | $3.76 \mathrm{e}-11$ | $1.14 \mathrm{e}-12$ | $3.52 \mathrm{e}-14$ | $\approx 5.03$ (5.00) |
|  | 6 | $3.01 \mathrm{e}-5$ | $2.79 \mathrm{e}-9$ | $9.85 \mathrm{e}-13$ | $1.47 \mathrm{e}-14$ | $2.25 \mathrm{e}-16$ | $\approx 6.05$ (6.00) |
| 0.75 | 2 | $4.84 \mathrm{e}-4$ | $1.19 \mathrm{e}-4$ | $2.93 \mathrm{e}-5$ | $7.30 \mathrm{e}-6$ | $1.82 \mathrm{e}-6$ | $\approx 2.00$ (2.00) |
|  | 3 | $2.55 \mathrm{e}-5$ | $3.04 \mathrm{e}-6$ | $3.72 \mathrm{e}-7$ | $4.60 \mathrm{e}-8$ | $5.71 \mathrm{e}-9$ | $\approx 3.01$ (3.00) |
|  | 4 | $1.94 \mathrm{e}-6$ | $1.11 \mathrm{e}-7$ | $6.68 \mathrm{e}-9$ | $4.09 \mathrm{e}-10$ | $2.53 \mathrm{e}-11$ | $\approx 4.02$ (4.00) |
|  | 5 | $2.95 \mathrm{e}-7$ | $5.30 \mathrm{e}-9$ | $1.55 \mathrm{e}-10$ | $4.70 \mathrm{e}-12$ | $1.45 \mathrm{e}-13$ | $\approx 5.03$ (5.00) |
|  | 6 | $1.67 \mathrm{e}-3$ | $3.01 \mathrm{e}-7$ | $4.53 \mathrm{e}-12$ | $6.61 \mathrm{e}-14$ | $1.01 \mathrm{e}-15$ | $\approx 6.07$ (6.00) |

To illustrate the impact of initial correction, we present in Table 5 the numerical results by the uncorrected BDF scheme (2.3), and two popular finite difference schemes, i.e., L1 scheme [19] and L1-2 scheme [11, 24]. The uncorrected BDF $k$ scheme can only achieve a first-order convergence, and all BDF schemes have almost identical accuracy, irrespective of the order $k$. This low-order convergence is due to the poor approximation at the initial steps, which persists in the numerical solutions at later steps. Meanwhile, for sufficiently smooth solutions, the L1 and L1-2 schemes converge at a rate $O\left(\tau^{2-\alpha}\right)$ and $O\left(\tau^{3-\alpha}\right)$, respectively. For general problem data, the L1 scheme converges at an $O(\tau)$ rate [14]. The L1 and L1-2 schemes can only deliver an empirical $O(\tau)$ rate for case (a), due to insufficient solution regularity for general problem data. Although not presented, it is noted that the numerical results for other fractional orders are similarly. Therefore, the correction is necessary in order to retain the desired rate, even for smooth initial data.

Next we consider the inhomogeneous problem in case (b). Since the source term $f$ is smooth in time, Theorem 2.2 is applicable, which predicts an $O\left(\tau^{k}\right)$ rate for the corrected $\mathrm{BDF} k$ scheme $\sqrt{2.4}$. This is fully supported by the numerical results in Table 6 . Like before, the uncorrected scheme 2.3 and the L1 and L1-2 schemes can only achieve an $O(\tau)$ rate, despite the smoothness of the problem data, cf. Table 7.


Figure 2: Numerical solution and error profile for case (a), with $\alpha=0.5, h=1 / 100, \tau=0.002$ and BDF5.

Table 5: The $L^{2}$-norm error $e^{N}$ for case (a) at $t_{N}=1$, by the uncorrected scheme $(2.3)$ with $h=1 / 100$.

| $\alpha$ | $N$ | 50 | 100 | 200 | 400 | 800 | rate |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | BDF3 | $4.98 \mathrm{e}-3$ | $2.48 \mathrm{e}-3$ | $1.24 \mathrm{e}-3$ | $6.19 \mathrm{e}-4$ | $3.09 \mathrm{e}-4$ | $\approx 1.00(1.00)$ |
|  | BDF4 | $4.97 \mathrm{e}-3$ | $2.48 \mathrm{e}-3$ | $1.24 \mathrm{e}-3$ | $6.19 \mathrm{e}-4$ | $3.09 \mathrm{e}-4$ | $\approx 1.00(1.00)$ |
| 0.5 | BDF5 | $4.97 \mathrm{e}-3$ | $2.48 \mathrm{e}-3$ | $1.24 \mathrm{e}-3$ | $6.19 \mathrm{e}-4$ | $3.09 \mathrm{e}-4$ | $\approx 1.00(1.00)$ |
|  | BDF6 | $4.94 \mathrm{e}-3$ | $2.48 \mathrm{e}-3$ | $1.24 \mathrm{e}-3$ | $6.19 \mathrm{e}-4$ | $3.09 \mathrm{e}-4$ | $\approx 1.00(1.00)$ |
|  | L1 | $5.10 \mathrm{e}-3$ | $2.52 \mathrm{e}-3$ | $1.25 \mathrm{e}-3$ | $6.24 \mathrm{e}-4$ | $3.11 \mathrm{e}-4$ | $\approx 1.04(1.00)$ |
|  | L1-2 | $3.57 \mathrm{e}-3$ | $1.73 \mathrm{e}-3$ | $8.40 \mathrm{e}-4$ | $4.08 \mathrm{e}-4$ | $1.99 \mathrm{e}-4$ | $\approx 1.04(--)$ |

Table 6: The $L^{2}$-norm error $e^{N}$ for case (b) at $t_{N}=1$, by the corrected scheme (2.4) with $h=1 / 100$.

| $\alpha$ | $k \backslash N$ | 50 | 100 | 200 | 400 | 800 | rate |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.25 | 2 | $6.67 \mathrm{e}-6$ | $1.65 \mathrm{e}-6$ | 4.10e-7 | $1.02 \mathrm{e}-7$ | $2.55 \mathrm{e}-8$ | $\approx 2.00$ (2.00) |
|  | 3 | $2.68 \mathrm{e}-7$ | $3.20 \mathrm{e}-8$ | $3.91 \mathrm{e}-9$ | $4.83 \mathrm{e}-10$ | $6.00 \mathrm{e}-11$ | $\approx 3.01$ (3.00) |
|  | 4 | $2.14 \mathrm{e}-8$ | $1.25 \mathrm{e}-9$ | $7.57 \mathrm{e}-11$ | $4.65 \mathrm{e}-12$ | $2.88 \mathrm{e}-13$ | $\approx 4.02$ (4.00) |
|  | 5 | $1.90 \mathrm{e}-9$ | $5.11 \mathrm{e}-11$ | $1.51 \mathrm{e}-12$ | $4.61 \mathrm{e}-14$ | $1.42 \mathrm{e}-15$ | $\approx 5.03$ (5.00) |
|  | 6 | $1.63 \mathrm{e}-6$ | $2.40 \mathrm{e}-10$ | $3.79 \mathrm{e}-14$ | $5.68 \mathrm{e}-16$ | $8.67 \mathrm{e}-18$ | $\approx 6.05$ (6.00) |
| 0.5 | 2 | $1.76 \mathrm{e}-5$ | $4.35 \mathrm{e}-6$ | $1.08 \mathrm{e}-6$ | $2.70 \mathrm{e}-7$ | $6.62 \mathrm{e}-8$ | $\approx 2.00$ (2.00) |
|  | 3 | $6.35 \mathrm{e}-7$ | $7.56 \mathrm{e}-8$ | $9.22 \mathrm{e}-9$ | $1.14 \mathrm{e}-9$ | $1.42 \mathrm{e}-10$ | $\approx 3.01$ (3.00) |
|  | 4 | $5.23 \mathrm{e}-8$ | $3.03 \mathrm{e}-9$ | $1.83 \mathrm{e}-10$ | $1.12 \mathrm{e}-11$ | $6.95 \mathrm{e}-13$ | $\approx 4.02$ (4.00) |
|  | 5 | $4.94 \mathrm{e}-9$ | 1.33e-10 | $3.91 \mathrm{e}-12$ | $1.19 \mathrm{e}-13$ | $3.66 \mathrm{e}-15$ | $\approx 5.03$ (5.00) |
|  | 6 | $3.14 \mathrm{e}-6$ | 2.91e-10 | $1.02 \mathrm{e}-13$ | $1.52 \mathrm{e}-15$ | $2.32 \mathrm{e}-17$ | $\approx 6.05$ (6.00) |
| 0.75 | 2 | $3.03 \mathrm{e}-5$ | $7.47 \mathrm{e}-6$ | $1.86 \mathrm{e}-6$ | $4.63 \mathrm{e}-7$ | $1.16 \mathrm{e}-7$ | $\approx 2.00$ (2.00) |
|  | 3 | $1.10 \mathrm{e}-6$ | $1.31 \mathrm{e}-7$ | $1.59 \mathrm{e}-8$ | $1.96 \mathrm{e}-9$ | $2.43 \mathrm{e}-10$ | $\approx 3.01$ (3.00) |
|  | 4 | $9.98 \mathrm{e}-8$ | $5.72 \mathrm{e}-9$ | $3.43 \mathrm{e}-10$ | $2.10 \mathrm{e}-11$ | $1.30 \mathrm{e}-12$ | $\approx 4.02$ (4.00) |
|  | 5 | $1.57 \mathrm{e}-8$ | 2.81e-10 | $8.24 \mathrm{e}-12$ | $2.50 \mathrm{e}-13$ | $7.68 \mathrm{e}-15$ | $\approx 5.03$ (5.00) |
|  | 6 | $8.95 \mathrm{e}-5$ | $1.61 \mathrm{e}-8$ | $2.40 \mathrm{e}-13$ | $3.50 \mathrm{e}-15$ | $5.33 \mathrm{e}-17$ | $\approx 6.07$ (6.00) |

### 4.2 Numerical results for diffusion-wave

Now we illustrate the corrected scheme 3.3 on the following 1D diffusion-wave example:
(c) $v(x)=x(1-x), b(x)=\sin (2 \pi x)$ and $f=e^{t}\left(1+\chi_{(0,1 / 2)}(x)\right)$

Table 7: The $L^{2}$-norm error $e^{N}$ for case (b) at $t_{N}=1$, by the uncorrected scheme (2.3) with $h=1 / 100$.

| $\alpha$ | $N$ | 50 | 100 | 200 | 400 | 800 | rate |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | BDF2 | $5.14 \mathrm{e}-4$ | $2.57 \mathrm{e}-4$ | $1.29 \mathrm{e}-4$ | $6.45 \mathrm{e}-5$ | $3.22 \mathrm{e}-5$ | $\approx 1.00(1.00)$ |
|  | BDF3 | $5.19 \mathrm{e}-4$ | $2.59 \mathrm{e}-4$ | $1.29 \mathrm{e}-4$ | $6.45 \mathrm{e}-5$ | $3.23 \mathrm{e}-5$ | $\approx 1.00(1.00)$ |
|  | BDF4 | $5.18 \mathrm{e}-4$ | $2.59 \mathrm{e}-4$ | $1.29 \mathrm{e}-4$ | $6.45 \mathrm{e}-5$ | $3.23 \mathrm{e}-5$ | $\approx 1.00(1.00)$ |
| 0.5 | BDF5 | $5.19 \mathrm{e}-4$ | $2.59 \mathrm{e}-4$ | $1.29 \mathrm{e}-4$ | $6.45 \mathrm{e}-5$ | $3.23 \mathrm{e}-5$ | $\approx 1.00(1.00)$ |
|  | BDF6 | $5.15 \mathrm{e}-4$ | $2.59 \mathrm{e}-4$ | $1.29 \mathrm{e}-4$ | $6.45 \mathrm{e}-5$ | $3.23 \mathrm{e}-5$ | $\approx 1.00(1.00)$ |
|  | L1 | $5.98 \mathrm{e}-4$ | $2.86 \mathrm{e}-4$ | $1.39 \mathrm{e}-4$ | $6.80 \mathrm{e}-5$ | $3.35 \mathrm{e}-5$ | $\approx 1.02(1.00)$ |
|  | L1-2 | $3.71 \mathrm{e}-4$ | $1.80 \mathrm{e}-4$ | $8.76 \mathrm{e}-5$ | $4.25 \mathrm{e}-5$ | $2.07 \mathrm{e}-5$ | $\approx 1.04(--)$ |

For the diffusion-wave model, the scheme (3.3) is only conditionally stable for $\alpha \geq \alpha^{*}(k)=\pi /\left(\pi-\vartheta_{k}\right)$, with a stability threshold $\tau_{0}=(c(\alpha, k) / r(A))^{1 / \alpha}$, according to Condition 3.1. To illustrate the sharpness of the threshold $\tau_{0}$ or equivalently the CFL constant $c(\alpha, k)$, we consider case (c) with $k=5, \alpha=1.5$, $h=1 / M=1 / 100$. The eigenvalues of the discrete Laplacian $A$ are available in closed form [13]:

$$
\lambda_{j}^{h}=\bar{\lambda}_{j}^{h} /\left(1-\frac{h^{2}}{6} \bar{\lambda}_{j}^{h}\right), \quad \text { with } \bar{\lambda}_{j}^{h}=-\frac{4}{h^{2}} \sin ^{2} \frac{\pi j}{2(N+1)}, \quad j=1,2, \ldots, M-1 .
$$

Thus the numerical radius $r(A)=\max _{j}\left(\lambda_{j}^{h}\right) \approx 1.2 \times 10^{5}$, which together with the value $c(\alpha, k)=1.58$ from Fig. 1 gives a stability threshold $\tau_{0} \approx 5.60 \times 10^{-4}$. In Figs. 3 (a) and (b), we plot the numerical solutions computed by the corrected scheme (3.3) with $N=1700$ (i.e., $\tau=5.88 \times 10^{-4}$ ) and with $N=1800$ (i.e., $\tau=5.55 \times 10^{-4}$ ), respectively. The scheme $(3.3)$ gives an unstable solution for $N=1700$ but a stable one for $N=1800$. This observation fully confirms the sharpness of the CFL constant $c(\alpha, k)$ in Condition 3.1. In Table 8, we present the $L^{2}$ error for $\alpha>\alpha^{*}$ and small $\tau$ (such that it satisfies the CFL condition). The numerical results indicate the desired $O\left(\tau^{k}\right)$ rate, supporting the theory.

For $\alpha<\alpha^{*}(k)=\pi /\left(\pi-\vartheta_{k}\right)$, with $\alpha^{*}$ being the critical value, the corrected scheme (3.3) based on $\mathrm{BDF} k$ is unconditionally stable. Numerically, the corrected scheme 3.3) converges at an $O\left(\tau^{k}\right)$ rate steadily, cf. Table 9, which agrees well with Theorem 3.2.


Figure 3: The numerical solutions for case (c) at $t=1$, with $N=1700\left(\tau=5.88 \times 10^{-4}\right)$ and $N=1800$ $\left(\tau=5.56 \times 10^{-4}\right), h=1 / 100$. The theoretical stability threshold is $\tau_{0}=5.60 \times 10^{-4}$.

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Table 8: The $L^{2}$-norm error $e^{N}$ for case (c) at $t_{N}=1$, by the corrected scheme (3.3), with $h=1 / 10$.

| $k\left(\alpha^{*}\right)$ | $N$ | 100 | 200 | 400 | 800 | 1600 | rate |
| :---: | :---: | :--- | :--- | :--- | :--- | :--- | :---: |
| $3(1.91)$ | 1.95 | $2.96 \mathrm{e}-5$ | $3.84 \mathrm{e}-6$ | $5.00 \mathrm{e}-7$ | $6.40 \mathrm{e}-8$ | $8.27 \mathrm{e}-9$ | $\approx 2.96(3.00)$ |
| $4(1.68)$ | 1.75 | $2.08 \mathrm{e}-6$ | $1.43 \mathrm{e}-7$ | $9.29 \mathrm{e}-9$ | $5.92 \mathrm{e}-10$ | $3.74 \mathrm{e}-11$ | $\approx 3.98(4.00)$ |
| $5(1.40)$ | 1.5 | $7.29 \mathrm{e}-8$ | $2.49 \mathrm{e}-10$ | $6.22 \mathrm{e}-12$ | $1.72 \mathrm{e}-13$ | $5.05 \mathrm{e}-15$ | $\approx 5.14(5.00)$ |
| $6(1.11)$ | 1.5 | $5.67 \mathrm{e}-2$ | $2.56 \mathrm{e}-10$ | $6.88 \mathrm{e}-13$ | $1.05 \mathrm{e}-14$ | $1.62 \mathrm{e}-16$ | $\approx 6.03(6.00)$ |

Table 9: The $L^{2}$-norm error $e^{N}$ for case (c) at $t_{N}=1$, by the corrected scheme (3.3), $h=1 / 100$.

| $k\left(\alpha^{*}\right)$ | $N$ | 100 | 200 | 400 | 800 | 1600 | rate |
| :---: | :---: | :--- | :--- | :--- | :--- | :--- | :---: |
|  | 1.25 | $2.34 \mathrm{e}-5$ | $5.85 \mathrm{e}-6$ | $1.46 \mathrm{e}-6$ | $3.65 \mathrm{e}-7$ | $9.14 \mathrm{e}-8$ | $\approx 2.00(2.00)$ |
| $2(2.00)$ | 1.5 | $6.87 \mathrm{e}-5$ | $1.69 \mathrm{e}-5$ | $4.18 \mathrm{e}-6$ | $1.04 \mathrm{e}-6$ | $2.59 \mathrm{e}-7$ | $\approx 2.00(2.00)$ |
|  | 1.75 | $3.15 \mathrm{e}-4$ | $8.55 \mathrm{e}-5$ | $2.21 \mathrm{e}-5$ | $5.62 \mathrm{e}-6$ | $1.42 \mathrm{e}-6$ | $\approx 1.98(2.00)$ |
| $3(1.91)$ | 1.25 | $1.54 \mathrm{e}-8$ | $1.66 \mathrm{e}-9$ | $3.20 \mathrm{e}-10$ | $4.80 \mathrm{e}-11$ | $6.33 \mathrm{e}-12$ | $\approx 3.00(3.00)$ |
|  | 1.5 | $4.22 \mathrm{e}-6$ | $5.12 \mathrm{e}-7$ | $6.30 \mathrm{e}-8$ | $7.82 \mathrm{e}-9$ | $9.74 \mathrm{e}-10$ | $\approx 3.00(3.00)$ |
|  | 1.75 | $5.27 \mathrm{e}-5$ | $6.43 \mathrm{e}-6$ | $7.93 \mathrm{e}-7$ | $9.78 \mathrm{e}-8$ | $1.15 \mathrm{e}-8$ | $\approx 3.02(3.00)$ |
| $4(1.68)$ | 1.25 | $2.74 \mathrm{e}-8$ | $1.64 \mathrm{e}-9$ | $1.00 \mathrm{e}-10$ | $6.20 \mathrm{e}-12$ | $3.63 \mathrm{e}-13$ | $\approx 4.00(4.00)$ |
|  | 1.5 | $1.88 \mathrm{e}-7$ | $1.27 \mathrm{e}-8$ | $8.22 \mathrm{e}-10$ | $5.19 \mathrm{e}-11$ | $3.07 \mathrm{e}-12$ | $\approx 4.00(4.00)$ |
| $5(1.40)$ | 1.1 | $3.3 \mathrm{e}-10$ | $9.52 \mathrm{e}-12$ | $2.85 \mathrm{e}-13$ | $8.71 \mathrm{e}-15$ | $2.69 \mathrm{e}-16$ | $\approx 5.00(5.00)$ |
|  | 1.3 | $2.38 \mathrm{e}-7$ | $1.28 \mathrm{e}-10$ | $1.08 \mathrm{e}-12$ | $3.40 \mathrm{e}-14$ | $1.06 \mathrm{e}-15$ | $\approx 5.00(5.00)$ |
| $6(1.11)$ | 1.05 | $3.31 \mathrm{e}-5$ | $1.94 \mathrm{e}-7$ | $1.28 \mathrm{e}-10$ | $7.58 \mathrm{e}-17$ | $7.39 \mathrm{e}-19$ | $\approx 6.68(6.00)$ |

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## A An alternative view on the correction scheme (2.4)

In this appendix, we discuss the connection between our approach and the approach studied in 22. The observation of this connection is due to Professor Christian Lubich.

For the following integral and its convolution quadrature approximation

$$
\begin{equation*}
u(t)=\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}} F(z) e^{-t z} d z \quad \text { and } \quad U^{n}=\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau}} \tau F\left(\delta_{\tau}\left(e^{-\tau z}\right)\right) e^{-t_{n} z} d z, \tag{A.1}
\end{equation*}
$$

Lubich [22, Theorem 2.1] showed the following error estimate away from $t=0$ :

$$
\begin{equation*}
\left|U^{n}-u\left(t_{n}\right)\right| \leq C t_{n}^{\nu-k-1} \tau^{k}, \tag{A.2}
\end{equation*}
$$

where $\nu \in \mathbb{R}$ is a parameter in the kernel estimate $\left|\frac{d^{m}}{d z^{m}} F(z)\right| \leq C|z|^{-\nu-m}, m \geq 0$. In particular, if we choose $F(z)=\left(z^{\alpha}-A\right)^{-1} z^{-\ell-1} \partial_{t}^{\ell} f(0)$ in A.11, then

$$
\begin{gathered}
u(t)=\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}}\left(z^{\alpha}-A\right)^{-1} z^{-\ell-1} \partial_{t}^{\ell} f(0) e^{-t z} d z, \\
U^{n}=\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau}} \tau\left(\delta_{\tau}\left(e^{-\tau z}\right)^{\alpha}-A\right)^{-1} \delta_{\tau}\left(e^{-\tau z}\right)^{-\ell-1} \partial_{t}^{\ell} f(0) e^{-t_{n} z} d z
\end{gathered}
$$

and
are the integral representations of the solutions of

$$
\begin{array}{lll}
\partial_{t}^{\alpha} u(t)-A u(t)=\frac{t^{\ell}}{\ell!} \partial_{t}^{\ell} f(0), & \text { with } & u(0)=0 \\
\bar{\partial}_{\tau}^{\alpha} U^{n}-A U^{n}=\omega_{n}^{(\ell)} \partial_{t}^{\ell} f(0), & \text { with } \quad U^{0}=0, \tag{A.4}
\end{array}
$$

respectively, which are solutions and approximations of 1.1 corresponding to a single component in the source splitting (2.5). The weights $\left\{\omega_{n}^{(\ell)}\right\}_{n=0}^{\infty}$ are the coefficients in the power series expansion $\delta_{\tau}(\zeta)^{-\ell-1}=\sum_{n=0}^{\infty} \omega_{n}^{(\ell)} \zeta^{n}$. By [22, Theorem 2.1], the approximation $\left\{U^{n}\right\}$ has the desired accuracy A.2). Our scheme 2.4 is connected to A.4 as follows: we replace $\delta(\zeta)^{-\ell-1}$ by an $O\left(|\zeta-1|^{k-\ell-1}\right)$ close approximation $\frac{\gamma_{\ell}(\zeta)}{\ell!}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} \zeta^{j}$, cf. 2.14). Our choice of the kernel leads to

$$
\begin{equation*}
\bar{\partial}_{\tau}^{\alpha} U^{n}-A U^{n}=\frac{t_{n}^{\ell}}{\ell!} \partial_{t}^{\ell} f(0)+b_{\ell, n}^{(k)} \tau^{\ell} \partial_{t}^{\ell} f(0) \tag{A.5}
\end{equation*}
$$

which formally differs from A.3 only in the starting $k-1$ steps, as $b_{\ell, n}^{(k)}=0$ for $n \geq k$ in our scheme. Further, A.5 is minimal (or optimal) in the sense that it is the unique correction scheme that only modifies the starting $k-1$ steps while having an accuracy of $O\left(\tau^{k}\right)$.

## B Proof of Theorem 2.1

We need the following estimates on the function $\delta_{\tau}\left(e^{-z \tau}\right)$.
Lemma B.1. Let $\alpha \in(0,2)$. For any $\varepsilon$, there exists $\theta_{\varepsilon} \in(\pi / 2, \pi)$ such that for any fixed $\theta \in\left(\pi / 2, \theta_{\varepsilon}\right)$, there exist positive constants $c, c_{1}, c_{2}$ (independent of $\tau$ ) such that

$$
\begin{array}{ll}
c_{1}|z| \leq\left|\delta_{\tau}\left(e^{-z \tau}\right)\right| \leq c_{2}|z|, & \delta_{\tau}\left(e^{-z \tau}\right) \in \Sigma_{\pi-\vartheta_{k}+\varepsilon} \\
\left|\delta_{\tau}\left(e^{-z \tau}\right)-z\right| \leq c \tau^{k}|z|^{k+1}, & \left|\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-z^{\alpha}\right| \leq c \tau^{k}|z|^{k+\alpha}, \quad \forall z \in \Gamma_{\theta, \delta}^{\tau}
\end{array}
$$

Proof. Since the function $\delta(\zeta) /(1-\zeta)$ has no zero in a neighborhood $\mathcal{N}$ of the unit circle 4. Proof of Lemma 2] and for $\theta$ sufficiently close to $\pi / 2, e^{-z \tau}$ lies in the neighborhood $\mathcal{N}$, there are positive constants $c_{1}^{\prime}$ and $c_{2}^{\prime}$ such that

$$
c_{1}^{\prime} \leq \frac{\left|\delta\left(e^{-z \tau}\right)\right|}{\left|1-e^{-z \tau}\right|}=\frac{\left|\delta_{\tau}\left(e^{-z \tau}\right)\right|}{\left|\left(1-e^{-z \tau}\right) / \tau\right|} \leq c_{2}^{\prime}, \quad \forall z \in \Gamma_{\theta, \delta}^{\tau}
$$

Since $\widetilde{c}_{1}|z \tau| \leq\left|1-e^{-z \tau}\right| \leq \widetilde{c}_{2}|z \tau|$ for $z \in \Gamma_{\theta, \delta}^{\tau}$, the first estimate follows.
When $|\zeta| \leq 1$ and $\zeta \neq 0$, we have $\delta_{\tau}(\zeta) \in \Sigma_{\pi-\vartheta_{k}}$ for the $A\left(\vartheta_{k}\right)$ stable BDF $k$ 12. Hence, by expressing $e^{-z \tau}$ as $e^{-|z| \tau \cos (\theta)} e^{-\mathrm{i}|z| \tau \sin (\theta)}$, we have

$$
\begin{aligned}
\left|\delta_{\tau}\left(e^{-z \tau}\right)-\delta_{\tau}\left(e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)\right| & =\left|\delta_{\tau}\left(e^{-|z| \tau \cos (\theta)} e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)-\delta_{\tau}\left(e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)\right| \\
& \leq c e^{-\sigma|z| \tau \cos (\theta)}\left|\delta_{\tau}^{\prime}\left(e^{-\sigma|z| \tau \cos (\theta)} e^{-\mathrm{i}|z| \tau \sin (\theta)}\right) z \tau \cos (\theta)\right|
\end{aligned}
$$

for some $\sigma \in(0,1)$, by the mean value theorem. For $\theta$ close to $\pi / 2$ and $z \in \Gamma_{\theta, \delta}^{\tau}$, by Taylor expansion, $|z| \tau \leq \pi / \sin \theta$ and the first estimate, we have

$$
\tau\left|\delta_{\tau}^{\prime}\left(e^{-\sigma|z| \tau \cos (\theta)} e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)\right| \leq c \quad \text { and } \quad\left|\delta_{\tau}\left(e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)\right| \geq c|z|
$$

Consequently, we deduce

$$
\begin{align*}
\left|\delta_{\tau}\left(e^{-|z| \tau \cos (\theta)} e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)-\delta_{\tau}\left(e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)\right| & \leq c|\cos (\theta)|\left|\delta_{\tau}\left(e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)\right| \\
& \leq c|\theta-\pi / 2|\left|\delta_{\tau}\left(e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)\right| \tag{B.1}
\end{align*}
$$

Hence, $\delta_{\tau}\left(e^{-\tau z}\right)$ is in a sector $\Sigma_{\pi-\vartheta_{k}+c|\theta-\pi / 2|}$. If $\theta>\pi / 2$ is sufficiently close to $\pi / 2$, then $c|\theta-\pi / 2|<\varepsilon$. This proves the second estimate.

The third estimate is given in [34, eq. (10.6)]. The last estimate follows from

$$
\begin{equation*}
\left|\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-z^{\alpha}\right|=\alpha\left|\int_{z}^{\delta_{\tau}\left(e^{-z \tau}\right)} \xi^{\alpha-1} d \xi\right| \leq \max _{\xi}|\xi|^{\alpha-1}\left|\delta_{\tau}\left(e^{-z \tau}\right)-z\right| \tag{B.2}
\end{equation*}
$$

where $\xi$ lies in the line segment with end points $\delta_{\tau}\left(e^{-z \tau}\right)$ and $z$. Since $\Im \delta_{\tau}\left(e^{-z \tau}\right)>0$ for $z \in \Gamma_{\theta, \delta}^{\tau}$ with $\Im z>0$, we have by the first estimate that

$$
|\xi|^{\alpha-1} \leq \max \left(|z|,\left|\delta_{\tau}\left(e^{-z \tau}\right)\right|\right)^{\alpha-1} \leq c|z|^{\alpha-1}
$$

This inequality and $(\overline{B .2}$ yield the last estimate.
Proof of Theorem 2.1. The functions $W^{n}, n=1, \ldots, N$, satisfy (with $W^{0}=0$ ):

$$
\begin{array}{ll}
\bar{\partial}_{\tau}^{\alpha} W^{n}-A W^{n}=\left(1+a_{n}^{(k)}\right)(A v+f(0))+\sum_{\ell=1}^{k-2}\left(\frac{t_{n}^{\ell}}{\ell!}+b_{\ell, n}^{(k)} \tau^{\ell}\right) \partial_{t}^{\ell} f(0)+R_{k}\left(t_{n}\right), & 1 \leq n \leq k-1 \\
\bar{\partial}_{\tau}^{\alpha} W^{n}-A W^{n}=A v+f(0)+\sum_{\ell=1}^{k-2} \frac{t_{n}^{\ell}}{\ell!} \partial_{t}^{\ell} f(0)+R_{k}\left(t_{n}\right), & k \leq n \leq N
\end{array}
$$

By multiplying both sides by $\zeta^{n}$, summing over $n$ and collecting terms, we obtain

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \zeta^{n} \bar{\partial}_{\tau}^{\alpha} W^{n}-\sum_{n=1}^{\infty} A W^{n} \zeta^{n} \\
& =\left(\sum_{n=1}^{\infty} \zeta^{n}+\sum_{j=1}^{k-1} a_{j}^{(k)} \zeta^{j}\right)(A v+f(0))+\sum_{\ell=1}^{k-2}\left(\sum_{n=1}^{\infty} \frac{t_{n}^{\ell}}{\ell!} \zeta^{n}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} \tau^{\ell} \zeta^{j}\right) \partial_{t}^{\ell} f(0)+\widetilde{R}_{k}(\zeta) \\
& =\left(\frac{\zeta}{1-\zeta}+\sum_{j=1}^{k-1} a_{j}^{(k)} \zeta^{j}\right)(A v+f(0))+\sum_{\ell=1}^{k-2}\left(\frac{\gamma_{\ell}(\zeta)}{\ell!}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} \zeta^{j}\right) \tau^{\ell} \partial_{t}^{\ell} f(0)+\widetilde{R}_{k}(\zeta)
\end{aligned}
$$

where $\widetilde{R}_{k}(\zeta)=\sum_{n=1}^{\infty} R_{k}\left(t_{n}\right) \zeta^{n}$ and elementary identities

$$
\begin{equation*}
\sum_{n=1}^{\infty} \zeta^{n}=\frac{\zeta}{1-\zeta} \quad \text { and } \quad \sum_{n=1}^{\infty} n^{\ell} \zeta^{n}=\left(\zeta \frac{d}{d \zeta}\right)^{\ell} \frac{1}{1-\zeta}:=\gamma_{\ell}(\zeta) \tag{B.3}
\end{equation*}
$$

Next we simplify the summations on both sides. Since $W^{0}=0$, by the convolution rule, $\sum_{n=1}^{\infty} \zeta^{n} \bar{\partial}_{\tau}^{\alpha} W^{n}=$ $\delta_{\tau}(\zeta)^{\alpha} \widetilde{W}(\zeta)$, and consequently, we obtain

$$
\widetilde{W}(\zeta)=K\left(\delta_{\tau}(\zeta)\right)\left[\tau^{-1} \mu(\zeta)(A v+f(0))+\sum_{\ell=1}^{k-2} \delta_{\tau}(\zeta)\left(\frac{\gamma_{\ell}(\zeta)}{\ell!}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} \zeta^{j}\right) \tau^{\ell} \partial_{t}^{\ell} f(0)+\delta_{\tau}(\zeta) \widetilde{R}_{k}(\zeta)\right]
$$

where the operator $K$ is given by (2.9), and the two polynomials $\mu(\zeta)$ and $\gamma_{\ell}(\zeta)$ are given by 2.12 . Since $\widetilde{W}(\zeta)$ is analytic with respect to $\zeta$ in the unit disk on the complex plane, thus Cauchy's integral formula implies the following representation for arbitrary $\varrho \in(0,1)$

$$
\begin{equation*}
W^{n}=\frac{1}{2 \pi \mathrm{i}} \int_{|\zeta|=\varrho} \zeta^{-n-1} \widetilde{W}(\zeta) d \zeta=\frac{\tau}{2 \pi \mathrm{i}} \int_{\Gamma^{\tau}} e^{z t_{n}} \widetilde{W}\left(e^{-z \tau}\right) d z \tag{B.4}
\end{equation*}
$$

where the second equality follows from the change of variable $\zeta=e^{-z \tau}$, and $\Gamma^{\tau}$ is given by

$$
\Gamma^{\tau}:=\{z=-\ln (\varrho) / \tau+\mathrm{i} y: y \in \mathbb{R} \text { and }|y| \leq \pi / \tau\}
$$

Note that
(1) $\eta(\zeta):=\delta_{\tau}(\zeta) /(1-\zeta)$ is a polynomial without roots in a neighborhood $\mathcal{N}$ of the unit circle [4]. Thus, $\eta(\zeta)^{\alpha}$ is analytic in $\mathcal{N}$.
(2) By choosing the angle $\theta$ sufficiently close to $\pi / 2, \varrho$ sufficiently close to 1 and $0<\delta<-\ln (\varrho / \tau)$, the function $e^{-\tau z}$ lies in $\mathcal{N}$ for

$$
z \in \Sigma_{\theta, \delta}^{\tau}=\left\{z \in \Sigma_{\theta}:|z| \geq \delta,|\operatorname{Im}(z)| \leq \tau / \pi, \operatorname{Re}(z) \leq-\ln (\varrho) / \tau\right\}
$$

(3) $\left(1-e^{-\tau z}\right)^{\alpha}$ is analytic for $z \in \mathbb{C} \backslash(-\infty, 0] \supset \Sigma_{\theta, \delta}^{\tau}$.

These properties ensure that $\delta_{\tau}\left(e^{-\tau z}\right)^{\alpha}=\tau^{-\alpha}\left(1-e^{-\tau z}\right)^{\alpha} \eta\left(e^{-\tau z}\right)^{\alpha}$ is analytic for $z \in \Sigma_{\theta, \delta}^{\tau}$. By choosing $\varepsilon$ small enough, Lemma B.1 implies that $0 \neq \delta_{\tau}\left(e^{-\tau z}\right)^{\alpha} \in \Sigma_{\alpha\left(\vartheta_{k}+\varepsilon\right)} \subset \Sigma_{\pi-\varepsilon}$ for $z \in \Sigma_{\theta, \delta}^{\tau}$. Thus $K\left(\delta_{\tau}\left(e^{-\tau z}\right)\right)=\delta_{\tau}\left(e^{-\tau z}\right)^{-1}\left(\delta_{\tau}\left(e^{-\tau z}\right)^{\alpha}-A\right)^{-1}$ is analytic for $z \in \Sigma_{\theta, \delta}^{\tau}$, which is a region enclosed by $\Gamma^{\tau}$, $\Gamma_{\theta, \delta}^{\tau}$ and the two lines $\Gamma_{ \pm}^{\tau}:=\mathbb{R} \pm \mathrm{i} \pi / \tau$ (oriented from left to right). Since the values of $e^{z t_{n}} \widetilde{W}\left(e^{-z \tau}\right)$ on $\Gamma_{ \pm}^{\tau}$ coincide, Cauchy's theorem allows deforming the contour $\Gamma^{\tau}$ to $\Gamma_{\theta, \delta}^{\tau}$ in the integral (B.4) to obtain the desired representation.

## C Proof of Theorem 2.2

Lemma C.1. Let Criteria 2.13 and 2.14 hold. Then for $z \in \Gamma_{\theta, \delta}^{\tau}$, there hold

$$
\begin{align*}
\left\|\mu\left(e^{-z \tau}\right) K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right)-K(z)\right\| & \leq c \tau^{k}|z|^{k-1-\alpha}  \tag{C.1}\\
\left\|\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1}\left(\frac{1}{\ell!} \gamma_{\ell}\left(e^{-z \tau}\right)+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} e^{-j z \tau}\right) \tau^{\ell+1}-z^{-\ell} K(z)\right\| & \leq c \tau^{k}|z|^{k-\ell-1-\alpha} \tag{C.2}
\end{align*}
$$

Proof. Since $\left|1-e^{-z \tau}\right| \leq c \tau|z|$ for $z \in \Gamma_{\theta, \delta}^{\tau}$, by Criterion 2.13 , there holds $\left|\mu\left(e^{-z \tau}\right)-1\right| \leq c \mid 1-$ $\left.e^{-z \tau}\right|^{k} \leq c \tau^{k}|z|^{k}$. Meanwhile, by the triangle inequality, we have

$$
\begin{aligned}
\left\|K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right)-K(z)\right\|= & \left\|\delta_{\tau}\left(e^{-z \tau}\right)^{-1}\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1}-z^{-1}\left(z^{\alpha}-A\right)^{-1}\right\| \\
\leq \mid & \delta_{\tau}\left(e^{-z \tau}\right)^{-1}-z^{-1} \mid\left\|\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1}\right\| \\
& +|z|^{-1}\left\|\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1}-\left(z^{\alpha}-A\right)^{-1}\right\|
\end{aligned}
$$

The identity $\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1}-\left(z^{\alpha}-A\right)^{-1}=\left(z^{\alpha}-\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}\right)\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1}\left(z^{\alpha}-A\right)^{-1}$, Lemma B. 1 and the resolvent estimate (2.10) imply directly $\left\|K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right)-K(z)\right\| \leq c|\tau|^{k}|z|^{k-1-\alpha}$. Consequently, we obtain the estimate C.1) by

$$
\begin{aligned}
\| \mu\left(e^{-z \tau}\right) K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right) & -K(z)\left\|\leq\left|\mu\left(e^{-z \tau}\right)-1\right|\right\| K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right) \| \\
& +\left\|K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right)-K(z)\right\| \leq c \tau^{k}|z|^{k-1-\alpha} \quad \forall z \in \Gamma_{\theta, \delta}^{\tau}
\end{aligned}
$$

Next we show the estimate (C.2). By Lemma B.1, there holds

$$
\left|\delta_{\tau}\left(e^{-z \tau}\right)^{\ell+1}-z^{\ell+1}\right| \leq c\left|\delta_{\tau}\left(e^{-z \tau}\right)-z\right||z|^{\ell} \leq c \tau^{k}|z|^{k+\ell+1} \quad \forall z \in \Gamma_{\theta, \delta}^{\tau}
$$

By Criterion 2.14, there holds

$$
\left|\frac{\gamma_{\ell}\left(e^{-z \tau}\right)}{\ell!}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} e^{-j z \tau}-\frac{1}{\delta\left(e^{-z \tau}\right)^{\ell+1}}\right| \leq c \tau^{k-\ell-1}|z|^{k-\ell-1} \quad \forall z \in \Gamma_{\theta, \delta}^{\tau}
$$

Consequently, for any $z \in \Gamma_{\theta, \delta}^{\tau}$, we have

$$
\begin{aligned}
& \left\|\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1}\left(\frac{1}{\ell!} \gamma_{\ell}\left(e^{-z \tau}\right)+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} e^{-j z \tau}\right) \tau^{\ell+1}-z^{-\ell} K(z)\right\| \\
& \leq\left\|\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1}\left[\left(\frac{1}{\ell!} \gamma_{\ell}\left(e^{-z \tau}\right)+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} e^{-j z \tau}\right) \tau^{\ell+1}-\delta_{\tau}\left(e^{-z \tau}\right)^{-\ell-1}\right]\right\| \\
& \quad+\left\|\delta_{\tau}\left(e^{-z \tau}\right)^{-\ell} K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right)-z^{-\ell} K(z)\right\| \leq c \tau^{k}|z|^{k-\ell-1-\alpha} .
\end{aligned}
$$

This completes the proof of the lemma.

Proof of Theorem 2.2. By (2.8) and 2.11, we split $U^{n}-u\left(t_{n}\right)=W^{n}-w\left(t_{n}\right)$ into

$$
W^{n}-w\left(t_{n}\right)=I_{1}+\sum_{\ell=1}^{k-2} I_{2, \ell}-I_{3}+I_{4}
$$

where the terms $I_{1}, \ldots, I_{4}$ are given by

$$
\begin{aligned}
I_{1} & =\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau}} e^{z t_{n}}\left(\mu\left(e^{-z \tau}\right) K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right)-K(z)\right)(A v+f(0)) d z \\
I_{2, \ell} & =\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau}} e^{z t_{n}}\left[\delta_{\tau}\left(e^{-z \tau}\right)\left(\frac{\gamma_{\ell}\left(e^{-z \tau}\right)}{\ell!}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} e^{-z \tau j}\right) \tau^{\ell+1} K\left(\delta_{\tau}\left(e^{-z \tau}\right)\right)-z^{-\ell} K(z)\right] \partial_{t}^{\ell} f(0) d z \\
I_{3} & =\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta} \backslash \Gamma_{\theta, \delta}^{\tau}} e^{z t_{n}} K(z)\left(A v+f(0)+z^{-\ell} \sum_{j=1}^{k-1} \partial_{t}^{\ell} f(0)\right) d z \\
I_{4} & =\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau}} e^{z t}\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1} \tau \widetilde{R}_{k}\left(e^{-z \tau}\right) d z-\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}}\left(z^{\alpha}-A\right)^{-1} \widehat{R}_{k}(z) d z
\end{aligned}
$$

It suffices to bound these terms separately. By Lemma C.1. and choosing $\delta=t_{n}^{-1}$ in the contour $\Gamma_{\theta, \delta}^{\tau}$, we bound the first term $I_{1}$ by

$$
\begin{aligned}
\left\|I_{1}\right\|_{L^{2}(\Omega)} & \leq c \tau^{k}\|A v+f(0)\|_{L^{2}(\Omega)}\left(\int_{\delta}^{\pi /(\tau \sin \theta)} e^{r t_{n} \cos \theta} r^{k-1-\alpha} d r+\int_{-\theta}^{\theta} e^{\delta t_{n}|\cos \psi|} \delta^{k-\alpha} d \psi\right) \\
& \leq c \tau^{k}\left(t_{n}^{\alpha-k}+\delta^{k-\alpha}\right)\|A v+f(0)\|_{L^{2}(\Omega)} \leq c \tau^{k} t_{n}^{\alpha-k}\|A v+f(0)\|_{L^{2}(\Omega)} .
\end{aligned}
$$

By appealing to Lemma C. 1 again and choosing $\delta=t_{n}^{-1}$ in $\Gamma_{\theta, \delta}^{\tau}$, we bound the terms $I_{2, \ell}$ by

$$
\begin{aligned}
\left\|I_{2, \ell}\right\|_{L^{2}(\Omega)} & \leq c \tau^{k}\left\|f^{(\ell)}\right\|_{L^{2}(\Omega)}\left(\int_{\delta}^{\pi /(\tau \sin \theta)} e^{r t_{n} \cos \theta} r^{k-\ell-1-\alpha} d r+\int_{-\theta}^{\theta} e^{\delta t_{n}|\cos \psi|} \delta^{k-\ell-\alpha} d \psi\right) \\
& \leq c \tau^{k} t_{n}^{\alpha+\ell-k}\left\|\partial_{t}^{\ell} f(0)\right\|_{L^{2}(\Omega)}, \quad \ell=1,2 \ldots, k-1
\end{aligned}
$$

Direct computation yields the following estimate on $I_{3}$ :

$$
\left\|I_{3}\right\|_{L^{2}(\Omega)} \leq c \tau^{k}\left(t_{n}^{\alpha-k}\|A v+f(0)\|_{L^{2}(\Omega)}+\sum_{\ell=1}^{k-2} t_{n}^{\alpha+\ell-k}\left\|\partial_{t}^{\ell} f(0)\right\|_{L^{2}(\Omega)}\right)
$$

The term $I_{4}$ is the error of the numerical solution with a compatible right-hand side $R_{k}$. Upon recalling the definition of $R_{k}$ in (2.6), we use the splitting $R_{k}=\frac{t^{k-1}}{(k-1)!} \partial_{t}^{k-1} f(0)+\frac{t^{k-1}}{(k-1)!} * \partial_{t}^{k} f(t)=: R_{k}^{1}+R_{k}^{2}$. Then we have $I_{4}=I_{4}^{1}+I_{4}^{2}$ with

$$
I_{4}^{i}=\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}^{\tau}} e^{z t_{n}}\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1} \tau \widetilde{R}_{k}^{i}\left(e^{-z \tau}\right) d z-\frac{1}{2 \pi \mathrm{i}} \int_{\Gamma_{\theta, \delta}} e^{z t_{n}}\left(z^{\alpha}-A\right)^{-1} \widehat{R}_{k}^{i}(z) d z
$$

By repeating the preceding argument and (2.17), we have the estimate for $I_{4}^{1}$ :

$$
\left\|I_{4}^{1}\right\|_{L^{2}(\Omega)} \leq c \tau^{k} t_{n}^{\alpha-1}\left\|\partial_{t}^{k-1} f(0)\right\|_{L^{2}(\Omega)}
$$

and using the argument in [16, Lemma 3.7],

$$
\left\|I_{4}^{2}\right\|_{L^{2}(\Omega)} \leq c \tau^{k} \int_{0}^{t_{n}}\left(t_{n}-s\right)^{\alpha-1}\left\|\partial_{s}^{k} f(s)\right\|_{L^{2}(\Omega)} d s
$$

This completes the proof of the theorem.

## D Proof of Theorem 3.1

Using the splitting (3.4), the functions $W^{n}, n=1, \ldots, N$, satisfy ( with $W^{0}=0$ ):

$$
\begin{array}{lr}
\bar{\partial}_{\tau}^{\alpha} W^{n}-A W^{n}=\left(1+a_{n}^{(k)}\right) A v+\left(t_{n}+\tau c_{n}^{(k)}\right) A b+\sum_{\ell=1}^{k-2}\left(\frac{\bar{\partial}_{\tau} t_{n}^{\ell}}{\ell!}+b_{\ell, n}^{(k)} \tau^{\ell-1}\right) \partial_{t}^{\ell-1} f(0)+\bar{\partial}_{\tau} R_{k}\left(t_{n}\right) \\
\bar{\partial}_{\tau}^{\alpha} W^{n}-A W^{n}=A v+t_{n} A b+\sum_{\ell=1}^{k-2} \frac{\bar{\partial}_{\tau} t_{n}^{\ell}}{\ell!} \partial_{t}^{\ell-1} f(0)+\bar{\partial}_{\tau} R_{k}\left(t_{n}\right), & k \leq n \leq 1
\end{array}
$$

By multiplying both sides by $\zeta^{n}$ and summing over $n$, we obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty} \zeta^{n} \bar{\partial}_{\tau}^{\alpha} W^{n}-\sum_{n=1}^{\infty} A W^{n} \zeta^{n}= & \left(\sum_{n=1}^{\infty} \zeta^{n}+\sum_{j=1}^{k-1} a_{j}^{(k)} \zeta^{j}\right) A v+\left(\sum_{n=1}^{\infty} \tau n \zeta^{n}+\sum_{j=1}^{k-1} \tau c_{j}^{(k)} \zeta^{j}\right) A b \\
& +\sum_{\ell=1}^{k-2}\left(\sum_{n=1}^{\infty} \frac{\bar{\partial}_{\tau} t_{n}^{\ell}}{\ell!} \zeta_{n}^{n}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} \tau^{\ell-1} \zeta^{j}\right) \partial_{t}^{\ell-1} f(0)+\sum_{n=1}^{\infty} \bar{\partial}_{\tau} R_{k}\left(t_{n}\right) \zeta^{n}
\end{aligned}
$$

Using the elementary identities in (B.3), the convolution rule $\sum_{n=1}^{\infty} \zeta^{n} \bar{\partial}_{\tau}^{\alpha} W^{n}=\delta_{\tau}(\zeta)^{\alpha} \widetilde{W}$, and $\sum_{n=1}^{\infty} \zeta^{n} \bar{\partial}_{\tau} \frac{t_{n}^{\ell}}{\ell!}=$ $\delta_{\tau}(\zeta) \sum_{n=0}^{\infty} \frac{t_{n}^{\ell}}{\ell!} \zeta^{n}=\delta(\zeta) \frac{\tau^{\ell-1}}{\ell!} \gamma_{\ell}(\zeta)$, we derive

$$
\begin{aligned}
\widetilde{W}(\zeta)= & K\left(\delta_{\tau}(\zeta)\right)\left[\tau^{-1} \mu(\zeta) A v+\delta_{\tau}(\zeta)\left(\gamma_{1}(\zeta)+\sum_{j=1}^{k-1} c_{j}^{(k)} \zeta^{j}\right) \tau A b\right. \\
& \left.+\sum_{\ell=1}^{k-2} \delta_{\tau}(\zeta)\left(\delta(\zeta) \frac{\gamma_{\ell}(\zeta)}{\ell!}+\sum_{j=1}^{k-1} b_{\ell, j}^{(k)} \zeta^{j}\right) \tau^{\ell-1} g^{(\ell)}+\delta_{\tau}(\zeta)^{2} \widetilde{R}_{k}(\zeta)\right]
\end{aligned}
$$

Under Condition 3.1 (i), by choosing $\varepsilon$ small enough, Lemma B.1 implies that $0 \neq \delta_{\tau}\left(e^{-\tau z}\right)^{\alpha} \in \Sigma_{\alpha\left(\vartheta_{k}+\varepsilon\right)} \subset$ $\Sigma_{\pi-\varepsilon}$ for $z \in \Sigma_{\theta, \delta}^{\tau}$. Under Condition 3.1 (ii), we have $\operatorname{dist}\left(\delta\left(e^{-z \tau}\right)^{\alpha}, \tau^{\alpha} S(A)\right)>0$ (cf. Appendix E), where $S(A)$ denotes the closure of the spectrum of $A$ in the complex plane $\mathbb{C}$. In either case, the operator $K\left(\delta_{\tau}\left(e^{-\tau z}\right)\right)=\delta_{\tau}\left(e^{-\tau z}\right)^{-1}\left(\delta_{\tau}\left(e^{-\tau z}\right)^{\alpha}-A\right)^{-1}$ is analytic for $z \in \Sigma_{\theta, \delta}^{\tau}$, which is the region enclosed by the four curves $\Gamma_{\theta, \delta}^{\tau},-\ln (\varrho) / \tau+i \mathbb{R}$ and $\mathbb{R} \pm \mathrm{i} \pi / \tau$ (for $\theta$ and $\varrho$ sufficiently close to $\pi / 2$ and 1 , respectively). Then, like in the proof of Theorem 2.1, the assertion follows from Cauchy's integral formula and the change of variables $\zeta=e^{-z \tau}$.

## E Proof of Theorem 3.2

Under Condition 3.1 (i), Theorem 3.2 can be proved in the same way as Theorem 2.2, using 3.5 and (3.6). Under Condition 3.1(ii), it can be proved analogously, provided that the following resolvent estimate holds:

$$
\begin{equation*}
\left\|\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1}\right\| \leq c|z|^{-\alpha}, \quad \forall z \in \Gamma_{\theta, \delta}^{\tau} \tag{E.1}
\end{equation*}
$$

To prove (E.1), we use the following estimate (cf. [28, Theorem 3.9, Chapter 1, pp. 12]):

$$
\begin{align*}
\left\|\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1}\right\| & =\tau^{\alpha}\left\|\left(\delta\left(e^{-z \tau}\right)^{\alpha}-\tau^{\alpha} A\right)^{-1}\right\| \\
& \leq c \tau^{\alpha} \operatorname{dist}\left(\delta\left(e^{-z \tau}\right)^{\alpha}, \tau^{\alpha} S(A)\right)^{-1}, \quad \forall z \in \Gamma_{\theta, \delta}^{\tau} \tag{E.2}
\end{align*}
$$

where $S(A)$ denotes the closure of the spectrum of $A$ in $\mathbb{C}$. For the discrete Laplacian $A=\Delta_{h}$, we have $S(A)=[-r(A), 0]$. Since the angle between the contour $\delta\left(e^{-\mathrm{i} \xi \tau}\right)^{\alpha}, \xi \in\left[-\frac{\pi}{\tau}, \frac{\pi}{\tau}\right]$, and the segment $[-r(A), 0]$ is $(1-\alpha / 2) \pi>0$, it follows that, for small $\kappa$,

$$
\operatorname{dist}\left(\delta\left(e^{-\mathrm{i} \xi \tau}\right)^{\alpha}, \tau^{\alpha} S(A)\right) \approx\left|\delta\left(e^{-\mathrm{i} \xi \tau}\right)^{\alpha}\right| \sin [(1-\alpha / 2) \pi] \geq c\left|\delta\left(e^{-\mathrm{i} \xi \tau}\right)^{\alpha}\right| \quad \text { if }|\xi| \tau \leq \kappa
$$

Furthermore, CFL condition 3.1 (ii) implies

$$
\operatorname{dist}\left(\delta\left(e^{-\mathrm{i} \xi \tau}\right)^{\alpha}, \tau^{\alpha} S(A)\right) \geq c \geq c|\xi \tau|^{\alpha} \quad \text { if } \kappa \leq|\xi| \tau \leq \pi
$$

Let $\Gamma_{\theta}^{\tau}=\left\{z \in \mathbb{C}: \arg (z)=\theta,-\frac{\pi}{\tau} \leq|z| \sin (\theta) \leq \frac{\pi}{\tau}\right\}$. Then the angle between the contour $\delta\left(e^{-z \tau}\right)^{\alpha}$, $z \in \Gamma_{\theta}^{\tau}$, and the segment $[-r(A), 0]$ is $\pi-\alpha \theta>0$ (if $\theta$ is close to $\pi / 2$ ). For small $\kappa$ and $z \in \Gamma_{\theta}^{\tau}$, $|z| \tau \sin (\theta) \leq \kappa$, we have

$$
\operatorname{dist}\left(\delta\left(e^{-z \tau}\right)^{\alpha}, \tau^{\alpha} S(A)\right) \approx\left|\delta\left(e^{-z \tau}\right)^{\alpha}\right| \sin (\pi-\alpha \theta) \geq c\left|\delta\left(e^{-z \tau}\right)^{\alpha}\right| \geq c|z \tau|^{\alpha}
$$

Estimate (B.1) implies $\left|\delta\left(e^{-z \tau}\right)-\delta\left(e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)\right| \leq c|\theta-\pi / 2|$, and thus

$$
\begin{aligned}
\left|\delta\left(e^{-z \tau}\right)^{\alpha}-\delta\left(e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)^{\alpha}\right| & \leq c|\theta-\pi / 2| \min \left(\left|\delta\left(e^{-z \tau}\right)\right|^{\alpha-1}, \mid \delta\left(\left.e^{-\mathrm{i}|z| \tau \sin (\theta)}\right|^{\alpha-1}\right)\right. \\
& \leq c|\theta-\pi / 2||z \tau|^{\alpha-1}, \quad z \in \Gamma_{\theta}^{\tau}
\end{aligned}
$$

Hence, if $z \in \Gamma_{\theta}^{\tau}$ and $\kappa \leq|z| \tau \sin (\theta) \leq \pi$, with $\theta$ close to $\pi / 2$, we have

$$
\begin{aligned}
\operatorname{dist}\left(\delta\left(e^{-z \tau}\right)^{\alpha}, \tau^{\alpha} S(A)\right) & \geq \operatorname{dist}\left(\delta\left(e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)^{\alpha}, \tau^{\alpha} S(A)\right)-\left|\delta\left(e^{-z \tau}\right)^{\alpha}-\delta\left(e^{-\mathrm{i}|z| \tau \sin (\theta)}\right)^{\alpha}\right| \\
& \geq c-c|\theta-\pi / 2||z \tau|^{\alpha-1} \geq c-c|\theta-\pi / 2||z \tau \sin (\theta)|^{\alpha-1} \\
& \geq c-c|\theta-\pi / 2| \max (\kappa, \pi)^{\alpha-1} \geq c \geq c|z \tau|^{\alpha} .
\end{aligned}
$$

Thus we have $\operatorname{dist}\left(\delta\left(e^{-z \tau}\right)^{\alpha}, \tau^{\alpha} S(A)\right) \geq c|z \tau|^{\alpha}$ for $z \in \Gamma_{\theta}^{\tau}$. This inequality and (E.2) yield (E.1) for $z \in \Gamma_{\theta, \delta}^{\tau} \cap \Gamma_{\theta}^{\tau}$. Further, if $z \in \Gamma_{\theta, \delta}^{\tau} \backslash \Gamma_{\theta}^{\tau}$, then $|z|=\delta$ and $-\theta<\arg (z)<\theta$, and Taylor expansion yields $\left|\delta\left(e^{-z \tau}\right)\right|^{\alpha} \leq|z \tau|^{\alpha} \leq \delta^{\alpha} \tau^{\alpha}$. By choosing $\delta$ small, we have

$$
\operatorname{dist}\left(\delta\left(e^{-z \tau}\right)^{\alpha}, \tau^{\alpha} S(A)\right) \geq \lambda_{\min } \tau^{\alpha}-\delta^{\alpha} \tau^{\alpha} \geq c \tau^{\alpha}
$$

where $\lambda_{\min }$ is the smallest positive eigenvalue of the operator $A$ (which can be made independent of $h$ ). This and (E.2) yield

$$
\left\|\left(\delta_{\tau}\left(e^{-z \tau}\right)^{\alpha}-A\right)^{-1}\right\| \leq c \leq c \delta^{-\alpha}=c|z|^{-\alpha}, \quad \forall z \in \Gamma_{\theta, \delta}^{\tau} \backslash \Gamma_{\theta}^{\tau}
$$

This completes the proof of E.1).

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[^1]:    ${ }^{1}$ http://www.advanpix.com/, last accessed on January 11, 2017.

