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# SHAPE DIFFERENTIABILITY OF LAGRANGIANS AND APPLICATION TO STOKES PROBLEM

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ABSTRACT. A class of convex constrained minimization problems over polyhedral cones for geometry-dependent quadratic objective functions is considered in a functional analysis framework. Shape differentiability of the primal minimization problem needs a bijective property for mapping of the primal cone. This restrictive assumption is relaxed to bijection of the dual cone within the Lagrangian formulation as a primal-dual minimax problem. In this paper, we give results on primal-dual shape sensitivity analysis that extends the class of shape-differentiable problems supported by explicit formula of the shape derivative. We apply the results to the Stokes problem under mixed Dirichlet–Neumann boundary conditions subject to the divergence-free constraint.

## 1. INTRODUCTION

We aim at shape differentiability for a class of convex constrained minimization problems over polyhedral cones, where the objective functions are assumed quadratic and depend on a geometry.

Typical examples are contact problems in solid mechanics, see [25, 27], and other elliptic partial differential equations in variable domains with equality and inequality type constraints, see [15, 30, 38, 41]. Our special interest concerns nonlinear crack problems in fracture mechanics due to non-penetration between crack faces, which are developed in [20, 21, 22] and other works by the authors. By this, shape variations may imply regular perturbations along a predefined crack path, see [2, 19, 26], as well as singular perturbations due to kink of the crack,

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see [23, 24]. A recent result of [32] concerns shape-topological control by posing a small defect in the cracked domain.

From the point of view of shape and topology optimization, a shape sensitivity analysis of the problem is performed with the help of the velocity method. Introducing a proper kinematic velocity, see e.g. [31], a general perturbation of quadratic constrained minimization problems over convex cones in Hilbert spaces is established in [17]. An explicit formula of the shape derivative is provided by bijective properties of the velocity-based diffeomorphic flow of a geometry. However, this result restricts the primal cone to be a bijection within the flow. The bijection fails for constraints involving normal on curves (e.g. Signorini conditions), having integral, gradient, divergence operator, etc. This is rather restrictive, even not a complete list.

In the case of Signorini-type constraints imposed on curvilinear manifolds implying cracks, the shape differentiability result is improved in [35, 45, 46] relying on a  $\Gamma$ -convergence of the primal cones. For this specific problem, in [28, 29] the assumption of bijection is relaxed further to the dual cone within a Lagrangian formalism. See another specific example of shape sensitivity of a Lagrangian associated with inhomogeneous Dirichlet problem in [11], and the general Lagrangian method together with related primal-dual minimax problems in [18].

For other example of such a non-bijective primal cone, in the present work we consider a Stokes problem under mixed Dirichlet–Neumann boundary conditions subject to the divergence-free constraint. We refer to [8, 14, 34] for the Stokes problems, and to [7, 16] for its shape sensitivity. It is worth to stress that the divergence-free constraint is not preserved by transport. The treatment of the incompressibility within the dynamical shape control of Navier–Stokes equations is discussed in [39, Section 5]. It employs special transforms (Piola transformation, transverse map), a hold-all domain assumption, but has a lack of rigorous mathematical justification [39, p.142].

In Section 2 we develop our concept of the shape differentiability of Lagrangians in a functional analysis framework. Based on the Lagrangian setting which implies a primal-dual minimax problem, we relax the bijection assumption from the primal cone K (in the space of primal variable) to the dual cone  $K^*$  (in the space of dual variable) (see (2.20c)). This relaxation allows us to lead the primal-dual shape sensitivity analysis and to obtain the shape derivative explicitly. The improvement of the previous shape sensitivity results is attained with respect to non-bijective primal cones, thus extending the class of shape-differentiable problems. It is important to put our investigation in the classic context of optimal value functions adopted in optimization. The directional differentiability of optimal value Lagrangians in abstract formulation was established in [9] (see also [4, Chapter 4.3.2]), and extended to the shape optimization framework in [10]. For a concept of directional differentiability of metric projections onto polyhedric sets corresponding to shape derivatives we refer to [36] and references therein.

The abstract optimal value Lagrangian function used for shape optimization in a time-dependent domain  $\Omega_t$  with parameter t can be defined by a general map of the form:

(OVF) 
$$\mathbb{R} \mapsto \mathbb{R}, \quad t \mapsto \mathcal{L}(u_t, \lambda_t; \Omega_t),$$

where a saddle point  $(u_t, \lambda_t) \in V(\Omega_t) \times K^*(\Omega_t)$  satisfies

(SP) 
$$\mathcal{L}(u_t, p; \Omega_t) \leq \mathcal{L}(u_t, \lambda_t; \Omega_t) \leq \mathcal{L}(w, \lambda_t; \Omega_t)$$
  
 $\forall (w, p) \in V(\Omega_t) \times K^*(\Omega_t)$ 

for a Lagrangian

(L) 
$$(u, \lambda) \mapsto \mathcal{L}(u, \lambda; \Omega_t) : V(\Omega_t) \times K^*(\Omega_t) \mapsto \mathbb{R}$$

defined over topological vector spaces  $V(\Omega_t)$  and  $K^*(\Omega_t)$  (the upper star to be explained later on). The aim is to find the *directional derivative*:

(DD) 
$$\partial_t \mathcal{L}(u_t, \lambda_t; \Omega_t) := \lim_{s \to 0} \frac{\mathcal{L}(u_{t+s}, \lambda_{t+s}; \Omega_{t+s}) - \mathcal{L}(u_t, \lambda_t; \Omega_t)}{s}.$$

Since the perturbed optimal value function  $\mathcal{L}(u_{t+s}, \lambda_{t+s}; \Omega_{t+s})$  in (DD) is given by the perturbed Lagrangian

(PL) 
$$(v,\mu) \mapsto \mathcal{L}(v,\mu;\Omega_{t+s}) : V(\Omega_{t+s}) \times K^{\star}(\Omega_{t+s}) \mapsto \mathbb{R},$$

which is defined over s-dependent spaces  $V(\Omega_{t+s}) \times K^*(\Omega_{t+s})$ , then the usual trick in shape optimization is to use a coordinate transformation

(CT) 
$$\phi_s : \Omega_t \mapsto \Omega_{t+s}, \quad \phi_s^{-1} : \Omega_{t+s} \mapsto \Omega_t$$

that maps (PL) to a transformed perturbed Lagrangian

(TPL) 
$$(s, u, \lambda) \mapsto \mathcal{L}_s(u, \lambda; \Omega_t) : \mathbb{R} \times V(\Omega_t) \times K^*(\Omega_t) \mapsto \mathbb{R}$$

over fixed spaces  $V(\Omega_t) \times K^*(\Omega_t)$  such that  $\mathcal{L}_0 = \mathcal{L}$  and

(BL) 
$$\mathcal{L}_s(v \circ \phi_s, \mu \circ \phi_s; \Omega_t) = \mathcal{L}(v, \mu; \Omega_{t+s})$$

for all  $(v, \mu) \in V(\Omega_{t+s}) \times K^*(\Omega_{t+s})$ . This needs the fulfillment of bijective property between the function spaces

(BS) 
$$[v \mapsto v \circ \phi_s] : V(\Omega_{t+s}) \mapsto V(\Omega_t),$$
  
 $[\mu \mapsto \mu \circ \phi_s] : K^*(\Omega_{t+s}) \mapsto K^*(\Omega_t)$ 

and allows to rewrite (DD) in the equivalent form:

(DD') 
$$\partial_s \mathcal{L}_s(0, u_t, \lambda_t; \Omega_t) = \lim_{s \to 0} \frac{\mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t) - \mathcal{L}(u_t, \lambda_t; \Omega_t)}{s}.$$

The bijection (BS) is central in this work.

In the constrained optimization context,  $K^*$  is associated to a dual cone compared with its primal counterpart K. For the divergence-free constraint, in Section 3 we give an example of the space  $K^*(\Omega_{t+s})$  where the bijection of dual cones (see (2.20c)) fails. Namely, considering Stokes problem under no-slip Dirichlet condition, the integral identity  $\int_{\Omega_{t+s}} v(y) dy = 0$  characterizing the space  $L_0^2(\Omega_{t+s})$  (see (3.42)) is not preserved by the transport  $y = \phi_s(x)$  in general, thus, the equivalence between (DD) and (DD') is not true. A possible remedy is to use special area-preserving maps. In the current paper, we suggest to consider the Stokes problem under mixed Dirichlet–Neumann boundary conditions such that the bijection property (BS) holds true.

### 2. Shape derivative of Lagrangians for polyhedral cones

We start the investigation with a family of *time-dependent geometric* sets  $t \mapsto \Omega_t \subset \mathbb{R}^d, d \in \mathbb{N}$ .

For every fixed time  $t \in \mathbb{R}$ , we consider two geometry-dependent Hilbert spaces  $V(\Omega_t)$  and  $H(\Omega_t)$  with the dual spaces  $V^*(\Omega_t)$  and  $H^*(\Omega_t)$ . Let a linear operator  $A: V(\Omega_t) \mapsto V^*(\Omega_t)$  be strongly monotone such that

(2.1) 
$$\langle Au, u \rangle_{\Omega_t} \ge \underline{c}_A ||u||^2_{V(\Omega_t)}, \quad \underline{c}_A > 0, \quad u \in V(\Omega_t)$$

with the duality pairing  $\langle \cdot, \cdot \rangle_{\Omega_t}$  between  $V^*(\Omega_t)$  and  $V(\Omega_t)$ , and continuous such that

(2.2) 
$$\|Au\|_{V^{\star}(\Omega_t)} \leq \overline{c}_A \|u\|_{V(\Omega_t)}, \quad \overline{c}_A \geq \underline{c}_A > 0, \quad u \in V(\Omega_t)$$

uniformly in a time interval  $t \in (t_0, t_1)$  with fixed  $t_0 < t_1$ . Let a linear operator  $B : V(\Omega_t) \mapsto H(\Omega_t)$  be surjective (i.e. for every  $\zeta \in H(\Omega_t)$ there is at least one  $u \in V(\Omega_t)$  such that  $Bu = \zeta$ ) and continuous with the following estimate

(2.3) 
$$\|Bu\|_{H(\Omega_t)} \le \overline{c}_B \|u\|_{V(\Omega_t)}, \quad \overline{c}_B > 0, \quad u \in V(\Omega_t)$$

that holds uniformly for all  $t \in (t_0, t_1)$ .

Using the order relation for measured functions in  $H(\Omega_t)$ , we define the *primal cone* as a polyhedral cone as follows

(2.4) 
$$K(\Omega_t) := \{ u \in V(\Omega_t) | \quad Bu \ge 0 \}$$

which is convex and closed. For a stationary right-hand side f such that  $f \in \bigcap_{t \in (t_0, t_1)} V^*(\Omega_t)$ , let the geometry-dependent objective function  $\mathcal{E} : V(\Omega_t) \mapsto \mathbb{R}$  be given by

(2.5) 
$$\mathcal{E}(u;\Omega_t) := \langle \frac{1}{2}Au - f, u \rangle_{\Omega_t}$$

that is quadratic, bounded due to (2.2), and coercive due to (2.1).

We consider the primal constrained minimization problem: Find  $u_t \in K(\Omega_t)$  such that

(2.6) 
$$\mathcal{E}(u_t; \Omega_t) = \min_{w \in K(\Omega_t)} \mathcal{E}(w; \Omega_t).$$

The unique solution to (2.6) exists and satisfies the first order *optimality* condition in the form of a variational inequality due to (2.5) and (2.6):

(2.7) 
$$\langle Au_t - f, w - u_t \rangle_{\Omega_t} \ge 0 \quad \forall w \in K(\Omega_t)$$

which is a necessary and sufficient condition for (2.6). For a general theory of pseudo-monotone variational inequalities see [42].

Now we define the *dual cone* (in the space of dual variable) as follows

(2.8) 
$$K^{\star}(\Omega_t) := \{ \lambda \in H^{\star}(\Omega_t) | \quad (\lambda, Bu)_{\Omega_t} \ge 0 \quad \forall u \in K(\Omega_t) \}$$

where  $(\cdot, \cdot)_{\Omega_t}$  stands for the duality pairing between  $H^*(\Omega_t)$  and  $H(\Omega_t)$ . It is important to note that, due to surjection of B, the dual cone in (2.8) can be restated equivalently in the form

(2.8') 
$$K^{\star}(\Omega_t) = \{\lambda \in H^{\star}(\Omega_t) | \quad (\lambda, \zeta)_{\Omega_t} \ge 0 \quad \forall \zeta \in H(\Omega_t), \ \zeta \ge 0 \}.$$

The corresponding *primal-dual minimax problem* reads: Find the pair  $(u_t, \lambda_t) \in V(\Omega_t) \times K^*(\Omega_t)$  such that

(2.9) 
$$\mathcal{L}(u_t, \lambda_t; \Omega_t) = \min_{w \in V(\Omega_t)} \max_{p \in K^*(\Omega_t)} \mathcal{L}(w, p; \Omega_t)$$

with the Lagrangian function  $\mathcal{L}: V(\Omega_t) \times H^*(\Omega_t) \mapsto \mathbb{R}$  given by

(2.10) 
$$\mathcal{L}(u,\lambda;\Omega_t) := \mathcal{E}(u;\Omega_t) - (\lambda, Bu)_{\Omega_t}$$

Well-posedness and optimality properties of (2.9) are gathered in the following theorem.

**Theorem 2.1.** (i) There exists a solution of the minimax problem (2.9) which implies that  $(u_t, \lambda_t) \in V(\Omega_t) \times K^*(\Omega_t)$  is a saddle point:

(2.9') 
$$\mathcal{L}(u_t, p; \Omega_t) \leq \mathcal{L}(u_t, \lambda_t; \Omega_t) \leq \mathcal{L}(w, \lambda_t; \Omega_t)$$
  
 $\forall (w, p) \in V(\Omega_t) \times K^{\star}(\Omega_t)$ 

and satisfies the primal-dual optimality conditions:

(2.11a) 
$$\langle Au_t - f, w \rangle_{\Omega_t} - (\lambda_t, Bw)_{\Omega_t} = 0 \quad \forall w \in V(\Omega_t)$$

(2.11b) 
$$(p - \lambda_t, Bu_t)_{\Omega_t} \ge 0 \quad \forall p \in K^*(\Omega_t).$$

The primal component  $u_t \in K(\Omega_t)$  is unique solution of the primal problem (2.6). If the Ladyzhenskaya–Babuška–Brezzi (LBB) condition holds for  $\lambda \in H^*(\Omega_t)$ :

(2.12) 
$$\sup_{u \in V(\Omega_t)/\{0\}} \frac{(\lambda, Bu)_{\Omega_t}}{\|u\|_{V(\Omega_t)}} \ge \underline{c}_B \|\lambda\|_{H^*(\Omega_t)}, \quad 0 < \underline{c}_B \le \overline{c}_B$$

then the dual component  $\lambda_t$  is unique.

(ii) The optimal value objective function  $t \mapsto \mathcal{E}(u_t; \Omega_t)$  defined by (2.6) and the optimal value Lagrangian function  $t \mapsto \mathcal{L}(u_t, \lambda_t; \Omega_t)$  given in (2.9) are equal:

(2.13) 
$$\min_{w \in V(\Omega_t)} \mathcal{E}(w; \Omega_t) = \min_{w \in V(\Omega_t)} \max_{p \in K^*(\Omega_t)} \mathcal{L}(w, p; \Omega_t)$$

Proof. Indeed, based on (2.1)–(2.10), existence of a solution to the minimax problem follows from e.g. [27, Theorem 3.11]. The inclusion  $u_t \in K(\Omega_t)$  is a consequence of the bipolar theorem, see e.g. [44, Theorem 14.1], due to surjection of B. The optimality conditions (2.11) and the uniqueness assertion under LBB condition (2.12) are stated e.g. in [27, Theorem 3.14]. The cone  $K^*(\Omega_t)$  is convex and  $V(\Omega_t)$  is linear, the Lagrangian  $\mathcal{L}$  is convex-concave and Gâteaux differentiable, so that (2.11) is equivalent to (see [13, Proposition 1.5]):

$$\begin{aligned} \langle \partial_u \mathcal{L}(u_t, \lambda_t; \Omega_t), w \rangle_{\Omega_t} &= 0 \quad \forall w \in V(\Omega_t), \\ (\partial_\lambda \mathcal{L}(u_t, \lambda_t; \Omega_t), p - \lambda_t)_{\Omega_t} \geq 0 \quad \forall p \in K^*(\Omega_t), \end{aligned}$$

and the pair  $(u_t, \lambda_t) \in V(\Omega_t) \times K^*(\Omega_t)$  also satisfies (2.9') implying the saddle point (see [13, Definition 1.1]).

To proof the assertion (ii), we test (2.11b) with p = 0 and  $p = 2\lambda_t$  yielding  $(\lambda_t, Bu_t)_{\Omega_t} = 0$ , hence  $\mathcal{E}(u_t; \Omega_t) = \mathcal{L}(u_t, \lambda_t; \Omega_t)$  in turn implying (2.13).

In the following we lead a shape sensitivity analysis of the problem.

2.1. **Primal-dual shape sensitivity analysis.** For fixed  $t \in (t_0, t_1)$  and a small perturbation parameter  $s \in (t_0 - t, t_1 - t)$ , let given vector-functions

(2.14a) 
$$[s \mapsto \phi_s], [s \mapsto \phi_s^{-1}] \in C^1([t_0 - t, t_1 - t]; W^{1,\infty}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d))$$

associate the *coordinate transformation*  $y = \phi_s(x)$  and the inverse mapping  $x = \phi_s^{-1}(y)$  such that its composition satisfies:

(2.14b) 
$$(\phi_s^{-1} \circ \phi_s)(x) = x, \quad (\phi_s \circ \phi_s^{-1})(y) = y.$$

Then the shape perturbation

(2.15) 
$$\Omega_{t+s} := \{ y \in \mathbb{R}^d | \quad y = \phi_s(x), \ x \in \Omega_t \}$$

builds the *diffeomorphism* 

(2.16)  $\phi_s: \Omega_t \mapsto \Omega_{t+s}, x \mapsto y; \quad \phi_s^{-1}: \Omega_{t+s} \mapsto \Omega_t, y \mapsto x.$ 

We reset the perturbed primal constrained minimization problem: Find  $u_{t+s} \in K(\Omega_{t+s})$  such that

(2.17) 
$$\mathcal{E}(u_{t+s};\Omega_{t+s}) = \min_{v \in K(\Omega_{t+s})} \mathcal{E}(v;\Omega_{t+s})$$

and the corresponding perturbed primal-dual minimax problem: Find the pair  $(u_{t+s}, \lambda_{t+s}) \in V(\Omega_{t+s}) \times K^*(\Omega_{t+s})$  such that

(2.18) 
$$\mathcal{L}(u_{t+s}, \lambda_{t+s}; \Omega_{t+s}) = \min_{v \in V(\Omega_{t+s})} \max_{\mu \in K^{\star}(\Omega_{t+s})} \mathcal{L}(v, \mu; \Omega_{t+s})$$

with the *perturbed Lagrangian and objective functions*, respectively:

(2.19a) 
$$\mathcal{L}(v,\mu;\Omega_{t+s}) = \mathcal{E}(v;\Omega_{t+s}) - (\mu,Bv)_{\Omega_{t+s}}$$

(2.19b) 
$$\mathcal{E}(v;\Omega_{t+s}) = \langle \frac{1}{2}Av - f, v \rangle_{\Omega_{t+s}}.$$

They are defined for  $v \in V(\Omega_{t+s})$  and  $\mu \in H^*(\Omega_{t+s})$  with the duality pairings  $\langle \cdot, \cdot \rangle_{\Omega_{t+s}}$  between  $V^*(\Omega_{t+s})$  and  $V(\Omega_{t+s})$ , and  $(\cdot, \cdot)_{\Omega_{t+s}}$ between  $H^*(\Omega_{t+s})$  and  $H(\Omega_{t+s})$ .

Within the kinematic flow (2.14)–(2.16), we employ the assumptions: The map  $[v \mapsto v \circ \phi_s]$  is bijective in the function spaces

(2.20a) 
$$V(\Omega_{t+s}) \mapsto V(\Omega_t), \quad V^*(\Omega_{t+s}) \mapsto V^*(\Omega_t),$$

(2.20b) 
$$H(\Omega_{t+s}) \mapsto H(\Omega_t), \quad H^*(\Omega_{t+s}) \mapsto H^*(\Omega_t),$$

and  $[\mu \mapsto \mu \circ \phi_s]$  is bijective in the dual cones

(2.20c) 
$$K^{\star}(\Omega_{t+s}) \mapsto K^{\star}(\Omega_t)$$

As  $s \to 0$ , let the asymptotic representations hold for the operator A:

(2.20d) 
$$\langle Av, \chi \rangle_{\Omega_{t+s}} = \langle [A + sA^1 + A_s^2](v \circ \phi_s), \chi \circ \phi_s \rangle_{\Omega_t}$$

with linear bounded operators  $A^1, A_s^2 : V(\Omega_t) \mapsto V^*(\Omega_t)$  and the residual  $A_s^2$  such that

(2.20e) 
$$||A_s^2 u||_{V^*(\Omega_t)} \le c_{RA}(s) ||u||_{V(\Omega_t)}, \quad 0 \le c_{RA}(s) = o(s);$$

for the operator B:

(2.20f) 
$$(\mu, Bv)_{\Omega_{t+s}} = (\mu \circ \phi_s, [B + sB^1 + B_s^2](v \circ \phi_s))_{\Omega_t}$$

with linear bounded operators  $B^1, B_s^2 : V(\Omega_t) \mapsto H(\Omega_t)$  such that  $B + sB^1 + B_s^2$  is surjective and the residual  $B_s^2$  satisfies

(2.20g) 
$$||B_s^2 u||_{H(\Omega_t)} \le c_{RB}(s) ||u||_{V(\Omega_t)}, \quad 0 \le c_{RB}(s) = o(s);$$

and for the right-hand side f:

(2.20h) 
$$\langle f, v \rangle_{\Omega_{t+s}} = \langle f + sf^1 + f_s^2, v \circ \phi_s \rangle_{\Omega_t}$$

with  $f^1, f^2_s \in V^*(\Omega_t)$  and the residual  $f^2_s$  such that

(2.20i) 
$$||f_s^2||_{V^*(\Omega_t)} \le c_{Rf}(s), \quad 0 \le c_{Rf}(s) = o(s)$$

for test-functions  $v, \chi \in V(\Omega_{t+s}), \mu \in H^*(\Omega_{t+s}), u \in V(\Omega_t)$ , uniformly for all  $s \in (t_0 - t, t_1 - t)$  and  $t \in (t_0, t_1)$ .

**Theorem 2.2.** Under the assumptions (2.20), the optimal value function  $\mathbb{R} \mapsto \mathbb{R}, t \mapsto \mathcal{E}(u_t; \Omega_t)$  of the objective  $\mathcal{E}$  given in (2.5) and (2.6) is shape differentiable such that

(2.21) 
$$\frac{d}{dt}\mathcal{E}(u_t;\Omega_t) := \lim_{s \to 0} \frac{\mathcal{E}(u_{t+s};\Omega_{t+s}) - \mathcal{E}(u_t;\Omega_t)}{s} = \mathcal{L}^1(u_t,\lambda_t;\Omega_t)$$

with the shape derivative  $\mathcal{L}^1(u_t, \lambda_t; \Omega_t)$  determined by

(2.22a) 
$$\mathcal{L}^{1}(u,\lambda;\Omega_{t}) := \mathcal{E}^{1}(u;\Omega_{t}) - (\lambda,B^{1}u)_{\Omega_{t}}$$

(2.22b) 
$$\mathcal{E}^1(u;\Omega_t) := \langle \frac{1}{2}A^1u - f^1, u \rangle_{\Omega_t}.$$

*Proof.* We apply to (2.18) the asymptotic formula (2.20d), (2.20f), (2.20h) and use the assumptions (2.20a)–(2.20c) to get the transformed solution pair  $(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s) \in V(\Omega_t) \times K^*(\Omega_t)$  which solves the minimax problem

(2.23) 
$$\mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t) = \min_{w \in V(\Omega_t)} \max_{p \in K^*(\Omega_t)} \mathcal{L}_s(w, p; \Omega_t)$$

implying a saddle point (see (2.9')):

(2.23') 
$$\mathcal{L}_s(u_{t+s} \circ \phi_s, p; \Omega_t) \leq \mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t)$$
$$\leq \mathcal{L}_s(w, \lambda_{t+s} \circ \phi_s; \Omega_t) \quad \forall (w, p) \in V(\Omega_t) \times K^*(\Omega_t).$$

The transformed Lagrangian  $\mathcal{L}_s: V(\Omega_t) \times H(\Omega_t) \mapsto \mathbb{R}$  is defined via

(2.24a) 
$$\mathcal{L}_s(v \circ \phi_s, \mu \circ \phi_s; \Omega_t) = \mathcal{L}(v, \mu; \Omega_{t+s}) \quad (\text{with } \mathcal{L}_0 = \mathcal{L})$$

for all  $(v, \mu) \in V(\Omega_{t+s}) \times K^*(\Omega_{t+s})$ , and yields the expansion

(2.24b) 
$$\mathcal{L}_s(u,\lambda;\Omega_t) := \mathcal{L}(u,\lambda;\Omega_t) + s\mathcal{L}^1(u,\lambda;\Omega_t) + \mathcal{L}_s^2(u,\lambda;\Omega_t)$$

where the first asymptotic terms  $\mathcal{L}^1(u, \lambda; \Omega_t)$  is given in (2.22a), and the residual

(2.24c) 
$$\mathcal{L}_{s}^{2}(u,\lambda;\Omega_{t}) := \langle \frac{1}{2}A_{s}^{2}u - f_{s}^{2}, u \rangle_{\Omega_{t}} - (\lambda, B_{s}^{2}u)_{\Omega_{t}}$$

Based on Theorem 2.1, optimality conditions for (2.23) are

(2.25a) 
$$\langle [A + sA^1 + A_s^2](u_{t+s} \circ \phi_s) - (f + sf^1 + f_s^2), w \rangle_{\Omega_t}$$
  
  $- (\lambda_{t+s} \circ \phi_s, [B + sB^1 + B_s^2]w)_{\Omega_t} = 0 \quad \forall w \in V(\Omega_t)$ 

(2.25b) 
$$(p - \lambda_{t+s} \circ \phi_s, [B + sB^1 + B_s^2](u_{t+s} \circ \phi_s))_{\Omega_t} \ge 0$$
  
 $\forall p \in K^*(\Omega_t).$ 

Taking the test function  $w = u_{t+s} \circ \phi_s$  in (2.25a), using the complementarity

(2.26) 
$$(\lambda_{t+s} \circ \phi_s, [B+sB^1+B_s^2]u_{t+s} \circ \phi_s)_{\Omega_t} = 0$$

which follows from (2.25b), the strong monotony (2.1) of A, and the residual estimates (2.20e), (2.20g), (2.20i), for  $|s| \in (0, s_0)$  with sufficiently small  $s_0 > 0$  and  $t \in (t_0, t_1)$  we get the uniform estimate:

(2.27a) 
$$\|u_{t+s} \circ \phi_s\|_{V(\Omega_t)} \le \text{const.}$$

Similarly, from (2.25a) we derive the uniform estimate in the dual space:

(2.27b) 
$$\|\lambda_{t+s} \circ \phi_s\|_{H^*(\Omega_t)} \le \text{const}$$

for  $|s| \in (0, s_1)$  with sufficiently small  $0 < s_1 \le s_0$  and  $t \in (t_0, t_1)$ .

From (2.27) it follows the existence of  $(\overline{u}, \overline{\lambda}) \in V(\Omega_t) \times H^*(\Omega_t)$  and a subsequence denoted by  $s_k$  such that as  $s_k \to 0$ :

(2.28a) 
$$u_{t+s_k} \circ \phi_{s_k} \rightharpoonup \overline{u}$$
 weakly in  $V(\Omega_t)$ 

(2.28b) 
$$\lambda_{t+s_k} \circ \phi_{s_k} \rightharpoonup \overline{\lambda} \quad \star\text{-weakly in } H^{\star}(\Omega_t)$$

Every linear and continuous operator B is weak-to-weak continuous (see [5, Theorem 3.10]), therefore

(2.28c) 
$$B(u_{t+s_k} \circ \phi_{s_k}) \rightharpoonup B\overline{u}$$
 weakly in  $H(\Omega_t)$ .

In accordance with (2.20c) the inclusion  $\lambda_{t+s} \circ \phi_s \in K^*(\Omega_t)$  holds, the convex closed set  $K^*(\Omega_t)$  is  $\star$ -weakly closed, hence  $\overline{\lambda} \in K^*(\Omega_t)$ . Since a quadratic form is weakly lower semi-continuous, we pass to the limit in (2.23') using the weak convergences in (2.28) and get

$$\mathcal{L}(\overline{u}, p; \Omega_t) \leq \liminf_{s_k \to 0} \mathcal{L}_{s_k}(u_{t+s_k} \circ \phi_{s_k}, p; \Omega_t) \leq \mathcal{L}(\overline{u}, \overline{\lambda}; \Omega_t)$$
$$\leq \limsup_{s_k \to 0} \mathcal{L}_{s_k}(w, \lambda_{t+s_k} \circ \phi_{s_k}; \Omega_t) \leq \mathcal{L}(w, \overline{\lambda}; \Omega_t)$$

for arbitrary  $(w, p) \in V(\Omega_t) \times K^*(\Omega_t)$ . Therefore,  $(\overline{u}, \overline{\lambda}) = (u_t, \lambda_t)$  is a saddle point satisfying (2.9'), thus solves (2.9).

In order to estimate the solution difference in the norm, we start with the inequality (2.1) and rearrange the terms such that

$$\frac{\underline{c}_{A}}{2} \| u_{t+s} \circ \phi_{s} - u_{t} \|_{V(\Omega_{t})}^{2} \leq \frac{1}{2} \langle A (u_{t+s} \circ \phi_{s} - u_{t}), u_{t+s} \circ \phi_{s} - u_{t} \rangle_{\Omega_{t}} \\ = - \langle A (u_{t+s} \circ \phi_{s} - u_{t}), u_{t} \rangle_{\Omega_{t}} - \frac{1}{2} \langle A u_{t}, u_{t} \rangle_{\Omega_{t}} + \frac{1}{2} \langle A (u_{t+s} \circ \phi_{s}), u_{t+s} \circ \phi_{s} \rangle_{\Omega_{t}} \\ = - \langle A (u_{t+s} \circ \phi_{s} - u_{t}), u_{t} \rangle_{\Omega_{t}} + \langle f, u_{t+s} \circ \phi_{s} - u_{t} \rangle_{\Omega_{t}} \\ + \mathcal{L} (u_{t+s} \circ \phi_{s}, \lambda_{t+s} \circ \phi_{s}; \Omega_{t}) - \mathcal{L} (u_{t}, \lambda_{t}; \Omega_{t}) + (\lambda_{t+s} \circ \phi_{s}, [sB^{1} + B_{s}^{2}]u_{t+s} \circ \phi_{s})_{\Omega_{t}} \\ \text{due to the orthogonality relations } (\lambda - P_{t}u) = 0 \text{ and } (2.26) \text{.} \text{ Using}$$

due to the orthogonality relations  $(\lambda_t, Bu_t)_{\Omega_t} = 0$  and (2.26). Using further

$$\lim_{s_k \to 0} \sup_{s_k \to 0} \left\{ \mathcal{L}(u_{t+s_k} \circ \phi_{s_k}, \lambda_{t+s_k} \circ \phi_{s_k}; \Omega_t) - \mathcal{L}(u_t, \lambda_t; \Omega_t) \right\}$$
$$= \lim_{s_k \to 0} \sup_{s_k \to 0} \left\{ \mathcal{L}_{s_k}(u_{t+s_k} \circ \phi_{s_k}, \lambda_{t+s_k} \circ \phi_{s_k}; \Omega_t) - \mathcal{L}_{s_k}(u_t, \lambda_{t+s_k} \circ \phi_{s_k}; \Omega_t) \right\} \le 0$$

because of (2.23') with  $w = u_t$  and (2.28), we conclude that

(2.29a) 
$$\frac{\underline{c}_A}{2} \limsup_{s_k \to 0} \|u_{t+s_k} \circ \phi_{s_k} - u_t\|_{V(\Omega_t)}^2 \le 0.$$

Therefore, from (2.3) it follows that as  $s_k \to 0$ 

(2.29b) 
$$||B(u_{t+s_k} \circ \phi_{s_k} - u_t)||_{H(\Omega_t)} \to 0.$$

From (2.11a) and (2.25a) we arrive at

$$(\lambda_{t+s} \circ \phi_s - \lambda_t, Bw)_{\Omega_t} = \langle A(u_{t+s} \circ \phi_s - u_t), w \rangle_{\Omega_t} + \mathcal{O}(s)$$

for all  $w \in V(\Omega_t)$ , henceforth the surjection of B provides that

(2.29c) 
$$\|\lambda_{t+s_k} \circ \phi_{s_k} - \lambda_t\|_{H^*(\Omega_t)} \to 0$$

The relations (2.29) imply the strong convergences in (2.28).

Based on the asymptotic formula (2.24) we find the lower bound:

(2.30a) 
$$\mathcal{L}_{s}(u_{t+s} \circ \phi_{s}, \lambda_{t+s} \circ \phi_{s}; \Omega_{t}) - \mathcal{L}(u_{t}, \lambda_{t}; \Omega_{t})$$
$$\geq \mathcal{L}_{s}(u_{t+s} \circ \phi_{s}, \lambda_{t}; \Omega_{t}) - \mathcal{L}(u_{t+s} \circ \phi_{s}, \lambda_{t}; \Omega_{t})$$
$$= s\mathcal{L}^{1}(u_{t+s} \circ \phi_{s}, \lambda_{t}; \Omega_{t}) + \mathcal{L}^{2}_{s}(u_{t+s} \circ \phi_{s}, \lambda_{t}; \Omega_{t})$$

using the maximum in (2.23') with the test function  $p = \lambda_t$ , and the minimum in (2.9') with the test function  $w = u_{t+s} \circ \phi_s$ . Similarly, we calculate the upper bound:

(2.30b) 
$$\mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t) - \mathcal{L}(u_t, \lambda_t; \Omega_t)$$
  
$$\leq \mathcal{L}_s(u_t, \lambda_{t+s} \circ \phi_s; \Omega_t) - \mathcal{L}(u_t, \lambda_{t+s} \circ \phi_s; \Omega_t)$$
  
$$= s\mathcal{L}^1(u_t, \lambda_{t+s} \circ \phi_s; \Omega_t) + \mathcal{L}_s^2(u_t, \lambda_{t+s} \circ \phi_s; \Omega_t)$$

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utilizing the minimum in (2.23') with the test function  $w = u_t$ , and the maximum in (2.9') with the test function  $p = \lambda_{t+s} \circ \phi_s$ . The strong convergences (2.29) provide the asymptotic order of the residuals:

$$\mathcal{L}^{2}_{s_{k}}(u_{t},\lambda_{t+s_{k}}\circ\phi_{s_{k}};\Omega_{t})=\mathrm{o}(s_{k}),\quad\mathcal{L}^{2}_{s_{k}}(u_{t+s_{k}}\circ\phi_{s_{k}},\lambda_{t};\Omega_{t})=\mathrm{o}(s_{k})$$

hence from (2.30) divided with s it follows existence of the limit

(2.31) 
$$\lim_{s_k \to 0} \frac{\mathcal{L}(u_{t+s_k}, \lambda_{t+s_k}; \Omega_{t+s_k}) - \mathcal{L}(u_t, \lambda_t; \Omega_t)}{s_k} = \mathcal{L}^1(u_t, \lambda_t; \Omega_t)$$

because of the identity  $\mathcal{L}(u_{t+s}, \lambda_{t+s}; \Omega_{t+s}) = \mathcal{L}_s(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s; \Omega_t)$ due to (2.24a). The optimal value Lagrangian and objective functions are equal, see (2.13) and the similar identity  $\mathcal{L}(u_{t+s}, \lambda_{t+s}; \Omega_{t+s}) = \mathcal{E}(u_{t+s}; \Omega_{t+s})$ , then (2.31) coincides with formula (2.21) of the shape derivative and completes the proof.

**Remark 2.1.** Theorem 2.2 presents a direct proof of the shape differentiability. Since the bijection (2.20a)-(2.20c) holds, then the Correa-Seeger theorem on directional differentiability can be applied by checking hypotheses (H1)-(H4) in [12, Chapter 10, Theorem 5.1].

To formulate the hypotheses, let us define the optimal values

$$l_t := \sup_{p \in K^{\star}(\Omega_t)} \inf_{w \in V(\Omega_t)} \mathcal{L}(w, p; \Omega_t) \le \inf_{w \in V(\Omega_t)} \sup_{p \in K^{\star}(\Omega_t)} \mathcal{L}(w, p; \Omega_t) =: l^t,$$

and the solution sets

$$V_t = \{ u \in V(\Omega_t) | \sup_{p \in K^*(\Omega_t)} \mathcal{L}(u, p; \Omega_t) = l^t \},$$
  
$$K_t^* = \{ \lambda \in K^*(\Omega_t) | \inf_{w \in V(\Omega_t)} \mathcal{L}(w, \lambda; \Omega_t) = l_t \} \text{ for } t \in (t_0, t_1).$$

(H1) The solution sets are nonempty due to Theorem 2.1. Moreover,  $l_t = l^t$  and  $V_t = \{u_t\}, K_t^* = \{\lambda_t\}$  are singleton.

(H2) For  $t \in (t_0, t_1)$  there exists the partial derivative:

(2.32a) 
$$\lim_{s \to 0} \frac{\mathcal{L}_s(u,\lambda;\Omega_t) - \mathcal{L}(u,\lambda;\Omega_t)}{s} = \mathcal{L}^1(u,\lambda;\Omega_t)$$
$$\forall (u,\lambda) \in \left( \bigcup_{\tau \in (t_0,t_1)} V_\tau \times K_t^\star \right) \cup \left( V_t \times \bigcup_{\tau \in (t_0,t_1)} K_\tau^\star \right)$$

within the asymptotic expansion (2.24b) which is uniform with respect to  $(u, \lambda)$ . This hypothesis holds due to assumptions (2.20d)–(2.20i).

(H3) There exist an accumulation point  $\overline{u} \in V_t$  and a subsequence  $u_{t+s_k} \circ \phi_{s_k} \in V_t$  denoted by  $s_k$  such that

(2.32b) 
$$||u_{t+s_k} \circ \phi_{s_k} - \overline{u}||_{V(\Omega_t)} \to 0 \text{ as } s_k \to 0,$$

which is proved in (2.28a) with  $\overline{u} = u_t$ , and

(2.32c) 
$$\liminf_{s_k \to 0} \mathcal{L}^1(u_{t+s_k} \circ \phi_{s_k}, p; \Omega_t) \ge \mathcal{L}^1(\overline{u}, p; \Omega_t) \quad \forall p \in K_t^\star,$$

that holds due to continuity in the strong topology of the bilinear mapping  $w \mapsto \mathcal{L}^1(w, p; \Omega_t)$ .

(H4) There exist an accumulation point  $\overline{\lambda} \in K_t^*$  and a subsequence  $\lambda_{t+s_k} \circ \phi_{s_k} \in K_t^*$  denoted by  $s_k$  such that

(2.32d) 
$$\|\lambda_{t+s_k} \circ \phi_{s_k} - \overline{\lambda}\|_{H^*(\Omega_t)} \to 0 \text{ as } s_k \to 0,$$

with  $\overline{\lambda} = \lambda_t$  according to (2.28c), and

(2.32e) 
$$\limsup_{s_k \to 0} \mathcal{L}^1(w, \lambda_{t+s_k} \circ \phi_{s_k}; \Omega_t) \le \mathcal{L}^1(w, \overline{\lambda}; \Omega_t) \quad \forall w \in V_t,$$

provided by the weak continuity of the linear mapping  $p \mapsto \mathcal{L}^1(w, p; \Omega_t)$ . Indeed, testing (2.23') with  $(w, p) = (u_t, \lambda_t)$  and (2.9') with (w, p) =

 $(u_{t+s} \circ \phi_s, \lambda_{t+s} \circ \phi_s)$  gives

$$\frac{\mathcal{L}_{s}(u_{t+s}\circ\phi_{s},\lambda_{t};\Omega_{t})-\mathcal{L}(u_{t+s}\circ\phi_{s},\lambda_{t};\Omega_{t})}{s} \leq \frac{\mathcal{L}_{s}(u_{t+s}\circ\phi_{s},\lambda_{t+s}\circ\phi_{s};\Omega_{t})-\mathcal{L}(u_{t},\lambda_{t};\Omega_{t})}{s} =: \Delta(s)$$
$$\leq \frac{\mathcal{L}_{s}(u_{t},\lambda_{t+s}\circ\phi_{s};\Omega_{t})-\mathcal{L}(u_{t},\lambda_{t+s}\circ\phi_{s};\Omega_{t})}{s}.$$

Since we show that the expansion (2.24b) holds, we get

$$\mathcal{L}^{1}(u_{t+s} \circ \phi_{s}, \lambda_{t}; \Omega_{t}) + \frac{1}{s} \mathcal{L}^{2}_{s}(u_{t+s} \circ \phi_{s}, \lambda_{t}; \Omega_{t}) \leq \Delta(s)$$
$$\leq \mathcal{L}^{1}(u_{t}, \lambda_{t+s} \circ \phi_{s}; \Omega_{t}) + \frac{1}{s} \mathcal{L}^{2}_{s}(u_{t}, \lambda_{t+s} \circ \phi_{s}; \Omega_{t})$$

and use (2.32c) and (2.32e) to pass it to the limit as  $s_k \to 0$ , which is essentially the idea of the theorem of Correa–Seeger.

**Remark 2.2.** The assumptions (2.20d)–(2.20i) on the asymptotic expansion can be relaxed in Theorem 2.2 to the abstract conditions (2.32).

We note the important special cases in two corollaries. The first corollary relates the assumption (2.20c) of the dual cones to the primal cones, see [17, Theorem 3.4].

**Corollary 2.1.** If the primal cone (2.4) is such that  $K^*(\Omega_t) = K(\Omega_t)$ , then the assumption (2.20c) is equivalent to bijection of the primal cones

(2.33) 
$$K(\Omega_t) \mapsto K(\Omega_{t+s})$$

and formula of the shape derivative (2.21) implies the equality

$$\frac{d}{dt}\mathcal{E}(u_t;\Omega_t) = \mathcal{E}^1(u_t;\Omega_t), \quad (\lambda_t, B^1 u_t)_{\Omega_t} = 0$$

under the assumptions (2.20) used in Theorem 2.2.

The second corollary extends the result to equality constraints.

**Corollary 2.2.** The inequality constraint in (2.4) can be replaced with the equality constraint resulting in the following primal and dual cones

(2.34) 
$$K(\Omega_t) = \{ u \in V(\Omega_t) | Bu = 0 \}, \quad K^*(\Omega_t) = H^*(\Omega_t).$$

Then the assumption (2.20c) is satisfied within (2.20b), thus Theorem 2.2 holds true under the made assumptions.

In the next section we realize an application of Corollary 2.2 to the Stokes problem with the divergence-free equality constraint, that mapping is not a bijection again.

### 3. Example of shape derivative: Stokes problem

Let  $\Omega_t$  be a domain with Lipschitz continuous boundary, denote by  $n_t$  the outward unit normal vector, and let the boundary  $\partial \Omega_t$  consist of two disjoint sets  $\Gamma_t^D$  and  $\Gamma_t^N$ . For a given stationary external force  $f \in H^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ , we consider the *Stokes problem* finding a vector-valued field of flow velocity  $u_t = ((u_t)_1, \ldots, (u_t)_d)$  and a scalar-valued  $\lambda_t$  implying the pressure such that

(3.35a) 
$$-\Delta u_t + \nabla \lambda_t = f \quad \text{in } \Omega_t$$

(3.35b) 
$$\operatorname{div} u_t = 0 \quad \text{in } \Omega_t$$

(3.35c) 
$$u_t = 0 \quad \text{on } \Gamma_t^L$$

(3.35d) 
$$\frac{\partial}{\partial n_t} u_t - \lambda_t n_t = 0 \quad \text{on } \Gamma_t^N.$$

The mixed boundary conditions imply no-slip (3.35c) and a Neumanntype condition (3.35d). For mixed boundary conditions appropriate for the Stokes equation see [6], [33, Chapter 6].

Corresponding to (3.35) primal minimization problem reads: Find  $u_t \in V(\Omega_t)$  such that  $\operatorname{div} u_t = 0$  and

(3.36) 
$$\mathcal{E}(u_t; \Omega_t) = \min_{w \in K(\Omega_t)} \mathcal{E}(w; \Omega_t).$$

minimizing the objective function of the energy:

(3.37) 
$$\mathcal{E}(w;\Omega_t) = \int_{\Omega_t} \sum_{i=1}^d \left(\frac{1}{2} |\nabla w_i|^2 - f_i w_i\right) dx$$

over the primal cone determined by the divergence-free constraint:

(3.38) 
$$K(\Omega_t) = \{ w \in V(\Omega_t) | \text{ div} w = 0 \text{ a.e. } \Omega_t \}$$

in the function space

(3.39) 
$$V(\Omega_t) = \{ w \in H^1(\Omega_t; \mathbb{R}^d) | w = 0 \text{ a.e. } \Gamma_t^D \}.$$

The operators  $A = -\Delta$  and B = div constitute the respective duality pairings:

(3.40a) 
$$\langle Au, w \rangle_{\Omega_t} = \int_{\Omega_t} \sum_{i=1}^d (\nabla u_i)^\top \nabla w_i \, dx, \quad u, w \in V(\Omega_t)$$

(3.40b) 
$$(\lambda, Bu)_{\Omega_t} = \int_{\Omega_t} \lambda \operatorname{div} u \, dx, \quad \lambda \in H(\Omega_t),$$

and the dual cone

(3.41) 
$$K^{\star}(\Omega_t) = \{\lambda \in H^{\star}(\Omega_t) | \quad (\lambda, Bu)_{\Omega_t} = 0 \quad \forall u \in K(\Omega_t)\},$$

where  $H(\Omega_t) = H^*(\Omega_t) = L^2(\Omega_t; \mathbb{R}).$ 

If the surface measure meas( $\Gamma_t^N$ ) > 0, then the LBB condition (2.12) holds [27, Theorem 7.2], which means that  $B : V(\Omega_t) \mapsto H(\Omega_t)$  is surjective and  $K^*(\Omega_t) = H^*(\Omega_t)$ . So we can apply Corollary 2.2.

If meas( $\Gamma_t^N$ ) = 0, then  $B: H_0^1(\Omega_t; \mathbb{R}^d) \mapsto L_0^2(\Omega_t; \mathbb{R})$ , where

(3.42) 
$$L_0^2(\Omega_t; \mathbb{R}) = \{\lambda \in H(\Omega_t) | \quad (\lambda, 1)_{\Omega_t} = 0\}$$

and its dual space excludes constants. In this case we cannot apply Corollary 2.2. In fact, the bijection in (2.20c) between  $L^2_0(\Omega_t; \mathbb{R})$  and

$$L_0^2(\Omega_{t+s}; \mathbb{R}) = \{ \mu \in H(\Omega_{t+s}) | \quad (\mu, 1)_{\Omega_{t+s}} = 0 \}$$

fails because  $(\mu, 1)_{\Omega_{t+s}} \neq (\mu \circ \phi_s, 1)_{\Omega_t}$  according to the transformation formula (2.20f).

The primal-dual formulation of (3.36) consists in finding the pair  $(u_t, \lambda_t) \in V(\Omega_t) \times L^2(\Omega_t; \mathbb{R})$  which is a saddle-point:

(3.43) 
$$\mathcal{L}(u_t, \lambda_t; \Omega_t) = \min_{w \in V(\Omega_t)} \max_{p \in L^2(\Omega_t; \mathbb{R})} \mathcal{L}(w, p; \Omega_t)$$

of the Lagrangian

(3.44) 
$$\mathcal{L}(w, p; \Omega_t) = \mathcal{E}(w; \Omega_t) - \int_{\Omega_t} p \operatorname{div} w \, dx$$

where the dual cone  $K^{\star}(\Omega_t) = L^2(\Omega_t; \mathbb{R})$  according to (3.41). The optimality conditions (2.11) for the problems (3.37) and (3.44) have the form:

(3.45a) 
$$\int_{\Omega_t} \left( \sum_{i=1}^d (\nabla(u_t)_i^\top \nabla w_i - f_i w_i) - \lambda_t \operatorname{div} w \right) dx = 0 \quad \forall w \in V(\Omega_t)$$

(3.45b) 
$$\int_{\Omega_t} p \operatorname{div} u_t \, dx = 0 \quad \forall p \in L^2(\Omega_t; \mathbb{R}).$$

The solution pair is unique since the LBB condition holds in this case.

For a stationary kinematic velocity  $\Lambda \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$  the unique solutions  $[s \mapsto \phi_s], [s \mapsto \phi_s^{-1}] \in C^1([-T,T]; W^{1,\infty}_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d))$  of the autonomous ODE systems with some T > 0:

$$\begin{cases} \frac{d}{ds}\phi_s = \Lambda(\phi_s) & \text{for } s \neq 0 \\ \phi_s = x & \text{for } s = 0, \end{cases} \begin{cases} \frac{d}{ds}\phi_s^{-1} = -\Lambda(\phi_s^{-1}) & \text{for } s \neq 0 \\ \phi_s^{-1} = y & \text{for } s = 0 \end{cases}$$

satisfy (2.14) and build the diffeomorphism (2.16), see [17, Lemma 2.2]. In the non-stationary case, the velocity  $\Lambda \in C([-T, T]; W_{\text{loc}}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d))$  is defined by  $\Lambda(t + s, y) = \frac{d}{ds}\phi_s(\phi_s^{-1}(y))$ , see [47, Section 2.9]. By this, the transformation matrix  $\nabla_y \phi_s^{-1} := \{(\phi_s^{-1})_{i,j}\}_{i,j=1}^d$ , where  $(\phi_s^{-1})_{i,j} = \frac{\partial(\phi_s^{-1})_i}{\partial y_j}$ , and the Jacobian determinant  $\det(\nabla \phi_s)$  of the matrix  $\nabla \phi_s := \{(\phi_s)_{i,j}\}_{i,j=1}^d$ , where  $(\phi_s)_{i,j} = \frac{\partial(\phi_s)_i}{\partial x_j}$ , admit the following asymptotic expansion as  $s \to 0$ :

(3.46) 
$$\nabla_y \phi_s^{-1}(\phi_s) = I - s \nabla \Lambda + r_s^1, \quad |\nabla \phi_s| = 1 + s \operatorname{div} \Lambda + r_s^2,$$

with the uniform estimate of the residuals  $||r_s^1||_{C([-T,T];L_{loc}^{\infty}(\mathbb{R}^{d\times d}))} = o(s)$ and  $||r_s^2||_{C([-T,T];L_{loc}^{\infty}(\mathbb{R}))} = o(s)$ , where  $\nabla \Lambda = \{\frac{\partial \Lambda_i}{\partial x_j}\}_{i,j=1}^d$ , and I stands for the *d*-by-*d*-identity matrix.

We apply the coordinate transformation  $y = \phi_s(x)$  to the duality pairings in (3.40) rewritten over the perturbed domain  $\Omega_{t+s}$  according to (2.15). As the result, using the chain rule  $\nabla_y = (\nabla_y \phi_s^{-1}(\phi_s))^\top \nabla_x$  and (3.46) we derive the following asymptotic expansions corresponding to the assumptions (2.20d)–(2.20i). Indeed, the operator A is expanded as follows for  $v, \chi \in H^1(\Omega_{t+s}; \mathbb{R}^d)$ :

$$(3.47a) \quad \langle Av, \chi \rangle_{\Omega_{t+s}} = \int_{\Omega_{t+s}} \sum_{i=1}^{d} (\nabla_y v_i)^\top \nabla_y \chi_i \, dy$$
$$= \int_{\Omega_t} \sum_{i=1}^{d} (\nabla(v_i \circ \phi_s))^\top \nabla_y \phi_s^{-1}(\phi_s) (\nabla_y \phi_s^{-1}(\phi_s))^\top \nabla(\chi_i \circ \phi_s)) \det(\nabla \phi_s) dx$$
$$= \int_{\Omega_t} \sum_{i=1}^{d} (\nabla(v_i \circ \phi_s))^\top (I + s\{(\operatorname{div}\Lambda)I - \nabla\Lambda - (\nabla\Lambda)^\top\}) \nabla(\chi_i \circ \phi_s)) \, dx + o(s)$$

implying (2.20d) and (2.20e) with the first asymptotic term

(3.47b) 
$$\langle A^1 u, w \rangle_{\Omega_t} = \int_{\Omega_t} \sum_{i=1}^d (\nabla u_i)^\top \{ (\operatorname{div} \Lambda) I - \nabla \Lambda - (\nabla \Lambda)^\top \} \nabla w_i \, dx.$$

Accordingly, for  $\mu \in L^2(\Omega_{t+s}; \mathbb{R})$  the operator B is expanded as

$$(3.47c) \quad (\mu, Bv)_{\Omega_{t+s}} = \int_{\Omega_{t+s}} \mu \operatorname{div}_y v \, dy$$
$$= \int_{\Omega_t} (\mu \circ \phi_s) \sum_{i,j=1}^d (\phi_s^{-1})_{j,i} (v \circ \phi_s)_{i,j} \det(\nabla \phi_s) \, dx$$

which implies (2.20f) and (2.20g) with

(3.47d) 
$$(\lambda, B^1 u)_{\Omega_t} = \int_{\Omega_t} \lambda\{(\operatorname{div} \Lambda)(\operatorname{div} u) - \sum_{i,j=1}^d \Lambda_{j,i} u_{i,j}\} dx$$

for  $u, w \in H^1(\Omega_t; \mathbb{R}^d)$  and  $\lambda \in L^2(\Omega_t; \mathbb{R})$ . And the transformation

(3.47e) 
$$\langle f, v \rangle_{\Omega_{t+s}} = \int_{\Omega_{t+s}} \sum_{i=1}^d f_i v_i \, dy$$
  
$$= \int_{\Omega_t} \sum_{i=1}^d (f_i \circ \phi_s) (v_i \circ \phi_s) \det(\nabla \phi_s) dx$$

due to (3.46) and  $f_i \circ \phi_s = f_i + s\Lambda^{\top} \nabla f_i + o(s)$  follows (2.20h) and (2.20i) with the first asymptotic term

(3.47f) 
$$\langle f^1, u \rangle_{\Omega_t} = \int_{\Omega_t} \sum_{i=1}^d ((\operatorname{div} \Lambda) f_i + \Lambda^\top \nabla f_i) u_i \, dx.$$

The decompositions (3.47) agree the assumptions (2.20a) and (2.20b).

The assumption of bijection (2.33) is not true for the primal cone (3.38) because of the transformation of the divergence (see formula (3.47c)). Nevertheless, the bijection of the dual cone allows us to apply Theorem 2.2 in the form of Corollary 2.2. The shape differentiability of the Stokes problem based on (3.47) and using div $u_t = 0$  is established in the next theorem.

**Theorem 3.1.** The Stokes problem given in (3.36)–(3.38) has the shape derivative  $\frac{d}{dt}\mathcal{E}(u_t;\Omega_t) = \mathcal{L}^1(u_t,\lambda_t;\Omega_t)$  which is defined in (2.21) and calculated according to formula (2.22) as follows

(3.48a) 
$$\mathcal{L}^{1}(u_{t}, \lambda_{t}; \Omega_{t}) = \mathcal{E}^{1}(u_{t}; \Omega_{t}) + \int_{\Omega_{t}} \lambda_{t} \sum_{i,j=1}^{d} \Lambda_{j,i}(u_{t})_{i,j} dx$$

(3.48b) 
$$\mathcal{E}^{1}(u_{t};\Omega_{t}) = \int_{\Omega_{t}} \sum_{i=1}^{d} \left( \frac{1}{2} (\operatorname{div}\Lambda) |\nabla(u_{t})_{i}|^{2} - \sum_{k,j=1}^{d} (u_{t})_{i,k} \Lambda_{k,j}(u_{t})_{i,j} - \left( (\operatorname{div}\Lambda) f_{i} + \Lambda^{\top} \nabla f_{i} \right) (u_{t})_{i} \right) dx.$$

We remark the singularity at the intersection  $\overline{\Gamma_t^D} \cap \overline{\Gamma_t^N}$  (see e.g. [3]) such that  $(u_t, \lambda_t)$  is generally not in  $H^2(\Omega_t; \mathbb{R}^d) \times H^1(\Omega_t; \mathbb{R})$  as shown in [40, Theorem 1.3.2]. Let the singular points are contained locally in a domain  $\overline{\omega_t} \subset \overline{\Omega_t}$  such that  $(u_t, \lambda_t) \in H^2(\Omega_t \setminus \omega_t; \mathbb{R}^d) \times H^1(\Omega_t \setminus \omega_t; \mathbb{R})$ , and  $f, \Lambda \equiv \text{const in } \omega_t$ . In this case, using integration of (3.48) by parts we get the following expression over the boundary of  $\Omega_t \setminus \omega_t$ :

$$\mathcal{L}^{1}(u_{t},\lambda_{t};\Omega_{t}) = \int_{\partial(\Omega_{t}\setminus\omega_{t})} \sum_{i=1}^{d} \left( (\Lambda^{\top}n_{t}) \left(\frac{1}{2} |\nabla(u_{t})_{i}|^{2} - f_{i}(u_{t})_{i} \right) - (\Lambda^{\top}\nabla(u_{t})_{i}) \left(\frac{\partial}{\partial n_{t}}(u_{t})_{i} - \lambda_{t}(n_{t})_{i}\right) \right) dS_{x},$$

which implies the generalized J-integral (see [2, 40]).

In the case of  $\Gamma_t^N = \emptyset$ , to preserve the integral (see (3.42)), this needs special area-preserving maps that form special linear group SL(d) as stated in the last result.

**Corollary 3.1.** Let the problem (3.35) be stated under solely no-slip Dirichlet condition  $u_t = 0$  on  $\partial \Omega_t = \Gamma_t^D$ . If the transformation  $y = \phi_s(x)$  is characterized by the Jacobian determinant  $\det(\nabla \phi_s) = 1$ , then formula (3.48) in Theorem 3.1 still holds true with  $\operatorname{div} \Lambda = 0$ .

Examples of such area-preserving bijection are translation and rotation of bodies obeying circular or cylindrical symmetry that maps the body into itself.

### 4. CONCLUSION

The result of the shape sensitivity analysis is useful in structure optimization, see e.g. [1]. In particular, a positive/ negative sign of the shape derivative forces respectively either increase or decay of the objective function  $\mathcal{E}$  of the energy.

For further development in the shape differentiability of Lagrangians, we may suggest to combine Theorem 2.2 together with Corollary 2.2 in order to account simultaneously for both equality and inequality type constraints within polyhedral cones. The example is the Stokes problem under the threshold slip boundary condition, see [37, 43]. Acknowledgment. V.A.K. is supported by the Austrian Science Fund (FWF) project P26147-N26: "Object identification problems: numerical analysis" (PION) and the Austrian Academy of Sciences (OeAW). K.O. is supported by the JSPS KAKENHI Grant Number 16K05285. The joint work began in CoMFoS15 that is the workshop by the Activity group MACM (Mathematical Aspects of Continuum Mechanics) of JSIAM. The authors thank two referees for the comments which helped to improve the manuscript.

### References

- G. Allaire, F. Jouve and A.-M. Toader, Structural optimization using sensitivity analysis and a level-set method, J. Comput. Phys. 194 (2004), 363–393.
- [2] H. Azegami, K. Ohtsuka and M. Kimura, Shape derivative of cost function for singular point: evaluation by the generalized J integral, JSIAM Lett. 6 (2014), 29–32.
- [3] M. Beneš, The qualitative properties of the Stokes and Navier–Stokes system for the mixed problem in a nonsmooth domain, *Math. Comput. Simulation* 76 (2007), 8–12.
- [4] J.F. Bonnans and A. Shapiro, Perturbation Analysis of Optimization Problems, Springer, New York, 2000.
- [5] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2010.
- [6] R. Brown, I. Mitrea, M. Mitrea and M. Wright, Mixed boundary value problems for the Stokes system, *Trans. Amer. Math. Soc.* 362 (2010), 1211–1230.
- [7] M. Bulíček, J. Haslinger, J. Málek and J. Stebel, Shape optimization for Navier-Stokes equations with algebraic turbulence model: existence analysis, *Appl. Math. Optim.* **60** (2009), 185–212.
- [8] G.P. Ciarlet, Linear and Nonlinear Functional Analysis with Applications, SIAM, Philadelphia, 2013.
- [9] R. Correa and A. Seeger, Directional derivative of a minimax function, Nonlinear Anal. Theory Methods Appl. 9 (1985), 834–862.
- [10] M.C. Delfour and J.-P. Zolésio, Shape sensitivity analysis via min max differentiability, SIAM J. Control Optim. 26 (1988), 1414–1442.
- [11] M.C. Delfour and J.-P. Zolésio, Velocity method and Lagrangian formulation for the computation of the shape Hessian, SIAM J. Control Optim. 29 (2006), 1414–1442.
- [12] M.C. Delfour and J.-P. Zolésio, Shape and Geometries: Metrics, Analysis, Differential Calculus, and Optimization, SIAM, Philadelphia, 2011.
- [13] I. Ekeland and R. Temam, Convex Analysis and Variational Problems, North-Holland, Amsterdam, 1976.
- [14] V. Girault and P.-A. Raviart, Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms, Springer, Berlin, 1986.
- [15] J. Haslinger, K. Ito, T. Kozubek, K. Kunisch and G. Peichl, On the shape derivative for problems of Bernoulli type, *Interfaces Free Bound.* **11** (2009), 317–330.
- [16] J. Haslinger, J. Stebel and T. Sassi, Shape optimization for Stokes problem with threshold slip, Appl. Math. 59 (2014), 631–652.

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- [17] M. Hintermüller and V.A. Kovtunenko, From shape variation to topology changes in constrained minimization: a velocity method-based concept, *Opti*mization Meth. Software 26 (2011), 513–532.
- [18] K. Ito and K. Kunisch, Lagrange Multiplier Approach to Variational Problems and Applications, SIAM, Philadelphia, PA, 2008.
- [19] H. Itou, A. M. Khludnev, E.M. Rudoy and A. Tani, Asymptotic behaviour at a tip of a rigid line inclusion in linearized elasticity, Z. Angew. Math. Mech. 92 (2012), 716–730.
- [20] H. Itou, V.A. Kovtunenko and K.R. Rajagopal, Nonlinear elasticity with limiting small strain for cracks subject to non-penetration, *Math. Mech. Solids* 22 (2017), 1334–1346.
- [21] H. Itou, V.A. Kovtunenko and A. Tani, The interface crack with Coulomb friction between two bonded dissimilar elastic media, *Appl. Math.* 56 (2011), 69–97.
- [22] A.M. Khludnev and V.A. Kovtunenko, Analysis of Cracks in Solids, WIT-Press, Southampton, Boston, 2000.
- [23] A.M. Khludnev, V.A. Kovtunenko, A. Tani, Evolution of a crack with kink and non-penetration, J. Math. Soc. Japan 60 (2008), 1219–1253.
- [24] A.M. Khludnev, V.A. Kovtunenko, A. Tani, On the topological derivative due to kink of a crack with non-penetration. Anti-plane model, J. Math. Pures Appl. 94 (2010), 571–596.
- [25] A.M. Khludnev and J. Sokolowski, Modelling and Control in Solid Mechanics, Birkhäuser, Basel, 1997.
- [26] A.M. Khludnev, K. Ohtsuka and J. Sokolowski, On derivative of energy functional for elastic bodies with cracks and unilateral conditions, *Quart. Appl. Math.* **60** (2002), 99–109.
- [27] N. Kikuchi and J.T. Oden, Contact Problems in Elasticity: a Study of Variational Inequalities and Finite Element Methods, SIAM, Philadelphia, PA, 1988.
- [28] V.A. Kovtunenko, Primal-dual methods of shape sensitivity analysis for curvilinear cracks with non-penetration, IMA J. Appl. Math. 71 (2006), 635–657.
- [29] V.A. Kovtunenko and K. Kunisch, Problem of crack perturbation based on level sets and velocities. Z. angew. Math. Mech. 87 (2007), 809–830.
- [30] V.A. Kovtunenko and K. Kunisch, High precision identification of an object: optimality conditions based concept of imaging, SIAM J. Control Optim. 52 (2014), 773–796.
- [31] V.A. Kovtunenko, K. Kunisch and W. Ring, Propagation and bifurcation of cracks based on implicit surfaces and discontinuous velocities, *Comput. Visual Sci.* 12 (2009), 397–408.
- [32] V.A. Kovtunenko and G. Leugering, A shape-topological control problem for nonlinear crack - defect interaction: the anti-plane variational model, SIAM J. Control. Optim. 54 (2016), 1329–1351.
- [33] V.A. Kozlov, V.G. Mazya and J. Rossmann, Spectral Problems Associated with Corner Singularities of Solutions to Elliptic Equations, AMS, Providence, 2001.
- [34] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Science Publishers, New York, 1969.

- [35] N.P. Lazarev and E.M. Rudoy, Shape sensitivity analysis of Timoshenko's plate with a crack under the nonpenetration condition. Z. angew. Math. Mech. 94 (2014), 730–739.
- [36] G. Leugering, J. Sokolowski and A. Zochowski, Shape-topological differentiability of energy functionals for unilateral problems in domains with cracks and applications, In: *Optimization with PDE Constraints; ESF Networking Pro*gram 'OPTPDE', R. Hoppe, ed. (2014), 203–221.
- [37] C. Le Roux and A. Tani, Steady solutions of the Navier–Stokes equations with threshold slip boundary conditions, *Math. Meth. Appl. Sci.* **30** (2007), 595–624.
- [38] A.U. Maharani, M. Kimura, H. Azegami, K. Ohtsuka and I. Armanda, Shape optimization approach to a free boundary problem, *Recent Development Comput. Sci.* 6, Kanazawa e-Publishing (2015), 42–55.
- [39] M. Moubachir and J.-P. Zolésio, Moving Shape Analysis and Control, Chapman & Hall/CRC, Boca Raton, 2006.
- [40] K. Ohtsuka, Shape optimization by generalized J-integral in Poissons equation with a mixed boundary condition, In: *Math. Anal. Cont. Mech. Ind. Appl. II* (*Proc. CoMFoS16*), P. van Meurs, M. Kimura, H. Notsu, eds., Springer (2018), 73–83.
- [41] K. Ohtsuka and M. Kimura, Differentiability of potential energies with a parameter and shape sensitivity analysis for nonlinear case: the p-Poisson problem, Jpn. J. Ind. Appl. Math. 29 (2012), 23–35.
- [42] N. Ovcharova and J. Gwinner, From solvability and approximation of variational inequalities to solution of nondifferentiable optimization problems in contact mechanics, *Optimization* 64 (2015), 1683–1702.
- [43] I.J. Rao and K.R. Rajagopal, The effect of the slip boundary condition on the flow of fluids in a channel, Acta Mechanica 135 (1999), 113–126.
- [44] R.T. Rockafellar, Convex Analysis, Princeton Univ. Press, 1970.
- [45] E.M. Rudoy, Differentiation of energy functionals in two-dimensional elasticity theory for solids with curvilinear cracks, J. Appl. Mech. Techn. Phys. 54 (2004), 843–852.
- [46] V.V. Shcherbakov, Shape derivative of the energy functional for the bending of elastic plates with thin defects. J. Phys.: Conf. Ser. 894 (2017), 012084.
- [47] J. Sokolowski and J.-P. Zolésio, Introduction to Shape Optimization. Shape Sensitivity Analysis, Springer, Berlin, Heidelberg, 1992.