GLOBAL SUPERCONVERGENCE OF THE LOWEST ORDER MIXED FINITE ELEMENT ON MILDLY STRUCTURED MESHES

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Abstract. In this paper, we develop global superconvergence estimates for the lowest order Raviart–Thomas mixed finite element method for second order elliptic equations with general boundary conditions on triangular meshes, where most pairs of adjacent triangles form approximate parallelograms. In particular, we prove the L^2-distance between the numerical solution and canonical interpolant for the vector variable is of order $1+\rho$, where $\rho \in (0,1]$ is dependent on the mesh structure. By a cheap local postprocessing operator G_h, we prove the L^2-distance between the exact solution and the postprocessed numerical solution for the vector variable is of order $1+\rho$. As a byproduct, we also obtain the superconvergence estimate for Crouzeix–Raviart nonconforming finite elements on triangular meshes of the same type.

Key words. superconvergence, mildly structured grids, mixed methods, Raviart–Thomas elements, Crouzeix–Raviart elements, a posteriori error estimation

AMS subject classifications. 65N50, 65N30

1. Introduction.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. For simplicity of presentation, we assume Ω is a polygon. The Sobolev seminorms and norms are defined by

$$|u|_{k,p,\Omega} = \left(\sum_{|\alpha|=k} \int_\Omega |\partial^\alpha u|^p\right)^{\frac{1}{p}}, \quad ||u||_{k,p,\Omega} = \left(\sum_{l=0}^{k} |u|^p_{l,p,\Omega}\right)^{\frac{1}{p}},$$
$$|u|_{k,\Omega} = |u|_{k,2,\Omega}, \quad ||u||_{k,\Omega} = ||u||_{k,2,\Omega}.$$

Sobolev norms with ∞-index, norms of vector/matrix-valued functions, and fractional order norms are generalized in usual ways.

We consider the following second order elliptic equation:

$$-\operatorname{div}(\boldsymbol{A}(\boldsymbol{x})\nabla u + \boldsymbol{b}(\boldsymbol{x})u) + c(\boldsymbol{x})u = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.1a}$$
$$u = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega, \tag{1.1b}$$

where $\boldsymbol{A}$ is symmetric and uniformly elliptic, $\boldsymbol{A}, \boldsymbol{b}, c$ are sufficiently smooth on $\overline{\Omega}$. Let

$$\boldsymbol{p} = \boldsymbol{A}(\boldsymbol{x})\nabla u + \boldsymbol{b}(\boldsymbol{x})u$$

and set

$$\boldsymbol{\alpha} = \boldsymbol{A}(\boldsymbol{x})^{-1}, \quad \boldsymbol{\beta} = \boldsymbol{\alpha}(\boldsymbol{x})\boldsymbol{b}(\boldsymbol{x}).$$

(1.1) can be written in the form of the first order system:

$$\boldsymbol{\alpha}\boldsymbol{p} - \boldsymbol{\beta}u - \nabla u = 0, \quad \boldsymbol{x} \in \Omega, \tag{1.2a}$$
$$-\operatorname{div}\boldsymbol{p} + cu = f, \quad \boldsymbol{x} \in \Omega, \tag{1.2b}$$
$$u = g, \quad \boldsymbol{x} \in \partial\Omega. \tag{1.2c}$$

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Denote

$$\mathcal{Q} = \{\boldsymbol{q} \in L^2(\Omega)^2 : \operatorname{div} \boldsymbol{q} \in L^2(\Omega)\}, \quad \mathcal{V} = L^2(\Omega).$$

Let $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote the $L^2(\Omega)$ and $L^2(\partial\Omega)$ inner product, respectively. Let $\boldsymbol{n}$ denote the outward unit normal to $\partial\Omega$. The mixed formulation for (1.2) is to find $\{\boldsymbol{p}, u\} \in \mathcal{Q} \times \mathcal{V}$, such that

$$(\boldsymbol{\alpha p}, \boldsymbol{q}) - (\boldsymbol{q}, \boldsymbol{\beta} u) + (\operatorname{div} \boldsymbol{q}, u) = \langle \boldsymbol{q} \cdot \boldsymbol{n}, g\rangle, \tag{1.3a}$$

$$-(\operatorname{div} \boldsymbol{p}, v) + (cu, v) = (f, v), \tag{1.3b}$$

for each pair $\{\boldsymbol{q}, v\} \in \mathcal{Q} \times \mathcal{V}$. Let $\{\mathcal{T}_h\}$ be a family of triangulations of Ω, where $0 < h < 1$ is the mesh size. Let $\mathcal{P}_p(\tau)$ denote the set of polynomials of degree $\le p$ on τ. Denote

$$\mathcal{RT}_0(\tau) := \{\boldsymbol{a} + a\boldsymbol{x} : \boldsymbol{a} \in \mathbb{R}^2,\ a \in \mathbb{R}\}. \tag{1.4}$$

The lowest order Raviart–Thomas (RT) finite element spaces are defined by

$$\begin{aligned}\mathcal{Q}_h &:= \{\boldsymbol{q}_h \in \mathcal{Q} : \boldsymbol{q}_h|_\tau \in \mathcal{RT}_0(\tau),\ \forall \tau \in \mathcal{T}_h\},\\ \mathcal{V}_h &:= \{v_h \in \mathcal{V} : v_h|_\tau \in \mathcal{P}_0(\tau),\ \forall \tau \in \mathcal{T}_h\},\end{aligned}$$

The mixed finite element approximation to the problem (1.3) is to find $\{\boldsymbol{p}_h, u_h\} \in \mathcal{Q}_h \times \mathcal{V}_h$, such that

$$(\boldsymbol{\alpha p}_h, \boldsymbol{q}_h) - (\boldsymbol{q}_h, \boldsymbol{\beta} u_h) + (\operatorname{div} \boldsymbol{q}_h, u_h) = \langle \boldsymbol{q}_h \cdot \boldsymbol{n}, g\rangle, \quad \boldsymbol{q}_h \in \mathcal{Q}_h, \tag{1.5a}$$

$$-(\operatorname{div} \boldsymbol{p}_h, v_h) + (cu_h, v_h) = (f, v_h), \quad v_h \in \mathcal{V}_h. \tag{1.5b}$$

Under the assumption that (1.2) is solvable for $\{f, g\} \in L^2(\Omega) \times H^{\frac{3}{2}}(\Omega)$ and that

$$||u||_{2,\Omega} \lesssim ||f||_{0,\Omega} + ||g||_{\frac{3}{2},\Omega}, \tag{1.6}$$

Douglas and Roberts [13] proved the well-posedness and a priori error estimates for the method (1.5).

In this paper, we shall prove supercloseness/superconvergence results for $||\Pi_h\boldsymbol{p} - \boldsymbol{p}_h||_{0,\Omega}$ and $||\operatorname{div}(\Pi_h\boldsymbol{p} - \boldsymbol{p}_h)||_{0,\Omega}$, where Π_h and P_h are the interpolation operators for the lowest order RT element. In particular, we shall prove that

$$||\Pi_h\boldsymbol{p} - \boldsymbol{p}_h||_{0,\Omega} \lesssim h^{1+\rho}||u||_{4+\varepsilon,\Omega}, \quad \varepsilon > 0, \tag{1.7a}$$

$$||\operatorname{div}(\Pi_h\boldsymbol{p} - \boldsymbol{p}_h)||_{0,\Omega} \lesssim h^2||u||_{3,\Omega}, \tag{1.7b}$$

where $\rho = \min(1, \alpha, \sigma/2)$. (1.7b) holds on general shape regular meshes while (1.7a) holds on quasi-uniform $\{\mathcal{T}_h\}$ satisfying the piecewise (α,σ)-condition. The (α,σ)-grid or its simplified versions have been considered by many authors (cf. [21, 19, 3, 18, 28] and references therein). Roughly speaking, $\mathcal{T}_h$ is said to be an (α,σ)-grid if most pairs of adjacent triangles in $\mathcal{T}_h$ form $\mathcal{O}(h^{1+\alpha})$ approximate parallelograms except for a region of measure $\mathcal{O}(h^\sigma)$ (cf. Definition 3.5). (1.7a) has several generalizations. For example, the quasi-uniformity assumption can be removed under the pure Neumann boundary condition or at the expense of a slower superconvergence rate ρ, see section 3 and Theorem 4.7 for details.

(1.7) is closely related to the superconvergence of the finite element solution to the exact solution. For example, by postprocessing $\boldsymbol{p}_h$ by a simple local averaging operator G_h proposed in [8], we achieve the following superconvergence estimate:

$$||\boldsymbol{p} - G_h\boldsymbol{p}_h||_{0,\Omega} \lesssim h^{1+\rho}||u||_{4+\varepsilon,\Omega}. \tag{1.8}$$

The recovered flux $G_h\boldsymbol{p}_h$ can be used to develop a posteriori error estimates. Due to the superconvergence (1.8), $||G_h\boldsymbol{p}_h - \boldsymbol{p}_h||_{0,\Omega}$ is known to be an asymptotically exact a posteriori estimator for $||\boldsymbol{p} - \boldsymbol{p}_h||_{0,\Omega}$ (cf. [8, 4, 1]), that is,

$$\lim_{h\to 0} \frac{||G_h\boldsymbol{p}_h - \boldsymbol{p}_h||_{0,\Omega}}{||\boldsymbol{p} - \boldsymbol{p}_h||_{0,\Omega}} = 1.$$

As a byproduct, (1.8) also gives the following superconvergence estimate for Crouzeix–Raviart (CR) nonconforming finite elements (cf. Theorem 5.6):

$$||\nabla u - G_h\nabla_h u_h^{CR}||_{0,\Omega} \lesssim h^{1+\rho}||u||_{4+\varepsilon,\Omega}, \tag{1.9}$$

where u_h^{CR} is the CR finite element solution of Poisson's equation.

The study of supercloseness between the finite element interpolant and finite element solution has a long history. For the analogue of (1.7a) for standard Lagrange elements on mildly structured grids, see [3, 28, 18] and references therein. For superconvergence of the scalar variable u in mixed methods, see [2, 10, 24] and references therein. In practice, it is frequently the case that the vector variable $\boldsymbol{p}$ is more important than the scalar u. Superconvergence results of rectangular/quadrilateral mixed finite elements for the vector variable $\boldsymbol{p}$ are well established (cf. [14, 15, 16]). However, corresponding superconvergence theory of triangular mixed finite elements are much less sophisticated. To our best knowledge, the only proven superconvergence estimate of triangular mixed elements for the vector variable are in [12, 8, 7]. In [12], the authors postprocessed $\boldsymbol{p}_h$ and achieved interior superconvergence by convolution with a Bramble–Schatz kernel [6] which is constructed on uniform grids, i.e. in the case of $\alpha = \sigma = \infty$. For the lowest order RT element on uniform grids in the case that $\boldsymbol{b} = \boldsymbol{0}, c = 0$ in (1.1), Brandts [8] proved

$$||\Pi_h\boldsymbol{p} - \boldsymbol{p}_h||_{0,\Omega} \lesssim h^{\frac{3}{2}}(||\boldsymbol{p}||_{\frac{3}{2},\Omega} + h^{\frac{1}{2}}|\boldsymbol{p}|_{1,\Omega} + h^{\frac{1}{2}}|\boldsymbol{p}|_{2,\Omega}), \tag{1.10}$$

In [7], he also proved an analogue of (1.10) for second order RT elements on uniform grids in the case that $\boldsymbol{A} = I_{2\times 2}, \boldsymbol{b} = \boldsymbol{0}, c = 0$.

Our result (1.7) improves existing results significantly in several ways. First, our estimate holds on general mildly structured grids instead of uniform grids. As pointed out in [3, 28], the (α, σ)-condition is very flexible and satisfied by many mature finite element codes. Second, in the best case that $\rho = 1$, (1.7a) becomes

$$||\Pi_h\boldsymbol{p} - \boldsymbol{p}_h||_{0,\Omega} \lesssim h^2||u||_{4+\varepsilon,\Omega},$$

which shows that the estimate (1.10) is suboptimal. This improvement results from carefully handling the boundary error, which is usually the trickiest part in global superconvergence estimates if test functions have nonzero trace. In addition, due to the cancellation of errors on boundary elements, (1.7a) holds on not only (α, σ)-grids but also piecewise (α, σ)-grids (cf. Definition 3.6 and Remark 3.8). Third, our superconvergence results allow the convection term $\boldsymbol{b}(x)\cdot\nabla u$ and reaction term $c(x)u$. Unlike the case of the standard variational formulation for elliptic equations, the error analysis of mixed methods with nonvanishing $\boldsymbol{b}, c$ is much more involved than the case $\boldsymbol{b} = \boldsymbol{0}, c = 0$ (cf. [13] and Remark 4.6). Last, the superconvergence estimate (1.9) for CR nonconforming elements is obtained. Since u_h^{CR} has jump on each interior edge, it is very difficult to prove superconvergence of nonconforming methods on triangular grids directly (cf. [17] and references therein).

The key ingredient of the proof of (1.7a) is two fold. First, we develop the variational error expansion for RT elements on a local triangle in terms of $\boldsymbol{q}_h \cdot \boldsymbol{n}_k$, the normal trace of $\boldsymbol{q}_h \in \mathcal{Q}_h$ on e_k, where $\{e_k\}_{k=1}^3$ are three edges of the triangle and $\boldsymbol{n}_k$ is the outward unit normal to e_k. Due to the continuity of $\boldsymbol{q}_h \cdot \boldsymbol{n}_k$ on e_k and the (α, σ)-condition, the lower order global variational error associated with interior edges is canceled in a very delicate and transparent way instead of using soft analysis tools (the Bramble–Hilbert lemma etc., cf. [8]). The aforementioned basic idea is motivated by Bank and Xu [3]. But the technicality here is quite different because of the apparent difference between Lagrange elements and RT elements. Second, we split $\Pi_h \boldsymbol{p} - \boldsymbol{p}_h$ into two parts by the discrete Helmholtz decomposition Lemma 4.4. The norm of one part can be estimated by (1.7b) and Theorem 4.1. To obtain optimal order global superconvergence, the error associated with another part occurring on triangles near the boundary is treated carefully by the Sobolev and discrete Sobolev inequalities, see section 3 for details.

The rest of this paper is organized as follows: section 2 contains technical geometric identities and local error expansions. In section 3, we estimate the global variational error that forms a basis for the estimate (1.7). The superconvergence result (1.7) and related results are presented in section 4. In section 5, we develop the superconvergence estimate (1.8) and the related estimate (1.9) for CR nonconforming elements. In section 6 we present a few numerical examples illustrating the optimality and flexibility of our estimates.

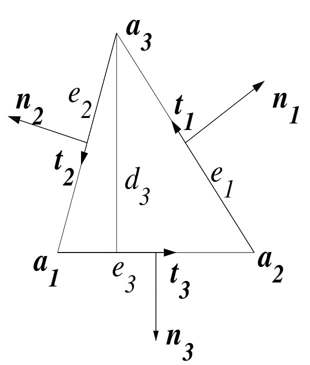


FIG. 1. *a local triangle and associated quantities*

2. Preliminaries. We begin with geometric identities on a local element τ. It has three vertices $\{\boldsymbol{a}_k\}_{k=1}^3$, oriented counterclockwise, and corresponding barycentric coordinates $\{\lambda_k\}_{k=1}^3$. Let e_k denote the edge opposite to $\boldsymbol{a}_k$, θ_k the angle opposite to e_k, ℓ_k the length of e_k, d_k the distance from $\boldsymbol{a}_k$ to e_k, $\boldsymbol{t}_k$ the unit tangent to e_k, oriented counterclockwise, $\boldsymbol{n}_k$ the unit outward normal to e_k, see Figure 1. Corresponding quantities on τ' and τ'' have superscripts $'$ and $''$ respectively. The subscripts are equivalent mod 3. From Bank and Xu [3], we have the following identities:

$$\boldsymbol{t}_k = \frac{\cos\theta_{k+1}}{\sin\theta_k}\boldsymbol{n}_{k+1} - \frac{\cos\theta_{k-1}}{\sin\theta_k}\boldsymbol{n}_{k-1}, \tag{2.1a}$$

$$\boldsymbol{n}_{k-1} = -\sin\theta_{k+1}\boldsymbol{t}_k - \cos\theta_{k+1}\boldsymbol{n}_k, \quad \boldsymbol{t}_{k-1} = -\cos\theta_{k+1}\boldsymbol{t}_k + \sin\theta_{k+1}\boldsymbol{n}_k, \tag{2.1b}$$

$$\boldsymbol{n}_{k+1} = \sin\theta_{k-1}\boldsymbol{t}_k - \cos\theta_{k-1}\boldsymbol{n}_k, \quad \boldsymbol{t}_{k+1} = -\cos\theta_{k-1}\boldsymbol{t}_k - \sin\theta_{k-1}\boldsymbol{n}_k, \tag{2.1c}$$

$$\sin\theta_k \int_{e_{k+1}} v\lambda_k\lambda_{k-1} = \sin\theta_{k+1}\int_{e_k} v\lambda_{k+1}\lambda_{k-1} - \int_\tau \frac{\partial v}{\partial \boldsymbol{t}_{k-1}}(1-\lambda_{k-1})\lambda_{k-1}, \tag{2.1d}$$

$$\sin\theta_k \int_{e_{k-1}} v\lambda_k\lambda_{k+1} = \sin\theta_{k-1}\int_{e_k} v\lambda_{k+1}\lambda_{k-1} + \int_\tau \frac{\partial v}{\partial \boldsymbol{t}_{k+1}}(1-\lambda_{k+1})\lambda_{k+1}, \tag{2.1e}$$

$$\nabla\lambda_k = -\boldsymbol{n}_k/d_k. \tag{2.1f}$$

In addition, we have two planar curl operators

$$\overrightarrow{\nabla}\times v = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\right)^t, \quad \nabla\times\boldsymbol{q} = \frac{\partial q_2}{\partial x_1} - \frac{\partial q_1}{\partial x_2}.$$

For convenience, we define the matrix

$$\mathbf{rot} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.2}$$

It's clear that **rot** rotates a vector by degree $\pi/2$ counterclockwise. By direct calculation, we have the following identities:

$$\mathbf{rot}\,\boldsymbol{n}_k = \boldsymbol{t}_k, \quad \mathbf{rot}\,\boldsymbol{t}_k = -\boldsymbol{n}_k, \tag{2.3a}$$

$$\nabla = \mathbf{rot}\,\overrightarrow{\nabla}\times, \quad \nabla\times = \operatorname{div}\mathbf{rot}^{-1}, \tag{2.3b}$$

$$\overrightarrow{\nabla}\times(vw) = v\overrightarrow{\nabla}\times w + w\overrightarrow{\nabla}\times v, \tag{2.3c}$$

$$\nabla\times(v\boldsymbol{q}) = -(\overrightarrow{\nabla}\times v)\cdot\boldsymbol{q} + v\nabla\times\boldsymbol{q}, \tag{2.3d}$$

$$\int_\tau v\nabla\times\boldsymbol{q} = \sum_{k=1}^{3}\int_{e_k} v\boldsymbol{q}\cdot\boldsymbol{t}_k + \int_\tau \overrightarrow{\nabla}\times v\cdot\boldsymbol{q}, \tag{2.3e}$$

$$\overrightarrow{\nabla}\times\lambda_i = \boldsymbol{t}_i/d_i. \tag{2.3f}$$

Now we introduce basic definitions for RT elements. On the element τ, the degrees of freedom of the lowest order RT elements are defined by

$$\mathcal{N}_k(\boldsymbol{q}) = \int_{e_k}\boldsymbol{q}\cdot\boldsymbol{n}_k, \quad 1 \le k \le 3.$$

For $\boldsymbol{q} \in \mathcal{Q}$, the interpolant $\Pi_h\boldsymbol{q}$ is the element in $\mathcal{Q}_h$ whose restriction to τ is the unique element in $\mathcal{RT}_0(\tau)$ such that

$$\mathcal{N}_k(\Pi_h\boldsymbol{q}) = \mathcal{N}_k(\boldsymbol{q}), \quad 1 \le k \le 3. \tag{2.4}$$

For $v \in \mathcal{V}$, the interpolant $P_h v$ is the $L^2(\Omega)$-projection of v onto $\mathcal{V}_h$. P_h and Π_h are connected by the following commuting diagram, which is crucial to the stability and error analysis of mixed methods (cf. [22]).

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "\operatorname{div}"] \arrow[d, "\Pi_h"'] & \mathcal{V} \arrow[d, "P_h"] & \\ \mathcal{Q}_h \arrow[r, "\operatorname{div}"] & \mathcal{V}_h \arrow[r] & 0 \end{tikzcd} \tag{2.5}$$

In addition, the following approximation properties hold:

$$||\boldsymbol{q} - \Pi_h \boldsymbol{q}||_{0,\Omega} \lesssim h||\nabla_h \boldsymbol{q}||_{0,\Omega}, \tag{2.6a}$$

$$||\operatorname{div}(\boldsymbol{q} - \Pi_h \boldsymbol{q})||_{0,\Omega} \lesssim h||\nabla_h \operatorname{div} \boldsymbol{q}||_{0,\Omega}, \tag{2.6b}$$

$$||v - P_h v||_{0,\Omega} \lesssim h||\nabla_h v||_{0,\Omega}. \tag{2.6c}$$

where ∇_h is the piecewise gradient.

The following facts can be checked in a straightforward way:

$$\phi_k = \lambda_{k+1} \overrightarrow{\nabla} \times \lambda_{k-1} - \lambda_{k-1} \overrightarrow{\nabla} \times \lambda_{k+1}, \quad 1 \le k \le 3,$$

is the dual basis of $\{\mathcal{N}_k\}_{k=1}^3$. $\{\phi_k\}_{k=1}^3$ in Cartesian coordinate are

$$\phi_k = \frac{\boldsymbol{x} - \boldsymbol{a}_k}{2|\tau|}, \quad 1 \le k \le 3. \tag{2.7}$$

$\{\phi_k\}_{k=1}^3$ together with

$$\psi_k = \lambda_{k+1} \overrightarrow{\nabla} \times \lambda_{k-1} + \lambda_{k-1} \overrightarrow{\nabla} \times \lambda_{k+1}, \quad 1 \le k \le 3, \tag{2.8}$$

form a basis of $\mathcal{P}_1(\tau)^2$. $\mathcal{N}_i$ vanishes at ψ_j for $1 \le i, j \le 3$.

It turns out that the CR interpolation is very useful in the analysis of RT elements. For $\boldsymbol{q} \in H^1(\tau)^2$, the local CR interpolant $\mathrm{I}_h^{CR} \boldsymbol{q}$ on τ is the unique element in $\mathcal{P}_1(\tau)^2$ such that

$$\int_{e_k} \mathrm{I}_h^{CR} \boldsymbol{q} = \int_{e_k} \boldsymbol{q}, \quad 1 \le k \le 3. \tag{2.9}$$

In addition, I_h^{CR} and Π_h are connected by the following lemma.

Lemma 2.1.

$$\Pi_h \mathrm{I}_h^{CR} \boldsymbol{q} = \Pi_h \boldsymbol{q}.$$

Proof. It follows from (2.4) and (2.9) that

$$\mathcal{N}_k(\Pi_h \mathrm{I}_h^{CR} \boldsymbol{q} - \Pi_h \boldsymbol{q}) = \int_{e_k} (\mathrm{I}_h^{CR} \boldsymbol{q} - \boldsymbol{q}) \cdot \boldsymbol{n}_k = 0.$$

Lemma 2.1 is then from the unisolvence of RT elements. ∎

Now, we expand the interpolation error for linear functions.

Lemma 2.2. *For $\boldsymbol{p}_L \in \mathcal{P}_1(\tau)^2$,*

$$\boldsymbol{p}_L - \Pi_h \boldsymbol{p}_L = \overrightarrow{\nabla} \times r,$$

where

$$r = -\sum_{k=1}^{3} \frac{\ell_k^2}{2} \boldsymbol{n}_k \cdot \frac{\partial \boldsymbol{p}_L}{\partial \boldsymbol{t}_k} \lambda_{k-1} \lambda_{k+1}.$$

Proof. First, it follows from $\mathcal{N}_i(\psi_k) = 0$ and $\mathcal{N}_i(\boldsymbol{p}_L - \Pi_h \boldsymbol{p}_L) = 0$ that

$$\boldsymbol{p}_L - \Pi_h \boldsymbol{p}_L = \sum_{k=1}^{3} \alpha_k \psi_k.$$

Then by (2.8), (2.3b), and (2.3c), we arrive at

$$\mathbf{rot}(\boldsymbol{p}_L - \Pi_h \boldsymbol{p}_L) = \nabla \left(\sum_{k=1}^{3} \alpha_k \lambda_{k-1} \lambda_{k+1} \right) = \nabla r. \tag{2.10}$$

It remains to verify that

$$\alpha_k = -\frac{\ell_k^2}{2} \boldsymbol{n}_k \cdot \frac{\partial \boldsymbol{p}_L}{\partial \boldsymbol{t}_k}. \tag{2.11}$$

Taking inner products with $\boldsymbol{t}_k$ and then taking the directional derivative along $\boldsymbol{t}_k$ on both sides of (2.10) leads to

$$\boldsymbol{n}_k \cdot \left(\frac{\partial \boldsymbol{p}_L}{\partial \boldsymbol{t}_k} - \frac{\partial \Pi_h \boldsymbol{p}_L}{\partial \boldsymbol{t}_k} \right) = \frac{\partial^2 r}{\partial \boldsymbol{t}_k^2}. \tag{2.12}$$

The definition of $\mathcal{RT}_0(\tau)$ (1.4) implies that $\partial \Pi_h \boldsymbol{p}_L / \partial \boldsymbol{t}_k$ is parallel to $\boldsymbol{t}_k$ and therefore

$$\boldsymbol{n}_k \cdot \frac{\partial \Pi_h \boldsymbol{p}_L}{\partial \boldsymbol{t}_k} = 0. \tag{2.13}$$

For the right hand side,

$$\frac{\partial^2 r}{\partial \boldsymbol{t}_k^2} = 2\alpha_k \frac{\partial \lambda_{k-1}}{\partial \boldsymbol{t}_k} \frac{\partial \lambda_{k+1}}{\partial \boldsymbol{t}_k} = -\frac{2\alpha_k}{\ell_k^2}. \tag{2.14}$$

Combing (2.12), (2.13) and (2.14), we obtain (2.11). ∎

3. Variational error expansions. The following is our main technical lemma for estimating the global variational error of mixed methods.

LEMMA 3.1. *For $\boldsymbol{q}_h \in \mathcal{P}_0(\tau)$,*

$$\int_\tau (\boldsymbol{p}_L - \Pi_h \boldsymbol{p}_L) \cdot \boldsymbol{q}_h = \sum_{k=1}^{3} \cot \theta_k \int_{e_k} \lambda_{k-1} \lambda_{k+1} \left(\sum_{j=1}^{3} \alpha_k^{(j)} \mathcal{A}_k^{(j)} \boldsymbol{p}_L \right) \boldsymbol{q}_h \cdot \boldsymbol{n}_k, \tag{3.1}$$

where

$$\alpha_k^{(1)} = |\tau|, \quad \alpha_k^{(2)} = -|\tau|, \quad \alpha_k^{(3)} = \frac{1}{2}(\ell_{k-1}^2 - \ell_{k+1}^2), \tag{3.2}$$

and $\mathcal{A}_k^{(j)}$ are operators defined by

$$\mathcal{A}_k^{(1)} = \boldsymbol{t}_k \cdot \frac{\partial}{\partial \boldsymbol{t}_k}, \quad \mathcal{A}_k^{(2)} = \boldsymbol{n}_k \cdot \frac{\partial}{\partial \boldsymbol{n}_k}, \quad \mathcal{A}_k^{(3)} = \boldsymbol{n}_k \cdot \frac{\partial}{\partial \boldsymbol{t}_k}. \tag{3.3}$$

Proof. Using (2.3e) and Lemma 2.2, we have

$$\int_\tau (\boldsymbol{p}_L - \Pi_h \boldsymbol{p}_L) \cdot \boldsymbol{q}_h = -\sum_{k=1}^{3} \int_{e_k} r \boldsymbol{q}_h \cdot \boldsymbol{t}_k.$$

Therefore, it follows from (2.1a), (2.1d), and (2.1e) that

$$\begin{aligned}
&\int_\tau (\boldsymbol{p}_L - \Pi_h \boldsymbol{p}_L)\cdot \boldsymbol{q}_h \\
&= \sum_{k=1}^{3} \int_{e_k} \frac{\ell_k^2}{2}\boldsymbol{n}_k \cdot \frac{\partial \boldsymbol{p}_L}{\partial \boldsymbol{t}_k}\lambda_{k-1}\lambda_{k+1}\boldsymbol{q}_h\cdot \boldsymbol{t}_k \\
&= \sum_{k=1}^{3} \int_{e_k} \frac{\ell_k^2}{2}\boldsymbol{n}_k \cdot \frac{\partial \boldsymbol{p}_L}{\partial \boldsymbol{t}_k}\lambda_{k-1}\lambda_{k+1}\left(\frac{\cos\theta_{k+1}}{\sin\theta_k}\boldsymbol{q}_h\cdot\boldsymbol{n}_{k+1} - \frac{\cos\theta_{k-1}}{\sin\theta_k}\boldsymbol{q}_h\cdot\boldsymbol{n}_{k-1}\right) \\
&= \sum_{k=1}^{3}\left\{\int_{e_{k-1}} \frac{\ell_{k-1}^2}{2}\boldsymbol{n}_{k-1}\cdot\frac{\partial \boldsymbol{p}_L}{\partial \boldsymbol{t}_{k-1}}\lambda_k\lambda_{k+1}\frac{\cos\theta_k}{\sin\theta_{k-1}}\right. \\
&\qquad\qquad \left. - \int_{e_{k+1}} \frac{\ell_{k+1}^2}{2}\boldsymbol{n}_{k+1}\cdot\frac{\partial \boldsymbol{p}_L}{\partial \boldsymbol{t}_{k+1}}\lambda_k\lambda_{k-1}\frac{\cos\theta_k}{\sin\theta_{k+1}}\right\}\boldsymbol{q}_h\cdot\boldsymbol{n}_k \\
&= \sum_{k=1}^{3}\cot\theta_k \int_{e_k}\lambda_{k+1}\lambda_{k-1}\left(\frac{\ell_{k-1}^2}{2}\boldsymbol{n}_{k-1}\cdot\frac{\partial \boldsymbol{p}_L}{\partial \boldsymbol{t}_{k-1}} - \frac{\ell_{k+1}^2}{2}\boldsymbol{n}_{k+1}\cdot\frac{\partial \boldsymbol{p}_L}{\partial \boldsymbol{t}_{k+1}}\right)\boldsymbol{q}_h\cdot\boldsymbol{n}_k.
\end{aligned}$$

Then by (2.1b) and (2.1c) and following identities:

$$\begin{aligned}
&\ell_{k-1}\sin\theta_{k+1} = \ell_{k+1}\sin\theta_{k-1} = d_k, \\
&\ell_{k-1}^2\cos^2\theta_{k+1} - \ell_{k+1}^2\cos^2\theta_{k-1} = \ell_{k-1}^2 - \ell_{k+1}^2, \\
&\ell_{k+1}\cos\theta_{k-1} + \ell_{k-1}\cos\theta_{k+1} = \ell_k,
\end{aligned}$$

and direct calculation, we obtain (3.1). ∎

Now we state definitions of $\mathcal{O}(h^{1+\alpha})$ approximate parallelograms and mildly structured grids in [3] below with a little generalization.

DEFINITION 3.2. *Let e be an edge in the triangulation $\mathcal{T}_h$. Let τ and τ' be the two adjacent elements sharing e. We say that τ and τ' form an $\mathcal{O}(h^{1+\alpha})$ approximate parallelogram if the lengths of any two opposite edges differ only by $\mathcal{O}(h^{1+\alpha})$.*

The boundary elements need more delicate treatment.

DEFINITION 3.3. *Let x be a vertex in $\mathcal{T}_h$ on $\partial\Omega$. Let e and e' be the two boundary edges sharing x as an endpoint, and let $\boldsymbol{t}$ and $\boldsymbol{t}'$ be the unit tangents, oriented counterclockwise. Let τ and τ' be the two adjacent elements having e and e' as edges respectively. Number e and e' as a pair of corresponding edges. By going along the boundaries of τ and τ' counterclockwise, we have other two pairs of corresponding edges. We say that τ and τ' form an $\mathcal{O}(h^{1+\alpha})$ approximate parallelogram if the lengths of any two corresponding edges differ only by $\mathcal{O}(h^{1+\alpha})$, and $|\boldsymbol{t}-\boldsymbol{t}'| = \mathcal{O}(h^\alpha)$.*

Remark 3.4. τ and τ' in Definition 3.3 don't form an approximate parallelogram in the *usual* sense, since they have no common edge.

DEFINITION 3.5. *The triangulation $\mathcal{T}_h$ satisfies the (α,σ)-condition if the following hold:*

1. *Let $\mathcal{E} = \mathcal{E}_1 \biguplus \mathcal{E}_2$ be the set of interior edges. For each $e \in \mathcal{E}_1$, τ and τ' form an $\mathcal{O}(h^{1+\alpha})$ approximate parallelogram, while $\sum_{e\in\mathcal{E}_2}|\tau| + |\tau'| = \mathcal{O}(h^\sigma)$.*
2. *Let $\mathcal{P} = \mathcal{P}_1 \biguplus \mathcal{P}_2$ be the set of boundary vertices. The adjacent boundary elements τ, τ' in Definition 3.3 associated with each $x \in \mathcal{P}_1$ form an $\mathcal{O}(h^{1+\alpha})$*

approximate parallelogram, and $|\mathcal{P}_2| = \kappa$ *is a finite number independent of* h.

For example, we have $\alpha = \sigma = \infty, \mathcal{E}_2 = \emptyset, \kappa = 4$ for the uniform grid in Figure 3(a).

DEFINITION 3.6. *Let* Ω *be decomposed into* N *subdomains, where* N *is independent of* h. $\mathcal{T}_h$ *is said to satisfy the piecewise* (α,σ)*-condition if the restriction of* $\mathcal{T}_h$ *to each subdomain satisfies the* (α,σ)*-condition.*

With the above definitions, we can present the main lemma.

LEMMA 3.7. *Let* $\mathcal{T}_h$ *be quasi-uniform and satisfy the* (α,σ)*-condition. Let* $\boldsymbol{q}_h \in \vec{\nabla}\times\mathcal{S}_h$*, where* $\mathcal{S}_h$ *consists of continuous piecewise linear polynomials on* $\mathcal{T}_h$*. Then*

$$|(\boldsymbol{p}-\Pi_h\boldsymbol{p},\boldsymbol{q}_h)| \lesssim h^{1+\rho}|\log h|^{\frac{1}{2}}||\nabla\boldsymbol{p}||_{1,\infty,\Omega}||\boldsymbol{q}_h||_{0,\Omega}, \tag{3.4}$$

where

$$\rho = \min(1,\alpha,\frac{\sigma}{2}).$$

Proof. By Lemmas 2.1 and 3.1 and passing through $\mathrm{I}_h^{CR}\boldsymbol{p}$, we have

$$\begin{aligned}
&(\boldsymbol{p}-\Pi_h\boldsymbol{p},\boldsymbol{q}_h)\\
&= (\boldsymbol{p}-\mathrm{I}_h^{CR}\boldsymbol{p},\boldsymbol{q}_h) + \sum_{\tau\in\mathcal{T}_h}\int_\tau(\mathrm{I}_h^{CR}\boldsymbol{p}-\Pi_h\,\mathrm{I}_h^{CR}\boldsymbol{p})\cdot\boldsymbol{q}_h\\
&= (\boldsymbol{p}-\mathrm{I}_h^{CR}\boldsymbol{p},\boldsymbol{q}_h)\\
&+ \sum_{\tau\in\mathcal{T}_h}\sum_{k=1}^{3}\cot\theta_k\int_{e_k}\lambda_{k-1}\lambda_{k+1}\left(\sum_{j=1}^{3}\alpha_k^{(j)}\mathcal{A}_k^{(j)}(\mathrm{I}_h^{CR}\boldsymbol{p}-\boldsymbol{p})\right)\boldsymbol{q}_h\cdot\boldsymbol{n}_k\\
&+ \sum_{\tau\in\mathcal{T}_h}\sum_{k=1}^{3}\cot\theta_k\int_{e_k}\lambda_{k-1}\lambda_{k+1}\left(\sum_{j=1}^{3}\alpha_k^{(j)}\mathcal{A}_k^{(j)}\boldsymbol{p}\right)\boldsymbol{q}_h\cdot\boldsymbol{n}_k\\
&= I + II + III.
\end{aligned}\tag{3.5}$$

Parts I and II can be simply estimated by the standard finite element interpolation theory:

$$|I| \lesssim h^2|\boldsymbol{p}|_{2,\Omega}||\boldsymbol{q}_h||_{0,\Omega}, \tag{3.6}$$

and

$$\begin{aligned}
|II| &\lesssim \sum_{\tau\in\mathcal{T}_h} h\int_\tau|\nabla(\mathrm{I}_h^{CR}\boldsymbol{p}-\boldsymbol{p})|\cdot|\boldsymbol{q}_h| + h^2\int_\tau|\nabla^2\boldsymbol{p}|\cdot|\boldsymbol{q}_h|\\
&\lesssim h^2|\boldsymbol{p}|_{2,\Omega}||\boldsymbol{q}_h||_{0,\Omega}.
\end{aligned}\tag{3.7}$$

The main task is to bound part III. For $e\subset\partial\Omega$, let τ be the element having e as an edge. For $e\in\mathcal{E}$, let τ and τ' be the two elements sharing e. Let $\boldsymbol{t}_e$ and $\boldsymbol{n}_e$ denote the unit tangent and normal to e whose directions are consistent with τ. Let $\mathcal{A}_e^{(j)}$ denote the operators in (3.3) corresponding to $\boldsymbol{t}_e$ and $\boldsymbol{n}_e$, θ_e the angle opposite to e in τ, ℓ_e the length of e, $\alpha_e^{(j)}$ the quantity associated with e on τ in (3.2). Corresponding quantities on τ' have a superscript $\prime$. Denote

$$b_e = \lambda_{k-1}\lambda_{k+1},\quad \beta_e^{(j)} = \alpha_e^{(j)}\cot\theta_e - \alpha_e^{(j)\prime}\cot\theta_e'.$$

$\boldsymbol{q}_h \in \mathcal{Q}_h$ implies that $\boldsymbol{q}_h|_\tau \cdot \boldsymbol{n}_e = \boldsymbol{q}_h|_{\tau'} \cdot \boldsymbol{n}_e$ on e. Thus we can transform III from element-wise summation to edge-wise summation:

$$III = III_1 + III_2 + III_3,$$

where

$$III_i = \sum_{e\in\mathcal{E}_i} \int_e b_e \left(\sum_{j=1}^{3} \beta_e^{(j)} \mathcal{A}_e^{(j)} \boldsymbol{p} \right) \boldsymbol{q}_h \cdot \boldsymbol{n}_e, \quad i = 1, 2,$$
$$III_3 = \sum_{e\subset\partial\Omega} \cot\theta_e \int_e b_e \left(\sum_{j=1}^{3} \alpha_e^{(j)} \mathcal{A}_e^{(j)} \boldsymbol{p} \right) \boldsymbol{q}_h \cdot \boldsymbol{n}_e.$$

For $e \in \mathcal{E}_1$, the fact that τ and τ' form an $\mathcal{O}(h^{1+\alpha})$ approximate parallelogram implies

$$|\beta_e^{(j)}| \lesssim h^{2+\alpha}.$$

Combining the above inequality with the trace inequality

$$\int_{\partial\tau} |f| \lesssim h^{-1} \int_\tau |f| + \int_\tau |\nabla f| \tag{3.8}$$

leads to

$$\begin{aligned}|III_1| &\lesssim \sum_{e\in\mathcal{E}_1} h^{2+\alpha} \left\{ h^{-1} \int_\tau |\nabla \boldsymbol{p}| \cdot |\boldsymbol{q}_h| + \int_\tau |\nabla^2 \boldsymbol{p}| \cdot |\boldsymbol{q}_h| \right\} \\ &\lesssim h^{1+\alpha} \sum_{e\in\mathcal{E}_1} \int_\tau (|\nabla \boldsymbol{p}| + |\nabla^2 \boldsymbol{p}|) \cdot |\boldsymbol{q}_h| \\ &\lesssim h^{1+\alpha} ||\nabla \boldsymbol{p}||_{1,\Omega} ||\boldsymbol{q}_h||_{0,\Omega}.\end{aligned} \tag{3.9}$$

For $e \in \mathcal{E}_2$, there is no cancellation. Due to the small measure of the region covered by elements near $e \in \mathcal{E}_2$, III_2 is estimated by

$$\begin{aligned}|III_2| &\lesssim h^3 \sum_{e\in\mathcal{E}_2} ||\nabla \boldsymbol{p}||_{0,\infty,\tau} ||\boldsymbol{q}_h||_{0,\infty,\tau} \\ &\lesssim h ||\nabla \boldsymbol{p}||_{0,\infty,\Omega} \sum_{e\in\mathcal{E}_2} \int_\tau |\boldsymbol{q}_h| \\ &\lesssim h^{1+\frac{\sigma}{2}} ||\nabla \boldsymbol{p}||_{0,\infty,\Omega} ||\boldsymbol{q}_h||_{0,\Omega}.\end{aligned} \tag{3.10}$$

The trickiest part of this proof is to bound III_3. Let $\boldsymbol{q}_h = \overrightarrow{\nabla} \times w_h$, where $w_h \in \mathcal{S}_h$. We can assume that $\int_\Omega w_h = 0$ by subtracting a constant from w_h. Then by the Poincaré inequality, we have

$$||w_h||_{1,\Omega} \lesssim ||\boldsymbol{q}_h||_{0,\Omega}. \tag{3.11}$$

Denote

$$B_e^{(j)} = \alpha_e^{(j)} \mathcal{A}_e^{(j)} \boldsymbol{p} \cot\theta_e, \quad \bar{B}_e^{(j)} = \frac{1}{\ell_e} \int_e B_e.$$

By (2.3a) and (2.3b),

$$\boldsymbol{q}_h \cdot \boldsymbol{n}_e = \frac{\partial w_h}{\partial \boldsymbol{t}_e}.$$

Then part III_3 can be split into:

$$\begin{aligned} III_3 &= \sum_{e\subset\partial\Omega} \int_e b_e \sum_{j=1}^{3} (B_e^{(j)} - \bar{B}_e^{(j)}) \frac{\partial w_h}{\partial \boldsymbol{t}_e} \\ &+ \sum_{e\subset\partial\Omega} \int_e b_e \sum_{j=1}^{3} \bar{B}_e^{(j)} \frac{\partial w_h}{\partial \boldsymbol{t}_e} = III_3^{(1)} + III_3^{(2)}. \end{aligned}$$

The first term can be estimated by (3.11):

$$\begin{aligned} |III_3^{(1)}| &\lesssim h^3 |\boldsymbol{p}|_{2,\infty,\Omega} \sum_{e\subset\partial\Omega} \int_e |\nabla w_h| \\ &\lesssim h^2 |\boldsymbol{p}|_{2,\infty,\Omega} \sum_{e\subset\partial\Omega} \int_\tau |\nabla w_h| \\ &\lesssim h^2 |\boldsymbol{p}|_{2,\infty,\Omega} ||\boldsymbol{q}_h||_{0,\Omega}. \end{aligned} \tag{3.12}$$

For $x \in \mathcal{P}$, let e, e' be the two edges on $\partial\Omega$ sharing x as an ending point. Then the second term becomes

$$\begin{aligned} III_3^{(2)} &= \sum_{e\subset\partial\Omega} \frac{\ell_e}{6} \sum_{j=1}^{3} \bar{B}_e^{(j)} \frac{\partial w_h}{\partial \boldsymbol{t}_e} \\ &= \sum_{x\in\mathcal{P}} \frac{1}{6} \sum_{j=1}^{3} \left(\bar{B}_e^{(j)} - \bar{B}_{e'}^{(j)} \right) w_h(x). \end{aligned}$$

For $x \in \mathcal{P}_1$, definitions (3.2) and (3.3) together with the (α, σ)-condition along the boundary imply cancellation and thus

$$\left| \bar{B}_e^{(j)} - \bar{B}_{e'}^{(j)} \right| \lesssim h^{2+\alpha} ||\nabla \boldsymbol{p}||_{1,\infty,\Omega}. \tag{3.13}$$

For $x \in \mathcal{P}_2$,

$$\left| \bar{B}_e^{(j)} - \bar{B}_{e'}^{(j)} \right| \le \left| \bar{B}_e^{(j)} \right| + \left| \bar{B}_{e'}^{(j)} \right| \lesssim h^2 ||\nabla \boldsymbol{p}||_{0,\infty,\Omega}. \tag{3.14}$$

It follows from the discrete Sobolev inequality

$$||w_h||_{0,\infty,\Omega} \lesssim |\log h|^{\frac{1}{2}} ||w_h||_{1,\Omega}, \tag{3.15}$$

the quasi-uniformity, (3.11), (3.13), and (3.14) that

$$\begin{aligned} |III_3^{(2)}| &\lesssim \left(\sum_{x\in\mathcal{P}_1} h^{2+\alpha} ||\nabla \boldsymbol{p}||_{1,\infty,\Omega} + \sum_{x\in\mathcal{P}_2} h^2 ||\nabla \boldsymbol{p}||_{0,\infty,\Omega} \right) ||w||_{0,\infty,\partial\Omega} \\ &\lesssim h^{1+\alpha} |\log h|^{\frac{1}{2}} ||\nabla \boldsymbol{p}||_{1,\infty,\Omega} ||w_h||_{1,\Omega} \\ &\lesssim h^{1+\alpha} |\log h|^{\frac{1}{2}} ||\nabla \boldsymbol{p}||_{1,\infty,\Omega} ||\boldsymbol{q}_h||_{0,\Omega}. \end{aligned} \tag{3.16}$$

Now, combining (3.5)–(3.7), (3.9), (3.10), (3.12), and (3.16), we obtain Lemma 3.7. ∎

In the rest of this paper, ρ *always* denotes $\min(1, \alpha, \sigma/2)$ unless confusion arises.

Remark 3.8. The proof of Lemma 3.7 is completely local and thus (3.4) holds on piecewise (α, σ)-grids.

Lemma 3.7 can be easily generalized. By checking the proof of Lemma 3.7, one can see that the quasi-uniformity is only used to guarantee the discrete Sobolev inequality and bound the number of vertices lying on $\partial\Omega$. By test function $\boldsymbol{q}_h \in \overrightarrow{\nabla} \times S_h$ for the Neumann boundary condition having zero normal trace, we have the following estimate.

COROLLARY 3.9. *Let $\mathcal{T}_h$ be an (α, σ)-grid without assuming adjacent boundary elements form approximate parallelograms in Definition 3.5. Then for $\boldsymbol{q}_h \in \overrightarrow{\nabla} \times \mathcal{S}_h$ and $\boldsymbol{q}_h \cdot \boldsymbol{n} = 0$,*

$$|(\boldsymbol{p} - \Pi_h \boldsymbol{p}, \boldsymbol{q}_h)| \lesssim h^{1+\rho} \left(||\nabla \boldsymbol{p}||_{0,\infty,\Omega} + ||\boldsymbol{p}||_{2,\Omega} \right) ||\boldsymbol{q}_h||_{0,\Omega}.$$

Proof. The proof is basically the same as Lemma 3.7. But we get rid of bounding III_3, the error occuring on boundary elements. Then the regularity assumption is weakened. ∎

Let diamτ denote the diameter of τ. The quasi-uniformity in Lemma 3.7 can be replaced by

$$\min_{\tau \in \mathcal{T}_h} \mathrm{diam}\tau \gtrsim h^\gamma, \quad \gamma \geq 1. \tag{3.17}$$

COROLLARY 3.10. *Assume the condition (3.17) instead of quasi-uniformity in Lemma 3.7. Then (3.4) holds with a smaller ρ:*

$$\rho = \min(1, 1 + \alpha - \gamma, \frac{\sigma}{2}).$$

Proof. The proof is basically the same as Lemma 3.7. (3.17) ensures the discrete Sobolev inequality (cf. [9]). The number of boundary vertices is bounded by

$$|\mathcal{P}| \lesssim h^{-\gamma}.$$

Therefore, the only difference is that the bound for $III_3^{(2)}$ becomes

$$|III_3^{(2)}| \lesssim h^{1+\min(1+\alpha-\gamma,1)} |\log h|^{\frac{1}{2}} ||\nabla \boldsymbol{p}||_{1,\infty,\Omega} ||\boldsymbol{q}_h||_{0,\Omega}.$$

∎

Remark 3.11. The condition (3.17) also appears in pointwise a posteriori error estimation (cf. [1]).

At the end of this section, we show that the logarithmic factor in Lemma 3.7 can be removed by using the following lemma proved by Brandts [8].

LEMMA 3.12. *Let $\Omega_h = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \leq h\}$. Then*

$$||u||_{0,\Omega_h} \lesssim h^s ||u||_{s,\Omega}, \quad 0 \leq s \leq \frac{1}{2}.$$

COROLLARY 3.13. *Assume the same conditions in Lemma 3.7. Then $\forall \varepsilon > 0$,*

$$|(\boldsymbol{p} - \Pi_h \boldsymbol{p}, \boldsymbol{q}_h)| \lesssim h^{1+\rho} ||\nabla \boldsymbol{p}||_{2+\varepsilon,\Omega} ||\boldsymbol{q}_h||_{0,\Omega},$$

Proof. The bounds for $I, II, III_1, III_2, III_3^{(1)}$ are the same as Lemma 3.7. The bound for $III_3^{(2)}$ is improved by the Sobolev embedding $H^{2+\frac{\varepsilon}{2}}(\Omega_h) \subset W^1_\infty(\Omega_h)$ and Lemma 3.12:

$$\begin{aligned}
|III_3^{(2)}| &\lesssim \left(\sum_{x\in\mathcal{P}_1} h^{2+\alpha} ||\nabla \boldsymbol{p}||_{1,\infty,\Omega_h} + \sum_{x\in\mathcal{P}_2} h^2 ||\nabla \boldsymbol{p}||_{0,\infty,\Omega_h} \right) ||w||_{0,\infty,\partial\Omega} \\
&\lesssim h^{1+\alpha} |\log h|^{\frac{1}{2}} ||\nabla \boldsymbol{p}||_{1,\infty,\Omega_h} ||w_h||_{1,\Omega} \\
&\lesssim h^{1+\alpha} |\log h|^{\frac{1}{2}} ||\nabla \boldsymbol{p}||_{2+\frac{\varepsilon}{2},\Omega_h} ||\boldsymbol{q}_h||_{0,\Omega} \\
&\lesssim h^{1+\alpha+\frac{\varepsilon}{2}} |\log h|^{\frac{1}{2}} ||\nabla \boldsymbol{p}||_{2+\varepsilon,\Omega} ||\boldsymbol{q}_h||_{0,\Omega}.
\end{aligned}$$

Therefore, $|\log h|^{1/2}$ is compensated by $h^{\varepsilon/2}$. ∎

Remark 3.14. The factor $|\log h|^{1/2}$ appearing in Lemma 2.5 and Theorem 3.1 in [3] can be removed in the same way.

4. Superconvergence results. From (1.3) and (1.5), we have the error equation

$$(\boldsymbol{\alpha}(\boldsymbol{p} - \boldsymbol{p}_h), \boldsymbol{q}_h) - (\boldsymbol{q}_h, \boldsymbol{\beta}(u - u_h)) + (\operatorname{div} \boldsymbol{q}_h, u - u_h) = 0, \quad \boldsymbol{q}_h \in \mathcal{Q}_h, \tag{4.1a}$$

$$-(\operatorname{div}(\boldsymbol{p} - \boldsymbol{p}_h), v_h) + (c(u - u_h), v_h) = 0, \quad v_h \in \mathcal{V}_h. \tag{4.1b}$$

From [13], we have the following a priori error estimates:

$$||\boldsymbol{p} - \boldsymbol{p}_h||_{0,\Omega} \lesssim h||u||_{2,\Omega}, \tag{4.2a}$$

$$||\operatorname{div}(\boldsymbol{p} - \boldsymbol{p}_h)||_{0,\Omega} \lesssim h||u||_{3,\Omega}, \tag{4.2b}$$

$$||u - u_h||_{0,\Omega} \lesssim h||u||_{2,\Omega}. \tag{4.2c}$$

The following is the well-known superconvergence result for $||P_h u - u_h||_{0,\Omega}$ on general unstructured meshes (cf. [13]).

THEOREM 4.1.

$$||P_h u - u_h||_{0,\Omega} \lesssim h^2 ||u||_{3,\Omega}.$$

Then we prove the superconvergence for $||\operatorname{div}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h)||_{0,\Omega}$.

THEOREM 4.2.

$$||\operatorname{div}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h)||_{0,\Omega} \lesssim h^2 ||u||_{3,\Omega}.$$

Proof. From (2.5), (2.6), (4.1), and (4.2) and Theorem 4.1, it follows that for $v_h \in \mathcal{V}_h$,

$$\begin{aligned}
(\operatorname{div}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h), v_h) &= (P_h \operatorname{div} \boldsymbol{p} - \operatorname{div} \boldsymbol{p}_h, v_h) \\
&= (\operatorname{div}(\boldsymbol{p} - \boldsymbol{p}_h), v_h) \\
&= (u - P_h u, c v_h) + (P_h u - u_h, c v_h) \\
&= (u - P_h u, c v_h - P_h(c v_h)) + \mathcal{O}(h^2)||u||_{3,\Omega} ||v_h||_{0,\Omega} \\
&= \mathcal{O}(h^2) ||u||_{3,\Omega} ||v_h||_{0,\Omega}.
\end{aligned} \tag{4.3}$$

Therefore, Theorem 4.2 follows from setting $v_h = \operatorname{div}(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h)$ in (4.3). ∎

Before proving the superconvergence result for $||\Pi_h \boldsymbol{p} - \boldsymbol{p}_h||_{0,\Omega}$, it is necessary to state two lemmas. The first lemma is due to Raviart and Thomas [22].

LEMMA 4.3. *For $v_h \in \mathcal{V}_h$, there exists $\boldsymbol{q}_h \in \mathcal{Q}_h$, such that*

$$\operatorname{div} \boldsymbol{q}_h = v_h, \quad ||\boldsymbol{q}_h||_{0,\Omega} \lesssim ||v_h||_{0,\Omega}.$$

Another useful lemma is the discrete Helmholtz decomposition (cf. [11] and references therein).

LEMMA 4.4. *$\mathcal{Q}_h$ has the following orthogonal decomposition with respect to $(\cdot,\cdot)$:*

$$\mathcal{Q}_h = \operatorname{grad}_h \mathcal{V}_h \oplus \vec{\nabla} \times \mathcal{S}_h,$$

where $\operatorname{grad}_h : \mathcal{V}_h \to \mathcal{Q}_h^*$ is defined by

$$(\operatorname{grad}_h v_h, \boldsymbol{q}_h) = -(v_h, \operatorname{div} \boldsymbol{q}_h), \quad \boldsymbol{q}_h \in \mathcal{Q}_h.$$

The following is a result from Lemmas 4.3 and 4.4, Corollary 3.13, and Theorems 4.1 and 4.2.

THEOREM 4.5. *Let $\mathcal{T}_h$ be a quasi-uniform and piecewise (α,σ)-grid. Then $\forall \varepsilon > 0$,*

$$||\Pi_h \boldsymbol{p} - \boldsymbol{p}_h||_{0,\Omega} \lesssim h^{1+\rho} ||u||_{4+\varepsilon,\Omega}.$$

Proof. Let $\boldsymbol{\xi}_h = \Pi_h \boldsymbol{p} - \boldsymbol{p}_h$. Lemma 4.4 gives

$$\boldsymbol{\xi}_h = \operatorname{grad}_h v_h \oplus \vec{\nabla} \times w_h, \tag{4.4}$$

where $(v_h, w_h) \in \mathcal{V}_h \times \mathcal{S}_h$, and

$$|| \operatorname{grad}_h v_h||_{0,\Omega} \lesssim ||\boldsymbol{\xi}_h||_{0,\Omega}, \tag{4.5a}$$

$$||w_h||_{0,\Omega} \lesssim || \vec{\nabla} \times w_h||_{0,\Omega} \lesssim ||\boldsymbol{\xi}_h||_{0,\Omega}. \tag{4.5b}$$

Let $\tilde{\boldsymbol{q}}_h \in \mathcal{Q}_h$ be the preimage of v_h under div in Lemma 4.3. Then

$$\begin{aligned} ||v_h||_{0,\Omega}^2 &= -(\operatorname{grad}_h v_h, \tilde{\boldsymbol{q}}_h) \\ &\lesssim || \operatorname{grad}_h v_h||_{0,\Omega} ||v_h||_{0,\Omega}, \end{aligned}$$

and thus

$$||v_h||_{0,\Omega} \lesssim || \operatorname{grad}_h v_h||_{0,\Omega}. \tag{4.6}$$

By (4.4) and (4.6), we have

$$\begin{aligned} || \operatorname{grad}_h v_h||_{0,\Omega}^2 &= -(v_h, \operatorname{div} \operatorname{grad}_h v_h) \\ &= -(v_h, \operatorname{div} \boldsymbol{\xi}_h) \\ &\lesssim || \operatorname{grad}_h v_h||_{0,\Omega} || \operatorname{div} \boldsymbol{\xi}_h||_{0,\Omega}. \end{aligned}$$

Then it follows from Theorem 4.2 that

$$|| \operatorname{grad}_h v_h||_{0,\Omega} \lesssim h^2 ||u||_{3,\Omega}. \tag{4.7}$$

It remains to bound $\overrightarrow{\nabla}\times w_h$. By the orthogonality in (4.4),

$$
\begin{aligned}
\|\overrightarrow{\nabla}\times w_h\|_{0,\Omega}^2 &= (\Pi_h \boldsymbol{p}-\boldsymbol{p}_h, \overrightarrow{\nabla}\times w_h)\\
&= -(\boldsymbol{p}-\Pi_h\boldsymbol{p}, \overrightarrow{\nabla}\times w_h)\\
&+ (\boldsymbol{\alpha}(\boldsymbol{p}-\boldsymbol{p}_h), \boldsymbol{A}\overrightarrow{\nabla}\times w_h - \Pi_h\boldsymbol{A}\overrightarrow{\nabla}\times w_h)\\
&+ (\boldsymbol{\alpha}(\boldsymbol{p}-\boldsymbol{p}_h), \Pi_h\boldsymbol{A}\overrightarrow{\nabla}\times w_h)\\
&= I + II + III.
\end{aligned}
\tag{4.8}
$$

I is estimated by Corollary 3.13

$$
|I| \lesssim h^{1+\rho}\|\nabla \boldsymbol{p}\|_{2+\varepsilon,\Omega}\|\overrightarrow{\nabla}\times w_h\|_{0,\Omega}. \tag{4.9}
$$

II is estimated by (2.6) and (4.2)

$$
|II| \lesssim h^2\|u\|_{2,\Omega}\|\overrightarrow{\nabla}\times w_h\|_{0,\Omega}. \tag{4.10}
$$

As for III, setting $\boldsymbol{q}_h = \Pi_h\boldsymbol{A}\overrightarrow{\nabla}\times w_h$ in (4.1) leads to

$$
\begin{aligned}
III &= (\boldsymbol{q}_h, \boldsymbol{\beta}(u-u_h)) - (\operatorname{div}\boldsymbol{q}_h, u-u_h)\\
&= III_1 + III_2.
\end{aligned}
$$

By (1.4), (2.5), and (2.6a), we have

$$
\|\boldsymbol{q}_h\|_{0,\Omega} \lesssim \|\overrightarrow{\nabla}\times w_h\|_{0,\Omega}, \tag{4.11}
$$

and

$$
\|\nabla_h \boldsymbol{q}_h\|_{0,\Omega} \lesssim \|\operatorname{div}\boldsymbol{q}_h\|_{0,\Omega} \lesssim \|\overrightarrow{\nabla}\times w_h\|_{0,\Omega}. \tag{4.12}
$$

Then III_1 can be estimated by (4.2), (4.11), and (4.12) and Theorem 4.1:

$$
\begin{aligned}
III_1 &= (\boldsymbol{\beta}\cdot\boldsymbol{q}_h, u-P_hu+P_hu-u_h)\\
&= (\boldsymbol{\beta}\cdot\boldsymbol{q}_h - P_h\boldsymbol{\beta}\cdot\boldsymbol{q}_h, u-P_hu) + \mathcal{O}(h^2)\|u\|_{3,\Omega}\|\boldsymbol{q}_h\|_{0,\Omega}\\
&= \mathcal{O}(h^2)\|\nabla_h(\boldsymbol{\beta}\cdot\boldsymbol{q}_h)\|_{0,\Omega}\|u\|_{1,\Omega} + \mathcal{O}(h^2)\|u\|_{3,\Omega}\|\boldsymbol{q}_h\|_{0,\Omega}\\
&= \mathcal{O}(h^2)\|u\|_{3,\Omega}\|\overrightarrow{\nabla}\times w_h\|_{0,\Omega}.
\end{aligned}
\tag{4.13}
$$

III_2 can be estimated by Theorem 4.2 and (4.12):

$$
\begin{aligned}
III_2 &= (\operatorname{div}\boldsymbol{q}_h, P_hu-u_h)\\
&= \mathcal{O}(h^2)\|u\|_{3,\Omega}\|\overrightarrow{\nabla}\times w_h\|_{0,\Omega}.
\end{aligned}
\tag{4.14}
$$

Combining (4.8)–(4.10), (4.13), and (4.14), we obtain

$$
\|\overrightarrow{\nabla}\times w_h\|_{0,\Omega} \lesssim h^{1+\rho}\|u\|_{4+\varepsilon,\Omega}. \tag{4.15}
$$

Then Theorem 4.5 follows from (4.4), (4.7), and (4.15). ∎

Remark 4.6. If $\boldsymbol{b}=\boldsymbol{0}, c=0$ in (1.1), then $\operatorname{div}(\Pi_h\boldsymbol{p}-\boldsymbol{p}_h)=0$, which implies that $\Pi_h\boldsymbol{p}-\boldsymbol{p}_h$ is a piecewise constant function. Then the superconvergence analysis in this section simplifies. In particular, to prove Theorem 4.5, it is not necessary to employ Lemma 4.4 and Theorems 4.1 and 4.2 in this simplified case.

Theorem 4.5 can be easily generalized by Lemma 3.7 and Corollaries 3.9 and 3.10 respectively. Here we present a theorem from Corollary 3.9.

THEOREM 4.7. *Let $\mathcal{T}_h$ be an (α,σ)-grid without assuming adjacent boundary elements form approximate parallelograms in Definition 3.5. Then under the Neumann boundary condition $\boldsymbol{p}\cdot\boldsymbol{n} = g$ on $\partial\Omega$,*

$$||\Pi_h \boldsymbol{p} - \boldsymbol{p}_h||_{0,\Omega} \lesssim h^{1+\rho}(||\nabla \boldsymbol{p}||_{0,\infty,\Omega} + ||u||_{3,\Omega}).$$

Proof. The proof is basically the same as Theorem 4.5. The difference is from function spaces. For the Neumann boundary condition, the test function spaces in (1.3a) and (1.5a) are $\mathcal{Q}_0 = \{\boldsymbol{q} \in \mathcal{Q} : \boldsymbol{q}\cdot\boldsymbol{n} = 0\}$ and $\mathcal{Q}_{0h} = \{\boldsymbol{q} \in \mathcal{Q}_h : \boldsymbol{q}_h\cdot\boldsymbol{n} = 0\}$, respectively. The numerical solution $\boldsymbol{p}_h$ is in $\mathcal{Q}_h$ with the constraint $\boldsymbol{p}_h\cdot\boldsymbol{n} = g_h$, where

$$g_h|_e = \frac{1}{\ell_e}\int_e g, \quad e \subset \partial\Omega.$$

Thus

$$\int_e (\Pi_h \boldsymbol{p} - \boldsymbol{p}_h)\cdot \boldsymbol{n}_e = 0. \tag{4.16}$$

The form of $\mathcal{RT}_0(\tau)$ (1.4) implies

$$\left(\frac{\partial \Pi_h \boldsymbol{p}}{\partial \boldsymbol{t}_e} - \frac{\partial \boldsymbol{p}_h}{\partial \boldsymbol{t}_e}\right)\cdot \boldsymbol{n}_e = 0. \tag{4.17}$$

Combining (4.16) and (4.17), we have $\boldsymbol{\xi}_h = \Pi_h\boldsymbol{p} - \boldsymbol{p}_h \in \mathcal{Q}_{0h}$. By the discrete Helmholtz decomposition for the Neumann boundary condition (cf. [7]), we have

$$\boldsymbol{\xi}_h = \operatorname{grad}_h v_h \oplus \overrightarrow{\nabla} \times w_h, \tag{4.18}$$

where $(v_h, w_h) \in \mathcal{V}_h \times \mathcal{S}_h$, and $\operatorname{grad}_h v_h \in \mathcal{Q}_{0h}, w_h|_{\partial\Omega} = 0$. Then by following the proof of Theorem 4.5 and using Corollary 3.9 instead of Corollary 3.13, we prove Theorem 4.7. ∎

5. Postprocessing operator and connection with CR nonconforming elements. In the gradient recovery framework, once supercloseness between the interpolant $\Pi_h\boldsymbol{p}$ and numerical solution $\boldsymbol{p}_h$ are available, one can construct postprocessing operator G_h to achieve superconvergence of $G_h\boldsymbol{p}_h$ to $\boldsymbol{p}$. Of course the construction and analysis of G_h is of independent interest (cf. [29, 4, 28, 5]). In this section, we first discuss a cheap recovery operator G_h proposed in [8] and use it to achieve the superconvergence (1.8). Then we prove the superconvergence estimate (1.9) for CR nonconforming elements.

5.1. Postprocessing operator. To define G_h, we introduce the nonconforming finite element space:

$$\begin{aligned}\mathcal{V}_h^{CR} := \{v : v|_\tau \text{ is linear on } \tau \in \mathcal{T}_h, v \text{ is continuous}\\ \text{at the midpoints of interior edges of } \mathcal{T}_h\}.\end{aligned}$$

DEFINITION 5.1. *The operator $G_h : \mathcal{Q}_h \to \mathcal{V}_h^{CR} \times \mathcal{V}_h^{CR}$ is defined as follows:*

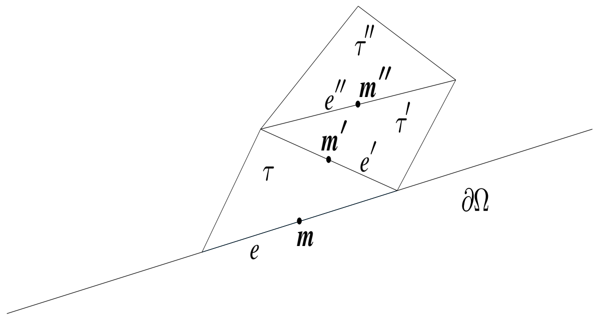


FIG. 2. *a local patch ω near the boundary*

1. *For each interior edge e, let τ and τ' be the pair of elements sharing e. Then the value of $G_h\boldsymbol{q}_h$ at the midpoint $\boldsymbol{m}$ of e is*

$$G_h\boldsymbol{q}_h(\boldsymbol{m}) = \frac{1}{2}(\boldsymbol{q}_h|_\tau(\boldsymbol{m}) + \boldsymbol{q}_h|_{\tau'}(\boldsymbol{m})).$$

2. *For each boundary edge $e\subset \partial\Omega$, let τ be the element having e as an edge. Let τ' be an element sharing an edge e' with τ. Let e'' denote the edge of τ' that does not meet with e, τ'' the element sharing e'' with τ'. Then the value of $G_h\boldsymbol{q}_h$ at the midpoint $\boldsymbol{m}$ of e is*

$$G_h\boldsymbol{q}_h(\boldsymbol{m}) = 2G_h\boldsymbol{q}_h(\boldsymbol{m}') - G_h\boldsymbol{q}_h(\boldsymbol{m}''),$$

where $\boldsymbol{m}'$ and $\boldsymbol{m}''$ is the midpoint of e' and e'', respectively, see Figure 2.

We have the following lemma.

LEMMA 5.2. *Let ω be the patch $\tau\cup\tau'$ associated with the interior edge or $\tau\cup\tau'\cup\tau''$ associated with the boundary edge having the midpoint $\boldsymbol{m}$ in Definition 5.1. Assume each pair of adjacent elements in ω forms an $\mathcal{O}(h^{1+\alpha})$ approximate parallelogram. Then we have*

$$|(\boldsymbol{q}_L - G_h\Pi_h\boldsymbol{q}_L)(\boldsymbol{m})| \lesssim h^\alpha||\nabla\boldsymbol{q}_L||_{0,\omega},$$

where $\boldsymbol{q}_L \in \mathcal{P}_1(\omega)^2$.

Proof. First consider the case of interior edges. By the fact that $\Pi_h\boldsymbol{q}_C = \boldsymbol{q}_C$ for $\boldsymbol{q}_C \in \mathcal{P}_0(\omega)$, we can assume $\boldsymbol{q}_L(\boldsymbol{m}) = 0$ without loss of generality. Let $\boldsymbol{m}_k$ be the midpoint of e_k. Then by (2.7),

$$\begin{aligned}
&(G_h\Pi_h\boldsymbol{q}_L - \boldsymbol{q}_L)(\boldsymbol{m})\\
&= \frac{1}{2}\sum_{k=1}^{3}\left(\frac{\boldsymbol{m}-\boldsymbol{a}_k}{2|\tau|}\int_{e_k}\boldsymbol{q}_L\cdot\boldsymbol{n}_k + \frac{\boldsymbol{m}-\boldsymbol{a}'_k}{2|\tau'|}\int_{e'_k}\boldsymbol{q}_L\cdot\boldsymbol{n}'_k\right)\\
&= \frac{1}{2}\sum_{k=1}^{3}\left(\frac{\boldsymbol{m}-\boldsymbol{a}_k}{2|\tau|}\ell_k\boldsymbol{q}_L(\boldsymbol{m}_k)\cdot\boldsymbol{n}_k + \frac{\boldsymbol{m}-\boldsymbol{a}'_k}{2|\tau'|}\ell'_k\boldsymbol{q}_L(\boldsymbol{m}'_k)\cdot\boldsymbol{n}'_k\right),
\end{aligned} \tag{5.1}$$

where $\boldsymbol{m}_k$ is the midpoint of e_k. From

$$\boldsymbol{q}_L(\boldsymbol{m}_k) = \nabla\boldsymbol{q}_L(\boldsymbol{m})(\boldsymbol{m}_k - \boldsymbol{m}), \quad \boldsymbol{q}_L(\boldsymbol{m}'_k) = \nabla\boldsymbol{q}_L(\boldsymbol{m})(\boldsymbol{m}'_k - \boldsymbol{m}),$$

and the $\mathcal{O}(h^{1+\alpha})$ approximate parallelogram condition, it follows that

$$|(G_h\Pi_h\boldsymbol{q}_L - \boldsymbol{q}_L)(\boldsymbol{m})| \lesssim h^{\alpha}||\nabla\boldsymbol{q}_L||_{0,\omega}. \tag{5.2}$$

As for the case of boundary edge, again assume $\boldsymbol{q}_L(\boldsymbol{m}') = 0$ without loss of generality. Then by

$$\boldsymbol{q}_L(\boldsymbol{m}) = \nabla\boldsymbol{q}_L(\boldsymbol{m}')(\boldsymbol{m} - \boldsymbol{m}'), \quad \boldsymbol{q}_L(\boldsymbol{m}'') = \nabla\boldsymbol{q}_L(\boldsymbol{m}')(\boldsymbol{m}'' - \boldsymbol{m}'),$$

the $\mathcal{O}(h^{1+\alpha})$ approximate parallelogram condition and (5.2), we have

□

$$\begin{aligned}|(\boldsymbol{q}_L - G_h\Pi_h\boldsymbol{q}_L)(\boldsymbol{m})| &= |\boldsymbol{q}_L(\boldsymbol{m}) + G_h\Pi_h\boldsymbol{q}_L(\boldsymbol{m}'') - 2G_h\Pi_h\boldsymbol{q}_L(\boldsymbol{m}')| \\ &\le |\boldsymbol{q}_L(\boldsymbol{m}) + \boldsymbol{q}_L(\boldsymbol{m}'')| + |G_h\Pi_h\boldsymbol{q}_L(\boldsymbol{m}'') - \boldsymbol{q}_L(\boldsymbol{m}'')| \\ &+ 2|G_h\Pi_h\boldsymbol{q}_L(\boldsymbol{m}') - \mathbf{q}_L(\boldsymbol{m}')| \lesssim h^{\alpha}||\nabla\boldsymbol{q}_L||_{0,\omega}.\end{aligned}$$

THEOREM 5.3. *Assume the triangulation $\mathcal{T}_h$ satisfies the (α,σ)-condition. Then*

$$||\boldsymbol{q} - G_h\Pi_h\boldsymbol{q}||_{0,\Omega} \lesssim h^{1+\rho}(||\nabla\boldsymbol{q}||_{1,\Omega} + |\boldsymbol{q}|_{1,\infty,\Omega}).$$

Proof. Because G_h is defined locally, we only need to estimate $\boldsymbol{q} - G_h\Pi_h\boldsymbol{q}$ element by element. We partition the domain into three disjoint parts $\{\Omega_i\}_{i=1}^3$. Ω_1 is covered by interior elements whose three edges belong to $\mathcal{E}_1$. Ω_2 is covered by boundary elements τ that forms an approximate parallelogram with one of its interior adjacent element τ' and τ' forms an approximate parallelogram with one of its interior adjacent elements τ'', as in Definition 5.1, see the pattern Figure 2. Ω_3 is the complement of $\Omega_1 \cup \Omega_2$. Then

$$||\boldsymbol{q} - G_h\Pi_h\boldsymbol{q}||_{0,\Omega}^2 = \sum_{i=1}^{3}\sum_{\tau\subset\Omega_i}||\boldsymbol{q} - G_h\Pi_h\boldsymbol{q}||_{0,\tau}^2 = \sum_{i=1}^{3} I_i. \tag{5.3}$$

For each element τ, let $\tilde{\tau}$ denote the union of elements sharing a side with τ. For $\tau \subset \Omega_1$ or Ω_2, $||\boldsymbol{q} - G_h\Pi_h\boldsymbol{q}||_{0,\tau}$ is estimated by passing through a linear polynomial $\boldsymbol{q}_L \in P_1(\tilde{\tau})^2$:

$$\begin{aligned}||\boldsymbol{q} - G_h\Pi_h\boldsymbol{q}||_{0,\tau} &\lesssim ||\boldsymbol{q} - \boldsymbol{q}_L||_{0,\tau} \\ &+ ||G_h\Pi_h(\boldsymbol{q} - \boldsymbol{q}_L)||_{0,\tau} + ||\boldsymbol{q}_L - G_h\Pi_h\boldsymbol{q}_L||_{0,\tau}.\end{aligned} \tag{5.4}$$

By the Bramble–Hilbert lemma and scaling argument, there exists $\boldsymbol{q}_L \in \mathcal{P}_1(\tilde{\tau})^2$ such that

$$||\boldsymbol{q} - \boldsymbol{q}_L||_{s,\tilde{\tau}} \lesssim h^{2-s}|\boldsymbol{q}|_{2,\tilde{\tau}}, \quad s = 0, 1, \tag{5.5}$$

and

$$\begin{aligned}||G_h\Pi_h(\boldsymbol{q} - \boldsymbol{q}_L)||_{0,\tau} &\lesssim h||G_h\Pi_h(\boldsymbol{q} - \boldsymbol{q}_L)||_{0,\infty,\tau} \\ &\lesssim h||\boldsymbol{q} - \boldsymbol{q}_L||_{0,\infty,\tilde{\tau}} \lesssim h^2|\boldsymbol{q}|_{2,\tilde{\tau}}.\end{aligned} \tag{5.6}$$

Then by Lemma 5.2 and (5.4)–(5.6), we have

$$
\begin{aligned}
I_1 + I_2 &\lesssim \sum_{i=1}^{2} \sum_{\tau\subset\Omega_i} \left\{h^4|\boldsymbol{q}|_{2,\tilde\tau}^2 + h^2||\boldsymbol{q}_L - G_h\Pi_h\boldsymbol{q}_L||_{0,\infty,\tau}^2\right\} \\
&\lesssim \sum_{i=1}^{2} \sum_{\tau\subset\Omega_i} \left\{h^4|\boldsymbol{q}|_{2,\tilde\tau}^2 + h^2 \max_{1\le k\le 3}(\boldsymbol{q}_L(\boldsymbol{m}_k) - G_h\Pi_h\boldsymbol{q}_L(\boldsymbol{m}_k))^2\right\} \\
&\lesssim h^{2+2\min(1,\alpha)} \sum_{i=1}^{2} \sum_{\tau\subset\Omega_i} (|\boldsymbol{q}|_{2,\tilde\tau}^2 + ||\nabla\boldsymbol{q}_L||_{\tilde\tau}^2) \\
&\lesssim h^{2+2\min(1,\alpha)}||\nabla\boldsymbol{q}||_{1,\Omega}^2.
\end{aligned}
\tag{5.7}
$$

By the (α,σ)-condition and local quasi-uniformity of $\mathcal{T}_h$, $|\Omega_3|$ is forced to be of the size $\mathcal{O}(h^\sigma)$. Since $G_h\Pi_h\boldsymbol{q}_C = \boldsymbol{q}_C$ for $\boldsymbol{q}_C \in \mathcal{P}_0(\tilde\tau)^2$, I_3 can be estimated by the Bramble–Hilbert lemma with a scaling argument and the small measure of Ω_3:

$$
\begin{aligned}
I_3 &= \sum_{\tau\subset\Omega_3} ||\boldsymbol{q} - G_h\Pi_h\boldsymbol{q}||_{0,\tau}^2 \\
&\lesssim \sum_{\tau\subset\Omega_3} h^2|\boldsymbol{q}|_{1,\tilde\tau}^2 \lesssim h^2\int_{\Omega_3}|\nabla\boldsymbol{q}|^2 \lesssim h^{2+\sigma}|\boldsymbol{q}|_{1,\infty,\Omega}^2.
\end{aligned}
\tag{5.8}
$$

By (5.3), (5.7), and (5.8), we obtain Theorem 5.3. ∎

The supercovnergence of $||\boldsymbol{p} - G_h\boldsymbol{p}_h||_{0,\Omega}$ is a direct result from Theorems 4.5 and 5.3.

THEOREM 5.4. *Let $\mathcal{T}_h$ be quasi-uniform and satisfy the (α,σ)-condition. Then*

$$||\boldsymbol{p} - G_h\boldsymbol{p}_h||_{0,\Omega} \lesssim h^{1+\rho}||u||_{4+\varepsilon,\Omega}.$$

Proof. For $\boldsymbol{q}_h \in \mathcal{Q}_h$ and $\tau \in \mathcal{T}_h$,

$$
\begin{aligned}
||G_h\boldsymbol{q}_h||_{0,\tau} &\lesssim h||G_h\boldsymbol{q}_h||_{0,\infty,\tau} \\
&\lesssim h\max_{1\le k\le 3}|G_h\boldsymbol{q}_h(\boldsymbol{m}_k)| \\
&\lesssim h||\boldsymbol{q}_h||_{0,\infty,\tilde\tau} \lesssim ||\boldsymbol{q}_h||_{0,\tilde\tau},
\end{aligned}
$$

and then

$$||G_h\boldsymbol{q}_h||_{0,\Omega} \lesssim ||\boldsymbol{q}_h||_{0,\Omega}, \tag{5.9}$$

that is, G_h is bounded in L^2 norm. Combining Theorems 4.5 and 5.3 and (5.9), we have

$$
\begin{aligned}
||\boldsymbol{p} - G_h\boldsymbol{p}_h||_{0,\Omega} &\lesssim ||\boldsymbol{p} - G_h\Pi_h\boldsymbol{p}_h||_{0,\Omega} + ||G_h(\Pi_h\boldsymbol{p} - \boldsymbol{p}_h)||_{0,\Omega} \\
&\lesssim h^{1+\rho}||u||_{4+\varepsilon,\Omega}.
\end{aligned}
$$

∎

5.2. Superconvergence for CR nonconforming elements. As one can see in section 3 and subsection 5.1, the CR nonconforming method is closely related to the RT mixed method. In this subsection, we prove the superconvergence estimate for CR nonconforming methods on (α,σ)-grids by Theorem 5.4. For simplicity, we only consider Poisson's equation with the homogeneous Dirichlet boundary condition, i.e. $\boldsymbol{A} = \boldsymbol{I}_{2\times 2}, \boldsymbol{b} = \boldsymbol{0}, c = 0, g = 0$ in (1.1). In this case, $\boldsymbol{p} = \nabla u$ in (1.2). The corresponding CR nonconforming method is to find $u_h^{CR} \in \mathcal{V}_{0h}^{CR}$, such that

$$(\nabla_h u_h^{CR}, \nabla_h v_h) = (f, v_h), \quad v_h \in \mathcal{V}_{0h}^{CR}, \tag{5.10}$$

where

$$\mathcal{V}_{0h}^{CR} := \{v \in \mathcal{V}_h^{CR} : v = 0 \text{ at the midpoints of boundary edges of } \mathcal{T}_h.\}.$$

Marini [23] proved the following theorem.

THEOREM 5.5. *Let $\bar{u}_h^{CR} \in \mathcal{V}_{0h}^{CR}$ solve*

$$(\nabla_h \bar{u}_h^{CR}, \nabla_h v_h) = (P_h f, v_h), \quad v_h \in \mathcal{V}_{0h}^{CR}. \tag{5.11}$$

Let $\{\bar{\boldsymbol{p}}_h, \bar{u}_h\} \in \mathcal{Q}_h \times \mathcal{V}_h$ solve the following mixed problem:

$$\begin{aligned}(\bar{\boldsymbol{p}}_h, \boldsymbol{q}_h) + (\operatorname{div} \boldsymbol{q}_h, \bar{u}_h) = 0, \quad \boldsymbol{q}_h \in \mathcal{Q}_h,\\ (\operatorname{div} \bar{\boldsymbol{p}}_h, v_h) = -(f, v_h), \quad v_h \in \mathcal{V}_h.\end{aligned}$$

Then

$$\bar{\boldsymbol{p}}_h(\boldsymbol{x}) = \nabla \bar{u}_h^{CR} - \frac{P_h f|_\tau}{2}(\boldsymbol{x} - \boldsymbol{x}_\tau), \quad \boldsymbol{x} \in \tau,$$

where $\boldsymbol{x}_\tau$ is the barycenter of $\tau \in \mathcal{T}_h$.

By Theorem 5.5 and (1.10), Hu and Ma [17] proved the following estimate for Poisson's equation on uniform grids:

$$||\nabla u - G_h \nabla_h u_h^{CR}||_{0,\Omega} \lesssim h^{\frac{3}{2}}(||u||_{\frac{5}{2},\Omega} + h^{\frac{1}{2}}|u|_{3,\Omega} + h^{\frac{1}{2}}|f|_{1,\infty,\Omega}). \tag{5.13}$$

Based on the idea of [17], we can prove the following superconvergence estimate for CR elements by Theorems 5.4 and 5.5.

THEOREM 5.6. *Let $\mathcal{T}_h$ be a quasi-uniform (α, σ)-grid. Let u_h^{CR} solve (5.10). Then*

$$||\nabla u - G_h \nabla_h u_h^{CR}||_{0,\Omega} \lesssim h^{1+\rho}||u||_{4+\varepsilon,\Omega}.$$

Proof. Split $||\nabla u - G_h \nabla_h u_h^{CR}||_{0,\Omega}$ as

$$\begin{aligned}&||\nabla u - G_h \nabla_h u_h^{CR}||_{0,\Omega}\\ \lesssim&||\nabla u - G_h \bar{\boldsymbol{p}}_h||_{0,\Omega} + ||G_h(\bar{\boldsymbol{p}}_h - \nabla_h \bar{u}_h^{CR})||_{0,\Omega}\\ &+||G_h(\nabla_h \bar{u}_h^{CR} - \nabla_h u_h^{CR})||_{0,\Omega} = I + II + III.\end{aligned} \tag{5.14}$$

I can be estimated by Theorem 5.4:

$$I \lesssim h^{1+\rho}||u||_{4+\varepsilon,\Omega}. \tag{5.15}$$

For the second term, first consider the patch $\omega = \tau \cup \tau'$ associated with an interior edge e having midpoint $\boldsymbol{m}$ in Definition 5.1. By Theorem 5.5, we have

$$G_h(\bar{\boldsymbol{p}}_h - \nabla_h \bar{u}_h^{CR})(\boldsymbol{m}) = -\frac{1}{4}\left((\boldsymbol{m} - \boldsymbol{x}_\tau)P_h f|_\tau + (\boldsymbol{m} - \boldsymbol{x}_{\tau'})P_h f|_{\tau'}\right)$$

If τ and τ' form an $\mathcal{O}(h^{1+\alpha})$ approximate parallelogram, then

$$|G_h(\bar{\boldsymbol{p}}_h - \nabla_h \bar{u}_h^{CR})(\boldsymbol{m})| \lesssim h^{1+\alpha}||f||_{1,\infty,\Omega}. \tag{5.16}$$

Then consider the patch $\omega = \tau \cup \tau' \cup \tau''$ associated with a boundary edge e having midpoint $\boldsymbol{m}$ in Definition 5.1. (5.16) implies that

$$\begin{aligned} |G_h(\bar{\boldsymbol{p}}_h - \nabla_h \bar{u}_h^{CR})(\boldsymbol{m})| &\lesssim 2|G_h(\bar{\boldsymbol{p}}_h - \nabla_h \bar{u}_h^{CR})(\boldsymbol{m}')| \\ &\quad + |G_h(\bar{\boldsymbol{p}}_h - \nabla_h \bar{u}_h^{CR})(\boldsymbol{m}'')| \lesssim h^{1+\alpha}||f||_{1,\infty,\Omega}, \end{aligned} \tag{5.17}$$

provided τ, τ' and τ', τ'' form $\mathcal{O}(h^{1+\alpha})$ approximate parallelograms. Now we partition Ω into $\cup_{i=1}^3 \Omega_i$ as in the proof of Theorem 5.3. By following the proof of Theorem 5.3 and using (5.16) and (5.17), II can be estimated by

$$II \lesssim h^{1+\rho}||f||_{1,\infty,\Omega}. \tag{5.18}$$

As for III, it follows from (5.10) and (5.11) that for $v_h \in \mathcal{V}_{0h}^{CR}$,

$$\begin{aligned} (\nabla_h \bar{u}_h^{CR} - \nabla_h u_h^{CR}, \nabla_h v_h) &= (f - P_h f, v_h) \\ &= (f - P_h f, v_h - P_h v_h) \\ &\lesssim h^2|f|_{1,\Omega}||\nabla_h v_h||_{0,\Omega}, \end{aligned} \tag{5.19}$$

By setting $v_h = \bar{u}_h^{CR} - u_h^{CR}$ in (5.19) and using boundedness of G_h in L^2 norm, we have

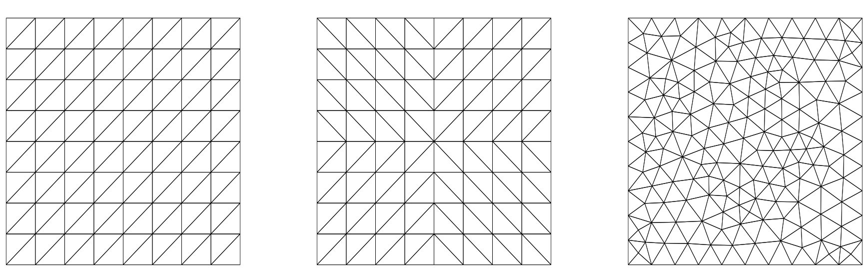
$$III \lesssim h^2|f|_{1,\Omega}. \tag{5.20}$$

Then Theorem 5.6 results from combining (5.14), (5.15), (5.18), and (5.20). ∎

In the case of uniform grids ($\alpha = \sigma = \infty$), Theorem 5.6 implies that

$$||\nabla u - G_h \nabla_h u_h^{CR}||_{0,\Omega} \lesssim h^2||u||_{4+\varepsilon,\Omega}.$$

which shows that is (5.13) suboptimal. However, Theorem 5.5 cannot be applied to (1.1) with nonvanishing $\boldsymbol{b}$ and c. It would be interesting to develop a formula similar to [23] in a more general setting.



(a) (b) (c)

FIG. 3. *Three different grids*

Table 1
Uniform grids

nu	$\|\|\boldsymbol{p}-\boldsymbol{p}_h\|\|_{0,\Omega}$		$\|\|\Pi_h\boldsymbol{p}-\boldsymbol{p}_h\|\|_{0,\Omega}$		$\|\|\boldsymbol{p}-G_h\boldsymbol{p}_h\|\|_{0,\Omega}$	
	Error	Order	Error	Order	Error	Order
336	7.281e-1	0.9911	1.033e-1	1.979	2.629e-1	2.094
1312	3.663e-1	0.9972	2.620e-2	1.995	6.157e-2	2.062
5184	1.835e-1	0.9998	6.574e-3	1.999	1.475e-2	2.035
20608	9.176e-2	0.9997	1.645e-3	1.999	3.598e-3	2.003
82176	4.589e-2		4.114e-4		8.976e-4	

Table 2
Piecewise uniform grids

nu	$\|\|\boldsymbol{p}-\boldsymbol{p}_h\|\|_{0,\Omega}$		$\|\|\Pi_h\boldsymbol{p}-\boldsymbol{p}_h\|\|_{0,\Omega}$		$\|\|\boldsymbol{p}-G_h\boldsymbol{p}_h\|\|_{0,\Omega}$	
	Error	Order	Error	Order	Error	Order
336	7.287e-1	0.9919	9.356e-2	1.937	2.898e-1	1.963
1312	3.664e-1	0.9976	2.449e-2	1.978	7.668e-2	1.790
5184	1.835e-1	0.9998	6.215e-3	1.993	2.217e-2	1.683
20608	9.176e-2	0.9997	1.561e-3	1.998	6.904e-3	1.607
82176	4.589e-2		3.907e-4		2.267e-3	

6. Numerical experiments. In this section, we test our superconvergence results for $||\Pi_h\boldsymbol{p}-\boldsymbol{p}_h||_{0,\Omega}$ and $||\boldsymbol{p}-G_h\boldsymbol{p}_h||_{0,\Omega}$ by the following equation:

$$\begin{aligned} -\Delta u + u &= f, \quad \boldsymbol{x}\in\Omega,\\ u &= 0, \quad \boldsymbol{x}\in\partial\Omega, \end{aligned}$$

where $\Omega=(0,1)\times(0,1)$ is the unit square. Let $u=\sin(2\pi x_1)\sin(\pi x_2)$ and f be the corresponding source term. The numerical experiments were performed using MATLAB, R2016. The linear system resulting from the mixed method (1.5) was solved by the operation \. The 'nu' in Tables 1 to 3 stands for the number of unknowns.

We began with the 8×8 uniform grid in Figure 3(a), and computed a sequence of meshes by regular refinement, i.e. partitioning an element into four similar subelements by connecting the midpoints of each edge. In this case, $\alpha=\sigma=\infty$, $\rho=1$. As shown in Table 1, the observed orders of convergence coincide with Theorem 4.5 and Theorem 5.4.

Then we considered the (α,σ)-grid with $(\alpha,\sigma)=(\infty,1)$ in Figure 3(b). The mesh was refined regularly. Although it is not globally uniform, it can be decomposed into four uniform subgrids. Hence the mesh is a piecewise (α,σ)-grid with $(\alpha,\sigma)=(\infty,\infty)$. By Theorem 4.5, we still obtain 2nd order of convergence for $||\Pi_h\boldsymbol{p}-\boldsymbol{p}_h||_{0,\Omega}$, which was confirmed by Table 2. However, Theorem 5.4 cannot be applied to piecewise (α,σ)-grid. Thus the order of convergence for $||\boldsymbol{p}-G_h\boldsymbol{p}_h||_{0,\Omega}$ approaches 3/2 in Table 2.

In the last experiment, we generated the initial mesh in Figure 3(c) by 'pdetool' in MATLAB and then refined it regularly. At first glance, it should be an unstructured grid or a mildly structured grid with unknown α and σ. Surprisingly, it is indeed a piecewise uniform mesh, since each element in the initial mesh was refined uniformly. On the other hand, it's not hard to see that the sequence of grids are globally (α,σ)-meshes with $\alpha=\infty$ and $\sigma=1$ asymptotically. As predicted by Theorems 4.5 and 5.4, the order of convergence for $||\Pi_h\boldsymbol{p}-\boldsymbol{p}_h||_{0,\Omega}$ approaches 2 while the order of convergence

TABLE 3
Unstructured grids

nu	$\|\|\boldsymbol{p}-\boldsymbol{p}_h\|\|_{0,\Omega}$		$\|\|\Pi_h\boldsymbol{p}-\boldsymbol{p}_h\|\|_{0,\Omega}$		$\|\|\boldsymbol{p}-G_h\boldsymbol{p}_h\|\|_{0,\Omega}$	
	Error	Order	Error	Order	Error	Order
800	4.585e-1	0.9934	5.152e-2	1.749	1.911e-1	1.724
3160	2.303e-1	0.9981	1.533e-2	1.823	5.783e-2	1.551
12560	1.153e-1	0.9990	4.334e-3	1.862	1.973e-2	1.532
50080	5.769e-2	0.9997	1.192e-3	1.885	6.824e-3	1.518
200000	2.885e-2		3.227e-4		2.383e-3	

for $||\boldsymbol{p}-G_h\boldsymbol{p}_h||_{0,\Omega}$ approaches 3/2. We didn't obtain exact 2nd order convergence for $||\Pi_h\boldsymbol{p}-\boldsymbol{p}_h||_{0,\Omega}$, since the linear system eventually became extremely large. In this case, both time cost and the condition number of the coefficient matrix became unacceptable.

7. Concluding remarks. In this paper, we proved optimal order global superconvergence for the lowest order RT element on mildly structured meshes for general second order elliptic equations. As a byproduct, we also proved superconvergence for the CR nonconforming method for Poisson's equations. The results in sections 2 and 3 are PDE-independent and applicable to other numerical PDEs using lowest order RT elements. The proof of Theorem 4.2 and most steps of the proof of Theorem 4.5 work for higher order mixed finite elements.

In practice, the solution u does not necessarily belong to $W^3_\infty(\Omega)$ in Lemma 3.7 or $H^{4+\varepsilon}(\Omega)$ in Theorem 4.5 if $\partial\Omega$ is not smooth enough. Hence our superconvergence estimates become questionable in this case. In fact, the high regularity requirement is a common issue shared by most superconvergence results (cf. [3, 8, 15, 18, 28]). There are several possible ways to fix it. First, by modifying the proof of Lemma 3.7, one can obtain smaller rate of superconvergence under weaker regularity assumptions. For example, one can easily show $||\boldsymbol{p}-G_h\boldsymbol{p}_h||_{0,\Omega}=\mathcal{O}(h^{1+\min(1/2,\alpha,\sigma/2)})$ on (α,σ)-grids for $u\in H^3(\Omega)\cap W^2_\infty(\Omega)$ if the error occuring on boundary triangles is not canceled in the proof of Lemma 3.7. Second, u is smooth on any compact subdomain in Ω. Hence it's meaningful to look for interior estimates (cf. [26, 27]). As far as we know, u should be at least in $W^2_\infty(\Omega)$ to prove interior superconvergence for linear Lagrange elements (cf. [28]). Of course, the assumption $u\in W^2_\infty(\Omega)$ may not hold on domains with corners. Third, for $u\in H^{1+\delta}$ with $\delta>0$, it's possible to prove superconvergence recovery for RT elements under adaptive meshes by following the framework of this paper and assuming certain mesh density function which is enough to resolve the singularity, see [25] for the case of Lagrange elements. However, it's difficult to prove that the adaptively refined sequence of meshes actually satisfies the mesh density pattern.

We also point out that $||\Pi_h\boldsymbol{p}-\boldsymbol{p}_h||_{0,\Omega}$ might not superconverge in the case of general mixed elements on triangular meshes, since part of finite element basis functions become more localized in higher order methods (cf. [7, 20]). We will present superconvergence results for higher order mixed elements on mildly structured meshes in another paper.

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