A PIECEWISE KORN INEQUALITY IN SBD AND APPLICATIONS TO EMBEDDING AND DENSITY RESULTS

MANUEL FRIEDRICH

Abstract. We present a piecewise Korn inequality for generalized special functions of bounded deformation ($GSBD^2$) in a planar setting generalizing the classical result in elasticity theory to the setting of functions with jump discontinuities. We show that for every configuration there is a partition of the domain such that on each component of the cracked body the distance of the function from an infinitesimal rigid motion can be controlled solely in terms of the linear elastic strain. In particular, the result implies that $GSBD^2$ functions have bounded variation after subtraction of a piecewise infinitesimal rigid motion. As an application we prove a density result in $GSBD^2$ in dimension two. Moreover, for all $d \geq 2$ we show $GSBD^2(\Omega) \subset (GBV(\Omega;\mathbb{R}))^d$ and the embedding $SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d) \hookrightarrow SBV(\Omega;\mathbb{R}^d)$ into the space of special functions of bounded variation (SBV). Finally, we present a Korn-Poincaré inequality for functions with small jump sets in arbitrary space dimension.

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1. Introduction

A natural framework for the investigation of damage and fracture models in a geometrically linear setting is given by the space of *special functions of bounded deformation* investigated in [4, 6]. The space $SBD^p(\Omega)$ consists of all functions $u \in L^1(\Omega;\mathbb{R}^d)$ whose symmetrized distributional derivative $Eu := \frac{1}{2}((Du)^T + Du)$ is a finite $\mathbb{R}^{d\times d}_{\rm sym}$-valued Radon measure which can be written as the sum of an $L^p(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ function $e(u) := \frac{1}{2}((\nabla u)^T + \nabla u)$ and a part concentrated on a rectifiable set J_u with finite $\mathcal{H}^{d-1}$ measure. Starting with the seminal paper [27], various variational problems for fracture mechanics in the realm of linearized elasticity have recently been investigated in the literature (see e.g. [7, 11, 12, 26, 29, 41]), where the common ground of all these models is that the main energy term is essentially of the form

$$\int_\Omega |e(u)|^2\, dx + \mathcal{H}^{d-1}(J_u) \tag{1.1}$$

for $u \in SBD^2(\Omega)$. These so-called Griffith functionals comprise elastic bulk contributions for the unfractured regions of the body represented by the linear elastic strain $e(u)$ and surface terms that assign energy contributions on the crack paths comparable to the size of the 'jump set' J_u.

For technical reasons models are often formulated in $SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d)$, e.g. in [6, 12, 41], since for this setting a compactness result in SBD was proved in [6, Theorem 1.1] and hereby the existence of solutions to minimization problems related to (1.1) is guaranteed. To overcome this restriction, the space of *generalized special functions of bounded deformation*, denoted by $GSBD(\Omega)$, was introduced in [20], admitting a compactness result under weaker assumptions and leading to the investigation of fracture models [29, 33, 36] in a more general

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setting. (For an exact definition of $GSBD$, which is based on properties of one-dimensional slices, we refer to Section 3.4.)

On the one hand, models of the form (1.1) with linearized elastic energies are in general easier to treat than their nonlinear counterparts since in the regime of finite elasticity the energy density of the elastic contributions is genuinely geometrically nonlinear due to frame indifference rendering the problem highly non-convex. On the other hand, a major difficulty of these models in contrast to nonlinear problems formulated in the space SBV of *special functions of bounded variation* (see [5]) is given by the fact that one controls only the linear elastic strain.

Indeed, as discussed in [17], it appears that various properties being well established in SBV are only poorly understood in SBD due to the lack of control on the skew symmetric part of the distributional derivative $(Du)^T - Du$. Especially, to the best of our knowledge, for the *coarea formula* in BV (see [5, Theorem 3.40]), being useful in various applications as [3, 8, 9, 21, 23, 28], no BD analog has been obtained in the literature.

Therefore, it is a natural and highly desirable issue to gain a deeper understanding of the relation between SBD and SBV or even to show that in certain circumstances SBD functions have bounded variation. In fact, one may expect that hereby various problems in SBD could be solvable by a reduction to corresponding results in SBV.

Clearly, a generic function of bounded deformation does not necessarily lie in SBV as one can already construct a function with $J_u = \emptyset$ such that $e(u) \in L^1(\Omega)$, but $\nabla u \notin L^1(\Omega)$ (cf. [16, 40]). Moreover, following the examples in [4, 20], one can define functions in $GSBD^2(\Omega)$ not having bounded variation (see Example 2.6 and Lemma 2.8 below), i.e., also the higher integrability for the elastic strain and the finiteness of the energy (1.1) is in general not sufficient. However, the examples involve unbounded configurations and it has therefore been conjectured that $SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d)$ is a subspace of $SBV(\Omega;\mathbb{R}^d)$.

Profound understanding of the connection between SBV and SBD functions is directly related to the validity of a Korn-type inequality in the setting of functions exhibiting jump discontinuities. In elasticity theory, Korn's inequality is the key estimate providing a relation between the symmetric and the full part of the gradient (see e.g. [39]). It states that the distance of ∇u from a skew symmetric matrix can be controlled solely in terms of the linear elastic strain $e(u)$. In the recently appeared contributions [18, 31], this well-know estimate has been generalized in a planar setting to configurations in SBD with small jump set controlling ∇u away from a small exceptional set.

However, for general functions of bounded deformation Korn's inequality in its basic form is doomed to fail. In fact, the domain may be disconnected into various parts by the jump set with completely different behavior on each component. In the special case that a brittle material does not store elastic energy, i.e., $e(u) = 0$, Chambolle, Giacomini, and Ponsiglione [15] showed that the body behaves piecewise rigidly, i.e., the only possibility that u is not an affine mapping is that the body is divided into at most countably many parts each of which subject to a different infinitesimal rigid motion.

Consequently, this observation already shows that an analogous statement of Korn's inequality in $(G)SBD$ has to be formulated in a considerably more complex way involving a partition of the domain. The problem is related to the result in [32], where a quantitative piecewise rigidity result in a geometrically nonlinear setting is established stating that in the planar case the distance of the deformation gradient from a piecewise rigid motion can be controlled.

The main goal of the present work is the derivation of an analogous result in the geometrically linear setting which we call a *piecewise Korn inequality*. We show that in *two dimensions* for each $u \in GSBD^2(\Omega)$ there is an associated partition whose boundary length is controlled by

$\mathcal{H}^1(J_u)$ and a corresponding piecewise infinitesimal rigid motion a, being constant on each connected component of the cracked body, such that the distance of u and ∇u from a can be estimated in terms of $\|e(u)\|_{L^2(\Omega)}$. This result for configurations storing elastic energy and exhibiting cracks may be seen as a suitable combination of Korn's inequality for elastic materials and the qualitative result in [15].

The estimate proves to be useful to gain a deeper understanding of the relation between SBV and SBD functions. Although, as discussed above, $GSBD^2$ functions do not have bounded variation in general, the piecewise Korn inequality yields (in dimension two) that after subtraction of a piecewise infinitesimal rigid motion the function lies in the space SBV^p for all $p < 2$. Hereby we particularly derive that each function in $GSBD^2$ has bounded variation away from an at most countable union of sets of finite perimeter with arbitrarily small Lebesgue measure.

It turns out that for the subspace of bounded functions this observation can be substantially improved. In this case it is possible to control the piecewise infinitesimal rigid motion in terms of $\|u\|_\infty$ and $\mathcal{H}^{d-1}(J_u)$ and we indeed obtain the embedding

$$SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d) \hookrightarrow SBV(\Omega;\mathbb{R}^d), \tag{1.2}$$

which, in contrast to the piecewise Korn inequality, is proved in *arbitrary space dimension* using a slicing technique. Moreover, similar arguments yield that without the L^∞-bound one may derive the inclusion $GSBD^2(\Omega) \subset (GBV(\Omega;\mathbb{R}))^d$ for $d \geq 2$. (See Section 3.4 below for the definition of *generalized functions of bounded variation*.) In particular, from (1.2) we deduce that in the space of bounded SBD^2 functions, being a natural and widely adopted space in the investigation of linear fracture models, the *coarea formula* is applicable.

Apart from the derivation of embeddings a major motivation for the derivation of the piecewise Korn inequality are applications to Griffith models. We show that the embedding result and the coarea formula allow to derive a Korn-Poincaré inequality in SBD stating that for a $GSBD^2$ function with small jump set the distance from a single infinitesimal rigid motion can be controlled outside a small exceptional set. This estimate, established in arbitrary space dimension, enhances a result recently obtained by Chambolle, Conti, and Francfort [13] in the sense that also the length of the boundary of the exceptional set can be bounded in terms of $\mathcal{H}^{d-1}(J_u)$ and therefore compactness results in $GSBD$ (see [20]) are applicable.

We also present an approximation result for $GSBD^2$ functions in a *planar setting* improving [36] in the sense that no L^2-bound on the function is needed. Hereby we can complete the Γ-convergence result for the linearization of Griffith energies [29] by providing recovery sequences for every $GSBD^2$ function. (We also refer to the recent contribution [14] where a generalization to arbitrary space dimension has been obtained.) Moreover, the main statement of this paper will be a fundamental ingredient to prove a general compactness theorem and to analyze quasistatic crack growth for energies of the form (1.1) (see [33]).

The major difficulties in the derivation of the result come from the fact we treat a full free discontinuity problem in the language of Ambrosio and De Giorgi [22] deriving an estimate without any a priori assumptions on the crack geometry. Already simplified situations, in which the jump set decomposes the body into a finite number of sets with Lipschitz boundary, are subtle since there are no uniform bounds on the constants in Poincaré's and Korn's inequality. In fact, in the basic case of simply connected sets, in [30] a lot of effort was needed to derive a decomposition result into domains for which uniform bounds on the constants can be derived. Even more challenging difficulties occur for highly irregular jump sets forming, e.g., infinite crack patterns on various mesoscopic scales.

At first sight the derivation of the piecewise Korn inequality appears to be easier than the related problem [32] due to the geometrical linearity. However, whereas in [32] the statement is established only for a suitable, arbitrarily small modification of the configuration, in the present context the estimates hold for the original function, where this stronger result is indispensable to prove the announced embedding and approximation results. In particular, the modification scheme for the deformation and jump set, which was iteratively applied in [32] on mesoscopic scales becoming gradually larger, is not adequate in the present context and we need to use comparably different proof techniques.

The general strategy will be to first derive an auxiliary piecewise Korn inequality, which similarly as in [18, 31] holds up to a small exceptional set. Then the main result follows by an iterative application of the estimate on various mesocopic scales. For the derivation of the auxiliary statement we first identify the regions where the jump set is too large. Hereby we can (1) construct a partition of the domain into simply connected sets and (2) use the Korn inequality [18, 31] for functions with small jump sets to find an approximation of the configuration, which is smooth on each component of the domain. To control the shape of the components we apply the main result of the paper [30] which allows to pass to a refined partition consisting of *John domains* with uniformly controlled John constant. The auxiliary statement then follows by using a Korn inequality for John domains (see e.g. [1]).

Let us remark that the piecewise Korn inequality is only proved in a planar setting since also the major proof ingredients, the Korn inequality for functions with small jump sets [18, 31] and the decomposition result [30], have only been shown in dimension two. Although the derivation of a higher dimensional analog seems currently out of reach, the majority of the applications in this contribution, namely the embedding (1.2) and the Korn-Poincaré inequality, hold in *arbitrary dimensions* due to slicing arguments.

The paper is organized as follows. In Section 2 we present the main results including the embedding and approximation for $GSBD^2$ functions. In Example 2.6 we also recall a standard construction for an SBD function not having bounded variation, which is based on the idea to cut out small balls from the bulk part. Hereby we see that in view of the piecewise Korn inequality, examples of this type essentially represent the only possibility to define SBD^2 functions not lying in SBV. Moreover, the construction shows that the embedding (1.2) is sharp in the sense that the result is false if (1) SBV is replaced by SBV^p, $p > 1$, and if (2) L^∞ is replaced by L^q, $q < \infty$.

Section 3 is devoted to some preliminaries. We first recall the definition of John domains and formulate the decomposition result established in [30]. Then we state some properties of affine mappings being especially important for the derivation of (1.2), where we need fine estimates how affine mappings and its derivatives can by controlled in terms of the volume and diameter of the underlying set. Afterwards, we recall the definition of $(G)SBV$ and $(G)SBD$ functions and discuss basic properties.

In Section 4 we prove a version of the piecewise Korn inequality outside a small exceptional set. Together with the construction of a partition into simply connected sets we also provide a covering of Whitney-type such that the jump set J_u in each element of the covering is small. This then allows us to apply the inequality [31] in order to define an approximation of the deformation being smooth in each part of the domain. We discuss properties of the covering and the exceptional set being crucial for the iterative application of the auxiliary result in Section 5.1.

Section 5 then contains the proof of the piecewise Korn inequality. In Section 5.1 we first prove the result for configurations in SBD with a regular jump set consisting of a finite number of segments. Then in Section 5.2 we derive the general case by an approximation argument similar to the one in [31]. Here we also prove our main approximation result.

Finally, Section 6 contains the proof of the embedding results and the Korn-Poincaré inequality. As an ingredient for the derivation of (1.2) we also provide a *piecewise Poincaré inequality*, which will allow to control the distance of configurations from piecewise infinitesimal rigid motions in terms of the L^∞-norm. For the latter as well as for the Korn-Poincaré inequality, the coarea formula in BV will turn out to be a key ingredient.

2. Main results

2.1. **Piecewise Korn inequality.** Before we formulate the main results of this article, let us introduce some notions. We say a partition $(P_j)_{j=1}^\infty$ of $\Omega \subset \mathbb{R}^d$ is a *Caccioppoli partition* if each P_j is a set of finite perimeter such that $\sum_{j=1}^\infty \mathcal{H}^{d-1}(\partial^* P_j) < +\infty$ (see Section 3.4 for details). Given corresponding affine mappings $a_j = a_{A_j,b_j} : \mathbb{R}^d \to \mathbb{R}^d$ with $a_j(x) = A_j\,x + b_j$ for $A_j \in \mathbb{R}^{d\times d}_{\rm skew}$, $b_j \in \mathbb{R}^d$, we say a_j is an *infinitesimal rigid motion* and the function $\sum_{j=1}^\infty a_j \chi_{P_j}$ is a *piecewise infinitesimal rigid motion*.

For the definition and properties of special functions of bounded variation (SBV) and deformation (SBD) we refer to Section 3.4. In particular, for $u \in SBD^2(\Omega)$ we denote by $e(u)$ the part of the strain $Eu = \frac{1}{2}((Du)^T + Du)$ which is absolutely continuous with respect to $\mathcal{L}^d$ and let J_u be the jump set. Our main goal is to show the following result.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary and $p \in [1,2)$. Then there is a constant $c = c(p) > 0$ and a constant $C > 0$ only depending on* $\mathrm{diam}(\Omega)$ *and p such that the following holds: for each $u \in GSBD^2(\Omega)$ there is a Caccioppoli partition $(P_j)_{j=1}^\infty$ of Ω with*

$$\sum\nolimits_{j=1}^\infty \mathcal{H}^1(\partial^* P_j) \le c(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega)) \tag{2.1}$$

and corresponding infinitesimal rigid motions $(a_j)_j = (a_{A_j,b_j})_j$ such that

$$v := u - \sum\nolimits_{j=1}^\infty a_j \chi_{P_j} \in SBV^p(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$$

and

$$\begin{aligned} &(i) \ \ \|v\|_{L^\infty(\Omega)} \le C(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega))^{-1}\|e(u)\|_{L^2(\Omega)}, \\ &(ii) \ \ \|\nabla v\|_{L^p(\Omega)} \le C\|e(u)\|_{L^2(\Omega)}. \end{aligned} \tag{2.2}$$

In its most general form the result holds in the generalized space $GSBD^2(\Omega)$, which loosely speaking arises from $SBD^2(\Omega)$ by requiring that one-dimensional slices have bounded variation. (We again refer to Section 3.4 for the exact definition.) The reader not interested in the generalized space may readily replace $GSBD^2(\Omega)$ by $SBD^2(\Omega)$ here and in the following (except for Lemma 2.8). The result will first be proved for configurations with a regular jump set consisting of a finite number of segments. Only at the very end we pass to the general case by using a density result (see Theorem 3.13, Lemma 5.4) and Ambrosio's compactness theorem in SBV. Consequently, the article is in large part accessible for readers without familiarity with the properties of the above mentioned function spaces.

Remark 2.2. Let us comment the results of Theorem 2.1.

(i) An important special case is given by functions not storing linearized elastic energy, i.e., $e(u) = 0$ a.e. Then (2.2) gives that u is a piecewise infinitesimal rigid motion and thus Theorem 2.1 can be interpreted as a *piecewise rigidity result*. It shows that the only way that global rigidity may fail is that the body is divided into at most countably many parts each of which subject to a different infinitesimal rigid motion. This result has been discussed for SBD functions in [15] and Theorem 2.1 may be seen as a quantititive extension in the $GSBD$ setting.

(ii) It is a well known fact that in general ∇u and u are not summable for $u \in GSBD^2(\Omega)$. The result shows that one may gain control over the function and its derivative after subtraction of a piecewise infinitesimal rigid motion. We hope that such an estimate will be useful to reduce various problems from the $(G)SBD$ to the SBV setting.
(iii) In contrast to [32], where a related result in a geometrically nonlinear setting is established, in Theorem 2.1 no modification of the configuration u is necessary. This will be crucial for the embedding results announced in Section 2.3.
(iv) In general one has $\sum_{j=1}^\infty \mathcal{H}^1(\partial^* P_j \setminus J_u) > 0$ even if J_u already induces a partition $(P_j')_j$ of Ω with $J_u = \bigcup_j \partial^* P_j' \cap \Omega$ up to a set of negligible $\mathcal{H}^1$-measure. Indeed, it is well known that in irregular domains, e.g. in domains with external cusps, Korn's inequality may fail (cf. [35]). Also in the case that each P_j' is a Lipschitz set, $\sum_{j=1}^\infty \mathcal{H}^1(\partial^* P_j \setminus J_u) > 0$ might be necessary since possibly the constants $C_{\mathrm{korn}}(P_j')$ involved in the $W^{1,p}$ Korn inequality tend to infinity as $j \to \infty$.
(v) Note that the constant in (2.2) depends on Ω only through its diameter and $p \in [1, 2)$. Consequently, the above result may be also interesting in the case of $u \in H^1(\Omega)$ and varying domains Ω.
(vi) At some points of the proof we have to pass to a slightly smaller exponent by Hölder's inequality and therefore we derive the result only for $p < 2$. In particular, we do not know if a similar result holds in the critical case $p = 2$. Note, however, that for the applications we have in mind, it is sufficient that (2.2) holds for $p = 1$.
(vii) In revising this paper, we remark that a generalized version of the result in the space $GSBD^q(\Omega)$ for $1 < q < \infty$ can be derived along similar lines, see Remark 5.6 for some details.

The result is first shown up to a small exceptional set (see Theorem 4.1) and then Theorem 2.1 for functions with regular jump set follows from an iterative application of the estimate in Theorem 4.1 (see Section 5.1). Finally, the general version is proved by approximation arguments given in Section 5.2. For a more detailed outline of the proof we refer to the beginning of Section 4 and Section 5.

In Remark 2.2(ii) we have discussed that in general $GSBD$ functions are not summable. A similar problem occurs in $GSBV$ (see Section 3.4 for the exact definition). However, we get that $GSBV$ functions are integrable after subtraction of a *piecewise constant function*.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Let $\rho > 0$ and $m \in \mathbb{N}$. For each $u \in (GSBV(\Omega;\mathbb{R}))^m$ with $\|\nabla u\|_{L^1(\Omega;\mathbb{R}^{d\times m})} + \mathcal{H}^{d-1}(J_u) < +\infty$ there is a Caccioppoli partition $(P_j)_{j=1}^\infty$ of Ω and corresponding translations $(b_j)_{j=1}^\infty \subset \mathbb{R}^m$ such that $v := u - \sum_{j=1}^\infty b_j \chi_{P_j} \in SBV(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ and*

$$\begin{aligned} &(i) \quad \sum\nolimits_{j=1}^\infty \mathcal{H}^{d-1}((\partial^* P_j \cap \Omega) \setminus J_u) \le c\rho, \\ &(ii) \quad \|v\|_{L^\infty(\Omega;\mathbb{R}^m)} \le c\rho^{-1} \|\nabla u\|_{L^1(\Omega;\mathbb{R}^{d\times m})} \end{aligned} \tag{2.3}$$

for a dimensional constant $c = c(m) > 0$.

Note that for the special choice $\rho = \theta(\mathcal{H}^{d-1}(J_u) + \mathcal{H}^{d-1}(\partial\Omega))$, $\theta > 0$, (2.3)(ii) is similar to (2.2)(i). In particular, in the proofs we will use Theorem 2.3 to derive (2.2)(i) once (2.2)(ii) is established. The estimate is essentially based on the *coarea formula* in BV (see [5, Theorem 3.40]) and the argumentation in the proof goes back to [3, Theorem 3.3] and [9, Proposition 6.2]. We remark that, in contrast to Theorem 2.1, the result is derived in arbitrary space dimension. The proof will be given in Section 6.1.

Remark 2.4. Before we proceed with some applications of Theorem 2.1, let us comment on the estimates (2.1) and (2.3)(i). In (2.3)(i) for the choice $\rho = \theta(\mathcal{H}^{d-1}(J_u) + \mathcal{H}^{d-1}(\partial\Omega))$ with θ small we obtain a fine estimate in the sense that the additional surface energy associated to the partition may be taken arbitrarily small with respect to the original jump set, where the constant in (2.3)(ii) blows up for $\theta \to 0$.

In contrast to the embedding and approximation results addressed in this article, for certain applications of the piecewise Korn inequality it is crucial to have also a refined version of (2.1), e.g. for a jump transfer theorem in SBD or a general compactness and existence result for Griffith energies in the realm of linearized elasticity (see [33]). The derivation of such an estimate needs additional techniques and we refer to [33] for a deeper analysis.

2.2. **Approximation of GSBD functions.** For each $u \in GSBD^2(\Omega)$ we can consider the sequence

$$v_n = u - \sum\nolimits_{j \geq n} a_j \chi_{P_j} \in SBV^p(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2) \tag{2.4}$$

with $(P_j)_j$ and $(a_j)_j$ as in Theorem 2.1. Hereby we can approximate $GSBD^2$ functions by bounded functions of bounded variation which coincide with the original function up to a set of arbitrarily small measure. In particular, combining this observation with the density result proved in [36] (see Theorem 3.13) we obtain the following improved approximation result for $GSBD^2$ functions in a planar setting (see Section 5.2 for the proof).

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Let $u \in GSBD^2(\Omega)$. Then there exists a sequence $(u_k)_k \subset SBV^2(\Omega;\mathbb{R}^2)$ such that each J_{u_k} is the union of a finite number of closed connected pieces of C^1-curves, each u_k belongs to $W^{1,\infty}(\Omega \setminus J_{u_k};\mathbb{R}^2)$ and the following properties hold:*

$$\begin{aligned} (i) &\quad u_k \to u \text{ in measure on } \Omega,\\ (ii) &\quad \|e(u_k) - e(u)\|_{L^2(\Omega)} \to 0,\\ (iii) &\quad \mathcal{H}^1(J_{u_k} \triangle J_u) \to 0. \end{aligned} \tag{2.5}$$

Here the symbol $\triangle$ denotes the symmetric difference of two sets. Observe that in [12, 36] the additional assumption $u \in L^2(\Omega)$ was needed which we circumvent here by (1) first approximating u by $(v_n)_n \subset GSBD^2(\Omega) \cap L^2(\Omega)$ as given in (2.4) and then (2) applying the original density result [12, 36]. In fact, in various applications in fracture mechanics approximation results are essential, e.g. for the construction of recovery sequences. Consequently, to establish complete Γ-limits it is indispensable to have a density result for all admissible functions. In particular, Theorem 2.5 allows us to complete the picture in the linearization result for Griffith energies presented in [29]. We also refer to the recent contribution [14] where with other techniques a generalization of the density result to $GSBD^p$ functions, $1 < p < \infty$, in arbitrary space dimension has been obtained.

Recall that we establish Theorem 2.1 first for functions with a regular jump set and the general case then follows by approximation of any $u \in GSBD^2(\Omega)$. Although Theorem 2.5 provides such a density result, in the proof we have to argue differently since we already use the general version of Theorem 2.1 to prove Theorem 2.5. The strategy in the proof of Theorem 2.1 is to provide an adaption of the arguments in [12, 36] and to approximate each $u \in GSBD^2(\Omega)$ (without $L^2(\Omega)$ assumption) by functions $(u_k)_k$ with regular jump set such that (2.5)(i) holds and $\|e(u_k)\|_{L^2(\Omega)} \leq c\|e(u)\|_{L^2(\Omega)}$, $\mathcal{H}^1(J_{u_k}) \leq c\mathcal{H}^1(J_u)$ (see Lemma 5.4), which is sufficient for (2.1) and (2.2).

2.3. **Relation between SBV and SBD: Embedding results.** The standard examples for functions of bounded deformation not having bounded variation are given by configurations where small balls are cut out from the bulk part with an appropriate choice of the functions on these specific sets (see e.g. [4, 17, 20]). We include the construction here for convenience of the reader.

Example 2.6. Let $B_k = B_{r_k}(x_k) \subset B_1(0) \subset \mathbb{R}^d$ be pairwise disjoint balls for $k \in \mathbb{N}$ with x_k converging to some limiting point x_∞. For $k \in \mathbb{N}$ let $A_k = (a^k_{ij}) \in \mathbb{R}^{d\times d}_{\rm skew}$ with $a^k_{12} = -a^k_{21} = d_k \in \mathbb{R}$ and $a^k_{ij} = 0$ otherwise. Define the piecewise infinitesimal rigid motion

$$u(x) = \sum\nolimits_{k=1}^\infty (A_k\,(x - x_k))\chi_{B_k}(x). \tag{2.6}$$

With the choice $r_k = (\frac{1}{k(\log(k))^2})^{\frac{1}{d-1}}$ and $d_k = r_k^{-1}$ we get $u \in L^\infty(B_1(0);\mathbb{R}^d)$ and $u \in SBV(B_1(0);\mathbb{R}^d) \cap SBD(B_1(0))$ with $e(u) = 0$ a.e. and $\mathcal{H}^{d-1}(J_u) < +\infty$. However, $\nabla u \notin L^p(B_1(0);\mathbb{R}^{d\times d})$ for $p > 1$.

In view of Theorem 2.1, we see that the above construction is essentially the only way to define $(G)SBD$ functions with $e(u) \in L^2(\Omega)$ and $\mathcal{H}^1(J_u) < +\infty$ not having bounded variation. In particular, by considering $u\chi_{\bigcup_{j=1}^n P_j}$, $n \in \mathbb{N}$, with $(P_j)_j$ as in (2.1), we observe that each $u \in GSBD^2(\Omega)$ has bounded variation away from an at most countable union of sets of finite perimeter with arbitrarily small Lebesgue measure.

Observe that $\nabla u \in L^1(B_1(0);\mathbb{R}^{d\times d})$ for the function given in Example 2.6. An L^∞ bound indeed implies that functions are of bounded variation as the following embedding result shows.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Then $SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d) \hookrightarrow SBV(\Omega;\mathbb{R}^d)$. More precisely, there is a constant $C > 0$ only depending on Ω such that*

$$\|\nabla u\|_{L^1(\Omega)} \le C\|e(u)\|_{L^2(\Omega)} + C\|u\|_{L^\infty(\Omega)}(\mathcal{H}^{d-1}(J_u) + 1). \tag{2.7}$$

Note that in contrast to Theorem 2.1, our main embedding result as well as the following theorems are valid in arbitrary space dimension, where we apply a slicing technique to extend the estimate from the planar to the d-dimensional setting. As a direct consequence, we get that one may apply the *coarea formula* in BV for functions in $SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d)$. This estimate can be potentially applied in many situations and will hopefully reveal useful.

Note that the property $u \in SBV(\Omega;\mathbb{R}^d)$ does not follow directly from (2.7). However, by first deriving Theorem 2.7 for functions with regular jump set and then using a density argument (see Theorem 3.13), we can show $SBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d) \subset BV(\Omega;\mathbb{R}^d)$. The claim then follows from Alberti's rank one property in BV (see [2]) implying $|D^c u|(\Omega) \le \sqrt{2}|E^c u|(\Omega)$.

The first term on the right hand side in (2.7) controls the distance of u from a piecewise infinitesimal rigid motion $(a_j)_j = (a_{A_j,b_j})_j$ (cf. (2.2)(ii)) and the last term allows to estimate the skew symmetric matrices $(A_j)_j$. Let us note that the above inequality is optimal in following sense: by Example 2.6 the L^1-norm on the left cannot be replaced by any other L^p-norm. Moreover, the bound $\mathcal{H}^{d-1}(J_u) < +\infty$ is essential by [17, Theorem 3.1], where a function in $u \in SBD(\Omega) \cap L^\infty(\Omega;\mathbb{R}^d)$ is constructed with $e(u) = 0$ a.e., $\mathcal{H}^{d-1}(J_u) = \infty$ and $\nabla u \notin L^1(\Omega;\mathbb{R}^{d\times d})$. Finally, the following lemma based on the construction given in Example 2.6 shows that the L^∞-norm cannot be replaced by any other L^q-norm.

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^d$ open, bounded. Then for all $1 \le q < \infty$ we find some $u \in GSBD^2(\Omega) \cap L^q(\Omega;\mathbb{R}^d)$ such that the approximate differential ∇u exists a.e. and $\nabla u \notin L^1(\Omega;\mathbb{R}^{d\times d})$.*

Consequently, $GSBD^2(\Omega) \cap L^q(\Omega;\mathbb{R}^d) \not\subset BV(\Omega;\mathbb{R}^d)$ for all $q < \infty$. In Theorem 2.1 we have seen that $v = u - a \in SBV^p(\Omega;\mathbb{R}^2) \subset GBV(\Omega;\mathbb{R}^2)$ for a piecewise infinitesimal rigid motion $a := \sum_{j=1}^\infty a_j \chi_{P_j}$. Moreover, one has that $a \in GBV(\Omega;\mathbb{R}^2)$ (see [17, Theorem 2.2]). Of course, herefrom we cannot directly deduce that $u = v + a \in GBV(\Omega;\mathbb{R}^2)$ since $GBV(\Omega;\mathbb{R}^2)$ is not a vector space. However, the property holds as the following inclusion shows.

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Then $GSBD^2(\Omega) \subset (GBV(\Omega;\mathbb{R}))^d \subset GBV(\Omega;\mathbb{R}^d)$. More precisely, there is a constant $C > 0$ only depending on Ω such that for all $i = 1, \dots, d$ and all $M > 0$*

$$|Du_i^M|(\Omega) \le CM(\mathcal{H}^{d-1}(J_u) + 1) + C\|e(u)\|_{L^2(\Omega)}. \tag{2.8}$$

Here for $u \in GSBD^2(\Omega)$ the functions $u_1, \dots, u_d$ denote the components of u and $u_i^M := \min\{\max\{u_i, -M\}, M\}$ for $M > 0$. Theorem 2.9 shows that the coarea formula in GBV (see [5, Theorem 4.34]) is applicable in $GSBD^2(\Omega)$. The results announced in this section will be proved in Section 6.2.

2.4. **A Korn-Poincaré inequality for functions with small jump set.** As an application of our embedding results and the coarea formula, we present a Korn-Poincaré inequality for functions with small jump set.

Theorem 2.10. *Let $Q \subset \mathbb{R}^d$ be a cube. Then there is a constant $C > 0$ only depending on* $\mathrm{diam}(Q)$ *and $d \in \mathbb{N}$ such that for all $u \in GSBD^2(Q)$ the following holds: there is a set of finite perimeter $E \subset Q$ with*

$$\mathcal{H}^{d-1}(\partial^* E \cap Q) \le C(\mathcal{H}^{d-1}(J_u))^{\frac{1}{2}}, \qquad |E| \le C(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}} \tag{2.9}$$

and an infinitesimal rigid motion $a = a_{A,b}$ such that

$$\begin{aligned} &(i) \quad \|u - a\|_{L^{\frac{2d}{d-1}}(Q \setminus E)} \le C\|e(u)\|_{L^2(Q)}, \\ &(ii) \quad \|u - a\|_{L^\infty(Q\setminus E)} \le C(\mathcal{H}^{d-1}(J_u))^{-\frac{1}{2}}\|e(u)\|_{L^2(Q)}. \end{aligned} \tag{2.10}$$

Moreover, if $\mathcal{H}^{d-1}(J_u) > 0$, $\bar{u} := (u-a)\chi_{Q\setminus E} \in SBV(Q;\mathbb{R}^d) \cap SBD^2(Q) \cap L^\infty(Q;\mathbb{R}^d)$ and

$$|D\bar{u}|(Q) \le C\|e(u)\|_{L^2(Q)}. \tag{2.11}$$

Note that the exceptional set E is associated to the parts of Q being detached from the bulk part of Q by J_u. A variant of the proof shows that $(\mathcal{H}^{d-1}(J_u))^{\frac{1}{2}}$ in (2.9), (2.10)(ii) can be replaced by $(\mathcal{H}^{d-1}(J_u))^{\frac{p}{2}}$ if in (2.10)(i) we replace $\frac{2d}{d-1}$ by $\frac{2d}{p(d-1)}$ for $1 \le p \le 2$.

This result improves an estimate recently obtained by Chambolle, Conti, and Francfort [13] in the sense that $\mathcal{H}^{d-1}(\partial^* E)$ can be controlled and therefore compactness results in $GSBD$ (see [20]) are applicable. In addition, we provide a Korn-type estimate in (2.11). Similar results in a planar setting have been investigated in [18, 31].

The statement essentially follows by combining (1) the result in [13] and (2) applying the coarea formula in BV together with (2.8). In particular, the argument uses a truncation at a specific level set (cf. (2.10)(ii)) which is reminiscent of the Poincaré inequality in SBV due to De Giorgi, Carriero, and Leaci (see [23]). The proof will be given in Section 6.3.

3. Preparations

3.1. **John domains.** A key step in our analysis will be the construction of a partition and a corresponding approximation of the configuration being in $W^{1,p}$ on each component of the partition. Then in the application of Poincaré's and Korn's inequality on each component, it is essential to provide uniform bounds for the constants involved in the inequalities. To this end, we introduce the notion of *John domains*.

Definition 3.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $x_0 \in \Omega$. We say Ω is a *ϱ-John domain* with respect to the *John center* x_0 and with the constant $\varrho \geq 1$ if for all $x \in \Omega$ there exists a rectifiable curve $\gamma : [0, l_\gamma] \to \Omega$, parametrized by arc length, such that $\gamma(0) = x$, $\gamma(l_\gamma) = x_0$ and $t \leq \varrho \operatorname{dist}(\gamma(t), \partial\Omega)$ for all $t \in [0, l_\gamma]$.

Domains of this form were introduced by John [37] to study problems in elasticity theory and the term was first used by Martio and Sarvas [38]. Roughly speaking, a domain is a John domain if it is possible to connect two arbitrary points without getting too close to the boundary of the set. This class is much larger than the class of Lipschitz domains and contains sets which may possess fractal boundaries or internal cusps (external cusps are excluded), e.g. the interior of Koch's *snow flake* is a John domain. Although in the following we will only consider domains with Lipschitz boundary, it is convenient to introduce the much more general notion of John domains as the constants in Poincaré's and Korn's inequality only depend on the John constant. More precisely, we have the following statement (see e.g. [1, 10, 24]).

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^d$ be a c-John domain. Let $p \in (1,\infty)$ and $q \in (1,d)$. Then there is a constant $C = C(c,p,q) > 0$ such that for all $u \in W^{1,p}(\Omega)$ there is some $A \in \mathbb{R}^{d\times d}_{\rm skew}$ such that*

$$\|\nabla u - A\|_{L^p(\Omega)} \leq C \|e(u)\|_{L^p(\Omega)}.$$

Moreover, for all $u \in W^{1,q}(\Omega)$ there is some $b \in \mathbb{R}^d$ such that

$$\|u - b\|_{L^{q^*}(\Omega)} \leq C \|\nabla u\|_{L^q(\Omega)},$$

where $q^* = \frac{dq}{d-q}$. The constant is invariant under rescaling of the domain.

We recall a result about the decomposition of sets into John domains.

Theorem 3.3. *There is a universal constant $c > 0$ such that for all simply connected, bounded domains $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary and all $\varepsilon > 0$ there is a partition $\Omega = \Omega_0 \cup \ldots \cup \Omega_N$ (up to a set of negligible measure) with $|\Omega_0| \leq \varepsilon$ and the sets $\Omega_1, \ldots, \Omega_N$ are c-John domains with Lipschitz boundary satisfying*

$$\sum\nolimits_{j=0}^N \mathcal{H}^1(\partial\Omega_j) \leq c\mathcal{H}^1(\partial\Omega). \tag{3.1}$$

Observe that in general it is necessary to introduce an (arbitrarily) small exceptional set Ω_0 as can be seen, e.g., by considering polygons with very acute interior angles. The statement is given explicitly in [30, Theorem 6.4], but can also easily be derived from the main result of that paper ([30, Theorem 1.1]) since Lipschitz sets can be approximated from within by smooth sets.

The essential step in the proof of Theorem 3.3 is to consider polygonal domains. One observes that convex polygons can be (iteratively) separated into convex polygons such that (3.1) holds and each set contains a ball whose size is comparable to the diameter of the set, whereby Definition 3.1 (for fixed $\varrho > 0$) can be confirmed. For general, nonconvex polygons the property in Definition 3.1 may be violated if a concave vertex is 'too close to the opposite part of the boundary'. It is shown that in this case the set may be (iteratively) separated into smaller polygons with less concave vertices, for which Definition 3.1 holds. We refer to [30] for more details.

We will also use the following simple property (see e.g. [43, Lemma 2.3]).

Lemma 3.4. *Let Ω be a ϱ-John domain. Then for each $x \in \Omega$ and $r > 0$ with $\Omega \setminus B_r(x) \neq \emptyset$, there is $z \in \overline{B_r(x)}$ with $B_{r/(2\varrho)}(z) \subset \Omega$. Particularly, we have $|\Omega| \geq |B_1(0)|(2\varrho)^{-d}(\operatorname{diam}(\Omega))^d$.*

3.2. **An elementary lemma about the distance of sets.** In this short section we formulate an elementary lemma which will be needed in the construction of a partition in the proof of Theorem 4.3.

Lemma 3.5. *Let $(F_j)_{j=1}^n \subset \mathbb{R}^2$, $(R_j)_{j=1}^m \subset \mathbb{R}^2$ be families of pairwise disjoint, closed and connected sets. Let $F = \bigcup_{j=1}^n F_j$ and $R = \bigcup_{j=1}^m R_j$. Let $G \subset \mathbb{R}^2$ be a closed, path connected set satisfying $G \supset F$ and*

$$\mathrm{dist}(x, F) \le d \quad \textit{for all } x \in G \setminus R \tag{3.2}$$

for some $d > 0$. Then for each F_j we find another component F_k such that (a) $\mathrm{dist}(F_j, F_k) \le 4d$ or (b) there is a corresponding set R_l such that

$$\mathrm{dist}(F_j, R_l) \le 4d \qquad \textit{and} \qquad \mathrm{dist}(F_k, R_l) \le 4d. \tag{3.3}$$

In the proof of Theorem 4.3, we will use this lemma to show that two connected components of F can be connected by a line segment of length at most $4d$ (case (a)) or two connected components of F can be connected by a curve lying inside a component R_l and two line segments of length at most $4d$ (case (b)).

Proof. Fix F_j and assume that (a) does not hold, i.e., $\mathrm{dist}(F_j, F_k) > 4d$ for all $k \neq j$. As G is path connected, we can find F_k, $j \neq k$, and a (closed) curve γ in $\overline{G \setminus F}$ such that $F_j \cup F_k \cup \gamma$ is connected. We let γ' be a connected component of

$$\gamma \cap \{x \in \mathbb{R}^2 : \mathrm{dist}(x, F_j) \ge 2d\}$$

having nonempty intersection with F_k. By (3.2) and the fact that (a) does not hold, we then find some $x \in R \cap \gamma'$ with $2d \le \mathrm{dist}(x, F_j) \le 4d$.

Let R_l be the component containing x and define $\gamma'' = \gamma' \cap R_l$. If γ'' intersects F_k, we see that R_l has the desired property (3.3). Otherwise, we find $y \in \gamma''$ with $y \in \partial R_l$ and therefore by (3.2) and the definition of γ' we get some $F_{k'}$, $k' \neq j$ (and possibly $k' \neq k$), so that $\mathrm{dist}(y, F_{k'}) \le d$. Herefrom we again deduce (3.3) with $F_{k'}$ in place of F_k. □

3.3. **Properties of infinitesimal rigid motions.** In this section we collect some properties of infinitesimal rigid motions. As before $a = a_{A,b}$ stands for the mapping $a(x) = A\,x + b$ with $A \in \mathbb{R}^{2\times 2}_{\rm skew}$ and $b \in \mathbb{R}^2$. The following lemma is shown in [33, Lemma 2.3].

Lemma 3.6. *Let $M > 0$, $\delta > 0$, $F \subset \mathbb{R}^2$ bounded, measurable and $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ a continuous nondecreasing function satisfying $\lim_{s\to\infty} \psi(s) = +\infty$. Then there is a constant $C = C(M, \delta, \psi, F)$ such that for every Borel set $E \subset F$ with $|E| \ge \delta$ and every infinitesimal rigid motion $a_{A,b}$ one has*

$$\int_E \psi(|Ax + b|)\, dx \le M \quad \Rightarrow \quad |A| + |b| \le C.$$

In the following we denote by $d(E)$ the diameter of a set $E \subset \mathbb{R}^2$.

Lemma 3.7. *Let $q \in [1, \infty)$. There is a constant $c = c(q) > 0$ such that for every Borel set $E \subset \mathbb{R}^2$ and every infinitesimal rigid motion $a = a_{A,b}$ one has for all $x \in E$*

$$\begin{aligned} &(i) \quad |A| \le c|E|^{-\frac{1}{2}-\frac{1}{q}} \|a\|_{L^q(E)}, \quad |A| \le c|E|^{-\frac{1}{2}} \|a\|_{L^\infty(E)}, \\ &(ii) \quad |A\,x + b| \le c|E|^{-\frac{1}{2}-\frac{1}{q}} d(E) \|a\|_{L^q(E)}. \end{aligned} \tag{3.4}$$

Proof. Without restriction assume that $A \neq 0$ as otherwise the statement is clear. The assumption $A \in \mathbb{R}^{2\times 2}_{\rm skew}$ implies that A is invertible and that $|Ay| = \frac{\sqrt{2}}{2}|A||y|$ for all $y \in \mathbb{R}^2$. Setting $z := -A^{-1}b$ we find for all $x \in E$

$$\tfrac{1}{\sqrt{2}}|A||x-z| = |a(x)| \le \|a\|_{L^\infty(E)}. \tag{3.5}$$

This together with $|E \setminus B_\rho(z)| \ge \frac{1}{2}|E|$ for $\rho = (\frac{|E|}{2\pi})^{\frac{1}{2}}$ yields $\frac{1}{2}|E|\rho^q|A|^q \le (\sqrt{2})^q\|a\|^q_{L^q(E\setminus B_\rho(z))} \le (\sqrt{2})^q\|a\|^q_{L^q(E)}$ for $q < \infty$. This implies the first part of (3.4)(i). In the case $q = \infty$ we immediately get $|A| \le c|E|^{-\frac{1}{2}}\|a\|_{L^\infty(E)}$ from (3.5). As there is some $x_0 \in E$ with $|A\,x_0 + b| \le |E|^{-\frac{1}{q}}\|a\|_{L^q(E)}$, we conclude for all $x \in E$

$$|A\,x + b| \le |A\,x_0 + b| + \tfrac{1}{\sqrt{2}}|A||x - x_0| \le |E|^{-\frac{1}{q}}\|a\|_{L^q(E)} + cd(E)|E|^{-\frac{1}{2}-\frac{1}{q}}\|a\|_{L^q(E)}.$$

This concludes the proof since $|E|^{\frac{1}{2}} \le d(E)$ by the isodiametric inequality. $\square$

Remark 3.8. Analogous estimates hold on lines. If, e.g., $\Gamma \subset \mathbb{R} \times \{0\}$ with $l := \mathcal{H}^1(\Gamma) < \infty$, then for a constant $c = c(q) > 0$ we have for all infinitesimal rigid motions $a = a_{A,b}$

$$l^q|A|^q \le cl^{-1}\int_\Gamma |a(x)|^q\,d\mathcal{H}^1(x).$$

With Lemma 3.7 at hand, we now see that the L^q-norm on larger sets can be controlled.

Lemma 3.9. *Let $q \in [1,\infty)$. Then there is a constant $c = c(q) > 0$ such that for all $y \in \mathbb{R}^2$, $R > 0$, Borel sets $E \subset Q^y_R := y + (-R,R)^2$, and $a = a_{A,b}$ one has*

$$\|a\|_{L^q(Q^y_R)} \le c(R^2|E|^{-1})^{\frac{1}{2}+\frac{1}{q}}\|a\|_{L^q(E)}.$$

Proof. Define $F = Q^y_R \supset E$. We repeat the last estimate of the previous proof for each $x \in F$ and obtain

$$|A\,x + b| \le |E|^{-\frac{1}{q}}\|a\|_{L^q(E)} + cd(F)|E|^{-\frac{1}{2}-\frac{1}{q}}\|a\|_{L^q(E)}.$$

This implies with $|F| = 4R^2$ and $d(F) = 2\sqrt{2}R$

$$\|a\|^q_{L^q(F)} \le cR^2|E|^{-1}\|a\|^q_{L^q(E)} + cR^{2+q}|E|^{-\frac{q}{2}-1}\|a\|^q_{L^q(E)}.$$

In view of $R^q|E|^{-\frac{q}{2}} \ge 4^{-\frac{q}{2}}$, this concludes the proof. $\square$

3.4. **(G)SBV and (G)SBD functions.** In this section we collect the definitions and fundamental properties of the function spaces needed in this article. In the following let $\Omega \subset \mathbb{R}^d$ be an open, bounded set.

SBV- and GSBV-functions. The space $BV(\Omega;\mathbb{R}^d)$ consists of the functions $u \in L^1(\Omega;\mathbb{R}^d)$ such that the distributional gradient Du is a $\mathbb{R}^{d\times d}$-valued finite Radon measure on Ω. BV-functions have an approximate differential $\nabla u(x)$ at $\mathcal{L}^d$-a.e. $x \in \Omega$ ([5, Theorem 3.83]) and their jump set J_u is $\mathcal{H}^{d-1}$-rectifiable in the sense of [5, Definition 2.57]. The space $SBV(\Omega;\mathbb{R}^d)$, often abbreviated hereafter as $SBV(\Omega)$, of *special functions of bounded variation* consists of those $u \in BV(\Omega;\mathbb{R}^d)$ such that

$$Du = \nabla u\mathcal{L}^d + [u] \otimes \nu_u\mathcal{H}^{d-1}\lfloor J_u,$$

where ν_u is a normal of J_u and $[u] = u^+ - u^-$ (the 'crack opening') with $u^\pm$ being the one-sided limits of u at J_u. If in addition $\nabla u \in L^p(\Omega)$ for some $1 < p < \infty$ and $\mathcal{H}^{d-1}(J_u) < +\infty$, we write $u \in SBV^p(\Omega)$. Moreover, $(S)BV_{\rm loc}(\Omega)$ denotes the space of functions which belong to $(S)BV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

We define the space $GBV(\Omega;\mathbb{R}^d)$ of *generalized functions of bounded variation* consisting of all $\mathcal{L}^d$-measurable functions $u : \Omega \to \mathbb{R}^d$ such that for every $\phi \in C^1(\mathbb{R}^d;\mathbb{R}^d)$ with the support of $\nabla\phi$ compact, the composition $\phi \circ u$ belongs to $BV_{\mathrm{loc}}(\Omega)$ (see [22]). Likewise, we say $u \in GSBV(\Omega;\mathbb{R}^d)$ if $\phi \circ u$ belongs to $SBV_{\mathrm{loc}}(\Omega)$ and $u \in GSBV^p(\Omega;\mathbb{R}^d)$ for $u \in GSBV(\Omega;\mathbb{R}^d)$ if $\nabla u \in L^p(\Omega)$ and $\mathcal{H}^{d-1}(J_u) < +\infty$. As usual we write for shorthand $GSBV(\Omega) := GSBV(\Omega;\mathbb{R})$. See [5] for the basic properties of these function spaces.

We now state a version of Ambrosio's compactness theorem in $GSBV$ adapted for our purposes (see e.g. [5, 21]):

Theorem 3.10. *Let $1 < p < \infty$. Let $(u_k)_k$ be a sequence in $GSBV^p(\Omega;\mathbb{R}^d)$ with*

$$\|\nabla u_k\|_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{u_k}) + \|u_k\|_{L^1(\Omega)} \le C$$

for some $C > 0$ not depending on k. Then there is a subsequence (not relabeled) and a function $u \in GSBV^p(\Omega;\mathbb{R}^d)$ such that $u_k \to u$ a.e., $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$. If in addition $\|u_k\|_\infty \le C$ for all $k \in \mathbb{N}$, we find $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$.

Caccioppoli-partitions. We say a partition $\mathcal{P} = (P_j)_{j=1}^\infty$ of Ω is a *Caccioppoli partition* of Ω if $\sum_{j=1}^\infty \mathcal{H}^{d-1}(\partial^* P_j) < +\infty$, where $\partial^* P_j$ denotes the *essential boundary* of P_j (see [5, Definition 3.60]). We say a partition is *ordered* if $|P_i| \ge |P_j|$ for $i \le j$. We now state a compactness result for ordered Caccioppoli partitions (see [5, Theorem 4.19, Remark 4.20]).

Theorem 3.11. *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Let $\mathcal{P}_i = (P_{j,i})_{j=1}^\infty$, $i \in \mathbb{N}$, be a sequence of ordered Caccioppoli partitions of Ω such that $\sup_i \sum_{j=1}^\infty \mathcal{H}^{d-1}(\partial^* P_{j,i} \cap \Omega) < +\infty$. Then there exists a Caccioppoli partition $\mathcal{P} = (P_j)_{j=1}^\infty$ and a not relabeled subsequence such that $\chi_{P_{j,i}} \to \chi_{P_j}$ in measure for all $j \in \mathbb{N}$ as $i \to \infty$.*

SBD- and GSBD-functions. We say that a function $u \in L^1(\Omega;\mathbb{R}^d)$ is in $BD(\Omega)$ if the symmetrized distributional derivative $Eu := \frac{1}{2}((Du)^T + Du)$ is a finite $\mathbb{R}^{d\times d}_{\mathrm{sym}}$-valued Radon measure. Likewise, we say u is a *special function of bounded deformation* if Eu has vanishing Cantor part $E^c u$. Then Eu can be decomposed as

$$Eu = e(u)\mathcal{L}^d + [u] \odot \nu_u \mathcal{H}^{d-1}|_{J_u},$$

where $e(u)$ is the absolutely continuous part of Eu with respect to the Lebesgue measure $\mathcal{L}^d$, $[u]$, ν_u, J_u as before and $\odot$ denotes the symmetrized tensor product. If in addition $e(u) \in L^q(\Omega)$ for some $1 < q < \infty$ and $\mathcal{H}^{d-1}(J_u) < +\infty$, we write $u \in SBD^q(\Omega)$. For basic properties of this function space we refer to [4, 6].

We now introduce the space of *generalized functions of bounded variation.* Observe that it is not possible to follow the approach in the definition of $GSBV$ since for $u \in SBD(\Omega)$ the composition $\phi \circ u$ typically does not lie in $SBD(\Omega)$. In [20], another approach is suggested which is based on certain properties of one-dimensional slices. For fixed $\xi \in S^{d-1}$ we set

$$\Pi^\xi := \{y \in \mathbb{R}^d : y \cdot \xi = 0\}, \quad \Omega^\xi_y := \{t \in \mathbb{R} : y + t\xi \in \Omega\} \text{ for } y \in \Pi^\xi,$$
$$\Omega^\xi := \{y \in \Pi^\xi : \Omega^\xi_y \neq \emptyset\}.$$

Definition 3.12. *An $\mathcal{L}^d$-measurable function $u : \Omega \to \mathbb{R}^d$ belongs to $GBD(\Omega)$ if there exists a positive bounded Radon measure λ_u such that, for all $\tau \in C^1(\mathbb{R}^d)$ with $-\frac{1}{2} \le \tau \le \frac{1}{2}$ and $0 \le \tau' \le 1$, and all $\xi \in S^{d-1}$, the distributional derivative $D_\xi(\tau(u\cdot\xi))$ is a bounded Radon measure on Ω whose total variation satisfies*

$$|D_\xi(\tau(u\cdot\xi))|(B) \le \lambda_u(B)$$

for every Borel subset B of Ω. A function $u \in GBD(\Omega)$ belongs to the subset $GSBD(\Omega)$ if in addition for every $\xi \in S^{d-1}$ and $\mathcal{H}^{d-1}$-a.e. $y \in \Pi^\xi$, the function $u^\xi_y(t) := u(y + t\xi)$ belongs to $SBV_{\rm loc}(\Omega^\xi_y)$.

As before we say $u \in GSBD^q(\Omega)$ if in addition $e(u) \in L^q(\Omega)$ and $\mathcal{H}^{d-1}(J_u) < +\infty$. For later purpose we note that there is a compactness result in $GSBD^q(\Omega)$ similar to Theorem 3.10 (see [20, Theorem 11.3]).

Density results. We recall a density result in $GSBD^2$ (see [36] and also [12, Theorem 3, Remark 5.3]).

Theorem 3.13. *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Let $u \in GSBD^2(\Omega) \cap L^2(\Omega;\mathbb{R}^d)$. Then there exists a sequence $(u_k)_k \subset SBV^2(\Omega;\mathbb{R}^d)$ such that each J_{u_k} is the union of a finite number of closed connected pieces of C^1-hypersurfaces, each u_k belongs to $W^{1,\infty}(\Omega \setminus J_{u_k};\mathbb{R}^d)$ and*

$$\begin{aligned} &(i) \ \ \|u_k - u\|_{L^2(\Omega)} \to 0,\\ &(ii) \ \ \|e(u_k) - e(u)\|_{L^2(\Omega)} \to 0,\\ &(iii) \ \ \mathcal{H}^{d-1}(J_{u_k} \triangle J_u) \to 0. \end{aligned} \tag{3.6}$$

If in addition $u \in L^\infty(\Omega)$, one can ensure $\|u_k\|_\infty \le \|u\|_\infty$ for all $k \in \mathbb{N}$.

Remark 3.14. The result together with [19] shows that the approximating sequence $(u_k)_k$ can also be chosen such that J_{u_k} is a finite union of closed $(d-1)$-simplices intersected with Ω. In this case, (3.6)(iii) has to be replaced by $\mathcal{H}^{d-1}(J_{u_k}) \to \mathcal{H}^{d-1}(J_u)$.

Korn inequality in SBD. As a final preparation, we recall a Korn inequality in SBD^2 in a planar setting for functions with small jump set.

Theorem 3.15. *Let $p \in [1,2]$. Then there is a universal constant $c > 0$ such that for all squares $Q_\mu = (-\mu,\mu)^2$, $\mu > 0$, and all $u \in SBD^2(Q_\mu)$ there is a set of finite perimeter $E \subset Q_\mu$ with*

$$\mathcal{H}^1(\partial E) \le c\mathcal{H}^1(J_u), \qquad |E| \le c(\mathcal{H}^1(J_u))^2 \tag{3.7}$$

and $A \in \mathbb{R}^{2\times 2}_{\rm skew}$, $b \in \mathbb{R}^2$ such that

$$\begin{aligned} &(i) \ \ \|u - (A\,\cdot + b)\|_{L^p(Q_\mu\setminus E)} \le c\mu^{\frac{2}{p}}\|e(u)\|_{L^2(Q_\mu)},\\ &(ii) \ \ \|\nabla u - A\|_{L^p(Q_\mu\setminus E)} \le c\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}. \end{aligned} \tag{3.8}$$

Moreover, there is a Borel set $\Gamma \subset \partial Q_\mu$ such that

$$\mathcal{H}^1(\Gamma) \le c\mathcal{H}^1(J_u), \qquad \int_{\partial Q_\mu \setminus \Gamma} |Tu - (A\,x + b)|^2 \, d\mathcal{H}^1(x) \le c\mu\|e(u)\|^2_{L^2(Q_\mu)}, \tag{3.9}$$

where Tu denotes the trace of u on ∂Q_μ.

The fact that the constant is independent of μ and p follows from a standard scaling argument and Hölder's inequality. More generally, the result holds on connected, bounded Lipschitz sets Ω for a constant C also depending on Ω. Note that the result is indeed only relevant for functions with sufficiently small jump set, as otherwise one can choose $E = Q_\mu$, $\Gamma = \partial Q_\mu$ and (3.8)-(3.9) trivially hold.

Proof. The result stated in (3.7)-(3.8) has first been derived in [31] for $p \in [1,2)$ and was then improved in [18] by showing the estimate for $p = 2$. In [13], the trace estimate has been established. We need to confirm that in both estimates (3.8), (3.9) one can indeed take the same infinitesimal rigid motion.

Let E and $a_{A,b}$ be given such that (3.7)-(3.8) hold for $p = 2$. By [13] we find Borel sets $F \subset Q_\mu$, $\Gamma \subset \partial Q_\mu$ and $a' = a_{A',b'}$ such that (3.9) holds with a' in place of a and $|F| \le c(\mathcal{H}^1(J_u))^2$ as well as $\|u - a'\|_{L^2(Q_\mu \setminus F)} \le c\mu \|e(u)\|_{L^2(Q_\mu)}$. We can assume that $\mathcal{H}^1(J_u)$ is so small that $|E \cup F| \le 2c(\mathcal{H}^1(J_u))^2 \le \frac{1}{2}|Q_\mu|$ since otherwise passing to a larger $c > 0$ in (3.7) we could choose $E = Q_\mu$ and the statement was trivially satisfied. Therefore, we have $\|a - a'\|_{L^2(Q_\mu \setminus (E \cup F))} \le c\mu\|e(u)\|_{L^2(Q_\mu)}$. By Lemma 3.9 we get $\|a - a'\|_{L^2(Q_\mu)} \le c\mu \|e(u)\|_{L^2(Q_\mu)}$ and thus by Lemma 3.7 $\|a - a'\|_{L^\infty(Q_\mu)} \le c\|e(u)\|_{L^2(Q_\mu)}$. This yields (3.9) for the function a. $\square$

4. A piecewise Korn inequality up to a small exceptional set

As a key step for the proof of Theorem 2.1, we first derive a piecewise Korn inequality up to a small exceptional set. By iterative application of this result in Section 5 we then derive the main inequality. In this section we will consider configurations on a square with regular jump set consisting of a finite number of segments and defer the general case also to Section 5, where we will make use of a density result (see Theorem 3.13 and Lemma 5.4).

For $\Omega \subset \mathbb{R}^d$ open, bounded we let $\mathcal{W}(\Omega) \subset SBV^2(\Omega)$ be the functions u such that $J_u = \bigcup_{j=1}^n \Gamma_j^u$ is a finite union of closed $(d-1)$-simplices intersected with Ω and $u|_{\Omega \setminus J_u} \in W^{1,\infty}(\Omega \setminus J_u)$. (In dimension $d = 2$, each Γ^u_j is a closed segment.) In the following we say $(P_j)_{j=1}^n$ is a partition of Ω if the sets are open, pairwise disjoint and satisfy $|\Omega \setminus \bigcup_{j=1}^n P_j| = 0$. For convenience we often write $\Omega = \bigcup_{j=1}^n P_j$ although the identity only holds up to a set of negligible $\mathcal{L}^2$-measure.

Theorem 4.1. *Let $p \in [1,2)$ and $\theta > 0$. Then there is a universal constant $c > 0$ and some $C = C(p,\theta) > 0$ such that the following holds:*

(1) For each square $Q_\mu = (-\mu,\mu)^2$ for $\mu > 0$ and each $u \in \mathcal{W}(Q_\mu)$ one finds an exceptional set $E \subset Q_\mu$ with

$$|E| \le c\mu\theta^2\, \mathcal{H}^1(J_u), \qquad \mathcal{H}^1(\partial E) \le C\mathcal{H}^1(J_u), \tag{4.1}$$

a partition $Q_\mu = \bigcup_{j=1}^n P_j$ and corresponding infinitesimal rigid motions $(a_j)_j = (a_{A_j,b_j})_j$ such that the function $v := u - \sum_{j=1}^n a_j \chi_{P_j}$ satisfies

$$\begin{aligned} &(i) \quad \mathcal{H}^1\Big(\bigcup_{j=1}^n \partial P_j \cap Q_\mu\Big) \le C\mathcal{H}^1\Big(\bigcup_{j=1}^n \partial P_j \cap J_u\Big),\\ &(ii) \quad \|\nabla v\|_{L^p(Q_\mu \setminus E)} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}. \end{aligned} \tag{4.2}$$

(2) One can choose E, $(P_j)_{j=1}^n$ and v such that we also have

$$\|v\|_{L^\infty(Q_\mu)} \le C\|u\|_{L^\infty(Q_\mu)}. \tag{4.3}$$

Note that (4.2)(ii) is similar to (2.2)(ii) with the correct scaling, where the estimate only holds outside of an exceptional set which, roughly speaking, covers the jump set J_u in a specific sense. Since (2.2)(i) will eventually follow from (2.2)(ii) by Theorem 2.3, it is not necessary at this stage to derive an analog of (2.2)(i). However, we establish (4.3) which may be of independent interest.

Estimate (4.2)(i) differs from (2.1) in the sense that the length of the boundary of the partition can be controlled solely by the part of J_u contained in $\bigcup_{j=1}^n \partial P_j$. This property will be crucial for the iterative application of the arguments in Section 5 since hereby a blow up of the length of the boundary of the partition can be avoided. Also the fact the volume of the exceptional set in (4.1) is controlled in terms of a (small) parameter $\theta > 0$ will be convenient for the subsequent analysis. Note that therefore it will not be restrictive to concentrate on the case $\theta \le 1$ and

$$\mu\theta^2 \le \mathcal{H}^1(J_u) \le 2\sqrt{2}\mu\theta^{-2}. \tag{4.4}$$

In fact, for functions with smaller jump sets, Theorem 4.1 follows directly from Theorem 3.15 for a partition only consisting of one element, and if the jump set is too large, one can choose $E = Q_\mu$ and (4.2) trivially holds.

Let us mention that for the iterative scheme in Section 5 we will make use of a *refined version* of Theorem 4.1 which particularly takes a specific structure of the exceptional set E into account. As its formulation needs some of the notions and concepts developed during the proof of Theorem 4.1, we defer the exact statement to Lemma 4.13 in Section 4.4. For a detailed outline of the iterative application of Lemma 4.13, which is necessary to derive the main result without an exceptional set, we refer to the beginning of Section 5.1.

We now concentrate on the proof of Theorem 4.1. We start by describing the basic strategy. We first identify so-called *bad squares* Q of various mesoscopic sizes where $\mathcal{H}^1(J_u \cap Q)$ is 'too large', i.e., comparable to the diameter of the square. Suitably combining regions formed by such bad squares with segments, we will be able to construct an auxiliary partition of Q_μ consisting of simply connected sets. Moreover, we obtain a corresponding Whitney covering such that $\mathcal{H}^1(J_u \cap Q) \ll d(Q)$ for the squares of the covering (see Section 4.1). Then applying the Korn inequality for functions with small jump set (Theorem 3.15) on each of these squares, we can replace the configuration u by a *piecewise smooth approximation* $\bar{u}$ which is smooth on each component of the auxiliary partition and whose distance from u can be controlled outside a small exceptional set (see Section 4.2). Afterwards, we use Theorem 3.3 to find another refined partition consisting of John domains whose John constant can be uniformly controlled. We then apply the Korn inequality in John domains (Theorem 3.2) for the auxiliary function $\bar{u}$, which is smooth in each component of the partition. Hereby, as an intermediate step, we can establish an estimate on $\nabla\bar{u}$. Finally, Theorem 4.1 follows from the fact that $\bar{u}$ is a suitable approximation of u, in particular $\nabla u - \nabla\bar{u}$ can be controlled outside a small exceptional set (see Section 4.3).

Before we start with the construction of the auxiliary partition, we introduce some general notation which will be used throughout the paper. For $s > 0$ we partition $\mathbb{R}^2$ up to a set of measure zero into squares $Q^s(p) = p + s(-1,1)^2$ for $p \in s(1,1) + 2s\mathbb{Z}^2$ and write

$$\tilde{\mathcal{Q}}^s = \{Q = Q^s(p) : \ p \in s(1,1) + 2s\mathbb{Z}^2\}. \tag{4.5}$$

Let $\theta > 0$ be given with $\theta \in 2^{-\mathbb{N}}$. For all $i \in \mathbb{N}_0$ we define $s_i = \mu\theta^i$ and we consider the coverings of dyadic squares $\mathcal{Q}^i := \tilde{\mathcal{Q}}^{s_i}$. All following statements will be formulated under the assumption that $\theta > 0$ is *small*. More precisely, this means that there exists a universal constant $\theta_0 > 0$ such that all statements hold for $\theta \in 2^{-\mathbb{N}} \cap (0, \theta_0]$.

For each square Q of the above form we also introduce the corresponding enlarged squares $Q \subset Q' \subset Q'' \subset Q'''$ defined by

$$Q' = \tfrac{3}{2}Q, \quad Q'' = 3Q, \quad Q''' = 5Q, \tag{4.6}$$

where λQ denotes the square with the same center and orientation and λ-times the sidelength of Q. By $d(A)$ we again indicate the diameter of a set $A \subset \mathbb{R}^2$ and by $\mathrm{dist}(A, B)$ we denote the Euclidian distance between $A, B \subset \mathbb{R}^2$. Finally, we introduce the *saturation* of a bounded set $A \subset \mathbb{R}^2$ by $\mathrm{sat}(A) = \mathbb{R}^2 \setminus A'$, where A' denotes the (unique) unbounded connected component of $\mathbb{R}^2 \setminus A$. Loosely speaking, $\mathrm{sat}(A)$ arises from A by 'filling the holes of A'.

4.1. **Construction of an auxiliary partition.** In this section we start the proof of Theorem 4.1 by constructing an auxiliary partition consisting of simply connected sets and a covering of Whitney-type, both associated to the jump set of the function. At this stage the constructions only involve the structure and geometry of the jump set. We therefore formulate the result in terms of a general set $J = \bigcup_{j=1}^n \Gamma_j \subset Q_\mu$ which consists of a finite number of closed segments and satisfies (cf. (4.4))

$$\mu\theta^2 \leq \mathcal{H}^1(J) \leq 2\sqrt{2}\mu\theta^{-2}. \tag{4.7}$$

(For convenience, we will still call J the *jump set* in the following.) Estimates for ∇u will only play a role starting from Section 4.2. We begin with some definitions.

Auxiliary jump set: To ensure that we provide an estimate which is valid up to a small exceptional set (cf. (4.1)), it will be convenient to decompose Q_μ into smaller squares. To this end, we introduce the *auxiliary jump set*

$$J^* := J \cup J_0 \quad \text{with } J_0 = \bigcup\nolimits_{Q \in \mathcal{Q}^7, Q \subset Q_\mu} \partial Q \tag{4.8}$$

and observe that together with (4.7) we find

$$\mathcal{H}^1(J^*) \leq \mathcal{H}^1(J) + c\theta^{-7}\mu \leq c\theta^{-9}\mathcal{H}^1(J) \tag{4.9}$$

for a universal constant $c > 0$.

Bad squares: One of the main strategies will be the application of Theorem 3.15. As the result only provides an estimate for functions with small jump set, we introduce for $i \geq 1$ the set of *bad squares*

$$\mathcal{Q}^i_{\mathrm{bad}} = \{Q \in \mathcal{Q}^i : \mathcal{H}^1(J^* \cap Q') \geq \theta^3 s_i\}. \tag{4.10}$$

The squares $\mathcal{Q}^i \setminus \mathcal{Q}^i_{\mathrm{bad}}$ will sometimes be called *good squares*. In particular, note that each square with $Q' \cap J_0 \neq \emptyset$ satisfies $\mathcal{H}^1(Q' \cap J^*) \geq 2s_i$ and thus lies in $\mathcal{Q}^i_{\mathrm{bad}}$. For $i \geq 8$ we let

$$B^i = \bigcup\nolimits_{Q \in \mathcal{Q}^i_{\mathrm{bad}}} \overline{Q'''} \tag{4.11}$$

and observe that B^i is a union of squares in $\mathcal{Q}^i$ up to a set of negligible measure (see Figure 1(a) below). Since B^i is defined in terms of enlarged squares, the squares $Q \in \mathcal{Q}^i$ lying inside B^i and touching the boundary ∂B^i are good squares. This property will be crucial for the construction of the Whitney covering in Theorem 4.3. Moreover, we will frequently use the following lemma about unions of 'bad sets'.

Lemma 4.2. *There is a universal constant $c > 0$ such that for all $8 \leq i_1 \leq i_2 < \infty$ and for all $\mathcal{R}^j \subset \mathcal{Q}^j_{\mathrm{bad}}$ we have for $R := \bigcup_{j=i_1}^{i_2} \bigcup_{Q \in \mathcal{R}^j} \overline{Q'''}$*

$$\mathcal{H}^1(\partial R) \leq c\theta^{-3}\mathcal{H}^1(J^* \cap R).$$

Moreover, there is a set $\Gamma_R \supset \partial R$ being a finite union of closed segments with $\mathcal{H}^1(\Gamma_R) \leq c\theta^{-3}\mathcal{H}^1(J^* \cap R)$ such that for each connected component R_k of R the set $\Gamma_R \cap R_k$ is connected.

Proof. For $i_1 \leq j \leq i_2$ let $\hat{\mathcal{R}}^j = \{Q \in \mathcal{R}^j : \overline{Q'''} \not\subset \bigcup_{i=i_1}^{j-1} \bigcup_{Q \in \mathcal{R}^i} \overline{Q'''}\}$ and $\hat{R}^j = \bigcup_{Q \in \hat{\mathcal{R}}^j} Q'$. Note that in view of (4.6), for θ small the sets $(\hat{R}^j)_{j=i_1}^{i_2}$ are pairwise disjoint and by (4.10) we have for all $i_1 \leq j \leq i_2$

$$\#\hat{\mathcal{R}}^j \leq \theta^{-3}s_j^{-1} \sum\nolimits_{Q \in \hat{\mathcal{R}}^j} \mathcal{H}^1(J^* \cap Q') \leq 4\theta^{-3}s_j^{-1}\mathcal{H}^1(J^* \cap \hat{R}^j),$$

where we used that each $x \in \mathbb{R}^2$ is contained in at most four Q', $Q \in \mathcal{Q}^j$. Since $\bigcup_{j=i_1}^{i_2} \hat{R}^j \subset R$, $(\hat{R}^j)_j$ are pairwise disjoint and $\mathcal{H}^1(\partial Q''') = 40 s_j$ for $Q \in \hat{\mathcal{R}}^j$, we then derive

$$\mathcal{H}^1(\partial R) \le \sum\nolimits_{j=i_1}^{i_2} \sum\nolimits_{Q \in \hat{\mathcal{R}}^j} \mathcal{H}^1(\partial Q''') \le \sum\nolimits_{j=i_1}^{i_2} c\theta^{-3} \mathcal{H}^1(J^* \cap \hat{R}^j) \le c\theta^{-3}\mathcal{H}^1(J^* \cap R).$$

To see the second part of the statement, we define $\Gamma_R = \bigcup_{j=i_1}^{i_2} \bigcup_{Q \in \hat{\mathcal{R}}^j} \partial Q'''$ and note that $\Gamma_R \cap R_k$ is connected for each connected component R_k of R. $\square$

Components and isolated components: Let $\mathcal{B}^i := (B^i_k)_k$ be the connected components of B^i for $i \ge 8$. By the remark below (4.10), for each $i \ge 8$ there is a component B^i_k with $J_0 \subset B^i_k$ (see also Figure 1(a) below). A basic strategy in the construction of the auxiliary partition will lie in inductively combining such connected components with line segments. Fix $r \in (0, \frac{1}{8})$. We say a set $B^i_k \in \mathcal{B}^i$ is an *isolated component* if

$$d(B^i_k) \le \theta^{-ir} s_i, \tag{4.12}$$

where, as before, $d(\cdot)$ denotes the diameter of a set. Let $\mathcal{B}^i_{\mathrm{iso}} \subset \mathcal{B}^i$ be the subset consisting of the isolated components. Due to their small diameter, these components will play a special role in the construction of the auxiliary partition. For more details, in particular concerning the necessity to introduce these objects, we refer to Lemma 4.4 below and the comments thereafter. At this point, let us just introduce the associated sets

$$U_i := \mathrm{sat}\Big(\bigcup\nolimits_{B^i_k \in \mathcal{B}^i_{\mathrm{iso}}} B^i_k\Big), \quad i \ge 8, \tag{4.13}$$

where $\mathrm{sat}(\cdot)$ was defined below (4.6). Using the saturation ensures that the connected components of U_i are simply connected sets.

Definition of a final iteration step: For the iterative scheme applied below, we have to introduce a final step (or equivalently a smallest length scale) at which we stop the construction. Recall that J^* is a finite union of closed segments and denote the connected components of J^* by $(\Gamma^*_j)_{j=1}^N$, each of which being a union of finitely many closed segments. Choosing $I \in \mathbb{N}$ sufficiently large, we see that for $i \ge I$ each set $(B^i_k)_k$ contains exactly one connected component of J^*. In particular, we can assume that $I = I(u, \theta, r) \in \mathbb{N}$ is so large that

$$\begin{aligned} &(i) \quad \mathrm{dist}(\Gamma^*_{j_1}, \Gamma^*_{j_2}) \ge \theta^{-1} s_I \quad \text{for} \quad 1 \le j_1 < j_2 \le N,\\ &(ii) \quad \mathcal{B}^i_{\mathrm{iso}} = \emptyset \quad \text{for} \quad i \ge I,\\ &(iii) \quad N \le s_I^{-1}\mathcal{H}^1(J^*_u). \end{aligned} \tag{4.14}$$

Indeed, for (ii) we observe $d(B^i_k) \ge d(J^* \cap B^i_k) \ge \min_{j=1,\dots,N} d(\Gamma^*_j) > 0$ and $\theta^{-ir}s_i \to 0$ for $i \to \infty$.

We now formulate the main result of this section. We show that Q_μ can be partitioned into simply connected components $(P'_j)_j$ with a corresponding covering of Whitney-type in terms of squares in $\bigcup_{i \ge 8} \mathcal{Q}^i$ such that the jump set is controllable in the squares not associated to the isolated components U_i.

Theorem 4.3. *Let $\mu > 0$, $\theta > 0$ small and $r \in (0, \frac{1}{8})$. Then there is a universal constant $c > 0$ and a constant $C = C(\theta, r) > 0$ such that for each $J \subset Q_\mu$ which is a finite union of closed segments with* (4.7) *there is a partition $(P'_j)_{j=1}^m$ of Q_μ consisting of open, simply connected sets satisfying*

$$\mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^m \partial P'_j\Big) \le C\mathcal{H}^1(J) \tag{4.15}$$

with the following properties: the set $\bigcup_{j=1}^m \partial P'_j$ *is a finite union of closed segments. Moreover, there is a covering* $\mathcal{C} \subset \bigcup_{i=8}^\infty \mathcal{Q}^i$ *of* Q_μ *(up to a set of negligible measure) with pairwise disjoint dyadic squares, a closed set* $Z \subset Q_\mu$ *and an index* $I = I(u,\theta,r) \in \mathbb{N}$ *such that*

$$Z \subset \bigcup\nolimits_{i=8}^{I} U_i, \tag{4.16}$$

for U_i *as in* (4.13)*, and with* $\mathcal{C}^i := \mathcal{C} \cap \mathcal{Q}^i$ *we have*

$$\begin{aligned}
&(i) \quad Q_\mu \setminus \big(J \cap \bigcup\nolimits_{j=1}^{m} \partial P'_j\big) \subset \bigcup\nolimits_{Q \in \mathcal{C}} Q' \subset Q_\mu,\\
&(ii) \quad Q'_1 \cap Q'_2 \neq \emptyset \ \ \textit{for} \ \ Q_1, Q_2 \in \mathcal{C} \ \Rightarrow \ \theta d(Q_1) \le d(Q_2) \le \theta^{-1} d(Q_1),\\
&(iii) \quad \#\{Q \in \mathcal{C} : x \in Q'\} \le 12 \quad \textit{for all} \ \ x \in Q_\mu,\\
&(iv) \quad \mathcal{H}^1\big(J \cap Q'\big) \le \theta^2 s_i \quad \textit{for all} \ \ Q \in \mathcal{C}^i \ \textit{with} \ Q'' \not\subset Z,\\
&(v) \quad \mathcal{H}^1\big(J \cap Q'\big) \le \theta^2 s_i \quad \textit{for all} \ \ Q \in \mathcal{C}^i \ \textit{with} \ Q'' \cap \hat{Q}'' \neq \emptyset \ \textit{for some} \ \hat{Q} \in \mathcal{C}^{i-1} \cup \mathcal{C}^{i+1},\\
&(vi) \quad J \ \cap \ \bigcup\nolimits_{i \ge I+1}^{\infty} \bigcup\nolimits_{Q \in \mathcal{C}^i} Q' \ = \emptyset,
\end{aligned} \tag{4.17}$$

where Q', Q'' *denote the enlarged squares corresponding to* $Q \in \mathcal{C}$ *(cf.* (4.6)*).*

The fact that the auxiliary partition consists of simply connected sets is essential since then Theorem 3.3 is applicable. Hereby, we will be able to find a refined partition (the one appearing in Theorem 4.1) consisting of John domains with uniformly controlled John constant, see Section 4.3. In this context, also note that (4.15) is crucial for the proof of (4.2)(i).

Observe that (4.17)(ii),(iii) are the typical properties of a Whitney covering. The main relation between the covering and the partition is given in (4.17)(i). Specifically, the (enlarged) squares of $\mathcal{C}$ do possibly not cover the part of J contained in the boundary of the partition. In particular, note that the squares $\mathcal{C}$ cover Q_μ only up to set of negligible measure. For simplicity, we will nevertheless call $\mathcal{C}$ *a covering of* Q_μ, referring to (4.17)(i) for the exact property.

The crucial conditions (4.17)(iv)-(vi) state that outside of Z the jump set in each square is small compared to its diameter. (In particular, (4.17)(vi) shows that squares of sufficiently small size do not intersect the jump at all, which is related to the definition of I in (4.14).) Note that, to achieve this, it is in general indispensable that the enlarged squares do not cover certain parts of the jump set J, cf. (4.17)(i). The conditions will allow us to apply the Korn inequality for functions with small jump set (Theorem 3.15) and to approximate the configuration u on each square outside of Z by an infinitesimal rigid motion. By using a partition of unity associated to $\mathcal{C}$, this will be the starting point to construct a piecewise smooth approximation of u, which is smooth on each component P'_j (see Section 4.2).

In the set Z we will have to apply other techniques. Note that its definition is related to the isolated components U_i, but in general the inclusion (4.16) is strict. Before we proceed with the proof of Theorem 4.3, we give a more precise meaning to Z and discuss its role in the construction of the partition.

Lemma 4.4. *Let be given the situation of Theorem 4.3 with* $(P'_j)_j$ *and* Z *as in* (4.15)-(4.16)*. Then the covering* $\mathcal{C}$ *can be chosen such that* (4.17) *holds and that there is a partition* $Z = \bigcup_{i=8}^I Z^i$ *into pairwise disjoint, closed sets with the property that each* Z^i *is a union of squares in* $\mathcal{C}^i = \mathcal{C} \cap \mathcal{Q}^i$ *up to a set of measure zero. We have*

$$\begin{aligned}
&(i) \quad \big|Z\big| \le c\mu\theta^5 \mathcal{H}^1(J), \qquad |Z^i| \le C\theta^{-ir} s_i \mathcal{H}^1(J),\\
&(ii) \quad \sum\nolimits_{i=8}^{I} \mathcal{H}^1(\partial Z^i) \le C\mathcal{H}^1(J)
\end{aligned} \tag{4.18}$$

for a universal $c>0$ and $C=C(\theta)>0$ only depending on θ. Denote by $(X^i_k)_k$ the connected components of Z^i and let $\mathcal{X}^i_k = \{Q \in \mathcal{C}^i : Q \subset X^i_k,\ \partial Q \cap \partial X^i_k \neq \emptyset\}$. Each X^i_k is a simply connected set and we have

$$\begin{aligned} &(i) \quad d(X^i_k) \le \theta^{-ir} s_i, \quad \#\mathcal{X}^i_k \le c\theta^{-2ir},\\ &(ii) \quad Q' \cap Z \subset X^i_k \quad \textit{for all} \quad Q \in \mathcal{X}^i_k,\\ &(iii) \quad \sum\nolimits_k \#\mathcal{X}^i_k \le C\mathcal{H}^1(J) s_i^{-1}. \end{aligned} \tag{4.19}$$

Since the jump set in squares of $\mathcal{C}$ lying in Z is possibly large, we cannot use Theorem 3.15 to approximate u on Z by infinitesimal rigid motions. Consequently, Z will be a part of the exceptional set of Theorem 4.1. The control on the area and the boundary of Z stated in (4.18) will be needed to show (4.1). The fact that each connected component of Z is covered by squares in $\mathcal{C}$ of a specific size will be important for the formulation of a refined version of Theorem 4.1 (see Lemma 4.13) and its application in Section 5.

Let us comment on the necessity of Z. The construction of the auxiliary partition in Theorem 4.3 is based on a suitable combination of different connected components of B^i, $i \ge 8$, with small segments (recall (4.11)). In this context, it is essential that the total length of the added segments is controlled in terms of $\mathcal{H}^1(J)$, cf. (4.15). As the total length of the added segments crucially depends on the number of connected components which have to be combined, we are only allowed to combine components with sufficiently large diameter since their number can be controlled suitably. Components with diameter smaller than $\theta^{-ir}s_i$ (see (4.12) and (4.19)(i)), called *isolated components*, are neglected in the construction of the partition.

On the one hand, this neglection allows us to obtain (4.15). On the other hand, inside these isolated components we cannot confirm the crucial property that the jump set in squares of the Whitney covering is small (cf. (4.17)(iv)). Consequently, Theorem 3.15 cannot be applied inside Z. Therefore, in Section 4.2 we will obtain a piecewise smooth approximation $\bar{u}$ of u first only outside Z. Then other techniques will have to be applied to extend $\bar{u}$ inside Z. In this context, it will be fundamental that the diameter of each connected component of Z is controlled (see (4.19)(i)), that the components are simply connected and well separated in the sense of (4.19)(ii). Finally, we will also exploit that the set of 'boundary squares' $\mathcal{X}^i_k$ consists of good squares (see (4.17)(iv)).

We now proceed with the proofs of Theorem 4.3 and Lemma 4.4.

Proof of Theorem 4.3. For the proof recall the notions introduced in (4.8)-(4.14). We start by describing the proof strategy.

Outline of the proof: *Step I:* We first construct a sequence of 'bad sets' $(E^i)^I_{i=8}$ being the central object in the proof. In iteration step i the starting point of the construction is the set B^i defined in (4.11). We consider a suitable union of B^i with components of the set $\bigcup^I_{l=i+1} B_l$, which contains bad squares of smaller length scales. Considering only connected components with sufficiently large diameter (i.e. not isolated components), we then define iteratively sets E^i such that the sequence $(E^i)^I_{i=8}$ is *decreasing* in i. In this context, it is essential to consider bad squares of various length scales in order to obtain a decreasing sequence of sets. The fundamental property of each E^i is the fact that a small layer around E^i consists only of good squares $\mathcal{Q}^i \setminus \mathcal{Q}^i_{\mathrm{bad}}$ (we refer to (4.43) for details). This together with the monotonicity of $(E^i)^I_{i=8}$ will be essential in the definition of the Whitney covering in Step IV where regions around E^i are covered with squares of $\mathcal{Q}^i$.

Step II and III: The sets $(E^i)^I_{i=8}$ are also the starting point for the definition of the partition $(P'_j)_j$. In fact, we construct a family of sets consisting of closed segments, denoted by $\mathcal{S}_I$, such that $E^I \cup \bigcup_{S \in \mathcal{S}_I} S$ is connected and $\sum_{S \in \mathcal{S}_I} \mathcal{H}^1(S)$ is controlled in terms of $\mathcal{H}^1(J^*)$. As E^I

contains ∂Q_μ, we find that the connected components of $Q_\mu \setminus (E^I \cup \bigcup_{S\in\mathcal{S}_I} S)$ are simply connected. It turns out that each connected component of E^I contains exactly one component of J^* (see (4.38)). Using these components of J^* and the sets $\mathcal{S}_I$, we then construct the partition $(P'_j)_j$ consisting of simply connected sets (see Step III).

The iterative construction of the sets $\mathcal{S}_I$ is performed in Step II. Here, in each iteration step i, the essential idea is to combine the various connected components of E^i with small segments of length $\sim s_i$. Due to a suitable bound on the number of connected components, which is related to the definition of isolated components (see (4.32)), we are able to show that the length of added segments in iteration step i is of the order $\theta^{ir}\mathcal{H}^1(J^*)$ (see (4.37)). This guarantees a control on the total length of the segments which are added in all iterations steps (see left contribution in (4.30)).

In the construction an additional difficulty has to be faced: if $E^{i-1}\setminus E^i$ contains 'large' connected sets, so-called *removed sets* as defined in (4.25), possibly longer segments have to be added. The length of these segments, however, can be controlled in terms of the original jump set contained in the (pairwise disjoint) removed sets, see the right contribution in (4.30).

Step IV: In the final step of the proof we define a covering of Whitney-type by tessellating regions around E^i with squares of $\mathcal{Q}^i$. We observe that the area of the sets $(E^i)_{i=8}^I$ becomes gradually smaller and eventually each component of E^I is a small neighborhood of a component of J^*. We then finally tessellate $E^I \setminus J$ with squares of $\bigcup_{l\geq I+1}^\infty \mathcal{Q}^l$. This allows us to show (4.17)(i),(vi). For properties (4.17)(ii),(iii) we follow the usual construction of Whitney coverings. Finally, for (4.17)(iv),(v) we fundamentally use that small layers around each E^i consist only of good squares (see (4.43)). To establish the latter property, it is crucial that the sets B^i are defined in terms of enlarged squares (cf. (4.6) and (4.11)).

In this step of the proof we also introduce the set Z and comment on its definition.

We now start with the proof. In the following $c>0$ stands for a universal constant. We may suppose that θ is small, i.e., $\theta \leq \theta_0$ for some small universal constant $\theta_0 \in (0,\frac{1}{16})$ chosen in dependence of the appearing universal constants in the proof (cf. (4.31) and (4.49)). Having fixed the value of θ, we let $I = I(u,\theta,r)\in\mathbb{N}$ be the index introduced in (4.14).

Step I (Definition of bad sets): We define sets forming the starting point for the construction of the partition and the Whitney covering. Recall the definition of B^i in (4.11). For $8\leq i\leq I$ let $\mathcal{D}^i = (D^i_k)_k$ be the connected components of $\bigcup_{l=i}^I B^l$ having nonempty intersection with B^i and satisfying

$$d(D^i_k) > \theta^{-ir}s_i. \tag{4.20}$$

Observe that by definition of $\mathcal{B}^i_{\rm iso}$, in particular (4.12), each $B^i_k \in \mathcal{B}^i\setminus\mathcal{B}^i_{\rm iso}$ is contained in a component of $\mathcal{D}^i$. As each connected component is a finite union of squares, we find $d(D^i_k)\leq \mathcal{H}^1(\partial D^i_k)$ and thus by Lemma 4.2 (for $\mathcal{R}^j = \mathcal{Q}^j_{\rm bad}$, $i\leq j\leq I$) we obtain

$$\sum\nolimits_k d(D^i_k) \leq \mathcal{H}^1\Big(\partial \bigcup\nolimits_{l=i}^I B^l\Big) \leq c\theta^{-3}\mathcal{H}^1\big(J^*\cap \bigcup\nolimits_{l=i}^I B^l\big) \leq c\theta^{-3}\mathcal{H}^1(J^*) \tag{4.21}$$

independently of $8\leq i\leq I$.

For notational reasons, it is convenient to start with the set $E^7 := \bigcup_{Q\in\mathcal{Q}^7, Q\subset Q_\mu}\overline{Q'''}$ containing Q_μ. Assuming that E^k, $7\leq k\leq i-1$, have been constructed, we let

$$\mathcal{E}^i = \{D^i_k : D^i_k\cap E^{i-1}\neq\emptyset\} \quad \text{and} \quad E^i = \bigcup\nolimits_{D^i_k\in\mathcal{E}^i} D^i_k. \tag{4.22}$$

(See also Figure 1(b)-(d).) As mentioned above, it will turn out that for each $8\leq i\leq I$ a small layer around E^i consists of good squares $\mathcal{Q}^i\setminus\mathcal{Q}^i_{\rm bad}$. (This property is essential for the construction of the Whitney covering in Step IV and we refer to (4.43) for details.)

Consequently, in the definition of E^i it suffices to consider the components D^i_k having nonempty intersection with E^{i-1}. This is indeed crucial to obtain a sequence of sets which is decreasing in i. In fact, we have

$$E^i \subset E^j \qquad \text{for } 7 \le j < i \le I. \tag{4.23}$$

First, $E^8 \subset E^7$ is clear. Let $i \ge 9$. Note that each $D^i_k \in \mathcal{E}^i$ intersects E^{i-1} and thus we find some $D^{i-1}_{k'} \in \mathcal{E}^{i-1}$ with $D^i_k \cap D^{i-1}_{k'} \neq \emptyset$. As $D^{i-1}_{k'}$ is a connected component of $\bigcup_{l=i-1}^I B^l$ and D^i_k is contained in a connected component of $\bigcup_{l=i-1}^I B^l$, we get $D^i_k \subset D^{i-1}_{k'}$. Then the definition of E^{i-1} implies $D^i_k \subset D^{i-1}_{k'} \subset E^{i-1}$. This yields $E^i \subset E^{i-1}$, as desired.

Moreover, note that each E^i contains J_0 (recall (4.8)) and therefore particularly

$$\partial Q_\mu \subset J_0 \subset E^i \qquad \text{for} \quad 8 \le i \le I. \tag{4.24}$$

In fact, by the remark below (4.10), for each $i \ge 8$ there is a component D^i_k satisfying $J_0 \subset D^i_k$.

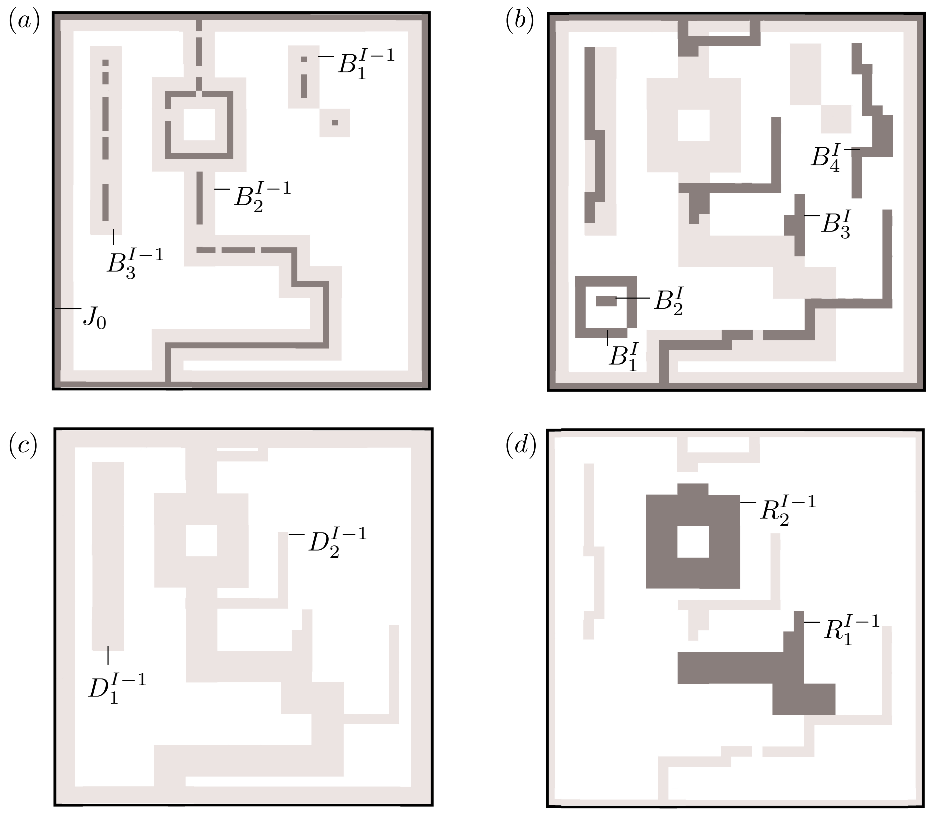

Figure 1. (a) A square of $\mathcal{Q}^7$ contained in Q_μ is depicted with $B^{I-1} = \bigcup_{k=1}^3 B_k^{I-1}$ in light gray and the squares $\mathcal{Q}^{I-1}_{\mathrm{bad}}$ in dark gray. Note that $B_1^{I-1} \in \mathcal{B}^{I-1}_{\mathrm{iso}}$ and $J_0 \subset B_2^{I-1}$. (b) In light gray we see B^{I-1} and in dark gray B^I. The sets B_1^I, B_2^I, B_3^I are contained in $\mathcal{B}^I_{\mathrm{iso}}$ with $B_2^I \subset \mathrm{sat}(B_1^I)$. We suppose $B_4^I \notin \mathcal{B}^I_{\mathrm{iso}}$ and observe $B_4^I \cap B^{I-1} = \emptyset$. (c) The set $E^{I-1} = D_1^{I-1} \cup D_2^{I-1}$ is sketched. Note that $B_4^I \cap E^{I-1} = \emptyset$ although $B_4^I \notin \mathcal{B}^I_{\mathrm{iso}}$. (d) In light gray E^I is depicted and in dark gray $R^{I-1} = R_1^{I-1} \cup R_2^{I-1}$. Observe that $E^I \subset E^{I-1}$.

For the construction of the partition we will also need the associated *removed sets*

$$\mathcal{P}^i = \Big\{Q \in \bigcup\nolimits_{l=i}^{I} \mathcal{Q}^l_{\rm bad} : \overline{Q'''} \subset E^i, \ \overline{Q'''} \cap E^{i+1} = \emptyset\Big\}, \quad R^i = \bigcup\nolimits_{Q\in\mathcal{P}^i} \overline{Q'''} \tag{4.25}$$

for $7 \le i \le I-1$, which will play a pivotal role for the construction of the partition in Step II. We observe that

$$R^j \cap R^i = \emptyset \quad \text{ for } 7 \le j < i \le I-1. \tag{4.26}$$

To see this, consider $j < i$ and note that $\overline{Q'''} \cap E^i = \emptyset$ for $Q \in \mathcal{P}^j$ since $E^i \subset E^{j+1}$. Thus, we find $R^j \cap E^i = \emptyset$ and therefore (4.26) holds as $R^i \subset E^i$.

Essentially, E^i arises from E^{i-1} by removing the squares $\mathcal{P}^{i-1}$. Moreover, E^i still intersects the squares $\{Q \in \mathcal{Q}^{i-1} \setminus \mathcal{P}^{i-1} : \overline{Q'''} \subset E^{i-1}\}$, but in general covers a smaller portion compared to E^{i-1} (cf. Figure 1(d)). More precisely, we get

$$\mathrm{dist}(x, E^i) \le 10\sqrt{2} s_{i-1} \quad \text{for all } \ x \in E^{i-1} \setminus R^{i-1}. \tag{4.27}$$

In fact, since $x \in E^{i-1}$, by (4.22) we find $Q \in \bigcup_{l=i-1}^{I} \mathcal{Q}^l_{\rm bad}$ such that $x \in \overline{Q'''} \subset E^{i-1}$. As $x \notin R^{i-1}$, we obtain $Q \notin \mathcal{P}^{i-1}$ and therefore $\overline{Q'''} \cap E^i \neq \emptyset$, which implies (4.27).

Step II (Construction of the partition, induction step): Note that the sets $(E^i)_{i=8}^I$ defined in Step I are in general not connected. The goal of this step is to construct a family of closed, connected sets $\mathcal{S}_I$ such that each $S \in \mathcal{S}_I$ is a finite union of closed segments, the set $E^I \cup \bigcup_{S\in\mathcal{S}_I} S$ is connected, and we have

$$\#\mathcal{S}_I \le \theta^{-4} s_I^{-1} \mathcal{H}^1(J^*), \qquad \mathcal{H}^1\big(\bigcup\nolimits_{S\in\mathcal{S}_I} S\big) \le C\mathcal{H}^1(J^*), \tag{4.28}$$

where $C = C(\theta, r) > 0$. In particular, since $\partial Q_\mu \subset E^I$, this implies that the connected components of $Q_\mu \setminus (E^I \cup \bigcup_{S\in\mathcal{S}_I} S)$ are simply connected. In Step III we will use this fact to construct the partition $(P'_j)_{j=1}^m$ consisting of simply connected sets. The family $\mathcal{S}_I$ is defined inductively, where in each iteration step i the strategy is to connect the components of E^i by suitable segments lying in E^{i-1}. The argument is mainly based on a suitable control on the number of components induced by (4.20), the definition of the removed sets (4.25), and an elementary lemma about the distance of sets (Lemma 3.5). As the details are not needed for the subsequent parts of the proof, the reader may prefer to proceed directly with Step III.

We will show by induction that for $i \ge 8$ there is a family of closed, connected sets $\mathcal{S}_i$ with

$$\#\mathcal{S}_i \le \theta^{-4}\theta^{ir} s_i^{-1}\mathcal{H}^1(J^*), \tag{4.29}$$

such that each $S \in \mathcal{S}_i$ is a finite union of closed segments and $E^i_{\mathcal{S}} := E^i \cup \bigcup_{S\in\mathcal{S}_i} S$ is connected. Moreover, we show (recall (4.25))

$$\mathcal{H}^1\big(\bigcup\nolimits_{S\in\mathcal{S}_i} S\big) \le c\theta^{-5}\mathcal{H}^1(J^*)\sum\nolimits_{j=8}^{i}\theta^{jr} + c\theta^{-3}\sum\nolimits_{j=7}^{i-1}\mathcal{H}^1(J^*\cap R^j). \tag{4.30}$$

Herefrom we then obtain (4.28). Indeed, it suffices to apply (4.29)-(4.30) for step I and to recall that the sets $(R^j)_j$ are pairwise disjoint (see (4.26)).

Let us assume that the above assertions, in particular (4.29)-(4.30), hold for $i-1$, $i \ge 8$. (Note that the assertions hold for $i = 7$ with $\mathcal{S}_7 := \emptyset$ since E^7 is connected, see before (4.22).) We now construct $\mathcal{S}_i$ and confirm (4.29)-(4.30).

Components: We denote the connected components of $E^i \cup \bigcup_{S\in\mathcal{S}_{i-1}} S$ by $\mathcal{F}^i := (F^i_k)_{k=1}^{M_i}$. The goal is to connect these sets by suitable segments. Observe that each F^i_k is a union of closed sets with Hausdorff-dimension one or two. Recall that each connected component $D^i_k \in \mathcal{E}^i$ satisfies $d(D^i_k) \ge \theta^{-ir} s_i$ by (4.20). Consequently, by (4.21) we get for θ small

$$\#\mathcal{E}^i \le \theta^{ir} s_i^{-1} \sum\nolimits_k d(D^i_k) \le c\theta^{-3}\theta^{ir} s_i^{-1}\mathcal{H}^1(J^*) \le \tfrac{1}{4}\theta^{-4}\theta^{ir} s_i^{-1}\mathcal{H}^1(J^*). \tag{4.31}$$

By (4.29) (for $i-1$) and (4.31) we have

$$M_i = \#\mathcal{F}^i \le \tfrac{1}{4}\theta^{-4}\theta^{ir} s_i^{-1}\mathcal{H}^1(J^*) + \theta^{-4}\theta^{(i-1)r} s_{i-1}^{-1}\mathcal{H}^1(J^*) \le \tfrac{1}{2}\theta^{-4}\theta^{ir} s_i^{-1}\mathcal{H}^1(J^*) \tag{4.32}$$

recalling that $r < \frac{1}{8}$, $\theta \le \frac{1}{16}$ and $s_i = \theta s_{i-1}$. By the definition of $\mathcal{F}^i$, $E^{i-1}_{\mathcal{S}} = E^{i-1} \cup \bigcup_{S \in \mathcal{S}_{i-1}} S$ and the fact that $E^i \subset E^{i-1}$ (see (4.23)) we get

$$E^{i-1}_{\mathcal{S}} = \bigcup\nolimits_{k=1}^{M_i} F^i_k \cup (E^{i-1} \setminus E^i). \tag{4.33}$$

We denote the connected components of R^{i-1} (see (4.25)) by $\mathcal{R}^{i-1} = (R^{i-1}_k)_k$ and let $\bar{c} = 10\sqrt{2}$ for shorthand. Observe that by (4.27), (4.33) and the fact that $E^i \subset \bigcup_{k=1}^{M_i} F^i_k$ we obtain

$$\operatorname{dist}\big(x, \bigcup\nolimits_{k=1}^{M_i} F^i_k\big) \le \bar{c} s_{i-1} \quad \text{for all } \ x \in E^{i-1}_{\mathcal{S}} \setminus R^{i-1}. \tag{4.34}$$

Recall that $E^{i-1}_{\mathcal{S}}$ is (path) connected by hypothesis. Using (4.33)-(4.34) we now apply Lemma 3.5 for the families $\mathcal{F}^i$ and $\mathcal{R}^{i-1}$ with $G = E^{i-1}_{\mathcal{S}}$ and $d = \bar{c}s_{i-1}$. We get that for each $F^i_j \in \mathcal{F}^i$ there is another component $F^i_k \in \mathcal{F}^i$ such that (a) $\operatorname{dist}(F^i_j, F^i_k) \le 4\bar{c}s_{i-1}$ or (b) there is a corresponding $R^{i-1}_l \in \mathcal{R}^{i-1}$ such that

$$\operatorname{dist}(F^i_j, R^{i-1}_l) \le 4\bar{c}s_{i-1}, \qquad \operatorname{dist}(F^i_k, R^{i-1}_l) \le 4\bar{c}s_{i-1}.$$

Due to (4.34) and the fact that $E^{i-1}_{\mathcal{S}}$ is connected, we further note that for each pair $F^i_{j_1}, F^i_{j_2} \in \mathcal{F}^i$ there is a chain $j_1 = k_1, k_2, \ldots, k_n = j_2$ such that each pair $F^i_{k_l}, F^i_{k_{l+1}}$, $l = 1, \ldots, n-1$, satisfies (a) or (b).

Construction of segments: Consider $F^i_j, F^i_k \in \mathcal{F}^i$ such that (a) or (b) holds. We observe that in case (a) one can choose a (closed) segment S with $\mathcal{H}^1(S) \le 4\bar{c}s_{i-1}$ such that $F^i_j \cup S \cup F^i_k$ is connected and in case (b) we can find two segments S^1, S^2 with $\mathcal{H}^1(S^1 \cup S^2) \le 8\bar{c}s_{i-1}$ such that $S^1 \cup S^2 \cup R^{i-1}_l \cup F^i_j \cup F^i_k$ is connected, where the component R^{i-1}_l is chosen as above. Thus, for a universal $c > 0$ large enough we can find $M_i - 1$ sets $(\hat{R}^{i-1}_l)_{l=1}^{M_i-1} \subset \mathcal{R}^{i-1} \cup \{\emptyset\}$ and sets $(S_l)_{l=1}^{M_i-1}$ with

$$\mathcal{H}^1(S_l) \le c s_{i-1}, \tag{4.35}$$

where each S_l consists of at most two segments, so that $\bigcup_{k=1}^{M_i} F^i_k \cup \bigcup_{l=1}^{M_i-1}(\hat{R}^{i-1}_l \cup S_l)$ is connected and each $\hat{R}^{i-1}_l \cup S_l$ is connected. Indeed, the construction of sets connecting two different components $(F^i_k)_k$ has been addressed above and the fact that it suffices to consider $M_i - 1 = \#\mathcal{F}^i - 1$ sets may be seen by induction over the number of components M_i.

We recall (4.25) and note $R^{i-1} = \bigcup_{j=i-1}^{I} \bigcup_{Q \in \mathcal{P}^{i-1} \cap \mathcal{Q}^j} \overline{Q'''}$. We apply Lemma 4.2 for the squares $\mathcal{P}^{i-1} \cap \mathcal{Q}^j \subset \mathcal{Q}^j_{\rm bad}$, $i-1 \le j \le I$, and obtain a set $\Gamma^{i-1} \supset \partial R^{i-1}$, being a finite union of closed segments, such that $\Gamma^{i-1} \cap \hat{R}^{i-1}_l$ is connected for $l = 1, \ldots, M_i - 1$ and

$$\mathcal{H}^1\big(\bigcup\nolimits_{l=1}^{M_i-1} \hat{R}^{i-1}_l \cap \Gamma^{i-1}\big) \le \mathcal{H}^1\big(\Gamma^{i-1}\big) \le c\theta^{-3}\mathcal{H}^1(R^{i-1} \cap J^*). \tag{4.36}$$

Then we introduce the new jump components

$$\mathcal{S}_* = \big(S_l \cup (\hat{R}^{i-1}_l \cap \Gamma^{i-1})\big)_{l=1}^{M_i-1}, \qquad \mathcal{S}_i = \mathcal{S}_{i-1} \cup \mathcal{S}_*.$$

We are now in the position to confirm (4.29)-(4.30). First, each component in $\mathcal{S}_i$ is connected and a finite union of closed segments. Moreover, we observe that $E^i_{\mathcal{S}} := E^i \cup \bigcup_{S\in\mathcal{S}_i} S$ is connected by the definition of $(F^i_k)_k$ and $\mathcal{S}_*$ (cf. Figure 2(b)). By (4.29) (for $i-1$) and (4.32) we find

$$\#\mathcal{S}_i \le \#\mathcal{S}_{i-1} + \#\mathcal{F}^i - 1 \le \theta^{-4}\theta^{r(i-1)} s_{i-1}^{-1}\mathcal{H}^1(J^*) + \tfrac{1}{2}\theta^{-4}\theta^{ir} s_i^{-1}\,\mathcal{H}^1(J^*) \le \theta^{-4}\theta^{ir} s_i^{-1}\mathcal{H}^1(J^*)$$

as $r < \frac{1}{8}$ and $\theta \le \frac{1}{16}$. This gives (4.29). Moreover, we use (4.32) as well as (4.35) to find

$$\sum\nolimits_{l=1}^{M_i-1} \mathcal{H}^1(S_l) \le c(M_i-1)s_{i-1} \le c\theta^{-5}\theta^{ir}\mathcal{H}^1(J^*). \tag{4.37}$$

This together with (4.36) and (4.30) for step $i-1$ yield (4.30) for step i.

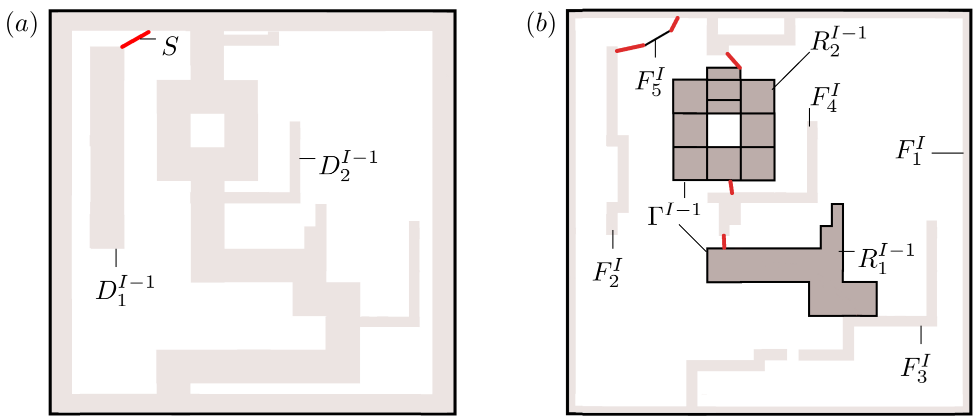

FIGURE 2. (a) The set E^{I-1} consists of two components D_1^{I-1} and D_2^{I-1} which have been connected by the red segment S. Thus, $E^{I-1}_{\mathcal{S}} = E^{I-1}\cup S$ is connected. (b) The set $E^I \cup S$ consists of five components $\mathcal{F}^I = \{F^I_1,\ldots,F^I_5\}$ where $F^I_5 = S$. The components of $\mathcal{S}_*$ are depicted, where the (union of) segments S_l, $l=1,\ldots,4$, connecting the sets $\mathcal{F}^I$ are highlighed in red and (a possible choice of) the set Γ^{I-1} is illustrated in black. While in the example we can choose $\Gamma^{I-1}\cap R_1^{I-1} = \partial R_1^{I-1}$, in R_2^{I-1} a more elaborated definition of Γ^{I-1} is necessary (cf. Lemma 4.2) since R_2^{I-1} is not simply connected.

Step III (Construction of the partition, final step): In Step II we have constructed a family of closed, connected sets $\mathcal{S}_I$ such that each $S\in\mathcal{S}_I$ is a finite union of closed segments, the set $E^I\cup\bigcup_{S\in\mathcal{S}_I} S$ is connected, and (4.28) holds. We now define the partition $(P'_j)_{j=1}^m$ with the desired properties. The argument will be based on the observation that, due to the choice of I in (4.14), each connected component of E^I contains exactly one component of J^*. These components will be connected suitably with $\bigcup_{S\in\mathcal{S}_I} S$.

First, note that by the definition of $\mathcal{D}^I$ in (4.20) and (4.14)(ii) we have $\mathcal{D}^I = \mathcal{B}^I$. Consequently, recalling (4.22) we see that E^I consists of the connected components of B^I having nonempty intersection with E^{I-1}. As before, by $\mathcal{E}^I \subset \mathcal{D}^I = \mathcal{B}^I$ we denote the connected components D^I_k with $D^I_k\cap E^{I-1}\neq\emptyset$.

In view of the remark before (4.14), we observe that each D^I_k contains exactly one connected component $\Gamma^*_{j_k}$ of J^* (cf. Figure 3(a)). More precisely, by (4.10) and (4.11) we have

$$\mathrm{dist}(x,\Gamma^*_{j_k}) \le \bar{c}s_I \qquad \text{for all } x \in D^I_k, \tag{4.38}$$

where as before we set $\bar{c} = 10\sqrt{2}$ for brevity. Moreover, since $\#\mathcal{E}^I$ is bounded from above by the number of components of J^*, we find $\#\mathcal{E}^I \le s_I^{-1}\mathcal{H}^1(J^*)$ by (4.14)(iii).

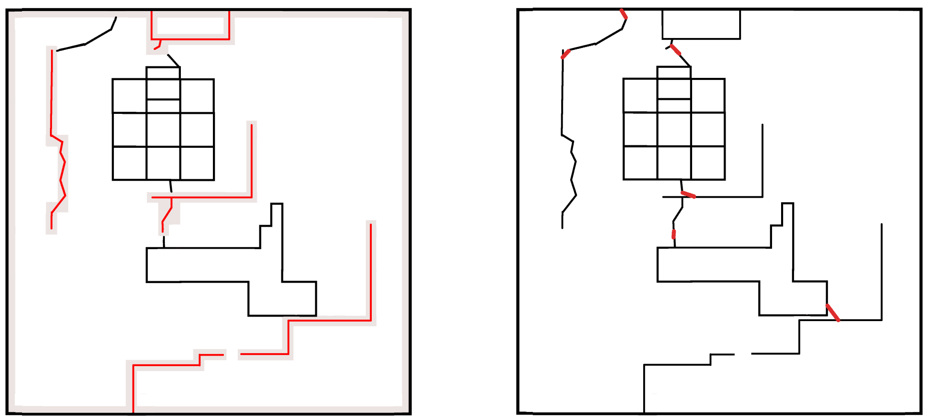

FIGURE 3. (a) The segments $\mathcal{S}_I$ are depicted in black and the set E^I in gray, where $E^I \cup \bigcup_{S\in\mathcal{S}_I} S$ is connected. Moreover, the components $(\Gamma^*_{j_k})_k$ contained in E^I are illustrated in red. (b) The components of $\mathcal{S}_I$ and $(\Gamma^*_{j_k})_k$ are combined to a connected set by the red segments.

We now proceed similarly as in Step II to connect the components $\mathcal{S}_I$ and $(\Gamma^*_{j_k})_k$ with segments (cf. Figure 3(b)). We apply Lemma 3.5 on the family $\mathcal{F} := \mathcal{S}_I \cup \{\Gamma^*_{j_k} : D^I_k \in \mathcal{E}^I\}$ with $R = 0$, $G = E^I \cup \bigcup_{S\in\mathcal{S}_I} S$ and $d = \bar{c}s_I$, where we note that (4.38) implies (3.2). We see that for each set in $\mathcal{F}$ there is another set in $\mathcal{F}$ with distance at most $4\bar{c}s_I$. Thus, we can find a family of closed segments $\hat{\mathcal{S}}$ with $\#\hat{\mathcal{S}} = \#\mathcal{F} - 1$ and $\mathcal{H}^1(\hat{S}) \le 4\bar{c}s_I$ for all $\hat{S} \in \hat{\mathcal{S}}$ such that

$$\bigcup\nolimits_{D^I_k\in\mathcal{E}^I} \Gamma^*_{j_k} \cup \bigcup\nolimits_{\hat{S}\in\hat{\mathcal{S}}} \hat{S} \cup \bigcup\nolimits_{S\in\mathcal{S}_I} S$$

is connected, where as before $\Gamma^*_{j_k}$ denotes the component contained in D^I_k. Note that $\partial Q_\mu \subset \bigcup_{D^I_k\in\mathcal{E}^I} \Gamma^*_{j_k}$ since $\partial Q_\mu \subset J_0 \subset E^I$ by (4.24). Thus, letting

$$\mathcal{S} = \mathcal{S}_I \cup \hat{\mathcal{S}} \cup \{\Gamma^*_{j_k} : D^I_k \in \mathcal{E}^I\} \tag{4.39}$$

we finally find that the connected components of $Q_\mu \setminus \bigcup_{S\in\mathcal{S}} S$, denoted by $(P'_j)_{j=1}^m$, are simply connected and form a partition of Q_μ (up to a set of negligible measure). By construction $\bigcup_{j=1}^m \partial P'_j$ is a finite union of closed segments. To conclude this step of the proof, it remains to show (4.15). Recall $\#\mathcal{E}^I \le s_I^{-1}\mathcal{H}^1(J^*)$ by (4.14)(iii). This together with (4.28) gives

$$\sum\nolimits_{\hat{S}\in\hat{\mathcal{S}}} \mathcal{H}^1(\hat{S}) \le 4\bar{c}s_I(\#\mathcal{F}-1) \le 4\bar{c}s_I(\#\mathcal{S}_I + \#\mathcal{E}^I) \le C\mathcal{H}^1(J^*)$$

for $C = C(\theta, r) > 0$. Then recalling (4.39) and using again (4.28) we conclude

$$\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^m \partial P'_j\big) \le \sum\nolimits_{j=1}^m \mathcal{H}^1(\partial P'_j) \le C\mathcal{H}^1(J^*) \le C\mathcal{H}^1(J), \tag{4.40}$$

where in the last step we employed (4.9).

Step IV (Covering): After having completed the construction of the partition $(P'_j)_{j=1}^m$, we now finally define an associated Whitney covering of Q_μ, introduce the set Z, and confirm (4.17). For this part of the proof, recall the definition of the sets E^i in (4.22) and the properties (4.23)-(4.24). Moreover, recall the relation between E^I, the components of J^*, and the partition $(P'_j)_{j=1}^m$ in (4.38)-(4.39).

As motivated above, we will cover certain regions around E^i with squares of $\mathcal{Q}^i$. Let us first introduce some relevant notation. For each $8 \le i \le I$ we define the sets

$$\begin{aligned} T^i &= \bigcup\nolimits_{Q \in \mathcal{T}^i} \overline{Q} && \text{with} \quad \mathcal{T}^i = \{Q \in \mathcal{Q}^i : \ Q \subset Q_\mu \setminus E^i\}, \\ T^i_- &= \bigcup\nolimits_{Q \in \mathcal{T}^i_-} \overline{Q} && \text{with} \quad \mathcal{T}^i_- = \{Q \in \mathcal{T}^i : Q'' \subset T^i\}, \\ T^i_{--} &= \bigcup\nolimits_{Q \in \mathcal{T}^i_{--}} \overline{Q} && \text{with} \quad \mathcal{T}^i_{--} = \{Q \in \mathcal{T}^i_- : Q'' \subset T^i_-\}, \end{aligned} \tag{4.41}$$

where, loosely speaking, T^i_-, T^i_{--} arise from T^i by removing 'layers' of squares. (We refer to Figure 4 for an illustration.) As a preparation, we observe that $T^j \supset T^i$ for $8 \le i < j \le I$ since $E^j \subset E^i$ (see (4.23)) and $s_j < s_i$. This particularly implies

$$T^{i-1}_- \subset T^i_-, \qquad \mathrm{dist}(\partial T^{i-1}_-, \, \partial T^i_-) \ge s_{i-1} \tag{4.42}$$

for $9 \le i \le I$. As mentioned above, for the proof of (4.17) it will be crucial that a small layer around E^i does not contain bad squares. Indeed, we have

$$\big(T^i \setminus T^i_{--}\big) \cap \bigcup\nolimits_{Q \in \mathcal{Q}^i_{\rm bad}} Q = \emptyset. \tag{4.43}$$

To see this, we note as a preparation that each $Q \in \mathcal{T}^i \setminus \mathcal{T}^i_{--}$ satisfies $\overline{Q'''} \cap E^i \neq \emptyset$. Indeed, since $Q \notin \mathcal{T}^i_{--}$, we find some $Q_* \in \mathcal{T}^i \setminus \mathcal{T}^i_-$ such that $Q_* \subset Q''$. As $Q_* \notin \mathcal{T}^i_-$, we find some $Q_{**} \in \mathcal{Q}^i \setminus \mathcal{T}^i$ with $Q_{**} \subset Q''_*$. Thus, $\overline{Q''_*} \cap E^i \neq \emptyset$ and it then suffices to note that $\overline{Q'''} \supset \overline{Q''_*}$ (we also refer to Figure 4(a)).

Now suppose by contradiction that $Q \in \mathcal{Q}^i_{\rm bad} \cap (\mathcal{T}^i \setminus \mathcal{T}^i_{--})$. Since $\overline{Q'''} \cap E^i \neq \emptyset$, by (4.22) we find some $D^i_k \in \mathcal{E}^i$ such that $\overline{Q'''} \cap D^i_k \neq \emptyset$. As D^i_k is a connected component of $\bigcup_{l=i}^I B^l$ (see before (4.20)), we get $\overline{Q'''} \subset D^i_k$ and thus $\overline{Q'''} \subset E^i$ by (4.22). This gives a contradiction as clearly $\overline{Q'''} \not\subset E^i$ due to $Q \in \mathcal{T}^i$.

Definition of the covering and the set Z: We now introduce the covering as follows. Recall that each set $T^i_- \setminus T^{i-1}_-$ for $8 \le i \le I$ is a union of squares in $\mathcal{Q}^i$ (up to a negligible set), where we set $T^7_- = \emptyset$. We define for $8 \le i \le I$

$$\mathcal{C}^i = \{Q \in \mathcal{Q}^i : Q \subset T^i_- \setminus T^{i-1}_-\}. \tag{4.44}$$

Hereby, we obtain a covering of T^I_-. More precisely, we have

$$T^I_- \subset \bigcup\nolimits_{Q \in \bigcup_{8 \le k \le I} \mathcal{C}^k} Q' \subset Q_\mu, \tag{4.45}$$

where the second inclusion follows from (4.41) and the fact that $\partial Q_\mu \subset E^i$ for all $i \in \mathbb{N}$ by (4.24). To complete the covering, we introduce for all $k \ge I+1$

$$\mathcal{G}_k = \{Q \in \mathcal{Q}^k : \ Q'' \subset Q_\mu, \ \ Q'' \cap (E^I \cap J) = \emptyset\}. \tag{4.46}$$

Assuming that we have already constructed $\mathcal{C}^I, \ldots, \mathcal{C}^n$ for $n \ge I$ we define

$$\mathcal{C}^{n+1} = \big\{Q \in \mathcal{G}_{n+1} : Q \cap \bigcup\nolimits_{k=8}^n \bigcup\nolimits_{\hat{Q} \in \mathcal{C}^k} \hat{Q} = \emptyset\big\}. \tag{4.47}$$

Clearly, $\mathcal{C} := \bigcup_{k=8}^\infty \mathcal{C}^k$ is a covering of Q_μ (up to a set of negligible measure) consisting of pairwise disjoint, dyadic squares. Finally, for $8 \le i \le I$ we define with U_i as given in (4.13)

$$\mathcal{Y}^i = \{Q \in \mathcal{C}^i : Q'' \subset U_i,\ \mathcal{H}^1(J \cap Q') > \theta^2 s_i\}, \qquad Y^i = \mathrm{sat}\Big(\bigcup\nolimits_{Q \in \mathcal{Y}^i} \overline{Q''}\Big), \tag{4.48}$$

where $\mathrm{sat}(\cdot)$ was defined below (4.6). Let $Z = \bigcup_{8 \le i \le I} Y^i$ and observe that (4.16) holds since $Y^i \subset U_i$. For some explanation and motivation about the definition of the set Z we refer to Remark 4.5(ii) below.

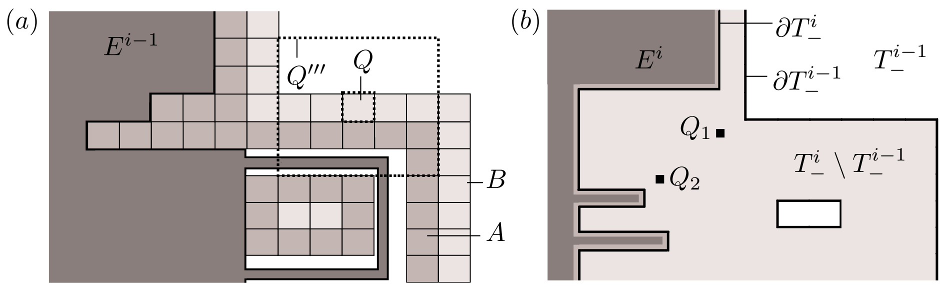

FIGURE 4. (a) The set E^{i-1} and the 'layers' $A = T^{i-1} \setminus T^{i-1}_-$, $B = T^{i-1}_- \setminus T^{i-1}_{--}$ are sketched, where in general E^{i-1} is not a union of squares in $\mathcal{Q}^{i-1}$. For the proof of (4.43), note that a square Q in $A \cup B$ is not in $\mathcal{Q}^{i-1}_{\mathrm{bad}}$ since $\overline{Q'''}$ would then belong to E^{i-1}. (b) In dark gray we have depicted E^i, which is a subset of E^{i-1}. The sets T^{i-1}_- and $T^i_- \setminus T^{i-1}_-$ are illustrated in white and light gray, respectively. Note that ∂T^{i-1}_-, ∂T^i_- satisfy (4.42). For the proof of (4.17)(iv), case (a), note that the black square $Q_1 \in \mathcal{Q}^i$ lies in A and thus satisfies (4.50). For case (c), observe that $Q_2 \subset E^{i-1}$ and thus $Q_2 \notin \mathcal{Q}^i_{\mathrm{bad}}$ or $Q_2'' \subset U_i$ since otherwise it would be contained in E^i.

Proof of (4.17)*(i):* First, we have already noticed (4.45). Then taking (4.46) into account, using $E^I \cap J = \bigcup_{D^I_k \in \mathcal{E}^I} \Gamma^*_{j_k} \cap J \subset \bigcup_{j=1}^m \partial P'_j \cap J$ (see (4.39)) and arguing as in the construction of a Whitney covering for open sets, we obtain (4.17)(i).

Proof of (4.17)*(ii):* We fix $Q_1 \in \mathcal{C}^i$ and $Q_2 \in \mathcal{C}^j$ with $j \ge i$ and $Q_1' \cap Q_2' \neq \emptyset$. (a) If $i \ge I+1$, we deduce $j \le i+1$ in view of (4.46)-(4.47), where one argues as in the construction of a Whitney covering. (b) If $i \le I-1$, we suppose that we had $j \ge i+2$. Then by (4.44) we get $Q_1 \subset T^i_-$ and $Q_2 \cap T^{i+1}_- = \emptyset$. Consequently, using (4.42) we obtain $\mathrm{dist}(Q_1, Q_2) \ge s_i$ and thus the contradiction $Q_1' \cap Q_2' = \emptyset$.

(c) Finally, for the case $i = I$ it suffices to show that all $Q \in \mathcal{Q}^{I+1}$ with $Q \subset Q_\mu \setminus T^I_-$ and $Q_1' \cap Q' \neq \emptyset$ fulfill $Q \in \mathcal{C}^{I+1}$. First, since $Q_1' \cap Q' \neq \emptyset$, Q satisfies $Q'' \subset T^I$ by (4.41) and (4.44). Then $Q'' \cap E^I = \emptyset$ by (4.41), which also implies $Q'' \subset Q_\mu$ since $\partial Q_\mu \subset E^I$ (see (4.24)). Consequently, $Q \in \mathcal{G}_{I+1}$ and thus $Q \in \mathcal{C}^{I+1}$ as $Q \subset Q_\mu \setminus T^I_-$.

Proof of (4.17)*(iii):* Property (4.17)(iii) follows directly from (4.17)(ii) recalling the fact that each $x \in Q_\mu$ is contained in at most four sets Q', $Q \in \mathcal{Q}^i$.

Proof of (4.17)*(vi):* We note that each $Q \in \mathcal{C}^i$ for $i \ge I+1$ satisfies $Q \cap (Q_\mu \setminus T^I_-) \neq \emptyset$ and thus $\mathrm{dist}(E^I, Q') \le cs_I$ by (4.41). Recalling (4.38), $E^I = \bigcup_{D^I_k \in \mathcal{E}^I} D^I_k$, and (4.46) we derive

$$\mathrm{dist}(Q', \bigcup\nolimits_{D^I_k \in \mathcal{E}^I} \Gamma^*_{j_k}) \le cs_I, \qquad \mathrm{dist}(Q', \bigcup\nolimits_{D^I_k \in \mathcal{E}^I} \Gamma^*_{j_k} \cap J) > 0 \tag{4.49}$$

for some universal $c > 0$. Consequently, (4.14)(i) for θ small enough yields (vi). It now remains to prove (4.17)(iv),(v), where by (4.17)(vi) it suffices to consider $8 \le i \le I$.

Proof of (4.17)*(iv):* Fix $Q \in \mathcal{C}^i$, $8 \le i \le I$, with $Q'' \not\subset Z$. By (4.48) we can even assume that $Q'' \not\subset U_i$ since $Q'' \not\subset Z$ together with $Q'' \subset U_i$ already implies (4.17)(iv). Moreover, we suppose that $Q \in \mathcal{Q}^i_{\rm bad}$ since otherwise the assertion follows directly from (4.10). Let $Q_* \in \mathcal{Q}^{i-1}$ with $Q \subset Q_*$. We distinguish three cases and refer also to Figure 4(b) for an illustration:

(a) Suppose that $Q \subset T^{i-1} \setminus T^{i-1}_-$. As T^{i-1} consists of squares in $\mathcal{Q}^{i-1}$, we have $Q_* \subset T^{i-1} \setminus T^{i-1}_-$ and then $Q_* \notin \mathcal{Q}^{i-1}_{\rm bad}$ by (4.43). Thus, by (4.10) we get

$$\mathcal{H}^1(J \cap Q') \le \mathcal{H}^1(J^* \cap Q'_*) \le \theta^3 s_{i-1} = \theta^2 s_i. \tag{4.50}$$

(b) Suppose that $Q \subset T^i_- \setminus T^{i-1}$ and $Q_* \notin \mathcal{Q}^{i-1}_{\rm bad}$. In this case, the assertion follows by repeating the argument in (4.50).

(c) Finally, suppose that $Q \subset T^i_- \setminus T^{i-1}$ and $Q_* \in \mathcal{Q}^{i-1}_{\rm bad}$. We will show that this assumption leads to a contradiction. First, note that $\overline{Q'''_*} \subset B^{i-1}$ by (4.11). As T^{i-1} consists of squares in $\mathcal{Q}^{i-1}$, we get $Q_* \cap T^{i-1} = \emptyset$ and then $Q_* \cap E^{i-1} \neq \emptyset$ by (4.41). Consequently, recalling (4.22) (for $i-1$) we get that $\overline{Q'''_*}$ is contained in a component of $\mathcal{E}^{i-1}$. This implies

$$Q \subset \overline{Q'''_*} \subset E^{i-1}. \tag{4.51}$$

Recall that $Q \in \mathcal{Q}^i_{\rm bad}$ and $Q'' \not\subset U_i$. Consequently, $\overline{Q'''}$ is contained in a component $\mathcal{B}^i \setminus \mathcal{B}^i_{\rm iso}$ and therefore by the remark below (4.20), $\overline{Q'''}$ is contained in a component of $\mathcal{D}^i$. Since $Q \subset E^{i-1}$ by (4.51), by (4.22) (for i) we derive $\overline{Q'''} \subset E^i$. This, however, contradicts the fact that $Q \in \mathcal{C}^i$. Indeed, by (4.44) we have $Q \subset T^i_-$, and since $T^i_- \cap E^i = \emptyset$, we get $Q \cap E^i = \emptyset$. This concludes the proof of (iv).

Proof of (4.17)*(v):* We suppose that for $Q \in \mathcal{C}^i$, $8 \le i \le I$, there is a neighbor $\hat{Q} \in \mathcal{C}^{i-1}$ such that $Q'' \cap \hat{Q}'' \neq \emptyset$. Then by construction we find $\hat{Q} \subset T^{i-1}_- \setminus T^{i-1}_{--}$ and a further square $Q_* \in \mathcal{Q}^{i-1}$ with $Q \subset Q_* \subset T^{i-1} \setminus T^{i-1}_-$. In view of (4.43), the desired property follows as in (4.50). Likewise, if there is a neighbor $\hat{Q} \in \mathcal{C}^{i+1}$ with $Q'' \cap \hat{Q}'' \neq \emptyset$, we see $Q \subset T^i_- \setminus T^i_{--}$ and then again by (4.43) we get the claim from (4.10). □

Remark 4.5. (i) For later reference, we remark that assumption (4.7) was necessary to obtain the estimate (4.15) on the length of the boundary of the partition (see (4.40) and (4.9)).

(ii) Let us comment briefly on the definition of Z in (4.48). In the definition of $\mathcal{Y}^i$, the condition on the amount of jump contained in a square, which differs from (4.10), is chosen in such a way that we will be able to use (4.17)(v) in the proof of Lemma 4.4 below. In contrast to the definition of U_i, we use squares $\overline{Q''}$ instead of $\overline{Q'''}$ in the definition of Y^i. This will be essential to prove property (4.19)(ii). We again use the saturation of sets to ensure that the connected components of Y^i are simply connected. Note that Y^i has empty intersection with the set E^i, which is the main object for the construction of the partition. Therefore, we call the components of Y^i *isolated*.

Proof of Lemma 4.4. Let $\mathcal{C}$ be given satisfying (4.17) and I as in (4.14). Recall the definition of U_i in (4.13) and the definitions of Y^i, $\mathcal{Y}^i$, and $Z = \bigcup_{8 \le i \le I} Y^i$ in (4.48). In the following proof we will particularly need property (4.17)(v). Moreover, at the end we will redefine the Whitney covering on the components of Z and in this context, we will exploit the construction of $\mathcal{C}$ in (4.44)-(4.47). We denote the (closed) connected components of Y^i by $(Y^i_k)_k$.

Properties of $(Y^i_k)_k$*:* We first show that

$$Q \in \mathcal{C}^i, Q \subset Y^i_k \quad \text{for all} \quad Q \in \hat{\mathcal{Y}}^i_k := \{Q \in \mathcal{C} : Q \cap Y^i_k \neq \emptyset,\ \partial Q \cap \partial Y^i_k \neq \emptyset\}. \tag{4.52}$$

In fact, for each $Q \in \hat{\mathcal{Y}}^i_k$ there is $\tilde{Q} \in \mathcal{Y}^i$, $\tilde{Q}'' \subset Y^i_k$ such that $|\tilde{Q}'' \cap Q| > 0$. Since $\tilde{Q}$ satisfies $\mathcal{H}^1(J \cap \tilde{Q}') > \theta^2 s_i$ by (4.48), property (4.17)(v) yields $\tilde{Q}'' \subset \bigcup_{Q_* \in \mathcal{C}^i} \overline{Q_*}$. As $\mathcal{C}$ consists of pairwise disjoint dyadic squares, this gives $Q \in \mathcal{C}^i$ and $Q \subset \tilde{Q}'' \subset Y^i_k$, as desired.

We now see that each pair Y^i_k, Y^j_l, $i < j$, either satisfies

$$\mathrm{dist}_\infty(Y^i_k, Y^j_l) \geq \tfrac{1}{2} s_i \tag{4.53}$$

or one set is contained in the other. (Here dist_∞ denotes the distance with respect to the maximum norm.) If not, we would find $Q_i \in \hat{\mathcal{Y}}^i_k$ and $Q_j \in \hat{\mathcal{Y}}^j_l$ such that $Q'_j \cap Q'_i \neq \emptyset$. Then $j = i+1$ by (4.17)(ii) and (4.52). Moreover, $Q_j \cap Q''_i \neq \emptyset$ and there would be $Q_* \in \mathcal{Y}^j$, $Q''_* \subset Y^j_l$, such that $Q''_* \cap Q''_i \neq \emptyset$. This, however, contradicts (4.17)(v) and the fact that $\mathcal{H}^1(J \cap Q'_*) > \theta^2 s_j$ by (4.48). (We also refer to Figure 5(a).)

Components $(X^i_k)_k$*:* Now for each $8 \leq i \leq I$, let $(X^i_k)_k \subset (Y^i_k)_k$ be the components such that for all $j \neq i$ we have $X^i_k \not\subset Y^j_l$ for all components Y^j_l. Accordingly, we denote the sets defined in (4.52) by $\hat{\mathcal{X}}^i_k$. Define $Z^i = \bigcup_k X^i_k$ and note that by (4.53) the sets $(Z^i)_i$ are pairwise disjoint with $Z = \bigcup_{i=8}^I Z^i$. Recalling the definition of $\mathcal{X}^i_k$ before (4.19), we then see that $\hat{\mathcal{X}}^i_k = \mathcal{X}^i_k$. As the sets Y^i are defined using the saturation (see (4.48)), we also get that the components $(X^i_k)_k$ are simply connected. We now show (4.18) and (4.19).

Proof of (4.19)*:* As by (4.48) each component $(X^i_k)_k$ of Z^i is contained in a component of U_i, (4.19)(i) follows directly from (4.12). By (4.53) and the fact that $\mathrm{dist}_\infty(X^i_k, X^i_l) \geq 2s_i$ for all i and $k \neq l$, we obtain (4.19)(ii). The proof of (4.52) showed that for each square in $\mathcal{X}^i_k$ there is an adjacent square in $\mathcal{Y}^i$ (cf. Figure 5(a)). Consequently, we obtain by (4.48)

$$\begin{aligned}\sum\nolimits_k \#\mathcal{X}^i_k &\leq 8 \sum\nolimits_k \#\{Q \in \mathcal{Y}^i : Q'' \subset X^i_k\} \leq c\theta^{-2} s_i^{-1} \sum\nolimits_{Q \in \mathcal{Y}^i} \mathcal{H}^1(Q' \cap Z^i \cap J) \\ &\leq c\theta^{-2} s_i^{-1} \mathcal{H}^1(J \cap Z^i),\end{aligned} \tag{4.54}$$

for $c > 0$ universal, where we used that each $x \in Q_\mu$ is contained in at most four enlarged squares. This gives (4.19)(iii).

Proof of (4.18)*:* Since $(Z^i)_i$ are pairwise disjoint, (4.54) also yields

$$\begin{aligned}\sum\nolimits_{i=8}^I \mathcal{H}^1(\partial Z^i) &\leq \sum\nolimits_{i=8}^I \sum\nolimits_k \mathcal{H}^1(\partial X^i_k) \leq \sum\nolimits_{i=8}^I \big(\sum\nolimits_{Q \in \bigcup_k \mathcal{X}^i_k} \mathcal{H}^1(\partial Q)\big) \\ &\leq \sum\nolimits_{i=8}^I c\theta^{-2}\mathcal{H}^1(J \cap Z^i) \leq c\theta^{-2}\mathcal{H}^1(J),\end{aligned} \tag{4.55}$$

i.e., (4.18)(ii) holds. Moreover, by (4.19)(i) and the fact that $d(X^i_k) \leq \mathcal{H}^1(\partial X^i_k)$

$$|Z^i| = \sum\nolimits_k |X^i_k| \leq \sum\nolimits_k d(X^i_k)^2 \leq \theta^{-ir} s_i \sum\nolimits_k \mathcal{H}^1(\partial X^i_k) \leq c\theta^{-ir} s_i \theta^{-2} \mathcal{H}^1(J).$$

Likewise, note that (4.19)(i) and the fact that $r < \frac{1}{8}$ yield $d(X^i_k) \leq \mu\theta^7$. We then conclude the proof of (4.18) by using (4.55) and calculating

$$|Z| = \big|\bigcup\nolimits_{i=8}^I Z^i\big| \leq \mu\theta^7 \sum\nolimits_{i=8}^I \sum\nolimits_k \mathcal{H}^1(\partial X^i_k) \leq c\mu\theta^5 \mathcal{H}^1(J).$$

Redefinition of the covering: Finally, we define a modification $\mathcal{C}_*$ of $\mathcal{C}$ such that each Z^i consists of squares $\mathcal{C}^i_* := \mathcal{C}_* \cap \mathcal{Q}^i$. To this end, consider a component X^i_k. By (4.44) and (4.52) we have $\bigcup_{Q \in \hat{\mathcal{X}}^i_k} Q \subset T^i_- \setminus T^{i-1}_-$, which by (4.41) has empty intersection with E^i. Since the diameter of each connected component of E^i exceeds $s_i \theta^{-ir}$ (see (4.20)), (4.19)(i) then yields $X^i_k \cap E^i = \emptyset$. Consequently, $X^i_k \subset T^i_-$ and again by (4.44) we then obtain $X^i_k \subset \bigcup_{j=8}^i \bigcup_{Q \in \mathcal{C}^j} \overline{Q}$ (cf. Figure 5(b)). This together with the fact that $\hat{\mathcal{X}}^i_k \subset \mathcal{Q}^i$ by (4.52), shows that we can replace the covering $\mathcal{C}$ in each component X^i_k by a covering $\mathcal{C}_*$ consisting exclusively of squares in $\mathcal{Q}^i$ such that all conditions in (4.17) remain true. $\square$

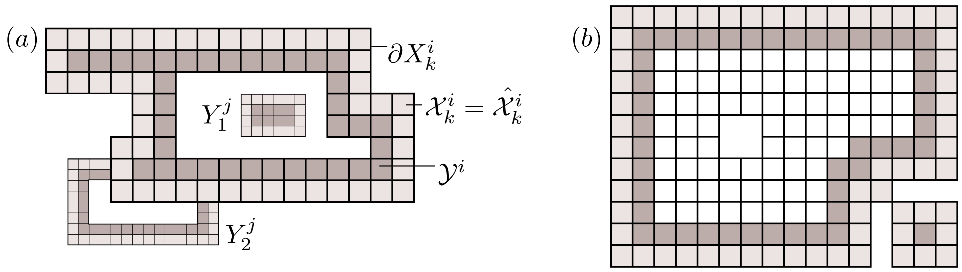

FIGURE 5. (a) The squares $\mathcal{X}^i_k$ are depicted in light gray and the dark gray squares are contained in $\mathcal{Y}^i$. It is possible that $Y^j_1 \subset X^i_k$, whereas Y^j_2, which overlaps with X^i_k, cannot exist since this contradics (4.17)(v). (b) We have illustrated the part of $\mathcal{C}$ contained in X^i_k, which may (in contrast to $\mathcal{C}_*$) contain squares in $\mathcal{Q}^j$, $j < i$.

4.2. **Construction of a piecewise smooth approximation.** In Theorem 4.3, given a finite union of closed segments J, we have constructed an auxiliary partition $(P'_j)^m_{j=1}$ consisting of simply connected sets and an associated covering $\mathcal{C}$ of Whitney-type such that in the squares of the covering the $\mathcal{H}^1$-norm of J is small compared to their diameter. Recall that this construction was exclusively based on the geometry of J. In this section, applying Theorem 4.3 for the jump set $J = J_u$ of $u \in \mathcal{W}(Q_\mu)$, we will see that, with the help of the Korn inequality for functions with small jump set (Theorem 3.15), we can approximate u on the squares of $\mathcal{C}$ by infinitesimal rigid motions. Proceeding similarly as in [31], by using a partition of unity associated to $\mathcal{C}$, this will allow us to define a piecewise smooth approximation $\bar{u}$ of u which is smooth on every component P'_j.

Note that an additional difficulty has to be faced: due to technical reasons in the construction of Theorem 4.3, there exists a set Z of isolated components given in Lemma 4.4 where Theorem 3.15 is not applicable and $\bar{u}$ has to be defined differently. Consequently, besides the main properties of the covering (4.17), we will also use the structure of Z described in (4.18)-(4.19). Note, however, that no other details of the construction from the previous section will be needed.

Theorem 4.6. *Let $\mu > 0, \theta > 0$ small and $p \in [1,2)$. Then there are a universal $c > 0$ and a constant $C = C(\theta, p) > 0$ such that for all $u \in \mathcal{W}(Q_\mu)$ with (4.4) the following holds: let $(P'_j)^m_{j=1}$ be the partition and $\mathcal{C}$ be the covering given by Theorem 4.3 and Lemma 4.4 applied for $J = J_u$ and $r = \frac{1}{24}(2-p)$. Then there is an approximation $\bar{u} : Q_\mu \to \mathbb{R}^2$, being smooth in $\bigcup_{Q\in\mathcal{C}} Q' \supset Q_\mu \setminus (J_u \cap \bigcup^m_{j=1} \partial P'_j)$, and an exceptional set $F \subset Q_\mu$ with*

$$|F| \le c\mu\theta^5\, \mathcal{H}^1(J_u), \qquad \mathcal{H}^1(\partial F) \le C\mathcal{H}^1(J_u) \tag{4.56}$$

such that $\bar{u} = u$ on ∂Q_μ (in the sense of traces) and

$$\begin{aligned} &(i) \quad \|e(\bar{u})\|_{L^p(Q_\mu)} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}, \\ &(ii) \quad \|\nabla\bar{u} - \nabla u\|_{L^p(Q_\mu\setminus F)} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}, \\ &(iii) \quad \|\bar{u} - u\|_{L^p(Q_\mu\setminus F)} \le C\mu^{\frac{2}{p}}\|e(u)\|_{L^2(Q_\mu)}. \end{aligned} \tag{4.57}$$

The approximation $\bar{u}$ is constructed by smoothing a piecewise affine approximation of u given on the covering $\mathcal{C}$ and it is therefore smooth on $\bigcup_{Q\in\mathcal{C}} Q'$. Recall $\bigcup_{Q\in\mathcal{C}} Q' \supset Q_\mu \setminus (J_u \cap \bigcup_{j=1}^m \partial P'_j)$ by (4.17)(i). This implies that a connected component of $\bigcup_{Q\in\mathcal{C}} Q'$ (i.e., a connected component of the set where $\bar{u}$ is smooth) contains sets $(P'_j)_{j=1}^m$. (It may contain indeed several of these sets if they are not completely disconnected by the jump set J_u.) Particularly, $\bar{u}$ is smooth on each of the sets P'_j, which will be the fundamental property used in the following. To highlight this fact, we will call $\bar{u}$ a *piecewise smooth approximation* of u, where the word *piecewise* refers to the partition $(P'_j)_{j=1}^m$.

Note that $\bar{u}$ plays the role of an auxiliary function and does not appear in the statement of Theorem 4.1. Later in Section 4.3, after passing to a refined partition consisting of John domains with uniformly controlled John constant, we will use the Korn inequality in John domains (Theorem 3.2) to obtain an estimate on $\nabla\bar{u}$. Finally, (4.57)(ii) will then yield control on ∇u outside of F.

In particular, F will be a part of the exceptional set of Theorem 4.1 and (4.56) will be needed to show (4.1). (We refer to Remark 4.7(i) below for more details on the structure of F.) Note that the assumption $p < 2$ is essential since at some point of the proof we need to reduce the exponent 2 in a Hölder-type estimate.

Proof. We apply Theorem 4.3 and Lemma 4.4 for $J = J_u$ and $r = \frac{1}{24}(2-p)$. Let the partition $(P'_j)_{j=1}^m$, the isolated components $Z = \bigcup_{l=8}^I Z^l = \bigcup_{l=8}^I \bigcup_k X_k^l$ and the Whitney-type covering $\mathcal{C}$ of $Q_\mu \setminus (J_u \cap \bigcup_{j=1}^m \partial P'_j)$ be given such that (4.17) holds and each X_k^l is a union of squares in $\mathcal{C}^l = \mathcal{C} \cap \mathcal{Q}^l$. By $\mathcal{X}_k^l$ we denote the squares at the boundary of X_k^l, see before (4.19).

Outline of the proof: We first use (4.17)(iv) and Theorem 3.15 to approximate u by infinitesimal rigid motions a_Q (also called affine mappings in the following) on the family of squares $Q \in \mathcal{C}$ with $Q'' \not\subset Z$, which particularly includes the squares $\mathcal{X}_k^l$ at the boundary of the isolated components. Inside each isolated component $X_k^l \subset Z$ we choose a single affine mapping a_k^l with $a_k^l = a_Q$ for some (arbitrary) $Q \in \mathcal{X}_k^l$.

We then estimate the difference of the infinitesimal rigid motions on adjacent squares. For the isolated components X_k^l, we have to face the additional difficulty that the affine mappings given on *all* $\mathcal{X}_k^l$ have to be compared against each other. To do so, we draw some ideas from [34, Section 4]. Hereby we see that, due to the control $d(X_k^l) \le \theta^{-lr} s_l$ (see (4.19)(i)), the involved constant may blow up in l, but is controlled in terms of $\theta^{-2(p+1)lr}$ (see (4.64)).

Finally, using a partition of unity associated to $\mathcal{C}$, we will construct a function being smooth in $\bigcup_{Q\in\mathcal{C}} Q'$. To confirm (4.57), we exploit the fact that the difference of the affine mappings on adjacent squares is controlled. In this context, we explicitly use $p < 2$ and Hölder's inequality, which together with the 'smallness' of $|Z^l|$ (see (4.18)(i)) allows us to compensate the fact that for isolated components the constants blow up in terms of $\theta^{-2(p+1)lr}$ (cf. (4.68)-(4.69)).

We now start with the proof. In the following $C = C(\theta, p) > 0$ denotes a generic constant and $c > 0$ is universal. We can suppose that θ is small with respect to c.

Step I (Definition of infinitesimal rigid motions): In this step we define infinitesimal rigid motions for each square of the Whitney covering.

Good squares: Let us first consider the subset of *good squares*

$$\mathcal{C}_{\mathrm{g}} := \big\{Q \in \mathcal{C} : Q'' \not\subset Z\big\} \subset \bigcup\nolimits_{i\ge 8} \big\{Q \in \mathcal{C}^i : \mathcal{H}^1(J_u \cap Q') \le \theta^2 s_i\big\}, \tag{4.58}$$

where the inclusion follows from (4.17)(iv). We apply Theorem 3.15 on Q', $Q \in \mathcal{C}_{\mathrm{g}}$, to find infinitesimal rigid motions $a_Q = a_{A_Q,b_Q}$ and exceptional sets $E_Q \subset Q'$ such that

$$d(Q)^{-\frac{2}{p}} \|u - a_Q\|_{L^p(Q'\setminus E_Q)} + d(Q)^{1-\frac{2}{p}} \|\nabla u - A_Q\|_{L^p(Q'\setminus E_Q)} \le c\|e(u)\|_{L^2(Q')}. \tag{4.59}$$

Moreover, taking (4.58) into account we find for all $Q \in \mathcal{C}_{\rm g}$

$$|E_Q| \le cd(Q)\theta^2\mathcal{H}^1(J_u \cap Q') \le c\theta^4 d(Q)^2, \quad \mathcal{H}^1(\partial E_Q) \le c\mathcal{H}^1(J_u \cap Q'). \tag{4.60}$$

As a preparation for Step II, we now estimate the difference of the infinitesimal rigid motions for neighboring squares in $\mathcal{C}_{\rm g}$. For $Q \in \mathcal{C}_{\rm g}$ we let

$$\mathcal{N}_{\rm g}(Q) = \{\hat{Q} \in \mathcal{C}_{\rm g} \setminus \{Q\} : Q' \cap \hat{Q}' \neq \emptyset\} \tag{4.61}$$

and observe that by (4.17)(ii) we have $\theta d(\hat{Q}) \le d(Q) \le \theta^{-1} d(\hat{Q})$ for all $\hat{Q} \in \mathcal{N}_{\rm g}(Q)$. This also implies $\#\mathcal{N}_{\rm g}(Q) \le c\theta^{-2}$. Moreover, as the covering consists of dyadic squares, $Q' \cap \hat{Q}'$ contains a ball B with radius larger than $c \min\{d(Q), d(\hat{Q})\} \ge c\theta d(Q)$ for some small universal $c > 0$. Consequently, by (4.60) we find $|\hat{E} \cap B| \le \frac{1}{2}|B|$ for θ sufficiently small, where $\hat{E} = E_{\hat{Q}} \cup E_Q$. Thus, $|B \setminus \hat{E}| \ge \frac{1}{2}|B| \ge c\theta^2 d(Q')^2$ and then we can apply Lemma 3.9 to find

$$\|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \le c'(d(Q')^2 |B \setminus \hat{E}|^{-1})^{\frac{p}{2}+1} \|a_Q - a_{\hat{Q}}\|^p_{L^p(B \setminus \hat{E})} \le c'\theta^{-(p+2)} \|a_Q - a_{\hat{Q}}\|^p_{L^p(B\setminus\hat{E})}$$

for all $\hat{Q} \in \mathcal{N}_{\rm g}(Q)$, where $c' = c'(p) > 0$. Then by (4.59), $\theta d(\hat{Q}) \le d(Q)$, and the triangle inequality we find

$$\|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \le cc'\theta^{-(p+2)}\theta^{-2} d(Q)^2 \|e(u)\|^p_{L^2(Q' \cup \hat{Q}')} \le Cd(Q)^2 \|e(u)\|^p_{L^2(Q'\cup\hat{Q}')} \tag{4.62}$$

for all $\hat{Q} \in \mathcal{N}_{\rm g}(Q)$, where $C = C(\theta, p) > 0$. Therefore, by $\#\mathcal{N}_{\rm g}(Q) \le c\theta^{-2}$ we get with $N_Q := \bigcup_{\hat{Q} \in \mathcal{N}_{\rm g}(Q) \cup \{Q\}} \hat{Q}'$

$$\sum\nolimits_{\hat{Q} \in \mathcal{N}_{\rm g}(Q)} \|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \le Cd(Q)^2 \|e(u)\|^p_{L^2(N_Q)}. \tag{4.63}$$

Squares in isolated components: To conclude Step I, it remains to define infinitesimal rigid motions for the squares in $\mathcal{C} \setminus \mathcal{C}_{\rm g}$. Consequently, in view of (4.58) and Lemma 4.4, we have to concern ourselves with the isolated components $(X^l_k)_k$ for $8 \le l \le I$. Let $\mathcal{X}^l_k$ as introduced before (4.19). For each $Q \in \mathcal{X}^l_k$ we have $Q' \not\subset X^l_k$ and thus $Q' \not\subset Z$ by (4.19)(ii). Therefore, $\mathcal{X}^l_k \subset \mathcal{C}_{\rm g}$ by the definition in (4.58). Thus, (4.63) holds for all $Q \in \mathcal{X}^l_k$.

Define $N^l_k = X^l_k \cup \bigcup_{Q \in \mathcal{X}^l_k} Q'$. Since $d(X^l_k) \le \theta^{-lr} s_l$ by (4.19)(i), we have by Lemma 3.9 for all $\tilde{Q} \in \mathcal{X}^l_k$ and $\hat{Q} \in \mathcal{N}_{\rm g}(\tilde{Q}) \cap \mathcal{X}^l_k$

$$\|a_{\tilde{Q}} - a_{\hat{Q}}\|_{L^p(N^l_k)} \le C\theta^{-(1+\frac{2}{p})lr} \|a_{\tilde{Q}} - a_{\hat{Q}}\|_{L^p(\tilde{Q}')}.$$

As X^l_k is simply connected, for all $Q_1, Q_2 \in \mathcal{X}^l_k$ we find a chain of squares in $\mathcal{X}^l_k$ connecting Q_1 with Q_2 (cf. Figure 5). The previous estimate and the triangle inequality then yield

$$\begin{aligned}\max_{Q_1, Q_2 \in \mathcal{X}^l_k} \|a_{Q_1} - a_{Q_2}\|_{L^p(N^l_k)} &\le \sum\nolimits_{\tilde{Q} \in \mathcal{X}^l_k} \sum\nolimits_{\hat{Q} \in \mathcal{N}_{\rm g}(\tilde{Q}) \cap \mathcal{X}^l_k} \|a_{\tilde{Q}} - a_{\hat{Q}}\|_{L^p(N^l_k)} \\ &\le C\theta^{-(1+\frac{2}{p})lr} \sum\nolimits_{\tilde{Q} \in \mathcal{X}^l_k} \sum\nolimits_{\hat{Q} \in \mathcal{N}_{\rm g}(\tilde{Q}) \cap \mathcal{X}^l_k} \|a_{\tilde{Q}} - a_{\hat{Q}}\|_{L^p(\tilde{Q}')}.\end{aligned}$$

Therefore, the discrete Hölder inequality together with $\#\mathcal{X}^l_k \le c\theta^{-2lr}$ (see (4.19)(i)), $\#\mathcal{N}_{\rm g}(\tilde{Q}) \cap \mathcal{X}^l_k \le 8$ for all $\tilde{Q} \in \mathcal{X}^l_k$ and (4.62) implies

$$\begin{aligned}\max_{Q_1, Q_2 \in \mathcal{X}^l_k} \|a_{Q_1} - a_{Q_2}\|^p_{L^p(N^l_k)} &\le C\theta^{-2(p+1)lr} \Big(\sum\nolimits_{\tilde{Q} \in \mathcal{X}^l_k} \sum\nolimits_{\hat{Q} \in \mathcal{N}_{\rm g}(\tilde{Q}) \cap \mathcal{X}^l_k} \|a_{\tilde{Q}} - a_{\hat{Q}}\|^2_{L^p(\tilde{Q}')} \Big)^{\frac{p}{2}} \\ &\le Cs_l^2 \theta^{-2(p+1)lr} \Big(\sum\nolimits_{\tilde{Q} \in \mathcal{X}^l_k} \sum\nolimits_{\hat{Q} \in \mathcal{N}_{\rm g}(\tilde{Q}) \cap \mathcal{X}^l_k} \|e(u)\|^2_{L^2(\tilde{Q}' \cup \hat{Q}')} \Big)^{\frac{p}{2}} \\ &\le Cs_l^2 \theta^{-2(p+1)lr} \|e(u)\|^p_{L^2(N^l_k)}.\end{aligned} \tag{4.64}$$

(See [34, Section 4] for similar arguments.) For each X_k^l we define an infinitesimal rigid motion by setting $a_k^l = a_Q$ for an arbitrary $Q \in \mathcal{X}_k^l$.

We can now define affine mappings associated to each $Q \in \mathcal{C} \setminus \mathcal{C}_g$. Given $Q \in \mathcal{C}^l$ with $Q'' \subset Z^l$ we choose the component X_k^l with $Q'' \subset X_k^l$ and define $a_Q = a_k^l$.

Step II (Difference of affine mappings on neighboring squares): In Step I we have already defined infinitesimal rigid motions for each square of the Whitney covering. In particular, $a_Q = a_k^l$ is constant for all $Q \in \mathcal{C}$ with $Q'' \subset X_k^l$, i.e., for squares lying well inside an isolated component. For good squares $\mathcal{C}_g$ defined in (4.58) we refer to (4.59). Similarly to (4.61), we introduce the neighbors of each $Q \in \mathcal{C}$ defined by

$$\mathcal{N}(Q) = \{\hat{Q} \in \mathcal{C} \setminus \{Q\} : Q' \cap \hat{Q}' \neq \emptyset\}.$$

Note that $\#\mathcal{N}(Q) \le c\theta^{-2}$ by (4.17)(ii). We now compare the infinitesimal rigid motions on neighboring squares by using (4.63) and (4.64). To this end, we distinguish three cases:

(a) First, consider $Q \in \mathcal{C}$ with $Q \cap Z = \emptyset$. Then $\mathcal{N}(Q) \subset \mathcal{C}_{\rm g}$ since each $\hat{Q} \in \mathcal{N}(Q)$ satisfies $\hat{Q}'' \not\subset Z$. Consequently, the results of Step I are applicable and we find by (4.63)

$$\sum\nolimits_{\hat{Q} \in \mathcal{N}(Q)} \|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \le C d(Q)^2 \|e(u)\|^p_{L^2(N_Q)}. \tag{4.65}$$

(b) Consider $Q \in \mathcal{C}$ with $Q \subset X_k^l$ and $Q''' \not\subset X_k^l$ (recall (4.6)). Then either $Q \in \mathcal{X}_k^l$, with $\mathcal{X}_k^l$ as introduced before (4.19), or Q is a neighbor of a square in $\mathcal{X}_k^l$ (see light and dark gray squares in Figure 5(b)). Recall that a_Q as in (4.59) if $Q \in \mathcal{X}_k^l$ and $a_Q = a_k^l$ otherwise. Moreover, by (4.19)(ii) we get $a_{\hat{Q}} = a_k^l$ for all $\hat{Q} \in \mathcal{N}(Q) \setminus \mathcal{N}_{\rm g}(Q)$. We apply (4.63)-(4.64) and find using $\#\mathcal{N}(Q) \le c\theta^{-2}$ and the definition of a_k^l

$$\sum\nolimits_{\hat{Q} \in \mathcal{N}(Q)} \|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \le C s_l^2 \|e(u)\|^p_{L^2(N_Q)} + C s_l^2 \theta^{-2(p+1)lr} \|e(u)\|^p_{L^2(N_k^l)}, \tag{4.66}$$

where we recall $N_Q = \bigcup_{\hat{Q} \in \mathcal{N}_{\rm g}(Q) \cup \{Q\}} \hat{Q}'$ and $N_k^l = X_k^l \cup \bigcup_{Q \in \mathcal{X}_k^l} Q'$.

(c) Finally, if $Q \subset X_k^l$, $Q''' \subset X_k^l$ (see the white squares in Figure 5(b)), we observe that all neighbors lie in the same isolated component and thus by definition

$$\sum\nolimits_{\hat{Q} \in \mathcal{N}(Q)} \|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} = 0. \tag{4.67}$$

We now sum over all squares and obtain collecting (4.65)-(4.67)

$$\begin{aligned} H &:= \sum\nolimits_{Q \in \mathcal{C}} \sum\nolimits_{\hat{Q} \in \mathcal{N}(Q)} d(Q)^{-p} \|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \\ &\le C \sum\nolimits_{l \ge 8} \sum\nolimits_k \#\mathcal{X}_k^l \; s_l^{2-p} \theta^{-2(p+1)lr} \|e(u)\|^p_{L^2(N_k^l)} + C \sum\nolimits_{Q \in \mathcal{C}} d(Q)^{2-p} \|e(u)\|^p_{L^2(N_Q)}, \end{aligned}$$

where we used that the number of squares in X_k^l of type (b) is bounded by $9\#\mathcal{X}_k^l$. Since $\#\mathcal{X}_k^l \le c\theta^{-2lr}$ by (4.19)(i), we get

$$H \le C \sum\nolimits_{l \ge 8} \sum\nolimits_k (\#\mathcal{X}_k^l)^{1-\frac{p}{2}} \; s_l^{2-p} \theta^{-(3p+2)lr} \|e(u)\|^p_{L^2(N_k^l)} + C \sum\nolimits_{Q \in \mathcal{C}} d(Q)^{2-p} \|e(u)\|^p_{L^2(N_Q)}.$$

Taking (4.17)(ii),(iii) into account we see that each $x \in Q_\mu$ is contained in a bounded number of different neighborhoods $N_Q = \bigcup_{\hat{Q} \in \mathcal{N}_{\rm g}(Q) \cup \{Q\}} \hat{Q}'$. Moreover, we note that for each $8 \le l \le I$ the sets $(N_k^l)_k$ are pairwise disjoint. We observe that $M_l := \sum_k \#\mathcal{X}_k^l \le C s_l^{-1} \mu$ by (4.19)(iii) and $\mathcal{H}^1(J_u) \le 2\sqrt{2}\theta^{-2}\mu$ (see (4.4)). Thus, using the discrete Hölder inequality we derive

$$\begin{aligned} H &\le C\sum_{l\ge 8} M_l^{1-\frac{p}{2}} s_l^{2-p}\theta^{-(3p+2)lr}\big(\sum_k \|e(u)\|^2_{L^2(N_k^l)}\big)^{\frac{p}{2}} + C\big(\sum_{Q\in\mathcal{C}} d(Q)^2\big)^{1-\frac{p}{2}}\big(\sum_{Q\in\mathcal{C}}\|e(u)\|^2_{L^2(N_Q)}\big)^{\frac{p}{2}} \\ &\le C\sum\nolimits_{l\ge 8}(\mu s_l)^{1-\frac{p}{2}}\theta^{-(3p+2)lr}\|e(u)\|^p_{L^2(Q_\mu)} + C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)}, \end{aligned} \tag{4.68}$$

where we used that $\sum_{Q\in\mathcal{C}} d(Q)^2 \le c|Q_\mu| \le c\mu^2$. Recalling $s_l = \mu\theta^l$, using $1-\frac{p}{2} = 12r \ge (4p+2)r$ as $r = \frac{1}{24}(2-p)$, and $\sum_{l\ge 8}\theta^{plr} \le C = C(\theta,p)$, we conclude

$$H \le C\mu^{2-p}\big(\sum\nolimits_{l\ge 8}\theta^{plr}+1\big)\|e(u)\|^p_{L^2(Q_\mu)} \le C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)}. \tag{4.69}$$

Step III (Definition of the approximation): We are now in the position to define the approximation $\bar{u}$ and the exceptional set F. Then we will use (4.59) and (4.69) to show (4.57). At the end we prove that the traces of u and the approximation coincide on ∂Q_μ.

Definition of $\bar{u}$ and F: First, we choose a partition of unity $(\varphi_Q)_{Q\in\mathcal{C}} \subset C^\infty(\mathbb{R}^2)$ with $\sum_{Q\in\mathcal{C}}\varphi_Q(x) = 1$ for $x \in \bigcup_{Q\in\mathcal{C}} Q'$ and

$$\begin{aligned} &(i) \ \ \mathrm{supp}(\varphi_Q) \subset Q' \text{ for all } Q\in\mathcal{C}, \\ &(ii) \ \ \|\nabla\varphi_Q\|_\infty \le c d(Q)^{-1} \text{ for all } Q\in\mathcal{C}. \end{aligned} \tag{4.70}$$

As the proof of the existence of such a partition is very similar to the construction of a partition of unity for Whitney coverings (see [25, 42]), we omit it here. Let us just briefly mention that the idea it to take a cut-off function $\bar{\varphi} \in C_c^\infty((0,1)^2)$ and to define the functions φ_Q as suitably rescaled versions of $\bar{\varphi}$ taking the fact into account that each point is contained in only a bounded number of different squares Q', $Q\in\mathcal{C}$. We now define the approximation $\bar{u}: Q_\mu \to \mathbb{R}^2$ by

$$\bar{u} = \sum\nolimits_{Q\in\mathcal{C}}\varphi_Q a_Q = \sum\nolimits_{Q\in\mathcal{C}}\varphi_Q(A_Q\,\cdot + b_Q) \tag{4.71}$$

in Q_μ. First, observe that $\bar{u}$ is smooth in $\bigcup_{Q\in\mathcal{C}} Q'$ and that $\bigcup_{Q\in\mathcal{C}} Q' \supset Q_\mu \setminus (J_u \cap \bigcup_{j=1}^m \partial P_j')$ by (4.17)(i). Recalling (4.60) we let

$$F := \bigcup\nolimits_{Q\in\mathcal{C}_{\rm g}} E_Q \cup Z. \tag{4.72}$$

By (4.17)(iii), (4.18), (4.60) and $d(Q) \le 2\sqrt{2}\mu\theta^8$ for all $Q\in\mathcal{C}\subset\bigcup_{j\ge 8}\mathcal{Q}^j$ we derive $|F| \le c\mu\theta^5\mathcal{H}^1(J_u)$ and $\mathcal{H}^1(\partial F) \le C\mathcal{H}^1(J_u)$. This gives (4.56).

Proof of (4.57)*:* We obtain $\nabla\bar{u} = \sum_{\hat{Q}\in\mathcal{C}}\big(\varphi_{\hat{Q}}A_{\hat{Q}} + a_{\hat{Q}}\otimes\nabla\varphi_{\hat{Q}}\big)$ and using that $\nabla(\sum_{\hat{Q}\in\mathcal{C}}\varphi_{\hat{Q}}) = 0$ on $\bigcup_{Q\in\mathcal{C}} Q'$ we find for $x\in Q$, $Q\in\mathcal{C}$

$$\nabla\bar{u}(x) = \sum\nolimits_{\hat{Q}\in\mathcal{C}}\big(\varphi_{\hat{Q}}(x)A_{\hat{Q}} + (a_{\hat{Q}}(x) - a_Q(x))\otimes\nabla\varphi_{\hat{Q}}(x)\big).$$

Then we get by (4.17)(ii),(iii), (4.69), (4.70), and the discrete Hölder inequality

$$\|e(\bar{u})\|^p_{L^p(Q_\mu)} \le C\sum\nolimits_{Q\in\mathcal{C}}\sum\nolimits_{\hat{Q}\in\mathcal{N}(Q)} d(Q)^{-p}\|a_Q - a_{\hat{Q}}\|^p_{L^p(Q')} \le C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)},$$

which implies (4.57)(i). Similarly, we find using (4.17)(ii),(iii), (4.59), (4.69), (4.70), (4.72), and the fact that $\bigcup_{Q\in\mathcal{C}_{\rm g}} Q' \supset Q_\mu\setminus F$

$$\begin{aligned} \|\nabla\bar{u} - \nabla u\|^p_{L^p(Q_\mu\setminus F)} &\le C\sum\nolimits_{Q\in\mathcal{C}_{\rm g}}\|\nabla u - A_Q\|^p_{L^p(Q'\setminus F)} + CH \\ &\le C\sum\nolimits_{Q\in\mathcal{C}_{\rm g}} d(Q)^{2-p}\|e(u)\|^p_{L^2(Q')} + CH \le C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)}, \end{aligned}$$

where we repeated the Hölder-type estimate in (4.68). Let $S^k = \{x \in Q_\mu : \mathrm{dist}(x, \partial Q_\mu) \le s_k\}$ and observe $d(Q) \le 2\sqrt{2}s_k$ for all $Q \in \mathcal{C}$ with $Q \cap S^k \neq \emptyset$ by (4.17)(i). Then we recall (4.71) and get by (4.17)(iii), (4.59), and again by a Hölder-type estimate (cf. (4.68))

$$\begin{aligned}\|\bar{u} - u\|^p_{L^p(S^k \setminus F)} &\le C \sum\nolimits_{l \ge k} \sum\nolimits_{Q \in \mathcal{C}_{\rm g} \cap \mathcal{C}^l} \|u - a_Q\|^p_{L^p(Q' \setminus F)} \\ &\le C s_k^p \sum\nolimits_{Q \in \mathcal{C}_{\rm g}} d(Q)^{2-p} \|e(u)\|^p_{L^2(Q')} \le C s_k^p \mu^{2-p} \|e(u)\|^p_{L^2(Q_\mu)}.\end{aligned} \tag{4.73}$$

In particular, for $k = 0$ this implies (4.57)(iii).

Traces of u and $\bar{u}$: Finally, to show that $\bar{u} = u$ on ∂Q_μ one may argue, e.g., as in the proof of [18, Theorem 2.1]. We briefly sketch the argument for the reader's convenience. Choose $\psi_k \in C^\infty(Q_\mu)$ such that $\psi_k = 1$ in a neighborhood of ∂Q_μ and $\psi_k = 0$ on $Q_\mu \setminus S^k$ with $\|\nabla \psi_k\|_\infty \le c s_k^{-1}$. Note that (4.17)(vi), (4.60) and (4.72) imply $F \cap S_k = \emptyset$ for k large enough. We define $v_k = \psi_k(u - \bar{u}) \in SBD(Q_\mu)$ and show $v_k \to 0$ strongly in BD which implies $v_k|_{\partial Q_\mu} \to 0$ in L^1 and implies the assertion. In fact, by (4.73) and Hölder's inequality we get for k sufficiently large

$$\|\bar{u} - u\|_{L^1(S^k)} \le C(\mu s_k)^{1 - \frac{1}{p}} \|\bar{u} - u\|_{L^p(S^k)} \le C \mu^{\frac{1}{p}} s_k^{2 - \frac{1}{p}} \|e(u)\|_{L^2(Q_\mu)}.$$

Now we derive that

$$|Ev_k|(Q_\mu) \le |Eu|(S^k) + |E\bar{u}|(S^k) + c s_k^{-1} \|\bar{u} - u\|_{L^1(S^k)}$$

vanishes for $k \to \infty$ and likewise $\|v_k\|_{L^1(Q_\mu)} \le \|u - \bar{u}\|_{L^1(S^k)} \to 0$. This implies $v_k \to 0$ in BD, as desired. □

Remark 4.7. (i) For later reference, we recall that the exceptional set F consists of the isolated components Z and the exceptional sets E_Q for $Q \in \mathcal{C}_{\rm g}$ (cf. (4.72)). In particular, by (4.17)(ii),(iii) and (4.60) we find (for θ small) that $|F \cap Q'| \le \theta |Q|$ for all $Q \in \mathcal{C}$ with $Q \cap Z = \emptyset$.

(ii) Moreover, in view of (4.58), each $Q \in \mathcal{C}$ with $Q \cap Z = \emptyset$ satisfies (see (4.59))

$$\|\nabla u - A_Q\|^p_{L^p(Q' \setminus F)} \le c d(Q)^{2-p} \|e(u)\|^p_{L^2(Q')}.$$

for some $A_Q \in \mathbb{R}^{2\times 2}_{\rm skew}$.

We close this section with the observation that also $\|\bar{u}\|_\infty$ can be controlled, which is necessary for the proof of (4.3). The reader not interested in the derivation of (4.3) (which will indeed not be needed in the sequel for the proof of Theorem 2.1) may readily skip the following lemma.

Lemma 4.8. *Let be given the situation of Theorem 4.6. Then the approximation $\bar{u} : Q_\mu \to \mathbb{R}^2$ can be chosen such that* (4.56)-(4.57) *hold and $\|\bar{u}\|_{L^\infty(Q_\mu \setminus F)} \le C \|u\|_\infty$ for $C = C(\theta, p) > 0$.*

Proof. The goal is to show that for each $Q \in \mathcal{C}_{\rm g}$ there is an infinitesimal rigid motion $\hat{a}_Q$ such that

$$\|\hat{a}_Q\|_{L^\infty(Q')} \le C \|u\|_\infty \tag{4.74}$$

and (4.59) still holds for a possibly larger constant with $\hat{a}_Q$ in place of a_Q. Then we can repeat the previous proof with $\hat{a}_Q$ in place of a_Q. Since for each $x \in Q_\mu \setminus F$ we have $Q \in \mathcal{C}_{\rm g}$ for all $Q \in \mathcal{C}$ with $x \in Q'$ (cf. (4.58)), we obtain $|\bar{u}(x)| \le C\|u\|_\infty$ in view of (4.71). Let us now show (4.74). Fix $Q \in \mathcal{C}_{\rm g}$. Using (4.59) for the affine mapping a_Q we find by (4.60) and Lemma 3.9

$$\|a_Q\|^p_{L^p(Q')} \le C \|a_Q\|^p_{L^p(Q' \setminus E_Q)} \le C d(Q)^2 \|e(u)\|^p_{L^2(Q')} + C|Q'| \|u\|^p_\infty.$$

Applying Lemma 3.7 we find for all $x \in Q'$

$$d(Q)|A_Q| + |A_Q\, x + b_Q| \le C(\|e(u)\|_{L^2(Q')} + \|u\|_\infty). \tag{4.75}$$

If now $\|e(u)\|_{L^2(Q')} \le \|u\|_\infty$, then (4.74) follows directly with $\hat{a}_Q = a_Q$. Otherwise, (4.74) is clearly satisfied with $\hat{a}_Q := 0$ and in view of (4.75), also (4.59) holds for a larger constant since

$$\|a_Q\|_{L^p(Q')} \le Cd(Q)^{\frac{2}{p}}\|e(u)\|_{L^2(Q')}, \quad \|A_Q\|_{L^p(Q')} \le Cd(Q)^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q')}.$$

□

4.3. **Partitions into John domains and proof of Theorem 4.1.** In Section 4.1 we have constructed an auxiliary partition $(P_j')_{j=1}^m$ consisting of simply connected sets. Subsequently, in Section 4.2 we have seen that the configuration u can be approximated, outside a small exceptional set, by an auxiliary function $\bar{u}$ which is smooth on each component of the partition $(P_j')_{j=1}^m$. We will now apply Theorem 3.3 to obtain a refined partition (the one appearing in Theorem 4.1) consisting of John domains such that we can apply Korn's inequality (see Theorem 3.2) with uniform constants. This allows to control the distance of the approximation $\bar{u}$ from an associated infinitesimal rigid motion on each component (see Theorem 4.9). Afterwards, using the properties of the approximation stated in (4.57), we can give the proof of Theorem 4.1.

In the following we essentially need the properties of $(P_j')_{j=1}^m$ and $\bar{u}$ given in Theorem 4.3 and Theorem 4.6, respectively. Details of other previously introduced objects, such as the covering of Whitney-type or the isolated components, are not relevant in this section.

Theorem 4.9. *Let $\mu, \eta > 0$, $\theta > 0$ small and $p \in (1,2)$. There are a universal $c > 0$ and a constant $C = C(\theta, p) > 0$ such that for all $u \in \mathcal{W}(Q_\mu)$ with (4.4) and for the corresponding partition $(P_j')_{j=1}^m$ and approximation $\bar{u}$ as constructed in Theorem 4.3 and Theorem 4.6, respectively, the following holds:*
There is a partition $(P_j)_{j=1}^n$ of Q_μ and a Borel set $R \subset Q_\mu$ such that each P_j with $P_j \not\subset R$ is a c-John domain with Lipschitz boundary. We have

$$\begin{aligned} &(i) \quad \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^n \partial P_j\Big) \le C\mathcal{H}^1(J_u), \quad \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^m \partial P_j' \setminus \bigcup\nolimits_{j=1}^n \partial P_j\Big) = 0,\\ &(ii) \quad |R| \le \eta, \qquad \mathcal{H}^1(\partial R) \le C\mathcal{H}^1(J_u), \end{aligned} \tag{4.76}$$

and there are infinitesimal rigid motions a_j, $j = 1, \ldots, n$, such that

$$\|\nabla \bar{v}\|_{L^p(Q_\mu \setminus R)} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}, \qquad \|\bar{v}\|_{L^p(P_j \setminus R)} \le Cd(P_j)\|\nabla \bar{v}\|_{L^p(P_j)}, \tag{4.77}$$

where $\bar{v} := \sum_{j=1}^n \chi_{P_j}(\bar{u} - a_j)$.

Observe that θ appears in the statement as the partition $(P_j')_{j=1}^m$ and the approximation $\bar{u}$ have been constructed in dependence of θ. The occurrence of a *rest set* R is unavoidable as discussed below Theorem 3.3. Note, however, that its area can be made arbitrarily small. Since in (4.77) we do not obtain any control on R, this set will be a part of the exceptional set of Theorem 4.1 and (4.76)(ii) will be needed to show (4.1).

Proof. We apply Theorem 4.3 (for $J = J_u$ and $r = \frac{1}{24}(2-p)$) and Theorem 4.6 to obtain a partition $(P_j')_{j=1}^m$ and an approximation $\bar{u}$ associated to u. Let us first note that Theorem 3.3 cannot be applied directly since the sets are possibly not Lipschitz. However, we can introduce a refined partition as follows. (Note that this is just a technical point and not the core of the argument.)

For fixed $i \in \mathbb{N}$ let $\mathcal{R}^i$ be the squares $\mathcal{Q}^i$ whose closure have nonempty intersection with $\bigcup_{j=1}^m \partial P_j'$. Since $\bigcup_{j=1}^m \partial P_j'$ consists of finitely many closed segments, we clearly get for $I \in \mathbb{N}$

large enough that $R_1 := \bigcup_{Q \in \mathcal{R}^I} \overline{Q''}$ satisfies $|R_1| \le \frac{\eta}{2}$, $\mathcal{H}^1(\partial R_1) \le c\mathcal{H}^1(\bigcup_{j=1}^m \partial P_j')$ and there is a partition $(P_j'')_{j=1}^M$ of Q_μ such that

$$\begin{aligned} &(i) \quad \bigcup\nolimits_{j=1}^M \partial P_j'' = \bigcup\nolimits_{j=1}^m \partial P_j' \cup \partial R_1, \\ &(ii) \quad \mathcal{H}^1\big(\bigcup\nolimits_{j=1}^M \partial P_j''\big) \le c\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^m \partial P_j'\big) \end{aligned} \tag{4.78}$$

for a universal $c > 0$. We order the partition such that $P_j'' = P_j' \setminus R_1$ for $j = 1, \ldots, m$, and note that (again for $I \in \mathbb{N}$ large enough) the sets $(P_j'')_{j=1}^m$ are simply connected domains with Lipschitz boundary, in particular unions of squares. Moreover, we have that $R_1 \cup \bigcup_{j=1}^m P_j''$ forms a partition of Q_μ.

Now we apply Theorem 3.3 for $\varepsilon = \frac{\eta}{2m}$ on each set in $(P_j'')_{j=1}^m$ and find a partition $Q_\mu = R_1 \cup R_2 \cup P_1''' \cup \ldots \cup P_N'''$ (up to a set of negligible measure) with $R_2 = \bigcup_{j=1}^m \Omega_0^j$ such that $|\Omega_0^j| \le \frac{\eta}{2m}$, $\mathcal{H}^1(\partial\Omega_0^j) \le c\mathcal{H}^1(\partial P_j'')$ for $j = 1, \ldots, m$ and $P_1''', \ldots, P_N'''$ are c-John domains with Lipschitz boundary. Then $|R_2| \le \frac{\eta}{2}$. Applying (3.1) for each P_j'', $j = 1, \ldots, m$, and summing over $j = 1, \ldots, m$, we obtain

$$\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^m \partial\Omega_0^j\big) + \mathcal{H}^1\big(\bigcup\nolimits_{k=1}^N \partial P_k'''\big) \le c\sum\nolimits_{j=1}^m \mathcal{H}^1\big(\partial P_j''\big).$$

Since $\sum_{j=1}^m \mathcal{H}^1(\partial P_j'') \le 2\mathcal{H}^1(\bigcup_{j=1}^m \partial P_j'')$, we get by (4.15) and (4.78)(ii)

$$\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^m \partial\Omega_0^j\big) + \mathcal{H}^1\big(\bigcup\nolimits_{k=1}^N \partial P_k'''\big) \le c\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^M \partial P_j''\big) \le c\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^m \partial P_j'\big) \le C\mathcal{H}^1(J_u)$$

for some $C = C(\theta, r) = C(\theta, p) > 0$. By $(P_j)_{j=1}^n$ we denote the partition consisting of $(P_k''')_{k=1}^N$, $(\Omega_0^j)_{j=1}^m$, and $(P_j'')_{j=m+1}^M$. Then in view of (4.78) and the previous estimate, (4.76)(i) follows. We let $R = R_1 \cup R_2$ and observe $|R| \le \eta$ as well as $\mathcal{H}^1(\partial R) \le C\mathcal{H}^1(J_u)$, i.e., (4.76)(ii) holds. Recall that $\bar{u}$ is smooth on each P_k''', $k = 1, \ldots, N$, by Theorem 4.6. Applying Theorem 3.2 on each P_k''' we obtain infinitesimal rigid motions $(a_k)_{k=1}^N$ such that by Hölder's inequality and the fact that $Q_\mu \setminus R = \bigcup_{k=1}^N P_k'''$

$$\|\nabla\bar{v}\|_{L^p(Q_\mu \setminus R)} \le C\|e(\bar{u})\|_{L^p(Q_\mu)} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}, \tag{4.79}$$

where $\bar{v} := \bar{u} - \sum_{k=1}^N \chi_{P_k'''} a_k$. By Poincaré's inequality and a scaling argument we also have $\|\bar{v}\|_{L^p(P_k''')} \le Cd(P_k''')\|\nabla\bar{v}\|_{L^p(P_k''')}$ for $k = 1, \ldots, N$. With $a_j = 0$ for all other components we get that $\bar{v}$ can be written as $\bar{v} = \sum_{j=1}^n \chi_{P_j}(\bar{u} - a_j)$. □

We are now in a position to give the proof of Theorem 4.1. The reader not interested in the derivation of (4.3) may readily skip the second part of the proof.

Proof of Theorem 4.1. (1) We can suppose that $p \in (1, 2)$. Once the result is proved in this case, the case $p = 1$ follows readily by Hölder's inequality. Moreover, we may without restriction assume that $\mathcal{H}^1(J_u) \le 2\sqrt{2}\theta^{-2}\mu$ as otherwise the claim is trivially satisfied (cf. (4.4)). If also $\mathcal{H}^1(J_u) \ge \mu\theta^2$, we let $(P_j')_{j=1}^m$ be the partition constructed in Theorem 4.3 (for $J = J_u$ and $r = \frac{1}{24}(2 - p)$) and let $\bar{u}$ be the approximation given by Theorem 4.6. We distinguish three cases:

(a) Suppose first that $\mathcal{H}^1(J_u) \ge \mu\theta^2$ and $\mathcal{H}^1(J_u \cap \bigcup_{j=1}^m \partial P_j') \ge \mu\theta^2$. We apply Theorem 4.9 to obtain the partition $Q_\mu = \bigcup_{j=1}^n P_j$ such that by $\mathcal{H}^1(J_u) \le 2\sqrt{2}\theta^{-2}\mu$ and (4.76)(i)

$$\mathcal{H}^1\big(\bigcup\nolimits_{j=1}^n \partial P_j\big) \le C\mathcal{H}^1(J_u) \le C\mu \le C\mathcal{H}^1(J_u \cap \bigcup\nolimits_{j=1}^m \partial P_j') \le C\mathcal{H}^1(J_u \cap \bigcup\nolimits_{j=1}^n \partial P_j), \tag{4.80}$$

where in the last step we used $\bigcup_{j=1}^m \partial P'_j \subset \bigcup_{j=1}^n \partial P_j$ up to a set of negligible $\mathcal{H}^1$-measure. This gives (4.2)(i). Let F be the exceptional set derived in Theorem 4.6 (see (4.56)), let R be the set in (4.76)(ii) and define

$$E = F \cup R. \tag{4.81}$$

Then (4.1) follows directly from (4.56) and (4.76)(ii) since η can be chosen arbitrarily small. To see (4.2)(ii), we apply (4.57)(ii) and (4.77) to find

$$\|\nabla v\|_{L^p(Q_\mu \setminus E)} \le \|\nabla \bar{v}\|_{L^p(Q_\mu \setminus R)} + \|\nabla u - \nabla \bar{u}\|_{L^p(Q_\mu \setminus F)} \le C\mu^{\frac{2}{p}-1}\|e(u)\|_{L^2(Q_\mu)}, \tag{4.82}$$

where $v = u - \sum_{j=1}^n \chi_{P_j} a_j$ and $\bar{v}$ as in (4.77) for infinitesimal rigid motions $(a_j)_{j=1}^n$.

(b)(i) We now suppose that $\mathcal{H}^1(J_u) \ge \mu\theta^2$ and $\mathcal{H}^1(J_u \cap \bigcup_{j=1}^m \partial P'_j) < \mu\theta^2$. As $\bar{u}$ is smooth on the complement of $J_u \cap \bigcup_{j=1}^m \partial P'_j$, this implies $\mathcal{H}^1(J_{\bar{u}}) < \mu\theta^2$. Consequently, we can apply Theorem 3.15 on $\bar{u}$ and define $E = \tilde{E} \cup F$ with $\tilde{E}$ as in (3.7) and F as in (4.56). Observe that also in this case (4.1) holds since particularly $|\tilde{E}| \le c(\mathcal{H}^1(J_{\bar{u}}))^2 \le c\mu\theta^2\mathcal{H}^1(J_u)$. Moreover, (4.2)(i) is trivially satisfied for the partition consisting only of Q_μ. Property (4.2)(ii) follows from (3.8) and (4.57)(ii).

(b)(ii) If $\mathcal{H}^1(J_u) < \mu\theta^2$, we proceed as in (b)(i), but apply Theorem 3.15 directly on u.

(2) We now show that we can find an exceptional set E' and a configuration $v' = u - \sum_{j=1}^n a'_j \chi_{P_j}$ such that (4.1)-(4.3) hold. We only treat case (a) where $\mathcal{H}^1(J_u \cap \bigcup_{j=1}^m \partial P'_j) \ge \mu\theta^2$ since (b)(i),(ii) are similar. With E as in part (1) we define $\mathcal{P} = \{P_j : |P_j \setminus E| \ge \frac{1}{2}|P_j|\}$ and let $E' = E \cup \bigcup_{P \notin \mathcal{P}} P$. Observe that $\mathcal{H}^1(\partial E') \le C\mathcal{H}^1(J_u)$ by (4.1), (4.76)(i) and $|E'| \le 2|E|$. This yields (4.1) for E'.

In view of (4.81), we find $\mathcal{P} \subset (P'''_k)_{k=1}^N$ for the components considered before (4.79). Since P'''_k is a c-John domain, we get $|P'''_k| \ge c' d(P'''_k)^2$ for $c' = c'(c)$, see Lemma 3.4. Therefore, $|P_j \setminus E| \ge \frac{1}{2}c'(d(P_j))^2$ for all $P_j \in \mathcal{P}$ and by Lemma 3.9 we get $\|a_j\|_{L^p(Q)} \le C\|a_j\|_{L^p(P_j \setminus E)}$, where Q is a square containing P_j with $|P_j \setminus E| \ge c'|Q|$ for a possibly smaller c'. By Lemma 3.7, (4.77), and the fact that $\|\bar{u}\|_{L^\infty(Q_\mu \setminus F)} \le C\|u\|_\infty$ (see Lemma 4.8) we then derive

$$\begin{aligned} d(P_j)|A_j| + \|a_j\|_{L^\infty(P_j)} &\le Cd(P_j)|P_j|^{-\frac{1}{2}-\frac{1}{p}}\|a_j\|_{L^p(P_j)} \le C|P_j|^{-\frac{1}{p}}\|a_j\|_{L^p(P_j)} \\ &\le C|P_j|^{-\frac{1}{p}}\|a_j\|_{L^p(P_j \setminus E)} \le C|P_j|^{-\frac{1}{p}}\|\bar{u} - a_j\|_{L^p(P_j \setminus E)} + C\|\bar{u}\|_{L^\infty(Q_\mu \setminus F)} \\ &\le C(d(P_j))^{1-\frac{2}{p}}\|\nabla \bar{v}\|_{L^p(P_j)} + C\|u\|_\infty. \end{aligned}$$

If $(d(P_j))^{1-\frac{2}{p}}\|\nabla \bar{v}\|_{L^p(P_j)} \le \|u\|_\infty$, we indeed obtain $\|v\|_{L^\infty(P_j)} \le C\|u\|_\infty$ and set $v' = v$ on P_j, i.e., $a'_j = a_j$. Otherwise, we set $v' = u$ on P_j, i.e., $a'_j = 0$, and derive by a short calculation using the previous estimate

$$\|\nabla v'\|^p_{L^p(P_j \setminus E)} \le C\|\nabla v\|^p_{L^p(P_j \setminus E)} + C|P_j||A_j|^p \le C\|\nabla v\|^p_{L^p(P_j \setminus E)} + C\|\nabla \bar{v}\|^p_{L^p(P_j)}.$$

For $P_j \notin \mathcal{P}$, let $a'_j = 0$. Thus, summing over all components $P_j \in \mathcal{P}$ and using (4.77), (4.82) we see that also v' satisfies (4.2)(ii) and additionally $\|v'\|_\infty \le C\|u\|_\infty$, which gives (4.3). $\square$

Remark 4.10. Let us comment on the distinction of the cases in the proof: (b) addresses the case where the jump set of the approximation $\bar{u}$ (or already of u itself, see (b)(ii)) is small in a certain sense and (a) deals with the case of larger jump sets.

Whereas in case (a) we establish a piecewise inequality by means of Theorem 4.9, the particularity of case (b) is that we use Theorem 3.15 to provide a Korn-type estimate for a *single* infinitesimal rigid motion. This is the case even if small pieces of the domain might be detached from the bulk part by J_u and the partition provided by Theorem 4.9 (in case (b)(i)) might be nontrivial. Intuitively, by the application of the Korn inequality for functions with small jump set (Theorem 3.15), these small detached pieces of the domain form part of the exceptional set given in (4.1) and (4.2)(ii) does not provide any information in theses pieces. (We refer to [13, 18, 31] for details.)

It may appear more natural to apply Theorem 4.9 also in case (b)(i) as by means of a piecewise inequality involving different infinitesimal rigid motions also the behavior in such small detached pieces could be controlled suitably. The distinction performed above, however, accounts not only for the Korn-type estimate (4.2)(ii) but also for the fact that we need a bound on the partition (4.2)(i). In view of (4.80), it appears to be impossible to derive such a bound in case (b)(i), where $\mathcal{H}^1(J_u \cap \bigcup_{j=1}^m \partial P_j')$ (and thus $\mathcal{H}^1(J_{\bar{u}})$) is small. Therefore, in case (b)(i) it is more convenient to derive an estimate for a trivial partition consisting only of one component as then (4.2)(i) follows immediately.

The problem is even more apparent in case (b)(ii): if already the jump set of the original function is small, Theorem 4.3, Theorem 4.6 and Theorem 4.9 are not applicable. Indeed, in this situation it would be difficult to construct a partition whose boundary is controlled suitably (cf. Remark 4.5(i)).

Remark 4.11. For later purpose, we recall the structure of the exceptional set E in case (a): by Remark 4.7 and (4.81) we have that $E = F \cup R$ consists of the isolated components Z introduced in Lemma 4.4, the exceptional sets related to the application of Theorem 3.15 on good squares, and the rest set R from Theorem 4.9. Note that the latter is less delicate since the area of R can be chosen arbitrarily small.

4.4. **A refined version of Theorem 4.1.** In the previous section we have concluded the proof of Theorem 4.1. We have seen that for $u \in \mathcal{W}(Q_\mu)$ we can find a partition $(P_j)_j$ of Q_μ such that on each component an estimate of Korn-type (4.2)(ii) holds outside a small exceptional set E. The derivation of the main result without exceptional set will be based on an iterative application of Theorem 4.1 on various mesoscopic scales. More precisely, one step of the algorithmic procedure will consist in considering the covering of Whitney-type $\mathcal{C}$ constructed in Section 4.1 (see Theorem 4.3) and using Theorem 4.1 on every square of the covering. (For details on the algorithm we refer to the beginning of Section 5.1.)

In order to derive the main estimate (2.2)(ii), it is fundamental to investigate the relation of infinitesimal rigid motions obtained by the application of Theorem 4.1 in different iteration steps. Therefore, as mentioned at the beginning of Section 4, we will need a refined version of Theorem 4.1 taking the following aspects into account:

(A) It is essential that the exceptional set E of Theorem 4.1 covers only a small portion of each square of $\mathcal{C}$. To this end, we have to investigate the *structure* of the exceptional set.

(B) It is essential that the estimate of Korn-type (4.2)(ii) does not only hold on each component P_j, but also on each square of the covering $\mathcal{C}$ intersecting P_j, i.e., on a set being in general larger than P_j. To this end, we possibly need to redefine the partition.

Before we come to the details, let us recall the relevant notions for this section. Besides the statement of Theorem 4.1, we need some properties of the Whitney covering $\mathcal{C}$ stated in Theorem 4.3. Moreover, recall the structure of the exceptional set $E = F \cup R$ described in Remark 4.11. We also recall that if the partition $(P_j)_j$ in Theorem 4.1 is not trivial (case (a) in the proof), it coincides with the one from Theorem 4.9 and we will use that the components with $P_j \not\subset R$ are c-John domains. Finally, at some point we need the approximation constructed in Theorem 4.6. We now address (A) and (B):

(A) By Remark 4.7(i) we get that F covers only a small portion of each square outside the isolated components Z. Although $|R|$ is small, its portion inside a square of the covering is not necessarily controllable. To this end, we introduce

$$R' = R \cup \bigcup\nolimits_{Q \in \mathcal{C}: |Q' \cap R| \geq \theta |Q|} Q \tag{4.83}$$

and note that $|R'| \leq \eta + 12\theta^{-1}\eta$ by (4.76) and (4.17)(iii). Although (A) does not hold on the squares contained in R', this will not affect our analysis as $|R'|$ can be chosen arbitrarily small. (Indeed, in Section 5 we will see that Theorem 2.1 already holds if it holds up to an exceptional set of arbitrarily small size.) Finally, we observe that isolated components always belong to the exceptional set E and (A) is not achievable for squares contained in Z. Here, other ideas will have to be applied using the properties in Lemma 4.4 and we refer to the description of the iteration scheme in Section 5.1 for more details.

(B) For the generalization of the Korn-type estimate it is essential that all squares of $\mathcal{C}$ intersecting P_j are not 'too large' with respect to the diameter of P_j. To this end, we have to redefine the partition consisting of *good components* (having exactly the desired property), *small components*, and *rest components*, which are contained in the set R' defined in (4.83).

Lemma 4.12. *Let $(P_j)_{j=1}^n$ be the partition from Theorem 4.9 and $\mathcal{C}$ be the covering from Theorem 4.3, Lemma 4.4. Let E as in Remark 4.11 and R' as in* (4.83)*. Then there is another partition $(P_j^*)_{j=1}^N = \mathcal{P}_R \cup \mathcal{P}_{\rm good} \cup \mathcal{P}_{\rm small}$ of Q_μ and $(A_j)_{j=1}^N \subset \mathbb{R}^{2\times 2}_{\rm skew}$ such that $\mathcal{P}_{\rm good}$ consists of c-John domains, $\mathcal{H}^1\big(\bigcup_{j=1}^N \partial P_j^*\big) \leq C\mathcal{H}^1(J_u)$, and with $P^*_{j,\mathcal{C}} := \bigcup_{Q \in \mathcal{C}, Q \cap P_j^* \neq \emptyset} Q$ we get*

$$\begin{aligned}
&(i) \quad \sum\nolimits_{j=1}^N \|\nabla u - A_j\|^p_{L^p(P_j^* \setminus E)} \leq C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)},\\
&(ii) \quad P_j^* \subset R' \text{ for all } P_j^* \in \mathcal{P}_R,\\
&(iii) \quad d(Q) \leq 4\sqrt{2}d(P_j^*) \text{ for all } P_j^* \in \mathcal{P}_{\rm good} \text{ and for all } Q \in \mathcal{C} \text{ with } Q \cap P_j^* \neq \emptyset,\\
&(iv) \quad \text{for all } P_k^* \in \mathcal{P}_{\rm small} \text{ there is } P_j^* \in \mathcal{P}_{\rm good} \text{ with } P^*_{k,\mathcal{C}} \subset P^*_{j,\mathcal{C}}.
\end{aligned} \tag{4.84}$$

Below in the proof of Lemma 4.13 we will exploit (4.84)(iii) to obtain refinement (B) and each small component will be combined with a good component by using (4.84)(iv).

Proof of Lemma 4.12. With $(P_j)_{j=1}^n$ from Theorem 4.9 and $\mathcal{C}$ from Theorem 4.3 we let

$$\mathcal{P}'_{\rm good} = \{P_j : d(Q) \leq 4\sqrt{2}d(P_j) \ \text{ for all } \ Q \in \mathcal{C} : Q \cap P_j \neq \emptyset\}. \tag{4.85}$$

By $\mathcal{P}'_{\rm small}$ we denote the remaining components. Recalling (4.6) we find for each $P_j \in \mathcal{P}'_{\rm small}$ some $Q \in \mathcal{C}$ such that $P_j \subset Q'$. Then (4.17)(ii) shows that $P_j \in \mathcal{P}'_{\rm small}$ intersects at most $c\theta^{-2}$ different squares of $\mathcal{C}$. Consequently, indicating by $\mathcal{P}''_{\rm small}$ the nonempty sets in $\{Q \cap P_j : P_j \in \mathcal{P}'_{\rm small},\ Q \in \mathcal{C}\}$ we derive for $c > 0$ large enough

$$\sum\nolimits_{P \in \mathcal{P}''_{\rm small}} \mathcal{H}^1(\partial P) \leq c\theta^{-2} \sum\nolimits_{P_j \in \mathcal{P}'_{\rm small}} \mathcal{H}^1(\partial P_j). \tag{4.86}$$

Note that $\mathcal{P}'_{\rm good} \cup \mathcal{P}''_{\rm small}$ forms a partition of Q_μ. We let $\mathcal{P}_R = \{P \in \mathcal{P}'_{\rm good} \cup \mathcal{P}''_{\rm small} : P \subset R'\}$ and let $\mathcal{P}''_{\rm good} = \mathcal{P}'_{\rm good} \setminus \mathcal{P}_R$, $\mathcal{P}'''_{\rm small} = \mathcal{P}''_{\rm small} \setminus \mathcal{P}_R$.

By $\mathcal{C}_{\rm small}$ we denote the squares $Q \in \mathcal{C}$ with $Q \not\subset R'$ and $Q \cap \bigcup_{P_j \in \mathcal{P}''_{\rm good}} P_j = \emptyset$. As then $|Q \cap \bigcup_{P \in \mathcal{P}'''_{\rm small}} P| \geq |Q \setminus R'| \geq (1-\theta)|Q| = \frac{1-\theta}{16}(\mathcal{H}^1(\partial Q))^2$ for $Q \in \mathcal{C}_{\rm small}$ by (4.83), the isoperimetric inequality and (4.86) yield for $C = C(\theta) > 0$

$$\sum_{Q \in \mathcal{C}_{\rm small}} \mathcal{H}^1(\partial Q) \leq C \sum_{Q \in \mathcal{C}_{\rm small}} \sum_{P \in \mathcal{P}'''_{\rm small}} \mathcal{H}^1(\partial P \cap \overline{Q}) \leq C \sum_{P_j \in \mathcal{P}'_{\rm small}} \mathcal{H}^1(\partial P_j). \tag{4.87}$$

We define $\mathcal{P}_{\rm good} = \mathcal{P}''_{\rm good} \cup (Q)_{Q \in \mathcal{C}_{\rm small}}$ and let $\mathcal{P}_{\rm small} \subset \mathcal{P}'''_{\rm small}$ be the components not contained in some $Q, Q \in \mathcal{C}_{\rm small}$. We denote the sets of the partition $\mathcal{P}_R \cup \mathcal{P}_{\rm good} \cup \mathcal{P}_{\rm small}$ by $(P_j^*)_{j=1}^N$. Note that by Theorem 4.9 and the definition of $\mathcal{P}_R$, the family $\mathcal{P}_{\rm good}$ consists of c-John domains. The assertion $\mathcal{H}^1\big(\bigcup_{j=1}^N \partial P_j^*\big) \le C\mathcal{H}^1(J_u)$ follows from (4.76)(i) and (4.86)-(4.87).

We show (4.84). First, (4.84)(ii) follows directly from the definition of $\mathcal{P}_R$, (4.84)(iii) is obvious for $(Q)_{Q\in\mathcal{C}_{\rm small}}$ and otherwise we recall (4.85). Likewise, (4.84)(iv) follows from the construction of $\mathcal{P}_{\rm small}$, in particular the definition of $\mathcal{C}_{\rm small}$. Indeed, we recall that each set in $\mathcal{P}_{\rm small}$ is contained in some $Q \in \mathcal{C}\setminus\mathcal{C}_{\rm small}$ with $Q \not\subset R'$ and thus $Q \cap \bigcup_{P_j\in\mathcal{P}''_{\rm good}} P_j \neq \emptyset$.

Finally, we show (4.84)(i). Recall the approximation $\bar{u}$ in Theorem 4.6. By (4.57)(i), the fact that $\bar{u}$ is smooth in $Q, Q\in\mathcal{C}_{\rm small}$, and Korn's inequality there are matrices $A_Q \in \mathbb{R}^{2\times 2}_{\rm skew}$ such that

$$\sum\nolimits_{Q\in\mathcal{C}_{\rm small}} \|\nabla\bar{u} - A_Q\|^p_{L^p(Q)} \le C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)}.$$

Then by (4.57)(ii) we also get (see (4.82) for a similar argument)

$$\sum\nolimits_{Q\in\mathcal{C}_{\rm small}} \|\nabla u - A_Q\|^p_{L^p(Q\setminus E)} \le C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)}$$

with the exceptional set E from Remark 4.11. For the components $\mathcal{P}_R \cup \mathcal{P}''_{\rm good}\cup\mathcal{P}_{\rm small}$, each of which being contained in a component of the original partition $(P_j)_{j=1}^n$, we apply (4.2)(ii) and then get that there are matrices $(A_j)_{j=1}^N$ such that (4.84)(i) holds. □

Given a covering $\mathcal{C}$ as in Theorem 4.3, a partition $(P_j)_j$ as in Theorem 4.9 (or Lemma 4.12) and an exceptional set R' as in (4.83) we define

$$\mathcal{Q}(P_j;\mathcal{C},R') = \{Q\in\mathcal{C} : Q\cap P_j \neq\emptyset,\ Q\cap P_j\not\subset R'\} \tag{4.88}$$

and the *covering-adapted sets* $P_{j,{\rm cov}} = \bigcup_{Q\in\mathcal{Q}(P_j;\mathcal{C},R')} Q$, which will play a crucial role in Section 5 (see refinement (B) above). Note that the definition of $P_{j,{\rm cov}}$ always depends on $\mathcal{C}, R'$ although not made explicit in the notation.

We now formulate the refined version of Theorem 4.1. Note that in Section 5 we will exclusively use Lemma 4.13 and no other details from Section 4 will be needed.

Lemma 4.13. *Let $\mu,\eta>0$, $\theta>0$ small, $p\in[1,2)$ and $r=\frac{1}{24}(2-p)$. Then there are a universal constant $c>0$ and $C=C(\theta,p)>0$ such that the following holds:*

(1) For each $u\in\mathcal{W}(Q_\mu)$ with $\mathcal{H}^1(J_u)\le 2\sqrt{2}\theta^{-2}\mu$ there is a partition $Q_\mu = \bigcup_{j=0}^n P_j$ with

$$P_0 = Q_\mu \quad \text{or} \quad P_0=\emptyset, \tag{4.89}$$

a covering $\mathcal{C}_u\subset\bigcup_{i\ge1}\mathcal{Q}^i$ of Q_μ consisting of pairwise disjoint dyadic squares, and pairwise disjoint, closed sets Z^l_u, $l\ge 8$, such that with $S_u := (J_u\cap Q_\mu)\setminus\bigcup_{Q\in\mathcal{C}_u} Q'$ we have

$$\begin{aligned} &(i)\ \ \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^n \partial P_j\Big) \le C\mathcal{H}^1(S_u),\\ &(ii)\ \ Q\subset Q_\mu\setminus S_u \text{ for all } Q\in\mathcal{C}_u,\ \ \overline{Q}\subset Q_\mu\setminus S_u \text{ for all } Q\subset \bigcup\nolimits_{l\ge8} Z^l_u,\\ &(iii)\ \ \partial X^l_k = X^l_k\cap\bigcup\nolimits_{j=1}^n\partial P_j\subset\bigcup\nolimits_{j=1}^n\partial P_j \quad \text{for all } X^l_k, \end{aligned} \tag{4.90}$$

where $(X^l_k)_k$ denote the connected components of Z^l_u. Each Z^l_u is a union of squares in $\mathcal{Q}^l\cap\mathcal{C}_u$ up to a set of measure zero. Moreover, there are exceptional sets E_u, R_u such that for $l\ge 8$

one has

$$\begin{aligned} &(i) \quad |E_u| \le c\mu\theta^2\mathcal{H}^1(J_u) \le cd(Q_\mu)\theta^2\mathcal{H}^1(J_u \cap Q_\mu), \qquad |R_u| \le \eta,\\ &(ii) \quad |Q' \cap E_u| \le c\theta|Q| \text{ for all } Q \in \mathcal{C}_u \text{ with } Q \not\subset R_u, \quad Q \cap \bigcup\nolimits_{l\ge 8} Z^l_u = \emptyset,\\ &(iii) \quad d(X^l_k) \le \theta^{-rl}s_l \quad \text{for all } X^l_k,\\ &(iv) \quad |Z^l_u| \le C\theta^{-lr}s_l\mathcal{H}^1(J_u) \le C\theta^{-rl}s_l\mu, \end{aligned} \tag{4.91}$$

and there are infinitesimal rigid motions $a_j = a_{A_j,b_j}$, $j = 0,\dots,n$, *such that*

$$\begin{aligned} &(i) \quad \sum\nolimits_{j=0}^n \|\nabla u - A_j\|^p_{L^p(P_{j,\mathrm{cov}}\setminus E_u)} \le C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)},\\ &(ii) \quad \#\{P_{j,\mathrm{cov}} : x \in P_{j,\mathrm{cov}}\} \le c \quad \text{for all} \quad x \in Q_\mu, \end{aligned} \tag{4.92}$$

where $P_{j,\mathrm{cov}}$ *as defined below (4.88) with respect to* $\mathcal{C}_u, R_u$.

(2) If $P_0 = Q_\mu$, *then* $P_{0,\mathrm{cov}} = Q_\mu$, $\mathcal{C}_u = \{Q \in \mathcal{Q}^1 : Q \subset Q_\mu\}$, $S_u = \emptyset$, $\bigcup_{l\ge 8} Z^l_u = \emptyset$, $R_u = \emptyset$ *and there is a set* $\Gamma_u \subset \partial Q_\mu$ *with* $\mathcal{H}^1(\Gamma_u) \le c\theta\mu$ *such that*

$$\int_{\partial Q_\mu \setminus \Gamma_u} |Tu - a_0|^2 \, d\mathcal{H}^1 \le c\mu\|e(u)\|^2_{L^2(Q_\mu)}, \tag{4.93}$$

where Tu *denotes the trace of* u *on* ∂Q_μ.

Remark 4.14. Two cases are distinguished depending on whether the partition is trivial or not. This is reflected in (4.89). In Section 5 we will say that the square is of *type 1* if $P_0 = \emptyset$ and of *type 2* otherwise. This distinction corresponds to the two different cases in the proof of Theorem 4.1 and we refer to Remark 4.10 for more explanation in that direction. If $P_0 = Q_\mu$, the situation is actually much easier and only (4.91)(i), (4.91)(ii) for a quite simple covering as well as (4.92)-(4.93) are relevant, where (4.92)(i) then simplifies to $\|\nabla u - A_0\|^p_{L^p(Q_\mu\setminus E_u)} \le C\mu^{2-p}\|e(u)\|^p_{L^2(Q_\mu)}$. However, we prefer to present both cases in one single statement since this is convenient for the application of the lemma in Section 5.

Property (4.90)(i) is a refinement of (4.2)(i) and shows that only a specific subset S_u of the jump set J_u is needed to control the length of the boundary of the partition. In particular, the squares of the covering do not intersect S_u (see (4.90)(ii)) which will be crucial in the algorithmic procedure in Section 5 to show that each part of J_u is only 'used' in *one single iteration step* to control the partition. We emphasize that estimate (4.90)(i) is the essential reason why a distinction into two different cases (1) and (2) is necessary (we refer to Remark 4.10 for an explanation).

Property (4.91)(ii) reflects refinement (A) above and (4.91)(i) is related to (4.1). Properties (4.91)(iii),(iv) on the isolated components have already been established in Lemma 4.4. Note that (4.90)(iii) states that each isolated component is a component of the partition. Finally, the fact that the Korn-type estimate (4.92) holds on the covering-adapted sets is a refinement of (4.2)(ii) and has has been addressed in (B) above.

Point (2) treats the special case of a trivial partition. In Section 5 we will use the trace estimate (4.93) to relate the different behavior of u on adjacent squares of type 2.

Proof. Similarly as in the proof of Theorem 4.1, we distinguish different cases depending on whether the jump set is large or not.

(1) First assume $\mathcal{H}^1(J_u) \ge \mu\theta^2$ and $\mathcal{H}^1(J_u \setminus \bigcup_{Q\in\mathcal{C}} Q') \ge \mu\theta^2$, where $\mathcal{C}$ is the covering from Theorem 4.3, Lemma 4.4 (applied for $J = J_u$) satisfying (4.17). We define $\mathcal{C}_u = \mathcal{C}$ and let $(Z^l_u)_{l\ge 8} = (Z^l)_{l\ge 8}$ be the isolated components from Lemma 4.4 satisfying (4.18)-(4.19). We let E_u as in Theorem 4.1 (see also Remark 4.11 for its definition) and $R_u = R'$ as in (4.83).

Denote the partition given in Lemma 4.12 by $(P_j^*)_{j=1}^N = \mathcal{P}_R \cup \mathcal{P}_{\rm good} \cup \mathcal{P}_{\rm small}$ and note that $\mathcal{H}^1\big(\bigcup_{j=1}^N \partial P_j^*\big) \le C\mathcal{H}^1(J_u)$.

Definition of the partition: For each $P_j^* \in \mathcal{P}_{\rm good}$ choose a set $\mathcal{T}(P_j^*) \subset \mathcal{P}_{\rm small}$ with $P_{k,\mathcal{C}}^* \subset P_{j,\mathcal{C}}^*$ for $P_k^* \in \mathcal{T}(P_j^*)$ (cf. (4.84)(iv)) such that $\mathcal{P}_{\rm small} = \dot{\bigcup}_{P_j^* \in \mathcal{P}_{\rm good}} \mathcal{T}(P_j^*)$ is a decomposition of $\mathcal{P}_{\rm small}$. Set $\mathcal{T}(P_j^*) = \emptyset$ for $P_j^* \in \mathcal{P}_R$. Let $(P_j)_{j=0}^n$ be the partition of Q_μ with $P_0 = \emptyset$ consisting of the connected components $(X_k^l)_{l,k}$ of $(Z_u^l)_{l\ge 8}$ and the sets

$$P_j := \big(P_j^* \cup \bigcup\nolimits_{P_k^* \in \mathcal{T}(P_j^*)} P_k^*\big) \setminus \bigcup\nolimits_{l \ge 8} Z_u^l, \quad \text{for } P_j^* \in \mathcal{P}_R \cup \mathcal{P}_{\rm good}. \tag{4.94}$$

Proof of (4.90)*:* First, the definition of the partition directly implies (4.90)(iii). By (4.18)(ii) and $\mathcal{H}^1(J_u) \le 2\sqrt{2}\mu\theta^{-2}$ we see $\mathcal{H}^1(\bigcup_{j=1}^n \partial P_j) \le C\mathcal{H}^1(J_u) \le C\mu$. Then (4.90)(i) follows from the assumption $\mathcal{H}^1(S_u) \ge \mu\theta^2$, where $S_u = J_u \setminus \bigcup_{Q \in \mathcal{C}_u} Q'$. Likewise, the definition of S_u and the fact that $Q' \subset Q_\mu$ for all $Q \in \mathcal{C}_u$ (see (4.17)(i)) imply $Q' \subset Q_\mu \setminus S_u$ and particularly (4.90)(ii).

Proof of (4.91)*:* The first part of (4.91)(i) follows from (4.1). Applying (4.76)(ii) with $\frac{\theta}{13}\eta$ in place of η, we get $|R_u| \le \eta$ by (4.83). Likewise, (4.91)(iii),(iv) are consequences of Lemma 4.4, see (4.18)(i) and (4.19)(i), and $\mathcal{H}^1(J_u) \le 2\sqrt{2}\mu\theta^{-2}$. Recalling Remark 4.11 we find $E_u \setminus R \subset F$ with F as defined in Theorem 4.6 and R as in (4.76)(ii). Thus, Remark 4.7(i) implies $|Q' \cap (E_u \setminus R)| \le \theta|Q|$ for all $Q \cap \bigcup_{l \ge 8} Z_u^l = \emptyset$. Then the fact that $|Q' \cap R| \le \theta|Q|$ for all $Q \in \mathcal{C}_u$ with $Q \not\subset R_u$ (see (4.83)) yields (4.91)(ii).

Proof of (4.92)*:* We distinguish the three cases of (a) isolated components, (b) components $\mathcal{P}_R$, and (c) components $\mathcal{P}_{\rm good}$.

(a) First, each component $P_j \in (X_k^l)_{l,k}$ of $Z_u := \bigcup_{l \ge 8} Z_u^l$ consists of a union of squares in $\mathcal{Q}^l \cap \mathcal{C}_u$ and thus $P_{j,{\rm cov}} \subset P_j \subset F \subset E_u$ (cf. Remark 4.7(i) and Remark 4.11).

(b) Now consider P_j such that $P_j^* \in \mathcal{P}_R$ with P_j^* as in (4.94). As $P_j \subset P_j^* \subset R_u$ by (4.84)(ii), (4.88) implies $P_{j,{\rm cov}} = \emptyset$.

(c) For P_j with $P_j^* \in \mathcal{P}_{\rm good}$ we recall that P_j^* is a c-John domain ($c \ge 1$). Let $Q \in \mathcal{Q}(P_j;\mathcal{C}_u,R_u)$ and note that by (4.84)(iv), (4.94), and the choice of $\mathcal{T}(P_j^*)$ we have $Q \cap P_j^* \neq \emptyset$. Since $d(Q) \le 4\sqrt{2}d(P_j^*)$ by (4.84)(iii), we find some $x \in P_j^* \cap Q$ such that $P_j^* \setminus B_r(x) \neq \emptyset$, where $r = d(Q)/(9\sqrt{2})$. Therefore, by Lemma 3.4 there exists $z \in \overline{B_r(x)}$ such that $B_{r/(2c)}(z) \subset P_j^*$. Also note that $B_{r/(2c)}(z) \subset Q'$. This implies that $|Q' \cap P_j^*| \ge |B_{r/(2c)}(z)| \ge c'd(Q)^2$ for some c' small only depending on the John constant c. As for $Q \in \mathcal{Q}(P_j;\mathcal{C}_u,R_u)$ we have $Q \cap Z_u = \emptyset$ (see (4.94)) and $Q \not\subset R_u$, this yields

$$|(P_j^* \cap Q') \setminus E_u| \ge c|Q| \tag{4.95}$$

for θ small by (4.91)(ii). In particular, this also implies that each $Q \in \mathcal{C}_u$ is contained only in a bounded number of different $\mathcal{Q}(P_j;\mathcal{C}_u,R_u)$. For $Q \in \mathcal{Q}(P_j;\mathcal{C}_u,R_u)$ we have $Q \cap Z_u = \emptyset$ and thus Remark 4.7(ii) holds for Q. This together with Lemma 3.9, $F \subset E_u$ (see Remark 4.11), and (4.95) yields $A_Q \in \mathbb{R}^{2\times 2}_{\rm skew}$ such that

$$\begin{aligned}\|\nabla u - A_j\|^p_{L^p(Q\setminus E_u)} &\le C\|\nabla u - A_Q\|^p_{L^p(Q\setminus E_u)} + C\|A_Q - A_j\|^p_{L^p((P_j^*\cap Q')\setminus E_u)}\\ &\le C\|\nabla u - A_Q\|^p_{L^p(Q'\setminus F)} + C\|\nabla u - A_j\|^p_{L^p((P_j^*\cap Q')\setminus E_u)}\\ &\le Cd(Q)^{2-p}\|e(u)\|^p_{L^2(Q')} + C\|\nabla u - A_j\|^p_{L^p((P_j^*\cap Q')\setminus E_u)}.\end{aligned}$$

Summing over all P_j with $P_j^* \in \mathcal{P}_{\rm good}$ and $Q \in \mathcal{Q}(P_j;\mathcal{C}_u,R_u)$ we finally get (4.92)(i). Indeed, for the term on the right we use (4.84)(i) and the fact that each $x \in Q_\mu$ is contained in at most

12 different Q', $Q \in \mathcal{C}_u$, see (4.17)(iii). For the left term we use that each square is contained only in a bounded number of different $\mathcal{Q}(P_j;\mathcal{C}_u,R_u)$ (see below (4.95)) and apply the discrete Hölder inequality to compute (cf. (4.68) for a similar argument)

$$\sum\nolimits_{Q\in\mathcal{C}_u}(d(Q))^{2-p}\|e(u)\|^p_{L^2(Q')} \le \big(\sum\nolimits_{Q\in\mathcal{C}_u}|Q|\big)^{1-\frac{p}{2}}\|e(u)\|^p_{L^2(Q_\mu)}. \tag{4.96}$$

This concludes the proof of (4.92)(i). Note that (a),(b) and (4.95) also show that each $Q\in\mathcal{C}_u$ intersects only a bounded number of different $(P_{j,\mathrm{cov}})_{j=1}^n$, which establishes (4.92)(ii).

(2) (i) Now suppose that $\mathcal{H}^1(J_u) \ge \mu\theta^2$ and $\mathcal{H}^1(J_u\setminus\bigcup_{Q\in\mathcal{C}}Q') < \mu\theta^2$. In particular, we have $\mathcal{H}^1(J_{\bar u}) \le \mu\theta^2$, where $\bar u$ is the approximation from Theorem 4.6. We now apply Theorem 3.15 on $\bar u$. Let $E_u = E'\cup F$ with E' as in (3.7) and F as given by Theorem 4.6. Moreover, we let $P_0 = Q_\mu$, a_0 as in (3.8), $R_u = \emptyset$, $\mathcal{C}_u = \{Q\in\mathcal{Q}^1 : Q\subset Q_\mu\}$ and $Z^l_u = \emptyset$ for all $l\ge 8$. This also implies $S_u = (J_u\cap Q_\mu)\setminus\bigcup_{Q\in\mathcal{C}_u}Q' = \emptyset$.

First, (4.90)(i),(iii) are trivially satisfied since the partition only consists of P_0. Moreover, (4.90)(ii) follows from the fact that $\bigcup_{l\ge8}Z^l_u = \emptyset$ and $S_u=\emptyset$. Properties (4.91)(iii),(iv) are trivial. By $\mathcal{H}^1(J_u)\le 2\sqrt{2}\mu\theta^{-2}$, $\mathcal{H}^1(J_{\bar u})\le\mu\theta^2$, (3.7) and (4.56) we get

$$|E_u| \le |E'| + |F| \le c(\mathcal{H}^1(J_{\bar u}))^2 + c\mu\theta^5\mathcal{H}^1(J_u) \le c\mu^2\theta^3.$$

Since $\mathcal{H}^1(J_{\bar u}) \le \mathcal{H}^1(J_u)$ and $\mathcal{H}^1(J_{\bar u})\le\mu\theta^2$, this implies (4.91)(i). Also (4.91)(ii) holds as $|Q| = 4\mu^2\theta^2$ for all $Q\in\mathcal{C}_u$. As $P_{0,\mathrm{cov}} = P_0$, (4.92) follows from (3.8) (applied on $\bar u$) and (4.57)(ii). Finally, (3.9) applied on $\bar u$ together with $\mathcal{H}^1(J_{\bar u})\le\mu\theta^2$ and $Tu = T\bar u$ on ∂Q_μ (see Theorem 4.6) yields a set $\Gamma_u\subset\partial Q_\mu$ with $\mathcal{H}^1(\Gamma_u)\le c\theta\mu$ such that (4.93) holds.

(ii) Finally, if $\mathcal{H}^1(J_u) < \mu\theta^2$, we proceed as in (i), but apply Theorem 3.15 directly on u. □

Let us conclude with a brief remark about the choice of $\mathcal{C}_u$ in case (2). In case (i) it would also have been natural to choose the covering from Theorem 4.3, but since this is not possible in case (ii) (Theorem 4.3 is not applicable), the present choice is the most convenient.

5. Derivation of the piecewise Korn inequality

This section is devoted to the proof of our piecewise Korn inequality. We first assume that $\Omega = (-\mu_0,\mu_0)^2$ is a square with $\mu_0 > 0$. Similarly as in (4.5)-(4.6), we define $\mathcal{Q}^i := \tilde{\mathcal{Q}}^{t_i}$ for $t_i = \mu_0\theta^i$, $\theta\in 2^{-\mathbb{N}}$, and denote by $Q' = \frac{3}{2}Q$ the enlarged square corresponding to $Q\in\mathcal{Q}^i$. (We will apply the results of Section 4 on squares $(-\mu,\mu)^2$ of different sizes and thus in general $s_i\neq t_i$, where s_i as defined below (4.5).) Recall the definition of the set $\mathcal{W}(\Omega)$ before Theorem 4.1. We first establish the main result for functions in $\mathcal{W}(\Omega)$ which additionally satisfy

$$\mathcal{H}^1(Q\cap J_u) \le \theta^{-1}d(Q) \qquad \text{for all } \ Q\in\bigcup\nolimits_{i\ge1}\mathcal{Q}^i. \tag{5.1}$$

Afterwards, we drop assumption (5.1) and show the result for all functions in $\mathcal{W}(\Omega)$. Finally, we pass to general domains with Lipschitz boundary and Theorem 2.1 will be derived using a density argument (see Section 5.2).

5.1. **Proof for configurations with regular jump set.** In this section we let $\Omega = Q_{\mu_0} = (-\mu_0,\mu_0)^2$. We first prove Theorem 2.1 for configurations in $\mathcal{W}(Q_{\mu_0})$ satisfying (5.1), which already represents the core of the argument. If (5.1) is violated on a square Q, we add ∂Q to the boundary of the partition and treat the problem on Q as a separate problem independent from the analysis on the rest of the domain (see Theorem 5.2 below for details).

Recall that in Section 4 we have already established a piecewise Korn inequality which holds outside of a small exceptional set. The derivation of the main result without exceptional set is based on an iterative application of Lemma 4.13 on various mesoscopic scales. The strategy is to apply first Lemma 4.13 on Q_{μ_0} to obtain a partition, corresponding infinitesimal rigid motions, and a covering of Q_μ with pairwise disjoint dyadic squares. Then Lemma 4.13 is again applied on each square of this covering. This procedure is iterated to obtain a progressively smaller exceptional set. For the following we will assume that the reader is already familiar with the statement and the notions of Lemma 4.13. Note, however, that in this section we will exclusively use Lemma 4.13 and no other details from Section 4 are needed.

Before we come to the rigorous proof, we first briefly give an intuition why an iteration scheme seems to be indispensable and we describe the main strategy of this procedure.

Necessity of an iteration scheme: First, we observe that it is basically enough to establish the result up to a small exceptional set whose area depends on the function u: Indeed, as $u \in \mathcal{W}(Q_{\mu_0})$ and thus $\nabla u \in L^p(Q_{\mu_0})$, we find $\varepsilon = \varepsilon(u) > 0$ small such that for all Borel sets $B \subset Q_{\mu_0}$ with $|B| \le \varepsilon$ one has $\|\nabla u\|_{L^p(B)} \le \mu_0^{2/p-1}\|e(u)\|_{L^2(Q_{\mu_0})}$. Based on that, one can show that it is enough to control ∇u on the complement of such a set of small measure.

The first naive idea could be to apply Theorem 4.1 for θ sufficiently small such that the exceptional set E in (4.1) indeed satisfies $|E| \le \varepsilon$ so that the above estimate applies. Observe, however, that then θ depends on ε (and thus on u), i.e., the constant $C = C(p, \theta)$ in (4.2) depends on u entailing nonuniform estimates for the partition and ∇u. Therefore, it appears that θ has to be fixed and the arguments have to be reliant upon an iterative application of the results presented in Section 4.

A first idea of an iterative scheme could be to apply Theorem 4.1 on Q_{μ_0}, then to cover suitably the resulting exceptional set with smaller squares and to again apply Theorem 4.1 in these squares. Without going into detail, we mention that an iterative application of this strategy will indeed lead to a sequence of exceptional sets $(E_k)_k$ (k denotes the iteration step) with $|E_k| \to 0$ as $k \to \infty$. At the same time, however, the constant for the estimate on the partition and ∇u (cf. (4.2)) might be of the form C^k for some $C > 1$, eventually blowing up as the number of iteration steps k tends to infinity.

Consequently, this strategy is non expedient in general and a more elaborated iteration scheme has to be applied, which is based on the refined version of Theorem 4.1 and ensures that the relevant constants do not blow up as $k \to \infty$. We will now proceed by describing the cornerstones of this iteration scheme, concentrating on the underlying ideas and without giving exact statements and notations. For the actual proof we refer to Theorem 5.1 below.

Description of the iteration scheme: To fix the main ideas and to give some intuition about the algorithmic procedure, we now illustrate one iteration step in detail. In particular, we describe how the partition, the infinitesimal rigid motions, the covering, and the exceptional sets are updated and we explain how uniform estimates on the boundary of the partition (2.1) and the Korn-type inequality (2.2)(ii) can be obtained. Recall that, due to some technical constructions in Section 4, in Lemma 4.13 we have to take a set of so-called isolated components into account (we refer to Lemma 4.4 and the comments thereafter for more details). Since in these sets a slightly different construction has to be performed, we first neglect them. The necessary adaptions will be described afterwards.

The case without isolated components: Let us suppose that in an iteration step we have given a partition $(P_j)_j$ of Q_{μ_0}, corresponding infinitesimal rigid motions $(a_j)_j$, as well as a covering $\mathcal{C}$, an exceptional set E, and a rest set R. We remark that in general squares of $\mathcal{C}$ might intersect various components of the partition $(P_j)_j$. We assume that for each square of $\mathcal{C}$ which is not completely contained in R only a small portion is covered by E. Moreover, suppose that there is a subset $S \subset J_u$ such that the length of the boundary of the partition can be controlled in terms of S and S does not intersect the squares of the covering. More precisely, we have

$$\mathcal{H}^1\Big(\bigcup\nolimits_j \partial P_j \setminus \partial Q_{\mu_0}\Big) \le C\mathcal{H}^1(S), \qquad S \cap Q = \emptyset \quad \text{for all} \quad Q \in \mathcal{C}. \tag{5.2}$$

Finally, we assume that we have a Korn-type estimate of the form

$$\sum\nolimits_j \|\nabla u - A_j\|^{p'}_{L^{p'}(P_{j,\mathrm{cov}} \setminus E)} \le C\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})}, \tag{5.3}$$

for $p' < 2$, where the covering-adapted sets $P_{j,\mathrm{cov}}$ are defined below (4.88) with respect to $\mathcal{C}$ and R. In particular, we remark that in the first iteration step all properties are guaranteed by an application of Lemma 4.13 on Q_{μ_0}.

We now pass to the next iteration step. To this end, we apply Lemma 4.13 on each $Q \in \mathcal{C}$. (Note that Lemma 4.13 is applicable due to (5.1).) In view of (4.89), we distinguish between two types of squares. For squares of *type 1*, denoted by $\mathcal{C}_1$, one has a partition $\mathcal{P}^Q$ of Q satisfying (4.90)(i) and associated infinitesimal rigid motions (see (4.92)). For squares of *type 2*, denoted by $\mathcal{C}_2$, the partition given by Lemma 4.13 is trivial and consists only of Q. Correspondingly, there is only one infinitesimal rigid motion a_Q, see also Remark 4.14. Apart from the partition and the affine mappings, Lemma 4.13 also yields a covering $\mathcal{C}^Q$ and sets E^Q, R^Q. The relevant notions are now updated as follows:

(a) *Covering:* We define $\mathcal{C}_{\mathrm{new}} = \bigcup_{Q\in\mathcal{C}} \mathcal{C}^Q$. In particular, $\mathcal{C}_{\mathrm{new}}$ is a refinement of $\mathcal{C}$.
(b) *Exceptional set:* We define $E_{\mathrm{new}} = \bigcup_{Q\in\mathcal{C}} E^Q$. Note that in general this set has no relation to the exceptional set E of the previous iteration step.
(c) *Rest set:* We define $R_{\mathrm{new}} = R \cup \bigcup_{Q\in\mathcal{C}} R^Q$.
(d) *Partition:* The new partition contains all partitions $\mathcal{P}^Q$ given by squares $Q \in \mathcal{C}_1$ of type 1 and it contains the components $(P_j^{\mathrm{new}})_j$ which are obtained from the partition of the previous iteration step by cutting away the squares of type 1, i.e., $P_j^{\mathrm{new}} := P_j \setminus \bigcup_{Q\in\mathcal{C}_1} Q$. We observe that each P_j^{new} is covered by squares in $\mathcal{C}_2$ but in general squares in $\mathcal{C}_2$ may intersect various components P_j^{new}.
(e) *Infinitesimal rigid motions:* On the components of the partitions given on squares of type 1 we use the corresponding infinitesimal rigid motions given by Lemma 4.13, see (4.92). For P_j^{new} we choose the mapping a_j from the previous iteration step.

As $\mathcal{C}_{\mathrm{new}}$ is a refinement of $\mathcal{C}$, we particularly have that the squares of the covering are smaller, i.e., $\max_{Q\in\mathcal{C}_{\mathrm{new}}} d(Q) \le \theta \max_{Q\in\mathcal{C}} d(Q)$. From (4.91)(i) we get that on each square $|E^Q|$ scales like $d(Q)\mathcal{H}^1(J_u \cap Q)$. Consequently, $|E_{\mathrm{new}}|$ scales like $\max_{Q\in\mathcal{C}_{\mathrm{new}}} d(Q)\mathcal{H}^1(J_u)$ and, due to the refinement of the covering, we will see that the area of the exceptional set will become progressively smaller along the iteration steps. Moreover, although the rest set increases during the iteration scheme, its area can still be controlled uniformly by some (arbitrarily small) $\varepsilon > 0$ since each $|R^Q|$ can be chosen arbitrarily small, cf. (4.91)(i). (Indeed, we will see that Theorem 2.1 already holds if it holds up to an exceptional set of arbitrarily small size.) We now come to the fundamental points (d), (e) and explain how a uniform control on the boundary of the partition and estimates of Korn-type can be obtained.

Partition: Let us first note that the procedure described in (d) leads to a partition of Q_{μ_0}: indeed, the components of $\mathcal{P}^Q$, $Q \in \mathcal{C}_1$, form a partition of $\bigcup_{Q\in\mathcal{C}_1} Q$ and the sets $(P_j^{\mathrm{new}})_j$ form a partition of $\bigcup_{Q\in\mathcal{C}_2} Q$ (up to sets of negligible measure). Concerning (5.2), the only delicate squares are those of type 1 since for them new components are added to the partition. In view of (4.90)(i), the length of the boundary of each partition $\mathcal{P}^Q$, $Q \in \mathcal{C}_1$, is controlled in terms of $S^Q := (J_u \cap Q) \setminus \bigcup_{\hat{Q}\in\mathcal{C}^Q} \hat{Q}'$. Since by (5.2) the set S^Q has empty intersection with S, we see that the parts of J_u contained in S are *not 'used'* to control the new parts of the partition. By (4.90)(ii) we also get that the squares of $\mathcal{C}_{\mathrm{new}}$ do not intersect S^Q, $Q \in \mathcal{C}_1$.

Thus, (5.2) also holds in the new iteration step for the new partition and the new covering with $S_{\rm new} = S \cup \bigcup_{Q \in \mathcal{C}_1} S^Q$. Consequently, we conclude that in the iteration scheme each part of J_u is only 'used' in *one single iteration step* to control the length of the boundary of the partition, which will eventually ensure a uniform bound independent of the number of iteration steps.

Korn inequalities: Here, the squares of type 1 are less delicate since by the choice of the infinitesimal rigid motions in (e), we obtain an estimate of Korn-type directly from (4.92). We therefore concentrate on a square $Q \in \mathcal{C}_2$ of type 2 which is not completely contained in the rest set $R_{\rm new}$ and intersects a component $P_j^{\rm new}$.

In view of the definition of the affine mappings in (e), our goal is to show that

$$\|\nabla u - A_j\|^{p'}_{L^{p'}((Q \cap P^{\rm new}_{j,\rm cov}) \setminus E_{\rm new})} \le c d(Q)^{2-p'} \|e(u)\|^{p'}_{L^2(Q)} + c\|\nabla u - A_j\|^{p'}_{L^{p'}((Q\cap P_{j,\rm cov})\setminus E)}. \tag{5.4}$$

Then by summing over all squares of type 2 and all components $P^{\rm new}_{j,\rm cov}$ we indeed obtain an estimate of the form (5.3) in the new iteration step, where the rightmost term in (5.4) is controlled by estimate (5.3) from the previous iteration step.

We now explain how to derive (5.4). By the definition of the partition and the covering we have $P^{\rm new}_{j,\rm cov} \subset P_{j,\rm cov}$. Thus, the control on $\|\nabla u - A_j\|_{L^{p'}((Q\cap P^{\rm new}_{j,\rm cov})\setminus E)}$ in terms of the right hand side of (5.4) is immediate and to establish (5.4) it suffices to control $\|\nabla u - A_j\|_{L^{p'}(Q \cap (E\setminus E_{\rm new}))}$.

To this end, from Lemma 4.13(2) applied on Q and the definition of $E_{\rm new}$ in (b) we get the estimate

$$\|\nabla u - A_Q\|^{p'}_{L^{p'}(Q\setminus E_{\rm new})} = \|\nabla u - A_Q\|^{p'}_{L^{p'}(Q\setminus E^Q)} \le c d(Q)^{2-p'}\|e(u)\|^{p'}_{L^2(Q)}. \tag{5.5}$$

(Since Q is of type 2, the partition is trivial and only consists of Q itself, see also Remark 4.14). We now aim at estimating the difference of A_j and A_Q. We recall that $Q \not\subset R_{\rm new}$. Thus, by the assumption stated before (5.2), E covers only a small part of Q. Likewise, E^Q covers only small portion of Q by (4.91)(ii). (This property was one of the motivations to establish Lemma 4.13, see refinement (A) in Section 4.4.) By the triangle inequality this allows to estimate the difference of A_j and A_Q and we are able to show (we refer to (5.26) below for details)

$$\begin{aligned}\|\nabla u - A_j\|^{p'}_{L^{p'}(Q\cap(E\setminus E_{\rm new}))} &\le c\|\nabla u - A_Q\|^{p'}_{L^{p'}((Q\cap E)\setminus E^Q)} + c|Q\setminus(E\cup E^Q)||A_j - A_Q|^{p'}\\ &\le c\|\nabla u - A_Q\|^{p'}_{L^{p'}(Q\setminus E^Q)} + c\|\nabla u - A_j\|^{p'}_{L^{p'}(Q\setminus E)}\\ &\le c d(Q)^{2-p'}\|e(u)\|^{p'}_{L^2(Q)} + c\|\nabla u - A_j\|^{p'}_{L^{p'}(Q\setminus E)}.\end{aligned} \tag{5.6}$$

where the last step follows from (5.5). This indeed yields the desired estimate (5.4) as by the definition of the *covering-adapted sets* we have $Q \subset P_{j,\rm cov}$. Note that at this point we fundamentally use the definition of $P_{j,\rm cov}$: it allows us to estimate the difference of A_j and A_Q in terms of $\nabla u - A_j$ and $\nabla u - A_Q$ on the set $Q \setminus (E \cup E^Q) = (Q \cap P_{j,\rm cov}) \setminus (E\cup E^Q)$ instead of only on $(Q\cap P_j)\setminus(E\cup E^Q)$. This is essential as $|(Q\cap P_j)\setminus(E\cup E^Q)|$ might be arbitrarily small. (Recall that this was one of the motivations to establish Lemma 4.13, see refinement (B) in Section 4.4.)

Although this procedure leads to an estimate outside $E_{\rm new}$, we have to face the additional difficulty that the constant C in (5.3) is not uniform but blows up when (5.6) is repeatedly applied in different iteration steps. In the proof, this issue is reflected by a decomposition $Q_{\mu_0} = \bigcup_l D_l$, where, roughly speaking, D_l stands for the set where estimates of the form (5.6) have already been applied (at most) l times. To be more precise, on each of the sets D_l the constant in (5.3) can be controlled in terms of $(\theta^{-\lambda})^l$ for some small $\lambda > 0$ to be specified in

(5.9), i.e., we have (cf. (5.13) below)

$$\sum\nolimits_j \|\nabla u - A_j\|^{p'}_{L^{p'}((P^{\rm new}_{j,\rm cov}\cap D_l)\setminus E_{\rm new})} \le C\theta^{-\lambda l}\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})}. \tag{5.7}$$

To overcome the difficulty that $\theta^{-\lambda l} \to \infty$ as $l \to \infty$, we use an idea from the proof of Theorem 4.6: the problem that the estimate on D_l blows up as $l \to \infty$ can be compensated by the fact that $|D_l| \to 0$ as $l \to \infty$. Indeed, as discussed above, in each iteration step we need to establish an estimate of the form (5.6) only on $E \setminus E_{\rm new}$. Consequently, as $|E|$ scales like $\max_{Q\in\mathcal{C}} d(Q)\mathcal{H}^1(J_u)$ and thus becomes smaller along the iteration steps, also $|D_l|$ becomes smaller as $l \to \infty$. More precisely, we will see that $|D_l|$ is controlled in terms of $\mu_0^2(\theta^{1-r})^l$ with r from Lemma 4.13, see (5.12).

As in the proof of Theorem 4.6, the strategy is then to reduce the exponent p' to some smaller p and to apply Hölder's inequality: using (5.7) and $|D_l| \le C\mu_0^2\theta^{l(1-r)}$ we calculate (see (5.31) below for details)

$$\begin{aligned}\sum\nolimits_l\sum\nolimits_j \|\nabla u - A_j\|^p_{L^p((P^{\rm new}_{j,\rm cov}\cap D_l)\setminus E_{\rm new})} &\le \sum\nolimits_l |D_l|^{1-\frac{p}{p'}}\Big(\sum\nolimits_j \|\nabla u - A_j\|^{p'}_{L^{p'}((P^{\rm new}_{j,\rm cov}\cap D_l)\setminus E_{\rm new})}\Big)^{\frac{p}{p'}}\\ &\le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})}\sum\nolimits_{l\ge 0}\theta^{l(1-r)(1-p/p')-l\lambda p/p'}.\end{aligned}$$

Thus, for $\lambda > 0$ sufficiently small such that $(1-r)(1-p/p') - \lambda p/p' > 0$ (see (5.9) for its choice), we obtain a uniform control *independently of* l, i.e., independently of the number of iteration steps in which estimates of the form (5.6) are applied. Here, we emphasize that this Hölder-type argument and the corresponding reduction of the exponent is not performed iteratively in each iteration step but *only once* after the final iteration step.

Adaptions for isolated components: As mentioned above, the construction is slightly different in the presence of isolated components $\bigcup_{l\ge 8} Z^l$ as given in (4.91). In this case, for the squares of the covering outside of these components we proceed exactly as before. Fix a component X^l_k of Z^l and recall that by (4.90)(iii) the set X^l_k is a component P_j of the partition. We indicate the necessary adaptions inside X^l_k. The covering, exceptional set and rest set are updated as in (a)-(c). For the partition and infinitesimal rigid motions we proceed as follows:

- (d') *Partition:* The partition of X^l_k contains all partitions given by squares of type 1 contained in X^l_k and the remaining components are given by the connected components of $X^l_k \setminus \bigcup_{Q\in\mathcal{C}_1}\overline{Q}$, denoted by $(P^{l,k}_j)_j$, which consist exclusively of squares of type 2.
- (e') *Infinitesimal rigid motions:* On the components of the partitions given on squares of type 1 we use the corresponding infinitesimal rigid motions given by Lemma 4.13, see (4.92). For $(P^{l,k}_j)_j$ we select an arbitrary square Q contained in $P^{l,k}_j$ and as infinitesimal rigid motion we choose the mapping a_Q, see Remark 4.14.

The control on the length of the boundary of the partition can be obtained as above. Moreover, for squares of type 1 we may repeat the above arguments for the Korn-type inequality. For the sets $P^{l,k}_j$, however, (5.6) cannot be repeated since E might cover X^l_k completely, cf. (4.91)(ii).

Recall that each square contained in $P^{l,k}_j$ is of type 2 and we have a corresponding infinitesimal rigid motion a_Q such that $\|\nabla u - A_Q\|_{L^{p'}(Q\setminus E^Q)}$ is controlled (cf. Remark 4.14). Proceeding as in the proof of Theorem 4.6, drawing some ideas from [34, Section 4], we estimate the difference of these affine mappings using the trace estimate (4.93). (In this context, it is convenient that all squares have the same sidelength $2t_l$.) Due to $d(X^l_k) \le \theta^{-lr}t_l$ (see (5.11)(iii) below), the involved constant may blow up in l, but as in the proof of Theorem 4.6 this can be compensated by using a Hölder-type estimate and the fact that the area of these components is small, cf. (5.22) and (5.25)(ii) below.

In order to carry out all Korn-type estimates simultaneously, in the proof below also the isolated components will be incorporated suitably in the sets $(D_l)_l$.

Theorem 5.1. *Let $p \in [1,2)$. Then Theorem 2.1 holds for all $u \in \mathcal{W}(Q_{\mu_0})$ satisfying* (5.1).

Proof. Let $p \in [1,2)$ and $u \in \mathcal{W}(Q_{\mu_0})$ be given satisfying (5.1). Note that this particularly implies $\mathcal{H}^1(J_u) \le 2\sqrt{2}\theta^{-2}\mu_0$. A classical result states that u is piecewise rigid if $\|e(u)\|_{L^2(Q_{\mu_0})} = 0$ (see also Remark 2.2(i)), so we can concentrate on the case $\|e(u)\|_{L^2(Q_{\mu_0})} > 0$. Since $\nabla u \in L^p(Q_{\mu_0})$, we can select ε sufficiently small such that for all Borel sets $B \subset Q_{\mu_0}$ with $|B| \le 4\varepsilon$ one has

$$\|\nabla u\|^p_{L^p(B)} \le \mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})}. \tag{5.8}$$

We introduce $p' = 1 + \frac{p}{2} \in (p,2)$. Later we will pass from p' to p by Hölder's inequality. We let

$$r = \frac{2-p}{24} \quad \text{and} \quad \lambda = \frac{(1-r)(2-p)}{3p+2}. \tag{5.9}$$

In the following $c > 0$ stands for a universal constant and $C = C(\theta,p) > 0$ for a generic constant independent of ε and μ_0. We can suppose that θ is small with respect to λ, particularly that $8\theta^\lambda \le \frac{1}{6}$ (cf. (5.27) below). At the end of the proof we will fix θ depending on p such that eventually the constant C depends only on p.

We first formulate the induction hypothesis (Step I) and then discuss the induction step as described above (Step II - Step V). The iteration is repeated a finite number of times until the exceptional set is smaller than ε. We then conclude the proof by exploiting (5.8) (Step VI).

Step I (Induction hypothesis): In this step we formulate the induction hypothesis for step i and by applying Lemma 4.13 we check that it holds for $i = 0$.

Hypothesis for step i: Assume that we have a partition $\mathcal{P}^i = (P^i_j)_{j=1}^\infty$ of Q_{μ_0} (up to a set of negligible measure), a covering $\mathcal{C}_i$ of Q_{μ_0} with $\mathcal{C}_i \subset \bigcup_{j \ge i+1} \mathcal{Q}^j$ consisting of pairwise disjoint dyadic squares, exceptional sets $E_i, R_i \subset Q_{\mu_0}$ with $|R_i| \le \frac{\varepsilon}{2}\sum_{j=0}^i 2^{-j}$, a set $S_i \subset J_u$ and pairwise disjoint, closed sets $(Z^l_i)_{l \ge i+8}$ such that

$$\begin{aligned}
&(i) \quad \mathcal{H}^1\Big(\bigcup\nolimits_{j=1}^\infty \partial P^i_j \setminus \partial Q_{\mu_0}\Big) \le C_1 \mathcal{H}^1(S_i),\\
&(ii) \quad Q \subset Q_{\mu_0} \setminus S_i \text{ for all } Q \in \mathcal{C}_i, \qquad \overline{Q} \subset Q_{\mu_0} \setminus S_i \text{ for all } Q \subset \bigcup\nolimits_{l \ge i+8} Z^l_i,\\
&(iii) \quad \bigcup\nolimits_{l \ge i+8}\bigcup\nolimits_k \partial X^{l,i}_k \subset \bigcup\nolimits_{j=1}^\infty \partial P^i_j
\end{aligned} \tag{5.10}$$

for some $C_1 = C_1(\theta,p) > 0$, where $(X^{l,i}_k)_k$ denote the connected components of Z^l_i. Each set Z^l_i is a union of squares in $\mathcal{Q}^l \cap \mathcal{C}_i$ up to a set of measure zero. Moreover, we have for $l \ge i+8$

$$\begin{aligned}
&(i) \quad |E_i| \le C_1 t_i \mu_0,\\
&(ii) \quad |Z^l_i| \le C_1 \theta^{-rl} t_l \mu_0,\\
&(iii) \quad d(X^{l,i}_k) \le \theta^{-rl} t_l \quad \text{for all } X^{l,i}_k,\\
&(iv) \quad |Q \cap E_i| \le c\theta|Q| \quad \text{for } Q \in \mathcal{C}_i \text{ with } \ Q \not\subset R_i, \quad Q \cap \bigcup\nolimits_{l \ge i+8} Z^l_i = \emptyset.
\end{aligned} \tag{5.11}$$

For each P^i_j we set $P^i_{j,\mathrm{cov}} = \bigcup_{Q \in \mathcal{Q}(P_j;\mathcal{C}_i,R_i)} Q$ with $\mathcal{Q}(P_j;\mathcal{C}_i,R_i)$ as defined in (4.88). We suppose that there is a decomposition $Q_{\mu_0} = \bigcup_{l=0}^i D^i_l$ of Q_{μ_0} satisfying

$$|D^i_l| \le C_2 \theta^{-rl} t_l \mu_0 \quad \text{for all } 0 \le l \le i \tag{5.12}$$

for some $C_2 = C_2(\theta, p) > 0$, and that for each P^i_j there is $A^i_j \in \mathbb{R}^{2\times 2}_{\rm skew}$ such that for all $0 \le l \le i$

$$\begin{aligned} &(i) \quad \sum\nolimits_{j=1}^{\infty} \|\nabla u - A^i_j\|^{p'}_{L^{p'}((P^i_{j,{\rm cov}} \cap D^i_l)\setminus E_i)} \le C_3 \theta^{-\lambda l} \mu_0^{2-p'} \|e(u)\|^{p'}_{L^2(Q_{\mu_0})}, \\ &(ii) \quad \#\{P^i_{j,{\rm cov}} : x \in P^i_{j,{\rm cov}}\} \le N_0 \quad \text{for all} \quad x \in Q_{\mu_0}, \end{aligned} \tag{5.13}$$

for some $N_0 \in \mathbb{N}$ and a constant $C_3 = C_3(\theta, p) > 0$ large enough, which will eventually be specified in (5.24) and (5.27).

Let us briefly comment on the conditions (5.10)-(5.13). Properties of the form (5.10)-(5.11) appeared already in (4.90)-(4.91) and we refer to the discussion below Lemma 4.13 for more details. Moreover, (5.13) is similar to (4.92) with an additional presence of the decomposition $(D^i_l)^i_{l=0}$. As motivated above, the sets $(D^i_l)^i_{l=0}$ represent regions where Korn-type estimates have been applied at most l times. Note that the constant $C_3\theta^{-\lambda l}$ in (5.13)(i) blows up for $l \to \infty$. This, however, will eventually be compensated by using a Hölder-type estimate and the fact that $|D^i_l| \to 0$ for $l \to \infty$.

Step $i = 0$: To see that the hypothesis holds for $i = 0$, we apply Lemma 4.13 on Q_{μ_0} with $\eta = \frac{\varepsilon}{2}$ to find a partition $Q_{\mu_0} = \bigcup_{j=0}^{n_0} P^0_j$ and define $\mathcal{C}_0 = \mathcal{C}_u$, $E_0 = E_u$, $R_0 = R_u$, $Z^l_0 = Z^l_u$ for $l \ge 8$ as well as $S_0 := J_u \setminus \bigcup_{Q \in \mathcal{C}_0} Q'$. (Observe that Lemma 4.13 is applicable as $\mathcal{H}^1(J_u) \le 2\sqrt{2}\theta^{-2}\mu_0$ by (5.1).)

Since $\eta = \frac{\varepsilon}{2}$, we get $|R_0| \le \frac{\varepsilon}{2}$ and (5.10)-(5.11) follow from (4.90)-(4.91) (for $\mu = \mu_0$ and $s_l = t_l = \mu_0\theta^l$). Moreover, (5.13) is a consequence of (4.92) with $D^0_0 := Q_{\mu_0}$ and p' in place of p if we choose C_3 and N_0 large enough.

Step II (Induction step: Application of Lemma 4.13): We now pass from step $i-1$ to i. In Step II we apply Lemma 4.13 on each square of the covering. In Step III - Step V we then define all relevant notions and show (5.10)-(5.13) for iteration step i. The reader may also first proceed with the conclusion in Step VI in order to see how the result can be deduced from the induction hypothesis.

For each $Q \in \mathcal{C}_{i-1}$ we apply Lemma 4.13 with p' in place of p, $\eta = \varepsilon 2^{-i-1}|Q_{\mu_0}|^{-1}|Q|$ and $\mu = t_k$ with $k \ge i$ such that $Q \in \mathcal{Q}^k$. Note that Lemma 4.13 is applicable since by (5.1) we have $\mathcal{H}^1(Q \cap J_u) \le \theta^{-1} d(Q) = 2\sqrt{2}\theta^{-1} t_k = 2\sqrt{2}\theta^{-1}\mu$. We obtain a partition $Q = \bigcup_{j=0}^{n_Q} P^Q_j$ up to a set of negligible measure satisfying (4.89), a set $R^Q \subset Q$ with

$$|R^Q| \le \varepsilon 2^{-i-1} |Q_{\mu_0}|^{-1}|Q|, \tag{5.14}$$

an exceptional set E^Q, a covering $\mathcal{C}^Q$ with $\mathcal{C}^Q \subset \bigcup_{j \ge i+1} \mathcal{Q}^j$, the set $S^Q = (J_u \cap Q) \setminus \bigcup_{\hat{Q} \in \mathcal{C}^Q} \hat{Q}'$, sets $(Z^l_Q)_{l \ge 8}$, and infinitesimal rigid motions $(a^Q_j)^{n_Q}_{j=0}$ such that (4.90)-(4.92) hold (with $s_l = t_k\theta^l$ for $l \in \mathbb{N}$ and $\mu = t_k$). Recall particularly that, depending on the case in (4.89), a partition of Q is formed either by $\bigcup_{j=1}^{n_Q} P^Q_j$ (square of *type 1*) or by P^Q_0 (square of *type 2*). In the following we will use the notions introduced here without further reference to Step II.

Step III (Induction step: Partition and covering): In this step we construct the partition and the covering and we confirm (5.10)(i),(ii). For the underlying ideas we refer to (d),(d') in the description of the iteration scheme.

Decomposition of the covering: We split up the covering $\mathcal{C}_{i-1}$ into different sets as follows. Let $\mathcal{C}'_{i-1}$ be the squares $Q \in \mathcal{C}_{i-1}$ with $P^Q_0 = \emptyset$, define

$$\mathcal{C}'''_{i-1} := \big\{Q \in \mathcal{C}_{i-1} \setminus \mathcal{C}'_{i-1} : Q \subset \bigcup\nolimits_{l \ge i+7} Z^l_{i-1}, \ \ \mathcal{H}^1(J_u \cap \partial Q) \ge \theta d(Q)\big\} \tag{5.15}$$

and $\mathcal{C}''_{i-1} = \mathcal{C}_{i-1} \setminus (\mathcal{C}'_{i-1} \cup \mathcal{C}'''_{i-1})$. The squares $\mathcal{C}'_{i-1}$ are of type 1. The squares $\mathcal{C}''_{i-1}$ are of type 2, i.e., the associated partition is trivial (see Lemma 4.13 (2) and Remark 4.14). We remark that the introduction of $\mathcal{C}'''_{i-1}$ is only a technical point needed for the application of a trace estimate in Step V below. The core of the argument lies in the separation of $\mathcal{C}'_{i-1}$ and $\mathcal{C}''_{i-1}$. For the construction of the partition as described in (d),(d') above, it is convenient to treat the squares $\mathcal{C}'''_{i-1}$ as squares of type 1 although the associated partition is trivial.

Definition of the partition: Recall that the connected components $(X_k^{l,i-1})_{k\ge1}$ of Z^l_{i-1} consist of squares $\mathcal{Q}^l \cap \mathcal{C}_{i-1}$, i.e. $\{Q \in \mathcal{Q}^l : Q \subset X_k^{l,i-1}\} = \{Q \in \mathcal{C}_{i-1} : Q \subset X_k^{l,i-1}\}$. Then by $\mathcal{P}^l_k$ we denote the connected components of the set

$$Y^l_k := \mathrm{int}\Big(\bigcup\nolimits_{Q\in\mathcal{C}''_{i-1}, Q\subset X_k^{l,i-1}} \overline{Q}\Big) = \mathrm{int}\Big(\bigcup\nolimits_{Q\in\mathcal{Q}^l\setminus(\mathcal{C}'_{i-1}\cup\mathcal{C}'''_{i-1}), Q\subset X_k^{l,i-1}} \overline{Q}\Big). \tag{5.16}$$

Note that by (4.90)(i) we have $\mathcal{H}^1(\partial Q) \le \mathcal{H}^1(\bigcup_{j=1}^{n_Q}\partial P_j^Q) \le C\mathcal{H}^1(S^Q)$ for all $Q \in \mathcal{C}'_{i-1}$. Then

$$\begin{aligned}\sum\nolimits_{P\in\mathcal{P}^l_k}\mathcal{H}^1(\partial P\setminus \partial X_k^{l,i-1}) &\le \sum\nolimits_{Q\in\mathcal{C}'_{i-1}\cup\mathcal{C}'''_{i-1}, Q\subset X_k^{l,i-1}}\mathcal{H}^1(\partial Q) \\ &\le C\sum_{Q\in\mathcal{C}'_{i-1}, Q\subset X_k^{l,i-1}}\mathcal{H}^1(S^Q) + \sum_{Q\in\mathcal{C}'''_{i-1}, Q\subset X_k^{l,i-1}}\mathcal{H}^1(\partial Q).\end{aligned} \tag{5.17}$$

We now introduce the partition $\mathcal{P}^i$ for iteration step i. We set

$$\begin{aligned}&\mathcal{P}_a = \{P_j^Q : Q\in\mathcal{C}'_{i-1},\ j=1,\dots,n_Q\}\cup\{P_0^Q = Q : Q\in\mathcal{C}'''_{i-1}\},\\ &\mathcal{P}_b = \{P\in\mathcal{P}^l_k : l\ge i+7, k\ge1\}\\ &P'_j := P_j^{i-1}\setminus \bigcup\nolimits_{P\in\mathcal{P}_a\cup\mathcal{P}_b}\overline{P} \quad\text{for all}\quad P_j^{i-1}\in\mathcal{P}^{i-1}, \quad \mathcal{P}_c = (P'_j)_{j=1}^\infty.\end{aligned} \tag{5.18}$$

Define $\mathcal{P}^i = \mathcal{P}_a\cup\mathcal{P}_b\cup\mathcal{P}_c$ and denote the sets also by $\mathcal{P}^i = (P^i_j)_{j=1}^\infty$. We observe that $\mathcal{P}^i$ is a partition of Q_{μ_0} up to a set of negligible measure and that the sets in $\mathcal{P}_c$ do not intersect $\bigcup_{l\ge i+7} Z^l_{i-1}$. We define the covering $\mathcal{C}_i = \bigcup_{Q\in\mathcal{C}_{i-1}}\mathcal{C}^Q \subset \bigcup_{j\ge i+1}\mathcal{Q}^j$ and now show (5.10)(i),(ii).

Proof of (5.10)*(i):* By (4.90)(i) we have $\mathcal{H}^1(\bigcup_{j=1}^{n_Q}\partial P_j^Q) \le C\mathcal{H}^1(S^Q)$ for all $Q\in\mathcal{C}'_{i-1}$. Moreover, by (5.15) with $\hat{S}^Q := J_u\cap\partial Q$ we find $\mathcal{H}^1(\partial P_0^Q) = \mathcal{H}^1(\partial Q) = 2\sqrt{2}d(Q) \le 2\sqrt{2}\theta^{-1}\mathcal{H}^1(\hat{S}^Q)$ for $Q\in\mathcal{C}'''_{i-1}$. Consequently, applying (5.10)(iii) for step $i-1$ as well as (5.17) we get

$$\begin{aligned}\mathcal{H}^1\Big(\bigcup_{j=1}^\infty\partial P^i_j\setminus\big(\bigcup_{j=1}^\infty\partial P_j^{i-1}\cup\partial Q_{\mu_0}\big)\Big) &\le \mathcal{H}^1\Big(\bigcup_{P\in\mathcal{P}_a}\partial P\Big) + \mathcal{H}^1\Big(\bigcup_{P\in\mathcal{P}_b}\partial P\setminus\bigcup_{l\ge i+7}\bigcup_k\partial X_k^{l,i-1}\Big)\\ &\le C\sum\nolimits_{Q\in\mathcal{C}'_{i-1}}\mathcal{H}^1(S^Q) + C\sum\nolimits_{Q\in\mathcal{C}'''_{i-1}}\mathcal{H}^1(\hat{S}^Q).\end{aligned}$$

Letting $S_i = S_{i-1}\cup S^*_i$ with $S^*_i := \bigcup_{Q\in\mathcal{C}'_{i-1}} S^Q \cup\bigcup_{Q\in\mathcal{C}'''_{i-1}}\hat{S}^Q$ we obtain $S_i\subset J_u$ and confirm (5.10)(i) as follows. Since $S^Q\subset Q\subset Q_{\mu_0}\setminus S_{i-1}$ for all $Q\in\mathcal{C}'_{i-1}$ and $\hat{S}^Q\subset\overline{Q}\subset Q_{\mu_0}\setminus S_{i-1}$ for all $Q\in\mathcal{C}'''_{i-1}$ by (5.10)(ii) for step $i-1$, we get $S_{i-1}\cap S^*_i=\emptyset$. Moreover, each $x\in Q_{\mu_0}$ is contained in at most four different $\hat{S}^Q$. Then the claim follows from (5.10)(i) for step $i-1$ (if we choose $C_1\ge C$).

Proof of (5.10)*(ii):* Consider $Q\in\mathcal{C}_i$. First, the fact that $\mathcal{C}_i$ is a refinement of $\mathcal{C}_{i-1}$ yields some $\hat{Q}\in\mathcal{C}_{i-1}$ such that $Q\subset\hat{Q}$. Then (5.10)(ii) for step $i-1$ implies $Q\subset Q_{\mu_0}\setminus S_{i-1}$. Likewise, recalling the definition of the set S^*_i, we get $Q\cap S^*_i=\emptyset$ by (4.90)(ii) and therefore $Q\cap S_i=\emptyset$. This shows the left statement in (5.10)(ii).

Now consider the special case that $Q\subset\bigcup_{l\ge8}Z^l_{\hat{Q}}$. (Recall that this particularly implies $\hat{Q}\in\mathcal{C}'_{i-1}$, see Lemma 4.13 (2).) We get $\overline{Q}\subset\hat{Q}$ by (4.90)(ii) and thus as before by (5.10)(ii), (4.90)(ii) we also derive $\overline{Q}\subset Q_{\mu_0}\setminus S_{i-1}$ and $\overline{Q}\cap S^*_i=\emptyset$. Anticipating the definition of $(Z^l_i)_{l\ge i+8}$ from (5.19), we see that the right statement in (5.10)(ii) holds.

Step IV (Induction step: Exceptional sets and isolated components): In this step we define the other relevant sets and show (5.10)(iii), (5.11)-(5.12). Let $R_i = R_{i-1} \cup \bigcup_{Q \in \mathcal{C}_{i-1}} R^Q$ and observe that $|R_i| \le \frac{\varepsilon}{2} \sum_{j=0}^i 2^{-j}$ by (5.14). For $l \ge i+8$ we set

$$Z_i^l = \bigcup\nolimits_{k=i}^{l-8} \bigcup\nolimits_{Q \in \mathcal{C}'_{i-1} \cap \mathcal{Q}^k} Z_Q^{l-k}, \qquad E_i = \bigcup\nolimits_{Q \in \mathcal{C}_{i-1}} E^Q. \tag{5.19}$$

Each Z_i^l is a union of squares in $\mathcal{Q}^l \cap \mathcal{C}_i$ up to a set of negligible measure since, as noted below (4.90), for each $Q \in \mathcal{C}_{i-1} \cap \mathcal{Q}^k$ the sets Z_Q^j consist of squares with sidelength $2s_j$, where according to the notation in the previous section we have $s_j = t_k\theta^j$ for $j \in \mathbb{N}$.

Proof of (5.10)*(iii):* From (4.90)(iii) we get that $\bigcup_{l \ge 8} \partial Z_Q^l \subset \bigcup_{j=1}^{n_Q} \partial P_j^Q$ for each $Q \in \mathcal{C}'_{i-1}$. Thus, property (5.10)(iii) follows from the definition of $\mathcal{P}_a$ in (5.18).

Proof of (5.11)*:* By (5.19) we get for each square $Q \in \mathcal{C}_{i-1} \cap \mathcal{Q}^k$, $k \ge i$, and $l \ge i+8$ that $Z_i^l \cap Q = Z_Q^{l-k}$ and thus by (4.91)(iv)

$$|Z_i^l \cap Q| \le C\theta^{-(l-k)r} s_{l-k} \mathcal{H}^1(J_u \cap Q) = C t_k \theta^{(l-k)(1-r)} \mathcal{H}^1(J_u \cap Q) \le C t_l \theta^{-rl} \mathcal{H}^1(J_u \cap Q),$$

where as before we have $s_j = t_k\theta^j$ for $j \in \mathbb{N}$. Then taking the sum over all squares we derive (5.11)(ii), where we use the fact that $\mathcal{H}^1(J_u) \le 2\sqrt{2}\theta^{-2}\mu_0$. Likewise, (5.11)(iii) follows from (4.91)(iii) and a similar calculation. Moreover, (4.91)(ii) yields (5.11)(iv) by the definition in (5.19) and the fact that $R^Q \subset R_i$. Finally, to see (5.11)(i), we recall that for all $Q \in \mathcal{C}_{i-1}$

$$|E^Q| \le c\theta^2 d(Q) \mathcal{H}^1(J_u \cap Q) \le C t_i \mathcal{H}^1(J_u \cap Q)$$

by (4.91)(i). Consequently, in view of (5.19) and $\mathcal{H}^1(J_u) \le 2\sqrt{2}\theta^{-2}\mu_0$, the result follows when we take the sum over all $Q \in \mathcal{C}_{i-1}$ (if we choose $C_1 \ge C$).

Decomposition $(D_l^i)_l$*:* Recalling the definition of the partition in (5.18), we define

$$D_i^i = E_{i-1} \cup \bigcup\nolimits_{P \in \mathcal{P}_a \cup \mathcal{P}_b} P \quad \text{and} \quad D_l^i = D_l^{i-1} \setminus D_i^i \quad \text{for} \quad 0 \le l \le i-1, \tag{5.20}$$

and note that $(D_l^i)_{l=0}^i$ is a decomposition of Q_{μ_0}. As motivated in the description of the iteration scheme, D_i^i represents the region of the domain where we will apply estimates of Korn-type in the current iteration step. We conclude this step with the proof of (5.12). First, (5.12) for $0 \le l \le i-1$ follows directly. Note that each $Q \in \mathcal{C}_{i-1}$ satisfies $|Q| = \frac{1}{16}\mathcal{H}^1(\partial Q)^2 \le \frac{1}{2} t_i \mathcal{H}^1(\partial Q) \le t_i \mathcal{H}^1(\partial Q \setminus \partial Q_{\mu_0})$. Recalling (5.10)(i), (5.18) and $\mathcal{H}^1(J_u) \le 2\sqrt{2}\theta^{-2}\mu_0$ we compute

$$\begin{aligned} \big|\bigcup\nolimits_{P \in \mathcal{P}_a} P\big| &= \big|\bigcup\nolimits_{Q \in \mathcal{C}'_{i-1} \cup \mathcal{C}'''_{i-1}} Q\big| \le t_i \sum\nolimits_{Q \in \mathcal{C}'_{i-1} \cup \mathcal{C}'''_{i-1}} \mathcal{H}^1(\partial Q \setminus \partial Q_{\mu_0}) \\ &\le C_1 t_i \mathcal{H}^1(S_i) \le C_1 t_i \mathcal{H}^1(J_u) \le CC_1 t_i \mu_0. \end{aligned}$$

Likewise, by (5.18) and (5.11)(ii) we see $|\bigcup_{P \in \mathcal{P}_b} P| \le \sum_{l \ge i+7} |Z_{i-1}^l| \le CC_1\theta^{-ri} t_i \mu_0$. This and the estimate for $|E_{i-1}|$ in (5.11)(i) yields (5.12) for D_i^i if one chooses $C_2 = C_2(C_1)$ large enough.

Step V (Induction step: Korn inequalities): In the previous steps we have already defined the partition in (5.18), the exceptional set and the isolated components (see (5.19)). We also recall the definition of the squares $\mathcal{C}'_{i-1}, \mathcal{C}''_{i-1}, \mathcal{C}'''_{i-1}$ in (5.15) and the decomposition in (5.20). It remains to define matrices $(A_j^i)_{j=1}^\infty \subset \mathbb{R}^{2\times 2}_{\rm skew}$ and to show (5.13). For the underlying ideas we refer to (e),(e') in the description of the iteration scheme.

We recall that by (4.92) we have for all $Q \in \mathcal{C}_{i-1}$

$$\sum\nolimits_{j=0}^{n_Q} \|\nabla u - A_j^Q\|^{p'}_{L^{p'}(P^Q_{j,\rm cov} \setminus E^Q)} \le C(d(Q))^{2-p'} \|e(u)\|^{p'}_{L^2(Q)} \tag{5.21}$$

for suitable $(a_j^Q)_j = (a_{A_j^Q, b_j^Q})_j$, where $P^Q_{j,\mathrm{cov}} = \bigcup_{\hat{Q} \in \mathcal{Q}(P_j^Q; \mathcal{C}^Q, R^Q)} \hat{Q}$ is defined below (4.88). (Note that for squares of type 2 the estimate simplifies as described in Remark 4.14.)

Isolated components: As mentioned above in (e'), we need a special construction inside of isolated components, which we discuss here as a preparatory step. Fix some $X_k^{l,i-1}$ with $l \geq i+7$ and recalling (5.16) we consider a corresponding connected component $P \in \mathcal{P}_k^l$. By definition there are squares $\mathcal{Q}(P) \subset \{Q \in \mathcal{Q}^l : Q \subset X_k^{l,i-1}\}$ such that $P = \mathrm{int}(\bigcup_{Q \in \mathcal{Q}(P)} \overline{Q})$.

As $Q \in \mathcal{C}''_{i-1}$ for $Q \in \mathcal{Q}(P)$, a single infinitesimal rigid motion a_0^Q is given such that (5.21) holds with $P^Q_{0,\mathrm{cov}} = Q$ (see Lemma 4.13 (2) and Remark 4.14) and by (4.93)

$$\int_{\partial Q \setminus \Gamma^Q} |Tu - a_0^Q|^2 \, d\mathcal{H}^1 \leq c t_l \|e(u)\|^2_{L^2(Q)}$$

for a set Γ^Q with $\mathcal{H}^1(\Gamma^Q) \leq c\theta t_l$. We can estimate the difference of the infinitesimal rigid motions on adjacent squares as follows. Consider $Q_1, Q_2 \in \mathcal{Q}(P)$ such that ∂Q_1 and ∂Q_2 have a common edge. Then the previous estimate implies with $\Gamma := \Gamma^{Q_1} \cup \Gamma^{Q_2} \cup (J_u \cap \partial Q_1 \cap \partial Q_2)$

$$\int_{(\partial Q_1 \cap \partial Q_2) \setminus \Gamma} |a_0^{Q_1} - a_0^{Q_2}|^2 \, d\mathcal{H}^1 \leq c \sum\nolimits_{k=1,2} \int_{\partial Q_k \setminus \Gamma} |Tu - a_0^{Q_k}|^2 \, d\mathcal{H}^1 \leq c t_l \|e(u)\|^2_{L^2(Q_1 \cup Q_2)},$$

where we used that the traces Tu on ∂Q_1 and ∂Q_2 coincide on $(\partial Q_1 \cap \partial Q_2) \setminus \Gamma$. By (5.15) we get $\mathcal{H}^1(\Gamma) \leq c\theta t_l$ as $Q_1, Q_2 \notin \mathcal{C}'''_{i-1}$. Therefore, by Remark 3.8 we get for θ small $t_l^2 |A_0^{Q_1} - A_0^{Q_2}|^2 \leq C\|e(u)\|^2_{L^2(Q_1 \cup Q_2)}$. Consequently, recalling that P is connected, $\#\mathcal{Q}(P) \leq c\theta^{-2rl}$ by (5.11)(iii) and using the discrete Hölder inequality we find

$$\max_{Q_1, Q_2 \in \mathcal{Q}(P)} t_l^2 |A_0^{Q_1} - A_0^{Q_2}|^{p'} \leq c t_l^{2-p'} \theta^{-2rl(p'-1)} \sum\nolimits_{Q \in \mathcal{Q}(P)} \|e(u)\|^{p'}_{L^2(Q)}.$$

Applying (5.21), $\#\mathcal{Q}(P) \leq c\theta^{-2rl}$, $p' \leq 2$ and recalling (5.19) we derive for each $\hat{Q} \in \mathcal{Q}(P)$

$$\|\nabla u - A_0^{\hat{Q}}\|^{p'}_{L^{p'}(P \setminus E_i)} \leq (C + c\theta^{-4rl}) t_l^{2-p'} \sum\nolimits_{Q \in \mathcal{Q}(P)} \|e(u)\|^{p'}_{L^2(Q)}. \tag{5.22}$$

Definition of infinitesimal rigid motions: We now define matrices associated to $\mathcal{P}^i$. Recalling the definition of the partition $\mathcal{P}^i$ in (5.18), we distinguish the following cases: (a) For $P_j^i = P_k^Q$ for some $Q \in \mathcal{C}'_{i-1}$ and $k = 1, \ldots, n_Q$ we let $A_j^i = A_k^Q$ and for $P_j^i = P_0^Q$ for some $Q \in \mathcal{C}'''_{i-1}$ we let $A_j^i = A_0^Q$ as given by (5.21). (b) If $P_j^i \in \mathcal{P}_k^l$, we set $A_j^i = A_0^{\hat{Q}}$ for an (arbitrary) $\hat{Q} \in \mathcal{Q}(P_j^i)$, see also (5.22). (c) Finally, for all P_j^i with $P_j^i = P'_k$ for some $P_k^{i-1} \in \mathcal{P}^{i-1}$ we let $A_j^i = A_k^{i-1}$.

Proof of (5.13)*(i):* We first consider the sets $(D_l^i)_{l=0}^{i-1}$. By the definition in (5.20), we see $D_l^i \subset \bigcup_{P \in \mathcal{P}_c} P$ for $0 \leq l \leq i-1$. Consider a component $P_j^i = P'_k \in \mathcal{P}_c$. By (5.18) note $P'_k \subset P_k^{i-1}$ with $P_k^{i-1} \in \mathcal{P}^{i-1}$. Recall the definition of the sets $P^i_{j,\mathrm{cov}}, P^{i-1}_{k,\mathrm{cov}}$ before (5.12). As $\mathcal{C}_i$ is a refinement of $\mathcal{C}_{i-1}$ and $R_i \supset R_{i-1}$, we obtain

$$P^i_{j,\mathrm{cov}} \subset P_j^* := \bigcup \{Q \in \mathcal{C}_{i-1} : \; Q \cap P_j^i \neq \emptyset, \; Q \subset P^{i-1}_{k,\mathrm{cov}}\} \subset P^{i-1}_{k,\mathrm{cov}}. \tag{5.23}$$

Recall also from (5.20) that $D_l^i \setminus E_i \subset D_l^{i-1} \setminus E_{i-1}$ for $0 \leq l \leq i-1$. Consequently, (5.13)(i) for $0 \leq l \leq i-1$ follows directly from the corresponding estimates in step $i-1$ and (5.23).

It remains to consider the set D_i^i as defined in (5.20). It suffices to show that for $k = a, b, c$

$$\sum\nolimits_{P_j^i \in \mathcal{P}_k} \|\nabla u - A_j^i\|^{p'}_{L^{p'}((P^i_{j,\mathrm{cov}} \cap D_i^i) \setminus E_i)} \leq \tfrac{1}{3} C_3 \theta^{-\lambda i} \mu_0^{2-p'} \|e(u)\|^{p'}_{L^2(Q_{\mu_0})}. \tag{5.24}$$

As a preparation, we recall that the discrete Hölder inequality together with $\#\{Q \in \mathcal{C}_i : Q \subset Z_i^l\} \le t_l^{-2}|Z_i^l| \le C_1 t_l^{-1}\theta^{-rl}\mu_0$ (see (5.11)(ii)), $r = \frac{1}{24}(2-p)$, and $t_l = \mu_0\theta^l$ implies

$$
\begin{aligned}
&(i) \quad \sum\nolimits_{Q\in\mathcal{C}_{i-1}} (d(Q))^{2-p'}\|e(u)\|^{p'}_{L^2(Q)} \le c\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})},\\
&(ii) \quad t_l^{2-p'}\theta^{-4rl}\sum\nolimits_{Q\in\mathcal{C}_i : Q\subset Z_i^l}\|e(u)\|^{p'}_{L^2(Q)} \le C_1\theta^{-4rl}(\theta^{-rl}t_l\mu_0)^{1-p'/2}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le C_1\theta^{rl}\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})}.
\end{aligned}
\tag{5.25}
$$

For (i) we also refer to (4.96) and for (ii) to (4.68)-(4.69) for similar arguments, where we particularly use $1 - \frac{p'}{2} = \frac{1}{2}(1-\frac{p}{2}) = 6r$ and $t_l = \mu_0\theta^l$.

Proof of (5.24) *for* $\mathcal{P}_a$*:* For the components $\mathcal{P}_a$ the property in (5.24) follows directly from (5.21) and (5.25)(i) provided that C_3 is chosen large enough, where in this case the additional factor $\theta^{-\lambda i}$ is not needed. In fact, for all $P_j^i \in \mathcal{P}_a$ we have $P_j^i = P_k^Q$ for some $Q \in \mathcal{C}'_{i-1} \cup \mathcal{C}'''_{i-1}$ and $P^i_{j,\mathrm{cov}} \subset P^Q_{k,\mathrm{cov}}$ due to the definition of $\mathcal{C}_i$ and $R_i \supset R^Q$ with $P^i_{j,\mathrm{cov}}$ as defined before (5.12).

Proof of (5.24) *for* $\mathcal{P}_b$*:* Likewise, for sets $P_j^i \in \mathcal{P}_b$ we get (5.24) by (5.22) and (5.25)(ii) using $P^i_{j,\mathrm{cov}} \subset \bigcup_{Q\in\mathcal{Q}(P_j^i)} Q \subset P_j^i$ as well as the fact that the sets $\mathcal{Q}(P)$ for $P \in \mathcal{P}_b$ are pairwise disjoint. Note that again the additional factor $\theta^{-\lambda i}$ is not needed, but the choice of C_3 also depends on C_1 and $\sum_l \theta^{rl} < +\infty$. (A very similar computation has been performed in (4.69).)

Proof of (5.24) *for* $\mathcal{P}_c$*:* Consider a component $P_j^i = P'_k \in \mathcal{P}_c$ and note $P'_k \subset P_k^{i-1}$ with $P_k^{i-1} \in \mathcal{P}^{i-1}$. Recall (5.23). Fix $Q \in \mathcal{C}_{i-1} \cap \mathcal{Q}^l$ with $Q \subset P_j^* \subset P^{i-1}_{k,\mathrm{cov}}$. As $P_j^i \cap Q \neq \emptyset$, (5.18) implies $Q \cap Z^l_{i-1} = \emptyset$ and that Q is a square of type 2, i.e., $P_0^Q = P^Q_{0,\mathrm{cov}} = Q$. We observe

$$(i) \quad |E_{i-1} \cap Q| \le c\theta t_l^2, \qquad (ii) \quad |E^Q| \le c\theta t_l^2.$$

In fact, (i) follows from (5.11)(iv) for step $i-1$ and the fact that $Q \not\subset R_{i-1}$ since $Q \subset P^{i-1}_{k,\mathrm{cov}}$ (cf. (4.88)). By (4.91)(i) and (5.1) we obtain (ii). Thus, for θ sufficiently small with respect to c we obtain $|Q \setminus (E_{i-1} \cup E^Q)| \ge \frac{1}{2}|Q|$ and thus by the triangle inequality

$$|Q||A_0^Q - A_k^{i-1}|^{p'} \le 4\|\nabla u - A_0^Q\|^{p'}_{L^{p'}(Q\setminus(E^Q\cup E_{i-1}))} + 4\|\nabla u - A_k^{i-1}\|^{p'}_{L^{p'}(Q\setminus(E^Q\cup E_{i-1})},$$

where A_0^Q is the matrix given in (5.21) for the square Q, which is of type 2. By (5.19) and the triangle inequality we find

$$
\begin{aligned}
\|\nabla u - A_k^{i-1}\|^{p'}_{L^{p'}(Q\setminus E_i)} &\le 2\|\nabla u - A_0^Q\|^{p'}_{L^{p'}(Q\setminus E^Q)} + 2|Q||A_0^Q - A_k^{i-1}|^{p'}\\
&\le 10\|\nabla u - A_0^Q\|^{p'}_{L^{p'}(Q\setminus E^Q)} + 8\|\nabla u - A_k^{i-1}\|^{p'}_{L^{p'}(Q\setminus E_{i-1})},
\end{aligned}
$$

and therefore, using (5.21) and recalling $P^Q_{0,\mathrm{cov}} = Q$, we derive

$$\|\nabla u - A_k^{i-1}\|^{p'}_{L^{p'}(Q\setminus E_i)} \le C(d(Q))^{2-p'}\|e(u)\|^{p'}_{L^2(Q)} + 8\|\nabla u - A_k^{i-1}\|^{p'}_{L^{p'}(Q\setminus E_{i-1})}. \tag{5.26}$$

We now confirm (5.24). Note that each $Q \in \mathcal{C}_{i-1}$ is contained in at most N_0 different P_j^* by (5.23) and (5.13)(ii) for step $i-1$. Summing over all squares $Q \subset P_j^*$ and all $P_j^i = P'_k \in \mathcal{P}_c$, we deduce using (5.13)(i) for step $i-1$ as well as (5.23), (5.25)(i), and (5.26)

$$\begin{aligned}
\sum\nolimits_{P^i_j = P'_k \in \mathcal{P}_c} \|\nabla u - A^{i-1}_k\|^{p'}_{L^{p'}((P^i_{j,\mathrm{cov}} \cap D^i_i)\setminus E_i)} &\le \sum\nolimits_{P^i_j = P'_k \in \mathcal{P}_c} \|\nabla u - A^{i-1}_k\|^{p'}_{L^{p'}((P^*_j \cap D^i_i)\setminus E_i)} \\
&\le \sum\nolimits_{P^i_j = P'_k \in \mathcal{P}_c} \sum\nolimits_{Q \in \mathcal{C}_{i-1}, Q \subset P^*_j} \big(C(d(Q))^{2-p'}\|e(u)\|^{p'}_{L^2(Q)} + 8\|\nabla u - A^{i-1}_k\|^{p'}_{L^{p'}(Q\setminus E_{i-1})}\big) \\
&\le N_0 \cdot C\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})} + 8\sum\nolimits_{P^{i-1}_k \in \mathcal{P}^{i-1}} \|\nabla u - A^{i-1}_k\|^{p'}_{L^{p'}(P^{i-1}_{k,\mathrm{cov}}\setminus E_{i-1})} \\
&\le (CN_0 + 8C_3\theta^{-(i-1)\lambda})\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})} \le \tfrac{1}{3}C_3\theta^{-i\lambda}\mu_0^{2-p'}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})}, \qquad (5.27)
\end{aligned}$$

where the last step holds for $C_3 = C_3(\theta, p)$ large enough and θ small (depending on λ) such that $CN_0 \le \frac{1}{6}C_3$ and $8\theta^\lambda \le \frac{1}{6}$. This concludes the proof of (5.24).

Proof of (5.13)*(ii):* It suffices to treat the cases (a)-(c) separately. Indeed, by the definition of $\mathcal{C}_i$ we have $P^i_{j_1,\mathrm{cov}}$ and $P^i_{j_2,\mathrm{cov}}$ are disjoint if they lie in different sets $\mathcal{P}_a, \mathcal{P}_b, \mathcal{P}_c$. For $\mathcal{P}_a$ the desired property follows directly from (4.92)(ii). For $\mathcal{P}_b$ it is obvious as the sets $P^i_{j,\mathrm{cov}} \subset P^i_j \in \mathcal{P}_b$ are pairwise disjoint. Finally, by (5.23) for each $P^i_j \in \mathcal{P}_c$ we have a (different) P^{i-1}_k with $P^i_{j,\mathrm{cov}} \subset P^{i-1}_{k,\mathrm{cov}}$ and the property follows from (5.13)(ii) for step $i-1$.

Step VI (Conclusion): We finally define the partition and the infinitesimal rigid motions such that the assertion of Theorem 2.1 holds. For this step of the proof we need the properties stated in Step I. As $\limsup_{i\to\infty}|E_i| = 0$ by (5.11)(i), we can choose $I \in \mathbb{N}$ so large that $|E_I| \le \varepsilon$. By $\mathcal{P}^I = (P^I_j)_{j=1}^\infty$ we denote the partition given in iteration step I and get by (5.10)(i) that

$$\sum\nolimits_{j=1}^\infty \mathcal{H}^1(\partial P^I_j \setminus \partial Q_{\mu_0}) \le C\mathcal{H}^1(S_I) \le C\mathcal{H}^1(J_u), \qquad (5.28)$$

where in the last step we used $S_I \subset J_u$. In view of the definition (4.88), we get

$$\sum\nolimits_{j=1}^\infty |P^I_j \setminus P^I_{j,\mathrm{cov}}| \le \sum\nolimits_{j=1}^\infty \sum\nolimits_{Q\in\mathcal{C}_I} |P^I_j \cap Q \cap R_I| \le |R_I| \le \varepsilon.$$

Thus, letting $E' = E_I \cup \bigcup_{j=1}^\infty (P^I_j \setminus P^I_{j,\mathrm{cov}})$ we get $|E'| \le 2\varepsilon$. We set $\mathcal{P}^* = \{P^I_j \in \mathcal{P}^I : |P^I_j \setminus E'| \ge \frac{1}{2}|P^I_j|\}$ and define

$$v' := u - \sum\nolimits_{P^I_j \in \mathcal{P}^*} (A^I_j\, x)\chi_{P^I_j}, \qquad E'' = E' \cup \bigcup\nolimits_{P^I_j \notin \mathcal{P}^*} P^I_j, \qquad (5.29)$$

where $(A^I_j)_j \subset \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ are the matrices corresponding to $(P^I_j)_j$ (cf. (5.13)). We now show (2.2)(ii) for v'. We deduce from (5.13) and the definitions of E', E'' that

$$\|\nabla v'\|^{p'}_{L^{p'}((Q_{\mu_0}\setminus E'')\cap D^I_l)} \le C\mu_0^{2-p'}\theta^{-l\lambda}\|e(u)\|^{p'}_{L^2(Q_{\mu_0})} \qquad (5.30)$$

for all $0 \le l \le I$ for a constant $C = C(\theta, p) > 0$. Moreover, by (5.12)

$$|D^I_l| \le C\theta^{-rl}t_l\mu_0 \le C\theta^{l(1-r)}\mu_0^2$$

for all $0 \le l \le I$. Recall the definition $p' = 1 + \frac{p}{2}$, $r = \frac{2-p}{24}$ and $\lambda = \frac{(1-r)(2-p)}{3p+2}$. Passing from p' to p and using (5.29), (5.30) we obtain by Hölder's inequality

$$\begin{aligned}
\|\nabla v'\|^p_{L^p(Q_{\mu_0}\setminus E'')} &= \sum\nolimits_{l=0}^I \|\nabla v'\|^p_{L^p((Q_{\mu_0}\setminus E'')\cap D^I_l)} \le \sum\nolimits_{l=0}^I |D^I_l|^{1-p/p'}\|\nabla v'\|^p_{L^{p'}((Q_{\mu_0}\setminus E'')\cap D^I_l)} \\
&\le C\mu_0^{2-p}\sum\nolimits_{l\ge 0}\theta^{l(1-r)(1-p/p')-l\lambda p/p'}\|e(u)\|^p_{L^2(Q_{\mu_0})} \\
&\le C\mu_0^{2-p}\sum\nolimits_{l\ge 0}\theta^{l\lambda}\|e(u)\|^p_{L^2(Q_{\mu_0})} \le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})} \qquad (5.31)
\end{aligned}$$

with $C = C(\theta, p, \lambda) = C(\theta, p)$. Now we have to analyze the behavior in E''. To this end, we fix some $P_j^I \in \mathcal{P}^I$. If $P_j^I \in \mathcal{P}^*$, we can choose a measurable set F_j with $P_j^I \cap E' \subset F_j \subset P_j^I$ and $|F_j| = 2|P_j^I \cap E'|$. We find by $|F_j| = 2|F_j \setminus E'| = 2|F_j \setminus E''|$ and the triangle inequality

$$|F_j||A_j^I|^p \le c\|\nabla u - A_j^I\|^p_{L^p(F_j \setminus E'')} + c\|\nabla u\|^p_{L^p(F_j\setminus E'')}. \tag{5.32}$$

If $P_j^I \notin \mathcal{P}^*$, we set $F_j = P_j^I$. Then we see that $F := \bigcup_{j=1}^\infty F_j$ satisfies $|F| \le 2|E'| \le 4\varepsilon$ and thus by (5.8) we derive $\|\nabla u\|^p_{L^p(F)} \le \mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})}$. Consequently, as $E'' \subset F$ we obtain by (5.29) and (5.31)-(5.32)

$$\begin{aligned}\|\nabla v'\|^p_{L^p(Q_{\mu_0})} &\le \|\nabla v'\|^p_{L^p(Q_{\mu_0}\setminus E'')} + \|\nabla v'\|^p_{L^p(F)} \\ &\le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})} + c\|\nabla u\|^p_{L^p(F)} + c\sum\nolimits_{P_j^I\in\mathcal{P}^*}|F_j||A_j^I|^p \\ &\le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})} + c\|\nabla v'\|^p_{L^p(Q_{\mu_0}\setminus E'')} \le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})}.\end{aligned} \tag{5.33}$$

All arguments hold for a fixed θ sufficiently small and thus the constant C eventually only depends on p. To obtain (2.2)(i), we apply Theorem 2.3 on v' and $\rho = \mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q_{\mu_0})$ to get a Caccioppoli partition $(P'_j)_j$ and corresponding translations $(b_j)_j$ such that with $v := v' - \sum_j b_j\chi_{P'_j}$ by (5.28), (5.29), and Hölder's inequality

$$\begin{aligned}&\sum\nolimits_j \mathcal{H}^1(\partial^* P'_j) \le \mathcal{H}^1(J_{v'}) + \mathcal{H}^1(\partial Q_{\mu_0}) + c\rho \le C(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q_{\mu_0})),\\ &\|v\|_{L^\infty(Q_{\mu_0})} \le c\rho^{-1}\|\nabla v'\|_{L^1(Q_{\mu_0})} \le c\mu_0\rho^{-1}\|e(u)\|_{L^2(Q_{\mu_0})}.\end{aligned}$$

(For the proof of Theorem 2.3, which is completely independent from the arguments in Section 4 and Section 5, we refer to Section 6.1 below.) Let $(P_j)_j$ be the Caccioppoli partition consisting of the sets $(P_i^I \cap P'_j)_{i,j}$ and observe that (2.1) follows. Choosing corresponding $(a_j)_j$ such that $v = u - \sum_j a_j\chi_{P_j}$, we obtain (2.2) as $\nabla v' = \nabla v$. Finally, since $u \in \mathcal{W}(Q_{\mu_0})$, we also have that $v \in SBV^2(Q_{\mu_0}) \cap L^\infty(Q_{\mu_0})$. □

We now drop assumption (5.1). The idea is to add the boundary of the squares violating (5.1) to the boundary of the partition. As in each of these squares the jump set is large, a careful bookkeeping of their number eventually shows that the length of the added boundary can be controlled, cf. (5.40) below. On each of these squares we then apply separately Theorem 5.1. For the following proof we introduce the notation $\mathcal{F}^k := \{F = \mathrm{int}(\bigcup_{Q\in\mathcal{Q}_F}\overline{Q}) : \mathcal{Q}_F \subset \mathcal{Q}^k\}$ for $k \in \mathbb{N}$.

Theorem 5.2. *Let $p \in [1,2)$. Then Theorem 2.1 holds for all $u \in \mathcal{W}(Q_{\mu_0})$.*

Proof. We reduce the problem to Theorem 5.1 by passing to suitable modifications of u. Since $u \in \mathcal{W}(Q_{\mu_0})$, we have that $J_u = \bigcup_{j=1}^n \Gamma_j^u$ consists of closed segments and $\nabla u \in L^p(Q_{\mu_0})$. As in Theorem 5.1, it suffices to treat the case $\|e(u)\|_{L^2(Q_{\mu_0})} > 0$. We choose $I \in \mathbb{N}$ such that for $\mathcal{R}_0 = \{Q \in \mathcal{Q}^I : Q \cap J_u \neq \emptyset\}$

$$\begin{aligned}&(i)\quad \|\nabla u\|^p_{L^p(A)} \le \mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})} \quad \text{for } A \subset \mathbb{R}^2 \text{ with } |A| \le \theta^{-1}\mathcal{H}^1(J_u)t_I,\\ &(ii)\quad \sum\nolimits_{Q\in\mathcal{R}_0}\mathcal{H}^1(\partial Q) \le c\mathcal{H}^1(J_u).\end{aligned} \tag{5.34}$$

Let $R_0 = \bigcup_{Q\in\mathcal{R}_0} Q$. Clearly, we have $|R_0| \le c\mathcal{H}^1(J_u)t_I \le \theta^{-1}\mathcal{H}^1(J_u)t_I$ for θ small (with respect to c). Moreover, $J_u \cap Q = \emptyset$ for each $Q \in \bigcup_{j\ge I}\mathcal{Q}^j$ with $Q \cap R_0 = \emptyset$. For later purpose, we also define $J'_u = J_u \cup \partial R_0$.

Identification of bad squares: Assume that $\mathcal{R}_j \subset \mathcal{Q}^{I-j}$, $0 \le j \le i$, have already been constructed and let $R_j := \bigcup_{Q \in \mathcal{R}_j} Q$ and $u_j := u\chi_{Q_{\mu_0} \setminus \bigcup_{k=0}^j \overline{R_k}} \in \mathcal{W}(Q_{\mu_0})$. For all $j = 1, \ldots, i$ we define the sets S^j_τ, $\tau = (\tau_0, \ldots, \tau_{j-1}) \in J_j := \{0,1\}^j$, by

$$S^j_\tau = \big(\overline{R_j} \cap \bigcup\nolimits_{k:\tau_k = 1} \overline{R_k}\big) \setminus \bigcup\nolimits_{k:\tau_k=0} \overline{R_k} \text{ for } \tau \neq 0, \qquad S^j_0 = \overline{R_j} \setminus \bigcup\nolimits_{k=0}^{j-1} \overline{R_k}.$$

For $j = 0$ we let $J_0 = \{0\}$ and $S^0_0 = \overline{R_0}$. Denoting $\#\tau = \sum_{k=0}^{j-1} \tau_k$ for $\tau \in J_j$ we assume that for all $j = 0, \ldots, i$

$$\begin{aligned} &(i) \ \ J_{u_j} \subset \Gamma_j := \bigcup\nolimits_{k=0}^j \big(\partial R_k \setminus \bigcup\nolimits_{l=k+1}^j \overline{R_l}\big) \cup \big(J_u \setminus \bigcup\nolimits_{k=0}^j \overline{R_k}\big),\\ &(ii) \ \ \mathcal{H}^1(J_{u_j} \cap Q) \le \theta^{-1} d(Q) \quad \text{for all } Q \in \bigcup\nolimits_{k=I-j}^\infty \mathcal{Q}^k, \\ &(iii) \ \ \mathcal{H}^1(\partial R_j \cap F) \le \sum\nolimits_{k=0}^j \sum\nolimits_{\tau \in J_j : \#\tau = k} 2^{-k}\, \mathcal{H}^1\big(J'_u \cap S^j_\tau \cap F\big) \text{ for all } F \in \mathcal{F}^{I-(j+1)}. \end{aligned} \tag{5.35}$$

From the above discussion we get that the properties hold for $i = 0$ where particularly (iii) follows from the fact that $J'_u \supset \partial R_0$. We pass to step $i+1$ as follows. Let

$$\mathcal{R}_{i+1} = \{Q \in \mathcal{Q}^{I-i-1} : \mathcal{H}^1(\Gamma_i \cap Q) > 16 t_{I-i-1}\} \tag{5.36}$$

and $R_{i+1} = \bigcup_{Q \in \mathcal{R}_{i+1}} Q$. Set $u_{i+1} = u_i \chi_{Q_{\mu_0} \setminus \overline{R_{i+1}}}$. Then (for θ small) (5.35)(i),(ii) hold by construction and (5.35)(i),(ii) for step i. We now show (iii). Fix $F \in \mathcal{F}^{I-(i+2)}$ and recall that F is open. Then (5.36) and the fact that $\mathcal{H}^1(\partial Q) = 8 t_{I-i-1}$ yield by (5.35)(i)

$$\begin{aligned} \mathcal{H}^1(\partial R_{i+1} \cap F) \le \frac{1}{2}\mathcal{H}^1(\Gamma_i \cap F \cap \overline{R_{i+1}}) &\le \frac{1}{2}\mathcal{H}^1\Big(J_u \cap F \cap \Big(\overline{R_{i+1}} \setminus \bigcup\nolimits_{k=0}^i \overline{R_k}\Big)\Big) \\ &\quad + \frac{1}{2}\sum\nolimits_{j=0}^i \mathcal{H}^1\Big((\partial R_j \cap F) \setminus \bigcup\nolimits_{l=j+1}^i \overline{R_l}\Big). \end{aligned} \tag{5.37}$$

We then note that by definition of S^{i+1}_0

$$\mathcal{H}^1\big(J_u \cap F \cap \big(\overline{R_{i+1}} \setminus \bigcup\nolimits_{k=0}^i \overline{R_k}\big)\big) = \mathcal{H}^1(J_u \cap F \cap S^{i+1}_0). \tag{5.38}$$

Moreover, for each $\tau \in J_j$ we have $S^j_\tau \setminus \bigcup_{l=j+1}^i \overline{R_l} = S^{i+1}_{\tau'} = S^{i+1}_{\tau'} \cap (\overline{R_j} \setminus \bigcup_{l=j+1}^i \overline{R_l})$ with $\tau' = (\tau, 1, 0 \ldots, 0) \in J_{i+1}$. Consequently, since $\#\tau' = \#\tau + 1$ we find by (5.35)(iii) for step $i-1$ and the fact that $F \setminus \bigcup_{l=j+1}^i \overline{R_l} \in \mathcal{F}^{I-(j+1)}$

$$\begin{aligned} &\frac{1}{2}\sum\nolimits_{j=0}^i \mathcal{H}^1\big((\partial R_j \cap F) \setminus \bigcup\nolimits_{l=j+1}^i \overline{R_l}\big) \\ &\quad \le \sum\nolimits_{j=0}^i \sum\nolimits_{k=0}^j \sum\nolimits_{\tau \in J_j : \#\tau = k} 2^{-k-1}\, \mathcal{H}^1\big(J'_u \cap S^j_\tau \cap (F \setminus \bigcup\nolimits_{l=j+1}^i \overline{R_l})\big) \\ &\quad \le \sum\nolimits_{j=0}^i \sum\nolimits_{k=1}^{j+1} \sum\nolimits_{\tau' \in J_{i+1} : \#\tau' = k} 2^{-k}\, \mathcal{H}^1\big(J'_u \cap S^{i+1}_{\tau'} \cap F \cap (\overline{R_j} \setminus \bigcup\nolimits_{l=j+1}^i \overline{R_l})\big) \\ &\quad \le \sum\nolimits_{k=1}^{i+1} \sum\nolimits_{\tau' : \#\tau' = k} 2^{-k}\, \mathcal{H}^1\big(J'_u \cap S^{i+1}_{\tau'} \cap F\big). \end{aligned}$$

This together with (5.37)-(5.38) yields (5.35)(iii).

For later purpose, note that $\big|Q \cap \bigcup_{k \le i} R_k\big| \le 2 t_{I-i} \mathcal{H}^1(\Gamma_i \cap \overline{Q})$ for all $Q \in \mathcal{Q}^{I-i-1}$, where we used that $\bigcup_{k \le i} R_k$ consists of squares with sidelength smaller than $2t_{I-i}$ whose boundaries are contained in Γ_i. Thus, for all $Q \in \mathcal{Q}^{I-i-1} \setminus \mathcal{R}_{i+1}$ we get for θ small by (5.36)

$$\big|Q \cap \bigcup\nolimits_{k \le i} R_k\big| \le 2t_{I-i}\mathcal{H}^1(\Gamma_i \cap \overline{Q}) \le 2t_{I-i}(16 t_{I-i-1} + 8 t_{I-i-1}) \le 2t^2_{I-i-1} = \tfrac{1}{2}|Q|. \tag{5.39}$$

We proceed in this way until step $i = I-1$ and finally let $R_I = Q_{\mu_0}$, $\mathcal{R}_I = \{Q_{\mu_0}\}$. By (5.35)(iii) and the fact that $(S^I_{\tau'})_{\tau' \in J_I}$ is a partition of Q_{μ_0} we derive

$$\begin{aligned}\sum\nolimits_{j=0}^{I-1} \mathcal{H}^1(\partial R_j \cap Q_{\mu_0}) &\le \sum\nolimits_{j=0}^{I-1} \sum\nolimits_{k=0}^{j} \sum\nolimits_{\tau \in J_j: \#\tau = k} 2^{-k}\ \mathcal{H}^1\big(J'_u \cap S^j_\tau\big) \\ &= \sum\nolimits_{\tau' \in J_I} \sum\nolimits_{j=0}^{I-1} \sum\nolimits_{k=0}^{j} \sum\nolimits_{\tau \in J_j: \#\tau = k} 2^{-k}\ \mathcal{H}^1\big(J'_u \cap S^j_\tau \cap S^I_{\tau'}\big).\end{aligned}$$

Note that $S^j_\tau \cap S^I_{\tau'} \neq \emptyset$ if and only if τ coincides with the first j entries of τ' and $\tau'_j = 1$. Consequently, with $\mathcal{H}^1(J'_u) \le c\mathcal{H}^1(J_u)$ we find

$$\begin{aligned}\sum\nolimits_{j=0}^{I-1} \mathcal{H}^1(\partial R_j \cap Q_{\mu_0}) &\le \sum\nolimits_{\tau' \in J_I} \sum\nolimits_{j: \tau'_j = 1} 2^{-(\sum_{l=0}^{j-1} \tau'_l)}\ \mathcal{H}^1\big(J'_u \cap S^I_{\tau'}\big) + \mathcal{H}^1\big(J'_u \cap S^I_0\big) \\ &\le c \sum\nolimits_{\tau' \in J_I} \mathcal{H}^1\big(J'_u \cap S^I_{\tau'}\big) \le c\mathcal{H}^1(J_u). \qquad (5.40)\end{aligned}$$

Application of Theorem 5.1: We now apply Theorem 5.1 separately on each square $Q \in \mathcal{R}_j$, $j = 1, \ldots, I$, for the function u_{j-1}. In fact, in view of (5.35)(ii), condition (5.1) is satisfied (replacing Q_{μ_0} by Q and u by u_{j-1}). Hereby, by (5.28) we obtain a partition of the square $Q \in \mathcal{R}_j$, which we can restrict to $Q \setminus \bigcup_{k=0}^{j-1} \overline{R_k}$. Consequently, taking the union over all partitions defined on each $Q \in \bigcup_{j=1}^I \mathcal{R}_j$ we obtain a partition $(P'_j)_{j=0}^\infty$ of Q_{μ_0} with $\bigcup_{j=0}^I \partial R_j \subset \bigcup_{j=0}^\infty \partial P'_j$, where we set $P'_0 = R_0$. By (5.35)(i), (5.40) and the fact that $\partial R_k \cap \bigcup_{l=0}^k R_l = 0$ we get

$$\begin{aligned}&\sum\nolimits_{j=1}^{I-1} \sum\nolimits_{Q \in \mathcal{R}_j} \big(\mathcal{H}^1(\partial Q \cap Q_{\mu_0}) + \mathcal{H}^1(J_{u_{j-1}} \cap Q)\big) + \mathcal{H}^1(J_{u_{I-1}} \cap Q_{\mu_0}) \\ &\quad\le \sum\nolimits_{j=1}^{I-1} \mathcal{H}^1(\partial R_j \cap Q_{\mu_0}) + \sum\nolimits_{j=1}^{I} \mathcal{H}^1(\Gamma_{j-1} \cap R_j) \\ &\quad\le c\mathcal{H}^1(J_u) + \sum_{j=1}^I \mathcal{H}^1\Big(R_j \cap \bigcup\nolimits_{k=0}^{j-1} \big(\partial R_k \setminus \bigcup\nolimits_{l=k+1}^{j-1} \overline{R_l}\big)\Big) + \sum_{j=1}^I \mathcal{H}^1\Big(J_u \cap \big(R_j \setminus \bigcup\nolimits_{k=0}^{j-1} \overline{R_k}\big)\Big) \\ &\quad\le c\mathcal{H}^1(J_u) + \sum_{j=1}^I \mathcal{H}^1\Big(\bigcup\nolimits_{k=0}^{j-1} \partial R_k \cap \big(R_j \setminus \bigcup\nolimits_{l=0}^{j-1} R_l\big)\Big) + \sum_{j=1}^I \mathcal{H}^1\Big(J_u \cap \big(R_j \setminus \bigcup\nolimits_{k=0}^{j-1} \overline{R_k}\big)\Big) \\ &\quad\le \sum\nolimits_{j=0}^{I-1} \mathcal{H}^1(\partial R_j \cap Q_{\mu_0}) + c\mathcal{H}^1(J_u) \le c\mathcal{H}^1(J_u).\end{aligned}$$

Consequently, applying (5.28) for each $Q \in \bigcup_{j=1}^I \mathcal{R}_j$, we find that $\sum_{j=0}^\infty \mathcal{H}^1(\partial P'_j \setminus \partial Q_{\mu_0}) \le C\mathcal{H}^1(J_u)$. Moreover, Theorem 5.1 yields piecewise constant $\mathbb{R}^{2\times 2}_{\rm skew}$-valued functions $\bar{A}_Q$ on each $Q \in \bigcup_{j=1}^I \mathcal{R}_j$ (being constant on each component of the corresponding partition) such that by (5.33) and the fact that $u_{j-1} = u\chi_{Q_{\mu_0} \setminus \bigcup_{k=0}^{j-1} \overline{R_k}}$ we have for $Q \in \mathcal{R}_j$

$$\begin{aligned}\|\nabla u - \bar{A}_Q\|^p_{L^p(Q \setminus \bigcup_{k<j} \overline{R_k})} &\le \|\nabla u_{j-1} - \bar{A}_Q\|^p_{L^p(Q)} \le Ct_{I-j}^{2-p} \|e(u_{j-1})\|^p_{L^2(Q)} \\ &= Ct_{I-j}^{2-p} \|e(u)\|^p_{L^2(Q \setminus \bigcup_{k<j} \overline{R_k})}.\end{aligned}$$

If $|Q \setminus \bigcup_{k<j} \overline{R_k}| > 0$, we find some $\hat{Q} \in \mathcal{Q}^{I-j+1} \setminus \mathcal{R}_{j-1}$ with $\hat{Q} \subset Q$. Then (5.39) (for $i = j-2$) implies $|Q \setminus \bigcup_{k<j} \overline{R_k}| \ge |\hat{Q} \setminus \bigcup_{k<j-1} \overline{R_k}| = |\hat{Q} \setminus \bigcup_{k \le j-2} R_k| \ge 2t^2_{I-j+1} = 2\theta^2 t^2_{I-j}$. Thus, for each $Q \in \mathcal{R}_j$, $j = 1, \ldots, I$,

$$\|\nabla u - \bar{A}_Q\|^p_{L^p(Q \setminus \bigcup_{k<j} \overline{R_k})} \le C|Q \setminus \bigcup\nolimits_{k<j} \overline{R_k}|^{1-\frac{p}{2}} \|e(u)\|^p_{L^2(Q \setminus \bigcup_{k<j} \overline{R_k})},$$

where $C = C(\theta, p) > 0$. Therefore, for each $(P'_j)_{j\ge1}$ we find a corresponding $A'_j \in \mathbb{R}^{2\times 2}_{\rm skew}$ such that for $v' := u - \sum_{j=1}^\infty (A'_j\, x)\chi_{P'_j}$ we have by summing over all squares $Q \in \bigcup_{j=1}^I \mathcal{R}_j$

$$\|v'\|^p_{L^p(Q_{\mu_0}\setminus R_0)} \le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})},$$

where similarly as before we have used the discrete Hölder inequality. (See, e.g., (4.96), (5.25). We omit the details.) It remains to analyze the behavior on R_0. Observe that Theorem 5.1 is not applicable in the squares contained in R_0 since property (5.1) might not hold. However, we may argue similarly as in Step VI of the previous proof. By (5.34)(i) and the fact that $|R_0| \le \theta^{-1}\mathcal{H}^1(J_u)t_I$ we find $\|\nabla u\|^p_{L^p(R_0)} \le \mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})}$. Since $u = v'$ on $R_0 = P'_0$, we get $\|\nabla v'\|^p_{L^p(Q_{\mu_0})} \le C\mu_0^{2-p}\|e(u)\|^p_{L^2(Q_{\mu_0})}$ and therefore have re-derived (5.33). To conclude the proof of (2.2), it now remains to apply Theorem 2.3 on v' as in the previous proof. ☐

Remark 5.3. The proof of Theorem 5.1 and Theorem 5.2, in particular (5.28), show that applying Theorem 2.3 with $\rho = \mathcal{H}^1(J_u)$ instead of $\rho = \mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q_{\mu_0})$ yields a partition $(P_j)_j$ and a function v such that (2.2)(ii) still holds and (2.1), (2.2)(i) are replaced by

$$\sum\nolimits_{j=1}^\infty \mathcal{H}^1(\partial^* P_j \cap Q_{\mu_0}) \le C\mathcal{H}^1(J_u), \qquad \|v\|_{L^\infty(Q_{\mu_0})} \le C(\mathcal{H}^1(J_u))^{-1}\|e(u)\|_{L^2(\Omega)}.$$

5.2. **Density arguments.** This section is devoted to the proof of the general version of Theorem 2.1 and to the proof of Theorem 2.5. The strategy is to approximate a given function $u \in GSBD^2(\Omega)$ by a sequence $(u_n)_n \subset \mathcal{W}(\Omega)$. (Recall the definition before Theorem 4.1.) We then apply Theorem 5.2 on $(u_n)_n$ and show that the desired properties can be recovered for the limiting function u. We first present a variant of Theorem 3.13 which yields an approximation result for every $GSBD^2$ function at the expense of a nonoptimal estimate for the surface energy. The reader only interested in the main result for functions in $(G)SBD^2(\Omega) \cap L^2(\Omega)$ may skip the following lemma and corollary and may replace Corollary 5.5 by Theorem 3.13 in the proof of Theorem 2.1 below. We also refer to the recent generalization [14] where density results in $GSBD^p$ are obtained for all $1 < p < \infty$.

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^d$ open, bounded. Let $u \in GSBD^2(\Omega)$ and $\Omega' \subset\subset \Omega$ with Lipschitz boundary. Then there exists a sequence $(u_k)_k \subset \mathcal{W}(\Omega')$ such that for a universal constant $c > 1$ one has*

$$\begin{aligned} &(i) \ \ u_k \to u \ \text{a.e. in } \Omega' \text{ as } k \to \infty, \\ &(ii) \ \ \|e(u_k)\|_{L^2(\Omega')} \le c\|e(u)\|_{L^2(\Omega)},\ \mathcal{H}^{d-1}(J_u) \le \mathcal{H}^{d-1}(J_{u_k}) \le c\mathcal{H}^{d-1}(J_u),\ k \in \mathbb{N}. \end{aligned} \tag{5.41}$$

Proof. We follow closely the proofs in [12, Theorem 1], [36, Theorem 3.5] and only indicate the necessary changes. By [36, Lemma 2.2] one can find a basis $\mathbf{e}_1, \ldots, \mathbf{e}_d$ of $\mathbb{R}^d$ such that

$$\mathcal{H}^{d-1}(\{x \in J_u : [u](x)\cdot e = 0\}) = 0$$

for all $e \in D := \{\mathbf{e}_i, 1 \le i \le d, \mathbf{e}_i + \mathbf{e}_j, 1 \le i < j \le d\}$. For $h > 0$ small and each $y \in [0,1)^d$ we introduce the discretized function of u defined by

$$u_h^y(\xi) = u(hy + \xi), \quad \xi \in h\mathbb{Z}^d \cap (\Omega - hy).$$

Moreover, letting $\triangle(x) = \prod_{i=1}^d \max\{(1 - |x_i|), 0\}$ we define the continuous interpolation

$$w_h^y(x) = \sum\nolimits_{\xi \in h\mathbb{Z}^d\cap\Omega} u_h^y(\xi)\triangle\Big(\frac{x - (\xi + hy)}{h}\Big).$$

Note that $w_h^y \in W^{1,\infty}(\Omega')$ and for h small enough w_h^y is indeed well defined for all $x \in \Omega'$ since $\Omega' \subset\subset \Omega$. We now show that $w_h^y \to u$ in measure on Ω' as $h \to 0$ for a.e. $y \in [0,1)^d$ which is equivalent to

$$\int_{y\in[0,1)^d} \|\varphi_1(|w_h^y - u|)\|_{L^1(\Omega')} \to 0, \tag{5.42}$$

where $\varphi_t(s) = \min\{s, \frac{1}{t}\}$ for $s \geq 0$ and $t > 0$. Using the subadditivity of φ_1, a change of variables and the fact that $\sum_{\xi\in h\mathbb{Z}^d\cap\Omega} \triangle\Big(\frac{x-\xi}{h} - y\Big) = 1$ for all $x \in \Omega'$, $y \in [0,1)^d$, we deduce

$$\begin{aligned}
&\int_{y\in[0,1)^d} dy \int_{\Omega'} \varphi_1(|w_h^y(x) - u(x)|)\, dx \\
&= \int_{y\in[0,1)^d} dy \int_{\Omega'} \varphi_1\Big(\Big|\sum\nolimits_{\xi\in h\mathbb{Z}^d\cap\Omega} \triangle\Big(\frac{x-\xi}{h} - y\Big)(u(x) - u(hy+\xi))\Big|\Big)\, dx \\
&\leq \int_{y\in[0,1)^d} dy \int_{\Omega'} \sum\nolimits_{\xi\in h\mathbb{Z}^d\cap\Omega} \varphi_1\Big(\Big|\triangle\Big(\frac{x-\xi}{h} - y\Big)(u(x) - u(hy+\xi))\Big|\Big)\, dx \\
&\leq \sum\nolimits_{\xi\in h\mathbb{Z}^d\cap\Omega} \int_{\frac{x-\xi}{h}-[0,1)^d} \int_{\Omega'} \varphi_1(|\triangle(z)(u(x) - u(x-hz))|)\, dx\, dz \\
&\leq \int_{(-1,1)^d} \int_{\Omega'} \varphi_1(|\triangle(z)(u(x) - u(x-hz))|)\, dx\, dz.
\end{aligned}$$

In the last step we used that the sets $\frac{x-\xi}{h} - [0,1)^d$ are pairwise disjoint for $\xi \in h\mathbb{Z}^d \cap \Omega$. Since

$$\int_{\Omega'} \varphi_1(|\triangle(z)(u(x) - u(x-hz))|)\, dx = \int_{\Omega'} \triangle(z)\varphi_{\triangle(z)}(|u(x) - u(x-hz)|)\, dx \to 0$$

for $h \to 0$ and is uniformly bounded by $|\Omega'|$ for all $z \in (-1,1)^d$, we obtain (5.42) by dominated convergence. We now follow closely the proof in [36]. For a specific choice of $y \in [0,1)^d$ and a set of 'bad cubes' (see (3.13)-(3.15) in [36] for details)

$$\mathcal{Q}_h \subset \{Q_\xi = \xi + hy + [0,h)^d : \xi \in h\mathbb{Z}^d\} \tag{5.43}$$

with $\#\mathcal{Q}_h \leq Ch^{-(d-1)}$ one defines the function $u_h = w_h^y \chi_{\Omega' \setminus \bigcup_{Q\in\mathcal{Q}_h} Q}$. Clearly, $u_h \in \mathcal{W}(\Omega')$ as the jump set is given by the boundary of the cubes $\mathcal{Q}_h$. (Strictly speaking J_{u_h} is only contained in the boundary of the cubes. However, the desired property may always be achieved by an infinitesimal perturbation of u_h, see [12, Remark 5.3]). In view of (5.42) and $|\bigcup_{Q\in\mathcal{Q}_h} Q| \leq Ch$, we get, possibly passing to a not relabeled subsequence, $u_h \to u$ a.e. on Ω' which gives (5.41)(i).

Following the lines of [36], for a suitable choice of $y \in [0,1)^d$ we get $\|e(u_h)\|_{L^2(\Omega')} \leq c\|e(u)\|_{L^2(\Omega)}$ and $\mathcal{H}^{d-1}(J_{u_h}) \leq c\mathcal{H}^{d-1}(J_u)$ for a universal constant $c > 1$ after passage to a not relabeled subsequence. In fact, in the estimates for the elastic energy, which are based on a slicing technique, the assumption $u \in L^2(\Omega)$ is not needed. (This assumption was only needed to define an extension of u, which is not necessary in our setting.) To conclude the proof of (5.41)(ii), we remark that the inequality $\mathcal{H}^{d-1}(J_u) \leq \mathcal{H}^{d-1}(J_{u_h})$ has not been stated explicitly in [36], but can always be achieved by introducing arbitrary additional 'bad squares' in (5.43). □

The proof of the following corollary is now straightforward.

Corollary 5.5. *Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary and let $Q \subset \mathbb{R}^d$ be a cube with $\Omega \subset\subset Q$. Let $u \in GSBD^2(\Omega)$. Then there exists a sequence $(u_k)_k \subset \mathcal{W}(Q)$ such that for* *a universal constant* $c > 1$ *one has*

$$\begin{aligned} &(i) \quad u_k \to u \text{ a.e. in } \Omega \text{ as } k \to \infty, \\ &(ii) \quad \|e(u_k)\|_{L^2(Q)} \le c\|e(u)\|_{L^2(\Omega)}, \\ &(iii) \quad \mathcal{H}^{d-1}(J_u) + \mathcal{H}^{d-1}(\partial\Omega) \le \mathcal{H}^{d-1}(J_{u_k}) \le c\mathcal{H}^{d-1}(J_u) + c\mathcal{H}^{d-1}(\partial\Omega). \end{aligned} \tag{5.44}$$

Proof. Choose a sequence of Lipschitz sets $\Omega_n \subset\subset \Omega$ with $\mathcal{H}^{d-1}(\partial\Omega) \le \mathcal{H}^{d-1}(\partial\Omega_n) \le c\mathcal{H}^{d-1}(\partial\Omega)$ and $|\Omega \setminus \Omega_n| \le \frac{1}{n}$ for all $n \in \mathbb{N}$ such that $\partial\Omega_n$ consists of a finite number of closed $(d-1)$-simplices. For each $n \in \mathbb{N}$ we apply Lemma 5.4 on $\Omega_n \subset\subset \Omega$ and obtain sequences $(v^n_l)_l \subset \mathcal{W}(\Omega_n)$ converging in measure to u on Ω_n such that $\|e(v^n_l)\|_{L^2(\Omega_n)}$ and $\mathcal{H}^{d-1}(J_{v^n_l})$ are uniformly controlled by $c\|e(u)\|_{L^2(\Omega)}$ and $c\mathcal{H}^{d-1}(J_u)$, respectively. Define the extensions $\hat{v}^n_l = v^n_l \chi_{\Omega_n} \in \mathcal{W}(Q)$. Possibly replacing $\hat{v}^n_l$ by $\hat{v}^n_l + t^n_l$ for a suitable $t^n_l \in \mathbb{R}^d$, $|t^n_l| \le \frac{1}{l}$, we obtain (not relabeled) sequences still converging to u on Ω_n such that $\mathcal{H}^{d-1}(\partial\Omega_n \setminus J_{\hat{v}^n_l}) = 0$ and thus

$$\mathcal{H}^{d-1}(J_u) + \mathcal{H}^{d-1}(\partial\Omega) \le \mathcal{H}^{d-1}(J_{\hat{v}^n_l}) \le c\mathcal{H}^{d-1}(J_u) + c\mathcal{H}^{d-1}(\partial\Omega).$$

Consequently, by a standard diagonal sequence argument taking into account that convergence in measure is metrizable (take $(f,g) \mapsto \int_\Omega \min\{|f-g|, 1\}$) we get a sequence $(w_k)_k \subset \mathcal{W}(Q)$ satisfying (5.44). □

We are now in the position to give the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $p \in (1,2)$ and let $u \in GSBD^2(\Omega)$ be given. Without restriction we assume that Ω is connected as otherwise the following arguments are applied on each connected component of Ω. Choose a square $Q_{\mu_0} \supset\supset \Omega$ with $\mathcal{H}^1(\partial Q_{\mu_0}) \le c\mathcal{H}^1(\partial\Omega)$ for a universal constant $c > 0$.

By Corollary 5.5 we find a sequence $(u_k)_k \subset \mathcal{W}(Q_{\mu_0})$ satisfying (5.44). (The reader only interested in the case $(G)SBD^2(\Omega) \cap L^2(\Omega)$ can instead apply Theorem 3.13 and Remark 3.14.) We apply Theorem 5.2 on each u_k and obtain a sequence of (ordered) Caccioppoli partitions $(P^k_j)_j$ of Q_{μ_0} and corresponding infinitesimal rigid motions $(a^k_j)_j = (a_{A^k_j, b^k_j})_j$ such that $v_k := u_k - \sum_{j=1}^\infty a^k_j \chi_{P^k_j} \in SBV^p(Q_{\mu_0}) \cap L^\infty(Q_{\mu_0})$ satisfies by (2.1) and (2.2)

$$\begin{aligned} &(i) \quad \sum\nolimits_{j=1}^\infty \mathcal{H}^1(\partial^* P^k_j) \le c(\mathcal{H}^1(J_{u_k}) + \mathcal{H}^1(\partial Q_{\mu_0})) \le c(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega)), \\ &(ii) \quad \|\nabla v_k\|_{L^p(Q_{\mu_0})} \le C\|e(u_k)\|_{L^2(Q_{\mu_0})} \le C\|e(u)\|_{L^2(\Omega)}, \\ &(iii) \quad \|v_k\|_{L^\infty(Q_{\mu_0})} \le C(\mathcal{H}^1(J_{u_k}))^{-1}\|e(u_k)\|_{L^2(Q_{\mu_0})} \le C(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega))^{-1}\|e(u)\|_{L^2(\Omega)}. \end{aligned} \tag{5.45}$$

Possibly passing to a (not relabeled) refinement of the partition (consisting of the sets $(P^k_j \cap \Omega)_j \cup (P^k_j \setminus \Omega)_j$) we may assume that each component P^k_j satisfies $P^k_j \subset \Omega$ or $P^k_j \cap \Omega = \emptyset$ and (5.45)(i) still holds. Therefore, Theorem 3.11 implies the existence of a Caccioppoli partition $(P_j)_{j=1}^\infty$ of Ω and a (not relabeled) subsequence such that $\chi_{P^k_j} \to \chi_{P_j}$ in $L^1(\Omega)$, when $k \to \infty$, for all $j \in \mathbb{N}$ and such that, passing to the limit in (5.45)(i) via the lower semicontinuity of the perimeter

$$\sum\nolimits_{j=1}^\infty \mathcal{H}^1(\partial^* P_j) \le c(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega)).$$

This gives (2.1). Applying Ambrosio's compactness theorem (Theorem 3.10) on the sequence $(v_k)_k$ we find $v \in SBV^p(Q_{\mu_0}) \cap L^\infty(Q_{\mu_0})$ such that $v_k \to v$ a.e. and $\nabla v_k \rightharpoonup \nabla v$ weakly in $L^p(Q_{\mu_0})$ up to a not relabeled subsequence. In particular, we have considering the restriction of v to Ω

$$\|v\|_{L^\infty(\Omega)} \le C(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega))^{-1}\|e(u)\|_{L^2(\Omega)}, \qquad \|\nabla v\|_{L^p(\Omega)} \le C\|e(u)\|_{L^2(\Omega)}. \tag{5.46}$$

Since $(P_j)_j$ is a partition of Ω, it now suffices to show the existence of infinitesimal rigid motions $(a_j)_j = (a_{A_j,b_j})_j$ such that

$$(u-v)\chi_{P_j} = a_j\chi_{P_j} \tag{5.47}$$

a.e. in P_j for all $j \in \mathbb{N}$. Indeed, (2.2) then is immediate by (5.46). Clearly, if $|P_j| = 0$ it suffices to set $a_j = 0$. If instead $|P_j| > 0$, then it exists $\delta > 0$ independently of k such that $|P_j^k| \geq \delta$. As $u_k\chi_{P_j^k} \to u\chi_{P_j}$ a.e. and $v_k\chi_{P_j^k} \to v\chi_{P_j}$ a.e., the sequence

$$a_j^k\chi_{P_j^k} = (u_k - v_k)\chi_{P_j^k}$$

converges in measure to $(u-v)\chi_{P_j}$. Therefore, it exists a positive nondecreasing continuous function ψ with $\lim_{s\to\infty}\psi(s) = +\infty$ such that $\sup_{k\geq 1}\int_{P_j^k}\psi(|a_j^k|)\,\mathrm{d}x \leq 1$ (see e.g. [33, Remark 2.2]). By Lemma 3.6 we infer that $(a_j^k)_k$ are bounded in $W^{1,\infty}(\Omega)$ for a constant independent of k. Consequently, we find an infinitesimal rigid motion a_j such that $a_j^k\chi_{P_j^k} \to a_j\chi_{P_j}$ in $L^1(\Omega)$. By the convergence of $a_j^k\chi_{P_j^k}$ to $(u-v)\chi_{P_j}$, this implies (5.47) and concludes the proof. □

Remark 5.6 (Piecewise Korn inequality for $q \neq 2$)**.** Having concluded the proof of Theorem 2.1, we remark that the result can be extended to a version in the space $GSBD^q(\Omega)$ for general $1 < q < \infty$ and exponents $1 \leq p < q$. Specifically, for each $u \in GSBD^q(\Omega)$ there is a Caccioppoli partition $(P_j)_{j=1}^\infty$ of Ω with $\sum_{j=1}^\infty \mathcal{H}^1(\partial^* P_j) \leq c(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega))$ and infinitesimal rigid motions $(a_j)_j = (a_{A_j,b_j})_j$ such that $v := u - \sum_{j=1}^\infty a_j\chi_{P_j} \in SBV^p(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$ and

$$\|v\|_{L^\infty(\Omega)} \leq C(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega))^{-1}\|e(u)\|_{L^q(\Omega)}, \quad \|\nabla v\|_{L^p(\Omega)} \leq C\|e(u)\|_{L^q(\Omega)}.$$

We emphasize that we do not formulate this as a main result of the paper in Section 2, but only on a level of a remark since we do not give a complete proof. We only provide some hints on how the arguments in the proof of Theorem 2.1 have to be adapted to the general case. Note that the main adaptions concern the replacement of L^2 by L^q norms and adjusting corresponding Hölder-type estimates. (Setting $q = 2$ below, we see that the parameters and exponents are consistent to the ones given along the proof of Theorem 2.1.) Moreover, we need an L^q-version of Theorem 3.15 (see [13, 18]) and use the density result [14] instead of Corollary 5.5. More specifically, we have the following adaptions, ordered by sections:

Section 4.1: The results in Section 4.1 (Theorem 4.3 and Lemma 4.4) remain completely unchanged as they rely exclusively on the geometry of J_u.

Section 4.2: Theorem 4.6 holds with $r = (q-p)/(6q^2)$ in place of $(2-p)/24$ and in (4.57) we need to replace $\mu^{2/p}$ by $\mu^{2/p-2/q+1}$ and $\|e(u)\|_{L^2(Q_\mu)}$ by $\|e(u)\|_{L^q(Q_\mu)}$. In Step I of the proof, the application of the Korn inequality for SBD^2 functions with small jump set (see (4.59)) has to be replaced by the corresponding version for SBD^q functions [18]. (Again replace the exponent $2/p$ by $2/p - 2/q + 1$.) In the rest of the proof we only need to adjust some exponents as we indicate now: In Step I-II, in the estimates on the differences of rigid motions (4.62)-(4.67) we replace $\|e(u)\|_2$ by $\|e(u)\|_q$ and $d(Q)^2, s_l^2$ by $d(Q)^{2-2p/q+p}, s_l^{2-2p/q+p}$. The argument leading to (4.69) is essentially the same, where we have two slightly different Hölder-type estimates: First, we need

$$\begin{aligned}\sum\nolimits_{Q\in\mathcal{C}}(d(Q))^{2-\frac{2p}{q}}\|e(u)\|^p_{L^q(N_Q)} &\leq \Big(\sum\nolimits_{Q\in\mathcal{C}}(d(Q))^{(2-\frac{2p}{q})\frac{q}{q-p}}\Big)^{1-\frac{p}{q}}\Big(\sum\nolimits_{Q\in\mathcal{C}}\|e(u)\|^q_{L^q(N_Q)}\Big)^{\frac{p}{q}}\\ &\leq C\mu^{2-\frac{2p}{q}}\|e(u)\|^p_{L^q(Q_\mu)}.\end{aligned} \tag{5.48}$$

Moreover, using $M_l = \sum_k \#\mathcal{X}_k^l \le C\mu s_l^{-1}$ and $\#\mathcal{X}_k^l \le c\theta^{-2lr}$, we have

$$\begin{aligned}\sum_{l\ge 8}\sum_k \#\mathcal{X}_k^l\, s_l^{2-\frac{2p}{q}}\theta^{-2(p+1)lr}\|e(u)\|^p_{L^q(N_k^l)} &\le C\sum_{l\ge 8}\sum_k (\#\mathcal{X}_k^l)^{1-\frac{p}{q}}\; s_l^{2-\frac{2p}{q}}\theta^{-2(p+1+\frac{p}{q})lr}\|e(u)\|^p_{L^q(N_k^l)}\\ &\le C\sum_{l\ge 8} M_l^{1-\frac{p}{q}} s_l^{2-\frac{2p}{q}}\theta^{-2(p+1+\frac{p}{q})lr}\big(\sum_k \|e(u)\|^q_{L^q(N_k^l)}\big)^{\frac{p}{q}}\\ &\le C\mu^{2-\frac{2p}{q}}\|e(u)\|^p_{L^q(Q_\mu)}\sum_{l\ge 8}\theta^{l(1-\frac{p}{q})-2(p+1+\frac{p}{q})lr}.\end{aligned} \tag{5.49}$$

Recalling $r = (q-p)/(6q^2)$, one can check that $1 - p/q - 2(p+1+p/q)r \ge 2(q-p)r$, i.e., the sum indeed converges. Finally, we note that the construction of $\bar{u}$ in Step III and the derivation of (4.56)-(4.57) remain unchanged.

Section 4.3: In the statement of Theorem 4.9 we again replace $\mu^{2/p-1}$ by $\mu^{2/p-2/q}$ and $\|e(u)\|_{L^2(Q_\mu)}$ by $\|e(u)\|_{L^q(Q_\mu)}$. The proof remains unchanged as it only relies on geometric arguments and an application of the Korn inequality in John domains (Theorem 3.2) for $1 < p < q$.

Section 4.4: In the statement of Lemma 4.12 we only need to adapt the exponent in (4.84) replacing μ^{2-p} by $\mu^{2-2p/q}$. The proof relying entirely on geometric arguments remains the same. In Lemma 4.13 we define $r = (q-p)/(6q^2)$, the exponents in (4.92) are adjusted as before and instead of (4.93) we have

$$\int_{\partial Q_\mu\setminus\Gamma_u}|Tu - a_0|^q\, d\mathcal{H}^1 \le c\mu^{q-1}\|e(u)\|^q_{L^q(Q_\mu)}. \tag{5.50}$$

In part (1) of the proof we only need to change the Hölder-type estimate (4.96) in the sense of (5.48). In part (2) of the proof, in the application of Theorem 3.15 we observe that there is also a general version of (3.9) in terms of L^q-norms, see [13].

Section 5.1: At the beginning of the proof of Theorem 5.1 we define the parameters

$$r = (q-p)/(6q^2), \qquad p' = (q+p)/2, \qquad \lambda = (1-r)(q-p)/(q+3p)$$

and make the usual adaptions of the exponents replacing $2-p'$ by $2-2p'/q$ (see e.g. (5.8), (5.13) and (5.21) among others). Step II - Step IV of the proof do not address Korn-type estimates and remain unchanged. In Step V we need to adapt the estimate for isolated components (5.22) and the Hölder-type estimates (5.25). The derivation of (5.22) relies on the scaling in the trace estimate (5.50) and an application of Remark 3.8 for exponent q. To see (5.25), we follow the lines of the adaptions given in (5.48)-(5.49). Finally, the essential change in Step VI concerns the Hölder-type estimate (5.31). Using $|D_l^I| \le C_2\theta^{-rl}t_l\mu_0$ and $\|\nabla v'\|^{p'}_{L^{p'}((Q_{\mu_0}\setminus E'')\cap D_l^I)} \le C\mu_0^{2-2p'/q}\theta^{-\lambda l}\|e(u)\|^{p'}_{L^q(Q_{\mu_0})}$ (cf. (5.12) and (5.30)) we calculate

$$\begin{aligned}\|\nabla v'\|^p_{L^p(Q_{\mu_0}\setminus E'')} &= \sum\nolimits_{l=0}^I \|\nabla v'\|^p_{L^p((Q_{\mu_0}\setminus E'')\cap D_l^I)} \le \sum\nolimits_{l=0}^I |D_l^I|^{1-\frac{p}{p'}}\|\nabla v'\|^p_{L^{p'}((Q_{\mu_0}\setminus E'')\cap D_l^I)}\\ &\le C\mu_0^{2-\frac{2p}{q}}\sum\nolimits_l \theta^{l(1-r)(1-\frac{p}{p'})-l\lambda\frac{p}{p'}}\|e(u)\|^p_{L^q(Q_{\mu_0})} = C\mu_0^{2-\frac{2p}{q}}\sum\nolimits_l \theta^{l\lambda}\|e(u)\|^p_{L^q(Q_{\mu_0})}\end{aligned}$$

where in the last step we used the definition of p' and λ. Theorem 5.2 addresses the modification of the jump set and remains unchanged.

Section 5.2: Finally, to extend the result from $\mathcal{W}(\Omega)$ to general functions $GSBD^q(\Omega)$, we proceed as in the proof of Theorem 2.1 by employing the recently proved density result [14] instead of Corollary 5.5. □

Now with Theorem 2.1 at hand, the proof of our density result Theorem 2.5 is straightforward.

Proof of Theorem 2.5. Let $u \in GSBD^2(\Omega)$. By Theorem 2.1 applied for some $p \in [1,2)$ we obtain a Caccioppoli partition $(P_j)_j$ of Ω and corresponding infinitesimal rigid motions $(a_j)_j$ such that $v := u - \sum_j a_j \chi_{P_j} \in SBV^p(\Omega) \cap L^\infty(\Omega)$. As motivated in (2.4), we consider the sequence

$$v_k = u - \sum\nolimits_{j \geq k} a_j \chi_{P_j} \in SBV^p(\Omega) \cap L^\infty(\Omega)$$

and observe that $v_k \to u$ in measure on Ω, $e(v_k) = e(u_k)$ for all $k \in \mathbb{N}$ and $\mathcal{H}^1(J_{v_k} \triangle J_u) \to 0$ when $k \to \infty$. Using Theorem 3.13, each function v_k can be approximated in $L^2(\Omega)$ by a sequence with the properties stated in Theorem 3.13 such that (3.6) holds. Now the assertion follows from a diagonal sequence argument. □

6. Proof of further results

6.1. **Piecewise Poincaré inequality.** We start with the proof of the piecewise Poincaré inequality which is essentially based on the coarea formula for BV functions and can be derived completely independently from the results discussed in the previous sections.

Proof of Theorem 2.3. Without restriction we assume $\|\nabla u\|_{L^1(\Omega)} > 0$ as otherwise u is piecewise constant (see [5, Theorem 4.23], [15]) and there is nothing to show. We start with the case $m = 1$ and $u \in SBV(\Omega; \mathbb{R})$. Following ideas in [3, 9, 28] we can use the coarea formula in BV (see [5, Theorem 3.40]) to write

$$\|\nabla u\|_{L^1(\Omega)} = |Du|(\Omega \setminus J_u) = \int_{-\infty}^{\infty} \mathcal{H}^{d-1}\big((\Omega \setminus J_u) \cap \partial^* \{u > t\}\big)\, dt.$$

Thus, for $\rho > 0$ and $M := \rho^{-1} \|\nabla u\|_{L^1(\Omega)}$ we find $t_i \in (iM, (i+1)M]$ for all $i \in \mathbb{Z}$ such that

$$\mathcal{H}^{d-1}\big((\Omega \setminus J_u) \cap \partial^* \{u > t_i\}\big) \leq \frac{1}{M} \int_{iM}^{(i+1)M} \mathcal{H}^{d-1}\big((\Omega \setminus J_u) \cap \partial^* \{u > t\}\big)\, dt. \tag{6.1}$$

Let $E_i = \{u > t_i\} \setminus \{u > t_{i+1}\}$ and note that each set has finite perimeter in Ω since it is the difference of two sets of finite perimeter. Now (6.1) implies

$$\sum\nolimits_{i \in \mathbb{Z}} \mathcal{H}^{d-1}\big((\Omega \cap \partial^* E_i) \setminus J_u\big) \leq \frac{2}{M} \|\nabla u\|_{L^1(\Omega)} = 2\rho. \tag{6.2}$$

Since $|\Omega \setminus \bigcup_{i \in \mathbb{Z}} E_i| = 0$, $(E_i)_i$ is a Caccioppoli partition of Ω. Moreover, we note that the function $v := u - \sum_i t_i \chi_{E_i}$ lies in $L^\infty(\Omega)$ and satisfies $\|v\|_{L^\infty(E_i)} \leq 2M$. This implies (2.3). Consider the sequence $v_n = \sum_{|i| \leq n} (u - t_i)\chi_{E_i}$ for $n \in \mathbb{N}$ with $v_n \in SBV(\Omega; \mathbb{R})$ by [5, Theorem 3.84]. Since $v_n \to v$ in BV norm and SBV is a closed subspace of BV, we conclude $v \in SBV(\Omega; \mathbb{R}) \cap L^\infty(\Omega)$. This concludes the proof in the case $m = 1$ and $u \in SBV(\Omega; \mathbb{R})$.

If $u \in GSBV(\Omega; \mathbb{R})$, we apply the analog of the coarea for $GSBV$ functions (see [5, Theorem 4.34]) and again obtain $(t_i)_{i \in \mathbb{Z}}$, $(E_i)_{i \in \mathbb{Z}}$ such that $v := u - \sum_i t_i \chi_{E_i} \in L^\infty(\Omega)$ and (2.3) holds. (Note that ∇u, J_u have to be understood in a weaker sense, cf. [5, Section 4.5].) The characterization of scalar $GSBV$ functions (see [5, Remark 4.27]) together with $\nabla u \in L^1(\Omega)$, $\mathcal{H}^{d-1}(J_u) < \infty$, yields $\min\{\max\{-n, u\}, n\} \in SBV(\Omega; \mathbb{R})$ for all $n \in \mathbb{N}$. Consequently, similarly as before, the sequence $v_n = \sum_{|i| \leq n} (u - t_i)\chi_{E_i}$ lies in $SBV(\Omega; \mathbb{R})$, converges to v in BV norm and thus $v \in SBV(\Omega; \mathbb{R}) \cap L^\infty(\Omega)$.

Now let $u \in (GSBV(\Omega; \mathbb{R}))^m$ for $m \geq 2$. We repeat the above argumentation for each component u^j, $j = 1, \ldots, m$, and obtain corresponding $(E_i^j)_{i \in \mathbb{Z}}$ and $(t_i^j)_{i \in \mathbb{Z}}$ such that (6.2) holds with E_i^j in place of E_i and $v^j := u^j - \sum_i t_i^j \chi_{E_i^j}$ satisfies $\|v^j\|_\infty \leq 2\rho^{-1} \|\nabla u\|_{L^1(\Omega)}$. We introduce the Caccioppoli partition of Ω, denoted by $(P_j)_{j\ge1}$, consisting of the (nonempty) sets in

$$\{E^1_{i_1}\cap\ldots\cap E^m_{i_m} : i_1,\ldots,i_m\in\mathbb{Z}\}.$$

In view of (6.2), we get that (2.3)(i) holds for a constant also depending on the dimension m. Fix $P_j = E^1_{i_1}\cap\ldots\cap E^m_{i_m}$ and define the corresponding translation $b_j\in\mathbb{R}^m$ by $b_j = (t_{i_1},\ldots,t_{i_m})$. Then we conclude that $v := u - \sum_j b_j\chi_{P_j}$ satisfies (2.3)(ii) for a constant depending on m. Arguing as before for each component, we find $v \in SBV(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$. □

6.2. **Embedding results.** We now prove the results stated in Section 2.3.

Proof of Theorem 2.7. We first prove (2.7) for $u\in\mathcal{W}(Q)$ for a square $Q\subset\mathbb{R}^2$ in dimension two. Afterwards, we use a slicing and density argument to derive the result for domains in $\mathbb{R}^d$. By Theorem 5.2 for $p=1$ we find a Caccioppoli partition $(P_j)_j$ of $Q\subset\mathbb{R}^2$ and corresponding infinitesimal rigid motions $(a_j)_j$ such that $v := u-\sum_j a_j\chi_{P_j}\in SBV(Q)\cap L^\infty(Q)$ and

$$\begin{aligned}&\|v\|_\infty \le C(\mathcal{H}^1(J_u)+\mathcal{H}^1(\partial Q))^{-1}\|e(u)\|_{L^2(Q)}, \quad \|\nabla v\|_{L^1(Q)}\le C\|e(u)\|_{L^2(Q)},\\ &\sum\nolimits_j \mathcal{H}^1(\partial^* P_j)\le C\mathcal{H}^1(J_u)+C\mathcal{H}^1(\partial Q),\end{aligned} \tag{6.3}$$

where C only depends on the diameter of Q. To conclude the proof of (2.7) for $d=2$, it now suffices to show that

$$\sum\nolimits_j |A_j||P_j| \le C(\mathcal{H}^1(J_u)+1)\|u\|_\infty + C\|e(u)\|_{L^2(Q)} \tag{6.4}$$

for $C=C(Q)$. To this end, we use Lemma 3.7 and the isoperimetric inequality to obtain

$$\sum\nolimits_j |A_j||P_j| \le c\sum\nolimits_j |P_j|^{\frac12}\|a_j\|_{L^\infty(P_j)} \le c(\|u\|_\infty+\|v\|_\infty)\sum\nolimits_j \mathcal{H}^1(\partial^*P_j).$$

Then (6.4) follows from (6.3).

We now treat the case $Q=(-\mu,\mu)^d$ and $u\in\mathcal{W}(Q)\subset SBV(Q)\cap L^\infty(Q)$. To prove the assertion, we need to control $\|\partial_j u_i\|_{L^1(\Omega)}$ for each $1\le i,j\le d$. For notational convenience we only treat the case $i,j\in\{1,2\}$. The other terms follow analogously due to the symmetry of the problem. For $x\in Q$ we write $x=(x_1,x_2,y)$ with $y\in(-\mu,\mu)^{d-2}$ and introduce the functions $w^y : (-\mu,\mu)^2\to\mathbb{R}^2$, $w^y(x_1,x_2) = (u_1(x),u_2(x))$ for $y\in(-\mu,\mu)^{d-2}$. Applying the result in $d=2$ we obtain for a.e. $y\in(-\mu,\mu)^{d-2}$

$$\sum\nolimits_{1\le i,j\le 2}\|\partial_j w^y_i\|_{L^1((-\mu,\mu)^2;\mathbb{R})}\le C\|e(w^y)\|_{L^2((-\mu,\mu)^2;\mathbb{R}^{2\times2}_{\rm sym})} + C\|u\|_\infty(\mathcal{H}^1(J_{w^y})+1),$$

where $C=C(\mu)$. Once we have proved

$$\int_{(-\mu,\mu)^{d-2}}\mathcal{H}^1(J_{w^y})\,d\mathcal{H}^{d-2}(y)\le C\mathcal{H}^{d-1}(J_u), \tag{6.5}$$

we take the integral over $(-\mu,\mu)^{d-2}$, use Fubini's theorem and Hölder's inequality to conclude

$$\sum\nolimits_{1\le i,j\le2}\|\partial_j u_i\|_{L^1(Q;\mathbb{R})}\le C\|e(u)\|_{L^2(Q;\mathbb{R}^{d\times d}_{\rm sym})}+C\|u\|_\infty(\mathcal{H}^{d-1}(J_u)+1).$$

To see (6.5), we apply slicing techniques for BV functions (see [5, Section 3.11]): for $f\in SBV((-\mu,\mu)^n;\mathbb{R}^m)$ and $j=1,\ldots,n$ we define $f_{j,s}:(-\mu,\mu)\to\mathbb{R}^m$ by $f_{j,s}(t) = f(s+t\mathbf{e}_j)$ for $s\in\Pi^n_j := \{s\in(-\mu,\mu)^n : s\cdot\mathbf{e}_j = 0\}$. We obtain

$$\int_{\Pi^n_j}\#J_{f_{j,s}}\,d\mathcal{H}^{n-1}(s) = \int_{J_f}|\nu_f\cdot\mathbf{e}_j|\,d\mathcal{H}^{n-1},$$

where ν_f denotes a normal of the jump set J_f. First, applying this estimate on $w^y \in SBV((-\mu,\mu)^2;\mathbb{R}^2)$ for $j=1,2$ and $y \in (-\mu,\mu)^{n-2}$ a.e. we obtain

$$\int_{\Pi_1^2} \# J_{w^y_{1,s}}\, d\mathcal{H}^1(s) + \int_{\Pi_2^2} \# J_{w^y_{2,s}}\, d\mathcal{H}^1(s) = \int_{J_{w^y}} (|\nu_{w^y}\cdot \mathbf{e}_1| + |\nu_{w^y}\cdot \mathbf{e}_2|)\, d\mathcal{H}^1 \ge \mathcal{H}^1(J_{w^y}).$$

Repeating the argument for the function $w=(u_1,u_2) \in SBV((-\mu,\mu)^d;\mathbb{R}^2)$ for $j=1,2$, we also get

$$\begin{aligned}\int_{\Pi_1^d} \# J_{w_{1,s}}\, d\mathcal{H}^{d-1}(s) + \int_{\Pi_2^d} \# J_{w_{2,s}}\, d\mathcal{H}^{d-1}(s) &= \int_{J_w} (|\nu_w\cdot \mathbf{e}_1| + |\nu_w\cdot \mathbf{e}_2|)\, d\mathcal{H}^{d-1} \\ &\le 2\mathcal{H}^{d-1}(J_w) \le 2\mathcal{H}^{d-1}(J_u).\end{aligned}$$

Taking the integral over $(-\mu,\mu)^{d-2}$ we derive (6.5) from the last two estimates.

It remains to consider general Lipschitz domains $\Omega \subset \mathbb{R}^d$ and $u \in SBD^2(\Omega)\cap L^\infty(\Omega)$. First, we choose a cube Q containing Ω and define the extension $\bar{u} = u\chi_\Omega \in SBD^2(Q)\cap L^\infty(Q)$. Clearly, the choice of Q depends only on Ω. Note that $\mathcal{H}^{d-1}(J_{\bar{u}}) \le \mathcal{H}^{d-1}(J_u) + \mathcal{H}^{d-1}(\partial\Omega)$. By Theorem 3.13, Remark 3.14 we find a sequence $(u_k)_k \subset \mathcal{W}(Q)$ with $u_k \to \bar{u}$ in $L^2(Q)$ and $\|u_k\|_\infty \le \|u\|_\infty$ such that by the above arguments

$$|Du_k|(Q) = \|\nabla u_k\|_{L^1(Q)} + \int_{J_{u_k}} |[u_k]|\, d\mathcal{H}^{d-1} \le C\|e(u)\|_{L^2(\Omega)} + C\|u\|_\infty(\mathcal{H}^{d-1}(J_u)+1) \tag{6.6}$$

for $C=C(\Omega)$, where we used $|[u_k]| \le 2\|u\|_\infty$ almost everywhere. As $(u_k)_k$ is uniformly bounded in BV norm, we deduce $u \in BV(\Omega)$ and that (2.7) holds by lower semicontinuity. Finally, by Alberti's rank one property $|D^c u| \le \sqrt{2}|E^c u|$ and the fact that $u \in SBD(\Omega)$ we conclude $u \in SBV(\Omega)$. □

Proof of Theorem 2.9. Let $u \in GSBD^2(\Omega)$. We show that each component u_i, $i=1,\dots,d$, satisfies (2.8) for the truncation $u_i^M = \min\{\max\{u_i,-M\},M\}$. Herefrom we particularly deduce $u_i \in GBV(\Omega;\mathbb{R})$ since in the scalar case this property is equivalent to $u_i^M \in BV_{\mathrm{loc}}(\Omega)$ for all $M>0$ (see [5, Remark 4.27]). As in the previous proof it essentially suffices to show

$$|Du_i^M|(Q) \le \|\nabla u_i^M\|_{L^1(Q)} + 2M\mathcal{H}^{d-1}(J_{u_i^M}) \le CM(\mathcal{H}^{d-1}(J_u)+1) + C\|e(u)\|_{L^2(Q)}, \tag{6.7}$$

where $\Omega = Q$ is a cube, $C=C(Q)$ and $u \in \mathcal{W}(Q) \subset SBV(Q)$. In fact, we then establish the general case using the approximation of u given by Corollary 5.5 and repeating the argument in (6.6). (Note, however, that in contrast to (6.6), we cannot apply Alberti's theorem and therefore only obtain $u_i^M \in BV(\Omega)$.) Finally, in view of the slicing argument (6.5), it is enough to treat the planar case $u \in \mathcal{W}(Q)$ for a square $Q \subset \mathbb{R}^2$.

By Theorem 5.2 for $p=1$ we get that $v := u - \sum_j a_j\chi_{P_j} \in SBV(Q)\cap L^\infty(Q)$ satisfying $\|v\|_{L^\infty(Q)} \le M'$ with $M' := C(\mathcal{H}^1(J_u)+\mathcal{H}^1(\partial Q))^{-1}\|e(u)\|_{L^2(Q)}$ by (2.2). Let $a = \sum_j a_j\chi_{P_j}$ and a_i be the i-th component for $i=1,2$. From [17, Theorem 2.2] we obtain $a \in (GSBV(Q))^2$, in particular it is shown that $|Da_i^M|(Q) \le cM\sum_j \mathcal{H}^1(\partial^* P_j)$ for a universal $c>0$. Thus,

$$|Da_i^M|(Q) \le cM\sum\nolimits_j \mathcal{H}^1(\partial^* P_j) \le cM(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q)) \tag{6.8}$$

by (2.1). Up to sets of negligible $\mathcal{L}^2$-measure we have

$$T := \{\nabla u_i^M \neq 0\} \subset \{|u_i| \le M\} \subset \{|a_i| \le M + M'\}$$

since $\|v\|_{L^\infty(Q)} \le M'$. Therefore, we compute using (6.8) and (2.2)

$$\begin{aligned} \|\nabla u_i\|_{L^1(T)} &\le \|\nabla v_i\|_{L^1(T)} + \|\nabla a_i\|_{L^1(T)} \le \|\nabla v_i\|_{L^1(Q)} + \|\nabla a_i^{M+M'}\|_{L^1(Q)} \\ &\le c(M+M')(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial Q)) + C\|e(u)\|_{L^2(Q)} \\ &\le CM(\mathcal{H}^1(J_u)+1) + C\|e(u)\|_{L^2(Q)}. \end{aligned} \tag{6.9}$$

As $\|\nabla u_i^M\|_{L^1(Q)} = \|\nabla u_i\|_{L^1(T)}$ and $\mathcal{H}^1(J_{u_i^M} \setminus J_u) = 0$, we obtain the second inequality in (6.7). The first inequality follows from the decomposition of the distributional derivative and the fact that $\|u_i^M\|_\infty \le M$. This concludes the proof. □

We also obtain the following variant of (2.8) needed in Section 6.3.

Corollary 6.1. *Let $Q \subset \mathbb{R}^d$ be a cube. Then there is a constant $C = C(Q) > 0$ such that for all $u \in \mathcal{W}(Q)$ and Borel sets $F \subset Q$ one has*

$$|Du_i^M|(Q) \le CM\big(\mathcal{H}^{d-1}(J_u) + |F|\big) + C\big(\|e(u)\|_{L^2(Q)} + \|u\|_{L^1(Q\setminus F)}\big).$$

for all $i = 1, \dots, d$ and $M > 0$, where $u_i^M =: \min\{\max\{u_i, -M\}, M\}$.

Proof. Similarly as in the proof of Theorem 2.9, it suffices to show this estimate in the planar setting $d = 2$ for a square $Q \subset \mathbb{R}^2$ as the general case then follows by Fubini and the slicing argument (6.5) for a constant depending on the dimension. Let $\bar{c} = \bar{c}(Q) > 0$ to be specified below. If $\mathcal{H}^1(J_u) + |F| \ge \bar{c}$, the result follows directly from (6.7) for a constant depending on $\bar{c}$.

Now let $\mathcal{H}^1(J_u) + |F| \le \bar{c}$. By Theorem 5.2 for $p = 1$ in the version of Remark 5.3 we obtain an ordered Caccioppoli partition $(P_j)_j$ and $v := u - \sum_j a_j \chi_{P_j}$ such that for $C = C(Q) > 0$

$$\|\nabla v\|_{L^1(Q)} + \mathcal{H}^1(J_u)\|v\|_{L^\infty(Q)} \le C\|e(u)\|_{L^2(Q)}, \qquad \sum\nolimits_j \mathcal{H}^1(\partial^* P_j \cap Q) \le C\mathcal{H}^1(J_u). \tag{6.10}$$

The essential step is now to show

$$\|\nabla a_i^{M+M'}\|_{L^1(Q)} \le C\big((M+M')\mathcal{H}^1(J_u) + \|u\|_{L^1(Q\setminus F)} + \|e(u)\|_{L^2(Q)}\big) \tag{6.11}$$

for $a := \sum_j a_j \chi_{P_j}$ and $M' := C(\mathcal{H}^1(J_u))^{-1}\|e(u)\|_{L^2(Q)}$. Indeed, the claim then follows by repeating the argument in (6.9) noting that $\{\nabla u_i^M \neq 0\} \subset \{|a_i| \le M + M'\}$ since $\|v\|_\infty \le M'$.

We now confirm (6.11). For notational convenience we set $\bar{M} = M + M'$. Since $\sum_j \mathcal{H}^1(\partial^* P_j \cap Q) \le C\bar{c}$, for $\bar{c}$ small enough the relative isoperimetric inequality implies $|P_1| > \frac{1}{2}|Q|$ and $|P_j| \le C(\mathcal{H}^1(\partial^* P_j \cap Q))^2$ for $j \ge 2$. Without restriction we assume that the sets $(P_j)_{j\ge 2}$ are connected (more precisely indecomposable, see [5, Example 4.18]) as otherwise we consider the indecomposable components. By [33, Lemma 4.6] we get that the diameter of each P_j, $j \ge 2$, is controlled in terms of $C\mathcal{H}^1(\partial^* P_j \cap Q)$ for $C = C(Q)$, which by [33, Lemma 4.3] then also yields $\mathcal{H}^1(\partial^* P_j) \le C\mathcal{H}^1(\partial^* P_j \cap Q)$ for all $j \ge 2$. Then again by [17, Theorem 2.2] (cf. also (6.8)) and (6.10)

$$\sum_{j\ge 2} \|\nabla a_i^{\bar{M}}\|_{L^1(P_j)} \le c\bar{M} \sum_{j \ge 2} \mathcal{H}^1(\partial^* P_j) \le C\bar{M} \sum_{j\ge 2} \mathcal{H}^1(\partial^* P_j \cap Q) \le C\bar{M}\mathcal{H}^1(J_u). \tag{6.12}$$

By Theorem 3.15 we find an infinitesimal rigid motion $a' = a_{A',b'}$ and an exceptional set $F' \subset Q$ with

$$\|\nabla u - A'\|_{L^1(Q\setminus F')} + \|u - a'\|_{L^1(Q\setminus F')} \le C\|e(u)\|_{L^2(Q)}, \qquad |F'| \le C(\mathcal{H}^1(J_u))^2 \le C\bar{c}^2.$$

Consequently, for $\bar{c}$ small we have $|F \cup F'| \le \bar{c} + C\bar{c}^2 \le \frac{1}{4}|Q|$ and derive using Lemma 3.7, (6.10) and the triangle inequality with $G := F \cup F'$

$$\begin{aligned}
\|\nabla a_i^{\bar{M}}\|_{L^1(P_1)} &\le |P_1||A_1| \le 2|P_1 \setminus G||A_1| \\
&\le 2|Q \setminus G||A'| + 2\|\nabla u - A_1\|_{L^1(P_1)} + 2\|\nabla u - A'\|_{L^1(Q\setminus F')} \\
&\le C|Q \setminus G|^{-\frac{1}{2}}\|a'\|_{L^1(Q\setminus G)} + C\|e(u)\|_{L^2(Q)} \\
&\le C\|u - a'\|_{L^1(Q\setminus F')} + C\|u\|_{L^1(Q\setminus F)} + C\|e(u)\|_{L^2(Q)} \\
&\le C\|u\|_{L^1(Q\setminus F)} + C\|e(u)\|_{L^2(Q)}.
\end{aligned}$$

This together with (6.12) concludes the proof of (6.11). □

We close this section with the proof of Lemma 2.8.

Proof of Lemma 2.8. Let $q \in [1, \infty)$. Consider the function defined in (2.6) with $r_k = k^{-p}$ and $d_k = k^{-1+dp}$, where $p = \frac{1}{d-1} + \frac{1}{q(d-1)^2}$. (Note that the existence of pairwise disjoint balls with this property is guaranteed by [20, Lemma 12.2] and the approximate differential of u in the sense [5, Definition 3.70] exists a.e.) We first see that $\mathcal{H}^{d-1}(J_u) < +\infty$ as $\sum_k r_k^{d-1} < \infty$ due to the fact that $p > \frac{1}{d-1}$. To see that $u \in L^q(\Omega)$ and $\nabla u \notin L^1(\Omega)$, it suffices to show

$$\sum\nolimits_k r_k^d (r_k d_k)^q < \infty, \qquad \sum\nolimits_k r_k^d d_k = \infty.$$

The latter is immediate since $-pd - 1 + pd = -1$. To see the first property we calculate $-p(d+q) - q + dpq = \frac{1}{d-1} - pd < -1$, where in the last step we used $p > \frac{1}{d-1}$.

It remains to show that $u \in GSBD^2(\Omega)$. To this end, we consider the sequence of functions

$$u_j = \sum\nolimits_{k=1}^{j} (A_k\,(x - x_k))\chi_{B_k}(x) \in GSBD^2(\Omega) \cap L^q(\Omega)$$

converging to u and by the compactness theorem for $GSBD$ (see [20, Theorem 11.3]) we see that $u \in GSBD^2(\Omega)$ since $e(u_j) = 0$ a.e., $\sup_j \mathcal{H}^{d-1}(J_{u_j}) < \infty$ and $\sup_j \|u\|_{L^q(\Omega)} < \infty$. □

6.3. **A Korn-Poincaré inequality for functions with small jump set.**

We finally give the proof of the Korn-Poincaré inequality for functions with small jump set.

Proof of Theorem 2.10. As in the previous sections we first treat the case $u \in \mathcal{W}(Q) \subset SBV(Q)$ with $\mathcal{H}^{d-1}(J_u) > 0$ and at the end of the proof we indicate the adaptions if $u \in GSBD^2(Q)$. We may assume that $\mathcal{H}^{d-1}(J_u) \le \mathcal{H}^{d-1}(\partial Q)$ since otherwise the theorem trivially holds with $E = Q$ for $C = C(Q) > 0$ sufficiently large. We apply the Korn-Poincaré inequality due to Chambolle, Conti, and Francfort (see [13, Theorem 1]) which is as the assertion of Theorem 2.10 with the difference that only the volume of the exceptional set can be controlled. We find $F \subset Q$ with $|F| \le C(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}}$ for $C = C(Q) > 0$ and an infinitesimal rigid motion a such that with $q = \frac{2d}{d-1}$ and $v := u - a$

$$\|v\|_{L^q(Q\setminus F)} \le C\|e(u)\|_{L^2(Q)}. \tag{6.13}$$

We now define

$$M := (\mathcal{H}^{d-1}(J_u))^{-\frac{d}{q(d-1)}}\|e(u)\|_{L^2(Q)} = (\mathcal{H}^{d-1}(J_u))^{-\frac{1}{2}}\|e(u)\|_{L^2(Q)}$$

and observe that by (6.13)

$$|\{|v| > M\}| \le |F| + \tfrac{1}{M^q}\|v\|^q_{L^q(Q\setminus F)} \le C(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}}. \tag{6.14}$$

We now consider the truncation $v_i' := \min\{\max\{v_i, -2M\}, 2M\}$ for $i = 1, \ldots, d$. By Corollary 6.1 we get for $i = 1, \ldots, d$

$$
\begin{aligned}
|Dv_i'|(Q) &\le CM(\mathcal{H}^{d-1}(J_u) + |F|) + C\|e(v)\|_{L^2(Q)} + C\|v\|_{L^1(Q\setminus F)} \\
&\le CM\mathcal{H}^{d-1}(J_u) + C\|e(u)\|_{L^2(Q)},
\end{aligned}
\tag{6.15}
$$

where in the last step we used (6.13) and $|F| \le C(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}} \le C\mathcal{H}^{d-1}(J_u)$. (Recall $\mathcal{H}^{d-1}(J_u) \le \mathcal{H}^{d-1}(\partial Q)$.) Now arguing similarly as in the proof of Theorem 2.3 (see Section 6.1), using the coarea formula [5, Theorem 3.40], we can find $t_i \in (M, 2M)$ such that $G_i := \{-t_i < v_i' < t_i\}$ is a set of finite perimeter and

$$
\begin{aligned}
\mathcal{H}^{d-1}(Q \cap \partial^* G_i) &\le \frac{1}{M}\int_{-\infty}^{\infty} \mathcal{H}^{d-1}\big(Q \cap \partial^*\{v_i' > t\}\big)\, dt \le \frac{1}{M}|Dv_i'|(Q) \\
&\le \frac{1}{M}\big(CM\mathcal{H}^{d-1}(J_u) + C\|e(u)\|_{L^2(Q)}\big) \le C(\mathcal{H}^{d-1}(J_u))^{\frac{1}{2}},
\end{aligned}
\tag{6.16}
$$

where in the last step we used $\mathcal{H}^{d-1}(J_u) \le C(\mathcal{H}^{d-1}(J_u))^{\frac{1}{2}}$ for a constant $C = C(Q)$.

We define $E = Q \setminus \bigcap_{i=1}^d G_i$. As $Q \setminus G_i \subset \{|v_i'| > M\}$ and $\{|v_i'| > M\} \subset \{|v_i| > M\}$, (6.14) yields the second part of (2.9). By (6.16) we get $\mathcal{H}^{d-1}(\partial^* E \cap Q) \le C(\mathcal{H}^{d-1}(J_u))^{\frac{1}{2}}$, which shows the first part of (2.9). Since $v_i' = v_i$ on $\{|v_i'| < 2M\}$ for $i = 1, \ldots, d$ and

$$
Q \setminus E = \bigcap\nolimits_{i=1}^d G_i \subset \bigcap\nolimits_{i=1}^d \{|v_i'| < 2M\},
$$

we observe $v' = v$ on $Q \setminus E$, where $v' := (v_1', \ldots, v_d')$. Consequently, we have $\|v\|_{L^\infty(Q\setminus E)} \le CM = C(\mathcal{H}^{d-1}(J_u))^{-\frac{1}{2}}\|e(u)\|_{L^2(Q)}$ by the definition of M. This gives (2.10)(ii) and together with (6.13), $|F| \le C(\mathcal{H}^{d-1}(J_u))^{\frac{d}{d-1}}$ we obtain

$$
\|v\|_{L^q(Q\setminus E)} \le \|v\|_{L^q(Q\setminus F)} + \|v\|_{L^q(F\setminus E)} \le C\|e(u)\|_{L^2(Q)} + CM|F|^{\frac{1}{q}} \le C\|e(u)\|_{L^2(Q)},
$$

which establishes (2.10)(i). Finally, $\bar{u} := (u-a)\chi_{Q\setminus E} = v\chi_{Q\setminus E} = v'\chi_{Q\setminus E}$ clearly lies in $SBV^2(Q) \cap L^\infty(Q)$ since $u \in \mathcal{W}(Q)$. We use (6.15) and [5, Theorem 3.84] to calculate

$$
|D\bar{u}|(Q) \le |Dv'|(Q) + CM\mathcal{H}^{d-1}(\partial^* E \cap Q) \le CM(\mathcal{H}^{d-1}(J_u))^{\frac{1}{2}} + C\|e(u)\|_{L^2(Q)},
$$

which by the definition of M gives (2.11). The variant announced below Theorem 2.10 with $p \in [1,2]$ follows by replacing $q = \frac{2d}{d-1}$ by $q' = \frac{2d}{p(d-1)}$ in the above proof.

Finally, we treat the case $u \in GSBD^2(Q)$ with $\mathcal{H}^{d-1}(J_u) > 0$. We proceed similarly as in the density argument in the proof of Theorem 2.1 (see Section 5.2) and refer therein for details. Let $(u^j)_j$ be a sequence of rescaled versions of u defined on squares $Q_j \supset\supset Q$ with $|Q_j \setminus Q| \to 0$ for $j \to \infty$. Using Lemma 5.4 we approximate $(u^j)_j$ with a (diagonal) sequence $(u_k)_k \subset \mathcal{W}(Q)$ such that (5.41) holds with $u_k \to u$ a.e. on Q.

We then obtain infinitesimal rigid motions $(a_k)_k$ and exceptional sets $(E_k)_k$ such that (2.9)-(2.11) hold for each $k \in \mathbb{N}$. By compactness we find a set of finite perimeter E satisfying (2.9) such that $\chi_{E_k} \to \chi_E$ in measure after extracting a not relabeled subsequence. Moreover, applying Lemma 3.6 we find that also the infinitesimal rigid motions converge to some a and (2.10) follows by lower semicontinuity.

The sequence $\bar{u}_k := (u_k - a_k)\chi_{Q\setminus E_k}$ is uniformly bounded in BV norm and we therefore deduce $\bar{u} := (u-a)\chi_{Q\setminus E} \in BV(Q)$ satisfies (2.11) by lower semicontinuity. Likewise, a compactness result in SBD^2 together with the uniform bound on $\|\bar{u}_k\|_\infty$ gives $\bar{u} \in SBD^2(Q)$ (see [6]). Finally, Alberti's rank one property $|D^c\bar{u}| \le \sqrt{2}|E^c\bar{u}|$ also yields $\bar{u} \in SBV(Q)$. $\square$

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(Manuel Friedrich) Applied Mathematics Münster, University of Münster, Einsteinstrasse 62, 48149 Münster, Germany.

E-mail address: manuel.friedrich@uni-muenster.de

URL: https://www.uni-muenster.de/AMM/Friedrich/index.shtml