# Connectivity of the k-out Hypercube 

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#### Abstract

In this paper we study the connectivity properties of the random subgraph of the $n$ cube generated by the $k$-out model and denoted by $Q^{n}(k)$. Let $k$ be an integer, $1 \leq k \leq$ $n-1$. We let $Q^{n}(k)$ be the graph that is generated by independently including for every $v \in V\left(Q^{n}\right)$ a set of $k$ distinct edges chosen uniformly from all the $\binom{n}{k}$ sets of distinct edges that are incident to $v$. We study connectivity the properties of $Q^{n}(k)$ as $k$ varies. We show that w.h.p. $\mathbb{1}^{n}(1)$ does not contain a giant component i.e. a component that spans $\Omega\left(2^{n}\right)$ vertices. Thereafter we show that such a component emerges when $k=2$. In addition the giant component spans all but $o\left(2^{n}\right)$ vertices and hence it is unique. We then establish the connectivity threshold found at $k_{0}=\log _{2} n-2 \log _{2} \log _{2} n$. The threshold is sharp in the sense that $Q^{n}\left(\left\lfloor k_{0}\right\rfloor\right)$ is disconnected but $Q^{n}\left(\left\lceil k_{0}\right\rceil+1\right)$ is connected w.h.p. Furthermore we show that w.h.p. $Q^{n}(k)$ is $k$-connected for every $k \geq\left\lceil k_{0}\right\rceil+1$.


## 1 Introduction

The n-dimensional cube, denoted by $Q^{n}$, is the graph with vertex set $V=\{0,1\}^{n}$ in which two vertices are connected if and only if they differ into precisely one coordinate. Clearly $Q^{n}$ is an $n$-regular bipartite graph on $2^{n}$ vertices. In this paper we study the random subgraph of the $n$-cube generated by the $k$-out model and denoted by $Q^{n}(k)$. Let $k$ be an integer, $1 \leq k \leq n-1$. We let $Q^{n}(k)$ be the graph that is generated by independently including for every $v \in V\left(Q^{n}\right)$ a set of $k$ distinct edges chosen uniformly from all the $\binom{n}{k}$ sets of distinct edges that are incident to $v$.

Random subgraphs of $Q^{n}$ can be generated in various ways. The most usual way to generate such graphs is either using the $G\left(Q^{n}, p\right)$ model or the $\left(Q^{n}\right)_{t}$ random process. In the $G\left(Q^{n}, p\right)$ model every edge of $Q^{n}$ is included independently with probability $0<p<1$. On the other hand the random process $\left(Q^{n}\right)_{t}$ is generated by starting with $\left(Q^{n}\right)_{0}$, the empty

[^0]graph on $V$, and extending $\left(Q^{n}\right)_{i}$ to $\left(Q^{n}\right)_{i+1}$ by adding to $\left(Q^{n}\right)_{i}$ an edge from $Q^{n}$, that is not currently present, uniformly at random. Various results on connectivity have been establish in both models. Burtin [5] was the first to study the connectivity of $G\left(Q^{n}, p\right)$. He proved that $G\left(Q^{n}, p\right)$ connected w.h.p. when $p>\frac{1}{2}$ and is disconnected when $p<\frac{1}{2}$. His result was sharped by Erdös and Spencer who also conjectured that if $p \geq \frac{1+\epsilon}{n}, \epsilon>0$, then $G\left(Q^{n}, p\right)$ almost surely has a giant component. Their conjecture was verified by Ajtai, Komlós and Szemerédi [1]. The connectivity of the random process $\left(Q^{n}\right)_{t}$ was studied by Bollobás, Kohayakawa and Luczak [2, 3]. They established the following result. Let $\ell=O(1)$, and $\tau_{\ell}=\min \left\{t \in\left[n 2^{n}\right]: \delta_{t} \geq \ell\right\}$. Then w.h.p $\left(Q^{n}\right)_{\tau_{\ell}}$ is $\ell$-connected. Here by $\delta_{t}$ we denote the minimum degree of $\left(Q^{n}\right)_{t}$. Furthermore we say that a graph is $\ell$-connected if it has more than $\ell$ vertices and remains connected whenever fewer than $\ell$ vertices are removed.

Observe that $\ell$-connectivity requires that the minimum degree of a graph is at least $\ell$. In both of the above models one has to wait until the minimum degree is $\ell$. Once this requirement is fulfilled then the graph is $\ell$-connected w.h.p. One is therefore interested in models of a random graph which guarantee a certain minimum degree while not having too many edges. The $k$-out model meets this requirement.

There have already been studies on connectivity properties of random graphs that are generated by the k-out model. For an arbitrary graph $G$ let $G(k)$ denote the random subgraph of $G$ that is generated by the $k$-out model, $1 \leq k \leq \delta(G)$ (here $\delta(G)$ denotes the minimum degree of $G$ ). In the case that $G$ is the compete graph on $n$ vertices $K_{n}$ the following are known to hold w.h.p. (see [10]). First $K_{n}(1)$ is disconnected. Then $K_{n}(2)$ is connected, a proof of which can been found in the Scottish book [12]. Furthermore Fenner and Frieze [8] show that for $k \geq 2$ we have that $K_{n}(k)$ is $k$-connected. The last theorem has been recently generalized by Frieze and Johansson [9] in the case where $k=O(1)$. They showed that for an arbitrary graph $G$ of minimum degree $\delta(G) \geq\left(\frac{1}{2}+\epsilon\right) n$ we have that the random graph $G(k)$ is $k$-connected for $2 \leq k=O(1)$.

In this paper we study connectivity properties of $Q^{n}(k)$ as $n \rightarrow \infty$. As we vary $k$ we ask whether some specific connectivity properties hold. Our results are summarized in the three theorems given below.

Theorem 1.1. W.h.p. $Q^{n}(1)$ does not contain a component spanning $\Omega\left(2^{n}\right)$ vertices.
Theorem 1.2. W.h.p. $Q^{n}(2)$ contains a unique giant component that spans all but o $\left(2^{n}\right)$ vertices.

Theorem 1.3. Let $k_{0}=\log _{2} n-2 \log _{2} \log _{2} n$. Then w.h.p. the following hold,

1. if $k \leq\left\lfloor k_{0}\right\rfloor$ then $Q^{n}(k)$ is disconnected,
2. if $k \geq\left\lceil k_{0}\right\rceil+1$ then $Q^{n}(k)$ is $k$-connected.

The most surprisingly feature of our results, as opposed to what someone might expect based on the results concerning $K_{n}(2)$, is that $Q^{n}(2)$ is not connected. Furthermore even though $Q^{n}(2)$ contains a giant component that spans all but $o\left(2^{n}\right)$ vertices we have that as $k$ increases $Q^{n}(k)$ persists on not being connected until $k$ passes $k_{0}$. On the other hand, as is proved for $K_{n}(k)$, we are able to prove that if $Q^{n}(k)$ is connected then it is $k$-connected.

The rest of the paper is split as follows. In section 2 we give some notation and preliminary
results. Thereafter in Sections 3 and 4 we give the proofs of Theorem 1.1 and 1.2 respectively. We continue by proving the first part of Theorem 1.3 in Section 5. We give the proof of the second part in section 6 . We close with section 7 .

## 2 Notation-Preliminaries

In this section we give some definitions and basic results that are used throughout the paper. We use $V$ and $E$ in order to denote $V\left(Q^{n}\right)$ and $E\left(Q^{n}\right)$ respectively.

Definition 2.1. We say that a graph $G$ is $\ell$-connected if it has more than $\ell$ vertices and remains connected whenever fewer than $\ell$ vertices are removed.

Notation. For $v \in V$ and $A \subset V$ we set $E(v, A):=\{v w \in E: w \in A\}$. Furthermore for $A, B \subset V$ we set $E(A, B):=\{v w \in E(G): v \in A, w \in B\}$. Finally we denote the quantities $|E(v, A)|$ and $|E(A, B)|$ by $d(v, A)$ and $d(A, B)$ respectively.

In various places we are going use the following inequalities (see [6], [13]). By $\operatorname{Bin}(n, p)$ we mean the $\operatorname{Binomial}(n, p)$ random variable.

Lemma 2.2. (Chernoff's bounds) Let $X$ be distributed as a $\operatorname{Bin}(n, p)$ random variable. Then for any $0 \leq \epsilon \leq 1$,

$$
\mathbb{P}(X \geq(1+\epsilon) n p) \leq e^{-\frac{\epsilon^{2} n p}{3}}
$$

Lemma 2.3. (McDiarmid's Inequality) Let $X_{1}, X_{2}, \ldots, X_{n}$ be independent random variables with $X_{i}$ taking values in $A_{i}$. Further, let $f: \prod_{i \in[n]} A_{i} \mapsto \mathbb{R}$ and assume there exist $c_{1}, \ldots, c_{n} \in$ $\mathbb{R}$ such that whenever $x, x^{\prime}$ differ only in their $i$-th coordinate we have

$$
\begin{equation*}
\left|f(x)-f\left(x^{\prime}\right)\right| \leq c_{i} \tag{1}
\end{equation*}
$$

Then $\forall \epsilon>0$,

$$
\mathbb{P}[f-\mathbb{E}[f(x)] \geq \epsilon] \leq \exp \left\{\frac{-2 \epsilon^{2}}{\sum_{i=1}^{n} c_{i}^{2}}\right\}
$$

Remark 2.4. In the setup above if $f$ satisfies condition (1) then so does $-f$. Therefore by applying McDiarmid's Inequality twice, once with $f$ and once with $-f$, we get that $\forall \beta>0$

$$
\mathbb{P}\{f \notin[(1-\beta) \mathbb{E}(f(x)),(1+\beta) \mathbb{E}(f(x))]\} \leq 2 \exp \left\{\frac{-2[\beta \mathbb{E}(f(x))]^{2}}{\sum_{i}^{n} c_{i}^{2}}\right\}
$$

We will also use the following isoperimetric inequality, see for example Bollobás and Leader (4).

Lemma 2.5. Let $A \subset V$. Then,

$$
d(A, A) \leq \frac{|A| \log _{2}|A|}{2}
$$

Equivalently, since $d(A, V \backslash A)=n|A|-2 d(A, A)$, we have

$$
d(A, V \backslash A) \geq n|A|-|A| \log _{2}|A|
$$

Remark 2.6. The function $f(x)=n x-x \log _{2} x$ has a unique stationary point which is a maximum. Therefore $\forall a \leq b \in\left[0,2^{n}\right]$ if $A \subset V$ and $|A| \in[a, b]$ then Lemma 2.5 implies that $d(A, V \backslash A) \geq \min \{f(a), f(b)\}$.

Finally we are also going to use the following two results. For a proof of Lemma 2.7 in the case where the underline graph has maximum degree $D$ see Knuth [11].

Lemma 2.7. For $v \in V$ there are at most $\binom{s n}{s} /[(n-1) s+1]$ trees $T$ such that $|T|=s$ and $v \in V(T)$.

Corollary 2.8. For $v \in V$ there are at most (en) sets $S$ such that (i) $v \in S$, (ii) $|S|=s$ and (iii) $G[S]$ is connected.

Proof. It follows directly from Lemma 2.8 and the following inequality,

$$
\binom{s n}{s} /[(n-1) s+1] \leq\binom{ s n}{s} \leq\left(\frac{e n s}{s}\right)^{s}=(e n)^{s}
$$

## 3 Structural properties of $Q^{n}(1)$

We split this section into two parts. In the first part we prove Theorem 1.1, Thereafter we split $\left[0,2^{n}\right]$ into sub-intervals and for each interval we study the number of components in $Q^{n}(1)$ with size in that interval. We use this information to prove Theorem 1.2 in the next section.

We generate $Q^{n}(1)$ in the following manner. Every vertex $v \in V$ independently chooses a vertex $f(v)$ from those adjacent to it in $Q^{n}$ uniformly at random. Let $E_{D}$ be the set of arcs $\{(v, f(v)): v \in V\}$. We set $G_{D}:=\left(V, E_{D}\right)$. Lastly we set $Q^{n}(1)$ be the simple graph that we get from $G_{D}$ when we ignore orientation.

Remark 3.1. $G_{D}$ is the union of in-arborescences and directed cycles. Moreover the inarborescences can be chosen such that the root of every in-arborescence lies on a cycle.

### 3.1 The lack of a giant component

Lemma 3.2. W.h.p. $\nexists v, w \in V$ such that in $G_{D}$ there is a directed path from $u$ to $w$ of length larger than $n^{2}$.

Proof. Let $v \in V$. Explore, by sequentially revealing the out-edges, the vertices that $v$ can reach (i.e the vertices $v, f(v), f^{2}(v), \ldots$ ). Suppose that we have revealed the arcs $(v, f(v)), \ldots,\left(f^{i-1}(v), f^{i}(v)\right)$ and that these arcs do not span a cycle. Then $\left(f^{i}(v), f^{i+1}(v)\right)$ is still distributed uniformly at random over the $n$ arcs out of $f^{i}(v)$. Thus with probability $\frac{1}{n}$ we have $f^{i-1}(v)=f^{i+1}(v)$, in which case we say that $f^{i}(v)$ "closes the path". Let $B\left(v, n^{2}\right)$ be the event that there exists a directed path out of $v$ of size larger than $n^{2}$ and $A(v, i)$ be
the event that $(v, f(v)), \ldots,\left(f^{i-1}(v), f^{i}(v)\right)$ do not span a cycle. Then,

$$
\begin{aligned}
\mathbb{P}\left(\exists v \in V: B\left(v, n^{2}\right)\right) & \leq \sum_{v \in V} \prod_{i \in\left[n^{2}\right]} \mathbb{P}\left(f^{i}(v) \text { does not close the path } \mid A(v, i)\right) \\
& \leq \sum_{v \in V} \prod_{i \in\left[n^{2}\right]}\left(1-\frac{1}{n}\right)=2^{n}\left(1-\frac{1}{n}\right)^{n^{2}} \leq 2^{n} \cdot e^{-\frac{1}{n} \cdot n^{2}}=o(1) .
\end{aligned}
$$

Proof of Theorem 1.1, Let $Z$ be that number of unordered pairs $u, v \in V$ such that there is a path from $u$ to $v$ in $Q^{n}(1)$ (i.e. $u, v$ belong to the same component in $\left.Q^{n}(1)\right)$. Let $u, v$ be such a pair. Then in $G_{D}$ there exists a unique $w \in V$ such that there exists di-paths (directed paths) from both $u$ and $v$ to $w$ that share no vertices other than $w$. (here we use the convection that for every $v \in V$ there is a di-path from $v$ to $v$ of length 0 ). Set $Z=Z_{S}+Z_{L}$ where $Z_{S}$ counts the pairs of vertices where both of the corresponding paths have length at most $n^{2}$ and $Z_{L}$ counts the rest of the pairs. Lemma 3.2 implies that there do not exist any di-paths of size larger than $n^{2}$ thus $Z_{L}=0$ w.h.p.

In order to bound $Z_{S}$ for $v, w \in V$ let $\mathcal{P}_{u \mapsto w}$ be the set of all di-paths from $v$ to $w$ of length at most $n^{2}$. Furthermore for $u, v, w \in V$ let $\mathcal{P}_{u, v \mapsto w} \subset \mathcal{P}_{u \mapsto w} \times \mathcal{P}_{v \mapsto w}$ be the set of all the pairs of di-paths $\left(P_{1}, P_{2}\right) \in \mathcal{P}_{u \mapsto w} \times \mathcal{P}_{v \mapsto w}$ where $P_{1}$ and $P_{2}$ do not share any vertices other than $w$. We denote by $\mathbb{I}(\cdot)$ the indicator function. Therefore for a path $P$ we have that $\mathbb{I}(P)=1$ in the event that $E(P) \subset E\left(G_{D}\right)$ and 0 otherwise. In addition for a set of paths $\mathcal{P}$ we have that $\mathbb{I}(\mathcal{P})=1$ in the event that there exists some $P \in \mathcal{P}$ such that $\mathbb{I}(P)=1$ and 0 otherwise. Finally for a pair of paths $P_{1}, P_{2}$ we set $\mathbb{I}\left(P_{1} \wedge P_{2}\right)=\mathbb{I}\left(P_{1}\right) \mathbb{I}\left(P_{2}\right)$. Thus

$$
\begin{align*}
\mathbb{E}\left(Z_{S}\right) & =\mathbb{E}\left[\sum_{u, v, w \in V} \sum_{\left(P_{1}, P_{2}\right) \in \mathcal{P}_{u, v \mapsto w}} \mathbb{I}\left(P_{1} \wedge P_{2}\right)\right]=\sum_{u, v, w \in V} \sum_{\left(P_{1}, P_{2}\right) \in \mathcal{P}_{u, v \mapsto w}} \mathbb{E}\left[\mathbb{I}\left(P_{1}\right)\right] \mathbb{E}\left[\mathbb{I}\left(P_{2}\right)\right](2)  \tag{2}\\
& \left.\leq \sum_{u, v, w \in V} \sum_{\left(P_{1}, P_{2}\right) \in \mathcal{P}_{u \mapsto w} \times \mathcal{P}_{v \mapsto w}} \mathbb{I}\left(P_{1}\right)\right] \mathbb{E}\left[\mathbb{I}\left(P_{2}\right)\right]=\sum_{u, v, w \in V} \mathbb{E}\left[\mathbb{I}\left(\mathcal{P}_{u \mapsto w}\right)\right] \mathbb{E}\left[\mathbb{I}\left(\mathcal{P}_{v \mapsto w}\right)\right] \\
& =\sum_{w \in V} \mathbb{E}\left(\sum_{u \in V} \mathbb{I}\left(\mathcal{P}_{u \mapsto w}\right)\right) \mathbb{E}\left(\sum_{v \in V} \mathbb{I}\left(\mathcal{P}_{v \mapsto w}\right)\right)
\end{align*}
$$

In the second equality we used linearity of expectations and the fact that if $\left(P_{1}, P_{2}\right) \in \mathcal{P}_{u, v \mapsto w}$ then $P_{1}, P_{2}$ do not share any vertices other than $w$. Therefore since both $P_{1}, P_{2}$ are both directed towards $w$ we have that they appear independently in $G_{D}$. In the inequality we used that $\mathcal{P}_{u, v \mapsto w} \subset \mathcal{P}_{u \mapsto w} \times \mathcal{P}_{v \mapsto w}$. Observe that for every $w_{1}, w_{2} \in V$ we have

$$
\mathbb{E}\left(\sum_{v \in V} \mathbb{I}\left(\mathcal{P}_{v \mapsto w_{1}}\right)\right)=\mathbb{E}\left(\sum_{v \in V} \mathbb{I}\left(\mathcal{P}_{v \mapsto w_{2}}\right)\right) .
$$

Therefore for $w \in V$

$$
\begin{aligned}
\mathbb{E}\left(\sum_{v \in V} \mathbb{I}\left(\mathcal{P}_{v \mapsto w}\right)\right) & =\frac{1}{|V|} \sum_{w^{\prime} \in V} \mathbb{E}\left(\sum_{v \in V} \mathbb{I}\left(\mathcal{P}_{v \mapsto w^{\prime}}\right)\right)=\frac{1}{2^{n}} \sum_{v \in V} \mathbb{E}\left(\sum_{w^{\prime} \in V} \mathbb{I}\left(\mathcal{P}_{v \mapsto w^{\prime}}\right)\right) \\
& \leq 2^{-n} \sum_{v \in V}\left(n^{2}+1\right)=\left(n^{2}+1\right) .
\end{aligned}
$$

In the inequality we used that out of any vertex there are at most $n^{2}+1$ di-paths of length at most $n^{2}$ (counting the path of length 0 ). Substituting in (2) we get

$$
\mathbb{E}\left(Z_{S}\right) \leq \sum_{w \in V}\left(n^{2}+1\right)^{2}=\left(n^{2}+1\right)^{2} 2^{n}
$$

Thus the Markov inequality gives us

$$
\mathbb{P}\left(Z_{S} \geq n^{6} 2^{n}\right)=o(1)
$$

Hence w.h.p. $Z=Z_{S}+Z_{L} \leq n^{6} 2^{n}$.
Now let $C$ be the size of a largest component in $Q^{n}(1)$. By summing over all the unordered pairs of vertices that belong to a largest component of $Q^{n}(1)$, including pairs of repeated vertices, we have

$$
C^{2} \leq Z
$$

Therefore given that $Z \leq n^{6} 2^{n}$ we have that $C \leq n^{3} 2^{\frac{n}{2}}=o\left(2^{n}\right)$ and so there is no component spanning $\Omega\left(2^{n}\right)$ vertices.

### 3.2 Distribution of Cycles in $Q^{n}(1)$

The following lemma is the first step in proving that $Q^{n}(2)$ has a giant component.
Lemma 3.3. W.h.p. in $Q^{n}(1)$ the following hold.
(i) There are $(1+o(1)) 2^{n-1} / n$ cycles of length two.
(ii) There are $O\left(2^{n} / n^{1.5}\right)$ cycles of length greater than two.

Proof. Since $Q^{n}$ is bipartite it does not contain any odd cycles. For $\ell \in\left[2^{n-1}\right]$ let $X_{2 \ell}$ be the number of cycles of length $2 \ell$ and set $X=\sum_{i \in \mathbb{N}} X_{2 i}$. Also let $Y_{2 \ell}$ be the number of vertices that lie on cycles of length $2 \ell$. Clearly $X_{2 \ell}=\frac{1}{2 \ell} Y_{2 \ell} \leq \frac{1}{2} Y_{2 \ell}$. We start by proving (i).

For $v \in V$ we denote by $e_{1}(v)$ the edge that is chosen by $v$ in the generation of $Q^{n}(1)$. We first bound $\mathbb{E}\left(X_{2}\right)$. Then we use McDiarmid's inequality to establish concentration of $X_{2}$.

A vertex $v$ lies on a 2-cycle in $Q^{n}(1)$ if $v$ and one of its neighbors choose the edge between them, thus with probability $n \cdot 1 / n^{2}$. Therefore $\mathbb{E}\left(Y_{2}\right)=\frac{2^{n}}{n}$. Observe that $Y_{2}$ is a function of $e\left(v_{1}\right), \ldots, e\left(v_{2^{n}}\right)$. Moreover if we alter one of the edges then we may destroy or/and we may create at most one cycle of size 2 . Thus Lemma [2.3, with $c_{i}=1$ for $i \in\left[2^{n}\right]$, implies

$$
\mathbb{P}\left(\left|Y_{2}-\mathbb{E}\left(Y_{2}\right)\right| \geq 2^{\frac{3 n}{4}}\right) \leq 2 \exp \left\{\frac{2^{\frac{3 n}{2}}}{2^{n}}\right\}=o(1)
$$

Hence w.h.p. $X_{2}=Y_{2} / 2=(1+o(1)) \frac{2^{n-1}}{n}$.
In order to prove (ii) (i.e. to bound the number of cycles of size at least 4) for every $v \in V$ we define the sequence $\left\{S_{i}(v)\right\}_{i \in \mathbb{N}}$ with elements in $[n]$ as follows. For $i \in \mathbb{N}$ we set $S_{i}(v)$ to be the coordinate in which $f^{i-1}(v)$ and $f^{i}(v)$ differ. Note we can deduce $\left\{f^{i}(v)\right\}_{i \geq 0}$ from $\left\{S_{i}(v)\right\}_{i \in \mathbb{N}}$.

Remark 3.4. Given that $(v, f(v)), \ldots,\left(f^{i-1}(v), f^{i}(v)\right)$ do not span a cycle of any length, $f^{i+1}(v)$ is independent of $v, f(v), \ldots, f^{i}(v)$ and is distributed uniformly at random over all the neighbors of $f^{i}(v)$. Hence $S_{i+1}(v)$ is distributed uniformly over [ $n$ ].

We now reveal the terms of $\left\{S_{i}(v)\right\}_{i \in \mathbb{N}}$ one by one. Let $\mathcal{E}_{2 \ell}(v)$ be the event that $v$ belongs to a cycle of length $2 \ell$. If $\mathcal{E}_{2 \ell}(v)$ occurs then $f^{2 \ell}(v)=v$. Hence every element in $[n]$ appears an even number of times in the first $2 \ell$ terms of the sequence $\left\{S_{i}(v)\right\}_{v \in \mathbb{N}}$. That is because if $x \in[n]$ appears an odd number of times among those terms then $v$ and $f^{2 \ell}(v)$ differ in their $x$-th entry. Therefore in the event $\mathcal{E}_{2 \ell}(v)$ we can pair the first $2 \ell$ terms of $\left\{S_{i}(v)\right\}_{v \in \mathbb{N}}$ such that in every pair the two elements are the same. We can pair the $2 \ell$ terms by first choosing $\ell$ terms out of the $2 \ell$ and then pairing them with the remaining ones. That is in $\binom{2 \ell}{\ell} \ell$ ! ways. Assume that the $a^{\text {th }}$ term of the sequence, $S_{a}(v)$, is paired with the $b^{t h}$ term $S_{b}(v), a<b$. Then given that $(v, f(v)), \ldots,\left(f^{b-2}(v), f^{b-1}(v)\right)$ do not span a cycle of any length, $S_{b}(v)$ equals $S_{a}(v)$ with probability $\frac{1}{n}$. Hence,

$$
\mathbb{E}\left(\sum_{\ell=2}^{n / 4} X_{2 \ell}\right) \leq \mathbb{E}\left(\sum_{\ell=2}^{n / 4} Y_{2 \ell}\right) \leq 2^{n} \sum_{\ell=2}^{n / 4}\binom{2 \ell}{\ell} \ell!\left(\frac{1}{n}\right)^{\ell} \leq 2^{n} \sum_{\ell=2}^{n / 4}\left(\frac{2 \ell}{n}\right)^{\ell}=O\left(\frac{2^{n}}{n^{2}}\right)
$$

The Markov inequality implies that w.h.p. $\sum_{\ell=2}^{n / 4} X_{2 \ell} \leq \frac{2^{n}}{n^{1.5}}$.
A cycle of size $2 \ell$ induces a path in $G_{D}$ of length $2 \ell$. Therefore Lemma 3.2 implies that there does not exists a cycle of size larger than $n^{2}$. To bound the remaining variables $Y_{2 \ell}$, i.e. when $\ell \in\left[n / 4, n^{2} / 2\right]$, we define the sequence $\left\{L_{i}(v)\right\}_{i \in \mathbb{N}}$. For $i \in \mathbb{N}$ we set $L_{i}(v)=\{j \in[n]$ : $\left.(v)_{j} \neq\left(f^{i}(v)\right)_{j}\right\}$. By $\left(f^{i}(v)\right)_{j}$ we denote the $j$-th coordinate of $f^{i}(v)$, hence $L_{i}(v)$ records the entries in which $v$ and $f^{i}(v)$ differ.

Recall that given that the edges $(v, f(v)), \ldots,\left(f^{i-1}(v), f^{i}(v)\right)$ do not span a cycle of any length $f_{i+1}(v)$ is chosen uniformly at random from the $n$ neighbors of $f^{i}(v)$. Let $j=S^{i+1}(v)$. If $j \in L_{i}(v)$, then $L_{i+1}(v)=L_{i}(v) \backslash\{j\}$. On the other hand if $j \in[n] \backslash L_{i}(v)$, then $L_{i+1}(v)=$ $L_{i}(v) \cup\{j\}$. Thus, as $j$ is chosen uniformly at random from $[n]$ we have that $\left|L_{i+1}(v)\right|=$ $\left|L_{i}(v)-1\right|$ with probability $\frac{\left|L_{i}(v)\right|}{n}$ and $\left|L_{i+1}(v)\right|=\left|L_{i}(v)\right|$ with probability $\frac{n-\left|L_{i}(v)\right|}{n}$.

Now let $\left\{L_{i}\right\}_{i \geq 0}$ be the biased random walk defined by,

$$
L_{i}= \begin{cases}0 & \text { if } i=0 \\ L_{i-1}-1 & \text { with probability } \frac{L_{i-1}}{n} \\ L_{i-1}+1 & \text { with probability } \frac{n-L_{i-1}}{n} \\ \text { if } i \geq 1\end{cases}
$$

If $v=f^{0}(v)$ lies on a cycle of length $2 \ell$ then $f^{2 \ell}(v)=v$ and $L_{2 \ell}(v)=\emptyset$. Therefore we can couple the sequence $\left\{\left|L_{i}(v)\right|\right\}_{i \geq 0}$ with the bias random walk $\left\{L_{i}\right\}_{i \geq 0}$ such that if $v$ lies on a cycle of length $2 \ell$ then $L_{2 \ell}=0$. In order to finish the proof we use the following fact whose proof is given in the appendix.
Lemma 3.5. Let $\left\{L_{i}\right\}_{i \geq 0}$ be a biased random walk given by,

$$
L_{i}= \begin{cases}0 & \text { if } i=0 \\ L_{i-1}-1 & \text { with probability } \frac{L_{i-1}}{n} \\ L_{i-1}+1 & \text { with probability } i \geq 1 \\ \frac{n-L_{i-1}}{n} & \text { if } i \geq 1\end{cases}
$$

Then we have that $\mathbb{P}\left(\exists \ell \in\left[n / 4, n^{2}\right]: L_{2 \ell}=0\right) \leq n^{-4}$.
Given Lemma 3.5 we have that

$$
\begin{aligned}
\sum_{\ell=\frac{n}{4}}^{\frac{n^{2}}{2}} \mathbb{E}\left(X_{2 \ell}\right) & \leq \sum_{\ell=\frac{n}{4}}^{\frac{n^{2}}{2}} \mathbb{E}\left(Y_{2 \ell}\right)=2^{n} \mathbb{P}\left(v \text { belongs to a cycle of size } 2 \ell, \ell \in\left[n / 4, n^{2} / 3\right]\right) \\
& \leq 2^{n} \mathbb{P}\left(\exists \ell \in\left[n / 4, n^{2}\right]: L_{2 \ell}=0\right) \leq n^{-4} 2^{n}
\end{aligned}
$$

Finally the Markov inequality implies, $\mathbb{P}\left(\sum_{\ell=n / 4}^{n^{2} / 2} X_{2 \ell} \geq \frac{2^{n}}{n^{2}}\right)=o(1)$.
Corollary 3.6. W.h.p. $Q^{n}(1)$ consists of $(1+o(1)) 2^{n-1} / n$ connected components.
Proof. The corollary follows directly from Lemma 3.3 and the fact that every component in $Q^{n}(1)$ contains a unique cycle.

## 4 The emergence of the Giant Component

### 4.1 The construction of $Q^{n}(1.5)$

Let $V=V_{0} \cup V_{1}$ where $V_{0}=\left\{v \in V: \sum_{i \in[n]}(v)_{i}=0 \bmod 2\right\}$ and $V_{1}=V \backslash V_{0}$. For $v \in V$ denote the edge found in $Q^{n}(1)(v, f(v))$ by $e_{1}(v)$. Given $Q^{n}(1)$ we construct $Q^{n}(1.5)$ as follows. Every $v \in V_{0}$ independently chooses, uniformly at random, an edge $e_{1.5}(v)$ from those adjacent to it in $Q^{n}$, excluding $e_{1}(v)$. Let $E_{1.5}^{\prime}:=\left\{e_{1.5}(v): v \in V_{0}\right\}$ and define $Q^{n}(1.5)$ by $E_{1.5}:=E\left(Q^{n}(1.5)\right):=E_{1.5}^{\prime} \cup E\left(Q^{n}(1)\right)$ and $V\left(Q^{n}(1.5)\right):=V$.

Lemma 4.1. W.h.p. $Q^{n}(1.5)$ consist of at most $\frac{5 \cdot 2^{n}}{n \log _{2} \log _{2} n}$ components.
Remark 4.2. The extra $1 / \log _{2} \log _{2} n$ factor in the number of the components will be the catalyst in our proof for the size of the giant component in $Q^{n}(2)$.

Proof. We split our proof into two parts. In the first one we show that an at most $1 / n$ fraction of the vertices lie in a component of size at most $n^{2}$. After this we show that the number of components of size less than $\log _{2} \log _{2} n$ in $Q^{n}(1.5)$ is small.
Claim 1. W.h.p. in $Q^{n}(1.5)$ at most $\frac{2^{n+2}}{n}$ vertices lie on a component of size at most $n^{2}$.
Proof of Claim 1. Let $C_{1}, \ldots, C_{z}$ be the components of $Q^{n}(1)$. For a given partition of the components of $Q^{n}(1)$ into two sets $P_{1}, P_{2}$ define $E_{P_{1}, P_{2}}(1):=\left\{u v \in E: u \in C_{i}, v \in C_{j}, C_{i} \in\right.$ $P_{1}$ and $\left.C_{j} \in P_{2}\right\}$ and for $i=1,2 V\left(P_{i}\right)=\left\{v \in V: \exists C_{j} \in P_{i}\right.$ with $\left.v \in C_{j}\right\}$. Furthermore define the set of partitions $\mathcal{P}_{1}:=\left\{\left(P_{1}, P_{2}\right):\left|E_{P_{1}, P_{2}}(1)\right| \geq 2^{n-1}\right\}$. For $\left(P_{1}, P_{2}\right) \in \mathcal{P}_{1}$ let
$\mathcal{B}_{1.5}\left(P_{1}, P_{2}\right)$ be the event that $E_{P_{1}, P_{2}}(1) \cap E_{1.5}^{\prime}=\emptyset$. Then

$$
\begin{align*}
\mathbb{P}\left(\mathcal{B}_{1.5}\left(P_{1}, P_{2}\right)\right) & =\prod_{v \in V\left(P_{1}\right) \cap V_{0}}\left(1-\frac{d\left(v, V\left(P_{2}\right)\right)}{n-1}\right) \prod_{v \in V\left(P_{2}\right) \cap V_{0}}\left(1-\frac{d\left(v, V\left(P_{1}\right)\right)}{n-1}\right)  \tag{3}\\
& \leq \exp \left\{-\sum_{v \in V\left(P_{1}\right) \cap V_{0}} \frac{d\left(v, V\left(P_{2}\right)\right)}{n}-\sum_{v \in V\left(P_{2}\right) \cap V_{0}} \frac{d\left(v, V\left(P_{1}\right)\right)}{n}\right\} \\
& =\exp \left\{-\frac{\left.\mid E_{P_{1}, P_{2}(1) \mid}^{n}\right\} \leq \exp \left\{-\frac{2^{n-1}}{n}\right\} .}{} .\right.
\end{align*}
$$

To go from the second to the third line we used that every edge in $E_{P_{1}, P_{2}}(1)$ has one endpoint in each of $V\left(P_{1}\right), V\left(P_{2}\right)$ and that exactly one of those belongs to $V_{0}$. Corollary 3.6 implies that $\left|\mathcal{P}_{1}\right| \leq 2^{\frac{(1+o(1)) 2^{n-1}}{n}}$. Therefore

$$
\begin{equation*}
\mathbb{P}\left(\exists\left(P_{1}, P_{2}\right) \in \mathcal{P}_{1}: \mathcal{B}_{1.5}\left(P_{1}, P_{2}\right) \text { occurs }\right) \leq 2^{\frac{(1+o(1)) 2^{n-1}}{n}} \cdot \exp \left\{-\frac{2^{n-1}}{n}\right\}=o(1) \tag{4}
\end{equation*}
$$

Now let $C_{1}^{\prime}, . ., C_{w}^{\prime}$ be the components of $Q^{n}(1.5)$ where $C_{1}^{\prime}, \ldots, C_{s}^{\prime}, s \leq z$, are all the components of size at most $n^{2}$. Furthermore set $E_{B}(1.5)=\left\{u v \in E: u \in C_{i}^{\prime}, v \in C_{j}^{\prime}\right.$ and $\left.i \neq j\right\}$. Let $\mathcal{B}_{1.5}$ be the event that more than $\frac{2^{n+2}}{n}$ vertices lie on a component of size at most $n^{2}$. If $\mathcal{B}_{1.5}$ occurs then

$$
\begin{aligned}
\left|E_{B}(1.5)\right| & =\frac{1}{2} \sum_{i \in[z]} E\left(C_{i}, V \backslash C_{i}\right) \geq \frac{1}{2} \sum_{i \in[s]} E\left(C_{i}, V \backslash C_{i}\right) \geq \frac{1}{2} \sum_{i \in[s]}\left(n\left|C_{i}\right|-\left|C_{i}\right| \log _{2}\left|C_{i}\right|\right) \\
& \geq \frac{1}{2} \sum_{i \in[s]}\left(n-\log _{2} n^{2}\right)\left|C_{i}\right| \geq \frac{[1-o(1)] n}{2} \cdot \frac{2^{n+2}}{n}=[1-o(1)] 2^{n+1} .
\end{aligned}
$$

We now partition $C_{1}^{\prime}, \ldots, C_{z}^{\prime}$ into two sets $P_{1}^{\prime \prime}, P_{2}^{\prime \prime}$ by independently including each $C_{i}^{\prime}$ in $P_{1}^{\prime \prime}$ with probability 0.5 and into $P_{2}^{\prime \prime}$ otherwise. Let $E_{P_{1}^{\prime \prime}, P_{2}^{\prime \prime}}(1.5)=\left\{u v \in Q^{n}: u \in C_{i}^{\prime}, v \in\right.$ $C_{j}^{\prime}, C_{i}^{\prime} \in P_{1}^{\prime \prime}$ and $\left.C_{j}^{\prime} \in P_{2}^{\prime \prime}\right\}$. Then

$$
\mathbb{E}\left(\left|E_{P_{1}^{\prime \prime}, P_{2}^{\prime \prime}}(1.5)\right|\right)=\sum_{e \in E_{B}} \mathbb{P}\left(e \in E_{P_{1}^{\prime \prime}, P_{2}^{\prime \prime}}(1.5)\right)=\sum_{e \in E_{B}} 0.5 \geq[1-o(1)] 2^{n}
$$

Therefore if the event $\mathcal{B}_{1.5}$ occurs then there exists a partition $P_{1}^{\prime}, P_{2}^{\prime}$ of $C_{1}^{\prime}, \ldots, C_{w}^{\prime}$ such that $E_{P_{1}^{\prime}, P_{2}^{\prime}}(1.5) \geq[1-o(1)] 2^{n}$. Each of $C_{i}^{\prime}$ is a union sets in $\left\{C_{1}, C_{2}, \ldots, C_{w}\right\}$. In addition $C_{i}^{\prime}$ are disjoint. Thus $P_{1}^{\prime}, P_{2}^{\prime}$ induces a partition of $\left\{C_{1}, C_{2}, \ldots, C_{w}\right\}$ into two sets $P_{1}, P_{2}$ such that $\left|E_{P_{1}, P_{2}}(1)\right|=\left|E_{P_{1}^{\prime}, P_{2}^{\prime}}(1.5)\right| \geq[1-o(1)] 2^{n} \geq 2^{n-1}$ and $E_{P_{1}, P_{2}}(1) \cap E_{1.5}^{\prime}=\emptyset$. Hence, since $\mathcal{B}_{1.5}$ implies that there exists $\left(P_{1}, P_{2}\right) \in \mathcal{P}_{1}$ such that $\mathcal{B}_{1.5}\left(P_{1}, P_{2}\right)$ occurs, (4) implies that $\mathbb{P}\left(\mathcal{B}_{1.5}\right)=o(1)$.

Claim 2. W.h.p. in $Q^{n}(1.5)$ there are at most $\frac{2^{n}}{n^{1.5}}$ vertices that lie on a connected component $C_{T}$ which satisfies the following: i) $\left|C_{T}\right| \leq s=\log _{2} \log _{2} n$, ii) any cycle spanned by $C_{T}$ of size larger than 2 has a nonempty intersection with $E_{1.5}^{\prime}$.

Proof of Claim 2. Let $M$ be the number of connected components satisfying the above
conditions and let $H$ be one of them. Then the following are true. $H$ spans a component $H_{1}$ in $G_{D}$ of size at most $s$. Moreover $H_{1}$ spans a cycle $C_{H_{1}}$ of size 2 (due condition ii). In addition if we let $v \in C_{H_{1}} \cap V_{0}$ then $\exists w \in V$ and a component $H_{2}$ in $G_{D}$ such that $e_{1.5}(v)=(v, w), w \in H_{2},\left|H_{2}\right| \leq s$ and $H_{2}$ contains a 2-cycle.

We can specify an instance of the above configuration as follows. First we specify $H_{1}$ by choosing two vertices $v_{1}, v_{2}$ and two in-arborescences $T_{1}, T_{2}$ rooted at $v_{1}$ and $v_{2}$ respectively. Furthermore we request that $v_{2}$ is a neighbor of $v_{1}$ (w.l.o.g $v_{1} \in V_{0}$ ) so that is feasible for $\left\{v_{1}, v_{2}\right\}$ to span a cycle in $G_{D}$. In addition $T_{1}, T_{2}$ must satisfy $\left|T_{1}\right|=s_{1},\left|T_{2}\right|=s_{2}$ for some $s_{1}, s_{2} \leq s$. Thus in $G_{D}, H_{1}$ consists of the 2 in-arborescences $T_{1}, T_{2}$ and the directed cycle $v_{1}, v_{2}, v_{1}$. Thereafter we choose $\left(v_{1}, w\right)=e_{1.5}(v)$. In the case that $H_{1}=H_{2}$ we choose $w \in V\left(H_{1}\right)$ and we set $s_{3}=0$. Otherwise we choose $w \notin V\left(H_{1}\right)$ and we also choose a tree $T_{3}$ that contains $w$ such that it is vertex disjoint from $H_{1}$ and it has size $s_{3} \leq s$. We then choose a vertex $v_{3}$ in $V\left(T_{3}\right)$ and a neighbor of it $v_{4}$. Then we direct every arc on $T_{3}$ either towards $v_{3}$ or $v_{4}$ to create two in-arborescences $T_{4}, T_{5}$ rooted at $v_{3}$ and $v_{4}$ respectively. Thus in $G_{D}, H_{2}$ consists of the 2 in-arborescences $T_{3}, T_{4}$ and the directed cycle $v_{3}, v_{4}, v_{3}$. Furthermore $w$ belongs to one of $T_{3}, T_{4}$. Observe that the probability of all the arcs in $E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup\left\{\left(v_{1}, v_{2}\right),\left(v_{2}, v_{1}\right)\right\}$ occurring in $G_{D}$ is $\frac{1}{n^{s_{1}+s_{2}}}$. In addition $\left(v_{1}, w\right)$ occurs with probability $\frac{1}{n-1}$ and in the case that $H_{1} \neq H_{2}$ the $\operatorname{arcs}$ in $E\left(T_{4}\right) \cup E\left(T_{5}\right) \cup\left\{\left(v_{3}, v_{4}\right),\left(v_{4}, v_{3}\right)\right\}$ occur in $G_{D}$ with probability $\frac{1}{n^{s 3}}$.
There are $2^{n}$ choices for $v_{1}$ thereafter $n$ choices for $v_{2}$. Furthermore from Lemma 2.7 we have that there are at most $\binom{n}{s_{i}} /\left[(n-1) s_{i}+1\right] \leq n^{s_{i}} /\left[(n-1) s_{i}+1\right]$ choices for $T_{i}, i \in\{1,2,3\}$. In the case that $w \in V\left(H_{1}\right)$ or equivalently in the case that $s_{3}=0$ there are at most $s_{1}+s_{2} \leq 2 s$ choices for $w$. Otherwise there are at most $n$ of them. Therefore,

$$
\begin{aligned}
\mathbb{E}(M) & \leq \sum_{s_{1}, s_{2} \leq s} n 2^{n} \prod_{i \in[2]} \frac{n^{s_{i}}}{(n-1) s_{i}+1} \cdot \frac{1}{n^{s_{1}+s_{2}}} \cdot \frac{1}{n-1}\left(2 s+\sum_{s_{3}=1}^{s} \frac{n^{s_{3}}}{(n-1) s_{3}+1} \cdot n \cdot \frac{1}{n^{s_{3}}}\right) \\
& \leq \sum_{s_{1}, s_{2} \leq s} n 2^{n} \cdot \frac{1}{n^{2}(n-1)}(2 s+s) \leq \frac{s^{2} \cdot 3 s \cdot 2^{n}}{n(n-1)}=o\left(\frac{2^{n} \cdot \log _{2} n}{n^{2}}\right) .
\end{aligned}
$$

The Markov inequality implies that $\mathbb{P}\left(M \geq \frac{2^{n}}{n^{1.5}}\right)=o(1)$.
In $Q^{n}(1.5)$ there are w.h.p. at most $\frac{2^{n}}{n^{2}}$ components of size at least $n^{2}$. Furthermore Claim 1 implies that w.h.p. there are at most $\frac{2^{n+2}}{n \log _{2} \log _{2} n}$ components of size in $\left[\log _{2} \log _{2} n, n^{2}\right]$. Thereafter Claim 2 implies that w.h.p. there are at most $\frac{2^{n}}{n^{1.5}}$ components of size at most $s=\log _{2} \log _{2} n$ such that any cycle of length larger than 2 spanned by such a component has a nonempty intersection with $E_{1.5}^{\prime}$. Any component that we have not accounted for must span a cycle of size at least 4 whose edges lie in $E\left(Q^{n}(1)\right)$. The number of such components is bounded by the number of cycles of size larger than 4 in $Q^{n}(1)$ which is $O\left(\frac{2^{n}}{n^{1.5}}\right)$ (by Lemma (3.3). Summing up we have that $Q^{n}(1.5)$ consists of at most $\frac{5 \cdot 2^{n}}{n \log _{2} \log _{2} n}$ connected components.
Lemma 4.3. W.h.p $Q^{n}(2)$ has a connected component of size $[1-o(1)] 2^{n}$.
Proof. We generate $E_{2}^{\prime}$ by independently including for every $v \in V_{1}$ an edge that is adjacent to $v$ chosen uniformly at random from the $n-1$ edges adjacent to it in $Q^{n}$ excluding $e_{1}(v)$.

We then extend $Q^{n}(1.5)$ to $Q^{n}(2)$ by adding to its edge set all the edges in $E_{2}^{\prime}$. Henceforth we follow the same argument given in the prove of Lemma 4.1 Claim 1.

Let $W_{1}, \ldots, W_{z}$ be the components of $Q^{n}(1.5)$. For a given partition of the components of $Q^{n}(1.5)$ into two sets $P_{1}, P_{2}$ define $E_{P_{1}, P_{2}}(1.5)=\left\{u v \in E: u \in W_{i}, v \in W_{j}, W_{i} \in\right.$ $P_{1}$ and $\left.W_{j} \in P_{2}\right\}$. Furthermore define the set of partitions $\mathcal{P}_{1.5}:=\left\{\left(P_{1}, P_{2}\right):\left|E_{P_{1}, P_{2}}(1.5)\right| \geq\right.$ $\left.\frac{25 \cdot 2^{n}}{\log _{2} \log _{2} n}\right\}$. For $\left(P_{1}, P_{2}\right) \in \mathcal{P}_{1.5}$ let $\mathcal{B}_{2}\left(P_{1}, P_{2}\right)$ be the event that $E_{P_{1}, P_{2}}(1.5) \cap E_{2}^{\prime}=\emptyset$. Then similar calculations to those done for (3) imply

$$
\mathbb{P}\left(\mathcal{B}_{2}\left(P_{1}, P_{2}\right)\right) \leq \exp \left(-\frac{\left|E_{P_{1}, P_{2}}(1.5)\right|}{n}\right)=\exp \left(-\frac{25 \cdot 2^{n}}{n \log _{2} \log _{2} n}\right)
$$

As argued earlier $Q^{n}(1.5)$ w.h.p. consists of at most $\frac{5 \cdot 2^{n}}{n \log _{2} \log _{2} n}$ connected components and hence $\left|\mathcal{P}_{1.5}\right| \leq 2^{\frac{5.2^{n}}{\log _{2} \log _{2} n}}$. Therefore

$$
\begin{equation*}
\mathbb{P}\left(\exists\left(P_{1}, P_{2}\right) \in \mathcal{P}_{1.5}: \mathcal{B}_{2}\left(P_{1}, P_{2}\right) \text { occurs }\right) \leq 2^{\frac{5 \cdot 2^{n}}{\log _{2} \log _{2} n}} \cdot \exp \left(-\frac{25 \cdot 2^{n}}{n \log _{2} \log _{2} n}\right)=o(1) \tag{5}
\end{equation*}
$$

Now let $W_{1}^{\prime}, . ., W_{q}^{\prime}$ be the components of $Q^{n}(2)$ and let $\mathcal{B}_{2}$ be the event that $Q^{n}(2)$ has no connected component of size larger than $h=\left(1-\frac{100 \log 2}{\log _{2} \log _{2} n}\right) 2^{n}$. Furthermore define $E_{B}(2)=\left\{u v \in E: u \in W_{i}^{\prime}, v \in W_{j}^{\prime}\right.$ and $\left.i \neq j\right\}$. If $\mathcal{B}_{2}$ occurs then

$$
\begin{aligned}
\left|E_{B}(2)\right| & =\frac{1}{2} \sum_{i \in[z]} E\left(C_{i}, V \backslash C_{i}\right) \geq \frac{1}{2} \sum_{i \in[z]}\left(n\left|C_{i}\right|-\left|C_{i}\right| \log _{2}\left|C_{i}\right|\right) \geq \frac{1}{2} \sum_{i \in[z]}\left(n-\log _{2} h\right)\left|C_{i}\right| \\
& \geq-\frac{2^{n}}{2} \cdot \log _{2}\left\{1-\frac{100 \log 2}{\log _{2} \log _{2} n}\right\} \geq-\frac{2^{n}}{2} \frac{1}{\log 2} \cdot \frac{-100 \log 2}{\log _{2} \log _{2} n}=\frac{50 \cdot 2^{n}}{\log _{2} \log _{2} n} .
\end{aligned}
$$

For the last inequality we used that $-\frac{x}{\log 2} \leq-\frac{\log (1+x)}{\log 2}=-\log _{2}(1+x)$. We place independently and uniformly at random $W_{1}^{\prime}, \ldots, W_{q}^{\prime}$ into one of the two sets $P_{1}^{\prime \prime}, P_{2}^{\prime \prime}$. Hence $P_{1}^{\prime \prime}, P_{2}^{\prime \prime}$ form a random partition of $W_{1}^{\prime}, \ldots, W_{q}^{\prime}$. In expectation half of the edges in $E_{B}(2)$ would cross this random partition (i.e. their endpoints would belong to two components not found in the same set of the partition). Thus there exists a partition of $W_{1}^{\prime}, \ldots, W_{q}^{\prime}$ into two sets $P_{1}^{\prime}, P_{2}^{\prime}$ where the number of edges that cross the partition $P_{1}^{\prime}, P_{2}^{\prime}$ is at least $\left|\frac{E_{B}(2)}{2}\right| \geq \frac{25 \cdot 2^{n}}{\log _{2} \log _{2} n}$. Furthermore $P_{1}^{\prime}, P_{2}^{\prime}$ induces a partition $P_{1}, P_{2}$ on the components of $Q^{n}(1.5), W_{1}, \ldots, W_{z}$, where the same number of edges cross the partition $P_{1}, P_{2}$. In the event $\mathcal{B}_{2}$ that number is at least $\frac{25 \cdot 2^{n}}{n \log _{2} \log _{2} n}$. Hence the occurrence of the event $\mathcal{B}_{2}$ implies that $\exists\left(P_{1}, P_{2}\right) \in \mathcal{P}_{1.5}$ such that $\mathcal{B}_{2}\left(P_{1}, P_{2}\right)$ occurs. Thus (5) implies that $\mathbb{P}\left(\mathcal{B}_{2}\right)=o(1)$.

## 5 Connectivity of $Q^{n}(k)$ - The Lower Bound

We still have not shown that $Q^{n}(2)$ is disconnected. However this fact follows from the first part of Theorem 1.3 where we prove that $\mathbb{P}\left(Q^{n}\left(k_{0}\right)\right.$ is disconnected $)=1-o(1)$ plus the fact that $\mathbb{P}\left(Q^{n}(k)\right.$ is disconnected $)$ is decreasing with respect to $k$.

In order to prove that $Q^{n}\left(k_{0}\right)$ is disconnected we show that $k_{0}$-cubes appear as connected components of $Q^{n}\left(k_{0}\right)$. That is there exists at least one $k_{0}$-cube $A \subset Q^{n}$ such that in the $k_{0^{-}}$ out model every vertex in $A$ chooses its neighbors in $A$ (there are exactly $k$ such neighbors). Moreover no vertex in $V \backslash A$ chooses a neighbor in $A$. We use the following definition in order to describe the two aforementioned events.

Definition 5.1. For $A, B \subset V$ and $k \in \mathbb{N}$ we let $\mathcal{E}_{k}(A \rightarrow B)$ be the event that in $Q^{n}(k)$ every vertex in $A$ chooses its neighbors from $B$.

Lemma 5.2. Let $k_{0}=\left\lfloor\log _{2} n-2 \log _{2} \log _{2} n\right\rfloor$. Then w.h.p. $Q^{n}\left(k_{0}\right)$ is disconnected.
Proof. We say that a $k_{0}$-cube $H$ lies at the level $\ell$ of $Q^{n}$ if there exist a partition of $[n]$ into 3 sets $L_{H}, F_{H}$ and $U_{H}$ of size $\ell, k_{0}$ and $n-\ell-k_{0}$ respectively such that $\forall v \in V(H)$ the entries of $v$ corresponding to the elements in $L_{H}$ (and $U_{H}$ respectively) equal to 1 ( 0 respectively). The entries of $v$ corresponding to elements in $F_{H}$ may be either 0 or 1 i.e. they are free.

Let $H_{1}, H_{2}, \ldots H_{s}$ be the $k_{0}$-cubes that lie at level $\frac{n}{2}$ of $Q^{n}$. Furthermore let $X_{i}$ be the indicator of the event that $H_{i}$ spans a connected component of $Q^{n}\left(k_{0}\right)$ and define $X$ by $X:=\sum_{i \in[s]} X_{i}$.

In order to specify a $k_{0}$-cube $H$ that lies at level $\frac{n}{2}$ we can first specify $L_{H}$, which can be done in $\binom{n}{\frac{n}{2}}$ ways, and thereafter we can specify $F_{H}$ which can be done in $\binom{\frac{n}{2}}{k_{0}}$ ways. Thus there are $s=\binom{n}{\frac{n}{2}}\binom{\frac{n}{2}}{k_{0}} k_{0}$-cubes that lie at level $\frac{n}{2}$. Let $H$ be such a $k_{0}$-cube. We denote by $V_{H}, \overline{V_{H}}$ and $N\left(V_{H}\right)$ the sets $V(H), V \backslash V(H)$ and the neighborhood of $V(H)$ found in $V \backslash V(H)$ respectively. Observe that $X_{H}=1$ if and only if the events $\mathcal{E}_{k_{0}}\left(V_{H} \rightarrow V_{H}\right)$ and $\mathcal{E}_{k_{0}}\left(N\left(V_{H}\right) \rightarrow \overline{V_{H}}\right)$ occur. The probability that a given vertex in $V_{H}$ chooses its $k_{0}$ neighbors from $V_{H}$ in $Q^{n}\left(k_{0}\right)$ is $\binom{n}{k_{0}}^{-1}>n^{-k_{0}}$. Hence

$$
\mathbb{P}\left(\mathcal{E}_{k_{0}}\left(V_{H} \rightarrow V_{H}\right)\right) \geq\left(n^{-k_{0}}\right)^{\left|V_{H}\right|}=n^{-k_{0} 2^{k_{0}}} .
$$

On the other hand if $v \in N\left(V_{H}\right)$ then $v$ has exactly one entry, say $q$, in $L_{H} \cup U_{H}$ such that if $q \in L_{H}$ then $v_{q}=0$ otherwise $v_{q}=1$. Thus $v$ has exactly one neighbor $v^{\prime} \in V_{H}\left(v, v^{\prime}\right.$ differ only on their $q$-th entry). Therefore since there are $\left|V_{H}\right|\left(n-k_{0}\right)$ edges coming out of $V_{H}$ we have that $\left|N\left(V_{H}\right)\right|=\left|V_{H}\right|\left(n-k_{0}\right)=2^{k_{0}}\left(n-k_{0}\right)$ and that the probability that a given vertex in $N\left(V_{H}\right)$ does not choose its neighbor in $V_{H}$ in the $k_{0}$ out model is $\binom{n-1}{k_{0}} /\binom{n}{k_{0}}=\frac{n-k_{0}}{n}$. Thus

$$
\mathbb{P}\left(\mathcal{E}_{k_{0}}\left(N\left(V_{H}\right) \rightarrow \overline{V_{H}}\right)\right)=\left(\frac{n-k_{0}}{n}\right)^{\left|N\left(V_{H}\right)\right|} \geq\left(1-\frac{k_{0}}{n}\right)^{n 2^{k_{0}}} \geq 2^{-k_{0} 2^{k_{0}}}
$$

Observe that $k_{0} 2^{k_{0}} \leq\left(\log _{2} n-2 \log _{2} \log _{2} n\right) \frac{n}{\log _{2}^{2} n}$. Thus, as the events $\mathcal{E}_{k_{0}}\left(V_{H} \rightarrow V_{H}\right)$ and $\mathcal{E}_{k_{0}}\left(N\left(V_{H}\right) \rightarrow \overline{V_{H}}\right)$ are independent, we have the following.

$$
\begin{aligned}
\mathbb{E}(X) & =\binom{n}{\frac{n}{2}}\binom{\frac{n}{2}}{k_{0}} \mathbb{P}\left(\mathcal{E}_{k_{0}}\left(V_{H} \rightarrow V_{H}\right)\right) \mathbb{P}\left(\mathcal{E}_{k_{0}}\left(N\left(V_{H}\right) \rightarrow \overline{V_{H}}\right)\right) \\
& \geq \frac{2^{n}}{n} \cdot n^{-k_{0} 2^{k_{0}}} \cdot 2^{-k_{0} 2^{k_{0}}} \geq 2^{c_{1}},
\end{aligned}
$$

where

$$
\begin{aligned}
c_{1} & =n-\log _{2} n-\left(\log _{2} n-2 \log _{2} \log _{2} n\right) \cdot \frac{n}{\log _{2}^{2} n} \cdot\left(1+\log _{2} n\right) \\
& =-\log _{2} n+2 \log _{2} \log _{2} n \cdot \frac{\left(1+\log _{2} n\right) \cdot n}{\log _{2}^{2} n}-\frac{n}{\log _{2} n} \geq \frac{n \log _{2} \log _{2} n}{\log _{2} n} .
\end{aligned}
$$

Hence $\mathbb{E}(X) \rightarrow \infty$ as $n \rightarrow \infty$.
Now let $i \in[s]$. If $V_{H_{1}} \cap V_{H_{i}} \neq \emptyset$ then if $X_{1}=1$ then there are no edges from $V_{H_{1}} \backslash V_{H_{i}}$ to $V_{H_{i}} \backslash V_{H_{1}}$. Thus $X_{i}=0$ i.e. $\mathbb{P}\left(X_{i}=1 \mid X_{1}=1\right)=0$. On the other hand if $V_{H_{1}} \cap V_{H_{i}}=\emptyset$ and $N\left(V_{H_{1}}\right) \cap N\left(V_{H_{i}}\right)=\emptyset$ then $X_{1}, X_{i}$ are independent i.e $\mathbb{P}\left(X_{i}=1 \mid X_{1}=1\right)=\mathbb{P}\left(X_{i}=1\right)$. Finally if $V_{H_{1}} \cap V_{H_{i}}=\emptyset$ but $N\left(V_{H_{1}}\right) \cap N\left(V_{H_{i}}\right) \neq \emptyset$ we have that $\mathbb{P}\left(X_{i}=1 \mid X_{1}=1\right) \leq \mathbb{P}\left(X_{i}=1\right)$. That is because given $X_{1}=1$ every vertex $v \in N\left(V_{H_{1}}\right) \cap N\left(V_{H_{i}}\right)$ does not choose an edge with endpoint to $V_{H_{1}}$. Hence they choose their $k_{0}$ edges from the remaining ones, which include vertices in $V_{H_{i}}$, any one of which has now larger probability to be chosen. In addition in the event that $v$ chooses an edge with an endpoint in $V_{H_{i}}$ we have $X_{i}=0$. In all three cases for $i \in[s] \backslash\{1\}$ we have that $\mathbb{P}\left(X_{i}=1 \mid X_{1}=1\right) \leq \mathbb{P}\left(X_{i}=1\right)$. Hence

$$
\begin{aligned}
\mathbb{E}\left(X^{2}\right) & =\mathbb{E}\left[\left(\sum_{i=1}^{s} X_{i}\right)^{2}\right]=s \cdot\left[\mathbb{E}\left(X_{1}^{2}\right)+\sum_{i=2}^{s} \mathbb{E}\left(X_{i} X_{1}\right)\right] \\
& =s \cdot\left[\mathbb{P}\left(X_{1}=1\right)+\sum_{i=2}^{s} \mathbb{P}\left(X_{i}=1 \wedge X_{1}=1\right)\right] \\
& =s \cdot\left[\mathbb{P}\left(X_{1}=1\right)+\sum_{i=2}^{s} \mathbb{P}\left(X_{i}=1 \mid X_{1}=1\right) \mathbb{P}\left(X_{1}=1\right)\right] \\
& \leq s \cdot \mathbb{P}\left(X_{1}=1\right)\left[1+\sum_{i=2}^{s} \mathbb{P}\left(X_{i}=1\right)\right] \leq \mathbb{E}(X)[1+\mathbb{E}(X)]
\end{aligned}
$$

Therefore, since $\mathbb{E}(X) \rightarrow \infty$, we have

$$
\mathbb{P}(X>0) \geq \frac{\mathbb{E}(X)^{2}}{\mathbb{E}\left(X^{2}\right)} \geq \frac{\mathbb{E}(X)^{2}}{\mathbb{E}(X)[1+\mathbb{E}(X)]}=\frac{\mathbb{E}(X)}{1+\mathbb{E}(X)}=1-o(1)
$$

## $6 k$-Connectivity of $Q^{n}(k)$

The fact that the threshold for connectivity of $Q^{n}(k)$ is sharp follows from the second part of Theorem 1.3 which we restate as Lemma 6.1.

Lemma 6.1. Let $k \geq\left\lceil\log _{2} n-2 \log _{2} \log _{2} n\right\rceil+1$. Then w.h.p. $Q^{n}(k)$ is $k$-connected.
Let $k_{1}=\left\lceil\log _{2} n-2 \log _{2} \log _{2} n\right\rceil+1$ and let $k \geq k_{1}$. In order to prove that $Q^{n}(k)$ is $k$-connected we use the first moment method. However we are not able to show in one go that the expected number of pairs $S, L$ such that $L=k-1$ and $S$ is disconnected from $V \backslash(S \cup L)$ in $Q^{n}(k)$ tends to zero as $n$ tends to infinity. That is because the upper bounds
that we use on the number of such pairs and on the probability of the corresponding events occurring are not strong enough to yield the desired result. For deriving better bounds a better understanding of the geometry of the hypercube is essential (see Remark 6.9).

In order to circumvent this problem we generate $Q^{n}(k)$ in three steps. We start by generating $G_{0}$ which is distributed as a $Q^{n}(k-1)$. Thereafter we extend $G_{0}$ to $G_{2}$, which is distributed as a $Q^{n}(k)$, in two phases. In the first phase we only allow vertices found in a small set of vertices that can be easily disconnected in $G_{0}$ to choose their $k^{\text {th }}$ edge adjacent to them. The remaining vertices will choose their $k^{t h}$ edge in the second phase. We show that after the first phase every set of vertices that can be easily disconnected is of large size. In this calculation for upper bounding the number of pairs $S, L$ we use Corollary 2.8. Thereafter we argue that after the second phase no such set remains. Here, in order to bound the number of pairs $S, L$ we make the following crucial observation. Fix $L \subset V$. Then every set $S$ for which $\mathbb{P}(S$ is not connected to $V \backslash(S \cup L)) \neq 0$ is a union of components of the subgraph of $G_{1}$ induced by $G \backslash L$.

### 6.1 Generation of the random graph sequence $G_{0} \subseteq G_{1} \subseteq G_{2}$.

We first generate $G_{0}$. Every vertex $v \in V$ independently chooses uniformly at random a set $E_{0}(v)$, consisting of $k-1$ edges, out of the $n$ edges incident to it in $Q^{n}$. We then define $G_{0}$ by $V\left(G_{0}\right):=V, E\left(G_{0}\right):=\bigcup_{v \in V} E_{0}(v)$. Clearly $G_{0}$ is distributed as a $Q^{n}(k-1)$. The following definitions are going to be of use in the constructions of $G_{1}$ and $G_{2}$.

Definition 6.2. Let $G$ be a graph. We say that a set $S \subset V$ can be ( $k-1$ )-disconnected in $G$ if there exists a set $L \subset V$ with $|L|=k-1$ such that there is no edge from $S$ to $V \backslash(S \cup L)$. If such a set $L$ exists we also say that $S$ is $L$-disconnected in $G$.

Definition 6.3. For $i \in\{0,1,2\}$ define

$$
\mathcal{S}_{i}:=\left\{S \subset V: S \text { can be }(k-1) \text {-disconnected in } G_{i}\right\} .
$$

Furthermore for $L \subset V$ define

$$
\mathcal{S}_{i}(L):=\left\{S \subset V: S \text { is a minimal } L \text {-disconnected set in } G_{i}\right\} .
$$

Now let $n_{s}=2^{k_{1}-0.1}$ and $\mathcal{A}_{0}=\left\{v \in V: \exists L, S \subset V\right.$ s.t. $|L|=k-1, S \in \mathcal{S}_{0}(L),|S| \leq$ $n_{s}$ and $\left.v \in S\right\}$. In other words $\mathcal{A}_{0}$ consists of all the vertices that belong to some ( $k-1$ )disconnected set whose size is relatively small (we consider these vertices to be the active ones during the construction of $G_{1}$ ). Every vertex $v \in \mathcal{A}_{0}$ independently chooses uniformly at random an edge $e_{k}(v)$ out of the $n-(k-1)$ edges that are incident to it in $Q^{n}$ and do not belong to $E_{0}(v)$. We let the set of newly chosen edges be $E_{1}^{\prime}$ and we define $G_{1}$ by $V\left(G_{1}\right):=V, E\left(G_{1}\right):=E\left(G_{0}\right) \cup E_{1}^{\prime}$.

We finally extend $G_{1}$ to $G_{2}$. We let $\mathcal{A}_{1}=V \backslash \mathcal{A}_{0}$. Every vertex in $\mathcal{A}_{1}$ independently chooses uniformly at random an edge $e_{k}(v)$ out of the $n-(k-1)$ edges that are incident to it in $Q^{n}$ and do not belong to $E_{0}(v)$. We let the set of newly chosen edges be $E_{2}^{\prime}$. Finally we define $G_{2}$ by $V\left(G_{2}\right):=V, E\left(G_{2}\right):=E\left(G_{1}\right) \cup E_{2}^{\prime}$. Observe that once we construct $G_{2}$ we
have that every vertex $v$ has chosen a set of exactly $k$ edges uniformly at random from all the edges incident it. Since $E\left(G_{2}\right)=\bigcup_{v \in V} E_{2}(v)$ we have that $G_{2}$ is distributed as a $Q^{n}(k)$. The following remarks can be made concerning definition 6.3,

Remark 6.4. For $i \in\{0,1,2\}$, every set in $\mathcal{S}_{i}(L)$ is connected in $G_{i}$ hence in $Q^{n}$. Therefore its vertices span a connected subgraph of $Q^{n}$.

Remark 6.5. For $i \in\{0,1,2\}$, every $L$-disconnected set in $G_{i}$ is the union of sets in $\mathcal{S}_{i}(L)$ hence,

$$
\min \left\{|S|: S \in \mathcal{S}_{i}\right\}=\min _{L \in\binom{V}{\ell}}\left\{\min \left\{|S|: S \in \mathcal{S}_{i}(L)\right\}\right\}
$$

In addition if $S \in S_{i}(L)$ then for $0 \leq j \leq i$, since $G_{j} \subset G_{i}$, we have that $S$ is $L$-disconnected in $G_{j}$ hence it is a union of sets in $\mathcal{S}_{j}(L)$.
Lemma 6.6. W.h.p. $\exists S, L \subset V$ s.t. $|L|=k-1, S \in \mathcal{S}_{1}(L)$ and $|S|<n_{s}$.
Proof. Assume the claim is false and let $L, S$ be a contradicting pair. Since $S$ is $L$ disconnected in $G_{1}$ it is also $L$-disconnected in $G_{0}$. Hence every vertex in $S$ belongs to some minimal $(k-1)$-disconnected set in $G_{0}$ of size at most $n_{s}$. Due to the construction of $G_{1}$ every such vertex has degree at least $k$ in $G_{1}$. Thus $S \cup L$ span at least $\frac{k|S|}{2}$ edges in $G_{1}$. In addition, by Lemma [2.5, every set of $|S|+(k-1)$ vertices span at most $\frac{|S|+(k-1)}{2} \log _{2}(|S|+k-1) \leq \frac{|S|+k}{2} \log _{2}(|S|+k)$ edges in $Q^{n}$. Thus, since $G_{1} \subset Q_{n}$, we have

$$
\begin{equation*}
\frac{k|S|}{2} \leq \frac{|S|+k}{2} \log _{2}(|S|+k) . \tag{6}
\end{equation*}
$$

Consequently one of the following two inequalities holds. Either

$$
\begin{equation*}
\frac{k|S|}{4} \leq \frac{k}{2} \log _{2}(|S|+k) \quad \text { which implies } \quad|S| \leq 2 \log _{2}(|S|+k) \tag{7}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{k|S|}{4} \leq \frac{|S|}{2} \log _{2}(|S|+k) \quad \text { which implies } \quad k \leq 2 \log _{2}(|S|+k) \tag{8}
\end{equation*}
$$

If (77) holds then, since $k$ is larger than $0.5 \log _{2} n$, it must be the case that $|S|=o(k)$, in particular $|S| \leq \frac{k}{4}$. $S$ is $L$-disconnected in $G_{1}$ therefore every vertex in $S$ is adjacent, in $G_{1}$, to $k$ vertices in $S \cup L$. Therefore, since $|S| \leq \frac{k}{4}$, every vertex in $S$ is adjacent to at least $\frac{3 k}{4}$ vertices in $L$. Hence any 2 distinct vertices in $S$ share at least 3 neighbors in $L$ which contradicts the fact that any two vertices in $Q^{n}$ have at most two common neighbors (note $S$ is $(k-1)$-disconnected and every vertex in $S$ has degree at least $k$ hence $|S|>1)$. So it must be the case that (8) holds. (8) implies that $|S| \geq k$ which implies

$$
\begin{equation*}
k \leq 2 \log _{2}(2|S|) \leq 3 \log _{2}|S| . \tag{9}
\end{equation*}
$$

(6) can be rewritten as

$$
\frac{k|S|}{2} \leq \frac{|S|+k}{2} \log _{2}(|S|+k)=\frac{|S|+k}{2} \log _{2}\left(1+\frac{k}{|S|}\right)+\frac{|S|}{2} \log _{2}|S|+\frac{k}{2} \log _{2}|S| .
$$

Dividing throughout by $\frac{|S|}{2}$, setting $u=3 \log _{2}|S|$ and using that $k \leq u$ we get that in the case that the statement of our lemma is false the following inequality holds

$$
\begin{equation*}
k \leq\left(1+\frac{u}{|S|}\right) \log _{2}\left(1+\frac{u}{|S|}\right)+\log _{2}|S|+\frac{u}{|S|} \log _{2}|S|=\log _{2}|S|+o(1) \tag{10}
\end{equation*}
$$

Therefore a crude lower bound on $|S|$ is $|S| \geq 2^{0.9 k} \geq n^{0.8}$. When $|S| \geq n^{0.8}$ we have that $\left(1+\frac{u}{|S|}\right) \log _{2}\left(1+\frac{u}{|S|}\right)+\frac{u}{|S|} \log _{2}|S| \leq\left(1+\frac{u}{|S|}\right) \cdot \frac{1}{\log 2} \cdot \frac{u}{|S|}+\frac{u}{|S|} \log _{2}|S|=o\left(n^{-0.6}\right)$. Therefore we can replace the $o(1)$ term in (10) by $o\left(n^{-0.6}\right)$. Thus as $|S| \leq n_{s}$ (10) implies that

$$
k \leq \log _{2}|S|+o\left(n^{-6}\right) \leq \log _{2} n_{s}+n^{-0.6}=k_{1}-0.1+n^{-0.6}<k,
$$

which give us a contradiction.
Remark 6.7. Let $k \geq k_{1}$. If $S, L \subset V$ are such that $|L|=k-1$ and $S \in \mathcal{S}_{1}(L)$ then the same argument we used to derive (9) implies that $k \leq 3 \log _{2}|S|$.

Lemma 6.8. Let $n_{1}=2^{\frac{n}{5}}$. Then w.h.p. $\nexists S, L \subset V$ s.t. $|L|=k-1, S \in \mathcal{S}_{1}(L)$ and $|S| \in\left[n_{s}, n_{1}\right]$.

Proof. In proving the above statement we observe that for every $S, L \subset V$ such that $S \in$ $\mathcal{S}_{1}(L)$ and $|L|=k-1$ then we have the following. There exists some $L^{\prime} \subset L$ such that the induced subgraph of $Q^{n}$ on $S \cup L^{\prime}$ is connected and in $G_{1}$ every vertex in $S$ is adjacent to vertices only in $S \cup L^{\prime}$. For $s \in \mathbb{N}$ let $\mathcal{D}_{s}=\left\{\left(S, L^{\prime}\right): S, L^{\prime} \subset V,\left|L^{\prime}\right| \leq k-1,|S|=\right.$ $s$ and $S \cup L^{\prime}$ is connected in $\left.Q^{n}\right\}$. Corollary [2.8 implies that for fixed $v \in V$ there are at most $\sum_{j \in[k-1]}(e n)^{s+j}$ choices for $S \cup L^{\prime}$ such that $v \in S \cup L^{\prime}$ and $s+1 \leq|S \cup L| \leq s+(k-1)$. Thereafter there are at most $\binom{s+k-1}{k-1} \leq(2 s)^{k}$ ways to choose $L^{\prime}$ out of $S \cup L^{\prime}$. Thus when $k \leq 3 \log _{2} s$ and $s \leq n^{2}$ we have that

$$
\begin{equation*}
\left|\mathcal{D}_{s}\right| \leq \sum_{j \in[k-1]} 2^{n}(e n)^{s+j}(2 s)^{k} \leq k 2^{n}(e n)^{s+k}\left(n^{3}\right)^{k} \leq 2^{n}(e n)^{s+5 k} \leq 2^{n+2 s \log _{2} n} \tag{11}
\end{equation*}
$$

On the other hand when $k \leq 3 \log _{2} s$ and $n^{2}<s$ we have

$$
\begin{equation*}
\left|\mathcal{D}_{s}\right| \leq \sum_{j \in[k-1]} 2^{n}(e n)^{s+j}(2 s)^{k} \leq k 2^{n}(e n)^{s+k} 2^{2 k \log _{2} s} \leq k 2^{n}(e n)^{s+k} 2^{s} \leq 2^{n+2 s \log _{2} n} \tag{12}
\end{equation*}
$$

At the same time Lemma 2.5 implies that for every $\left(S, L^{\prime}\right) \in \mathcal{D}_{s}$ we have

$$
\frac{1}{s} \sum_{v \in S} d\left(v, S \cup L^{\prime}\right) \leq \frac{1}{s} \sum_{v \in S \cup L^{\prime}} d\left(v, S \cup L^{\prime}\right) \leq \frac{1}{s} \cdot(s+k) \log _{2}(s+k)=(1+o(1)) \log _{2} s
$$

Therefore by the arithmetic-geometric mean inequality we get

$$
\begin{equation*}
\left\{(1+o(1)) \log _{2} s\right\}^{s} \geq\left(\frac{1}{s} \sum_{v \in S} d\left(v, S \cup L^{\prime}\right)\right)^{s} \geq \prod_{v \in S} d\left(v, S \cup L^{\prime}\right) \tag{13}
\end{equation*}
$$

(13) implies

$$
\begin{equation*}
\prod_{v \in S}\left(\frac{\binom{d\left(v, S \cup L^{\prime}\right)}{k-1}}{\binom{n-1}{k-1}}\right) \leq \prod_{v \in S}\left(\frac{d\left(v, S \cup L^{\prime}\right)}{n}\right)^{k-1} \leq\left(\frac{(1+o(1)) \log _{2} s}{n}\right)^{s(k-1)} \tag{14}
\end{equation*}
$$

Hence, using that $k \leq 3 \log _{2} s$ (see Remark 6.7), we have

$$
\begin{aligned}
& \mathbb{P}\left(\exists L, S \subset V:|L|=k-1, S \in \mathcal{S}_{1}(L) \text { and }|S|=s \in\left[n_{s}, n^{2}\right)\right) \\
& \quad \leq \mathbb{P}\left(\exists s \in\left[n_{s}, n^{2}\right) \text { and } \exists S, L^{\prime} \subset V:\left(S, L^{\prime}\right) \in \mathcal{D}_{s} \text { and } E_{0}(v) \subset\left(S \cup L^{\prime}\right) \times\{v\}\right) \\
& \quad \leq \sum_{s=n_{s}}^{n^{2}} \sum_{\left(S, L^{\prime}\right) \in \mathcal{D}_{s}}\left(\frac{\binom{d\left(v, S \cup L^{\prime}\right)}{k-1}}{\binom{n}{k-1}}\right) \leq \sum_{s=n_{s}}^{n^{2}} 2^{n+2 s \log _{2} n}\left(\frac{(1+o(1)) \log _{2} s}{n}\right)^{s(k-1)} \\
& \quad \leq \sum_{s=n_{s}}^{n^{2}} 2^{n+2 s \log _{2} n}\left(\frac{2 \log _{2} n^{2}}{n}\right)^{s\left(k_{1}-1\right)} \leq \sum_{s=n_{s}}^{n^{2}} 2^{c_{2}(s)}=o(1) .
\end{aligned}
$$

In the second line we used (11) and (14). Furthermore in the last line we used that for $s \in\left[n_{s}, n^{2}\right]$,

$$
\begin{aligned}
c_{2}(s) & =n+2 s \log _{2} n-\left[\log _{2} n-\log _{2}\left(4 \log _{2} n\right)\right] s\left(k_{1}-1\right) \\
& \leq n+2 n_{s} \log _{2} n-\left[\log _{2} n-\log _{2}\left(4 \log _{2} n\right)\right] n_{s}\left(k_{1}-1\right) \\
& \leq n+2 \cdot \frac{4 n}{\log _{2}^{2} n} \log _{2} n-[1-o(1)] \cdot \log _{2} n \cdot \frac{2^{0.9} n}{\log _{2}^{2} n} \log _{2} n \\
& =-\left(2^{0.9}-1\right) n+o(n) .
\end{aligned}
$$

In the third line we used that $\frac{2^{0.9} n}{\log _{2}^{2} n} \leq n_{s} \leq \frac{4 n}{\log _{2}^{2} n}$. Similarly, we have

$$
\begin{aligned}
& \mathbb{P}\left(\exists L, S \subset V:|L|=k, S \in \mathcal{S}_{1}(L) \text { and }|S|=s \in\left[n^{2}, n_{1}\right)\right) \\
& \quad \leq \sum_{s=n^{2}}^{n_{1}} 2^{n+2 s \log _{2} n}\left(\frac{(1+o(1)) \log _{2} s}{n}\right)^{s(k-1)} \\
& \quad \leq \sum_{s=n^{2}}^{n_{1}} 2^{n+2 s \log _{2} n}\left(\frac{(1+o(1)) \log _{2} n_{1}}{n}\right)^{s\left(k_{1}-1\right)}=\sum_{s=n^{2}}^{n_{1}} 2^{n+2 s \log _{2} n}\left(\frac{(1+o(1))}{5}\right)^{s\left(k_{1}-1\right)} \\
& \quad \leq \sum_{s=n^{2}}^{n_{1}} 2^{c_{3}(s)} \leq \sum_{s=n^{2}}^{n_{1}} 2^{-\left(\log _{2} 5-2\right) n^{2}}=o(1) .
\end{aligned}
$$

In the last line we used that for $s \in\left[n^{2}, n_{1}\right]$

$$
\begin{aligned}
c_{3}(s) & =n+2 s \log _{2} n-(1+o(1)) \log _{2} 5 \cdot s\left(k_{1}-1\right) \\
& \leq-(1+o(1))\left(\log _{2} 5-2\right) s \log _{2} n \leq-\left(\log _{2} 5-2\right) n^{2}
\end{aligned}
$$

Remark 6.9. We can extend the above calculations to pairs of sets $S, L$ where $|S|$ satisfies $\frac{\log _{2}|S|}{n}-2=o\left(\frac{1}{n^{2}}\right)$. In order to implement similar calculations for larger sets $S$ we would
have to sharpen the bounds derived in (12), (13) and (14). Observe that (12) counts each set multiple times. Moreover we expect that as the size of the sets that we consider in (12) is increased the proportion of the upper bound derived over the true value is also increased. At the same time observe that for fixed $S, L$ in order to bound the probability that $S$ is $L$-disconnected we do not use any information about the sets $S, L$ other than their sizes. On the other hand (13), (14) indicate that we can relate this probability with the quantity $|E(S, V \backslash(S \cup L))|$.

In the proof of Lemma 6.1 we are also going to use the following lemma.
Lemma 6.10. W.h.p $\left|\mathcal{A}_{0}\right| \leq 2 \frac{n}{10}$.
Proof. Let $S, L \subset V$ be such that $|L|<k, S \in \mathcal{S}_{0}(L)$ and $|S| \leq n_{s}$. The same arguments used to derive (6) imply

$$
\begin{aligned}
\frac{(k-1)|S|}{2} & \leq \frac{|S|+(k-1)}{2} \log _{2}(|S|+(k-1)) \\
& \leq \frac{|S|+k}{2} \log _{2}(|S|+k)
\end{aligned}
$$

Let $a=2^{k-1}-|S| \geq 0$. By dividing throughout by $\frac{|S|}{2}$ and then substituting $|S|=2^{k-1}-a$ we have

$$
\begin{align*}
k-1 & \leq\left(1+\frac{k}{|S|}\right) \log _{2}(|S|+k)=\left(1+\frac{k}{2^{k-1}-a}\right) \log _{2}\left[2^{k-1}\left(1+\frac{k-a}{2^{k-1}}\right)\right] \\
& \leq(k-1)\left(1+\frac{k}{2^{k-1}-a}\right)+2 \log _{2}\left(1+\frac{k-a}{2^{k-1}}\right) \\
& \leq k-1+\frac{(k-1) k}{2^{k-1}-a}+4 \frac{k-a}{2^{k-1}} . \tag{15}
\end{align*}
$$

In the second line at the calculations above we used that $k=o(|S|)=o\left(2^{k-1}-a\right)$ (see Remark 6.7). Furthermore in the last inequality we used that $\forall x \in \mathbb{R}$ we have that $\log (1+x) \leq x$. Therefore $2 x>\frac{x}{\log 2} \geq \frac{\log (1+x)}{\log _{2}}=\log _{2}(1+x)$. (15) implies

$$
0 \leq \frac{2^{k-1} k(k-1)}{2^{k-1}\left(2^{k-1}-a\right)}+\frac{\left(2^{k-1}-a\right) 4(k-a)}{2^{k-1}\left(2^{k-1}-a\right)} \leq \frac{2^{k-1}\left(k^{2}+4 k-4 a\right)}{2^{k-1}\left(2^{k-1}-a\right)}
$$

Hence $0 \leq k^{2}+4 k-4 a$ which implies that $a \leq 2 k^{2}$. Thus, since $|S|=2^{k-1}-a$, we have $|S| \in\left[2^{k-1}-2 k^{2}, n_{s}\right] \subseteq\left[2^{k_{1}-1}-2 k_{1}^{2}, n_{s}\right]$. Let $n_{\ell}=2^{k_{1}-1}-2 k_{1}^{2}$. Using, in the second line of the calculations below, (14), (11) and that $(1+o(1)) \log _{2} s \leq \log _{2}^{2} n$ for $s \leq n_{s}$ we have

$$
\begin{aligned}
\mathbb{E}\left(\left|\mathcal{A}_{0}\right|\right) & \leq n_{s} \mathbb{E}\left(\mid\left\{S: \exists L \subset V,|L|=k-1, S \in \mathcal{S}_{0}(L) \text { and }|S| \in\left[n_{\ell}, n_{s}\right]\right\} \mid\right) \\
& \leq n_{s} \sum_{s=n_{\ell}}^{n_{s}} \sum_{\left(S, L^{\prime}\right) \in \mathcal{D}_{s}} \prod_{v \in S}\left(\frac{\left(\begin{array}{c|c|c|c|}
k-1
\end{array}\right)}{\binom{n}{k-1}}\right) \leq n_{s} \sum_{s=n_{\ell}}^{n_{s}} 2^{n+2 s \log _{2} n}\left(\frac{\log _{2}^{2} n}{n}\right)^{s\left(k_{1}-1\right)} \\
& \leq n_{s} \sum_{s=n_{\ell}}^{n_{s}} 2^{c_{4}(s)} \leq n_{s} \sum_{s=n_{\ell}}^{n_{s}} 2^{\frac{n}{12}} \leq 2^{\frac{n}{11}} .
\end{aligned}
$$

In the last line we used that for $s \in\left[n_{\ell}, n_{s}\right]$

$$
\begin{aligned}
c_{4}(s) & =n+2 s \log _{2} n-\left(\log _{2} n-2 \log _{2} \log _{2} n\right) s\left(k_{1}-1\right) \\
& \leq n+n_{\ell}\left[2 \log _{2} n-\left(\log _{2} n-2 \log _{2} \log _{2} n\right)\left(k_{1}-1\right)\right] \\
& \leq n+\left(\frac{n}{\log _{2}^{2} n}-2 \log _{2}^{2} n\right)\left(2 \log _{2} n-\log _{2}^{2} n+4 \log _{2} n \cdot \log _{2} \log _{2} n\right)=o(n) .
\end{aligned}
$$

In the last line we used that $\frac{n}{\log _{2}^{2} n}-2 \log _{2}^{2} n \leq 2^{k_{0}}-2 k_{0}^{2} \leq n_{\ell}$. Hence by Markov's inequality we have that $\mathbb{P}\left(\left|\mathcal{A}_{0}\right| \geq 2^{\frac{n}{10}}\right) \leq \mathbb{E}\left(\left|\mathcal{A}_{0}\right|\right) 2^{-\frac{n}{10}}=o(1)$.

In the proof of Lemma 6.1 we are going to use the following definition.
Definition 6.11. For $L \subset V,|L|=k-1$ and $2 \leq j \leq 9$, with $u_{j}=\min \left\{2^{\frac{(j+1) n}{10}}, 2^{n-1}\right\}$, we define the sets $\mathcal{U}_{1}^{j}(L)$ as follows,

$$
\mathcal{U}_{1}^{j}(L):=\left\{T \subset V: T \text { is a union of sets in } \mathcal{S}_{1}(L) \text { and }|T| \in\left[2^{\frac{j n}{10}}, u_{j}\right]\right\} .
$$

Observe that Lemmas 6.6, 6.8 imply that $\mathcal{S}_{1}(L)$ consists of disjoint sets of size at least $2^{\frac{n}{5}}$. Thus $\left|\mathcal{S}_{1}(L)\right| \leq 2^{\frac{4 n}{5}}$. Furthermore if $T \in \mathcal{U}_{1}^{j}(L)$, then $T$ is the union of at most $2^{\frac{(j-1) n}{10}}$ sets in $\mathcal{S}_{1}(L)$. Therefore

$$
\left|\mathcal{U}_{1}^{j}(L)\right| \leq \sum_{h=1}^{2^{\frac{(j-1) n}{10}}}\binom{|S|}{h} \leq \sum_{h=1}^{2 \frac{(j-1) n}{10}}\binom{2^{n}}{h} \leq \sum_{h=1}^{2^{\frac{(j-1) n}{10}}} \frac{2^{n h}}{h!} \leq 2^{n 2^{\frac{(j-1) n}{10}}} .
$$

Proof of Lemma 6.1. In the event that $G_{2}$ is not $k$-connected there is a set $L$ consisting of $k-1$ vertices whose removal partitions the rest of the vertices into connected components. The smallest one of those components is of size at most $2^{n-1}$. Therefore we have that there exists $T, L \subset V$ and $2 \leq j \leq 9$ such that $T \in \mathcal{U}_{1}^{j}(L)$ and no edge in $E(T, V \backslash(T \cup L))$ appears in $E_{2}^{\prime}$ i.e. $\left\{e_{k}(v): v \in T \cap \mathcal{A}_{1}\right\} \cap E(T, V \backslash(T \cup L))=\emptyset$. For a fixed such triple $T, L, j$ let $\mathcal{B}(T, L, j)$ be the event that $\left\{e_{k}(v): v \in T \cap \mathcal{A}_{1}\right\} \cap E(T, V \backslash(T \cup L))=\emptyset$. Then

$$
\begin{aligned}
\mathbb{P}(\mathcal{B}(T, L, j)) & \leq \prod_{v \in T \cap \mathcal{A}_{1}} \mathbb{P}\left[e_{k}(v) \notin E(v, V \backslash T \cup L)\right] \leq \prod_{v \in T \cap \mathcal{A}_{1}}\left[1-\frac{d(v, V \backslash T \cup L)}{n-(k-1)}\right] \\
& \leq \prod_{v \in T \cap \mathcal{A}_{1}} \exp \left\{-\frac{d(v, V \backslash T \cup L)}{n-(k-1)}\right\} \leq \exp \left\{-\frac{1}{n} \sum_{v \in T \cap \mathcal{A}_{1}} d(v, V \backslash T \cup L)\right\} \\
& \leq \exp \left\{-\frac{1}{n}\left(\sum_{v \in T} d(v, V \backslash T)-n|L|-n\left|T \backslash \mathcal{A}_{1}\right|\right)\right\} \\
& \leq \exp \left\{-\frac{1}{n}(n|T|-|T| \log |T|)+k+2^{\frac{n}{10}}\right\} \\
& \leq \exp \left\{-\frac{2^{\frac{j n}{10}}}{n}\left(n-\log _{2} 2^{\frac{j n}{10}}\right)+k+2^{\frac{n}{10}}\right\} \leq \exp \left\{-\frac{2^{\frac{j n}{10}}}{20 n}\right\} .
\end{aligned}
$$

To go from the third to the fourth line we used Lemma 2.5 and that $\left|T \backslash \mathcal{A}_{1}\right|=\left|T \cap \mathcal{A}_{0}\right| \leq 2^{\frac{n}{10}}$ (see Lemma 6.10). Thereafter we used Remark [2.6. In the last inequality we used that $2 \leq j \leq 9$. Finally we have,

$$
\begin{aligned}
& \mathbb{P}\left(G_{2} \text { is not k-connected }\right)=\mathbb{P}(\exists L, T \subset V \text { and } 2 \leq j \leq 9: \mathcal{B}(T, L, j) \text { occurs }) \\
& \quad \leq \sum_{j=2}^{9} \sum_{L \in\left(\begin{array}{c}
V \\
k-1
\end{array}\right.} \sum_{T \in \mathcal{U}_{1}^{j}(L)} \exp \left\{-\frac{2^{\frac{j n}{10}}}{20 n}\right\} \leq \sum_{j=2}^{9} \sum_{L \in\left(V_{k-1}^{V}\right)} 2^{n 2^{\frac{(j-1) n}{10}}} \exp \left\{-\frac{2^{\frac{j n}{10}}}{20 n}\right\}=o(1) .
\end{aligned}
$$

Since $G_{2}$ is distributed has the same distribution with $Q^{n}(k)$ the statement of Lemma 6.1 follows.

A question that now arises is the following. For $k \geq k_{1}$ can $Q^{n}(k)$ be $\ell$-connected, for some $\ell>k$ ? We answer this question negatively in our next lemma with which we close this section.

Lemma 6.12. Let $1 \leq k \leq n-1$. Then w.h.p. $Q^{n}(k)$ contains a vertex of degree $k$ and hence $Q^{n}(k)$ is not $(k+1)$-connected.

Proof. Let $v \in V$. In $Q^{n}(k), v$ has degree $k$ in the event that the $(n-k)$ edges not selected by $v$ do not belong to $Q^{n}(k)$. That is those $(n-k)$ edges are not selected by their other endpoint either. Each of those edges is selected by their other endpoint independently with probability $\frac{k}{n}$. Hence

$$
p_{k}=\mathbb{P}\left(v \text { has degree } k \text { in } Q^{n}(k)=\left(1-\frac{k}{n}\right)^{n-k}=\left[\left(1-\frac{k}{n}\right)^{1-\frac{k}{n}}\right]^{n} .\right.
$$

$p_{k}$ is minimized when $1-\frac{k}{n}=e^{-1}$ thus $p_{k} \geq e^{-\frac{n}{e}} \geq 0.6^{n}$. The degrees of any set of vertices which are at distance at least three from each other are independent. We can greedily select such a set $S$, of size at least $\frac{2^{n}}{n^{2}}$, by sequentially including a non-deleted vertex and then deleting all the vertices at distance at most 2 from it. For $v \in V$ let $d_{k}(v)$ be the degree of $v$ in $Q^{n}(k)$. Therefore

$$
\begin{aligned}
\mathbb{P}\left[Q^{n}(k) \text { is }(k+1) \text { connected }\right] & \leq \mathbb{P}\left(\nexists v \in S: d_{k}(v)=k\right)=\left(1-p_{k}\right)^{|S|} \leq\left(1-p_{k}\right)^{\frac{2^{n}}{n^{2}}} \\
& \leq e^{-\frac{p_{k} 2^{n}}{n^{2}}} \leq e^{-\frac{0.6^{n} \cdot 2^{n}}{n^{2}}}=o(1) .
\end{aligned}
$$

## $7 \quad$ Final Remarks

In this paper we have established the connectivity threshold for the random subgraph of the $n$-cube that is generated by the $k$-out model. When $k$ is below the threshold $k_{1}$ the giant components consists of all but $o\left(2^{n}\right)$ vertices. Furthermore a calculation similar to the one given at the proof of Lemma 6.8 give us that when $k$ is below this threshold $Q^{n}(k)$ does not have any components of size in $\left[\frac{2 n}{\log _{2} n}, 2^{\frac{n}{5}}\right]$. Hence it would be interesting to investigate the size of the second largest component.

On the other hand, when $k$ is at least $k_{1}$ we showed $Q^{n}(k)$ is far more than just connected, it is $k$-connected. In the proof of the $k$-connectivity we used the following properties of $Q^{n}(k)$. Let $N=2^{n}$ then $Q^{n}$ is a graph on $N$ vertices of maximum degree $\log _{2} N$ such that for any partition $S, V \backslash S$ there are at least $|S|\left(\log _{2} N-\log _{2}|S|\right)$ edges crossing the partition. In addition any two vertices have at most $0.25 \log _{2} \log _{2} N$ common neighbors. Therefore by repeating the arguments given in this paper we have the following. Every random subgraph of a graph on $N$ vertices that satisfies the aforementioned properties and is generated by the $k$-out model, where $k \geq k_{1}$, is $k$-connected. An interesting question would therefore be to state more general conditions of a similar flavor, such that the random subgraph of a graph that satisfies these conditions and is generated by the $k$-out model is $k$-connected (or even just connected).

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## A Proof of Lemma 3.5

Proof. We start by proving that

$$
\mathbb{P}\left(L_{\frac{n}{5}} \leq \frac{n}{20}\right) \leq e^{-10^{-3} n}
$$

Observe that for $i \leq \frac{n}{5}, L_{i} \leq \frac{n}{5}$ hence $\mathbb{P}\left(L_{i+1}=L_{i}+1\right) \geq 0.8$. In the event that $L_{\frac{n}{5}} \leq \frac{n}{20}$ we have that $\left|\left\{i \leq \frac{n}{5}: L_{i+1} \neq L_{i}+1\right\}\right| \geq 0.5\left(\frac{n}{5}-\frac{n}{20}\right)=\frac{3 n}{40}$. Equivalently we have $\left|\left\{i \leq \frac{n}{5}: L_{i+1}=L_{i}+1\right\}\right| \leq \frac{n}{5}-\frac{3 n}{40}=\frac{n}{8}$. Therefore,

$$
\mathbb{P}\left(L_{\frac{n}{5}} \leq \frac{n}{20}\right) \leq \mathbb{P}\left[\operatorname{Bin}\left(\frac{n}{5}, 0.8\right) \leq \frac{n}{8}\right] \leq \exp \left[\left(\frac{\frac{4}{25}-\frac{1}{8}}{\frac{4}{25}}\right)^{2} \frac{\frac{4}{25} n}{3}\right] \leq e^{-10^{-3} n}
$$

In the second inequality we used Lemma [2.2. Our second step is to show that for $i \in\left[n^{2}\right]$

$$
\mathbb{P}\left(\left.L_{i+\frac{n}{40}} \leq \frac{n}{20} \right\rvert\, L_{i} \geq \frac{n}{20}\right) \leq e^{-10^{-3} n} .
$$

Observe that $\left|L_{i+\frac{n}{40}}-L_{i}\right| \leq \frac{n}{40}$. Therefore

$$
\mathbb{P}\left(\left.L_{i+\frac{n}{40}} \leq \frac{n}{20} \right\rvert\, L_{i}>\frac{n}{20}+\frac{n}{40}\right)=0 .
$$

On the other hand if $L_{i} \leq \frac{n}{20}+\frac{n}{40}$ we have that $L_{j} \leq \frac{n}{20}+\frac{n}{40}+\frac{1 n}{40}=\frac{n}{10}$ for every $j \in\left[i, i+\frac{n}{40}\right]$ hence $\mathbb{P}\left(L_{j+1}=L_{j}+1\right) \geq \frac{9}{10}$. In the event that $L_{i+\frac{n}{40}} \leq \frac{n}{20}$ we have

$$
\left|\left\{j \in\left[i, i+\frac{n}{40}\right]: L_{j+1} \neq L_{j}+1\right\}\right| \geq 0.5 \cdot \frac{n}{40}
$$

or equivalently $\left|\left\{j \in\left[i, i+\frac{n}{40}\right]: L_{j+1}=L_{i}+1\right\}\right| \leq \frac{n}{80}$. Therefore,

$$
\begin{aligned}
& \mathbb{P}\left(\left.L_{i+\frac{n}{40}} \leq \frac{n}{20} \right\rvert\, L_{i} \geq \frac{n}{20}\right) \leq \mathbb{P}\left(L_{i+\frac{n}{40}} \leq \frac{n}{20} \left\lvert\, \frac{n}{20} \leq L_{i} \leq \frac{n}{20}+\frac{n}{40}\right.\right) \\
& \leq \mathbb{P}\left[\operatorname{Bin}\left(\frac{n}{40}, \frac{9}{10}\right) \leq \frac{n}{80}\right] \leq \exp \left[\left(\frac{\frac{9}{400}-\frac{1}{80}}{\frac{9}{400}}\right)^{2} \frac{9}{400} n\right. \\
& 3
\end{aligned} \leq e^{-10^{-3} n} .
$$

In the third inequality we once again used Lemma [2.2. In the event that $L_{2 i}=0$ for some $i \in\left[\frac{n}{4}, n^{2}\right]$ we have that either $L_{\frac{n}{5}} \leq \frac{n}{20}$ or there exist $\frac{n}{5} \leq i<n^{2}$ such that $L_{2 i-\frac{n}{40}} \geq \frac{n}{20}$ but $L_{2 i} \leq \frac{n}{20}$. Hence

$$
\mathbb{P}\left(\exists i \in\left[n / 4, n^{2}\right]: L_{2 i}=0\right) \leq n^{2} \cdot e^{-10^{-3} n}=o\left(n^{-4}\right)
$$


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    ${ }^{1}$ we say that a sequence of events $\left\{\mathcal{E}_{n}\right\}$ holds with high probability (w.h.p.) or equivalency almost surely if $\mathbb{P}\left(\mathcal{E}_{n}\right) \rightarrow 1$ as $n \rightarrow \infty$.

