DTU Library

Deciding parity of graph crossing number

Hliněný, Petr; Thomassen, Carsten

Deciding Parity of Graph Crossing Number

Petr Hliněný[1*] and Carsten Thomassen[2**]

[1] Faculty of Informatics, Masaryk University, Brno, Czech Republic
hlineny@fi.muni.cz
[2] Department of Applied Mathematics and Computer Science, Technical University of Denmark, Denmark
ctho@dtu.dk

Abstract. We prove that it is NP-hard to determine whether the crossing number of an input graph is even or odd.

1 Introduction

For many graph invariants, the complexity of determining the parity of the invariant is the same as that of determining the invariant itself. Suppose, for example that we have an algorithm for finding the parity of the *chromatic number* $\chi(G)$ of a graph G. Then we can apply the algorithm to the graphs $K_1 \cup G, K_2 \cup G, \ldots$ where K_n is the complete graph with n vertices. The sequence of parities is first constant and then alternating. The number of elements in the constant part of the sequence is the chromatic number of G. Similar arguments apply to the *clique number* $\omega(G)$ and the *independence number* $\alpha(G)$. It also applies to the *genus* $g(G)$ of a graph G, since one can construct, in polynomial time, a graph G' such that $g(G') = \alpha(G) - |E(G)|$, as proved in [16].

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise edge crossings in a drawing of G in the plane. The crossing number has a certain similarity to the genus: For planar graphs, the invariants agree, and for each fixed k the questions "Is $\mathrm{cr}(G) \leq k$", and "Is $g(G) \leq k$" are in P. The former can easily be reduced to a planarity problem (see also [6]). The latter is in P by the Robertson-Seymour theory. Both problems are NP-complete when k is part of the input [5],[16]. Both problems remain NP-complete even for very restricted graphs, and they may be hard to determine even for very simple classes of graphs such as complete graphs and complete bipartite graphs where the crossing numbers are still unknown. The crossing number problem is NP-complete even for cubic graphs [8] and for graphs that become planar after removing only one edge [4], and also for drawings where all local orientations are prescribed [13]. Even approximation is hard: There exists a number $c > 1$ such that the crossing number cannot be approximated within the factor c in polynomial (unless NP=P) time [3].

For some graphs we know the parity of the crossing number, i.e., the value $(\mathrm{cr}(G) \bmod 2)$, for example for $G = K_p$ and $G = K_{q,r}$ when all p, q, r are odd, see [10, 7, 1]. Knowledge of parity is sometimes useful for determining the crossing number.

It seems that the hardness results for crossing numbers in [3–5, 8, 13], do not answer to the associated parity question, and Schaefer [15] asks the question: What is the complexity of determining $(\mathrm{cr}(G) \bmod 2)$?

The purpose of this note is to point out that a recent hardness result on crossing numbers of tiles by Hliněný and Derňár [9] can be used to prove that that the parity question is NP-hard.
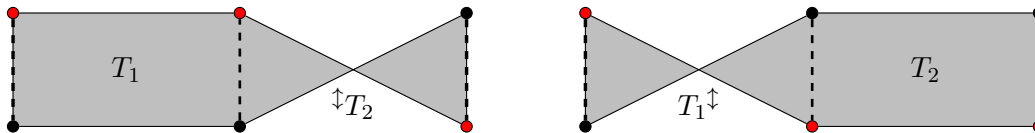
Fig. 1. Two possible drawings of a twisted join of two planar tiles T_1, T_2. Clearly $\mathrm{tcr}(T_1 \otimes {}^{\updownarrow}T_2) \leq \min\{\mathrm{tcr}(T_1{}^{\updownarrow}), \mathrm{tcr}(T_2{}^{\updownarrow})\}$, although strict inequality might be achieved by a different drawing.

2 Crossing number of graphs and tiles

We consider multigraphs (although we can subdivide loops and parallel edges in order to make the graphs simple if we wish so). We follow basic terminology of topological graph theory, see e.g. [12]. In a *drawing* of a graph G in the plane, the vertices of G are distinct points, and the edges are simple curves joining their endvertices. An edge contains no vertex, except its ends. Two edges are disjoint except for common ends. Finally, no three edges meet in a common point. A *crossing* is a point which is not a vertex and which belongs to two distinct edges.

The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of crossings in a drawing of G in the plane.

Hence, a graph G is planar if and only if $\mathrm{cr}(G) = 0$.

Inspired by [11, 14] we define a *tile* $T = (G, a, b, c, d)$ where G is a graph and a, b, c, d is a sequence of distinct vertices. We call a, b the *left wall* and c, d the *right wall* of T. The *right-inverted* tile $T^{\updownarrow}$ is the tile (G, a, b, d, c) and the *left-inverted* tile ${}^{\updownarrow}T$ is (G, b, a, c, d).

A *tile drawing* of a tile $T = (G, a, b, c, d)$ is a drawing of the underlying graph G in the unit square such that the vertices a, b, c, d are the upper left, lower left, lower right, and upper right corner, respectively.

The *tile crossing number* $\mathrm{tcr}(T)$ of a tile T is the minimum number of crossings over all tile drawings of T. A tile T is *planar* if $\mathrm{tcr}(T) = 0$.

The *join of two tiles* $T = (G, a, b, c, d)$ and $T' = (G', a', b', c', d')$ is defined as the tile $T \otimes T' := (G'', a, b, c', d')$, where G'' is the graph obtained from the disjoint union of G and G', by identifying c, b' and d, a'

Clearly, the join of two planar tiles is again a planar tile.

Let T_1, T_2 be planar tiles. Then $\mathrm{tcr}(T_1 \otimes {}^{\updownarrow}T_2) \leq \min\{\mathrm{tcr}(T_1{}^{\updownarrow}), \mathrm{tcr}(T_2{}^{\updownarrow})\}$. This is illustrated in Figure 1.

We now define a *diagonally separated planar tile* as a planar tile, which has the following additional property: there exists a path $Q \subseteq G$, called a *special diagonal path*, from a to c such that every tile drawing of $T^{\updownarrow}$ with $\mathrm{tcr}(T^{\updownarrow})$ crossings has no crossing on Q.

The definition of a diagonally separated planar tiles in [9, Definition 9] is more restricted that the definition above. Hence [9, Lemma 10 and Corollary 12] implies the following.

Theorem 1 ([9]). *Let T be a diagonally separated planar tile. Then computing $\mathrm{tcr}(T^{\updownarrow})$ is an NP-hard problem.*

3 Hardness reduction

Let S_k be the tile with 6 vertices and $5 + k$ edges as in Figure 2, where the edge between r and d consists of k parallel edges.
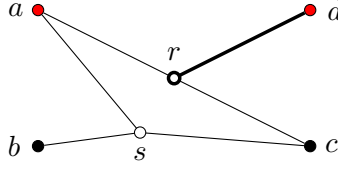
Fig. 2. A planar tile S_k used in Theorem 2. The thick edge rd consists of k parallel edges.

Theorem 2. *Let T be any diagonally separated planar tile with q edges and a special diagonal path Q, and let k be any natural number. Replace every edge in Q by q^2 parallel edges and call the resulting tile T_1. Similarly, replace every edge in the path arc of S_k by q^2 parallel edges and call the result $S'_k = T_2$. Then*

$$\mathrm{tcr}(T_1 \otimes {}^{\updownarrow}T_2) = \min\{\mathrm{tcr}(T_1^{\updownarrow}), \mathrm{tcr}(S'^{\updownarrow}_k)\} = \min\{\mathrm{tcr}(T^{\updownarrow}), k\}.$$

Proof. Clearly, $\mathrm{tcr}(T_1 \otimes {}^{\updownarrow}T_2) \leq \min\{\mathrm{tcr}(T_1^{\updownarrow}), \mathrm{tcr}(S'^{\updownarrow}_k)\} = \min\{\mathrm{tcr}(T^{\updownarrow}), k\}$.

Suppose now that there is a tile drawing of $T_1 \otimes {}^{\updownarrow}S'_k$ with fewer than $\min\{\mathrm{tcr}(T^{\updownarrow}), k\}$ crossings. Then the multiple edge rd is not involved in any crossing because that would imply at least k crossings. Also, no edge of the paths Q or arc is involved in any crossing since that would imply at least $q^2 > \mathrm{tcr}(T^{\updownarrow})$ crossings. So S'_k is drawn without crossings, and therefore the paths ard and bsc are disjoint. If necessary, we can redraw them so that they do not cross any edge of T. Using these paths we hence obtain a tile drawing of $T^{\updownarrow}$. However such a drawing has at least $\mathrm{tcr}(T^{\updownarrow})$ crossings, a contradiction which completes the proof. □

Using Theorems 1 and 2 we proceed to the main result.

Theorem 3. *The problem of determining the parity of the crossing number $(\mathrm{cr}(G) \bmod 2)$ for any given graph G is NP-hard in general.*

Proof. We prove that the problem of determining $(\mathrm{cr}(G) \bmod 2)$ for any graph G is at least as hard as the problem of computing $\mathrm{tcr}(T^{\updownarrow})$ for any diagonally separated planar tile T.

Consider therefore an algorithm $\mathcal{A}$ for determining $(\mathrm{cr}(G) \bmod 2)$ for any graph G. Let T be any diagonally separated planar tile T. Now we form a graph G_k as follows: We form the tile $T_1 \otimes {}^{\updownarrow}S'_k$ as in Theorem 2. We let p denote the number of edges (outside the special diagonal paths) in this tile and add p^2 edges between the four corners of the unit square in which the tile $T_1 \otimes {}^{\updownarrow}S'_k$ is drawn, more precisely, between the pairs $(a, b), (b, c'), (c', d'), (d', a)$; see Figure 3. The crossing number of the resulting graph G_k equals $\mathrm{tcr}(T_1 \otimes {}^{\updownarrow}S'_k)$ since none of the edges ab, bc', $c'd'$, $d'a$ are involved in crossings in an optimum drawing of G_k. By Theorem 2, $\mathrm{cr}(G_k) = \mathrm{tcr}(T_1 \otimes {}^{\updownarrow}S'_k) = \min\{\mathrm{tcr}(T^{\updownarrow}), k\}$.

We now apply the algorithm $\mathcal{A}$ to the graphs $G_1, G_2, \ldots$. This results in a sequence which is first alternating, and then constant. The number of entries in the maximal alternating subsequence equals $\mathrm{tcr}(T^{\updownarrow})$ which is hard to find, by Theorem 1. □

4 Conclusions

Our arguments can easily be extended to show that deciding, for any fixed integer $p \geq 2$, whether $\mathrm{cr}(G)$ is divisible by p is NP-hard.
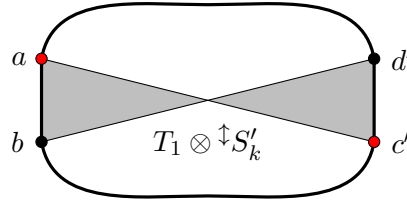
3

Fig. 3. A sketch of the construction of G_k from the tile T in the proof of Theorem 3. Each of the four thick lines represents many parallel edges which cannot be crossed in an optimal drawing.

The method in this note also extends to other variants of the crossing number. For example, it is NP-hard to determine the parity of the rectilinear crossing number since the crossing number of a graph G equals the rectilinear crossing number of an appropriate subdivision of G. On the other hand, it is shown in [8] that it is NP-hard to determine the so-called *minor crossing number* [2]. But, we do not know if it is equally hard to determine the parity.

References

1. Dan Archdeacon and R. Bruce Richter. On the parity of crossing numbers. *Journal of Graph Theory*, 12(3):307–310, 1988.
2. Drago Bokal, Gasper Fijavz, and Bojan Mohar. The minor crossing number. *SIAM J. Discrete Math.*, 20(2):344–356, 2006.
3. Sergio Cabello. Hardness of approximation for crossing number. *Discrete & Computational Geometry*, 49(2):348–358, 2013.
4. Sergio Cabello and Bojan Mohar. Adding one edge to planar graphs makes crossing number and 1-planarity hard. *SIAM J. Comput.*, 42(5):1803–1829, 2013.
5. Michael R. Garey and David S. Johnson. Crossing number is NP-complete. *SIAM J. Alg. Discr. Meth.*, 4:312–316, 1983.
6. Martin Grohe. Computing crossing numbers in quadratic time. *J. Comput. Syst. Sci.*, 68(2):285–302, 2004.
7. Richard K. Guy. Crossing numbers of graphs. In *Graph Theory and Applications: Proceedings of the Conference at Western Michigan University, May 10-13, 1972*, volume 303 of *Lecture Notes in Mathematics*, pages 111–124. Springer, 1972.
8. Petr Hliněný. Crossing number is hard for cubic graphs. *J. Comb. Theory, Ser. B*, 96(4):455–471, 2006.
9. Petr Hliněný and Marek Derňár. Crossing number is hard for kernelization. In *32nd International Symposium on Computational Geometry, SoCG 2016, June 14-18, 2016, Boston, MA, USA*, volume 51 of *LIPIcs*, pages 42:1–42:10. Schloss Dagstuhl – Leibniz-Zentrum fuer Informatik, 2016.
10. Daniel J. Kleitman. The crossing number of $K_{5,n}$. *Journal of Combinatorial Theory*, 9(4):315 – 323, 1970.
11. Martin Kochol. Construction of crossing-critical graphs. *Discrete Mathematics*, 66(3):311–313, 1987.
12. Bojan Mohar and Carsten Thomassen. *Graphs on Surfaces*. Johns Hopkins University Press, Baltimore, 2001.
13. Michael J. Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Crossing numbers of graphs with rotation systems. *Algorithmica*, 60(3):679–702, 2011.
14. Benny Pinontoan and R. Bruce Richter. Crossing numbers of sequences of graphs II: planar tiles. *Journal of Graph Theory*, 42(4):332–341, 2003.
15. Marcus Schaefer. The graph crossing number and its variants: A survey. *Electronic Journal of Combinatorics*, #DS21, May 15, 2014.
16. Carsten Thomassen. The graph genus problem is NP-complete. *J. Algorithms*, 10:568–576, 1989.