## STABILITY OF NONLOCAL DIRICHLET INTEGRALS AND IMPLICATIONS FOR PERIDYNAMIC CORRESPONDENCE MATERIAL MODELING\*

QIANG DU† AND XIAOCHUAN TIAN‡

**Abstract.** Nonlocal gradient operators are basic elements of nonlocal vector calculus that play important roles in nonlocal modeling and analysis. In this work, we extend earlier analysis on nonlocal gradient operators. In particular, we study a nonlocal Dirichlet integral that is given by a quadratic energy functional based on nonlocal gradients. Our main finding, which differs from claims made in previous studies, is that the coercivity and stability of this nonlocal continuum energy functional may hold for some properly chosen nonlocal interaction kernels but may fail for some other ones. This can be significant for possible applications of nonlocal gradient operators in various nonlocal models. In particular, we discuss some important implications for the peridynamic correspondence material models.

**Key words.** Nonlocal gradient, nonlocal models, peridynamics, elasticity, constitutive relation, stability, coercivity

AMS subject classifications. 45A05, 45K05, 47G10, 74G65

1. Introduction. Recently, there have been much interests developing nonlocal models for a variety of problems arising in physics, biology, materials and social sciences [6, 8, 15, 19, 32]. A nonlocal model made up by nonlocal integral operators can potentially allow more singular solutions than the classical differential equation counterpart, thus offering great promise in the effective modeling of singular defects and anomalous properties such as cracks and fractures [27]. Nonlocal gradient operators are basic elements of nonlocal vector calculus that play important roles in nonlocal modeling and analysis [9, 10, 24]. The development of a systematic mathematical framework for nonlocal problems, in parallel to that for local classical partial differential equations (PDEs), in turn has provided foundation and clarity to practical nonlocal modeling techniques such as peridynamics. In particular, the rigorous mathematical studies of nonlocal gradient operators have found successful applications ranging from nonlocal gradient recovery for robust a posteriori stress analysis in nonlocal mechanics to nonlocal in time modeling of anomalous diffusion [11, 13]. Asymptotically compatible schemes to discretize the nonlocal gradient operators have also been presented in [11], following the framework given in [33, 34].

In this work, we continue our analysis of nonlocal gradient operators initiated in [9, 10] and further explored in [11, 13, 24]. We address the coercivity and stability of energy functionals with the energy density formed by the nonlocal gradient. As a representative example, the functional considered here is what we refer to as a nonlocal Dirichlet integral, which is simply a quadratic energy of the nonlocal gradient. Given

<sup>\*</sup>This work is supported in part by the U.S. NSF grant DMS-1719699, and AFOSR MURI center for material failure prediction through peridynamics.

 $<sup>^\</sup>dagger \mbox{Department}$  of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027; qd2125@columbia.edu

<sup>&</sup>lt;sup>‡</sup>Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027; Present address: Department of Mathematics, University of Texas, Austin, TX 78712; xtian@math.utexas.edu

that the most popular forms of the nonlocal gradients are often exclusively determined by some nonlocal interaction kernels (micro-modulus functions), the issue of coercivity and stability of simple energies like the Dirichlet integral rests largely on properties of the underlying interaction kernel functions (beside possibly boundary conditions or nonlocal constraints). While there were claims made in the literature on the universal loss of coercivity and stability for all feasible kernels, we show that, contrary to such claims, there is a class of kernels that can assure the coercivity and stability. The coercivity result established here is quite strong in the sense that it holds uniformly with respect to the horizon parameter that measures the range of nonlocality, so that, in the local limit, we recover the well-known coercivity result of the classical, local Dirichlet integrals formed by the conventional local derivative. Providing conditions to characterize this class of kernels is the main contribution of this work.

Our main finding has a number of implications for nonlocal modeling. In particular, in the context of peridynamic correspondence material models (see a brief description in the next section and additional discussions in [27, 17] and [4, 11, 16, 35]), we can conclude that there may not be any loss of stability if proper kernels are used for the correspondence formulation. This is an encouraging news to the community that has found convenience in using peridynamic correspondence material models. However, identifying the right kernels is crucial, in addition to requiring their consistency to the underlying physical processes and principles, In simple terms, the requirement on the kernels to assure the energy coercivity is that one should suitably enforce stronger interactions as the undeformed bond length gets shorter. The specific form of the strengthening is given in the assumption and theorems in section 5, We also point out that, in contrast, the weakening of interactions (or the lack of sufficient strengthening) among materials points in closer proximity is likely going to cause a loss of coercivity of the correspondence formulation. In short, this means that the choice of nonlocal interaction kernels is a much more subtle issue for the correspondence theory, in comparison with, for example, other nonlocal formulations such as the bond-based or state-based peridynamics of linear elasticity [21, 22].

To avoid technical complications, we only present the derivation in one space dimension for a scalar field, though the extension to multidimensional cases and vector fields are immediate, based on similar calculations give in [12]. For simplicity, we also only consider periodic boundary conditions to avoid the discussion near physical boundary. By utilizing this special geometry, we can carry out the needed mathematical derivations using simple Fourier analysis and elementary calculations. The extensions to more general boundary conditions or more appropriate nonlocal volumetric constraints [9, 10] are more involved and will be left as future work. Further studies on the discrete level can also be carried out in a similar fashion but it is beyond the scope of the current work so that in this work we can focus on delivering a simple but importance message on nonlocal correspondence models on the continuum level. Indeed, as shown in [33, 34] and again in this work, delineating the effects on physical and continuum scales from those arising from numerical resolutions in order to better investigate their interplay has proven to be a helpful strategy to validate nonlocal modeling and simulations. Moreover, the conditions given later on the nonlocal interaction kernels rule out many popular choices used in existing simulation codes and applications. Again, this is another instance related to nonlocal modeling and simulations where popular practices in the past may need to be carefully scrutinized. As another example, simple mid-point quadrature and piecewise constant Galerkin finite element approximations are both popular discretizations of bond-based peridynamics but they are not robust and run the risk of converging to wrong physical solutions [33, 34]. Ensuring properly defined models and convergent algorithms is particularly important to subjects like peridynamics since their main goal is to deal with complex systems involving multiscale features, patterns and singular solutions potentially generated by inherent material instabilities so that one does not mix up model or numerical instability with the physically reality.

The remainder of the paper is organized as follows. In section 2, some general background on the subject is given. We then provide more discussions on the nonlocal gradient operator for one dimensional scalar field in section 3. Equivalent formulations are presented in section 4 to draw connections with nonlocal diffusion operators. The main stability analysis is in section 5. Different variants are considered in section 6. Finally, in section 7, we make some conclusions on the implications and generalizations.

2. Background. In nonlocal mechanical models and nonlocal diffusion equations, the primary quantities used are often displacement and density variables. Drawing analogy to classical and local models, the notion of nonlocal gradient is indispensable as it is related naturally to concepts of nonlocal strain and stress and nonlocal flux, see [17, 27] and [4, 11, 16, 35].

Indeed, it is a widely accepted practice of continuum mechanics to use classical local gradients to help defining constitutive relations that are central to the underlying mathematical models. Taking the peridynamic model in  $\mathbb{R}^d$  as an example, if we use **u** to denote the deformation vector field from  $\mathbb{R}^d$  to  $\mathbb{R}^d$ , the so-called nonlocal deformation gradient tensor  $\mathbf{F}(\mathbf{x}) = \mathbb{G}_o^{\delta} \mathbf{u}(\mathbf{x})$  is given by

(2.1) 
$$\mathbb{G}_{\rho}^{\delta}\mathbf{u}(\mathbf{x}) = \left(\int_{B_{\delta}(\mathbf{x})} \rho_{\delta}(|\mathbf{y} - \mathbf{x}|) \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|} \otimes \frac{\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|} d\mathbf{y}\right) \mathbf{K}^{-1}(\mathbf{x})$$

where  $\delta > 0$  represents the horizon measuring the range of nonlocal interactions,  $\rho_{\delta}$  is a scalar function (micromodulus function as named in [27]) representing the nonlocal interaction kernel, and  $\mathbf{K} = \mathbf{K}(\mathbf{x})$  is the shape tensor defined by

$$\mathbf{K}(\mathbf{x}) = \int_{B_{\delta}(\mathbf{x})} \rho_{\delta}(|\mathbf{y} - \mathbf{x}|) \, \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|} \otimes \, \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|} \, d\mathbf{y} \, .$$

It has been suggested that, corresponding to a well-defined local constitutive model  $\sigma = \sigma(\varepsilon)$  where  $\sigma$  is the stress tensor and  $\varepsilon$  the strain tensor, one can formally derive a nonlocal peridynamic analog as  $\sigma = \sigma(\bar{\mathbf{F}})$  where  $\bar{\mathbf{F}}$  is the symmetric part of  $\mathbf{F}$ . This in essence leads to the peridynamic correspondence material models or correspondence theory for short [29]. More discussions and additional references in the context of peridynamics can be found in [1, 2, 5, 14, 30, 31, 35], among others.

We note that in more mathematical generality, a nonlocal gradient operator for vector field  $\mathbf{u}$  defined on a domain  $\Omega$  may take on the form of a second order tensor given by

$$\mathbb{G}_{\rho}^{\delta}\mathbf{u}(\mathbf{x}) := \lim_{\epsilon \to 0} \int_{\Omega \setminus B_{\epsilon}(\mathbf{x})} \rho_{\delta}(|\mathbf{y} - \mathbf{x}|) \, \mathsf{M}_{\delta}(\mathbf{y} - \mathbf{x}) \, \frac{\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})}{|\mathbf{y} - \mathbf{x}|} \, d\mathbf{y},$$

where  $M_{\delta}$  is a 3rd-order odd tensor, thanks to the Schwartz kernel theorem [24]. The simpler version given in (2.1) is nevertheless sufficient to serve the purpose of our discussion here.

Though the form given in (2.1) is appealing and is formally consistent to the classical deformation gradient in the local limit with suitably normalized nonlocal interaction kernel  $\rho_{\delta}$ , there have been various issues concerning its use in the correspondence theory [3, 29, 35]. Similar issues have been noticed in other applications such as those involving particle discretizations [4]. Some of these issues are related to numerical implementations but there are also fundamental limitations on the level of the continuum models, see [29] for a recent study that provided a comprehensive summary on the topic. In particular, it has been made aware of the loss of coercivity (stability) of the nonlocal energy functionals constructed explicitly via the nonlocal deformation gradient. We may use Dirichlet integrals as representative examples of energy functionals given by a quadratic energy density corresponding to the linear elasticity. With the nonlocal deformation gradient  $\mathbf{F} = \mathbb{G}_{\rho}^{\delta}\mathbf{u}$ , the nonlocal Dirichlet integral is given by

(2.2) 
$$E_{\delta}(\mathbf{u}) = \int |\bar{\mathbf{F}}(\mathbf{x})|^2 d\mathbf{x} = \int \left| \frac{\mathbb{G}_{\rho}^{\delta} \mathbf{u}(\mathbf{x}) + (\mathbb{G}_{\rho}^{\delta} \mathbf{u}(\mathbf{x}))^T}{2} \right|^2 d\mathbf{x},$$

which is an nonlocal analog of the usual Dirichlet integral defined for the local gradient as follows:

$$E_0(\mathbf{u}) = \int \left| \frac{\nabla \mathbf{u}(\mathbf{x}) + (\nabla \mathbf{u}(\mathbf{x}))^T}{2} \right|^2 d\mathbf{x}.$$

The lack of coercivity in the nonlocal version is unfortunate as the the local version is well-known to be coercive subject to suitable boundary conditions or constraints. Despite the existing issues and improved understanding, formulations based correspondence theory continue to be popular among practitioners and, at the same time, they are also challenged by the community given the lingering debate on the relevant controversies. The latest attempt [29] has suggested the addition of a penalty term that does provide the needed stability to the elastic energy on the continuum level, however, the additional term to the elastic energy generically is not a null-Lagrangian, meaning that the equilibrium solutions, for example, of the associated energy may be different from its original form without the penalty unless linear deformation fields are obtained. Thus, the additional term does not vanish and its effect, unlike a null-Lagrangian, is present in general. A central question remains, that is, in what circumstances can one ensure the coercivity of the original energy functionals, such as (2.2), defined by the nonlocal deformation gradient.

3. Nonlocal gradient operator and Dirichlet integral in 1D. To illustrate the key concepts, we focus on a nonlocal gradient operator for a scalar function field defined on a one dimensional periodic cell given by  $\Omega=(0,1)$ . The simple setting allows us to more clearly present the central findings without much more tedious technical derivations for higher dimensional vector fields. It is expected that our approach works also for such a more general scenario.

To be more specific about the scalar nonlocal gradient operator (or nonlocal first derivative)  $\mathbb{G}_{\rho}^{\delta}$ , we consider suitably scaled kernels so that the local limit of  $\mathbb{G}_{\rho}^{\delta}$  as  $\delta$ 

goes to zero recovers exactly the first order derivative  $\frac{d}{dx}$  denoted by  $\mathbb{G}_0$ . A popular choice of the kernel  $\rho_{\delta}(s)$  is based on a rescaling  $\rho_{\delta}(s) = \delta^{-1}\rho(s/\delta)$  with the following assumptions on  $\rho$ :

(3.1) 
$$\rho(-s) = \rho(s), \quad \begin{cases} \rho(s) \ge 0, & \forall s \in (-1,1), \\ \rho(s) = 0, & \forall s \notin (-1,1), \end{cases} \text{ and } \int_{-1}^{1} \rho(s) ds = 1,$$

with further assumptions on  $\rho = \rho(s)$  to be discussed later.

Specializing (2.1) for a one dimensional scalar field u = u(x) that are assumed to be square integrable and periodic with a periodic cell  $\Omega$ , we focus on the nonlocal gradient operators defined below:

(3.2) 
$$\mathbb{G}_{\rho}^{\delta}u(x) = \int_{-\delta}^{\delta} \rho_{\delta}(|s|) \frac{u(x+s) - u(x)}{s} ds$$

$$= \int_{-\delta}^{\delta} \rho_{\delta}(|s|) \frac{u(x) - u(x-s)}{s} ds$$

$$= \int_{-\delta}^{\delta} \rho_{\delta}(|s|) \frac{u(x+s) - u(x-s)}{2s} ds$$

$$= \int_0^\delta \rho_\delta(s) \frac{u(x+s) - u(x-s)}{s} ds.$$

Since  $\rho_{\delta} = \rho_{\delta}(s)$  can be seen as a density function defined on  $(-\delta, \delta)$ ,  $\mathbb{G}_{\rho}^{\delta}$  is effectively a continuum weighted average of some discrete first order difference operators up to the scale  $\delta$ . If  $\rho_{\delta} = \rho_{\delta}(s)$  gets localized as  $\delta \to 0$  and behaves like a Dirac-delta measure at the origin in the limit, we may indeed see  $\mathbb{G}_0 = \frac{d}{dx}$  as the formal local limit of  $\mathbb{G}_{\rho}^{\delta}$ .

Detailed studies of operators defined by (2.1) and the specialized form (3.2) ad well as their local limits are the subject of recently developed nonlocal vector calculus, see [9] for formal derivations and [24] and [13] for more extended functional analysis. Nonlocal analog of integration by parts formula has also been rigorously derived [13, 24].

The one dimensional version of the Dirichlet integral associated with  $\mathbb{G}_{\rho}^{\delta}$  and its local form can be written as

(3.6) 
$$E_{\delta}(u) = \int_{\Omega} |\mathbb{G}_{\rho}^{\delta} u(x)|^2 dx \qquad E_{0}(u) = \int_{\Omega} |\nabla u(x)|^2 dx,$$

for any scalar periodic function u with the periodic cell  $\Omega$ .

Due to the periodic boundary condition, a constraint is needed to determine the constant shift in the deformation field. We thus only consider those functions that satisfy

$$\int_{\Omega} u(x)dx = 0.$$

There are also the one sided versions (see [11, 13] for related discussions)

(3.8) 
$$\mathbb{G}_{\delta}^{\pm} u(x) = \pm 2 \int_{0}^{\delta} \rho_{\delta}(s) \frac{u(x \pm s) - u(x)}{s} ds.$$

Similarly, nonlocal diffusion operators that are analogs of the classical diffusion or second derivative operators have also been a subject of extensive study [9, 10, 21, 22, 23]. The connections between these operators are to be further discussed in the next section.

4. Equivalent formulation of Dirichlet integral. In the one dimensional case, it is particularly convenient to connect the nonlocal Dirichlet integral given by  $E_{\delta}(u)$  in (3.6) with another popular nonlocal version of  $E_0(u)$ :

(4.1) 
$$\hat{E}_{\delta}(u) = \int_{\Omega} \int \omega_{\delta}(s) \left| \frac{u(x+s) - u(x)}{s} \right|^{2} ds dx.$$

In the context of peridynamics,  $\hat{E}_{\delta}(u)$  may be seen as the linearized bond-based elastic energy associated with a nonlocal interaction kernel  $\omega_{\delta} = \omega_{\delta}(s)$ .

We now discuss the relations between (3.6) and (4.1). Let  $D_{\delta}$  denote  $(-\delta, \delta)^2$  and  $H_{\delta} = \Omega \times D_{\delta}$ . Note first that in the principal value sense of the integrals, we have

$$\mathbb{G}_{\rho}^{\delta}u(x) = \int_{-\delta}^{\delta} \rho_{\delta}(|s|) \frac{u(x+s)}{s} ds = -\int_{-\delta}^{\delta} \rho_{\delta}(|s|) \frac{u(x-s)}{s} ds$$

Thus,

$$E_{\delta}(u) = \int_{\Omega} |\mathbb{G}_{\rho}^{\delta} u(x)|^{2} dx$$

$$= -\int_{H_{\delta}} \frac{\rho_{\delta}(|s|)\rho_{\delta}(|t|)}{st} u(x+s)u(x-t) ds dt dx$$

$$= -\int_{H_{\delta}} \frac{\rho_{\delta}(|s|)\rho_{\delta}(|t|)}{st} u(y+s+t)u(y) ds dt dy$$

where we have done a shift in the variable y = x - t but the domain of integration remains the same due to the periodicity of u = u(x) in x. From the above, we then notice that, after switching y back to x,

$$E_{\delta}(u) = -\int_{H_{\delta}} \frac{\rho_{\delta}(|s|)\rho_{\delta}(|t|)}{st} u(x+s+t)u(x)dsdtdx$$

$$= \int_{H_{\delta}} \frac{\rho_{\delta}(|s|)\rho_{\delta}(|t|)}{2st} [u(x+s+t)^{2} - 2u(x+s+t)u(x) + u(x)^{2}]dsdtdx$$

$$= \int_{H_{\delta}} \frac{\rho_{\delta}(|s|)\rho_{\delta}(|t|)}{2st} [u(x+s+t) - u(x)]^{2} dsdtdx.$$

Now, consider the transformation a = s + t and b = t - s wit x unchanged, we use  $\hat{H}_{\delta}$  to denote the region in the new variables obtained from the transformation of  $H_{\delta}$ , then

$$E_{\delta}(u) = \int_{\hat{H}_{\delta}} \rho_{\delta} \left( \left| \frac{a-b}{2} \right| \right) \rho_{\delta} \left( \left| \frac{a+b}{2} \right| \right) \frac{2a^{2}}{a^{2} - b^{2}} \left| \frac{u(x+a) - u(x)}{a} \right|^{2} dadbdx$$
$$= \int_{\Omega} \int_{-2\delta}^{2\delta} \omega_{\delta}(|a|) \left| \frac{u(x+a) - u(x)}{a} \right|^{2} dadx,$$

where

$$\omega_{\delta}(a) = \omega_{\delta}(|a|) = \int_{|a|-2\delta}^{-|a|+2\delta} \rho_{\delta}\left(\left|\frac{a-b}{2}\right|\right) \rho_{\delta}\left(\left|\frac{a+b}{2}\right|\right) \frac{2a^2}{a^2 - b^2} db$$

is a kernel function supported in  $a \in (-2\delta, 2\delta)$ .

This implies that  $E_{\delta}(u)$  is equivalent to  $\hat{E}_{\delta}(u)$  with the kernel  $\omega_{\delta} = \omega_{\delta}(|a|)$ . Consequently, the corresponding nonlocal diffusion operator (that is, the variation of the energy, or the bond force operator in peridynamics) is given by

$$-\mathcal{L}_{\delta}u(x) = \int_{-2\delta}^{2\delta} \omega_{\delta}(s) \frac{u(x+s) - 2u(x) + u(x-s)}{s^2} ds.$$

As we have elucidated in our earlier works, the nonlocal operator  $\mathcal{L}_{\delta}$  may be viewed as a nonlocal continuum weighted average of the classical second order central difference operator with the kernel  $\omega_{\delta}$  serving as the weight function. A direct calculation shows that

$$\int_{-2\delta}^{2\delta} \omega_{\delta}(a) da = 1,$$

which gives the correct normalization condition on  $\omega_{\delta}$ . However, it can also be seen that

$$\int_{-2\delta}^{2\delta} \frac{\omega_{\delta}(a)}{a^2} da = 0,$$

which implies that the nonlocal interaction kernel is a sign-changing one. That is, for example, in the context of linear bond-based peridynamics, we get both repulsive and attractive bond forces. Naturally, repulsive interaction with a positive sign of kernel more likely yields coercivity and stability. Having attractive interactions may cause the loss of coercivity but this is not always the case. In [20], well-posedness of linear bond-based peridynamics with a sign changing kernel has been established. In essence, as long as the repulsive effects are dominant, we could still expect a well-defined nonlocal model.

We take the moment to consider a couple of properties of  $\omega_{\delta}$  in connection with  $\rho_{\delta}$ . First, we note that if  $\rho_{\delta}$  is taken to be rescaled from a horizon  $(\delta)$  independent kernel  $\rho(s)$ , that is  $\rho_{\delta}(s) = \delta^{-1}\rho(s/\delta)$  with  $\rho$  satisfying (3.1). Then, by a change of variables  $a = \tilde{a}\delta$  and  $b = \tilde{b}\delta$ , we have

$$(4.2) \qquad \omega_{\delta}(a) = \frac{1}{\delta} \int_{|\tilde{a}|-2}^{-|\tilde{a}|+2} \rho(\left|\frac{\tilde{a}-\tilde{b}}{2}\right|) \rho(\left|\frac{\tilde{a}+\tilde{b}}{2}\right|) \frac{2\tilde{a}^2}{\tilde{a}^2 - \tilde{b}^2} d\tilde{b} = \frac{1}{\delta} \omega_1(\frac{a}{\delta}).$$

This means, not surprisingly, that  $\omega_{\delta}$  is also rescaled from a kernel  $\omega_1$  symmetrically defined on (-1,1). Next, we note that it is easy to see

$$\int_{-2\delta}^{2\delta} \frac{|\omega_{\delta}(a)|}{a^2} da \le \left( \int_{-\delta}^{\delta} \frac{\rho_{\delta}(s)}{|s|} ds \right)^2,$$

which tells us in particular that the singular behavior of  $\rho_{\delta}$  at the origin likely controls the singularity of  $\omega_{\delta}$  near the origin. Clearly, if  $|s|^{-1}\rho_{\delta}(s)$  is integrable, then so

is  $a^{-2}|\omega_{\delta}(a)|$ . This in turn implies that  $\mathbb{G}_{\rho}^{\delta}$  and the associated nonlocal diffusion operator  $\mathcal{L}_{\delta}$  are bounded operators on the space of square integrable functions. While kernels with integrable  $|s|^{-1}\rho_{\delta}(s)$  and  $a^{-2}|\omega_{\delta}(a)|$  are among the popular choices in applications, they are not necessarily good choices, as shown later, for correspondence formulations like (3.6) are to be adopted.

In the following, we provide more detailed calculations to give some explicit conditions on  $\rho_{\delta}$  under which the coercivity of (3.6) can be assured. The periodic setting allows us to use Fourier analysis, a very convenient and frequently used tool in studies of nonlocal models, see for example [37]. The type of calculations involved is similar to dispersion analysis, see for instance [3, 27, 36] for studies related to peridynamics.

5. Stability of nonlocal Dirichlet integral. Let us first clarify the stability or coercivity that we refer to, namely, we define a function space  $V_{\delta}$  that is the completion of  $C^{\infty}$  periodic functions with mean zero subject to the nonlocal norm

$$||u||_{\delta} = (E_{\delta}(u) + \int_{\Omega} |u(x)|^2 dx)^{1/2}.$$

The coercivity (or variational stability) of the Dirichlet integral  $E_{\delta}(u)$  refers to the fact that over the space  $V_{\delta}$ , we have a positive constant C > 0, such that

$$E_{\delta}(u) \ge C \|u\|_{\delta}^2, \quad \forall u \in V_{\delta}.$$

Obviously, this can be seen as a consequence of the so-called nonlocal Poincaré inequality [21, 22]: there exists a constant c > 0 such that

(5.1) 
$$E_{\delta}(u) = \int_{\Omega} |\mathbb{G}_{\rho}^{\delta} u(x)|^{2} dx \ge c \int_{\Omega} |u(x)|^{2} dx, \quad \forall u \in V_{\delta}.$$

We note that the argument presented in [29, Proposition 2] for the failure of stability of  $E_{\delta}(u)$  was based the choice of an increment in the deformation field by a Dirac-delta point measure. Such a choice is not feasible in the space  $V_{\delta}$ . We now attempt to specify some conditions on the kernel  $\rho_{\delta}$  so that (5.1) can be verified and thus leading to the stability of  $E_{\delta}(u)$ . We take advantage of the periodicity of u to adopt Fourier analysis.

Under periodic conditions and the constraint (3.7), we could write u in terms of their Fourier series, namely,

$$u(x) = \sum_{k=1}^{\infty} \widehat{u}(k)e^{i2\pi kx}$$
, where  $\widehat{u}(k) = \int_{\Omega} u(x)e^{-i2\pi kx}dx$ .

The rest of the technical derivations is focused on one dimensional scalar fields. For the vector field case in high dimensions, we refer to [12]. Let us first present the so-called Fourier symbols of the operator  $\mathbb{G}_{\rho}^{\delta}$ .

LEMMA 1. The Fourier transform of the nonlocal gradient operator  $\mathbb{G}_{\rho}^{\delta}$  is given by

(5.2) 
$$\widehat{\mathbb{G}_{\rho}^{\delta}}u(k) = ib_{\delta}(k)\widehat{u}(k)$$

where  $b_{\delta}(k)$  are the Fourier symbols of  $\mathbb{G}_{0}^{\delta}$  given by

(5.3) 
$$b_{\delta}(k) = 2 \int_0^{\delta} \frac{\rho_{\delta}(s)}{s} \sin(2\pi ks) ds.$$

*Proof.* Using (3.5) and the periodicity of u, we have

$$\begin{split} \widehat{\mathbb{G}_{\rho}^{\delta}}u(k) &= \int_{\Omega} \mathbb{G}_{\rho}^{\delta}u(x)e^{-i2\pi kx}dx \\ &= \int_{\Omega} \int_{0}^{\delta} \rho_{\delta}(s)\frac{u(x+s) - u(x-s)}{s}e^{-i2\pi kx}dsdx \\ &= \int_{\Omega} \int_{0}^{\delta} \frac{\rho_{\delta}(s)}{s}(e^{i2\pi ks} - e^{-i2\pi ks})u(x)e^{-i2\pi kx}dsdx \\ &= 2i\int_{0}^{\delta} \frac{\rho_{\delta}(s)}{s}\sin(2\pi ks)ds\int_{\Omega}u(x)e^{-i2\pi kx}dx \\ &= ib_{\delta}(k)\widehat{u}(k) \,. \end{split}$$

The following simple fact on the sine Fourier coefficient, a special form of Riemann-Stieltjes type integrals, is useful to our discussion.

LEMMA 2. Given a measurable, non-negative and non-increasing function g = g(x) with xg(x) integrable, we have

$$\int_0^{2\pi} g(x)\sin(x)dx \ge 0$$

with the equality holds only for g being a constant function. Consequently, for any h > 0 and a > 0, we have

(5.5) 
$$\int_0^h g(x)\sin(ax)dx \ge 0,$$

with the equality holds only for g being a constant function (and with value zero if ha is not an integer multiple of  $2\pi$ ).

*Proof.* The inequality (5.4) follows immediately from the observation that

$$\int_0^{2\pi} g(x)\sin(x)dx = \int_0^{\pi} [g(x) - g(x+\pi)]\sin(x)dx \ge 0.$$

By the non-increasing property, we see that the equality holds only for g being a constant function. The more general case follows by applying a change of variable and taking a zero extension of g outside (0,h) to cover complete periods of the scaled sine function.  $\square$ 

The expressions of Fourier symbols  $\{b_{\delta}(k)\}$  given in Lemma 1 have a number of immediate consequences upon simple observations. First, if as  $\delta \to 0$  we have  $2\rho_{\delta}(s)$ , which is a density on  $(0, \delta)$ , approaches to the Dirac-delta measure at the origin, then for any given k and  $\delta \to 0$ ,

$$b_{\delta}(k) = 2 \int_0^{\delta} \frac{\rho_{\delta}(s)}{s} \sin(2\pi ks) ds = 2\pi k \int_0^{\delta} 2\rho_{\delta}(s) \frac{\sin(2\pi ks)}{2\pi ks} ds \to 2\pi k ,$$

which again gives the obvious connection of the nonlocal derivative to the local derivative.

Another immediate consequence, coming from Lemma 2, is that  $b_{\delta}(k)$  stays positive for any finite wave number k if  $s^{-1}\rho_{\delta}(s)$  is non-increasing and not a constant. This provides a sufficient condition to assure  $b_{\delta}(k) \neq 0$ . While not necessary, violating such a condition may result in undesirable effects. We can see from the simple example that  $s^{-1}\rho_{\delta}(s)$  being a constant function leads to  $b_{\delta}(k) = 0$  for some wave number k. Indeed, for such a choice of  $\rho_{\delta}$ ,

$$b_{\delta}(k) = c \int_0^{\delta} \sin(2\pi ks) ds = \frac{c}{2\pi k} (1 - \cos(2\pi k\delta)),$$

which is zero if  $k\delta$  is an integer. The Fourier modes corresponding to such wave numbers may be interpreted as the so-called *zero-energy* modes, reinforcing the understanding that there are inherent instabilities associated with the correspondence theory at the continuum level [29]. However, such zero-energy modes do not exist for proper choices of the kernel described below.

Naturally, the coercivity of the Dirichlet integral is not only concerned with  $b_{\delta}(k)$  for finite k but also its asymptotic and uniform behavior as  $k \to \infty$ . The latter is a feature of the continuum theory that allows the wave number going to infinity, though, unfortunately, it has not been given adequate attention in the existing literature before. From the Riemann-Lebesgue lemma, we know that  $b_{\delta}(k) \to 0$  as  $k \to \infty$  if  $s^{-1}\rho_{\delta}(s)$  is integrable. Thus, requiring  $s^{-1}\rho_{\delta}(s)$  being non-integrable becomes necessary, a fact that should be taken into consideration seriously when working with the correspondence theory.

Note that requiring  $s^{-1}\rho_{\delta}(s)$  both non-increasing and non-integrable, in order to get the non-degeneracy of  $b_{\delta}(k)$  for finite k and as  $k \to \infty$ , would lead to a restricted growth of  $\rho_{\delta}(s)$  near the origin. That is, as the undeformed bond length approaches zero, the prescription of the nonlocal interaction is limited. A most general discussion about necessary and sufficient conditions on the kernel to assure the coercivity is beyond the scope of this work. Instead, to keep the mathematical discussions to a minimal level, we present a result on the coercivity of the nonlocal Dirichlet integral under the following sufficient conditions.

Assumption 1. We assume that the kernel  $\rho_{\delta}$  is given by a rescaled kernel  $\rho_{\delta}(s) = \rho(s/\delta)/\delta$  with  $\rho = \rho(s)$  satisfying (3.1) and the following:

- 1)  $\rho(s)/s$  is non-increasing for  $s \in (0,1)$ ;
- 2)  $\rho(s)$  is of fractional type in at least a small neighborhood of origin, namely, there exists some  $\epsilon > 0$  such that for  $s \in (0, \epsilon)$  we have

$$\rho(s) = \frac{c}{s^{\alpha}},$$

for some constant c > 0 and  $\alpha \in (0,1)$ .

Note that the condition associated with (5.6) automatically implies the non-integrability of  $s^{-1}\rho_{\delta}(s)$  in the one space dimensional case under consideration.

Theorem 3. Under the Assumption 1, the nonlocal Poincaré inequality holds, namely, there exists a constant C independent of u such that

(5.7) 
$$||u||_{L^{2}(\Omega)}^{2} \leq CE_{\delta}(u).$$

Moreover, C is also independent of  $\delta$  as  $\delta \to 0$ .

*Proof.* Under periodic conditions, (5.7) is equivalent to showing that  $b_{\delta}(k)$  is uniformly bounded from below. In fact, we can write

$$b_{\delta}(k) = 2 \int_{0}^{\delta} \frac{\rho_{\delta}(s)}{s} \sin(2\pi ks) ds = \frac{2}{\delta} \int_{0}^{1} \frac{\rho(s)}{s} \sin(as) ds$$

where  $a = 2\pi k\delta$ . Notice that the above quantity is positive with the assumption that  $\rho(s)/s$  is non-increasing and not a constant.

For a < 1, we use

$$\sin(as) \ge as - \frac{(as)^3}{6}$$

to get

$$b_{\delta}(k) \geq \frac{2a}{\delta} \int_{0}^{1} \rho(s)ds - \frac{a^{3}}{3\delta} \int_{0}^{1} \rho(s)s^{2}ds \geq \frac{5a}{3\delta} = \frac{10\pi}{3}k \geq \frac{10\pi}{3}.$$

For  $a \in [1, 2\pi/\epsilon]$ , where  $\epsilon$  is the parameter defined in Assumption 1, we know that

$$2\int_0^1 \frac{\rho(s)}{s} \sin(as) ds > 0$$

is a continuous function of a, thus it has a lower bound  $C_1$ , then

$$b_{\delta}(k) \ge \frac{C_1}{\delta} \ge C_1 \epsilon k$$
.

Now for  $a > 2\pi/\epsilon$ , we write

$$b_{\delta}(k) = \frac{2}{\delta} \int_0^1 \frac{\rho(s)}{s} \sin(as) ds = \frac{2}{\delta} \left( \int_0^{\frac{2\pi}{a}} \frac{\rho(s)}{s} \sin(as) ds + \int_{\frac{2\pi}{a}}^1 \frac{\rho(s)}{s} \sin(as) ds \right).$$

Using Lemma 2 and the assumption that  $\rho(s)/s$  is non-increasing on (0,1), we have

$$\int_{\frac{2\pi}{a}}^{1} \frac{\rho(s)}{s} \sin(as) ds = \int_{0}^{1 - \frac{2\pi}{a}} \frac{\rho(s + 2\pi/a)}{s + 2\pi/a} \sin(as) ds \ge 0.$$

Then

$$(5.8) b_{\delta}(k) \ge \frac{2}{\delta} \int_{0}^{\frac{2\pi}{a}} \frac{\rho(s)}{s} \sin(as) ds \ge \frac{ca^{\alpha}}{\delta} \int_{0}^{2\pi} \frac{1}{s^{1+\alpha}} \sin(s) ds = \frac{\tilde{C}k^{\alpha}}{\delta^{1-\alpha}},$$

where the Assumption 1 is used with  $\alpha \in (0,1)$ . Thus  $b_{\delta}(k)$  has a lower bound for all integer  $k \geq 1$  independent of  $\delta$ , which implies the Poincaré inequality (5.7).  $\square$ 

Theorem 3 shows the stability of the Dirichlet integral while the kernel  $\rho(s)$  a fractional-type kernel near origin with the fractional component  $\alpha \in (0,1)$ . We can further see from its proof that the generated function space is equivalent to a fractional Sobolev space.

THEOREM 4. Under the Assumption 1, for each fixed  $\delta > 0$ , the space  $V_{\delta}$  defined by the nonlocal norm  $\|\cdot\|_{\delta}$  is equivalent to the fractional Sobolev space  $H^{\alpha}(\Omega)$ , where  $\alpha \in (0,1)$  is the index defined in (5.6).

*Proof.* In the proof of Theorem 3, notice that in the case  $a > 2\pi/\epsilon$ , we first have the lower bound given by (5.8). Moreover, we could also write

$$b_{\delta}(k) = \frac{2}{\delta} \int_0^1 \frac{\rho(s)}{s} \sin(as) ds = \frac{2}{\delta} \left( \int_0^{\frac{\pi}{a}} \frac{\rho(s)}{s} \sin(as) ds + \int_{\frac{\pi}{a}}^1 \frac{\rho(s)}{s} \sin(as) ds \right).$$

Using Lemma 2 once again, we have

$$\int_{\frac{\pi}{a}}^{1} \frac{\rho(s)}{s} \sin(as) ds = \int_{0}^{1-\frac{\pi}{a}} \frac{\rho(s+\pi/a)}{s+\pi/a} \sin(as+\pi) ds$$
$$= -\int_{0}^{1-\frac{\pi}{a}} \frac{\rho(s+\pi/a)}{s+\pi/a} \sin(as) ds < 0.$$

Thus we obtain the upper bound given by

$$b_{\delta}(k) \leq \frac{2}{\delta} \int_{0}^{\frac{\pi}{a}} \frac{\rho(s)}{s} \sin(as) ds \leq \frac{ca^{\alpha}}{\delta} \int_{0}^{\pi} \frac{1}{s^{1+\alpha}} \sin(s) ds = \frac{\tilde{C}k^{\alpha}}{\delta^{1-\alpha}}$$

Combining both the lower and upper bounds, we get the equivalence to the fractional space and norm.  $\square$ 

We remark that an alternative way to see the above is to use the equivalent formulation given in (4.1) for the kernel  $\omega_{\delta}$  in (4.2). For the kernel  $\rho(s)$  in (5.6) that grows like  $s^{-\alpha}$  near the origin, a direct calculation shows that  $\omega_{\delta}(s)$  grows like  $s^{1-2\alpha}$ . Hence, the nonlocal Dirichlet integral gives a canonical form of the square of a  $H^{\alpha}$  semi-norm.

The equivalence of function spaces mentioned above is not simply a mathematical statement, it too bears significance in nonlocal modeling of physical processes involving singularities such as the peridynamic modeling of cracks. In practice, in order to allow discontinuous solutions in the underlying energy space  $V_{\delta}$ , we see that one should make  $\alpha < 1/2$  (in one space dimension) by the standard Sobolev space embedding result. Based on the above theorems and the characterization on the kernels given in the Assumption 1, we see that there are indeed reasonable choices of the nonlocal interaction kernels that provide coercive (and stable) forms of energy for the correspondence theory (to maintain a well-behaved mathematical model), while allowing the discontinuous deformation field (to keep a physically desirable feature). However, these possible choices may be limited, for example, they may be subject to conditions given in Assumption 1. Note again, we do not claim that the assumption here is the most general one possible since our objective is to establish some rigorous results with fairly elementary calculations without making the mathematical derivations too technical.

6. Other nonlocal variants of the Dirichlet integrals. In [29], an alternative formulation to the elastic energy is provided as a possible remedy to alleviate the loss of coercivity of the original correspondence peridynamic materials models. The main ingredient consists of an additional contribution involving the nonuniform part

of the deformation field u that is denoted by z. In the linearized theory and one space dimension, we have

$$z(y,x) = u(y+x) - u(y) - \mathbb{G}_{\rho}^{\delta}u(y)x.$$

Note that if taking z = z(y, x) as a peridynamic state [28] (with dependence on both y and x), we have

$$\mathbb{G}_{o}^{\delta}z(y,x) = \mathbb{G}_{o}^{\delta}u(y+x) - \mathbb{G}_{o}^{\delta}u(y),$$

which is in general nonzero for nonlinear deformation field u. The *stabilized* energy suggested in [29] is given by the following variant of the nonlocal Dirichlet integral:

$$(6.1) \quad \tilde{E}_{\delta}(u) = \int_{\Omega} |\mathbb{G}_{\rho}^{\delta} u(x)|^2 dx + \int_{\Omega} \int_{-\delta}^{\delta} \sigma_{\delta}(|s|) \left| \frac{u(x+s) - u(x)}{s} - \mathbb{G}_{\rho}^{\delta} u(x) \right|^2 ds dx \,,$$

where  $\sigma_{\delta} = \sigma_{\delta}(|s|)$  is assumed to be a compactly supported kernel for  $|s| \leq \delta$ . In this case, the coercivity of  $\tilde{E}_{\delta}$  can be established under some conditions on  $\sigma_{\delta}$  but without imposing the stronger conditions on the kernel  $\rho_{\delta} = \rho_{\delta}(s)$  given in the Assumption 1. Here, we present an argument that is different from that given in [29]. Let us denote

$$\int_{-\delta}^{\delta} \sigma_{\delta}(|s|) ds = \beta > 0.$$

Then

$$\tilde{E}_{\delta}(u) = \int_{\Omega} |\mathbb{G}_{\rho}^{\delta} u(x)|^{2} dx + \int_{\Omega} \int_{-\delta}^{\delta} \sigma_{\delta}(|s|) \left| \frac{u(x+s) - u(x)}{s} - \mathbb{G}_{\rho}^{\delta} u(x) \right|^{2} ds dx$$

$$= \int_{\Omega} (1+\beta) |\mathbb{G}_{\rho}^{\delta} u(x)|^{2} dx + \int_{\Omega} \int_{-\delta}^{\delta} \sigma_{\delta}(|s|) \left| \frac{u(x+s) - u(x)}{s} \right|^{2} ds dx$$

$$-2 \int_{\Omega} \int_{-\delta}^{\delta} \sigma_{\delta}(|s|) \frac{u(x+s) - u(x)}{s} \mathbb{G}_{\rho}^{\delta} u(x) ds dx.$$

Notice that

$$2\left|\frac{u(x+s)-u(x)}{s}\mathbb{G}_{\rho}^{\delta}u(x)\right| \leq \frac{2}{2+\beta}\left|\frac{u(x+s)-u(x)}{s}\right|^{2} + \frac{2+\beta}{2}|\mathbb{G}_{\rho}^{\delta}u(x)|^{2}.$$

So

$$\tilde{E}_{\delta}(u) \geq \frac{\beta}{2} \int_{\Omega} |\mathbb{G}_{\rho}^{\delta} u(x)|^{2} dx + \frac{\beta}{2+\beta} \int_{\Omega} \int_{-\delta}^{\delta} \sigma_{\delta}(|s|) \left| \frac{u(x+s) - u(x)}{s} \right|^{2} ds dx.$$

Now we can see that the second term represents a typical energy for a linear bond-based model and, for a positive  $\sigma_{\delta} = \sigma_{\delta}(s)$ , the term vanishes only for a constant field that is identically zero by (3.7).

In fact, under suitable assumptions on  $\sigma_{\delta}$  as those presented in [7] and [26] (and extended to vector fields in [20, 21, 22]), we have the Poincaré inequality

$$\int_{\Omega} \int_{-\delta}^{\delta} \sigma_{\delta}(|s|) \left| \frac{u(x+s) - u(x)}{s} \right|^{2} ds dx \ge c \int_{\Omega} |u(x)|^{2} dx$$

for a constant c > 0, hence the coercivity of the energy  $\tilde{E}_{\delta}$ . This line of argument has a striking similarity with the study of coercivity of the energy functionals for the linearized state-based peridynamic Navier equation. The state-based energy functional involves contributions from two terms similar to those in (6.1) but with different mechanical interpretations: one from the elongational part and the other from the deviatoric part, see [21] for more details.

While the coercivity of  $\tilde{E}_{\delta}$  can be guaranteed, such a variant presents other issues. For instance, one issue with (6.1) is that while for linear deformation field the energy remains independent of  $\beta$ , there are obvious discrepancies when more general nonlinear deformation fields are considered, leading to different physical responses that are dependent on the choices of  $\beta$ . The same can be said about the variational problems related to (6.1) subject to the work of an external force. The equilibrium solutions, for the same given external force, will be generically different when  $\beta$  changes. Similarly, the Fourier symbols associated with the modified operator (and thus the dispersion relations) also differ from their original and un-modified forms. Thus, the extra penalty term could play a significant role for nonzero  $\beta$ , which may become an undesirable feature in practice. In contrast, for more careful choices of the kernel  $\rho_{\delta}$ , our analysis in the previous section shows that there is no need to introduce the penalty term for the sake of coercivity or variational stability.

Besides the variant discussed above, there are also other options to get a nonlocal Dirichlet integral that is different from (3.6). For example, we may replace the nonlocal gradient operator in (3.2) by the one-sided versions given in (3.8). Similar discussions can be made with these new choices, along with extensions to vector fields defined in higher space dimensions (see [12] for related calculations).

7. Implications and generalizations. An important message taken from the investigation presented in this work is that one should evaluate carefully the choices of the nonlocal interaction kernels when correspondence models are adopted. An intuitive interpretation of our rigorous analysis is that strengthened interactions among close-by materials points tend to promote stability. This is perhaps not surprising as the well-posed local model, when physically valid, represents the extreme case that all interactions are concentrated at the same materials points. On the other hand, in order to allow solutions with defects and singularities like discontinuities in the deformation field for periydnamics, it is equally important to have the interactions spread over a nonlocal region. Moreover, the close-by interactions should not be too strong to disallow the formation of such singular behavior. Hence, adopting appropriate nonlocal interaction kernels becomes a subtle issue when one desires to take advantage of the correspondence theory to model complex systems. This finding sheds light on the range of applicability of peridynamic correspondence material models. The latter should not be adopted blindly but needs not be thrown out entirely. Via Fourier analysis, we are able to offer a rather precise characterization on the feasible choices of nonlocal interaction kernels that helps maintaining coercivity and stability on the continuum level.

We note that our discussion here is focused on the continuum level since nonlocal models like peridynamics are indeed fundamentally continuum theory in the first place. The stability and coercivity issues should thus be variational (or continuum model) properties, independent of discretizations. While the discussion is focused on one dimensional scalar fields, generalizations to high dimensional vector fields are

possible, and one can find relevant discussions in [12]. One may also consider geometric settings other than the periodic domain, in which case, Fourier analysis is no longer effective, but one may use techniques similar to those in [20] for peridynamic models with a sign-changing kernel to study the stability of energy with a more general class of kernels subject to various nonlocal constraints.

There are inevitably additional issues when numerical discretizations are concerned. In fact, a side-effect of the strengthened interaction for close-by materials points is that more accurate quadrature rules may have to be adopted in discretization. Still, one may follow similar ideas presented here to reach further understanding of the numerical stability and coercivity issues as well. This would allow us to delineate the roles of physical scale and level of numerical resolutions, a point that is worthwhile to be emphasized for nonlocal modeling. One can also attempt to develop asymptotically compatible discretizations [11, 33, 34] to the correspondence models so as to retain consistency and robustness of the numerical approximations. In addition, the current study can further enable us to introduce mixed formulation to numerically approximate the nonlocal models based on the correspondence material models whenever the latter are physically sound and mathematically valid. Lastly, the notion of nonlocal gradient may also be related to the use of kernel-based integral approximations to differential operators in methods like SPH and RKPM [12, 18, 25]. Future studies can help build a stronger connections between these similar subjects.

**Acknowledgement**: The authors would like to thank S. Silling, M. Gunzburger, R. Lehoucq, J. Foster, F. Bobaru, J.S. Chen, W.K. Liu, and in particular T. Mengesha, for discussions on related subjects.

## REFERENCES

- [1] A. Aguiar, and R. Fosdick. A constitutive model for a linearly elastic peridynamic body, *Mathematics and Mechanics of Solids*, **19**, (2014), 502-523.
- [2] E. Askari, F. Bobaru, R. B. Lehoucq, M. L. Parks, S. A. Silling, and O. Weckner, Peridynamics for multiscale materials modeling, J. Phys. Conf. Ser., 125 (2008), 12-78.
- [3] Z. Bazant, W. Luo, V. Chau and M. Bessa. Wave Dispersion and Basic Concepts of Peridynamics Compared to Classical Nonlocal Damage Models, *Journal of Applied Mechanics* 83 (2016), 111004.
- [4] M. Bessa, J. Foster, T. Belytschko, and W. K. Liu, A meshfree unification: reproducing kernel peridynamics, *Computational Mechanics*, **53** (2014), 1251-1264.
- [5] M. Breitenfeld, P. Geubelle, O. Weckner and S. Silling, Non-ordinary state-based peridynamic analysis of stationary crack problems, *Comput. Methods Appl. Mech. Engrg.*, 272 (2014), 233250.
- [6] F. Bobaru and M. Duangpanya, The peridynamic formulation for transient heat conduction, Internat. J. Heat Mass Transfer, 53 (2010), 4047-4059.
- [7] J. Bourgain, H. Brezis, and P. Mironescu. Another look at Sobolev spaces, 439–455, IOS Press, Amsterdam, 2001.
- [8] A. Buades, B. Coll, and J. M. Morel, Image denoising methods. A new nonlocal principle, SIAM Rev., 52 (2010), 113-147.
- [9] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou, Analysis and approximation of nonlocal diffusion problems with volume constraints, SIAM Rev., 56 (2012), 676-696.
- [10] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou, A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws, *Math. Models Methods Appl.* Sci., 23 (2013), 493-540.
- [11] Q. Du, Y. Tao, X. Tian and J. Yang, Robust a posteriori stress analysis for quadrature collocation approximations of nonlocal models via nonlocal gradients, Computer Methods in Applied Mechanics and Engineering, 310 (2016), 605-627
- [12] Q. Du and X. Tian, Mathematics of Smoothed Particle Hydrodynamics, Part I: a nonlocal Stokes equation, preprint (2017).

- [13] Q. Du, J. Yang and Z. Zhou, Analysis of a nonlocal-in-time parabolic equation, Discrete & Continuous Dynamical Systems - B, 22 (2017). 339-368,
- [14] J. Foster, Constitutive Modeling in Peridynamics, in Handbook of Peridynamic Modeling, (2016), CRC Press, 143.
- [15] G. Gilboa and S. Osher, Nonlocal operators with applications to image processing, Multiscale Model. Simul., 7 (2008), 1005-1028.
- [16] R. Lehoucq, P. Reu, and D. Turner. A Nonlocal Strain Measure for Digital Image Correlation, Strain, 51 (2015), 265-275.
- [17] R. Lehoucq, and S.A. Silling. Force flux and the peridynamic stress tensor, Journal of the Mechanics and Physics of Solids, 56.4 (2008), 1566-1577.
- [18] W.K. Liu, Y. Chen, S. Jun, J. Chen, T. Belytschko, C. Pan, R. Uras and C. Chang, Overview and applications of the reproducing kernel particle methods, Archives of Computational Methods in Engineering, 3 (1996), 3-80.
- [19] Y. Lou, X. Zhang, S. Osher, and A. Bertozzi, Image recovery via nonlocal operators, J. Sci. Comput., 42 (2010), 185-197.
- [20] T. Mengesha and Q. Du, Analysis of a scalar peridynamic model with a sign changing kernel, Disc. Cont. Dyn. Sys. B, 18 (2013), 1415-1437.
- [21] T. Mengesha and Q. Du, Nonlocal Constrained Value Problems for a Linear Peridynamic Navier Equation, Journal of Elasticity, 116 (2014), 27-51.
- [22] T. Mengesha and Q. Du, The bond-based peridynamic system with Dirichlet-type volume constraint, Proceedings of the Royal Society of Edinburgh, 144A (2014), 161-186.
- [23] T. Mengesha and Q. Du. On the variational limit of a class of nonlocal functionals related to peridynamics, *Nonlinearity*, 28 (2015), 3999-4035.
- [24] T. Mengesha and Q. Du, Characterization of function spaces of vector fields via nonlocal derivatives and an application in peridynamics, Nonlinear Analysis A: Theory, Methods and Applications, 140 (2016), 82-111.
- [25] J.J. Monaghan, Smoothed particle hydrodynamics, Rep. Prog. Phys., 68 (2005), 1703-1759.
- [26] A. C. Ponce. An estimate in the spirit of Poincaré's inequality. J. Eur. Math. Soc, 6 (2004), 1-15.
- [27] S.A. Silling, Reformulation of elasticity theory for discontinuities and long-range forces, J. Mech. Phys. Solids, 48 (2000), 175-209.
- [28] S.A. Silling, Linearized theory of peridynamic states. Journal of Elasticity, 99 (2010), 85-111.
- [29] S.A. Silling, Stability of peridynamic correspondence material models and their particle discretizations, Computer Methods in Applied Mechanics and Engineering, 322 (2017), 42-57,
- [30] S. A. Silling and R. B. Lehoucq, Peridynamic theory of solid mechanics, Adv. Appl. Mech., 44 (2010), 73-168.
- [31] S. A. Silling, O. Weckner, E. Askari and F. Bobaru, Crack nucleation in a peridynamic solid, Internat. J. Fracture, 162 (2010), 219-227.
- [32] E. Tadmor and C. Tan. Critical thresholds in flocking hydrodynamics with non-local alignment. Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 372 (2014), 20130401.
- [33] X. Tian and Q. Du, Analysis and comparison of different approximations to nonlocal diffusion and linear peridynamic equations, SIAM J. Numer. Anal., 51 (2013), 3458-3482.
- [34] X. Tian and Q. Du, Asymptotically Compatible Schemes and Applications to Robust Discretization of Nonlocal Models, SIAM J. Numer. Anal., 52 (2014), 1641-1665.
- [35] M. Tupek and R. Radovitzky, An extended constitutive correspondence formulation of peridynamics based on nonlinear bond-strain measures, *Journal of the Mechanics and Physics* of Solids, 65 (2014), 82-92.
- [36] O. Weckner and R. Abeyaratne, The Effect of Long-Range Forces on the Dynamics of a Bar, J. Mech. Phys. Solids, 53 (2005), 705728.
- [37] K. Zhou and Q. Du, Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions, SIAM J. Numer. Anal., 48 (2010), 1759-1780.