# Strong local optimality for a bang-bang-singular extremal: the fixed-free case 

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In this paper we give sufficient conditions for a Pontryagin extremal trajectory, consisting of two bang arcs followed by a singular one, to be a strong local minimizer for a Mayer problem. The problem is defined on a manifold $M$ and the end-points constraints are of fixed-free type. We use a Hamiltonian approach and its connection with the second order conditions in the form of an accessory problem on the tangent space to $M$ at the final point of the trajectory. Two examples are proposed.

## 1 Introduction

In this paper we consider a reference trajectory consisting of two bang arcs followed by a singular (or partially singular) one, for a Mayer problem with fixed final time $T$ and a control affine dynamics.

We give sufficient optimality conditions for the reference trajectory to be a strong local minimiser in the case when the end-point constraints are of fixed-free type.

A Bolza problem can be reduced to a Mayer one, hence sufficient optimality conditions can be also derived for a Bolza problem, see the examples in Section 4.5.

Control affine systems can be modelled in different ways; since we want to consider both bang-bang arcs and partially singular arcs, we model the system as follows.

Let $M$ be a finite dimensional manifold and let $X_{1}, \ldots, X_{m}$ be smooth vector fields on $M$. Let $\Delta:=\left\{u=\left(u_{1}, \ldots, u_{m}\right) \in \mathbb{R}^{m}: u_{i} \geq 0, i=1, \ldots, m, \sum_{i=1}^{m} u_{i}=1\right\}$ so that at each point $x \in M$ the closed convex hull $\mathcal{X}$ of the vector fields $X_{1}, \ldots, X_{m}$ is given by

$$
\mathcal{X}(x)=\left\{\sum_{i=1}^{m} u_{i} X_{i}(x): u=\left(u_{1}, \ldots, u_{m}\right) \in \Delta\right\} .
$$

Let $T>0$ and $x_{0} \in M$, we consider an optimal control problem of the following kind

$$
\begin{align*}
& \text { minimize } c(\xi(T)) \text { subject to }  \tag{1a}\\
& \dot{\xi}(t) \in \mathcal{X}(\xi(t)) \quad \text { a.e. } t \in[0, T]  \tag{1b}\\
& \xi(0)=x_{0} \tag{1c}
\end{align*}
$$

Equivalently, by Filippov's theorem, see e.g. [4], equation (1b) can also be written as

$$
\dot{\xi}(t)=\sum_{i=1}^{m} v_{i}(t) X_{i}(\xi(t)), \quad \text { a.e. } t \in[0, T], \quad v \in L^{\infty}([0, T], \Delta)
$$

Our aim is to give sufficient conditions for an extremal reference trajectory to be indeed a strong local optimiser of the problem in the following sense:
Definition 1.1. The trajectory $\widehat{\xi}:[0, T] \rightarrow M$ is a strong local minimiser of problem (1) if there exists a neighbourhood $\mathcal{U}$ of its graph in $\mathbb{R} \times M$ such that $\widehat{\xi}$ is a minimiser among the admissible trajectories whose graph is in $\mathcal{U}$, i.e. among the admissible trajectories which are in a neighborhood of $\widehat{\xi}$ with respect to the $C^{0}$ topology.

Here we assume that the control associated to the reference trajectory is the concatenation of two bang arcs and of a partially singular one, as explained below.

Remark 1.2. In this paper we consider the case when the final point is not constrained, in order to avoid some technical difficulties. In a future paper, [10], we shall extend the result to the case when the final point $\xi(T)$ is constrained to a smooth submanifold $N$ of $M$. The extension can be obtained by adding a penalty term and taking advantage of some classical results on quadratic forms due to Hestenes, see [7], which permit to reduce the problem to a problem with free final point.

In [10] we shall also give an explicit formulation of the sufficient conditions for a Bolza problem.

Assume $\widehat{\xi}$ is the reference trajectory and that there exist times $\widehat{\tau}_{1}, \widehat{\tau}_{2}, 0<\widehat{\tau}_{1}<\widehat{\tau}_{2}<T$, vector fields $h_{1}, h_{2}, h_{3} \in\left\{X_{1}, \ldots X_{m}\right\}$, (where $h_{1}$ and $h_{3}$ might be the same vector field) and a measurable function $\widehat{v} \in L^{\infty}\left(\left[\widehat{T}_{2}, T\right],(0,1)\right)$ such that the solution $\widehat{\xi}$ to

$$
\begin{aligned}
& \dot{\xi}(t)=h_{1}(\xi(t)) \\
& \dot{\xi}(t)=h_{2}(\xi(t)) \\
&\left.\dot{\xi}(t)=\left(\widehat{\tau}_{1}\right), \widehat{\tau}_{2}\right), \\
& \xi(0)=x_{0}, \\
& \text { a.e. } t \in\left(\widehat{\tau}_{2}, T\right], \\
& \hline(\xi(t))+(1-\widehat{v}(t)) h_{2}(\xi(t))
\end{aligned}
$$

satisfies Pontryagin Maximum Principle (PMP).
Setting $f_{1}:=h_{3}-h_{2}$ we can write the dynamic on the singular arc as

$$
\begin{equation*}
\dot{\xi}(t)=h_{2}(\xi(t))+\widehat{v}(t) f_{1}(\xi(t)), \quad t \in\left(\widehat{\tau}_{2}, T\right) . \tag{2}
\end{equation*}
$$

We shall also define the time-dependent reference vector field $\widehat{f_{t}}$ as

$$
\widehat{f_{t}}:= \begin{cases}h_{1} & t \in\left[0, \widehat{\tau}_{1}\right),  \tag{3}\\ h_{2} & t \in\left(\widehat{\tau}_{1}, \widehat{\tau}_{2}\right), \\ h_{2}+\widehat{v}(t) f_{1} & \text { a.e. } t \in\left(\widehat{\tau}_{2}, T\right]\end{cases}
$$

To get the sufficient conditions we use a Hamiltonian approach and its connection with the second order conditions, whose leading ideas are the following:

1. To use the symplectic properties of the cotangent bundle to compare the costs of neighbouring admissible trajectories by lifting them to the cotangent bundle.
2. To define a suitable Hamiltonian flow $\mathcal{H}_{t}$ in the cotangent bundle $T^{*} M$, emanating from a horizontal Lagrangian submanifold $\Lambda$. Since the final point is free, the flow is considered to have the final time $T$ as a starting time and to go backward in time, Sec. 4.1.
3. To obtain a suitable second order approximation (2 $2^{\text {nd }}$ variation) in the form of a coordinate-free linear-quadratic (LQ) problem and to require its coercivity, Sec. 3.2.
4. To show that the derivative of $\mathcal{H}_{t}$ along the reference extremal is, up to an isomorphism, the linear Hamiltonian flow associated to the LQ problem, see Sec. 4.2 and 4.3.
5. To deduce that the projection on $M$ of $\mathcal{H}_{t}$ emanating from $\Lambda$ is locally invertible (see Sec. 4.4), so that we can go back to the first issue and we can compare the costs of neighbouring admissible trajectories by lifting them to the cotangent bundle, Theorem 4.3.

In this paper we only give the main ideas of the constructions and some proofs of the main results, while all the details will be given in [10].

## 2 Notation and preliminaries

In this paper we use some basic element of the theory of symplectic manifolds referred to the cotangent bundle $T^{*} M$. For a general introduction see [3], for specific application to Control Theory we refer to [1]. Let us recall some basic facts and let us introduce some specific notations.

Denote by $\pi: T^{*} M \rightarrow M$ the canonical projection, for $\ell \in T^{*} M$ the space $T_{\pi \ell}^{*} M$ is canonically embedded in $T_{\ell} T^{*} M$ as the space of tangent vectors to the fibres.

The canonical Liouville one-form $s$ on $T^{*} M$ and the associated canonical symplectic two-form $\boldsymbol{\sigma}=\mathrm{d} \boldsymbol{s}$ allow associating to any, possibly time-dependent, smooth Hamiltonian $H_{t}: T^{*} M \rightarrow \mathbb{R}$, a Hamiltonian vector field $\overrightarrow{H_{t}}$, by

$$
\boldsymbol{\sigma}\left(v, \vec{H}_{t}(\ell)\right)=\left\langle\mathrm{d} H_{t}(\ell), v\right\rangle, \quad \forall v \in T_{\ell} T^{*} M
$$

In this paper we consider all the flows - both in $M$ and in $T^{*} M$ - as starting at the final time $T$, unless otherwise explicitly stated. We denote the flow of $\overrightarrow{H_{t}}$ from time $T$ to time $t$ by

$$
\mathcal{H}:(t, \ell) \mapsto \mathcal{H}(t, \ell)=\mathcal{H}_{t}(\ell) .
$$

We keep these notation throughout the paper, namely the overhead arrow denotes the vector field associated to a Hamiltonian and the script letter denotes its flow from time $T$, unless otherwise stated.

Finally we recall that any vector field $f$ on the manifold $M$ defines, by lifting to the cotangent bundle, a Hamiltonian

$$
F: \ell \in T^{*} M \mapsto\langle\ell, f(\pi \ell)\rangle \in \mathbb{R} .
$$

We denote by $F_{1}, H_{i}$ the Hamiltonians associated to $f_{1}, h_{i}, i=1,2,3$, respectively and by

$$
\begin{array}{ll}
H_{i j}:=\left\{H_{i}, H_{j}\right\}, & i, j \in 1,2,3 \\
H_{i j k}:=\left\{H_{i},\left\{H_{j}, H_{k}\right\}\right\}, & i, j, k \in 1,2,3
\end{array}
$$

the Poisson parenthesis and the iterated Poisson parenthesis between Hamiltonians. We recall that $H_{i j}$ is the Hamiltonian associated to the Lie bracket $h_{i j}:=\left[h_{i}, h_{j}\right]$.

In order to write the second order variation of the problem in an useful way we shall consider flows going backwards in time, i.e. starting at the final time $T$. The flow from time $T$ of the reference vector field $\widehat{f}_{t}$ is a map defined in a neighbourhood of $\widehat{x}_{f}:=\widehat{\xi}(T)$. We denote such flow as $\widehat{S}_{t}: M \rightarrow M, t \in[0, T]$, i.e.

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \widehat{S}_{t}(x)=\widehat{f}_{t} \circ \widehat{S}_{t}(x), \quad \widehat{S}_{T}(x)=x .
$$

We also denote $\widehat{x}_{1}:=\widehat{\xi}\left(\widehat{\tau}_{1}\right)=\widehat{S}_{\widehat{\tau}_{1}}\left(\widehat{x}_{f}\right), \widehat{x}_{2}:=\widehat{\xi}\left(\widehat{\tau}_{2}\right)=\widehat{S}_{\widehat{\tau}_{2}}\left(\widehat{x}_{f}\right)$.
The time-dependent Hamiltonian associated to $\widehat{f}_{t}$ is denoted by $\widehat{F}_{t}$ and its flow backwards in time starting at time $T$ is denoted by $\widehat{\mathcal{F}}_{t}$.

Also we use the following notation from differential geometry: $L_{f} \alpha(\cdot)$ is the Lie derivative of a function $\alpha$ with respect to the vector field $f$. Moreover, if $G$ is a $C^{1}$ map from a manifold $M_{1}$ in a manifold $M_{2}$, we denote its tangent map at a point $x \in M_{1}$ as $T_{x} G$. If the point $x$ is clear from the context, we also write $T_{x} G=G_{*}$.

### 2.1 The necessary conditions

We start by stating the necessary conditions of optimality, i.e. Pontryagin Maximum Principle (PMP) and the Legendre condition. Since there is no constraint on the final point, then PMP must hold in its normal form:
Assumption 1 (PMP). There exists a map $\widehat{\lambda}:[0, T] \rightarrow T^{*} M$, which is absolutely continuous and such that

$$
\begin{array}{lr}
\pi \widehat{\lambda}(t)=\widehat{\xi}(t) & t \in[0, T], \\
\dot{\widehat{\lambda}}(t)=\overrightarrow{\widehat{F}}_{t}(\widehat{\lambda}(t)) & \text { a.e. } t \in[0, T], \\
\widehat{\lambda}(T)=-\mathrm{d} c\left(\widehat{x}_{f}\right), & \\
\widehat{F}_{t}(\widehat{\lambda}(t))=\max \{\langle\widehat{\lambda}(t), v\rangle: v \in \mathcal{X}(\widehat{\xi}(t))\} & t \in[0, T] .
\end{array}
$$

We shall use the following notation for the end points and for the switching points of $\widehat{\lambda}(t):$

$$
\widehat{\ell}_{f}:=\widehat{\lambda}(T), \quad \widehat{\ell}_{2}:=\widehat{\lambda}\left(\widehat{\tau}_{2}\right)=\widehat{\mathcal{F}}_{\widehat{\tau}_{2}}\left(\widehat{\ell}_{f}\right), \quad \widehat{\ell}_{1}:=\widehat{\lambda}\left(\widehat{\tau}_{1}\right)=\widehat{\mathcal{F}}_{\widehat{\tau}_{1}}\left(\widehat{\ell}_{f}\right), \quad \widehat{\ell}_{0}:=\widehat{\lambda}(0)=\widehat{\mathcal{F}}_{0}\left(\widehat{\ell}_{f}\right) .
$$

Thanks to the structure of the reference trajectory, PMP gives the following necessary conditions:

1. On the first bang arc, $t \in\left[0, \widehat{\tau}_{1}\right]$, we get $H_{1}(\widehat{\lambda}(t)) \geq\langle\widehat{\lambda}(t), X\rangle, \quad \forall X \in \mathcal{X}(\widehat{\xi}(t))$.
2. On the second bang arc, $t \in\left[\widehat{\tau}_{1}, \widehat{\tau}_{2}\right]$, we get $H_{2}(\widehat{\lambda}(t)) \geq\langle\widehat{\lambda}(t), X\rangle, \quad \forall X \in \mathcal{X}(\widehat{\xi}(t))$, in particular $H_{1}\left(\widehat{\ell_{2}}\right)=H_{2}\left(\widehat{\ell}_{2}\right)$.
3. On the singular arc, $t \in\left[\widehat{\tau}_{2}, T\right]$, we get

$$
\left(H_{2}+a F_{1}\right)(\widehat{\lambda}(t)) \geq\langle\widehat{\lambda}(t), X\rangle, \quad \forall X \in \mathcal{X}(\widehat{\xi}(t)), \quad \forall a \in[0,1],
$$

which implies $F_{1}(\widehat{\lambda}(t)) \equiv 0$ and, by differentiation,

$$
\frac{\mathrm{d}}{\mathrm{~d} t} F_{1}(\widehat{\lambda}(t))=H_{23}(\widehat{\lambda}(t)) \equiv\left\{H_{2}, F_{1}\right\} \circ \widehat{\lambda}(t)=0
$$

and

$$
\begin{equation*}
-H_{232}(\widehat{\lambda}(t))+\widehat{v}(t) \mathbb{L}(\widehat{\lambda}(t))=0 \tag{4}
\end{equation*}
$$

where

$$
\mathbb{L}(\ell):=\left(H_{323}+H_{232}\right)(\ell)=\left\langle\ell,\left[f_{1},\left[h_{2}, f_{1}\right]\right](\pi \ell)\right\rangle, \quad \ell \in T^{*} M
$$

4. At the first switching time $\widehat{\tau}_{1}$ we get $H_{12}\left(\widehat{\ell}_{1}\right)=\left.\frac{\mathrm{d}}{\mathrm{d} t}\left(H_{2}-H_{1}\right) \circ \widehat{\lambda}(t)\right|_{t=\widehat{\tau}_{1}} \geq 0$, see for example [2].
5. At the second switching time $\widehat{\tau}_{2}$ we get $H_{232}\left(\widehat{\ell}_{2}\right)=-\left.\frac{\mathrm{d}^{2}}{\mathrm{~d} t^{2}} F_{1} \circ \widehat{\lambda}(t)\right|_{t=\widehat{\tau}_{2}^{-}} \geq 0$, see [11].

Moreover, other necessary conditions are known, namely the Goh condition (which in this case is automatically satisfied) and the generalised Legendre condition (GLC), see e.g. [1],

$$
R(t):=\mathbb{L}(\widehat{\lambda}(t)) \geq 0 \quad t \in\left[\widehat{\tau}_{2}, T\right] .
$$

## 3 Assumptions and main result

### 3.1 Regularity conditions

We now state regularity conditions by requiring strict inequalities to hold whenever necessary conditions yield mild inequalities.
Assumption 2 (Regularity along the bang arcs).

$$
\begin{array}{ll}
H_{1}(\widehat{\lambda}(t))>\langle\widehat{\lambda}(t), X\rangle, & \forall X \in \mathcal{X}(\widehat{\xi}(t)) \backslash\left\{h_{1}(\widehat{\xi}(t))\right\}, \\
H_{2}(\widehat{\lambda}(t))>\langle\widehat{\lambda}(t), X\rangle, \quad \forall X \in \mathcal{X}(\widehat{\xi}(t)) \backslash\left\{h_{2}(\widehat{\xi}(t))\right\}, & \forall t \in\left(\widehat{\tau}_{1}, \widehat{\tau}_{2}\right),
\end{array}
$$

i.e. we require that the reference control is the only maximising control along the given arc.

Assumption 3 (Regularity along the singular arc). For any $a, s \in[0,1]$ and any $t \in$ $\left[\widehat{\tau}_{2}, T\right]$

$$
H_{2}(\widehat{\lambda}(t))+a F_{1}(\widehat{\lambda}(t))>\langle\widehat{\lambda}(t), X(\widehat{\xi}(t))\rangle, \quad \forall X \in \mathcal{X}, \quad X \neq h_{2}+s f_{1},
$$

i.e. we require that the set of maximisers along the singular arc is the edge defined by $h_{2}$ and $h_{3}$.
Assumption 4 (Regularity at the switching points).

$$
\begin{equation*}
H_{12}\left(\widehat{\ell}_{1}\right)>0, \quad H_{232}\left(\widehat{\ell}_{2}\right)>0 \tag{5}
\end{equation*}
$$

Assumption 5 (Strong generalised Legendre condition).

$$
\begin{equation*}
R(t)=\mathbb{L}(\widehat{\lambda}(t))=\left\{F_{1},\left\{H_{2}, F_{1}\right\}\right\}(\widehat{\lambda}(t))>0 \quad t \in\left[\widehat{\tau}_{2}, T\right] \tag{SGLC}
\end{equation*}
$$

Thanks to (SGLC) we can recover the value of the control along the singular arc:

$$
\widehat{v}(t)=\frac{H_{232}}{\mathbb{L}}(\widehat{\lambda}(t)) \quad \forall t \in\left(\widehat{\tau}_{2}, T\right],
$$

so that, by recurrence, one can easily prove that $\widehat{v} \in C^{\infty}\left(\left[\widehat{\tau}_{2}, T\right],(0,1)\right)$.
Notice that under (SGLC), the second inequality in (5) is equivalent to the discontinuity of the reference vector field at $t=\widehat{\tau}_{2}$.

For $\ell$ in a neighborhood of the range of the singular $\operatorname{arc} \widehat{\lambda}\left(\left[\widehat{\tau}_{2}, T\right]\right)$ in $T^{*} M$ we can define the Hamiltonian feedback control

$$
\begin{equation*}
u_{\mathcal{S}}(\ell):=\frac{H_{232}}{\mathbb{L}}(\ell) \tag{6}
\end{equation*}
$$

Notice that $\widehat{\lambda}$ also satisfies the autonomous differential equation

$$
\begin{equation*}
\dot{\lambda}(t)=\left(\overrightarrow{H_{2}}+\overrightarrow{u_{\mathcal{S}} F_{1}}\right)(\lambda(t)) \tag{7}
\end{equation*}
$$

The condition $\widehat{v}(t) \in(0,1)$ reads

$$
\begin{equation*}
H_{232}(\widehat{\lambda}(t))>0, \quad H_{323}(\widehat{\lambda}(t))>0 \quad \forall t \in\left(\widehat{\tau}_{2}, T\right] \tag{8}
\end{equation*}
$$

### 3.2 The extended second variation

The sufficient conditions will be derived studying a sub problem of the given one. Namely we consider problem (1), the reference vector field $\widehat{f_{t}}$ and allow only for perturbations of $\widehat{v}$ on the singular interval $\left(\widehat{\tau}_{2}, T\right)$ and for perturbations of the switching time $\widehat{\tau}_{1}$. Following the ideas of [11] the subproblem can be written as

$$
\begin{align*}
& \text { Minimize } c(\xi(T)) \text { subject to }  \tag{9a}\\
& \dot{\xi}(t)= \begin{cases}v_{0}(t) h_{1}(\xi(t)) & t \in\left(0, \widehat{\tau}_{1}\right), \\
v_{0}(t) h_{2}(\xi(t)) & t \in\left(\widehat{\tau}_{1}, \widehat{\tau}_{2}\right), \\
h_{2}(\xi(t))+v(t) f_{1}(\xi(t)) & t \in\left(\widehat{\tau}_{2}, T\right),\end{cases}  \tag{9b}\\
& v_{0}(t)>0, \int_{0}^{\widehat{\tau}_{2}} v_{0}(t) \mathrm{d} t=\widehat{\tau}_{2},  \tag{9c}\\
& \xi(0)=x_{0} . \tag{9d}
\end{align*}
$$

Set

$$
\begin{equation*}
g_{t}:=\widehat{S}_{t *}^{-1} f_{1} \circ \widehat{S}_{t}, t \in\left[\widehat{\tau}_{2}, T\right], \quad k_{i}:=\widehat{S}_{\widehat{\tau}_{1} *}^{-1} h_{i} \circ \widehat{S}_{\widehat{\tau}_{1}}, i=1,2, \quad k:=k_{1}-k_{2}, \tag{10}
\end{equation*}
$$

i.e. $g_{t}$ is the push-forward of $f_{1}$ from time $t \in\left[\widehat{\tau}_{2}, T\right]$ to time $T$ while the $k_{i}$-s are the push-forward of the $h_{i}$-s from the first switching time $\widehat{\tau}_{1}$ to $T$. With this notation the second variation of (9) is given by

$$
\begin{equation*}
J^{\prime \prime}\left[\left(\delta x, \delta v_{0}(\cdot), \delta v(\cdot)\right)\right]^{2}=\int_{\widehat{\tau}_{2}}^{T} \delta v(t) L_{\delta \eta(t)} L_{g_{t}} c\left(\widehat{x}_{f}\right) d t+\frac{\varepsilon_{0}^{2}}{2}\left(L_{k}^{2} c\left(\widehat{x}_{f}\right)+H_{12}\left(\widehat{\ell}_{1}\right)\right) \tag{11}
\end{equation*}
$$

subject to

$$
\dot{\delta \eta}(t)=\delta v(t) g_{t}\left(\widehat{x}_{f}\right), \quad \delta \eta(T)=\delta x \in T_{\widehat{x}_{f}} M, \quad \delta \eta\left(\widehat{\tau}_{2}\right)=\varepsilon_{0} k\left(\widehat{x}_{f}\right)
$$

where

$$
\varepsilon_{0}=\int_{0}^{\widehat{\tau}_{1}} \delta v_{0}(t) d t=-\int_{\widehat{\tau}_{1}}^{\widehat{\tau}_{2}} \delta v_{0}(t) d t .
$$

The precise construction will appear in [10]. We point out that the perturbation at the switching time $\widehat{\tau}_{1}$ gives rise to a cost in the accessory problem.

We then extend the second variation to a new quadratic form called extended second variation. Following the same lines as in the appendix of [11] and setting

$$
w(t):=\int_{t}^{\widehat{\tau}_{2}} \delta u(s) \mathrm{d} s, \quad \varepsilon_{1}:=w(T)
$$

the extended second variation of (9) is given by the following singular LQ problem on the interval $\left[\widehat{\tau}_{2}, T\right]$.

$$
\begin{align*}
& J_{\text {ext }}^{\prime \prime}\left[\left(\delta x, \varepsilon_{0}, \varepsilon_{1}, w\right)\right]^{2}=-\varepsilon_{1} L_{\delta x} L_{f_{1}} c\left(\widehat{x}_{f}\right)-\frac{\varepsilon_{1}^{2}}{2} L_{f_{1}}^{2} c\left(\widehat{x}_{f}\right)+ \\
& +\frac{\varepsilon_{0}^{2}}{2}\left(L_{k}^{2} c\left(\widehat{x}_{f}\right)+H_{12}\left(\widehat{\ell}_{1}\right)\right)+\frac{1}{2} \int_{\widehat{\tau}_{2}}^{T}\left(2 w(t) L_{\zeta(t)} L_{\dot{g}_{t}} c\left(\widehat{x}_{f}\right)+w(t)^{2} R(t)\right) \mathrm{d} t \tag{12}
\end{align*}
$$

subject to

$$
\begin{equation*}
\dot{\zeta}(t)=w(t) \dot{g}_{t}\left(\widehat{x}_{f}\right), \quad \zeta\left(\widehat{\tau}_{2}\right)=\varepsilon_{0} k\left(\widehat{x}_{f}\right), \quad \zeta(T)=\delta x+\varepsilon_{1} f_{1}\left(\widehat{x}_{f}\right) \tag{13}
\end{equation*}
$$

This means that we consider the quadratic form $J_{\text {ext }}^{\prime \prime}$ defined by (12) on the linear space called space of admissible variations given by

$$
\mathcal{W}_{\text {ext }}:=\left\{\left(\delta x, \varepsilon_{0}, \varepsilon_{1}, w\right) \in T_{\widehat{x}_{f}} M \times \mathbb{R} \times \mathbb{R} \times L^{2}\left(\left[\widehat{\tau}_{2}, T\right]\right):\right.
$$

system (13) admits a solution\}.
Notice that

$$
\begin{equation*}
\dot{g}_{t}=\widehat{S}_{t *}^{-1} h_{23} \circ \widehat{S}_{t}, \quad t \in\left[\widehat{\tau}_{2}, T\right] . \tag{14}
\end{equation*}
$$

Choosing $\left(\delta x, \varepsilon_{0}, \varepsilon_{1}, w(\cdot)\right)=\left(-f_{1}\left(\widehat{x}_{f}\right), 0,1,0\right)$ in (12) we get $L_{f_{1}}^{2} c\left(\widehat{x}_{f}\right)>0$ as a necessary condition for the coercivity of the extended second variation (12) on $\mathcal{W}_{\text {ext }}$.

Let $\mathcal{O}\left(\widehat{x}_{f}\right)$ be a neighborhood of $\widehat{x}_{f}$ in $M$ and consider the set

$$
\widetilde{M}:=\left\{x \in \mathcal{O}\left(\widehat{x}_{f}\right): L_{f_{1}} c(x)=0\right\}
$$

If $L_{f_{1}}^{2} c\left(\widehat{x}_{f}\right)>0$, then $\widetilde{M}$ is a hypersurface such that

$$
T_{\widehat{x}_{f}} \widetilde{M}=\left\{\delta z \in T_{\widehat{x}_{f}} M: L_{\delta z} L_{f_{1}} c\left(\widehat{x}_{f}\right)=0\right\} .
$$

For $x=\exp \left(r f_{1}\right)(z), z \in \widetilde{M}$ set

$$
\widetilde{c}(x):=c(z)
$$

i.e. we extend $\left.c\right|_{\widetilde{M}}$ as a constant function along the integral lines of $f_{1}$. If $\mathcal{O}\left(\widehat{x}_{f}\right)$ is sufficiently small, then the function $\widetilde{c}: \mathcal{O}\left(\widehat{x}_{f}\right) \rightarrow \mathbb{R}$ is smooth and it enjoys the following properties

$$
\begin{array}{ll}
\widetilde{c}\left(\widehat{x}_{f}\right)=c\left(\widehat{x}_{f}\right), & \mathrm{d} \widetilde{c}\left(\widehat{x}_{f}\right)=\mathrm{d} c\left(\widehat{x}_{f}\right), \\
\widetilde{c}(x) \leq c(x), & L_{f_{1}} \widetilde{c}(x)=0 \quad \forall x \in \mathcal{O}\left(\widehat{x}_{f}\right) \tag{15}
\end{array}
$$

Following [11] it can be shown that the coercivity of (12) on $\mathcal{W}_{\text {ext }}$ is equivalent to $L_{f_{1}}^{2} c\left(\widehat{x}_{f}\right)>0$ plus the coercivity of

$$
\begin{align*}
\widetilde{J}_{\text {ext }}\left[\left(\delta x, \varepsilon_{0}, w\right)\right]^{2}= & \frac{\varepsilon_{0}^{2}}{2}\left(L_{k}^{2} \widetilde{c}\left(\widehat{x}_{f}\right)+H_{12}\left(\widehat{\ell}_{1}\right)\right)+ \\
& +\frac{1}{2} \int_{\widehat{\tau}_{2}}^{T}\left(2 w(t) L_{\zeta(t)} L_{\dot{g}_{t}} \widetilde{c}\left(\widehat{x}_{f}\right)+R(t) w(t)^{2}\right) \mathrm{d} t \tag{16}
\end{align*}
$$

subject to

$$
\begin{equation*}
\dot{\zeta}(t)=w(t) \dot{g}_{t}\left(\widehat{x}_{f}\right), \quad \zeta\left(\widehat{\tau}_{2}\right)=\varepsilon_{0} k\left(\widehat{x}_{f}\right), \quad \zeta(T)=\delta x \in T_{\widehat{x}_{f}} M \tag{17}
\end{equation*}
$$

In the case when $L_{f_{1}} c(\cdot) \equiv 0$ in $\mathcal{O}\left(\widehat{x}_{f}\right)$, we set $\widetilde{c}:=c$. Thus, also in this case, we end up with (16) subject to (17).

Assumption 6. We assume the following conditions hold

1. The quadratic form $\widetilde{J}_{\mathrm{ext}},(16)$, is coercive on

$$
\widetilde{\mathcal{W}}_{\mathrm{ext}}:=\left\{\left(\delta x, \varepsilon_{0}, w\right) \in T_{\widehat{x}_{f}} M \times \mathbb{R} \times L^{2}\left(\left[\widehat{\tau}_{2}, T\right], \mathbb{R}\right):\right.
$$

system (17) admits a solution $\}$.
2. Either $L_{f_{1}}^{2} c\left(\widehat{x}_{f}\right)>0$ or $L_{f_{1}} c(\cdot) \equiv 0$ in a neighborhood $\mathcal{O}\left(\widehat{x}_{f}\right)$ of $\widehat{x}_{f}$ in $M$.

### 3.3 The main result

We can now state the main result of this paper
Theorem 3.1. Let $\widehat{\xi}$ be the admissible trajectory defined in (2). Assume that $\widehat{\xi}$ satisfies Assumptions 1-6. Then $\widehat{\xi}$ is a strict strong local optimal trajectory of (1).

More precisely we prove that Assumptions 1-5 plus 1. of Assumption 6 imply that $\widehat{\xi}$ is a strict strong locally optimal trajectory for the cost $\widetilde{c}(\xi(T))$. This concludes the proof in the case $L_{f_{1}}^{2} c(\cdot) \equiv 0$.

When $L_{f_{1}}^{2} c\left(\widehat{x}_{f}\right)>0, c=\widetilde{c}$ on $\widetilde{M}$, hence we have to compute the difference $c-\widetilde{c}$ along the integral lines of $f_{1}$ starting at $z \in \widetilde{M}$, so that (15) easily gives the claim.

## 4 Hamiltonian approach

The first step in applying the Hamiltonian approach described in the Introduction, is the construction of an overmaximised Hamiltonian flow. Indeed the presence of a singular arc prevents us from using the maximized Hamiltonian (see [11]) which can be used in the classical case, i.e. when it is $C^{2}$, see [1]. The overmaximized Hamiltonian was introduced in [13] and then used in [11, 12]. In [15] the authors give a sistematic extension of the classical techniques to the case of an overmaximized Hamiltonian whose flow is only Lipshitz continuous.

### 4.1 The overmaximised flow

The (SGLC) condition (Assumption 5) implies that there exists a neighborhood $\mathcal{O}_{\mathrm{s}}$ of the range of the singular arc $\widehat{\lambda}\left(\left[\widehat{\tau}_{2}, T\right]\right)$ in $T^{*} M$ such that

$$
\Sigma:=\left\{\ell \in \mathcal{O}_{\mathrm{s}}: F_{1}(\ell)=0\right\}=\left\{\ell \in \mathcal{O}_{\mathrm{s}}: H_{2}(\ell)=H_{3}(\ell)\right\}
$$

and

$$
\mathcal{S}:=\left\{\ell \in \Sigma: H_{23}(\ell)=0\right\}=\left\{\ell \in \mathcal{O}_{\mathrm{s}}: H_{2}(\ell)=H_{3}(\ell), H_{23}(\ell)=0\right\}
$$

are smooth simply connected manifolds of codimension 1 and 2 , respectively. More precisely $\overrightarrow{H_{23}}$ is transverse to $\Sigma$ in $\mathcal{O}_{\mathrm{s}}$, while $\vec{F}_{1}$ is tangent to $\Sigma$ and transverse to $\mathcal{S}$, see [11].

Here we want to describe how the regularity conditions allow to define in a tubular neighborhood $\mathcal{O}$ of the graph of $\widehat{\lambda}$ in $[0, T] \times T^{*} M$, a time-dependent Hamiltonian function $H: \mathcal{O} \rightarrow \mathbb{R}$ whose flow satisfies the assumptions stated in [15]. The coercivity of the second variation will then guarantee the invertibility of the projected overmaximised flow of such Hamiltonian.

In [11] the authors prove that possibly restricting $\mathcal{O}_{\mathrm{s}}$, the following implicit function problem has a solution $\theta: \mathcal{O}_{\mathrm{s}} \rightarrow \mathbb{R}$ :

$$
\theta(\ell):\left\{\begin{array}{l}
H_{23} \circ \exp \theta \vec{F}_{1}(\ell)=0 \\
\theta(\ell)=0 \quad \text { if } H_{23}(\ell)=0
\end{array}\right.
$$

and

$$
\left\langle\mathrm{d} \theta\left(\ell_{\mathcal{S}}\right), \delta \ell\right\rangle=\frac{-\boldsymbol{\sigma}\left(\delta \ell, \overrightarrow{H_{23}}\left(\ell_{\mathcal{S}}\right)\right)}{\mathbb{L}\left(\ell_{\mathcal{S}}\right)} \quad \forall \ell_{\mathcal{S}}: H_{23}\left(\ell_{\mathcal{S}}\right)=0 .
$$

Let

$$
\widetilde{H}_{2}(\ell):=H_{2} \circ \exp \theta(\ell) \overrightarrow{F_{1}}(\ell) .
$$

From the results in [11] we can derive the following Lemma:
Lemma 4.1. Possibly restricting $\mathcal{O}_{\mathrm{s}}$ the following properties hold

1. $\widetilde{H}_{2}(\ell) \geq H_{2}(\ell)$ for any $\ell \in \Sigma$. Equality holds if and only if $\ell \in \mathcal{S}$.
2. For any $\ell_{\mathcal{S}} \in \mathcal{S}$

$$
\mathrm{D}\left(\widetilde{H}_{2}-H_{2}\right)\left(\ell_{\mathcal{S}}\right)=0, \quad \mathrm{D}^{2}\left(\widetilde{H}_{2}-H_{2}\right)\left(\ell_{\mathcal{S}}\right)=\frac{\left(\sigma\left(\delta \ell, \overrightarrow{H_{23}}\left(\ell_{\mathcal{S}}\right)\right)\right)^{2}}{\mathbb{L}\left(\ell_{\mathcal{S}}\right)} .
$$

3. $\vec{H}_{2}$ and hence $\vec{H}_{2}+\widehat{v}(t) \vec{F}_{1}$ are tangent to $\Sigma$ for any $t \in\left[\widehat{\tau}_{2}, T\right]$.

Set

$$
\begin{equation*}
H_{t}(\ell)=\widetilde{H}_{2}+\widehat{v}(t) F_{1} \quad \forall(t, \ell) \in\left[\widehat{\tau}_{2}, T\right] \times \mathcal{O}_{\mathrm{s}} \tag{18}
\end{equation*}
$$

4. $\left.\widehat{\lambda}\right|_{\left[\hat{\tau}_{2}, T\right]}$ is the solution of the Cauchy problem

$$
\dot{\lambda}(t)=\vec{H}_{t}(\lambda(t)), \quad \lambda(T)=\widehat{\ell}_{f} .
$$

Moreover the following invariant properties hold:
5. $\overrightarrow{H_{2}}$ is invariant with respect to the flow of $\overrightarrow{H_{t}}=\overrightarrow{\widetilde{H}}_{2}+\widehat{v}(t) \overrightarrow{F_{1}}$ along the singular arc of the reference trajectory:

$$
\overrightarrow{H_{2}}(\widehat{\lambda}(t))=\mathcal{H}_{t *} \overrightarrow{H_{2}}\left(\widehat{\ell}_{f}\right) \quad t \in\left[\widehat{\tau}_{2}, T\right] ;
$$

6. $\overrightarrow{F_{1}}$ is invariant on $\Sigma$ with respect to the flow of $\overrightarrow{H_{t}}$ :

$$
\overrightarrow{F_{1}} \circ \mathcal{H}_{t}(\ell)=\mathcal{H}_{t *} \vec{F}_{1}(\ell), \quad \forall \ell \in \Sigma, \quad t \in\left[\widehat{\tau}_{2}, T\right] .
$$

This Lemma is the main tool for handling the singular arc. The bang arcs present a different kind of problems. Namely we need to define the switching times near the reference switching points $\widehat{\ell}_{1}$ and $\widehat{\ell}_{2}$ of the Pontryagin extremal $\widehat{\lambda}$. In [11] it is shown that the flow of $\overrightarrow{H_{2}}$ is the maximised one in a left hand side neighborhood of $\widehat{\tau}_{2}$ if and only if $H_{23}(\ell) \geq 0$. In order to overcome this problem we introduce a correction of the backwards flow from time $\widehat{\tau}_{2}$ by keeping the flow on $\Sigma$ when $H_{23}(\ell)<0$.

By the implicit function theorem applied to the problem:

$$
\left\{\begin{array}{l}
H_{23} \circ \exp \left(t_{2}-\widehat{\tau}_{2}\right) \overrightarrow{\vec{H}}_{2}(\ell)=0 \\
t_{2}(\ell)=\widehat{\tau}_{2} \quad \text { if } H_{23}(\ell)=0
\end{array}\right.
$$

it is possible to define a function $t_{2}: \mathcal{O}\left(\widehat{\ell}_{2}\right) \rightarrow \mathbb{R}$ such that if $\ell \in \Sigma$, then $t_{2}(\ell)=\widehat{\tau}_{2}$ if and only if $\ell \in \mathcal{S}$; moreover

$$
\left\langle\mathrm{d} t_{2}\left(\widehat{\ell_{2}}\right), \delta \ell\right\rangle=\frac{-\boldsymbol{\sigma}\left(\delta \ell, \overrightarrow{H_{23}}\left(\widehat{\ell}_{2}\right)\right)}{H_{223}\left(\widehat{\ell_{2}}\right)} .
$$

We set

$$
\tau_{2}(\ell):=\min \left\{t_{2}(\ell), \widehat{\tau}_{2}\right\}= \begin{cases}t_{2}(\ell) & \text { if } H_{23}(\ell)<0 \\ \widehat{\tau}_{2} & \text { if } H_{23}(\ell) \geq 0\end{cases}
$$



Figure 1: The over-maximised Hamiltonian

The next step will be the definition of the switching time $\tau_{1}: \mathcal{O}\left(\widehat{\ell}_{2}\right) \rightarrow \mathbb{R}$. Actually, the implicit function theorem applies also to

$$
\left\{\begin{array}{l}
\left(H_{2}-H_{1}\right) \circ \exp \left(\tau_{1}-\tau_{2}(\ell)\right) \overrightarrow{H_{2}} \circ \exp \left(\tau_{2}(\ell)-\widehat{\tau}_{2}\right) \overrightarrow{\vec{H}_{2}}(\ell)=0, \\
\tau_{1}\left(\widehat{\ell_{2}}\right)=\widehat{\tau_{1}},
\end{array}\right.
$$

see [2] and

$$
\begin{equation*}
\left\langle\mathrm{d} \tau_{1}\left(\widehat{\ell}_{2}\right), \delta \ell\right\rangle=\frac{-\boldsymbol{\sigma}\left(\exp \left(\widehat{\tau}_{1}-\widehat{\tau}_{2}\right) \overrightarrow{H_{2}} * \delta \ell,\left(\overrightarrow{H_{2}}-\overrightarrow{H_{1}}\right)\left(\widehat{\ell}_{1}\right)\right)}{H_{12}\left(\widehat{\ell}_{1}\right)} \tag{19}
\end{equation*}
$$

We can now define the flow $(t, \ell) \mapsto \mathcal{H}_{t}(\ell)$ backwards in time emanating from a neighborhood $\mathcal{O}\left(\widehat{\ell}_{f}\right)$ of $\widehat{\ell}_{f}$ in $T^{*} M$ at time $T$. Namely, for any $t \in\left[\widehat{\tau}_{2}, T\right], \mathcal{H}_{t}(\ell)$ is the flow associated to the time-dependent Hamiltonian defined in (18). Let $\widetilde{\ell}:=\mathcal{H}_{\widehat{\tau}_{2}}(\ell)$. For $t<\widehat{\tau}_{2}$, $\mathcal{H}_{t}(\ell)$ is defined as

$$
\mathcal{H}_{t}(\ell):= \begin{cases}\exp \left(t-\widehat{\tau}_{2}\right) \overrightarrow{\vec{H}_{2}}(\widetilde{\ell}) & t \in\left[\tau_{2}(\widetilde{\ell}), \widehat{\tau}_{2}\right],  \tag{20}\\ \exp \left(t-\tau_{2}(\widetilde{\ell}) \overrightarrow{H_{2}} \circ \mathcal{H}_{\tau_{2}(\widetilde{\ell})}(\ell)\right. & t \in\left[\tau_{1}(\widetilde{\ell}), \tau_{2}(\widetilde{\ell})\right), \\ \exp \left(t-\tau_{1}(\widetilde{\ell})\right) \overrightarrow{H_{1}} \circ \mathcal{H}_{\tau_{1}(\widetilde{\ell})}(\ell) & t \in\left[0, \tau_{1}(\widetilde{\ell})\right),\end{cases}
$$

see Figure 4.1.
Remark 4.2. Notice that $\mathcal{H}$ is $C^{\infty}$ on $\left[\hat{\tau}_{2}^{+}, T\right] \times \mathcal{O}\left(\widehat{\ell}_{f}\right)$ and it is Lipschitz continuous on $\left[0, \widehat{\tau}_{2}^{-}\right] \times \mathcal{O}\left(\widehat{\ell}_{f}\right)$.

We now state and prove the main result obtained by the Hamiltonian approach, see [15]. After, we shall exploit the coercivity of $\widetilde{J}$ in order to obtain the required invertibility property.

Theorem 4.3. Let $\Lambda:=\left\{\mathrm{d}(-\widetilde{c})(x): x \in \mathcal{O}\left(\widehat{x}_{f}\right)\right\}$. Assume the projected overmaximised flow emanating from $\Lambda$ is locally Lipschitz invertible onto a neighborhood $\mathcal{U}$ of the graph of $\widehat{\xi}$ in $[0, T] \times M$ :

$$
\begin{equation*}
\text { id } \times \pi \mathcal{H}:(t, \ell) \in[0, T] \times \Lambda \mapsto\left(t, \pi \mathcal{H}_{t}(\ell)\right) \in \mathcal{U} \tag{21}
\end{equation*}
$$

Then $\widehat{\xi}$ is a strict strong locally optimal trajectory for the cost $\widetilde{c}(\xi(T))$ subject to (1b)-(1c).
Proof. Clearly $(\mathrm{id} \times \pi \mathcal{H})^{-1}(t, \widehat{\xi}(t))=\left(t, \widehat{\ell}_{f}\right)$ for any $t \in[0, T]$. Let $\xi:[0, T] \rightarrow M$ be an admissible trajectory for (1) whose graph is in $\mathcal{U}$ and let

$$
(t, \mu(t)):=(\operatorname{id} \times \pi \mathcal{H})^{-1}(t, \xi(t)), \quad \lambda(t):=\mathcal{H}_{t}(\mu(t)), \quad t \in[0, T] .
$$

Let $\varphi:[0,1] \rightarrow \Lambda$ be a smooth curve such that $\varphi(0)=\mu(T), \varphi(1)=\widehat{\ell}_{f}$. In $[0, T] \times \Lambda$ we can consider the closed path obtained by the concatenation of the curves $t \in[0, T] \mapsto(t, \mu(t))$, $s \in[0,1] \mapsto(T, \varphi(s))$ and of the curve $t \in[0, T] \mapsto\left(t, \widehat{\ell}_{f}\right)$ ran backwards in time.

Integrating the one-form $\omega:=\mathcal{H}^{*}\left(p \mathrm{~d} q-H_{t} \mathrm{~d} t\right)$ (which is exact on [ $\left.0, T\right] \times \Lambda$, see [15], we obtain

$$
\begin{align*}
0=\oint \omega & =\int_{\mathrm{id} \times \mu}\langle\lambda(t), \dot{\xi}(t)\rangle-H_{t}(\lambda(t)) \mathrm{d} t+\int_{\varphi} \mathcal{H}^{*} p \mathrm{~d} q  \tag{22}\\
& -\int_{\mathrm{id} \times \widehat{\widehat{\ell}_{f}}}\langle\widehat{\lambda}(t), \dot{\widehat{\xi}}(t)\rangle-H_{t}(\widehat{\lambda}(t)) \mathrm{d} t .
\end{align*}
$$

By construction of the overmaximised Hamiltonian $H_{t}$ the integrand is non positive along id $\times \mu$ and is identically zero along id $\times \widehat{\ell}_{f}$. Thus

$$
\begin{align*}
0 \leq \int_{\varphi} \mathcal{H}^{*} p \mathrm{~d} q & =\int_{0}^{1}\left\langle\varphi(s), \frac{\mathrm{d}}{\mathrm{~d} s}(\pi \varphi)(s)\right\rangle \mathrm{d} s \\
& =\int_{0}^{1}\left\langle\mathrm{~d}(-\widetilde{c})(\pi \varphi(s)), \frac{\mathrm{d}}{\mathrm{~d} s}(\pi \varphi)(s)\right\rangle \mathrm{d} s=\widetilde{c}(\xi(T))-\widetilde{c}\left(\widehat{x}_{f}\right) \tag{23}
\end{align*}
$$

Thus $\widetilde{c}(\xi(T)) \geq \widetilde{c}\left(\widehat{x}_{f}\right)$, i.e. the reference trajectory $\widehat{\xi}$ is a strong local minimiser for the cost $\widetilde{c}$. Let us show that in fact it is a strict one.

If $\widetilde{c}(\xi(T))=\widetilde{c}\left(\widehat{x}_{f}\right)$, then (22)-(23) imply that

$$
\begin{equation*}
\langle\lambda(t), \dot{\xi}(t)\rangle-H_{t}(\lambda(t))=0 \quad \text { a.e. } t \in[0, T] . \tag{24}
\end{equation*}
$$

Since $\xi(0)=x_{0}=\widehat{\xi}(0)$, we also have $\lambda(0)=\widehat{\ell}_{0}$ and from the regularity condition along the bang arcs, Assumption 2, we easily get $\lambda(t)=\widehat{\lambda}(t)$ for any $t \in\left[0, \widehat{\tau}_{2}\right]$, so that $\xi(t)=\pi \lambda(t)=\pi \widehat{\lambda}(t)=\widehat{\xi}(t)$ for any $t \in\left[0, \widehat{\tau}_{2}\right]$. In particular $\lambda\left(\widehat{\tau}_{2}\right)=\widehat{\ell}_{2}$.

Moreover, for $t \in\left[\widehat{\tau}_{2}, T\right]$, equation (24) yields $\widetilde{H}_{2}(\lambda(t))=H_{2}(\lambda(t))$, i.e. $\lambda(t) \in \mathcal{S}$. Let $\Sigma_{\widehat{\xi}(t)}$ be the intersection of $\Sigma$ with the fiber over $\widehat{\xi}(t)$ and consider the function

$$
\Delta: \ell \in \Sigma_{\widehat{\xi}(t)} \mapsto\langle\ell, \dot{\xi}(t)\rangle-H_{t}(\ell) \in \mathbb{R}
$$

By PMP the function $\Delta$ is non positive and by (24) it is null in $\lambda(t)$. Differentiating $\Delta$ with respect to the vertical fiber we thus obtain

$$
\begin{equation*}
\left\langle\delta p, \dot{\xi}(t)-\pi_{*} \vec{H}_{t}(\lambda(t))\right\rangle=0, \quad \forall \delta p \in T_{\xi(t)}^{*} M, \text { such that }\left\langle\delta p, f_{1}(\xi(t))\right\rangle=0 . \tag{25}
\end{equation*}
$$

Hence there exists $b(t) \in \mathbb{R}$ such that

$$
\dot{\xi}(t)=\pi_{*} \overrightarrow{H_{t}}(\lambda(t))+b(t) f_{1}(\xi(t)) \quad \forall t \in\left[\widehat{\tau}_{2}, T\right] .
$$

Hence, by Lemma 4.1, point 6,

$$
\dot{\mu}(t)=\left(\pi \mathcal{H}_{t}\right)_{*}^{-1}\left(\dot{\xi}(t)-\pi_{*} \vec{H}_{t}(\lambda(t))\right)=b(t)\left(\pi \mathcal{H}_{t}\right)_{*}^{-1} f_{1}(\xi(t))=b(t) \vec{F}_{1}(\mu(t)) .
$$

Thus

$$
\dot{\lambda}(t)=\overrightarrow{H_{t}}(\lambda(t))+\mathcal{H}_{t *} \dot{\mu}(t)=\overrightarrow{H_{2}}(\lambda(t))+(\widehat{v}(t)+b(t)) \overrightarrow{F_{1}}(\lambda(t)) .
$$

Finally, since $\lambda(t) \in \mathcal{S}$, we get

$$
\begin{equation*}
0=\boldsymbol{\sigma}\left(\dot{\lambda}(t), \overrightarrow{H_{23}}(\lambda(t))\right)=-H_{232}(\lambda(t))+(\widehat{v}(t)+b(t)) \mathbb{L}(\lambda(t)) . \tag{26}
\end{equation*}
$$

Comparing (26) with (6) we obtain

$$
\widehat{v}(t)+b(t)=u_{\mathcal{S}}(\lambda(t)),
$$

so that $\lambda(t)$ and $\hat{\lambda}(t)$ solve the same Cauchy problem on the interval $\left[\widehat{\tau}_{2}, T\right]$ :

$$
\dot{\lambda}=\overrightarrow{H_{2}}(\lambda)+u_{\mathcal{S}}(\lambda) \overrightarrow{F_{1}}(\lambda), \quad \lambda\left(\widehat{\tau}_{2}\right)=\widehat{\ell}_{2} .
$$

Hence $\lambda \equiv \widehat{\lambda}$ and $\xi \equiv \widehat{\xi}$. This proves that $\widehat{\xi}$ is a strict strong locally optimal trajectory for the cost $\widetilde{c}(\xi(T))$.

### 4.2 Consequences of the coercivity of $\widetilde{J}$

In this section we exploit the coercivity of the second variation, Assumption 6 a). Let $\Lambda:=\left\{\mathrm{d}(-\widetilde{c})(x): x \in \mathcal{O}\left(\widehat{x}_{f}\right)\right\}$.

Assume $k\left(\widehat{x}_{f}\right) \neq 0$, i.e. $h_{1}\left(\widehat{x}_{1}\right) \neq h_{2}\left(\widehat{x}_{1}\right)$. In order to rewrite the extended second variation (16) as a standard LQ form, choose $\omega \in T_{\widehat{x}_{f}}^{*} M$ such that $\left\langle\omega, k\left(\widehat{x}_{f}\right)\right\rangle=1$ and set

$$
\gamma_{\widehat{\tau}_{2}}^{\prime \prime}:=H_{12}\left(\widehat{\ell}_{1}\right) \omega \otimes \omega-\frac{1}{2}\left(\omega \otimes L_{(\cdot)} L_{k}(-\widetilde{c})\left(\widehat{x}_{f}\right)+L_{(\cdot)} L_{k}(-\widetilde{c})\left(\widehat{x}_{f}\right) \otimes \omega\right) .
$$

We obtain

$$
\begin{equation*}
\widetilde{J}[(\delta x, w)]^{2}=\frac{1}{2} \gamma_{\widehat{\tau}_{2}}^{\prime \prime}\left[\zeta\left(\widehat{\tau}_{2}\right)\right]^{2}+\frac{1}{2} \int_{\widehat{\tau}_{2}}^{T}\left(w(t)^{2} R(t)+2 w(t) L_{\zeta(t)} L_{\dot{g}_{t}}(-\widetilde{c})\left(\widehat{x}_{f}\right)\right) \mathrm{d} t \tag{27}
\end{equation*}
$$

subject to

$$
\begin{equation*}
\dot{\zeta}(t)=w(t) \dot{g}_{t}\left(\widehat{x}_{f}\right), \quad \zeta\left(\widehat{\tau}_{2}\right)=\delta y \in \mathbb{R} k\left(\widehat{x}_{f}\right), \quad \zeta(T)=\delta x \in T_{\widehat{x}_{f}} M \tag{28}
\end{equation*}
$$

If $k\left(\widehat{x}_{f}\right)=0$ then $\zeta\left(\widehat{\tau}_{2}\right)=0$, so that in (16) $\varepsilon_{0}$ is a decoupled variable and $J_{\text {ext }}$ is equivalent to the problem described by (27)-(28) whatever the quadratic form $\gamma_{\bar{\tau}_{2}}^{\prime \prime}$ is.

Consider the Lagrangian subspace of trasversality conditions

$$
L_{T}^{\prime \prime}=\left\{(0, \delta x): \delta x \in T_{\widehat{x}_{f}} M\right\} .
$$

Let

$$
\mathcal{W}:=\left\{(\delta x, w) \in T_{\widehat{x}_{f}} M \times L^{2}\left(\left[\widehat{\tau}_{2}, T\right]\right): \text { system (28) admits a solution }\right\}
$$

and consider the subspace of $\mathcal{W}$

$$
\mathcal{V}:=\left\{\delta e=(\delta x, w) \in \mathcal{W}: \zeta\left(\widehat{\tau}_{2}\right)=0\right\} .
$$

Notice that $\mathcal{V}=\mathcal{W}$ if and only if $k\left(\widehat{x}_{f}\right)=0$. It can be easily shown, see [7], that
Proposition 4.4. $\widetilde{J}$ is coercive if and only if $\widetilde{J}$ is coercive on $\mathcal{V}$ and $\widetilde{J}[\delta e]^{2}>0$ for any $\delta e \in \mathcal{W}$, $\delta e \neq 0$, which is $\widetilde{J}$-orthogonal to $\mathcal{V}$.

The Hamiltonian relative to (27)-(28) is given by the quadratic form

$$
\begin{equation*}
H_{t}^{\prime \prime}(\delta p, \delta x)=-\frac{1}{2 R(t)}\left(\left\langle\delta p, \dot{g}_{t}\left(\widehat{x}_{f}\right)\right\rangle+L_{\delta x} L_{\dot{g}_{t}}(-\widetilde{c})\left(\widehat{x}_{f}\right)\right)^{2} \tag{29}
\end{equation*}
$$

while the associated Hamiltonian linear system with initial conditions in $L_{T}^{\prime \prime}$ is given by

$$
\begin{cases}\dot{\mu}(t)=\frac{1}{R(t)}\left(\left\langle\mu(t), \dot{g}_{t}\left(\widehat{x}_{f}\right)\right\rangle+L_{\zeta(t)} L_{\dot{g}_{t}}(-\widetilde{c})\left(\widehat{x}_{f}\right)\right) L_{(\cdot)} L_{\dot{g}_{t}}(-\widetilde{c})\left(\widehat{x}_{f}\right), & \mu(T)=0  \tag{30}\\ \dot{\zeta}(t)=\frac{-1}{R(t)}\left(\left\langle\mu(t), \dot{g}_{t}\left(\widehat{x}_{f}\right)\right\rangle+L_{\zeta(t)} L_{\dot{g}_{t}}(-\widetilde{c})\left(\widehat{x}_{f}\right)\right) \dot{g}_{t}\left(\widehat{x}_{f}\right), & \zeta(T)=\delta x .\end{cases}
$$

$\widetilde{J}$ is coercive on $\mathcal{V}$ if and only if for any solution of the Hamiltonian system (30) where $\delta x \neq 0$, we have $\zeta(t) \neq 0$ for any $t \in\left[\widehat{\tau}_{2}, T\right]$, see for example [14]. This concludes the case $k\left(\widehat{x}_{f}\right)=0$.

Assume $k\left(\widehat{x}_{f}\right) \neq 0$ and consider the variations $\delta e \in \mathcal{W}$ which are $\widetilde{J}$-orthogonal to $\mathcal{V}$. In terms of system (30) the bilinear form associated to $\widetilde{J}(27)$ is given by

$$
\begin{equation*}
\widetilde{J}[\delta e, \overline{\delta e}]=\left\langle\bar{\mu}\left(\widehat{\tau}_{2}\right), \zeta\left(\widehat{\tau}_{2}\right)\right\rangle+\left\langle\mu\left(\widehat{\tau}_{2}\right), \bar{\zeta}\left(\widehat{\tau}_{2}\right)\right\rangle+\widetilde{\gamma}_{\widehat{\tau}_{2}}^{\prime \prime}\left[\zeta\left(\widehat{\tau}_{2}\right), \bar{\zeta}\left(\widehat{\tau}_{2}\right)\right], \tag{31}
\end{equation*}
$$

where $\delta e=(\delta x, w)$ and $\overline{\delta e}=(\overline{\delta x}, \bar{w})$ are in $\mathcal{W}$ and $\left(\mu\left(\widehat{\tau}_{2}\right), \zeta\left(\widehat{\tau}_{2}\right)\right)$ and $\left(\bar{\mu}\left(\widehat{\tau}_{2}\right), \bar{\zeta}\left(\widehat{\tau}_{2}\right)\right)$ are the solutions of the Hamiltonian system (30) with initial conditions $(0, \delta x)$ and $(0, \overline{\delta x})$, respectively. Thus $\delta e \in \mathcal{W} \cap \mathcal{V}^{\widetilde{J}^{\perp}}$ if and only if there exists $\delta p \in T_{\widehat{x}_{f}}^{*} M$ such that

$$
\widetilde{J}[\delta e, \overline{\delta e}]=\left\langle\delta p, \bar{\zeta}\left(\widehat{\tau}_{2}\right)\right\rangle \quad \forall \overline{\delta e} \in \mathcal{W}
$$

i.e. if and only if

$$
\begin{cases}\left\langle\bar{\mu}\left(\widehat{\tau}_{2}\right), \zeta\left(\widehat{\tau}_{2}\right)\right\rangle=0 & \forall \overline{\delta e} \\ \delta p=\mu\left(\widehat{\tau}_{2}\right)+\widetilde{\gamma}_{\widehat{\tau}_{2}}^{\prime \prime}\left[\zeta\left(\widehat{\tau}_{2}\right), \cdot\right] & \end{cases}
$$

Hence

$$
\begin{equation*}
0<\widetilde{J}[\delta e]^{2}=\left\langle\mu\left(\widehat{\tau}_{2}\right), \zeta\left(\widehat{\tau}_{2}\right)\right\rangle+\widetilde{\gamma}_{\widehat{\tau}_{2}}^{\prime \prime}\left[\zeta\left(\widehat{\tau}_{2}\right)\right]^{2} \quad \forall \delta e \in \mathcal{W} \cap \mathcal{V}^{\widetilde{J}^{\perp}} \tag{32}
\end{equation*}
$$

Since $\zeta\left(\widehat{\tau}_{2}\right) \in \mathbb{R} k\left(\widehat{x}_{f}\right)$, from equation (32) we get

$$
\begin{equation*}
0<\widetilde{\gamma}_{\widehat{\tau}_{2}}^{\prime \prime}\left[k\left(\widehat{x}_{f}\right)\right]^{2}+\left\langle\mu\left(\widehat{\tau}_{2}\right), k\left(\widehat{x}_{f}\right)\right\rangle=H_{12}\left(\widehat{\ell}_{1}\right)-L_{k}^{2}(-\widetilde{c})\left(\widehat{x}_{f}\right)+\left\langle\mu\left(\widehat{\tau}_{2}\right), k\left(\widehat{x}_{f}\right)\right\rangle \tag{33}
\end{equation*}
$$

### 4.3 The antisymplectic isomorphism

Define the linear mapping $\iota$ by

$$
\iota:(\delta p, \delta x) \in T_{\widehat{x}_{f}}^{*} M \otimes T_{\widehat{x}_{f}} M \mapsto \delta \ell:=-\delta p+\mathrm{d}(-\widetilde{c})_{*} \delta x \in T_{\widehat{\ell}_{f}} T^{*} M
$$

so that

$$
\iota^{-1}: \delta \ell \in T_{\widehat{\ell}_{f}} T^{*} M \mapsto\left(\mathrm{~d}(-\widetilde{c})_{*} \pi_{*} \delta \ell-\delta \ell, \pi_{*} \delta \ell\right) \in T_{\widehat{x}_{f}}^{*} M \otimes T_{\widehat{x}_{f}} M
$$

Moreover $\iota$ is an antisymplectic ismorphism, i.e.

$$
\boldsymbol{\sigma}(\iota(\delta p, \delta x), \iota(\overline{\delta p}, \overline{\delta x}))=\boldsymbol{\sigma}((\overline{\delta p}, \overline{\delta x}),(\delta p, \delta x)), \quad \forall(\delta p, \delta x),(\overline{\delta p}, \overline{\delta x}) \in T_{\widehat{x}_{f}}^{*} M \otimes T_{\widehat{x}_{f}} M
$$

With this notation we get

$$
\iota L_{T}^{\prime \prime}=\left\{\mathrm{d}(-\widetilde{c})_{*} \delta x: \delta x \in T_{\widehat{x}_{f}} M\right\}=T_{\widehat{\ell}_{f}} \Lambda
$$

Following the lines of Lemma 9 in [11] one can prove the following Lemma:
Lemma 4.5. Let $\mathcal{H}_{t}^{\prime \prime}$ and $\mathcal{H}_{t}$ be the Hamiltonian flows associated to the quadratic Hamiltonian $H_{t}^{\prime \prime}$ defined in (29) and to the overmaximised Hamiltonian $H_{t}$ defined in (18), respectively. Then

$$
\begin{equation*}
\iota \mathcal{H}_{t}^{\prime \prime} \iota^{-1}=\widehat{\mathcal{F}}_{t *}^{-1} \mathcal{H}_{t *} \quad \forall t \in\left[\widehat{\tau}_{2}, T\right] \tag{34}
\end{equation*}
$$

### 4.4 Proof of the main result

Applying Theorem 4.3, the proof of our main result, Theorem 3.1, is completed once we show that $\pi \mathcal{H}_{t}$ is locally Lipschitz one-to-one for each $t \in[0, T]$. In fact, as $[0, T]$ is a compact interval, the map id $\times \pi \mathcal{H}_{t}$ defined in (21) is locally Lipschitz invertible if and only if for all $t \in[0, T]$ the map

$$
\pi \mathcal{H}_{t}: \Lambda \mapsto \pi \mathcal{H}_{t}(\ell) \in \mathcal{O}(\widehat{\xi}(t))
$$

is locally Lipschitz invertible. We in fact show that $\pi_{*} \mathcal{H}_{t *}: T_{\widehat{\ell}_{f}} \Lambda \rightarrow T_{\widehat{\xi}(t)} M$ is one-to-one for $t \neq \widehat{\tau}_{1}$ and by means of Clarke inverse function theorem, see [5, 6], for $t=\widehat{\tau}_{1}$.

Since the Hamiltonian $\widehat{F}_{t}$ is the lift of a vector field, from the coercivity of $\widetilde{J}$ on $\mathcal{V}$ and (34), the claim holds for any $t \in\left[\widehat{\tau}_{2}, T\right]$. For $t \in\left(\widehat{\tau}_{1}, \widehat{\tau}_{2}\right)$ from the definition of the flow, equation (20), we get

$$
\left(\pi \mathcal{H}_{t}\right)_{*}=\exp \left(t-\widehat{\tau}_{2}\right) h_{2 *} \pi_{*} \mathcal{H}_{\widehat{\tau}_{2} *} ;
$$

hence we now have to prove the invertibility of $\left(\pi \mathcal{H}_{\widehat{\tau}_{1}}\right)_{*}$.
Let $\delta \ell \in T_{\widehat{\ell}_{f}} \Lambda$ and set $\widetilde{\delta \ell}:=\left(\pi \mathcal{H}_{\widehat{\tau}_{2}}\right)_{*} \delta \ell$. Notice that since $\left(\pi \mathcal{H}_{\widehat{\tau}_{2}}\right)_{*}$ is one-to-one, then $\pi_{*} \widetilde{\delta \ell}=0$ if and only if $\widetilde{\delta \ell}=0$.

Thus the linearization of $\pi \mathcal{H}_{\widehat{\tau}_{1}}(\ell)$ at $\widehat{\ell}_{f}$ is given by

$$
\begin{aligned}
\left(\pi \mathcal{H}_{\widehat{\tau}_{1}}\right)_{*} \delta \ell & = \begin{cases}\exp \left(\widehat{\tau}_{1}-\widehat{\tau}_{2}\right) h_{2 *} \pi_{*} \widetilde{\delta} \ell & \left\langle\mathrm{~d} \tau_{1}\left(\widehat{\ell}_{2}\right), \widetilde{\delta \ell}\right\rangle<0, \\
\left\langle\mathrm{~d} \tau_{1}\left(\widehat{\ell}_{2}\right), \widetilde{\delta \ell}\right\rangle\left(h_{2}-h_{1}\right)\left(\widehat{x}_{1}\right)+\exp \left(\widehat{\tau}_{1}-\widehat{\tau}_{2}\right) h_{2 *} \pi_{*} \widetilde{\delta \ell} & \left\langle\mathrm{~d} \tau_{1}\left(\widehat{\ell_{2}}\right), \widetilde{\delta \ell}\right\rangle>0 .\end{cases} \\
& = \begin{cases}\exp \left(\widehat{\tau}_{1}-\widehat{\tau}_{2}\right) h_{2 *} \pi_{*} \widetilde{\delta \ell} & \left\langle\mathrm{~d} \tau_{1}\left(\widehat{\ell}_{2}\right), \widetilde{\delta \ell}\right\rangle<0, \\
\exp \left(\widehat{\tau}_{1}-\widehat{\tau}_{2}\right) h_{2 *}\left(\pi_{*} \widetilde{\delta \ell}-\left\langle\mathrm{d} \tau_{1}\left(\widehat{\ell}_{2}\right), \widetilde{\delta} \ell\right) \widetilde{k}\left(\widehat{x}_{2}\right)\right) & \left.\left\langle\mathrm{d} \tau_{1}\left(\widehat{\ell}_{2}\right), \widetilde{\delta}\right\rangle\right\rangle 0 .\end{cases}
\end{aligned}
$$

where $\widetilde{k}:=\widehat{S}_{\widehat{\tau}_{2} *} k \circ \widehat{S}_{\widehat{\tau}_{2}}^{-1}=\exp \left(\left(\widehat{\tau}_{2}-\widehat{\tau}_{1}\right) h_{2}\right)_{*}\left(h_{1}-h_{2}\right) \circ \exp \left(\widehat{\tau}_{1}-\widehat{\tau}_{2}\right) h_{2}$. It thus suffices to prove that for any $a \in[0,1]$ and $\delta \ell \in T_{\widehat{\ell}_{f}} \Lambda, \delta \ell \neq 0$

$$
(1-a) \pi_{*}(\widetilde{\delta \ell})+a\left(\pi_{*}(\widetilde{\delta \ell})-\left\langle\mathrm{d} \tau_{1}\left(\widehat{\ell}_{2}\right), \widetilde{\delta} \ell \widetilde{k}\left(\widehat{x}_{2}\right)\right) \neq 0\right.
$$

If $\left\langle\mathrm{d} \tau_{1}\left(\widehat{\ell}_{2}\right), \widetilde{\delta}\right\rangle \widetilde{k}\left(\widehat{x}_{2}\right)=0$ there is nothing to prove. Otherwise assume by contradiction there exist $a \in[0,1], \delta \ell \in T_{\widehat{\ell}_{f}} \Lambda$ such that

$$
\begin{equation*}
\pi_{*} \widetilde{\delta \ell}-a\left\langle\mathrm{~d} \tau_{1}\left(\widehat{\ell}_{2}\right), \widetilde{\delta} \ell \widetilde{k}\left(\widehat{x}_{2}\right)=0\right. \tag{35}
\end{equation*}
$$

Since $\left(\pi \mathcal{H}_{\hat{\tau}_{2}}\right)_{*}$ is bijective, there exists a function $\alpha_{2}: \mathcal{O}\left(\widehat{x}_{2}\right) \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
\mathrm{d} \alpha_{2}\left(\widehat{x}_{2}\right)=\widehat{\ell}_{2} \text {, and } \mathcal{H}_{\widehat{\tau}_{2} *}\left(T_{\widehat{\ell}_{f}} \Lambda\right)=\mathrm{d} \alpha_{2 *}\left(\pi \mathcal{H}_{\widehat{\tau}_{2}}\right)_{*}\left(T_{\widehat{\ell}_{f}} \Lambda\right) . \tag{36}
\end{equation*}
$$

Thus, from (35) we get

$$
\begin{equation*}
0=\mathrm{d} \alpha_{2 *}\left(\pi_{*} \widetilde{\delta \ell}-a\left\langle\mathrm{~d} \tau_{1}\left(\widehat{\ell}_{2}\right), \delta \ell\right\rangle \widetilde{k}\left(\widehat{x}_{2}\right)\right)=\widetilde{\delta \ell}-a\left\langle\mathrm{~d} \tau_{1}\left(\widehat{\ell}_{2}\right), \widetilde{\delta \ell}\right\rangle \mathrm{d} \alpha_{2 *} \widetilde{k}\left(\widehat{x}_{2}\right) \tag{37}
\end{equation*}
$$

Computing $\mathrm{d} \tau_{1}\left(\widehat{\ell}_{2}\right)$ on each side of (37) we finally get

$$
\begin{aligned}
0 & =\left\langle\mathrm{d} \tau_{1}\left(\widehat{\ell}_{2}\right), \widetilde{\delta \ell}\right\rangle-a\left\langle\mathrm{~d} \tau_{1}\left(\widehat{\ell}_{2}\right), \widetilde{\delta \ell}\right\rangle\left\langle\mathrm{d} \tau_{1}\left(\widehat{\ell}_{2}\right), \mathrm{d} \alpha_{2 *} \widetilde{k}\left(\widehat{x}_{2}\right)\right\rangle= \\
& =\left\langle\mathrm{d} \tau_{1}\left(\widehat{\ell}_{2}\right), \widetilde{\delta \ell}\right\rangle\left(1-a\left\langle\mathrm{~d} \tau_{1}\left(\widehat{\ell}_{2}\right), \mathrm{d} \alpha_{2 *} \widetilde{k}\left(\widehat{x}_{2}\right)\right\rangle\right)
\end{aligned}
$$

i.e. $1-a\left\langle\mathrm{~d} \tau_{1}\left(\widehat{\ell}_{2}\right), \mathrm{d} \alpha_{2 *} \widetilde{k}\left(\widehat{x}_{2}\right)\right\rangle=0$ or, equivalently by (19),

$$
H_{12}\left(\widehat{\ell}_{1}\right)-a \boldsymbol{\sigma}\left(\mathrm{~d} \alpha_{2 *} \widetilde{k}\left(\widehat{x}_{2}\right), \vec{K}\left(\widehat{\ell}_{2}\right)\right)=0
$$

where $\overrightarrow{\widetilde{K}}=\exp \left(\widehat{\tau}_{2}-\widehat{\tau}_{1}\right) \overrightarrow{H_{2}}\left(\overrightarrow{H_{2}}-\overrightarrow{H_{1}}\right)$, so that

$$
\begin{equation*}
H_{12}\left(\widehat{\ell}_{1}\right)-a L_{\widetilde{k}}^{2} \alpha_{2}\left(\widehat{x}_{2}\right)=0 . \tag{38}
\end{equation*}
$$

We now use (33), i.e. the coercivity of $\widetilde{J}$, to get a contradiction. Let $(0, \delta x) \in L_{T}^{\prime \prime}$ be such that $\mathcal{H}_{\widehat{\tau}_{2}}^{\prime \prime}(0, \delta x)=\left(\mu\left(\widehat{\tau}_{2}\right), k\left(\widehat{x}_{f}\right)\right)$. Then

$$
\begin{aligned}
& \left\langle\mu\left(\widehat{\tau}_{2}\right), k\left(\widehat{x}_{f}\right)\right\rangle=\boldsymbol{\sigma}\left(\mathcal{H}_{\widehat{\tau}_{2}}^{\prime \prime}(0, \delta x),\left(0, k\left(\widehat{x}_{f}\right)\right)\right)=\boldsymbol{\sigma}\left(\mathcal{H}_{\widehat{\tau}_{2}}^{\prime \prime} \iota \iota^{-1}(0, \delta x), \iota^{-1}\left(0, k\left(\widehat{x}_{f}\right)\right)\right)= \\
& =\boldsymbol{\sigma}\left(\mathcal{H}_{\widehat{\tau}_{2}}^{\prime \prime} \iota \mathrm{d}(-\widetilde{c})_{*} \delta x, \iota \mathrm{~d}(-\widetilde{c})_{*} k\left(\widehat{x}_{f}\right)\right)=\boldsymbol{\sigma}\left(\iota \mathrm{d}(-\widetilde{c})_{*} k\left(\widehat{x}_{f}\right), \iota^{-1} \mathcal{H}_{\widehat{\tau}_{2}}^{\prime \prime} \iota(-\widetilde{c})_{*} \delta x\right)= \\
& =\boldsymbol{\sigma}\left(\mathrm{d}(-\widetilde{c})_{*} k\left(\widehat{x}_{f}\right), \widehat{\mathcal{F}}_{\widehat{\tau}_{2} *}^{-1} \mathcal{H}_{\widehat{\tau}_{2} *} \mathrm{~d}(-\widetilde{c})_{*} \delta x\right)=\boldsymbol{\sigma}\left(\mathrm{d}\left(-\widetilde{c} \circ \widehat{S}_{\widehat{\tau}_{2}}^{-1}\right)_{*} \widetilde{k}^{\left(x_{2}\right)}, \mathrm{d} \alpha_{2 *} \widetilde{k}\left(\widehat{x}_{2}\right)\right) .
\end{aligned}
$$

The last equality holds because $\widehat{F}_{t}$ is the lift of a vector field and thanks to (36). Moreover

$$
L_{k}^{2}(-\widetilde{c})\left(\widehat{x}_{f}\right)=L_{\widetilde{k}}^{2}\left(-\widetilde{c} \circ \widehat{S}_{\widehat{\tau}_{2}}^{-1}\right)\left(\widehat{x}_{2}\right)
$$

Substituting in (33) we finally get

$$
\begin{aligned}
0< & H_{12}\left(\widehat{\ell}_{1}\right)-L_{\widetilde{k}}^{2}\left(-\widetilde{c} \circ \widehat{S}_{\widehat{\tau}_{2}}^{-1}\right)\left(\widehat{x}_{2}\right)+\boldsymbol{\sigma}\left(\mathrm{d}\left(-\widetilde{c} \circ \widehat{S}_{\widehat{\tau}_{2}}^{-1}\right)_{*} \widetilde{k}\left(\widehat{x}_{2}\right), \mathrm{d} \alpha_{2 *} \widetilde{k}\left(\widehat{x}_{2}\right)\right)= \\
& =H_{12}\left(\widehat{\ell}_{1}\right)-L_{\widetilde{k}}^{2} \alpha_{2}\left(\widehat{x}_{2}\right),
\end{aligned}
$$

a contradiction to (38).

### 4.5 Examples

Van der Pol Oscillator. As an example consider the following Van der Pol Oscillator, studied in [9] where the author numerically shows that the optimal control is bang-bangsingular.

$$
\begin{align*}
& \operatorname{minimize} \frac{1}{2} \int_{0}^{4}\left(\xi_{1}^{2}+\xi_{2}^{2}\right)(t) \mathrm{d} t \quad \text { subject to }  \tag{39a}\\
& \dot{\xi}_{1}(t)=\xi_{2}(t) \\
& \dot{\xi}_{2}(t)=-\xi_{1}(t)+\xi_{2}(t)\left(1-\xi_{1}^{2}(t)\right)+u(t),  \tag{39b}\\
& \xi(0)=(0,1), \quad \xi(4) \in \mathbb{R}^{2} . \tag{39c}
\end{align*}
$$

The problem can be restated as a Mayer problem by substituting the state variable $\xi$ with a state variable (which we still denote as $\xi$ ) in $\mathbb{R}^{3}$ :

$$
\begin{align*}
& \operatorname{minimize} \xi_{3}(4) \quad \text { subject to }  \tag{40a}\\
& \dot{\xi}_{1}(t)=\xi_{2}(t), \\
& \dot{\xi}_{2}(t)=-\xi_{1}(t)+\xi_{2}(t)\left(1-\xi_{1}^{2}(t)\right)+u(t), \quad \text { a.e. } t \in[0,4],  \tag{40b}\\
& \dot{\xi}_{3}(t)=\frac{1}{2}\left(\xi_{1}^{2}(t)+\xi_{2}^{2}(t)\right), \\
& \xi(0)=(0,1,0), \quad \xi(4) \in \mathbb{R}^{3} . \tag{40c}
\end{align*}
$$

More precisely the author numerically shows that the optimal control has two bang arcs and a singular arc where the control can be written as a feedback control.

$$
\widehat{u}= \begin{cases}-1 & t \in\left[0, \widehat{\tau}_{1}\right), \\ 1 & t \in\left(\widehat{\tau}_{1}, \widehat{\tau}_{2}\right), \\ u_{\operatorname{sing}}(x)=2 x_{1}-x_{2}\left(1-x_{1}^{2}\right) & t \in\left(\widehat{\tau}_{2}, 4\right] .\end{cases}
$$

with $\widehat{\tau}_{1} \simeq 1.3667, \widehat{\tau}_{2} \simeq 2.4601$.
The problem fits in our setting defining

$$
\begin{gathered}
h_{1}(x)=h_{3}(x)=\left(\begin{array}{c}
x_{2} \\
-x_{1}+x_{2}\left(1-x_{1}^{2}\right)-1 \\
\frac{x_{1}^{2}+x_{2}^{2}}{2}
\end{array}\right), \quad h_{2}(x)=\left(\begin{array}{c}
x_{2} \\
-x_{1}+x_{2}\left(1-x_{1}^{2}\right)+1 \\
\frac{x_{1}^{2}+x_{2}^{2}}{2},
\end{array}\right), \\
\mathcal{X}=\overline{\operatorname{co}}\left\{h_{1}, h_{2}\right\}, \quad \widehat{v}(x)=\frac{1+u_{\text {sing }}(x)}{2} \in(0,1), \quad f_{1}(x)=\left(\begin{array}{c}
0 \\
-2 \\
0
\end{array}\right) .
\end{gathered}
$$

Bilinear systems. Consider the following example proposed in [8] with state-space $M:=\left\{N=\left(N_{1}, \ldots N_{n}\right) \in \mathbb{R}^{n}: N_{i}>0, i=1, \ldots n\right\}$ and control set $U:=\prod_{i=1}^{m}\left[0, u_{i}^{\max }\right]:$

$$
\text { minimize } C(u):=\langle r, N(T)\rangle+\int_{0}^{T}\langle q, N(T)\rangle+\langle s, u(t)\rangle \mathrm{d} t \text { subject to }
$$

$$
\dot{N}(t)=\left(A+\sum_{j=1}^{m} u_{j}(t) B_{j}\right) N(t) \quad \text { a.e. } t \in[0, T]
$$

$$
u \in L^{\infty}([0, T], U)
$$

$$
N(0)=N_{0}
$$

where $T>0$ is fixed and $A, B_{1}, \ldots, B_{m}$ are given $n \times n$ matrices,
The problem can be transformed into a Mayer one on $M \times \mathbb{R}$ and the control box can be normalised to the unit control box $\widetilde{U}:=[0,1]^{m}$ by setting

$$
\widetilde{s}_{j}=u_{j}^{\max } s_{j}, \quad C_{j}:=u_{j}^{\max } B_{j}, \quad \forall j=1, \ldots m
$$

as

$$
\begin{aligned}
& \operatorname{minimize} C(u):=\langle r, N(T)\rangle+N_{n+1}(T) \quad \text { subject to } \\
& \dot{N}(t)=\left(A+\sum_{j=1}^{m} u_{j}(t) C_{j}\right) N(t) \quad \text { a.e. } t \in[0, T] \\
& \dot{N}_{n+1}(t)=\langle q, N(t)\rangle+\langle\widetilde{s}, u(t)\rangle \\
& u \in L^{\infty}([0, T], \widetilde{U}) \\
& N(0)=N_{0}, \quad N_{n+1}(0)=0
\end{aligned}
$$

Denote as $\widetilde{x}=\left(x, x_{n+1}\right)$ the points in $M \times \mathbb{R}^{n}$ and set

$$
\widetilde{r}:=(r, 1), f_{0}(\widetilde{x}):=\left(\begin{array}{cc}
A & 0 \\
q & 0
\end{array}\right) \widetilde{x}, \quad f_{j}(\widetilde{x}):=\left(\begin{array}{cc}
C_{j} & 0 \\
0 & 1
\end{array}\right)\binom{x}{\widetilde{s}_{j}} \quad j=1, \ldots, m
$$

Then the problem can be written as

$$
\begin{aligned}
& \operatorname{minimize} \quad C(u):=\langle\widetilde{r}, \widetilde{\xi}(T)\rangle \quad \text { subject to } \\
& \dot{\widetilde{\xi}}(t)=f_{0}(\widetilde{\xi}(t))+\sum_{j=1}^{m} u_{j}(t) f_{j}(\widetilde{\xi}(t)) \quad \text { a.e. } t \in[0, T] \\
& u \in L^{\infty}([0, T], \widetilde{U}) \\
& \widetilde{\xi}(0)=\binom{N_{0}}{0}
\end{aligned}
$$

Thus the problem fits into our setting defining $X_{1}=f_{0}, X_{j+1}=f_{0}+f_{j}, j=1, \ldots, m$ $X_{m-1+j+k}=f_{0}+f_{j}+f_{k}, 1 \leq j<k \leq m, \ldots, X_{2^{m}}=f_{0}+f_{1}+f_{2}+\ldots+f_{m}$.

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