ON TANGENT CONES TO LENGTH MINIMIZERS IN CARNOT–CARATHÉODORY SPACES

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Abstract. We give a detailed proof of some facts about the blow-up of horizontal curves in Carnot–Carathéodory spaces.

1. Introduction

We give a detailed proof of some facts about the blow-up of horizontal curves in Carnot–Carathéodory spaces. These results are crucially used in [6, 7, 10]. The proof of a fraction of these results was already sketched, in a special case, in [13, Section 3.2].

Let M be a connected n-dimensional C^∞-smooth manifold and $\mathscr{X} = \{X_1, \dots, X_r\}$, $r \geq 2$, a system of C^∞-smooth vector fields on M that are pointwise linearly independent and satisfy the Hörmander condition introduced below. We call the pair $(M, \mathscr{X})$ a *Carnot–Carathéodory (CC) structure*. Given an interval $I \subseteq \mathbb{R}$, a Lipschitz curve $\gamma : I \to M$ is said to be *horizontal* if there exist functions $h_1, \dots, h_r \in L^\infty(I)$ such that for a.e. $t \in I$ we have

$$\dot{\gamma}(t) = \sum_{i=1}^{r} h_i(t) X_i(\gamma(t)). \tag{1.1}$$

The function $h \in L^\infty(I; \mathbb{R}^r)$ is called the *control* of γ. Letting $|h| := (h_1^2 + \dots + h_r^2)^{1/2}$, the length of γ is then defined as

$$L(\gamma) := \int_I |h(t)|\, dt.$$

Since M is connected, by the Chow–Rashevsky theorem (see e.g. [2, 12, 1]) for any pair of points $x, y \in M$ there exists a horizontal curve joining x to y. We can therefore define a distance function $d : M \times M \to [0, \infty)$ letting

$$d(x, y) := \inf\{L(\gamma) \mid \gamma : [0, T] \to M \text{ horizontal with } \gamma(0) = x \text{ and } \gamma(T) = y\}. \tag{1.2}$$

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The resulting metric space (M, d) is a *Carnot–Carathéodory space*. Since our analysis is local, our results apply in particular to *sub-Riemannian manifolds* $(M, \mathscr{D}, g)$, where $\mathscr{D} \subset TM$ is a completely non-integrable distribution and g is a smooth metric on $\mathscr{D}$.

If the closure of any ball in (M, d) is compact, then the infimum in (1.2) is a minimum, i.e., any pair of points can be connected by a length-minimizing curve. A horizontal curve $\gamma : [0, T] \to M$ is a *length minimizer* if $L(\gamma) = d(\gamma(0), \gamma(T))$.

The main contents of the paper are the following:

(i) we define a tangent Carnot–Carathéodory structure $(M^\infty, \mathscr{X}^\infty)$ at any point of M, using exponential coordinates of the first kind, see Section 2;

(ii) in Section 3, we define the tangent cone for a horizontal curve, at a given time, as the set of all possible blow-ups in $(M^\infty, \mathscr{X}^\infty)$ of the curve, and we show that this cone is always nonempty, see Proposition 3.2;

(iii) we show that, if the curve has a right derivative at the given time, the (positive) tangent cone consists of a single half-line, see Theorem 3.5;

(iv) if the curve is a length minimizer, in Theorem 3.6 we show that all the blow-ups are length minimizers in $(M^\infty, \mathscr{X}^\infty)$, as well;

(v) in Section 4, we show that a tangent Carnot–Carathéodory structure can be lifted to a free Carnot group, in a way that preserves length minimizers.

2. Nilpotent approximation: definition of a tangent structure

In this section we introduce some basic notions about Carnot–Carathéodory spaces. Then we describe the structure of a specific frame of vector fields $Y_1, \dots, Y_n$ (constructed below) in exponential coordinates, see Theorem 2.3. We also prove a lemma describing the infinitesimal behaviour of the Carnot–Carathéodory distance d near 0, with respect to suitable anisotropic dilations, see Lemma 2.4.

We denote by $\mathrm{Lie}(X_1, \dots, X_r)$ the real Lie algebra generated by $X_1, \dots, X_r$ through iterated commutators. The evaluation of this Lie algebra at a point $x \in M$ is a vector subspace of the tangent space T_xM. If, for any $x \in M$, we have

$$\mathrm{Lie}(X_1, \dots, X_r)(x) = T_xM,$$

we say that the system $\mathscr{X} = \{X_1, \dots, X_r\}$ satisfies the *Hörmander condition* and we call the pair $(M, \mathscr{X})$ a *Carnot–Carathéodory (CC) structure*.

Given a point $x_0 \in M$, let $\varphi \in C^\infty(U; \mathbb{R}^n)$ be a chart such that U is an open neighborhood of x_0 and $\varphi(x_0) = 0$. Then $V := \varphi(U)$ is an open neighborhood of $0 \in \mathbb{R}^n$ and the system of vector fields $Y_i := \varphi_* X_i$, with $i = 1, \dots, r$, still satisfies the Hörmander condition in V.

For a multi-index $J = (j_1, \dots, j_k)$ with $k \geq 1$ and $j_1, \dots, j_k \in \{1, \dots, r\}$, define the iterated commutator

$$Y_J := [Y_{j_1}, \dots, Y_{j_{k-1}}, Y_{j_k}]$$

where, here and in the following, for given vector fields $V_1, \dots, V_q$ we use the short notation $[V_1, \dots, V_q]$ to denote the commutator $[V_1, [\cdots, [V_{q-1}, V_q] \cdots]]$. We say that Y_J is a commutator of *length* $\ell(J) := k$ and we denote by L^j the linear span of $\{Y_J(0) \mid \ell(J) \le j\}$, so that

$$\{0\} = L^0 \subseteq L^1 \subseteq \cdots \subseteq L^s = \mathbb{R}^n$$

for some minimal $s \ge 1$. We select multi-indices $J_1 = (1), \dots, J_r = (r), J_{r+1}, \dots, J_n$ such that, for each $1 \le j \le s$,

$$\ell(J_{\dim L^{(j-1)}+1}) = \cdots = \ell(J_{\dim L^j}) = j$$

and such that, setting $Y_i := Y_{J_i}$, the vectors $Y_1(0), \dots, Y_{\dim L^j}(0)$ form a basis of L^j. In particular, we have $\dim L^1 = r$.

Possibly composing φ with a diffeomorphism (and shrinking U and V), we can assume that V is convex, that for any point $x = (x_1, \dots, x_n) \in V$ we have

$$x = \exp\Big(\sum_{i=1}^n x_i Y_i\Big)(0) \tag{2.3}$$

and that $Y_1, \dots, Y_n$ are linearly independent on V. Such coordinates $(x_1, \dots, x_n)$ are called *exponential coordinates of the first kind* associated with the frame $Y_1, \dots, Y_n$. To each coordinate x_i we assign the weight $w_i := \ell(J_i)$ and we define the anisotropic dilations $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$

$$\delta_\lambda(x) := (\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n), \qquad \lambda > 0. \tag{2.4}$$

Definition 2.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *δ-homogeneous of degree $w \in \mathbb{N}$* if $f(\delta_\lambda(x)) = \lambda^w f(x)$ for all $x \in \mathbb{R}^n$, $\lambda > 0$. We will refer to such a w as the *δ-degree* of f.

We will frequently use the anisotropic (pseudo-)norm

$$\|x\| := \sum_{i=1}^n |x_i|^{1/w_i}, \qquad x \in \mathbb{R}^n. \tag{2.5}$$

The norm function, $x \mapsto \|x\|$, is δ-homogeneous of degree 1.

We recall two facts about the exponential map, which are discussed e.g. in [11, pp. 141–147]. First, for any $\psi \in C^\infty(V)$, we have the Taylor expansion

$$\psi\Big(\exp\Big(\sum_{i=1}^n s_i Y_i\Big)(0)\Big) \sim \left(\mathrm{e}^{\sum_i s_i Y_i}\psi\right)(0) \tag{2.6}$$

where

- the left-hand side is a function of $s \in \mathbb{R}^n$ near 0;

- the right-hand side is a shorthand for the formal series

$$\sum_{k=0}^{\infty} \frac{1}{k!}((s_1 Y_1 + \cdots + s_n Y_n)^k \psi)(0) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1,\dots,i_k \in \{1,\dots,n\}} s_{i_1} \cdots s_{i_k} (Y_{i_1} \cdots Y_{i_k} \psi)(0);$$

- given a smooth function $f(x)$ and a formal power series $S(x)$, we define the relation $f(x) \sim S(x)$ if the formal Taylor series of $f(x)$ at 0 is $S(x)$.

Second, letting $S := \sum_{i=1}^n s_i Y_i$ and $T := \sum_{i=1}^n t_i Y_i$, the following formal Taylor expansions hold as well:

$$\psi\Big(\exp(S) \circ \exp(T)(0)\Big) \sim \left(\mathrm{e}^T \mathrm{e}^S \psi\right)(0) = (\mathrm{e}^{P(T,S)} \psi)(0), \tag{2.7}$$

where

$$P(T,S) := \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} \sum_{k_i+\ell_i \geq 1} \frac{[T^{k_1}, S^{\ell_1}, \dots, T^{k_p}, S^{\ell_p}]}{k_1! \cdots k_p!\, \ell_1! \cdots \ell_p!\, (k_1 + \ell_1 + \cdots + k_p + \ell_p)}. \tag{2.8}$$

Above, the notation T^k stands for $T, \dots, T$, k times.

Remark 2.2. The formal power series identity $\mathrm{e}^T \mathrm{e}^S = \mathrm{e}^{P(T,S)}$ is a purely algebraic fact which holds in any (noncommutative, graded, complete) associative real algebra, see e.g. [5, Sec. X.2]: this principle will be used in the proofs of Theorem 2.3 and Lemma 2.4.

In the case of exponential coordinates *of the second kind*, the following theorem is proved in [4].

Theorem 2.3. The vector fields $Y_1, \dots, Y_n$ are of the form

$$Y_i(x) = \sum_{j=i}^{n} a_{ij}(x) \frac{\partial}{\partial x_j}, \qquad x \in V, \quad i = 1, \dots, n, \tag{2.9}$$

where $a_{ij} \in C^\infty(V)$ are functions such that $a_{ij} = p_{ij} + r_{ij}$ and:

(i) for $w_j \geq w_i$, p_{ij} are δ-homogeneous polynomials in $\mathbb{R}^n$ of degree $w_j - w_i$;
(ii) for $w_j \leq w_i$, $p_{ij} = \delta_{ij}$ (in particular, $p_{ij} = 0$ for $w_j < w_i$);
(iii) $r_{ij} \in C^\infty(V)$ satisfy $r_{ij}(0) = 0$;
(iv) for $w_j \geq w_i$, $r_{ij}(x) = o(\|x\|^{w_j - w_i})$ as $x \to 0$.

Proof. Suppose for a moment that

$$a_{ij}(x) = O(\|x\|^{w_j - w_i}), \qquad i, j = 1, \dots, n, \ w_j \geq w_i. \tag{2.10}$$

Let p_{ij} be the sum of all monomials of δ-degree $w_j - w_i$ in the Taylor expansion of a_{ij}, with the convention that $p_{ij} = 0$ if $w_j < w_i$. Statements (i) and (iv) then hold by construction, while (ii) and (iii) follow from $a_{ij}(0) = \delta_{ij}$, which is a consequence of (2.3).

Let us show (2.10). We pullback the identity $Y_i(x) = \sum_j a_{ij}(x)\frac{\partial}{\partial x_j}$ to the origin using the map $\exp(-X)$ (locally defined near x), where $X := \sum_k x_k Y_k$, for a fixed $x \in V$. We have

$$\exp(-X)_*(Y_i(x)) = \sum_j a_{ij}(x)\exp(-X)_*\Big(\frac{\partial}{\partial x_j}(x)\Big), \tag{2.11}$$

where the sum ranges from 1 to n. The above equation reads

$$\sum_\ell b_{i\ell}(x)Y_\ell(0) = \sum_{j,\ell} a_{ij}(x)c_{j\ell}(x)Y_\ell(0)$$

for suitable smooth coefficients $b_{i\ell}(x), c_{j\ell}(x)$. We claim that

$$b_{i\ell}(x) = O(\|x\|^{w_\ell - w_i}), \quad c_{j\ell}(x) = O(\|x\|^{w_\ell - w_j}), \quad \text{and} \quad c_{j\ell}(0) = \delta_{j\ell}.$$

Then, defining $A := (a_{ij})$, $B := (b_{i\ell})$ and $C := 1 - (c_{j\ell})$ (1 denoting the identity matrix), we obtain three $n \times n$ matrices satisfying $B(x) = A(x)(1 - C(x))$ and $C(0) = 0$. In particular, $1 - C(x)$ is invertible for x close to 0 and $(1 - C(x))^{-1} = \sum_{p=0}^\infty C(x)^p$. This gives

$$A(x) = \sum_{p=0}^s B(x)C(x)^p + o(|x|^s) = \sum_{p=0}^s B(x)C(x)^p + o(\|x\|^s)$$

for any $s \in \mathbb{N}$, and (2.10) easily follows.

The proof of $c_{j\ell}(0) = \delta_{j\ell}$ follows from the definition of $c_{j\ell}$ and from $\frac{\partial}{\partial x_j} = Y_j(0)$, which in turn comes from (2.3), as already observed.

We prove the claim $b_{i\ell}(x) = O(\|x\|^{w_\ell - w_i})$. By (2.3), the left-hand side of (2.11) satisfies

$$\exp(-X)_*(Y_i(x)) = \frac{d}{dt}\exp(-X)\circ\exp(tY_i)\circ\exp(X)(0)\Big|_{t=0}.$$

Using (2.7) and Remark 2.2, for any smooth ψ we obtain

$$\psi(\exp(-X)\circ\exp(tY_i)\circ\exp(X)(0)) \sim \mathrm{e}^{P(P(X,tY_i),-X)}\psi(0),$$

the left-hand side being interpreted as a function of (x, t). We now differentiate this identity at $t = 0$. Since $W(t) := P(P(X, tY_i), -X)$ vanishes at $t = 0$, one has $\frac{d}{dt}(\mathrm{e}^{W(t)}\psi)(0)\Big|_{t=0} = \frac{d}{dt}(W(t)\psi)(0)\Big|_{t=0}$ and, letting ψ range among the coordinate functions, we deduce that any finite-order expansion in x of $\exp(-X)_*(Y_i(x))$ is a linear combination of terms of the form

$$x_{i_1}\cdots x_{i_p}[Y_{i_1}, \ldots, Y_{i_m}, Y_i, Y_{i_{m+1}}, \ldots, Y_{i_p}](0)$$

where $p \geq 1$ and $0 \leq m \leq p$. By Jacobi's identity, the iterated commutator $[Y_{i_1}, \ldots, Y_{i_m}, Y_i, Y_{i_{m+1}}, \ldots, Y_{i_p}](0)$ is a linear combination of the vectors $Y_J(0)$ with

$\ell(J) = \overline{w} := \sum_{q=1}^{p} w_{i_q} + w_i$ and so, by construction, it is a linear combination of the vectors $Y_\ell(0)$ with $w_\ell \le \overline{w}$. Hence, letting $w_\alpha := \sum_{q=1}^{n} \alpha_q w_q$ for all $\alpha \in \mathbb{N}^n$, we have

$$\exp(-X)_*(Y_i(x)) \sim \sum_{\ell} \sum_{\alpha : w_\alpha \ge w_\ell - w_i} d_{\alpha i \ell} x^\alpha Y_\ell(0),$$

for suitable coefficients $d_{\alpha i \ell} \in \mathbb{R}$. This gives the required estimate.

The proof of $c_{j\ell}(x) = O(\|x\|^{w_\ell - w_j})$ is analogous to the preceding argument, once we observe that

$$\exp(-X)_*\Big(\frac{\partial}{\partial x_j}(x)\Big) = \frac{d}{dt} \exp(-X) \circ \exp(X + tY_j)(0)\Big|_{t=0}.$$

We can omit the details. □

Lemma 2.4. For any compact set $K \subset \mathbb{R}^n$ and any $\varepsilon > 0$ there exist $\delta > 0$ and $\overline{\lambda} > 0$ such that $\lambda d(\delta_{1/\lambda}(x), \delta_{1/\lambda}(y)) < \varepsilon$ for all $x, y \in K$ with $|x - y| < \delta$ and all $\lambda \ge \overline{\lambda}$.

Proof. Let $\psi \in C^\infty(V)$ be an arbitrary smooth function. Using (2.6) and Remark 2.2, we have the following identity of formal power series in $(s,t) \in \mathbb{R}^n \times \mathbb{R}^n$: letting $S := \sum_{i=1}^n s_i Y_i$ and $T := \sum_{i=1}^n t_i Y_i$,

$$\psi(\exp(S)(0)) \sim (\mathrm{e}^S \psi)(0) = (\mathrm{e}^T \mathrm{e}^{-T} \mathrm{e}^S \psi)(0) = (\mathrm{e}^T \mathrm{e}^{P(-T,S)} \psi)(0). \tag{2.12}$$

The truncation $P_N(-T,S)$ of the series $P(-T,S)$ up to δ-degree $N := w_n$ is

$$P_N(-T,S) = \sum_{1 \le \ell(J) \le N} q_J(s,t) Y_J, \tag{2.13}$$

where the sum is over all J such that $1 \le \ell(J) \le N$ and q_J is a homogeneous polynomial with δ-degree $\ell(J)$, i.e., $q_J(\delta_\lambda s, \delta_\lambda t) = \lambda^{\ell(J)} q_J(s,t)$. This follows from the fact that any iterated commutator $[Y_{i_1}, \dots, Y_{i_k}]$ is a constant linear combination of the vector fields Y_J's with $\ell(J) = \sum_{j=1}^k w_{i_j}$ (which in turn is a consequence of Jacobi's identity).

Moreover, using (2.13) and applying (2.7) with the vector fields Y_J in place of $Y_1, \dots, Y_n$, we have the following formal Taylor expansion in (s,t) at $0 \in \mathbb{R}^{2n}$

$$\psi(\exp(P_N(-T,S)) \circ \exp(T)(0)) \sim \Big(\mathrm{e}^T \mathrm{e}^{P_N(-T,S)} \psi\Big)(0),$$

which, by (2.12), coincides with the one of $\psi(\exp(S)(0))$ up to δ-degree N. Since this holds for any ψ, we deduce (for instance letting ψ range among the coordinate functions) that

$$\exp(S)(0) = \exp(P_N(-T,S)) \circ \exp(T)(0) + o(|s|^N + |t|^N),$$

which by (2.3) gives

$$s = \exp(P_N(-T,S))(t) + o(|s|^N + |t|^N) =: f(s,t) + o(|s|^N + |t|^N).$$

Now let $s = \delta_{1/\lambda}(x)$ and $t = \delta_{1/\lambda}(y)$ with $x, y \in K$. Since

$$q_J(s,t) = \lambda^{-\ell(J)} q_J(x,y),$$

by [11, Theorem 4] we get

$$d(t, f(s,t)) \le C \sum_{1\le\ell(J)\le N} |q_J(s,t)|^{1/\ell(J)} = C\lambda^{-1} \sum_{1\le\ell(J)\le N} |q_J(x,y)|^{1/\ell(J)},$$

while, by [11, Lemma 2.20(b)],

$$d(s, f(s,t)) = O(|s - f(s,t)|^{1/w_n}) = o(|s|+|t|) = o(\lambda^{-1}),$$

provided λ is sufficiently large. Thus, by the triangle inequality,

$$\lambda d(\delta_{1/\lambda}(x), \delta_{1/\lambda}(y)) = \lambda d(s,t) \le C \sum_{1\le\ell(J)\le N} |q_J(x,y)|^{1/\ell(J)} + \frac{\varepsilon}{2}$$

for all $\lambda \ge \overline{\lambda}$, for a suitably large $\overline{\lambda} > 0$. Finally, since $P_N(S, -S) = 0$, we can assume that q_J vanishes on the diagonal of $K \times K$ (possibly replacing $q_J(s,t)$ with $q_J(s,t) - q_J(s,s)$). Hence, by compactness of K, we also have

$$C \sum_{1\le\ell(J)\le N} |q_J(x,y)|^{1/\ell(J)} < \frac{\varepsilon}{2}$$

whenever $x, y \in K$ are such that $|x - y| < \delta$, for a suitably small $\delta > 0$. □

We now introduce the vector fields $Y_1^\infty, \dots, Y_r^\infty$ in $\mathbb{R}^n$ defined by

$$Y_i^\infty(x) := \sum_{j=1}^n p_{ij}(x) \frac{\partial}{\partial x_j},$$

and we let $\mathscr{X}^\infty = \{Y_1^\infty, \dots, Y_r^\infty\}$. The vector fields $Y_1^\infty, \dots, Y_r^\infty$ are known as the *nilpotent approximation* of $Y_1, \dots, Y_r$ at the point 0. By Proposition 2.5 below, the pair $(\mathbb{R}^n, \mathscr{X}^\infty)$ is a Carnot–Carathéodory structure. We set $M^\infty := \mathbb{R}^n$ and we call $(M^\infty, \mathscr{X}^\infty)$ a *tangent* Carnot–Carathéodory structure to $(M, \mathscr{X})$ at the point $x_0 \in M$.

Proposition 2.5. The vector fields $Y_1^\infty, \dots, Y_r^\infty$ are pointwise linearly independent and satisfy the Hörmander condition in $\mathbb{R}^n$. Moreover, any iterated commutator $Y_J^\infty := [Y_{j_1}^\infty, [\dots, [Y_{j_{k-1}}^\infty, Y_{j_k}^\infty] \dots]]$ of length $\ell(J) = k > s$ vanishes identically.

Proof. We claim that Theorem 2.3 implies $Y_i^\infty = \lim_{\lambda\to\infty} \lambda^{-1}(\delta_\lambda)_* Y_i$, for all $i = 1, \dots, r$, in the (local) C^∞-topology (the vector field $\lambda^{-1}(\delta_\lambda)_* Y_i$ being defined on $\delta_\lambda(V)$). Indeed, since $Y_i(x) = Y_i^\infty(x) + \sum_j r_{ij}(x) \frac{\partial}{\partial x_j}$, we have

$$\lambda^{-1}((\delta_\lambda)_* Y_i)(x) = Y_i^\infty(x) + \sum_{j=1}^n \lambda^{w_j - 1} r_{ij}(\delta_{1/\lambda}(x)) \frac{\partial}{\partial x_j},$$

because $\lambda^{-1}(\delta_\lambda)_* Y_i^\infty = Y_i^\infty$. By Theorem 2.3, the monomials in the Taylor expansion of r_{ij} have δ-degree greater than $w_j - 1$. Thus, for any $\alpha \in \mathbb{N}^n$,

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}(\lambda^{w_j-1} r_{ij}(\delta_{1/\lambda}(x))) = \lambda^{w_j-1-w_\alpha} \frac{\partial^{|\alpha|} r_{ij}}{\partial x^\alpha}(\delta_{1/\lambda}(x)),$$

where $w_\alpha := \sum_\ell \alpha_\ell w_\ell$. The monomials in the expansion of $\frac{\partial^{|\alpha|} r_{ij}}{\partial x^\alpha}$ have δ-degree greater than $w_j - 1 - w_\alpha$, hence $\left|\frac{\partial^{|\alpha|} r_{ij}}{\partial x^\alpha}(\delta_{1/\lambda}(x))\right| = o(\lambda^{-(w_j-1-w_\alpha)})$ and the claim follows.

In particular, we deduce that for any multi-index J

$$Y_J^\infty = \lim_{\lambda\to\infty} \lambda^{-\ell(J)}(\delta_\lambda)_* Y_J, \tag{2.14}$$

in the local C^∞ topology. Hence, defining the $n \times n$ matrix $D_\lambda :=\mathrm{diag}[\lambda^{w_1}, \dots, \lambda^{w_n}]$ and recalling that $\ell(J_p) = w_p$, for all $p = 1, \dots, n$ we have

$$Y_{J_p}^\infty(x) = \lim_{\lambda\to\infty} \lambda^{-w_p} D_\lambda Y_{J_p}(\delta_{1/\lambda}(x)).$$

Now the first statement follows from

$$\begin{aligned}\det(Y_{J_1}^\infty, \dots, Y_{J_n}^\infty)(x) &= \lim_{\lambda\to\infty} \lambda^{-\sum_i w_i} \det(D_\lambda) \det(Y_{J_1}, \dots, Y_{J_n})(\delta_{1/\lambda}(x)) \\ &= \det(Y_{J_1}, \dots, Y_{J_n})(0) = \det(Y_1, \dots, Y_n)(0),\end{aligned}$$

which is a nonzero constant. This gives the first part of the statement.

In order to prove the last assertion, we use again the fact that $\lambda^{-1}(\delta_\lambda)_* Y_i^\infty = Y_i^\infty$ for $i = 1, \dots, r$. For any $x \in \mathbb{R}^n$ and any J with $\ell(J) > s = w_n$ we have, by (2.14),

$$Y_J^\infty(x) = \lim_{\lambda\to\infty} \lambda^{-\ell(J)}((\delta_\lambda)_* Y_J)(x) = \lim_{\lambda\to\infty} \lambda^{-\ell(J)} D_\lambda Y_J(\delta_{1/\lambda}(x)).$$

The right-hand side is bounded by $\lambda^{s-\ell(J)}|Y_J(\delta_{1/\lambda}(x))|$ (if $\lambda \geq 1$), which tends to 0 as $\lambda \to \infty$. This shows that $Y_J^\infty = 0$. □

Remark 2.6. Setting $Y_i^\infty := Y_{J_i}^\infty$ for $i = 1, \dots, n$, the coordinate functions on $M^\infty = \mathbb{R}^n$ are exponential coordinates of the first kind for $(Y_1^\infty, \dots, Y_n^\infty)$, namely

$$x = \exp\Big(\sum_{i=1}^n x_i Y_i^\infty\Big)(0). \tag{2.15}$$

for any $x \in \mathbb{R}^n$. This follows from the fact that, for λ large enough (depending on x), we have $y := \delta_{\lambda^{-1}}(x) \in V$ and, using (2.3) with y in place of x,

$$x = \delta_\lambda\Big(\exp\Big(\sum_i y_i Y_i\Big)(0)\Big) = \exp\Big(\sum_i x_i \lambda^{-w_i}(\delta_\lambda)_* Y_i\Big)(0) \to \exp\Big(\sum_i x_i Y_i^\infty\Big)(0)$$

as $\lambda \to \infty$, since (2.14) gives $\lambda^{-w_i}(\delta_\lambda)_* Y_i \to Y_i^\infty$ in the local C^∞ topology.

3. The tangent cone to a horizontal curve

Let $(M, \mathscr{X})$ be a CC structure and let $\gamma : [-T, T] \to M$ be a horizontal curve. Given $t \in (-T, T)$, let φ be a chart centered at $x_0 = \gamma(t)$, as in the previous section, together with the dilations δ_λ and the tangent CC structure $(M^\infty, \mathscr{X}^\infty)$ introduced above.

Definition 3.1. The *tangent cone* $\mathrm{Tan}(\gamma; t)$ to γ at $t \in (-T, T)$ is the set of all horizontal curves $\kappa : \mathbb{R} \to M^\infty$ such that there exists an infinitesimal sequence $\eta_i \downarrow 0$ satisfying, for any $\tau \in \mathbb{R}$,

$$\lim_{i\to\infty} \delta_{1/\eta_i} \varphi(\gamma(t + \eta_i \tau)) = \kappa(\tau),$$

with uniform convergence on compact subsets of $\mathbb{R}$.

We remark that any limit curve as above is automatically $(M^\infty, \mathscr{X}^\infty)$-horizontal: see e.g. the proof of Theorem 3.6.

The definition of $\mathrm{Tan}(\gamma; t)$ depends on the choice $Y_1, \dots, Y_n$ of linearly independent iterated commutators. When $\gamma : [0, T] \to M$, the tangent cones $\mathrm{Tan}^+(\gamma; 0)$ and $\mathrm{Tan}^-(\gamma; T)$ can be defined in a similar way: $\mathrm{Tan}^+(\gamma; 0)$ contains curves in M^∞ defined on $[0, \infty)$, while $\mathrm{Tan}^-(\gamma; T)$ contains curves defined on $(-\infty, 0]$.

When $M = M^\infty$ or $M = G$ is a Carnot group, there is already a group of dilations on M itself. In such cases, when $\gamma(t) = 0$, we define the tangent cone $\mathrm{Tan}(\gamma; t)$ as the set of horizontal limit curves of the form $\kappa(t) = \lim_{i\to\infty} \delta_{1/\eta_i} \gamma(t + \eta_i \tau)$.

The tangent cone is closed under uniform convergence of curves on compact sets.

Proposition 3.2. For any horizontal curve $\gamma : [-T, T] \to M$ the tangent cone $\mathrm{Tan}(\gamma; t)$ is nonempty for any $t \in (-T, T)$. The same holds for $\mathrm{Tan}^+(\gamma; 0)$ and $\mathrm{Tan}^-(\gamma; T)$, for a horizontal curve $\gamma : [0, T] \to M$.

Proof. We prove that $\mathrm{Tan}^+(\gamma; 0) \neq \emptyset$. The other cases are analogous.

We use exponential coordinates of the first kind centered at $\gamma(0)$. By (1.1), we have a.e.

$$\dot\gamma = \sum_{i=1}^r h_i Y_i(\gamma) = \sum_{j=1}^n \sum_{i=1}^r h_i a_{ij}(\gamma) \frac{\partial}{\partial x_j},$$

where $h_i \in L^\infty([0, T])$ and $a_{ij} = p_{ij} + r_{ij}$, as in Theorem 2.3. Letting $K := \gamma([0, T])$, we have $|\dot\gamma(t)| \leq C$ for some constant depending on $\|a_{ij}\|_{L^\infty(K)}$ and $\|h\|_{L^\infty}$. This implies that $|\gamma(t)| \leq Ct$ for all $t \in [0, T]$.

By induction on $k \geq 1$, we prove the following statement: for any j satisfying $w_j \geq k$ we have $|\gamma_j(t)| \leq Ct^k$. The base case $k = 1$ has already been treated. Now assume that $w_j \geq k > 1$ and that the statement is true for $1, \dots, k-1$. Since r_{ij} is smooth, we have $r_{ij} = q_{ij,k} + r_{ij,k}$, where $q_{ij,k}$ is a polynomial containing only terms with δ-homogeneous degree at least $w_j - w_i + 1 = w_j$ and $|r_{ij,k}(x)| \leq C|x|^{k-1}$ on K (here $|x|$ denotes the usual Euclidean norm).

Each monomial $c_\alpha x^\alpha$ of the polynomial $p_{ij} + q_{ij,k}$ has δ-degree $w_\alpha \geq w_j - 1$. If $\alpha_m = 0$ whenever $w_m \geq k$, then we can estimate

$$|\gamma(t)^\alpha| = \prod_{m : w_m \leq k-1} |\gamma_m(t)|^{\alpha_m} \leq C t^{w_\alpha} \leq C t^{k-1},$$

using the inductive hypothesis with k replaced by $w_m \leq k-1$. Otherwise, there exists some index m with $w_m \geq k$ and $\alpha_m > 0$, in which case

$$|\gamma(t)^\alpha| \leq C|\gamma_m(t)| \leq C t^{k-1},$$

using the inductive hypothesis with k replaced by $k-1$. Thus $|p_{ij}(\gamma(t)) + q_{ij,k}(\gamma(t))| \leq C t^{k-1}$. Combining this with the estimate $|r_{ij,k}(\gamma(t))| \leq C t^{k-1}$, we obtain $|a_{ij}(\gamma(t))| \leq C t^{k-1}$. So we finally have

$$|\gamma_j(t)| \leq \|h\|_{L^\infty} \sum_{i=1}^{r} \int_0^t |a_{ij}(\gamma(\tau))| \, d\tau \leq C t^k,$$

completing the inductive proof. Applying the above statement with $k = w_j$, we obtain

$$|\gamma_j(t)| \leq C t^{w_j}, \tag{3.16}$$

for a suitable constant C depending only on K, T and $\|h\|_{L^\infty}$.

Now we prove that $\mathrm{Tan}^+(\gamma; 0)$ is nonempty. For $\eta > 0$ consider the family of curves $\gamma^\eta(t) := \delta_{1/\eta}(\gamma(\eta t))$, defined for $t \in [0, T/\eta]$. The derivative of γ^η is a.e.

$$\dot\gamma^\eta(t) = \sum_{j=1}^{n} \sum_{i=1}^{r} h_i(\eta t) \eta^{1-w_j} a_{ij}(\gamma(\eta t)) \frac{\partial}{\partial x_j},$$

where, by Theorem 2.3 and the estimates (3.16), we have

$$|a_{ij}(\gamma(\eta t))| \leq C \|\gamma(\eta t)\|^{w_j - 1} \leq C (\eta t)^{w_j - 1}.$$

This proves that the family of curves $(\gamma^\eta)_{\eta > 0}$ is locally Lipschitz equicontinuous. So it has a subsequence $(\gamma^{\eta_i})_i$ that is converging locally uniformly as $\eta_i \to 0$ to a curve $\kappa : [0, \infty) \to \mathbb{R}^n$. □

Remark 3.3. The following result was obtained along the proof of Proposition 3.2. Let $(M, \mathscr{X})$ be a Carnot–Carathéodory structure. Using exponential coordinates of the first kind, we (locally) identify M with $\mathbb{R}^n$ and we assign to the coordinate x_j the weight w_j, as above. Given $T > 0$ and K compact, there exists a positive constant $C = C(K, T)$ such that the following holds: for any horizontal curve $\gamma : [0, T] \to K$ parametrized by arclength and such that $\gamma(0) = 0$, one has

$$|\gamma_j(t)| \leq C t^{w_j}, \quad \text{for any } j = 1, \dots, n \text{ and } t \in [0, T]. \tag{3.17}$$

In Carnot groups, by homogeneity, the constant C is independent of K and T.

Definition 3.4. We say that $v \in \mathbb{R}^n$ is a *right tangent vector* to a curve $\gamma : [0,T] \to \mathbb{R}^n$ at 0 if

$$\gamma(t) = tv + o(t), \quad \text{as } t \to 0^+.$$

The definition of a *left tangent vector* is analogous.

The next result is stated in exponential coordinates of the first kind.

Theorem 3.5. Let $\gamma : [0,T] \to V$ be a horizontal curve parametrized by arclength, with $\gamma(0) = 0$. If γ has a right tangent vector $v \in \mathbb{R}^n$ at 0, then:

(i) $v_j = 0$ for $j > r$ and $|v| \leq 1$;
(ii) $\mathrm{Tan}^+(\gamma; 0) = \{\kappa\}$, where $\kappa(t) = tv$ for $t \in [0,\infty)$;
(iii) $|v| = 1$ if γ is also length minimizing.

A similar statement holds if $\gamma : [-T, 0] \to V$ has a left tangent vector at 0.

Proof. (i) Since $Y_i(x) = \frac{\partial}{\partial x_i} + o(1)$ as $x \to 0$, we have

$$\gamma_j(t) = \int_0^t \sum_{i=1}^r h_i(s)\delta_{ij}\, ds + o(t). \tag{3.18}$$

We deduce that $v_j = 0$ for $j > r$ and

$$|v| = \lim_{t\to 0^+} \left|\frac{\gamma(t)}{t}\right| \leq \lim_{t\to 0^+} \frac{1}{t}\int_0^t |h(s)|\, ds = 1.$$

(ii) Since $\gamma_j(t) = v_j t + o(t)$ for $j \leq r$, it suffices to show that

$$\gamma_j(t) = o(t^{w_j}), \quad j > r. \tag{3.19}$$

Up to a rotation of the vector fields $Y_1, \dots, Y_r$, which by (2.3) corresponds to a rotation of the first r coordinates, we can assume that $v_2 = \ldots = v_r = 0$. Notice that Theorem 2.3 still applies in these new exponential coordinates. From (3.18) we get

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t h_i(s)\, ds = \begin{cases} v_1 & i = 1 \\ 0 & i = 2, \dots, r. \end{cases} \tag{3.20}$$

By Remark 3.3 we have $\|\gamma(t)\| = O(t)$. We now show (3.19) by induction on $j \geq r+1$.

Assume the claim holds for $r+1, \dots, j-1$. The coordinate γ_j, with $j > r$, is

$$\gamma_j(t) = \sum_{i=1}^r \int_0^t h_i(s)a_{ij}(\gamma(s))\, ds = \int_0^t h_1(s)a_{1j}(\gamma(s))\, ds + \sum_{i=2}^r \int_0^t h_i(s)a_{ij}(\gamma(s))\, ds.$$

By Theorem 2.3, $a_{ij} = p_{ij} + r_{ij}$ with $r_{ij}(x) = o(\|x\|^{w_j - 1})$, so we deduce that

$$a_{ij}(\gamma(s)) = p_{ij}(\gamma(s)) + r_{ij}(\gamma(s)) = p_{ij}(\gamma(s)) + o(s^{w_j-1}), \quad i = 1, \dots, r.$$

From (2.3) we deduce that for $i = 1, \dots, r$ we have $Y_i(0, \dots, x_i, \dots, 0) = \frac{\partial}{\partial x_i}$, hence

$$a_{ij}(0, \dots, x_i, \dots, 0) = 0, \quad j > r. \tag{3.21}$$

The polynomial $p_{ij}(x)$ is δ-homogeneous of degree $w_j - w_i = w_j - 1$ and so it contains no variable x_k with $k \geq j$. Condition (3.21) implies that $p_{ij}(x)$ does not contain the monomial $x_i^{w_j-1}$, either. Thus, when $i = 1$ each monomial in $p_{1j}(x)$ contains at least one of the variables $x_2, \dots, x_{j-1}$. By the inductive assumption, it follows that $p_{1j}(\gamma(s)) = o(s^{w_j-1})$, and thus $a_{1j}(\gamma(s)) = o(s^{w_j-1})$. This implies that

$$\int_0^t h_1(s) a_{1j}(\gamma(s))\, ds = o(t^{w_j}).$$

Now we consider the case $i = 2, \dots, r$. Letting $p_{ij} = c_{ij} x_1^{w_j-1} + \widehat{p}_{ij}$ with $c_{ij} \in \mathbb{R}$ and $\widehat{a}_{ij} := \widehat{p}_{ij} + r_{ij}$, we have $\widehat{a}_{ij}(\gamma(s)) = o(s^{w_j-1})$ as in the previous case and thus

$$\int_0^t h_i(s) \widehat{a}_{ij}(\gamma(s))\, ds = o(t^{w_j}).$$

We claim that, for $i = 2, \dots, m$, we also have

$$\int_0^t h_i(s) \gamma_1(s)^{w_j-1}\, ds = o(t^{w_j}).$$

Indeed, since $v_i = 0$ we have $H_i(s) := \int_0^s h_i(s')\, ds' = o(s)$, so integration by parts gives

$$\begin{aligned} \int_0^t h_i(s)\gamma_1(s)^{w_j-1}\, ds &= H_i(t)\gamma_1(t)^{w_j-1} - (w_j - 1)\int_0^t H_i(s)\gamma_1(s)^{w_j-2}\dot{\gamma}_1(s)\, ds \\ &= o(t^{w_j}) + \int_0^t o(s^{w_j-1})\, ds = o(t^{w_j}). \end{aligned}$$

This ends the proof of (3.19) and hence of (ii).

(iii) By Theorem 3.6 below, κ is parametrized by arclength. But $(v_1, \dots, v_r)$ equals its (continuous) control $h(t)$ at $t = 0$, so $|v| = 1$. $\square$

For $\lambda > 0$, we define the vector fields $Y_1^\lambda, \dots, Y_r^\lambda$ in $\delta_\lambda(V)$ by

$$Y_i^\lambda(x) := \lambda^{-1}((\delta_\lambda)_* Y_i)(x) = \sum_{j=1}^n \lambda^{w_j-1} a_{ij}(\delta_{1/\lambda}(x)) \frac{\partial}{\partial x_j}, \quad x \in \delta_\lambda(V).$$

In the proof of Proposition 2.5 it was shown that

$$Y_i^\lambda \to Y_i^\infty \tag{3.22}$$

locally uniformly in $\mathbb{R}^n$ as $\lambda \to \infty$, together with all the derivatives.

We denote by d^λ the Carnot–Carathéodory metric of $(\delta_\lambda(V), \mathscr{X}^\lambda)$, with $\mathscr{X}^\lambda := \{Y_1^\lambda, \dots, Y_r^\lambda\}$. The distance function d^λ is related to the distance function d via the formula

$$d^\lambda(x, y) = \lambda d(\delta_{1/\lambda}(x), \delta_{1/\lambda}(y)), \tag{3.23}$$

for all $x, y \in \delta_\lambda(V)$ and $\lambda > 0$. Indeed, let $\gamma : [0,1] \to V$ be a horizontal curve

$$\gamma(t) = \gamma(0) + \int_0^t \sum_{i=1}^r h_i(s) Y_i(\gamma(s))\, ds, \quad t \in [0,1], \tag{3.24}$$

and define the curve $\gamma^\lambda : [0,\lambda] \to \delta_\lambda(V)$

$$\gamma^\lambda(t) := \delta_\lambda \gamma(t/\lambda), \quad t \in [0,\lambda]. \tag{3.25}$$

Then we have

$$\gamma^\lambda(t) = \gamma^\lambda(0) + \int_0^t \sum_{i=1}^r h_i(s/\lambda) Y_i^\lambda(\gamma^\lambda(s))\, ds, \quad t \in [0,\lambda], \tag{3.26}$$

and therefore the length of γ^λ is

$$L^\lambda(\gamma^\lambda) = \int_0^\lambda |h(s/\lambda)|\, ds = \lambda \int_0^1 |h(s)|\, ds = \lambda L(\gamma). \tag{3.27}$$

If γ is length minimizing, then the curves in $\operatorname{Tan}(\gamma; t)$ are also locally length minimizing. This is the content of the next theorem.

Theorem 3.6. Let $\gamma : [-T, T] \to M$ be a length-minimizing curve in $(M, \mathscr{X})$, parametrized by arclength, and let $\gamma^\infty \in \operatorname{Tan}(\gamma; t_0)$ for some $t_0 \in (-T, T)$. Then γ^∞ is horizontal, parametrized by arclength and, when restricted to any compact interval, it is length minimizing in the tangent Carnot–Carathéodory structure $(M^\infty, \mathscr{X}^\infty)$.

Proof. We can assume $t_0 = 0$. We use exponential coordinates of the first kind centered at $\gamma(0)$. Given any $\overline{T} > 0$, for some sequence $\lambda_h \to \infty$ we have

$$\gamma^{\lambda_h}(t) := \delta_{\lambda_h} \gamma(t/\lambda_h) \to \gamma^\infty(t) \text{ in } L^\infty([-\overline{T}, \overline{T}]). \tag{3.28}$$

Up to a subsequence, we can assume that the functions $h(t/\lambda_h)$ weakly converge in $L^2([-\overline{T}, \overline{T}]; \mathbb{R}^r)$ to some $h^\infty \in L^2([-\overline{T}, \overline{T}]; \mathbb{R}^r)$ such that $|h^\infty| \leq 1$ almost everywhere. Then, using (3.26), we have

$$\gamma^\infty(t) = \lim_{h\to\infty} \int_0^t \sum_{i=1}^r h_i(s/\lambda_h) Y_i^{\lambda_h}(\gamma^{\lambda_h}(s))\, ds = \int_0^t \sum_{i=1}^r h_i^\infty Y_i^\infty(\gamma^\infty(s))\, ds,$$

so γ^∞ is $(M^\infty, \mathscr{X}^\infty)$-horizontal and, denoting by d^∞ the Carnot–Carathéodory distance on M^∞ induced by the family $\mathscr{X}^\infty$, its length satisfies

$$d^\infty(\gamma^\infty(-\overline{T}), \gamma^\infty(\overline{T})) \leq L^\infty\Big(\gamma^\infty|_{[-\overline{T}, \overline{T}]}\Big) = \int_{-\overline{T}}^{\overline{T}} |h^\infty|\, dt \leq 2\overline{T}. \tag{3.29}$$

We will see that, in fact, the converse inequality $d^\infty(\gamma^\infty(-\overline{T}), \gamma^\infty(\overline{T})) \geq 2\overline{T}$ holds as well, thus proving that γ^∞ is length minimizing on $[-\overline{T}, \overline{T}]$ and parametrized by arclength (with control h^∞).

Let $\kappa^\infty : [-\overline{T}, \overline{T}] \to \mathbb{R}^n$ be an $(M^\infty, \mathscr{X}^\infty)$-horizontal curve such that $\kappa^\infty(\pm\overline{T}) = \gamma^\infty(\pm\overline{T})$, with control $k^\infty \in L^\infty([-\overline{T}, \overline{T}]; \mathbb{R}^n)$. For all h large enough, the ordinary differential equation

$$\dot\kappa^{\lambda_h}(t) = \sum_{i=1}^r k_i^\infty(t) Y_i^{\lambda_h}(\kappa^{\lambda_h}(t)) \tag{3.30}$$

with initial condition $\kappa^{\lambda_h}(-\overline{T}) = \kappa^\infty(-\overline{T})$ has a (unique) solution defined on $[-\overline{T}, \overline{T}]$. Indeed, let K be a compact neighborhood of $\kappa^\infty([-\overline{T}, \overline{T}])$. For any $\varepsilon > 0$ we have $\|Y_i^{\lambda_h} - Y_i^\infty\|_{L^\infty(K)} \le \varepsilon$ eventually. If $-\overline{T} \in I \subseteq [-\overline{T}, \overline{T}]$ is the maximal (compact) subinterval such that κ^{λ_h} is defined on I and $\kappa^{\lambda_h}(I) \subseteq K$, we have

$$|\dot\kappa^{\lambda_h} - \dot\kappa^\infty| \le C\varepsilon + C\sum_i |Y_i^\infty(\kappa^{\lambda_h}) - Y_i^\infty(\kappa^\infty)| \le C\varepsilon + C|\kappa^{\lambda_h} - \kappa^\infty|$$

on I, for some C depending on $\|k^\infty\|_{L^\infty}$ and $\|\nabla Y_i^\infty\|_{L^\infty(K)}$. Hence, by Gronwall's inequality, $|\kappa^{\lambda_h} - \kappa^\infty| \le C\varepsilon$ on I. If ε is small enough, we deduce that $\kappa^{\lambda_h}(\max I)$ belongs to the interior of K, so $I = [-\overline{T}, \overline{T}]$. Since ε was arbitrary, we also get

$$\lim_{h\to\infty} \kappa^{\lambda_h}(\pm\overline{T}) = \kappa^\infty(\pm\overline{T}) = \gamma^\infty(\pm\overline{T}) = \lim_{h\to\infty} \gamma^{\lambda_h}(\pm\overline{T}). \tag{3.31}$$

From the length minimality of γ^{λ_h} in $(\delta_{\lambda_h}(V), \mathscr{X}^{\lambda_h})$ it follows that

$$\begin{aligned} 2\overline{T} = L^{\lambda_h}\Big(\gamma^{\lambda_h}\big|_{[-\overline{T},\overline{T}]}\Big) &\le L^{\lambda_h}(\kappa^{\lambda_h}) + d^{\lambda_h}\Big(\kappa^{\lambda_h}(-\overline{T}), \gamma^{\lambda_h}(-\overline{T})\Big) + d^{\lambda_h}\Big(\kappa^{\lambda_h}(\overline{T}), \gamma^{\lambda_h}(\overline{T})\Big) \\ &= \int_{-\overline{T}}^{\overline{T}} |k^\infty(t)|\, dt + \lambda_h d\Big(\delta_{1/\lambda_h}\kappa^{\lambda_h}(-\overline{T}), \delta_{1/\lambda_h}\gamma^{\lambda_h}(-\overline{T})\Big) \\ &\quad + \lambda_h d\Big(\delta_{1/\lambda_h}\kappa^{\lambda_h}(\overline{T}), \delta_{1/\lambda_h}\gamma^{\lambda_h}(\overline{T})\Big). \end{aligned}$$

By Lemma 2.4 and (3.31), we have

$$\lim_{h\to\infty} \lambda_h d(\delta_{1/\lambda_h}\kappa^{\lambda_h}(\pm\overline{T}), \delta_{1/\lambda_h}\gamma^{\lambda_h}(\pm\overline{T})) = 0.$$

Hence, $2\overline{T} \le \int_{-\overline{T}}^{\overline{T}} |k^\infty(t)|\, dt = L^\infty(\kappa^\infty)$. Since κ^∞ was arbitrary, we conclude that $d^\infty(\gamma^\infty(-\overline{T}), \gamma^\infty(\overline{T})) \ge 2\overline{T}$. □

The following fact is a special case of the general principle according to which the tangent to the tangent is (contained in the) tangent.

Proposition 3.7. Let $\gamma : [-T, T] \to M$ be a horizontal curve and $t \in (-T, T)$. If $\kappa \in \mathrm{Tan}(\gamma; t)$ and $\widehat\kappa \in \mathrm{Tan}(\kappa; 0)$, then $\widehat\kappa \in \mathrm{Tan}(\gamma; t)$.

Proof. We can assume without loss of generality that $t = 0$. We use exponential coordinates of the first kind centered at $\gamma(0)$. Let $N > 0$ be fixed. Since $\widehat\kappa \in \mathrm{Tan}(\kappa; 0)$, there exists an infinitesimal sequence $\xi_k \downarrow 0$ such that, for all $t \in [-N, N]$ and $k \in \mathbb{N}$, we have

$$\|\widehat\kappa(t) - \delta_{1/\xi_k}\kappa(\xi_k t)\| \le \frac{1}{2^k}.$$

Since $\kappa \in \mathrm{Tan}(\gamma; 0)$, there exists an infinitesimal sequence $\eta_k \downarrow 0$ such that, for all $t \in [-N, N]$ and $k \in \mathbb{N}$, we have

$$\|\kappa(\xi_k t) - \delta_{1/\eta_k}\gamma(\eta_k \xi_k t)\| \leq \frac{\xi_k}{2^k}.$$

It follows that for the infinitesimal sequence $\sigma_k := \xi_k \eta_k$ we have, for all $t \in [-N, N]$,

$$\|\widehat{\kappa}(t) - \delta_{1/\sigma_k}\kappa(\sigma_k t)\| \leq \|\widehat{\kappa}(t) - \delta_{1/\xi_k}\kappa(\xi_k t)\| + \|\delta_{1/\xi_k}\kappa(\xi_k t) - \delta_{1/\sigma_k}\gamma(\sigma_k t)\| \leq \frac{1}{2^{k-1}}.$$

The thesis now follows by a diagonal argument. □

When $\gamma : [0, T] \to M$, there are analogous versions of Propositions 3.6 and 3.7 for $\mathrm{Tan}^+(\gamma; 0)$ and $\mathrm{Tan}^-(\gamma; T)$.

Proposition 3.8. Let $\kappa : \mathbb{R} \to M^\infty$ be a horizontal curve in $(M^\infty, \mathscr{X}^\infty)$. The following statements are equivalent:

(i) there exist $c_1, \dots, c_r \in \mathbb{R}$ such that $\dot{\kappa} = \sum_{i=1}^r c_i Y_i^\infty(\kappa)$ and $\kappa(0) = 0$;
(ii) there exists $x_0 \in M^\infty$ such that $\kappa(t) = \delta_t(x_0)$ (here δ_t is defined by (2.4) also for $t < 0$).

Proof. We prove (i)⇒(ii). Since $(\delta_\lambda)_* Y_i^\infty = \lambda Y_i^\infty$ for $\lambda \neq 0$, the curve $\delta_\lambda \circ \kappa(\cdot/\lambda)$ satisfies the same differential equation, so $\delta_\lambda \circ \kappa(t/\lambda) = \kappa(t)$; choosing $\lambda = t$ we get $\kappa(t) = \delta_t(\kappa(1))$.

We check (ii)⇒(i). Up to rescaling time, we can assume that $\dot{\kappa}(1)$ exists and is a linear combination of $Y_1^\infty(\kappa(1)), \dots, Y_r^\infty(\kappa(1))$, so $\dot{\kappa}(1) = \sum_i \overline{h}_i Y_i^\infty(\kappa(1))$ for some $\overline{h} \in \mathbb{R}^r$. If h is the control of κ, for a.e. s we have

$$\sum_{i=1}^r \overline{h}_i Y_i^\infty(\kappa(1)) = \dot{\kappa}(1) = s\frac{d}{dt}\kappa(t/s)\Big|_{t=s} = s\frac{d}{dt}(\delta_{1/s} \circ \kappa(t))\Big|_{t=s} = \sum_{i=1}^r h_i(s) Y_i^\infty(\kappa(1)),$$

again because $s(\delta_{1/s})_* Y_i^\infty = Y_i^\infty$. Since $Y_1^\infty, \dots, Y_r^\infty$ are pointwise linearly independent (see Proposition 2.5), we get $h = \overline{h}$ a.e. □

Definition 3.9. We say that a horizontal curve κ in $(M^\infty, \mathscr{X}^\infty)$ is a *horizontal line* (through 0) if one of the conditions (i)–(ii) of Proposition 3.8 holds.

The definition of *positive and negative half-line* is similar, the formulas above being required to hold for $t \geq 0$ and $t \leq 0$, respectively.

Remark 3.10. Let us observe the following fact. Let $\gamma : [-T, T] \to M$ be a length minimizer parametrized by arclength with control $h = (h_1, \dots, h_r)$ and let $t \in (-T, T)$ be fixed. Then, the tangent cone $\mathrm{Tan}(\gamma; t)$ contains a horizontal line κ in M^∞ if and only if there exist an infinitesimal sequence $\eta_i \downarrow 0$ and a constant unit vector $c \in S^{r-1}$ such that

$$h(t + \eta_i \,\cdot) \to c \qquad \text{in } L^2_{loc}(\mathbb{R}).$$

As usual, an analogous version holds for $\mathrm{Tan}^+(\gamma;0)$ and $\mathrm{Tan}^-(\gamma;T)$ in case γ is a length minimizer parametrized by arclength on the interval $[0,T]$.

Let us prove our claim; we can set $t=0$. Assume that there exists a sequence $\eta_i \downarrow 0$ such that the curves $\gamma^i(\tau) := \delta_{1/\eta_i}\varphi(\gamma(\eta_i\tau))$ converge locally uniformly to a horizontal line κ in the tangent CC structure $(M^\infty, \mathscr{X}^\infty)$; we have

$$\gamma^i(\tau) = \int_0^\tau \sum_{j=1}^r h_j(\eta_i s) Y_j^{1/\eta_i}(\gamma^i(s))\, ds.$$

Up to subsequences we have $h(\eta_i\,\cdot) \rightharpoonup h_\infty$ in $L^2_{loc}(\mathbb{R})$, with $\|h_\infty\|_{L^\infty}\le 1$. Since $Y_j^{1/\eta_i} \to Y_j^\infty$ locally uniformly, we obtain

$$\kappa(\tau) = \int_0^\tau \sum_{j=1}^r h_\infty(s) Y_j^\infty(\kappa(s))\, ds.$$

By Proposition 3.6, κ is parametrized by arclength. So $|h_\infty|=1$ a.e. and, since κ is a horizontal line, h_∞ is constant. Finally, for any compact set $K\subset\mathbb{R}$, we trivially have $\|h(\eta_i\,\cdot)\|_{L^2(K)}\to \|h_\infty\|_{L^2(K)}$, which gives $h(\eta_i\,\cdot)\to h_\infty$ in $L^2(K)$. The reverse implication (if $h(t+\eta_i\,\cdot)\to c$ in $L^2_{loc}(\mathbb{R})$, then $\mathrm{Tan}(\gamma;t)$ contains a horizontal line) follows a similar argument.

4. Lifting the tangent structure to a free Carnot group

In this section we show how a tangent CC structure $(M^\infty, \mathscr{X}^\infty)$ can be lifted to a free Carnot group F, by means of a desingularization process. We also show that length minimizers in M^∞ lift to length minimizers in F.

Let $(M^\infty, \mathscr{X}^\infty)$ be a tangent CC structure as in Section 2. The Lie algebra $\mathfrak{g}$ generated by $\mathscr{X}^\infty = (Y_1^\infty,\dots,Y_r^\infty)$ is nilpotent because, by Proposition 2.5, any iterated commutator of length greater than s vanishes. The identity $(\delta_\lambda)_* Y_i^\infty = \lambda Y_i^\infty$ implies that $(\delta_\lambda)_* X \to 0$ pointwise as $\lambda\to 0$, for any $X\in\mathfrak{g}$. We deduce that the j-th component of X is a polynomial function depending only on the previous variables. It follows that the flow $(x,t)\mapsto \exp(tX)(x)$ is a polynomial function in $(x,t)\in M^\infty\times\mathbb{R}$ and X is therefore complete.

Let $\mathfrak{f}$ be the free Lie algebra of rank r and step s, with generators $W_1,\dots,W_r$. The connected, simply connected Lie group F with Lie algebra $\mathfrak{f}$ can be constructed explicitly as follows: we let $F := \mathfrak{f}$ and we endow F with the group operation $A\cdot B := P(A,B)$, where

$$P(A,B) = \sum_{p=1}^s \frac{(-1)^{p+1}}{p} \sum_{1\le k_i+\ell_i\le s} \frac{[A^{k_1}, B^{\ell_1},\dots,A^{k_p},B^{\ell_p}]}{k_1!\cdots k_p!\,\ell_1!\cdots\ell_p!\sum_i(k_i+\ell_i)}. \tag{4.32}$$

This is a finite truncation of the series in (2.8): the omitted terms vanish by the nilpotency of $\mathfrak{f}$. One readily checks that $P(A,0)=P(0,A)=A$ and $P(A,-A)=$

$P(-A, A) = 0$, while the associativity identity $P(P(A, B), C) = P(A, P(B, C))$ is shown in [5, Sec. X.2] for free Lie algebras and can be deduced for $\mathfrak{f}$ by truncation. For any $A \in F$, $t \mapsto tA$ is a one-parameter subgroup. From this, it is straightforward to check that $\mathfrak{f}$ identifies with the Lie algebra of F, with $\exp : \mathfrak{f} \to F$ given by the identity map. In particular, $\exp : \mathfrak{f} \to F$ is a diffeomorphism and we have

$$\exp(A)\exp(B) = \exp(P(A, B)), \quad A, B \in \mathfrak{f}. \tag{4.33}$$

The group F is a *Carnot group*, which means that it is a connected, simply connected and nilpotent Lie group whose Lie algebra is stratified, i.e., it has an assigned decomposition $\mathfrak{f} = \mathfrak{f}_1 \oplus \cdots \oplus \mathfrak{f}_s$ satisfying $[\mathfrak{f}_1, \mathfrak{f}_{i-1}] = \mathfrak{f}_i$ and $[\mathfrak{f}, \mathfrak{f}_s] = \{0\}$ (in this case $\mathfrak{f}_1$ is the linear span of $W_1, \dots, W_r$). The group F just constructed is called the *free Carnot group of rank r and step s*.

Proposition 4.1. The group F is generated by $\exp(\mathfrak{f}_1)$.

Proof. See [3, Lemma 1.40]. □

By the nilpotency of $\mathfrak{g}$, there exists a unique homomorphism $\psi : \mathfrak{f} \to \mathfrak{g}$ such that $\psi(W_i) = Y_i^\infty \in \mathfrak{g}$ for $i = 1, \dots, r$. The group F acts on M^∞ on the right. The action $M^\infty \times F \to M^\infty$ is given by $(x, f) \mapsto x \cdot f := \exp(\psi(A))(x)$, where $f = \exp(A)$. In fact, by (4.33), for any $f' = \exp(B)$ we have

$$x \cdot (ff') = \exp(P(\psi(A), \psi(B)))(x) = \exp(\psi(B)) \circ \exp(\psi(A))(x) = (x \cdot f) \cdot f'. \tag{4.34}$$

The second equality is a consequence of the formula $\exp(P(tY, tX))(x) = \exp(tX) \circ \exp(tY)(x)$ for $X, Y \in \mathfrak{g}$ (with P given by (4.32)), which holds since both sides are polynomial functions in t, with the same Taylor expansion (by (2.7)). We define the map

$$\pi^\infty : F \to M^\infty, \quad \pi^\infty(f) := 0 \cdot f,$$

where the dot stands for the right action of F on M^∞.

Let $\mathscr{W} := \{W_1, \dots, W_r\}$ and extend $\mathscr{W}$ to a basis $W_1, \dots, W_N$ of $\mathfrak{f}$ adapted to the stratification. Via the exponential map $\exp : \mathfrak{f} \to F$, the one-parameter group of automorphisms of $\mathfrak{f}$ defined by $W_k \mapsto \lambda^i W_k$ if and only if $W_k \in \mathfrak{f}_i$ induces a one-parameter group of automorphisms $(\widehat{\delta}_\lambda)_{\lambda>0}$ of F, called *dilations*.

If $A \in \mathfrak{f}_1$, for any $\lambda > 0$ and $x \in M^\infty$ we have the identity

$$\exp(\lambda\psi(A))(\delta_\lambda(x)) = \delta_\lambda(\exp(\psi(A))(x)), \tag{4.35}$$

which follows from $(\delta_\lambda)_*\psi(A) = \lambda\psi(A)$.

Definition 4.2. We call the CC structure $(F, \mathscr{W})$ the *lifting* of $(M^\infty, \mathscr{X}^\infty)$ with projection $\pi^\infty : F \to M^\infty$.

Proposition 4.3. The lifting $(F, \mathscr{W})$ of $(M^\infty, \mathscr{X}^\infty)$ has the following properties:

(i) for any $f \in F$ and $i = 1, \dots, r$ we have $\pi^\infty_*(W_i(f)) = Y_i^\infty(\pi^\infty(f))$;

(ii) the dilations of F and M^∞ commute with the projection: namely, for any $\lambda > 0$ we have
$$\pi^\infty \circ \widehat{\delta}_\lambda = \delta_\lambda \circ \pi^\infty.$$

Proof. (i) Using the action property (4.34), we find
$$\begin{aligned}\pi^\infty_*(W_i(f)) &= \frac{d}{dt}\pi^\infty(f\exp(tW_i))\Big|_{t=0} = \frac{d}{dt}0\cdot(f\exp(tW_i))\Big|_{t=0}\\ &= \frac{d}{dt}\pi^\infty(f)\cdot\exp(tW_i)\Big|_{t=0} = \psi(W_i)(\pi^\infty(f)) = Y_i^\infty(\pi^\infty(f)).\end{aligned}$$

(ii) Let $\lambda > 0$ and $x \in M^\infty$. By (4.35), for any $W \in \mathfrak{f}_1$ we have
$$\delta_\lambda(x)\cdot\exp(\lambda W) = \exp(\lambda\psi(W))(\delta_\lambda(x)) = \delta_\lambda(\exp(\psi(W))(x)) = \delta_\lambda(x\cdot\exp(W)). \tag{4.36}$$
We deduce that the claim holds for any $f = \exp(W)$ with $W \in \mathfrak{f}_1$, because
$$\pi^\infty(\widehat{\delta}_\lambda(f)) = \pi^\infty(\exp(\lambda W)) = \delta_\lambda(0)\cdot\exp(\lambda W) = \delta_\lambda(0\cdot\exp(W)) = \delta_\lambda(\pi^\infty(f)).$$

By Proposition 4.1, any $f \in F$ is of the form $f = f_1 f_2 \dots f_k$ with each $f_i \in \exp(\mathfrak{f}_1)$. Assume by induction that the claim holds for $\widehat{f} = f_1 f_2 \dots f_{k-1}$. By (4.36), letting $f_k = \exp(W)$ we have
$$\begin{aligned}\pi^\infty(\widehat{\delta}_\lambda(f)) &= \pi^\infty(\widehat{\delta}_\lambda(\widehat{f})\exp(\lambda W)) = \pi^\infty(\widehat{\delta}_\lambda(\widehat{f}))\cdot\exp(\lambda W)\\ &= \delta_\lambda(\pi^\infty(\widehat{f}))\cdot\exp(\lambda W) = \delta_\lambda(\pi^\infty(\widehat{f})\cdot\exp(W)) = \delta_\lambda(\pi^\infty(f)).\end{aligned}$$
□

Let $\kappa : I \to M^\infty$ be a horizontal curve in $(M^\infty, \mathscr{X}^\infty)$, with control $h \in L^\infty(I, \mathbb{R}^r)$. A horizontal curve $\overline{\kappa} : I \to F$ such that
$$\kappa = \pi^\infty \circ \overline{\kappa} \qquad \text{and} \qquad \dot{\overline{\kappa}}(t) = \sum_{i=1}^r h_i(t) W_i(\overline{\kappa}(t)) \quad \text{for a.e. } t \in I$$
is called a *lift* of κ to $(F, \mathscr{W})$.

Proposition 4.4. Let $(F, \mathscr{W})$ be the lifting of $(M^\infty, \mathscr{X}^\infty)$ with projection $\pi^\infty : F \to M^\infty$. Then the following facts hold:

(i) If κ is length minimizing in $(M^\infty, \mathscr{X}^\infty)$, then any horizontal lift $\overline{\kappa}$ of κ is length minimizing in $(F, \mathscr{W})$.

(ii) If $\overline{\kappa}$ is a horizontal (half-)line in F, then $\pi^\infty \circ \overline{\kappa}$ is a horizontal (half-)line in $(M^\infty, \mathscr{X}^\infty)$.

Proof. Claim (i) follows from $L(\overline{\kappa}) = L(\kappa)$ and from the inequality $L(\overline{\kappa}') = L(\kappa') \geq L(\kappa)$, whenever $\overline{\kappa}'$ is horizontal with the same endpoints as $\overline{\kappa}$ and $\kappa' = \pi^\infty \circ \overline{\kappa}'$. We now turn to Claim (ii). Let $\overline{\kappa}(t) = \exp(tW)$ for some $W \in \mathfrak{f}_1$. The projection $\pi^\infty \circ \overline{\kappa}$ is horizontal by part (i) of Proposition 4.3. The thesis follows from characterization (i) for horizontal lines, contained in Proposition 3.8.

□

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