

LONG RANGE SCATTERING FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH CRITICAL HOMOGENEOUS NONLINEARITY

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ABSTRACT. In this paper, we consider the final state problem for the nonlinear Schrödinger equation with a homogeneous nonlinearity which is of the long range critical order and is not necessarily a polynomial, in one and two space dimensions. As the nonlinearity is the critical order, the possible asymptotic behavior depends on the shape of the nonlinearity. The aim here is to give a sufficient condition on the nonlinearity to construct a modified wave operator. To deal with a non-polynomial nonlinearity, we decompose it into a resonant part and a non-resonant part via the Fourier series expansion. Our sufficient condition is then given in terms of the Fourier coefficients. In particular, we need to pay attention to the decay of the Fourier coefficients since the non-resonant part is an infinite sum in general.

1. INTRODUCTION

This paper is devoted to the study of long time behavior of solutions to the nonlinear Schrödinger equation

$$(NLS) \quad i\partial_t u + \Delta u = F(u).$$

Here, $(t, x) \in \mathbb{R}^{1+d}$ ($d = 1, 2$) and $u = u(t, x)$ is a \mathbb{C} -valued unknown function. We suppose that the nonlinearity F is homogeneous of degree $1 + 2/d$, that is, F satisfies

$$(1.1) \quad F(\lambda u) = \lambda^{1+\frac{2}{d}} F(u)$$

for any $\lambda > 0$ and $u \in \mathbb{C}$. Our aim here is to determine the asymptotic behavior of nontrivial small solutions to (NLS) with a general homogeneous nonlinearity. More specifically, we give a sufficient condition on the nonlinearity F to construct a modified wave operator.

A typical example of nonlinearity satisfying (1.1) is a gauge-invariant power type nonlinearity

$$(1.2) \quad F(u) = \mu |u|^{\frac{2}{d}} u,$$

where $\mu \in \mathbb{R} \setminus \{0\}$. As for the nonlinearity of the form $\mu |u|^p u$, the power $p = 2/d$ is known to be a threshold. The equation (NLS) with the nonlinearity $|u|^p u$ admits a nontrivial solution asymptotically behaves like a free solution for large time when $p > 2/d$. However, in the case $p = 2/d$, there is no nontrivial solution to the equation (NLS) with (1.2) belongs to L^2 and

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scatters in L^2 (see [1, 18, 20]). It is shown in [2, 15] that when the nonlinearity is of the form (1.2) then for given *final data* u_+ the equation admits a solution which asymptotically behaves like

$$(1.3) \quad u_p(t) = (2it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} \widehat{u_+} \left(\frac{x}{2t} \right) \exp \left(-i\mu \left| \widehat{u_+} \left(\frac{x}{2t} \right) \right|^{\frac{2}{d}} \log t \right)$$

as $t \rightarrow \infty$, where $\widehat{u_+}$ is the Fourier transform of u_+ .

When the nonlinearity is homogeneous of the critical order, asymptotic behavior of a solution depends on the shape of the nonlinearity. In [12], it is shown that if the nonlinearity is $F(u) = u^{2/d+1}$ then (NLS) admits an asymptotic free solution which is a solution behaves like

$$(1.4) \quad u_p = (2it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} \widehat{u_+} \left(\frac{x}{2t} \right).$$

Remark that this is nothing but the asymptotic behavior of the free solution $e^{it\Delta} u_+$. Hence, the behavior in this case is similar to the case $|u|^p u$ ($p > 2/d$). Let us now introduce the following terminology: We say a nonlinearity is *short range* if (NLS) admits a nontrivial solution behaves like (1.4), and is *long range* if (NLS) admits a nontrivial solution behaves like (1.3) with a suitable $\mu \in \mathbb{R} \setminus \{0\}$. It is shown in [8, 9, 17] that the nonlinearity $\mu|u|^{2/d}u + N_d(u)$ is short range if $\mu = 0$ and long range if $\mu \neq 0$, where

$$N_1(u) = \lambda_1 u^3 + \lambda_2 |u|^2 \bar{u} + \lambda_3 \bar{u}^3$$

if $d = 1$ and

$$N_2(u) = \lambda_1 u^2 + \lambda_2 \bar{u}^2$$

if $d = 2$, $\lambda_j \in \mathbb{C}$, and $\mu \in \mathbb{R} \setminus \{0\}$. Furthermore, if $\mu \neq 0$ then the asymptotic behavior of a solution is given by (1.3). Thus, the gauge-invariant term determines the asymptotic behavior.

In this paper, we handle more general nonlinearity satisfying (1.1) and give a sufficient condition on nonlinearity to become short range or long range. A special example in our mind is

$$(1.5) \quad F(u) = |\operatorname{Re} u| \operatorname{Re} u,$$

which satisfies (1.1) for $d = 2$. The nonlinearity appears, for instance, as a main part of a generalized version of Gross-Pitaevskii equation introduced in [10]. We restrict our attention to a solution corresponding to a given final data which has very small low-frequency part. We remark that if a final data has non-negligible low-frequency part then other kinds of asymptotic behavior take place (see [3–7, 13, 14]).

With the example (1.5), let us explain the main point of our argument to treat general homogeneous nonlinearity. To compare with, let us first consider the nonlinearity $F(u) = |\operatorname{Re} u|^2 \operatorname{Re} u$ in $d = 1$. As for the nonlinearity, a simple calculation shows

$$|\operatorname{Re} u|^2 \operatorname{Re} u = \left(\frac{u + \bar{u}}{2} \right)^3 = \frac{3}{8} |u|^2 u + \frac{1}{8} u^3 + \frac{3}{8} |u|^2 \bar{u} + \frac{1}{8} \bar{u}^3$$

Hence, this is of the form $F(u) = (3/8)|u|^2 u + N_1(u)$ and so it is included in the previous results [8, 9, 17]. One sees that the (NLS) admits a solution asymptotically behaves like (1.3) with $\mu = 3/8$. The term $\frac{3}{8}|u|^2 u$ is a resonant term which determines the asymptotic behavior, and the other terms

are non-resonant terms. We would emphasize that the non-resonant part is a *finite sum*. On the contrary, the resonant term of (1.5) is not extracted by such a simple calculation. Hence, we use the Fourier series expansion to obtain

$$|\operatorname{Re} u| \operatorname{Re} u = \frac{4}{3\pi} |u| u + \sum_{m \neq 0} \frac{4(-1)^{m+1}}{\pi(2m-1)(2m+1)(2m+3)} |u|^{-2m+1} u^{2m+1}.$$

A remarkable point is that the non-resonant part consists of *infinitely many terms*. The question now arises whether decay of Fourier coefficients in n is enough to sum up. One main respect of the present paper is to establish a sufficient condition to handle the non-resonant part. The condition is given in terms of the Fourier coefficients of the nonlinearity. It will turn out that $|\operatorname{Re} u| \operatorname{Re} u$ is long range and (NLS) admits a solution which has the asymptotic (1.3) with $\mu = 4/3\pi$.

To state our result precisely, we introduce some notation. A homogeneous nonlinearity is written as

$$(1.6) \quad F(u) = |u|^{1+\frac{2}{d}} F\left(\frac{u}{|u|}\right).$$

We introduce a 2π -periodic function $g(\theta)$ by

$$(1.7) \quad g(\theta) = F(e^{i\theta})$$

We identify a homogeneous nonlinearity F satisfying (1.1) with a 2π -periodic function g through (1.6) and (1.7). Namely, given nonlinearity F , we give a 2π -periodic function $g(\theta) = g_F(\theta)$ by the above procedure. Conversely, for a given 2π -periodic function g , we can construct a homogeneous nonlinearity $F = F_g : \mathbb{C} \rightarrow \mathbb{C}$ by

$$F_g(u) = \begin{cases} |u|^{1+\frac{2}{d}} g(\arg u), & u \neq 0, \\ 0 & u = 0. \end{cases}$$

We now apply the Fourier series expansion. Since $g(\theta)$ is 2π -periodic function, it holds, at least formally, that $g(\theta) = \sum_{n \in \mathbb{Z}} g_n e^{in\theta}$ with the coefficients

$$(1.8) \quad g_n := \frac{1}{2\pi} \int_0^{2\pi} g(\theta) e^{-in\theta} d\theta.$$

This expansion gives us

$$F(u) = |u|^{\frac{2}{d}+1} \sum_{n \in \mathbb{Z}} g_n \left(\frac{u}{|u|}\right)^n = \sum_{n \in \mathbb{Z}} g_n |u|^{1+\frac{2}{d}-n} u^n,$$

by means of (1.6) and (1.7). We then write

$$(1.9) \quad F(u) = g_0 |u|^{\frac{2}{d}+1} + g_1 |u|^{\frac{2}{d}} u + \sum_{n \neq 0,1} g_n |u|^{1+\frac{2}{d}-n} u^n.$$

The extraction of a resonant term via Fourier expansion is motivated by [11, 19]. We also remark that the Fourier coefficients are represented as the integral $g_n = \frac{1}{2\pi i} \int_{|z|=1} F(z) \frac{dz}{z^{1+n}}$, some of which are used in previous works such as [16].

In this paper, we suppose the following.

Assumption 1.1. Assume that F is a homogeneous nonlinearity such that a corresponding 2π -periodic function $g(\theta)$ satisfies

$$g_0 := \frac{1}{2\pi} \int_0^{2\pi} g(\theta) d\theta = 0, \quad g_1 := \frac{1}{2\pi} \int_0^{2\pi} g(\theta) e^{-i\theta} d\theta \in \mathbb{R},$$

and $\sum_{n \in \mathbb{Z}} |n|^{1+\eta} |g_n| < \infty$ for some $\eta > 0$, where g_n is given in (1.8). In particular, g is Lipschitz continuous.

1.1. Main result. Set $\langle a \rangle = (1 + |a|^2)^{1/2}$ for any $a \in \mathbb{C}$. The weighted Sobolev space on \mathbb{R}^d is defined by $H^{m,s} = \{u \in \mathcal{S}'(\mathbb{R}^d); \langle i\nabla \rangle^m \langle x \rangle^s u \in L^2\}$, and $\dot{H}^m = \{u \in \mathcal{S}'(\mathbb{R}^d); (-\Delta)^{\frac{m}{2}} u \in L^2\}$ denotes the homogeneous Sobolev space on \mathbb{R}^d , where $m, s \in \mathbb{R}$. Let us simply write $H^m = H^{m,0}$. Let $\|g\|_{\text{Lip}} := \sup_{\theta \neq \theta'} |g(\theta) - g(\theta')|/|\theta - \theta'|$ be the Lipschitz norm.

Our main result is as follows:

Theorem 1.2. Suppose that the nonlinearity F satisfies Assumption 1.1 for some $\eta > 0$. Fix $\delta \in (d/2, (d+1)/2)$ so that $\delta - d/2 < 2\eta$. Let $\gamma = \delta/2$ if $d = 1$ and $\gamma = (\delta+2)/6$ if $d = 2$. Take $b \in (d/4, \gamma)$. Then, there exists $\varepsilon_0 = \varepsilon_0(b, \|g\|_{\text{Lip}})$ such that for any $u_+ \in H^{0,d} \cap \dot{H}^{-\delta}$ satisfying $\|\widehat{u}_+\|_{L^\infty} < \varepsilon_0$, there exists $T > 0$ and a unique solution $u \in C([T, \infty); L^2(\mathbb{R}^d))$ of (NLS) such that

$$\sup_{t \in [T, \infty)} t^b \|u(t) - u_p(t)\|_{L^2} + \sup_{t \in [T, \infty)} t^b \left(\int_t^\infty \|u(s) - u_p(s)\|_{X_d}^4 ds \right)^{\frac{1}{4}} < \infty,$$

where

$$u_p(t) = (2it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} \widehat{u}_+ \left(\frac{x}{2t} \right) \exp \left(-ig_1 \left| \widehat{u}_+ \left(\frac{x}{2t} \right) \right|^{\frac{2}{d}} \log t \right),$$

$X_1 = L^\infty(\mathbb{R})$ and $X_2 = L^4(\mathbb{R}^2)$.

Remark 1.3. Our theorem include the example (1.5) in $d = 2$. The corresponding periodic function is $g(\theta) = |\cos \theta| \cos \theta$ and so

$$g_n = \begin{cases} -\frac{4}{\pi(n-2)n(n+2)} \sin \frac{\pi n}{2} & n: \text{ odd}, \\ 0 & n: \text{ even}. \end{cases}$$

Remark that it satisfies Assumption 1.1 for $0 < \eta < 1$.

Remark 1.4. The regularity assumption on the data is similar to that is made in [8] and stronger than in [9]. This is because we use regularity of the data to weaken the condition on the nonlinearity. Hence, if F is a sufficiently good one, for instance if it satisfies Assumption 1.1 with $\eta \geq d$, then the regularity assumption can be taken the same as in [9] by their argument.

Remark 1.5. Theorem 1.2 implies that (NLS) admits a nontrivial asymptotic free solution when F satisfies Assumption 1.1 and $g_1 = 0$. For example, $F(u) = |\operatorname{Re} u| \operatorname{Re} u - i|\operatorname{Im} u| \operatorname{Im} u$, $d = 2$, is short range. Indeed, the corresponding periodic function is $g(\theta) = |\cos \theta| \cos \theta - i|\sin \theta| \sin \theta$ and so

$$g_n = \begin{cases} \frac{8}{\pi(n-2)n(n+2)}, & n \equiv 3 \pmod{4}, \\ 0, & \text{otherwise.} \end{cases}$$

The rest of the paper is organized as follows. In the next section, we outline the proof. Then, it will turn out that the main step of the proof is the estimate of non-resonant part (Proposition 2.3). After summarizing several useful estimates in Section 3, we prove Proposition 2.3 in Section 4. Main theorem is then shown in Section 5.

2. OUTLINE OF THE PROOF

By the decomposition (1.9) and Assumption 1.1, we write

$$F(u) = g_1|u|^{\frac{2}{d}}u + \sum_{n \neq 0,1} g_n|u|^{\frac{2}{d}-n}u^n.$$

Denote

$$\mathcal{G}_d(u) = g_1|u|^{\frac{2}{d}}u, \quad \mathcal{N}_d(u) = \sum_{n \neq 0,1} g_n|u|^{1+\frac{2}{d}-n}u^n.$$

The heart of matter is that the expansion (1.9) successfully extracts a “resonant part” \mathcal{G}_d which determines the shape of asymptotic behavior u_p . The validity of the extraction is confirmed by proving the other part, a “non-resonant part” \mathcal{N}_d , enjoys better time decay. The decay comes from the fact that the phase of the non-resonant part is different from that of linear part. In the integral form of the equation, it can be seen that this disagreement of phase actually causes a time decay effect (cf. stationary phase). This kind of additional decay was known for the case where \mathcal{N}_d is a specific *finite sum* of $|u|^{1+2/d-n}u^n$ ($n \neq 0, 1$) (see [8, 9]). However, our non-resonant part is an *infinite sum*. In the technical point of view, a contribution of this paper is a treatment of the infinite sum under Assumption 1.1.

We introduce a formulation in [9] (see also [8, 17]). In what follows, we let $t > 1$ unless otherwise stated. Let $U(t) = e^{it\Delta}$. Introduce a multiplication operator $M(t)$ and a dilation operator $D(t)$ by

$$(2.1) \quad M(t) = e^{\frac{i|x|^2}{4t}}, \quad (D(t)f)(x) = (2it)^{-\frac{d}{2}}f\left(\frac{x}{2t}\right).$$

They are isometries on $L^2(\mathbb{R}^d)$. Then,

$$(2.2) \quad u_p(t) = M(t)D(t)\widehat{w}(t), \quad \widehat{w}(t) = \widehat{u_+} \exp(-ig_1|\widehat{u_+}|^{\frac{2}{d}} \log t).$$

We regard the equation (NLS) as

$$\mathcal{L}(u - u_p) = (F(u) - F(u_p)) - \mathcal{L}u_p + \mathcal{G}_d(u_p) + \mathcal{N}_d(u_p),$$

where $\mathcal{L} = i\partial_t + \Delta_x$. A computation shows that it is rewritten as the following integral equation;

$$(2.3) \quad \begin{aligned} u(t) - u_p(t) = & i \int_t^\infty U(t-s) (F(u) - F(u_p))(s) ds \\ & + \mathcal{R}(t)\widehat{w} - i \int_t^\infty U(t-s)\mathcal{R}(s)\mathcal{G}_d(\widehat{w})(s) \frac{ds}{s} \\ & + i \int_t^\infty U(t-s)\mathcal{N}_d(u_p)(s) ds, \end{aligned}$$

where

$$\mathcal{R}(t) = M(t)D(t) \left(U\left(-\frac{1}{4t}\right) - 1 \right)$$

(see [9]).

Let $X_1 = L^\infty(\mathbb{R})$ and $X_2 = L^4(\mathbb{R}^2)$. For $R, T, b > 0$, we define a complete metric space

$$X_{d,T,b,R} := \{v \in C([T, \infty); L^2(\mathbb{R}^d)); \|v - u_p\|_{X_{d,T,b}} \leq R\},$$

$$\|v\|_{X_{d,T,b}} := \sup_{t \in [T, \infty)} t^b \|v(t)\|_{L^2(\mathbb{R}^d)} + \sup_{t \in [T, \infty)} t^b \left(\int_t^\infty \|v(s)\|_{X_d}^4 ds \right)^{\frac{1}{4}},$$

$$d(u, v) := \|u - v\|_{X_{d,T,b}}.$$

We shall show that, under the assumption of Theorem 1.2, there exists $\varepsilon_0 > 0$ such that for any data $u_+ \in H^{0,d} \cap \dot{H}^{-\delta}$ with $\|\widehat{u}_+\|_{L^\infty} \leq \varepsilon_0$, we can choose $R, T > 0$ so that the map

$$\begin{aligned} \Phi(v)(t) &:= u_p(t) + i \int_t^\infty U(t-s) (F(v) - F(u_p))(s) ds \\ (2.4) \quad &+ \mathcal{R}(t) \widehat{w} - i \int_t^\infty U(t-s) \mathcal{R}(s) \mathcal{G}_d(\widehat{w})(s) \frac{ds}{s} \\ &+ i \int_t^\infty U(t-s) \mathcal{N}_d(u_p)(s) ds. \end{aligned}$$

is a contraction map on $X_{d,T,b,R}$.

To this end, we introduce three intermediate results. The first one is a consequence of Strichartz' estimate.

Lemma 2.1. *Let $\widehat{u}_+ \in L^\infty$. Assume that $g(\theta)$ is Lipschitz continuous. If $b > d/4$ then it holds that*

$$\begin{aligned} &\left\| \int_t^\infty U(t-s) (F(v) - F(u_p)) ds \right\|_{X_{d,T,b}} \\ &\leq C \|v - u_p\|_{X_{d,T,b}} \left(\|v - u_p\|_{X_{d,T,b}}^{\frac{2}{d}} T^{\frac{1}{2} - \frac{2}{d}b} + \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \right), \end{aligned}$$

where C depends on the Lipschitz constant of g .

The estimate is essentially the same as in [8, 9, 17]. Remark that Lipschitz continuity of g gives us

$$|F(v) - F(u_p)| \leq C \left(|v - u_p|^{1+\frac{2}{d}} + |u_p|^{\frac{2}{d}} |v - u_p| \right).$$

The detail is given in Appendix A.

The main step is the estimate of “external terms” on the right hand side of (2.4). The second one is due to Hayashi, Wang, and Naumkin [9, Lemma 2.1].

Lemma 2.2. *Let $u_+ \in H^{0,d}$ and $d/2 < \delta < d$. Then, the estimates*

$$\begin{aligned} &\|\mathcal{R}(t) \widehat{w}\|_{L_t^\infty(T, \infty; L^2)} + \|\mathcal{R}(t) \widehat{w}\|_{L_t^4(T, \infty; X_d)} \\ &\leq CT^{-\frac{\delta}{2}} \left\langle g_1 \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \log T \right\rangle^\delta \|u_+\|_{H^{0,d}}, \end{aligned}$$

and

$$\left\| \int_t^\infty U(t-s) \mathcal{R}(s) \mathcal{G}_d(\widehat{w}) \frac{ds}{s} \right\|_{L_t^\infty(T, \infty; L^2)} + \left\| \int_t^\infty U(t-s) \mathcal{R}(s) \mathcal{G}_d(\widehat{w}) \frac{ds}{s} \right\|_{L_t^4(T, \infty; X_d)}$$

$$\leq C|g_1|T^{-\frac{\delta}{2}} \left\langle g_1 \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \log T \right\rangle^\delta \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \|u_+\|_{H^{0,d}}$$

hold for all $T > 1$.

The last one, in which the main technical issue lies, is an estimate on the term $\mathcal{N}_d(u_p)$.

Proposition 2.3. *Let $u_+ \in H^{0,d} \cap \dot{H}^{-\delta}$ with $d/2 < \delta < (d+1)/2$. Assume that $\sum_{n \in \mathbb{Z}} |n|^{1+\eta} |g_n| < \infty$ for some $\eta > \frac{1}{2}(\delta - \frac{d}{2})$. Then, the estimate*

$$\begin{aligned} & \left\| \int_t^\infty U(t-s) \mathcal{N}_d(u_p) ds \right\|_{L_t^\infty(T, \infty; L^2)} + \left\| \int_t^\infty U(t-s) \mathcal{N}_d(u_p) ds \right\|_{L_t^4(T, \infty; X_d)} \\ & \leq CT^{-\gamma} \left\langle g_1 \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \log T \right\rangle^\delta \left\langle g_1 \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \right\rangle \\ & \quad \times \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \|u_+\|_{H^{0,d} \cap \dot{H}^{-\delta}} \sum_{n \neq 0,1} |n|^{1+\eta} |g_n| \end{aligned}$$

holds for all $T > 1$, where $\|u_+\|_{H^{0,d} \cap \dot{H}^{-\delta}} = \|u_+\|_{H^{0,d}} + \|u_+\|_{\dot{H}^{-\delta}}$.

As for the assumption on the nonlinearity, the assumption of Proposition 2.3 is stronger than that of Proposition 2.1 because if g satisfies $\sum_n |n|^{1+\eta} |g_n| < \infty$ then it is Lipschitz continuous. The assumption of the Theorem 1.2 comes from this proposition in order to estimate Sobolev norm of the nonlinearity. Roughly speaking, s time derivative of $|u|^{1+2/d-n} u^n$ produces $O(|n|^s)$. Hence, to weaken the assumption of the nonlinearity we shall use as less derivative as possible. We remark again that we have to pay attention to the above growth order just because we are working with the non-resonant term which consists of infinitely many terms. Our proof is in the same spirit as in [9]. However, the argument in [9] works only for large η . We introduce two techniques to handle small η . Especially, they are necessary to include the example (1.5). The detail of the technique is discussed in the forthcoming section.

3. KEY ESTIMATES

We introduce two techniques to weaken the assumption on the nonlinearity. The argument in [9] works only for large η .

3.1. Estimates on nonlinearity. The first one is related to estimation of $\| |\widehat{w}|^{1+\frac{2}{d}-n} \widehat{w}^n \|_{H^\delta}$. One easily obtains such an estimate via an equivalent difference characterization of the norm of the corresponding Besov space $B_{2,2}^\delta$. However, a straightforward calculation in this direction gives us no more than an upper bound of order $O(n^d)$ (remark that here d equals the minimum integer larger than δ). Hence, we use an interpolation inequality to improve the order into $O(n^\delta)$ in exchange for strengthening the regularity assumption on the data. This is the first technique.

Let us begin with a preliminary estimate.

Lemma 3.1. *For $n \in \mathbb{Z}$, it holds that*

$$\left\| |u|^{1+\frac{2}{d}-n} u^n \right\|_{H^d} \leq C \langle n \rangle^d \|u\|_{L^\infty}^{\frac{2}{d}} \|u\|_{H^d}$$

for any $u \in H^d(\mathbb{R}^d)$.

The lemma is obvious by $\|f\|_{H^d}^2 \sim \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^d, |\alpha| \leq d} \|\partial_x^\alpha f\|_{L^2}^2$. Then, we have the following estimate.

Lemma 3.2. *Let \widehat{w} be as in (2.2). Then, it holds that*

$$\begin{aligned} \|\widehat{w}\|_{H^d} &\leq C \|u_+\|_{H^{0,d}} \left\langle g_1 \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \log t \right\rangle^d, \\ \|\partial_t \widehat{w}\|_{H^d} &\leq C \frac{|g_1|}{t} \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \|u_+\|_{H^{0,d}} \left\langle g_1 \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \log t \right\rangle^d. \end{aligned}$$

Moreover,

$$\begin{aligned} \left\| |\widehat{w}|^{1+\frac{2}{d}-n} \widehat{w}^n \right\|_{H^\delta} &\leq C \langle n \rangle^\delta \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \|u_+\|_{H^{0,d}} \left\langle g_1 \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \log t \right\rangle^\delta, \\ \left\| \partial_t (|\widehat{w}|^{1+\frac{2}{d}-n} \widehat{w}^n) \right\|_{H^\delta} &\leq C \frac{\langle n \rangle^{1+\delta} |g_1|}{t} \|\widehat{u}_+\|_{L^\infty}^{\frac{4}{d}} \|u_+\|_{H^{0,d}} \left\langle g_1 \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \log t \right\rangle^\delta \end{aligned}$$

for any $0 \leq \delta \leq d$ and $t \geq 1$.

Proof. The first estimate is immediate. By interpolation inequality, Hölder's inequality, and Lemma 3.1, we have

$$\begin{aligned} \left\| |\widehat{w}|^{1+\frac{2}{d}-n} \widehat{w}^n \right\|_{H^\delta} &\leq \left\| |\widehat{w}|^{1+\frac{2}{d}-n} \widehat{w}^n \right\|_{L^2}^{1-\frac{\delta}{d}} \left\| |\widehat{w}|^{1+\frac{2}{d}-n} \widehat{w}^n \right\|_{H^d}^{\frac{\delta}{d}} \\ &\leq C \langle n \rangle^\delta \|\widehat{w}\|_{L^\infty}^{\frac{2}{d}} \|\widehat{w}\|_{L^2}^{1-\frac{\delta}{d}} \|\widehat{w}\|_{H^d}^{\frac{\delta}{d}}. \end{aligned}$$

Then, the third estimate is a consequence of the first.

Let us next prove the second. We only consider $d = 2$. The other case is similar. By definition, we have

$$\partial_t \widehat{w} = -\frac{ig_1}{t} |\widehat{u}_+| \widehat{u}_+ \exp(-ig_1 |\widehat{u}_+| \log t).$$

Hence, by the Schwarz inequality, one sees that

$$|\nabla^2 \partial_t \widehat{w}| \leq C \frac{|g_1|}{t} (|\widehat{u}_+| |\nabla^2 \widehat{u}_+| + |\nabla \widehat{u}_+|^2) + C \frac{|g_1|^3 (\log t)^2}{t} |\widehat{u}_+| |\nabla \widehat{u}_+|^2.$$

Then, a use of Gagliardo-Nirenberg inequality yields

$$\|\partial_t \widehat{w}\|_{\dot{H}^2} \leq C \frac{|g_1|}{t} \|\widehat{u}_+\|_{L^\infty} \|\widehat{u}_+\|_{\dot{H}^2} + C \frac{|g_1|^3 (\log t)^2}{t} \|\widehat{u}_+\|_{L^\infty}^3 \|\widehat{u}_+\|_{\dot{H}^2}.$$

Plugging this to the trivial L^2 estimate, we obtain the second estimate.

To prove the last estimate, we remark that $\partial_t (|\widehat{w}|^{1+\frac{2}{d}-n} \widehat{w}^n)$ is of the form

$$\frac{g_1}{t} (P_1(\widehat{u}_+) \exp(-ig_1 |\widehat{u}_+| \log t) + P_2(\widehat{u}_+) \exp(ig_1 |\widehat{u}_+| \log t))$$

with polynomials $P_j(z) = O(\langle n \rangle |z|^{\frac{4}{d}-n} z^n)$ and so that we can obtain

$$\left\| \partial_t (|\widehat{w}|^{1+\frac{2}{d}-n} \widehat{w}^n) \right\|_{\dot{H}^d} \leq C \frac{\langle n \rangle^{1+d} |g_1|}{t} \|\widehat{u}_+\|_{L^\infty}^{\frac{4}{d}} \|u_+\|_{H^{0,d}} \left\langle g_1 \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \log t \right\rangle^d$$

as in the second estimate. Then, mimicking the proof of the third estimate, we obtain the desired estimate. \square

3.2. Time-dependent regularizing operator. To obtain additional time decay property of non-resonant part $\mathcal{N}_d(u_p)$, we use integration by parts in time. However, the argument requires spatial regularity. In [9], Hayashi, Wang, and Naumkin introduce a *time-dependent* regularizing operator (or a *time-dependent* cutoff to low-frequency), and reduce required regularity by applying the above integration by parts only for a low-frequency part and by estimating the remaining high-frequency part with the fact that the operator converges to the identity operator as $t \rightarrow \infty$.

In this paper, we take this kind of regularizing operator *dependently on both t and n* . This is the second technique to weaken the assumption on the nonlinearity.

Let $\psi \in \mathcal{S}$. We introduce a regularizing operator $\mathcal{K}_\psi = \mathcal{K}_\psi(t, n)$ by

$$(3.1) \quad \mathcal{K}_\psi := \psi \left(\frac{i\nabla}{|n|t^{\sigma/2}} \right) := \mathcal{F}^{-1} \psi \left(\frac{\xi}{|n|t^{\sigma/2}} \right) \mathcal{F},$$

where $\sigma = 1$ if $d = 1$ and $\sigma = \frac{2+\delta}{3} > 1$ if $d = 2$. We have

$$\mathcal{K}_\psi f = C_d(|n|t^{\sigma/2})^d \mathcal{F}^{-1} \psi(|n|t^{\sigma/2} \cdot) * f(x).$$

Lemma 3.3 (Boundedness of \mathcal{K}). *Take $\psi \in \mathcal{S}$ and set \mathcal{K}_ψ as in (3.1). Let $s \in \mathbb{R}$ and $\theta \in [0, 1]$. For any $t > 0$ and $n \neq 0$, the followings hold.*

- (i) \mathcal{K}_ψ is a bounded linear operator on L^2 and satisfies $\|\mathcal{K}_\psi\|_{\mathcal{L}(L^2)} \leq \|\psi\|_{L^\infty}$. Further, \mathcal{K}_ψ commutes with ∇ . In particular, \mathcal{K}_ψ is a bounded linear operator on \dot{H}^s and satisfies $\|\mathcal{K}_\psi\|_{\mathcal{L}(\dot{H}^s)} \leq \|\psi\|_{L^\infty}$.
- (ii) $\mathcal{K} - \psi(0)$ is a bounded linear operator from \dot{H}^s to $\dot{H}^{s+\theta}$ with norm

$$\|\mathcal{K}_\psi - \psi(0)\|_{\mathcal{L}(\dot{H}^{s+\theta}, \dot{H}^s)} \leq Ct^{-\frac{\theta\sigma}{2}} |n|^{-\theta}.$$

Proof. The first item is obvious. Let us prove the second. It suffices to show the case $s = 0$. For $f \in \dot{H}^\theta$, one sees from the equivalent expression that

$$\begin{aligned} \|(\mathcal{K}_\psi - \psi(0))\phi\|_{L^2} &\leq C_d(|n|t^{\sigma/2})^d \int_{\mathbb{R}^d} |\mathcal{F}^{-1} \psi(|n|t^{\sigma/2} \eta)| \|\phi(\cdot - \eta) - \phi\|_{L^2} d\eta \\ &\leq C(|n|t^{\sigma/2})^d \int_{\mathbb{R}^d} |\mathcal{F}^{-1} \psi(|n|t^{\sigma/2} \eta)| \left\| \sin \frac{\xi \cdot \eta}{2} \mathcal{F} \phi \right\|_{L^2} d\eta \\ &\leq C(|n|t^{\sigma/2})^d \int_{\mathbb{R}^d} |\mathcal{F}^{-1} \psi(|n|t^{\sigma/2} \eta)| |\eta|^\theta \left\| |\xi|^\theta \mathcal{F} \phi \right\|_{L^2} d\eta \\ &\leq C_\psi t^{-\frac{\theta\sigma}{2}} |n|^{-\theta} \|\phi\|_{\dot{H}^\theta}. \end{aligned}$$

The proof is completed. \square

4. PROOF OF PROPOSITION 2.3

In this section, we prove Proposition 2.3. Using $u_p = M(t)D(t)\hat{w}(t) = D(t)E(t)\hat{w}(t)$ with $E(t) = e^{it|x|^2/2}$, we obtain

$$\mathcal{N}_d(u_p) = \sum_{n \neq 0, 1} g_n \left(\frac{1}{it} D(t) E^n(t) \phi_n(t) \right),$$

where

$$(4.1) \quad \phi_n(t) := |\hat{w}(t)|^{1+\frac{2}{d}-n} \hat{w}^n(t).$$

Let $\psi_0(x) = e^{-|x|^2/4} \in \mathcal{S}$ and set $\mathcal{K}(t, n) := \mathcal{K}_{\psi_0}(t, n)$ as in (3.1) with $\sigma = 1$ if $d = 1$ and $\sigma = \frac{2+\delta}{3} > 1$ if $d = 2$. We decompose $\mathcal{N}_d(u_p)$ into low frequency part and high frequency part,

$$\mathcal{N}(u_p) = \mathcal{P}_d + \mathcal{Q}_d,$$

where

$$\begin{aligned} \mathcal{P}_d &= \sum_{n \neq 0,1} g_n \left(\frac{1}{it} D(t) (E^n(t) \mathcal{K} \phi_n(t)) \right), \\ \mathcal{Q}_d &= - \sum_{n \neq 0,1} g_n \left(\frac{1}{it} D(t) (E^n(t) (\mathcal{K} - 1) \phi_n(t)) \right). \end{aligned}$$

We estimate high frequency part \mathcal{Q}_d . By Strichartz' estimate, it suffices to bound $\|\mathcal{Q}_d\|_{L^1(T, \infty; L^2)}$. By using Lemma 3.3 (ii) and Lemma 3.2, we have

$$\begin{aligned} \|\mathcal{Q}_d(t)\|_{L^2} &\leq C t^{-1} \sum_{n \neq 0,1} |g_n| \|(\mathcal{K} - 1) \phi_n\|_{L^2} \\ &\leq C t^{-1-\frac{\theta\sigma}{2}} \sum_{n \neq 0,1} |n|^{-\theta} |g_n| \|\phi_n\|_{\dot{H}^\theta} \\ &\leq C t^{-1-\frac{\theta\sigma}{2}} \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \|u_+\|_{H^{0,d}} \left\langle g_1 \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \log t \right\rangle^\theta \sum_{n \neq 0,1} |g_n|. \end{aligned}$$

We choose $\theta = \delta < 1$ if $d = 1$ and $\theta = 1$ if $d = 2$. Then, we obtain

$$\begin{aligned} (4.2) \quad \|\mathcal{Q}_d\|_{L^1(T, \infty; L^2)} &\leq C T^{-\gamma} \left\langle g_1 \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \log T \right\rangle^\delta \|\widehat{u}_+\|_{L^\infty}^{\frac{2}{d}} \|u_+\|_{H^{0,d}} \sum_{n \neq 0,1} |g_n|. \end{aligned}$$

Next, we estimate low frequency part $\left\| \int_t^\infty U(t-s) \mathcal{P}_d(u_p) ds \right\|_{X_d}$. By the factorization of $U(t)$,

$$U(t) = M(t) D(t) \mathcal{F} M(t) = M(t) D(t) U \left(-\frac{1}{4t} \right) \mathcal{F}.$$

Further, the Gagliardo-Nirenberg inequality implies $\|F\|_{L^p} \leq C \|F\|_{H^\nu}^a \|F\|_{L^2}^{1-a}$ for $p \geq 2$ and $a \in [0, 1)$ with $\frac{1}{p} = \frac{1}{2} - \frac{a\nu}{d}$. Hence,

$$\begin{aligned} (4.3) \quad &\left\| \int_t^\infty U(t-s) \mathcal{P}_d(s) ds \right\|_{X_d} \\ &= \left\| U(t) \mathcal{F}^{-1} \int_t^\infty \mathcal{F} U(-s) \mathcal{P}_d(s) ds \right\|_{X_d} \\ &= \left\| D(t) U \left(-\frac{1}{4t} \right) \int_t^\infty \mathcal{F} U(-s) \mathcal{P}_d(s) ds \right\|_{X_d} \\ &\leq C t^{-\frac{1}{2}} \left\| \int_t^\infty \mathcal{F} U(-s) \mathcal{P}_d(s) ds \right\|_{H^\nu}^a \left\| \int_t^\infty \mathcal{F} U(-s) \mathcal{P}_d(s) ds \right\|_{L^2}^{1-a} \end{aligned}$$

for $\nu = 1/2a > 1/2$. We fix ν so that

$$(4.4) \quad \frac{1}{2} < \nu < \min(\delta, 2 - \delta).$$

To choose such ν , we need $\delta < 3/2$.

By factorization of $U(t)$, we have

$$\mathcal{F}U(-t)D(t)E^\rho(t) = i^{\frac{d}{2}}E^{1-\frac{1}{\rho}}(t)U\left(\frac{\rho}{4t}\right)D\left(\frac{\rho}{2}\right)$$

for $\rho \neq 0$ (see [9]). Therefore, we further compute

$$\begin{aligned}\mathcal{F}U(-s)\mathcal{P}_d(s) &= \sum_{n \neq 0,1} g_n \mathcal{F}U(-s) \frac{1}{is} D(s) E^n(s) \mathcal{K}\phi_n(s) \\ &= \sum_{n \neq 0,1} g_n s^{-1} i^{\frac{d}{2}-1} E^{1-\frac{1}{n}}(s) U\left(\frac{n}{4s}\right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(s).\end{aligned}$$

Now, we have $E^{1-\frac{1}{n}}(s) = A(s)\partial_s(sE^{1-\frac{1}{n}}(s))$ for $n \neq 0,1$, where $A(s) = \left(1 + \frac{i(1-\frac{1}{n})s}{2}|\xi|^2\right)^{-1}$. Further,

$$\partial_s U\left(\frac{n}{4s}\right) = U\left(\frac{n}{4s}\right) \left(\partial_s - \frac{in}{4s^2}\Delta\right).$$

Therefore, an integration by parts gives us

$$\begin{aligned}(4.5) \quad & \int_t^\infty E^{1-\frac{1}{n}}(s) U\left(\frac{n}{4s}\right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(s) \frac{ds}{s} \\ &= -E^{1-\frac{1}{n}}(t) A(t) U\left(\frac{n}{4s}\right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(t) \\ & - \int_t^\infty E^{1-\frac{1}{n}}(s) s \partial_s (s^{-1} A(s)) U\left(\frac{n}{4s}\right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(s) ds \\ & - \int_t^\infty E^{1-\frac{1}{n}}(s) A(s) U\left(\frac{n}{4s}\right) \left(\partial_s - \frac{in}{4s^2}\Delta\right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(s) ds \\ & =: I_1 + I_2 + I_3.\end{aligned}$$

Thanks to (4.3), we shall estimate I_j ($j = 1, 2, 3$) in L^2 and H^ν . The following estimate is useful.

Lemma 4.1. *Let $d/2 < \delta < (d+1)/2$ and $\delta < d/2 + 2\eta$. Let ν satisfy either $\nu = 0$ or $1/2 < \nu < \min(\delta, 2 - \delta)$. Let $\beta = \max(1, \delta)$ and let $m = 1, 2$. Then, it holds for any $t \geq 1$ and $n \neq 0, 1$ that*

$$\begin{aligned}(4.6) \quad & \left\| E^{1-\frac{1}{n}}(t) A^m(t) U\left(\frac{n}{4t}\right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(t) \right\|_{H^\nu} \\ & \leq Ct^{\frac{\nu-\delta}{2}} |n|^{-\delta+\eta} \left(\|\phi(t)\|_{H^\delta} + \left\| |\xi|^{-\delta} \phi_n(t) \right\|_{L^2} \right) \\ & \quad + Ct^{\frac{\nu-\alpha}{2}} |n|^{-\alpha+\eta} \left(\|\phi_n(t)\|_{H^\delta} + \left\| |\xi|^{-\delta} \phi_n(t) \right\|_{L^2} \right)^{1-\nu} \|\phi_n(t)\|_{H^\beta}^\nu,\end{aligned}$$

where $\alpha = (1 - \nu)\delta + \nu\beta$.

We postpone the proof of this lemma and continue the proof of Proposition 2.3. For simplicity, we consider the case $d = 2$, in which case $\alpha = \beta = \delta$ in Lemma 4.1. Fix $\eta > \frac{1}{2}(\delta - \frac{d}{2})$. Using Lemma 4.1, we obtain

$$\begin{aligned}(4.7) \quad & \|I_1\|_{H^\nu} = \left\| E^{1-\frac{1}{n}}(t) A(t) U\left(\frac{n}{4t}\right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(t) \right\|_{H^\nu} \\ & \leq Ct^{\frac{\nu-\delta}{2}} |n|^{-\delta+\eta} \|\phi(t)\|_{H^\delta \cap H^{0,-\delta}}.\end{aligned}$$

Let us estimate $\|I_2\|_{H^\nu}$. By $\partial_s (s^{-1}A(s)) = -2s^{-2}A(s) + s^{-2}(A(s))^2$ and Lemma 4.1, we compute

(4.8)

$$\begin{aligned} \|I_2\|_{H^\nu} &\leq C \int_t^\infty \left\| E^{1-\frac{1}{n}}(s) s^{-1} A(s) U\left(\frac{n}{4s}\right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(s) \right\|_{H^\nu} ds \\ &\quad + C \int_t^\infty \left\| E^{1-\frac{1}{n}}(s) s^{-1} (A(s))^2 U\left(\frac{n}{4s}\right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(s) \right\|_{H^\nu} ds \\ &\leq C|n|^{-\delta+\eta} \int_t^\infty s^{\frac{\nu-\delta}{2}-1} \|\phi(s)\|_{H^\delta \cap H^{0,-\delta}} ds. \end{aligned}$$

Finally, we estimate $\|I_3\|_{H^\nu}$. We introduce the regularizing operators $\mathcal{K}_j := \mathcal{K}_{\psi_j}$ ($j = 1, 2$) by (3.1) with

$$\psi_1(x) = -\frac{\sigma}{2}x \cdot \nabla \psi_0 \in \mathcal{S}, \quad \psi_2(x) = \frac{i}{4}|x|^2 \psi_0(x) \in \mathcal{S}.$$

We then have an identity

$$\begin{aligned} \left(\partial_s - \frac{in}{4s^2} \Delta \right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n &= D\left(\frac{n}{2}\right) \mathcal{K}\partial_s \phi_n + s^{-1} D\left(\frac{n}{2}\right) \mathcal{K}_1 \phi_n \\ &\quad + s^{\sigma-2} n D\left(\frac{n}{2}\right) \mathcal{K}_2 \phi_n. \end{aligned}$$

Since \mathcal{K}_1 and \mathcal{K}_2 of the form (3.1), the estimate (4.6) is valid also for these regularizing operators. Then, we have

$$\begin{aligned} \|I_3\|_{H^\nu} &\leq C|n|^{-\delta+\eta} \int_t^\infty s^{\frac{\nu-\delta}{2}} \|\partial_s \phi_n(s)\|_{H^\delta \cap H^{0,-\delta}} ds \\ (4.9) \quad &\quad + C|n|^{-\delta+\eta} \int_t^\infty s^{\frac{\nu-\delta}{2}-1} \|\phi_n(s)\|_{H^\delta \cap H^{0,-\delta}} ds \\ &\quad + C|n|^{-\delta+1+\eta} \int_t^\infty s^{\frac{\nu-\delta}{2}+\sigma-2} \|\phi_n(s)\|_{H^\delta \cap H^{0,-\delta}} ds. \end{aligned}$$

By (4.7), (4.8), (4.9), Lemma 3.2 and the estimates

$$\begin{aligned} \|\phi_n\|_{H^{0,-\delta}} &\leq C \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \|u_+\|_{\dot{H}^{-\delta}}, \\ \|\partial_t \phi_n\|_{H^{0,-\delta}} &\leq C \frac{|g_1|}{t} \|\widehat{u_+}\|_{L^\infty}^{\frac{4}{d}} \|u_+\|_{\dot{H}^{-\delta}}, \end{aligned}$$

we find

$$\begin{aligned} &\left\| \int_t^\infty E^{1-\frac{1}{n}}(s) U\left(\frac{n}{4s}\right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(s) \frac{ds}{s} \right\|_{H^\nu} \\ &\leq C t^{\frac{\nu-\delta}{2}+\sigma-1} |n|^{1+\eta} \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \|u_+\|_{\dot{H}^{-\delta} \cap H^{0,d}} \left\langle g_1 \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \log t \right\rangle^\delta \\ &\quad + C t^{\frac{\nu-\delta}{2}} |n|^{1+\eta} |g_1| \|\widehat{u_+}\|_{L^\infty}^{\frac{4}{d}} \|u_+\|_{\dot{H}^{-\delta} \cap H^{0,d}} \left\langle g_1 \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \log t \right\rangle^\delta \end{aligned}$$

Thus, summing up with respect to n , we reach to the estimate

$$(4.10) \quad \left\| \int_t^\infty U(t-s) \mathcal{P}_d(s) ds \right\|_{H^\nu}$$

$$\begin{aligned}
&\leq C t^{\frac{\nu-\delta}{2}+\sigma-1} \left\langle g_1 \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \log t \right\rangle^\delta \left\langle g_1 \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \right\rangle \\
&\quad \times \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \|u_+\|_{\dot{H}^{-\delta} \cap H^{0,d}} \sum_{n \neq 0,1} |n|^{1+\eta} |g_n|.
\end{aligned}$$

In a similar way, one sees that (4.10) holds true for $\nu = 0$. Therefore, in light of (4.3), we obtain

$$\begin{aligned}
(4.11) \quad &\left\| \int_t^\infty U(t-s) \mathcal{P}_d(s) ds \right\|_{X_d} \\
&\leq C t^{-\frac{5}{4}-\frac{\delta}{2}+\sigma} \left\langle g_1 \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \log t \right\rangle^\delta \left\langle g_1 \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \right\rangle \\
&\quad \times \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \|u_+\|_{H^{0,\delta} \cap \dot{H}^{-\delta}} \sum_{n \neq 0,1} |n|^{1+\eta} |g_n|.
\end{aligned}$$

By (4.10) with $\nu = 0$ and (4.11), we finally obtain

$$\begin{aligned}
(4.12) \quad &\left\| \int_t^\infty U(t-s) \mathcal{P}_d(s) ds \right\|_{L^\infty(T,\infty;L^2)} + \left\| \int_t^\infty U(t-s) \mathcal{P}_d(s) ds \right\|_{L^4(T,\infty;X_d)} \\
&\leq C T^{-\frac{\delta}{2}+\sigma-1} \left\langle g_1 \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \log T \right\rangle^\delta \left\langle g_1 \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \right\rangle \\
&\quad \times \|\widehat{u_+}\|_{L^\infty}^{\frac{2}{d}} \|u_+\|_{H^{0,\delta} \cap \dot{H}^{-\delta}} \sum_{n \neq 0,1} |n|^{1+\eta} |g_n|
\end{aligned}$$

since $-\frac{\delta}{2} + \sigma - 1 = -\gamma < 0$. The result follows from (4.2) and (4.12).

To complete the proof we prove Lemma 4.1.

Proof of Lemma 4.1. It suffice to estimate \dot{H}^ν norm instead of H^ν norm because smaller ν gives better estimate and because the case $\nu = 0$ is included. Further, we only treat the case $m = 1$. We set $B = (1 + t|\xi|^2)^{-\frac{1}{2}}$, which yields $|A(t)| \leq C B^2$ for any $n \neq 0, 1$. Since $\nu < 2 - \delta < 2 - d/2$, we have $|\xi|^\delta B^{2-\nu} \leq C t^{-\frac{\delta}{2}}$ and $B^{2-\nu} \in L^2 \cap L^\infty(\mathbb{R}^d)$. Set $\psi = U\left(\frac{n}{4t}\right) D\left(\frac{n}{2}\right) \mathcal{K} \phi_n(t)$. By a standard argument, we have

$$(4.13) \quad \left\| E^{1-\frac{1}{n}}(t) A(t) \psi \right\|_{\dot{H}^\nu} \leq C \|\partial^\nu(A(t) \psi)\|_{L^2} + C t^{\frac{\nu}{2}} \|B^{2-\nu} \psi\|_{L^2}.$$

We first estimate the second term in (4.13). By the triangle inequality,

$$\begin{aligned}
\left\| B^{2-\nu} U\left(\frac{n}{4t}\right) D\left(\frac{n}{2}\right) \mathcal{K} \phi_n(t) \right\|_{L^2} &\leq \left\| B^{2-\nu} \left(U\left(\frac{n}{4t}\right) - 1 \right) D\left(\frac{n}{2}\right) \mathcal{K} \phi_n(t) \right\|_{L^2} \\
&\quad + \left\| B^{2-\nu} D\left(\frac{n}{2}\right) (\mathcal{K} - 1) \phi_n(t) \right\|_{L^2} \\
&\quad + \left\| B^{2-\nu} D\left(\frac{n}{2}\right) \phi_n(t) \right\|_{L^2} \\
&=: \text{I} + \text{II} + \text{III}
\end{aligned}$$

For any $p_1 \in (2, \infty]$, one sees from Sobolev embedding and Lemma 3.3 (i) that

$$\begin{aligned} \|I\|_{L^2} &\leq C \|B^{2-\nu}\|_{L^{p_1}} \left\| |\nabla|^{\frac{d}{p_1}} \left| \frac{n|\nabla|^2}{t} \right|^{\frac{1}{2}(\delta - \frac{d}{p_1})} D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(t) \right\|_{L^2} \\ &\leq Ct^{-\frac{\delta}{2}} |n|^{-\delta + (\frac{\delta}{2} - \frac{d}{2p_1})} \|\phi_n(t)\|_{\dot{H}^\delta}. \end{aligned}$$

By definition of η , we are able to choose p_1 so that

$$(4.14) \quad \frac{\delta}{2} - \frac{d}{2p_1} < \eta.$$

By Lemma 3.3 (ii), we estimate

$$\begin{aligned} \|II\|_{L^2} &\leq C \|B^{2-\nu}\|_{L^{p_2}} \left\| |\nabla|^{\frac{d}{p_2}} D\left(\frac{n}{2}\right) (\mathcal{K} - 1)\phi_n(t) \right\|_{L^2} \\ &\leq Ct^{-\frac{d}{2p_2}} |n|^{-\frac{d}{p_2}} \left\| |\nabla|^{\frac{d}{p_2}} (\mathcal{K} - 1)\phi_n(t) \right\|_{L^2} \\ &\leq Ct^{-\frac{1}{2}(\frac{d}{p_2} - \theta_2)} |n|^{-\frac{d}{p_2} - \theta_2} \|\phi_n(t)\|_{\dot{H}^{\frac{d}{p_2} + \theta_2}} \end{aligned}$$

for any $p_2 \in (2, \infty]$ and $\theta_2 \in [0, 1]$, where we have used the relation $\sigma \geq 1$. Taking p_2 and θ_2 so that $\theta_2 = \delta - \frac{d}{p_2} \leq 1$, we obtain desired estimate for II. We can choose such p_2 and θ_2 because $\nu < 2 - \delta$. Next, we have

$$\|III\|_{L^2} \leq Ct^{-\frac{\delta}{2}} \left\| |\xi|^{-\delta} D\left(\frac{n}{2}\right) \phi_n(t) \right\|_{L^2} \leq Ct^{-\frac{\delta}{2}} |n|^{-\delta} \left\| |\xi|^{-\delta} \phi_n(t) \right\|_{L^2}.$$

These estimates yield

$$(4.15) \quad t^{\frac{\nu}{2}} \|B^{2m-\nu}\psi\|_{L^2} \leq Ct^{\frac{\nu-\delta}{2}} |n|^{-\delta+\eta} \left(\|\phi_n(t)\|_{H^\delta} + \left\| |\xi|^{-\delta} \phi_n(t) \right\|_{L^2} \right).$$

Let us move on to the estimate of the first term in (4.13). By interpolation inequality,

$$\|\partial^\nu(A(t)\psi)\|_{L^2} \leq \|A(t)\psi\|_{L^2}^{1-\nu} \|\nabla(A(t)\psi)\|_{L^2}^\nu.$$

From $|\nabla A(t)| \leq Ct^{1/2} B^2$ and the Leibniz rule, we have

$$\|\nabla(A(t)\psi)\|_{L^2} \leq Ct^{1/2} \|B^2\psi\|_{L^2} + \|B^2\nabla\psi\|_{L^2}.$$

These implies that

$$(4.16) \quad \|\partial^\nu(A(t)\psi)\|_{L^2} \leq Ct^{\frac{\nu}{2}} \|B^2\psi\|_{L^2} + C \|B^2\psi\|_{L^2}^{1-\nu} \|B^2\nabla\psi\|_{L^2}^\nu.$$

The estimate of $\|B^2\psi\|_{L^2}$ is the same as in (4.15). To complete the proof, it then suffices to show that

$$(4.17) \quad \|B^2\nabla\psi\|_{L^2} \leq Ct^{\frac{1-\beta}{2}} |n|^{-\beta} \|\phi_n(t)\|_{\dot{H}^\beta},$$

where $\beta = 1$ if $d = 1$ and $\beta = \delta$ if $d = 2$.

Let us show (4.17). We estimate as

$$\begin{aligned} \|B^2\nabla U\left(\frac{n}{4t}\right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(t)\|_{L^2} &\leq \left\| B^2\nabla \left(U\left(\frac{n}{4t}\right) - 1 \right) D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(t) \right\|_{L^2} \\ &\quad + \left\| B^2\nabla D\left(\frac{n}{2}\right) (\mathcal{K} - 1)\phi_n(t) \right\|_{L^2} \\ &\quad + \left\| B^2\nabla D\left(\frac{n}{2}\right) \phi_n(t) \right\|_{L^2} \\ &=: \text{IV} + \text{V} + \text{VI}. \end{aligned}$$

For any $p_3 \in (4, \infty]$, one sees from Sobolev embedding and Lemma 3.3 (i) that

$$\begin{aligned} \|\text{IV}\|_{L^2} &\leq C \|B^2\|_{L^{p_3}} \left\| |\nabla|^{1+\frac{d}{p_3}} D\left(\frac{n}{2}\right) \mathcal{K}\phi_n(t) \right\|_{L^2} \\ &\leq C t^{\frac{1-\beta}{2}} |n|^{-\beta} \|\phi_n(t)\|_{\dot{H}^\beta}. \end{aligned}$$

Here, we take p_3 so that $1 + \frac{d}{p_3} = \beta$. By Lemma 3.3 (ii), we estimate

$$\begin{aligned} \|\text{V}\|_{L^2} &\leq C \|B^2\|_{L^{p_4}} \left\| |\nabla|^{1+\frac{d}{p_2}} D\left(\frac{n}{2}\right) (\mathcal{K}-1)\phi_n(t) \right\|_{L^2} \\ &\leq C t^{-\frac{d}{2p_2}} |n|^{-1-\frac{d}{p_4}} \left\| |\nabla|^{1+\frac{d}{p_4}} (\mathcal{K}-1)\phi_n(t) \right\|_{L^2} \\ &\leq C t^{\frac{1-\beta}{2}} |n|^{-\beta} \|\phi_n(t)\|_{\dot{H}^\beta}, \end{aligned}$$

where $\beta = 1 + \frac{1}{p_4}$. Finally, from the Hardy inequality, we have

$$\begin{aligned} \|\text{VI}\|_{L^2} &\leq C t^{\frac{1-\beta}{2}} \left\| |\xi|^{1-\delta} |\nabla| D\left(\frac{n}{2}\right) \phi_n(t) \right\|_{L^2} \\ &\leq C t^{\frac{1-\beta}{2}} |n|^{-\beta} \left\| |\nabla|^\beta \phi_n(t) \right\|_{L^2}. \end{aligned}$$

By these estimates, we obtain (4.17), which completes the proof of (4.6). \square

5. PROOF OF MAIN RESULT

We are now in a position to prove our main result.

Proof of Theorem 1.2. Let $\eta > 0$ and $\delta \in (d/2, (d+1)/2)$ be as in the assumption. Then, we have the relation $\eta > \frac{1}{2}(\delta - d/2)$. Take $b \in (d/4, \gamma)$. By Lemma 2.1, Lemma 2.2, and Proposition 2.3, we have

$$\begin{aligned} &\|\Phi(v)\|_{X_{d,T,b}} \\ &\leq C_1 \|g\|_{\text{Lip}} R \left(R^{\frac{2}{d}} T^{\frac{1}{2}-\frac{2}{d}b} + \varepsilon^{\frac{2}{d}} \right) \\ (5.1) \quad &+ C_2 (1 + |g_1|) T^{b-\frac{\delta}{2}} \left\langle g_1 \varepsilon^{\frac{2}{d}} \log T \right\rangle^\delta \left\langle \varepsilon^{\frac{2}{d}} \right\rangle \|u_+\|_{H^{0,d}} \\ &+ C_3 T^{b-\gamma} \left\langle g_1 \varepsilon^{\frac{2}{d}} \log T \right\rangle^\delta \left\langle g_1 \varepsilon^{\frac{2}{d}} \right\rangle \varepsilon^{\frac{2}{d}} \|u_+\|_{\dot{H}^{-\delta} \cap H^{0,d}} \sum_{n \neq 0,1} |n|^{1+\eta} |g_n|, \end{aligned}$$

for any $v \in X_{d,T,b,R}$, $R > 0$, $T > T_0$ and $\varepsilon > 0$.

We next see that

$$(5.2) \quad d(\Phi(u), \Phi(v)) \leq C_4 \|g\|_{\text{Lip}} \left(R^{\frac{2}{d}} T^{\frac{1}{2}-\frac{2}{d}b} + \varepsilon^{\frac{2}{d}} \right) d(u, v).$$

Indeed, by the integral equation of (NLS), we see that

$$\Phi(u) - \Phi(v) = i \int_t^\infty U(t-s) (F(u) - F(v)) (s) ds.$$

We then find

$$\begin{aligned} |F(u) - F(v)| &\leq C \|g\|_{\text{Lip}} \left(|u|^{\frac{2}{d}} + |v|^{\frac{2}{d}} \right) |u - v| \\ &\leq C \|g\|_{\text{Lip}} \left(|u - u_p|^{\frac{2}{d}} + |v - u_p|^{\frac{2}{d}} \right) |u - v| \\ &\quad + \|g\|_{\text{Lip}} |u_p|^{\frac{2}{d}} |u - v|. \end{aligned}$$

The rest of the proof is similar to that of Lemma 2.1.

Choose $\varepsilon = \varepsilon(\|g\|_{\text{Lip}})$ so small that

$$C_1 \|g\|_{\text{Lip}} \varepsilon^{\frac{2}{d}} \leq \frac{1}{2}, \quad C_4 \|g\|_{\text{Lip}} \varepsilon^{\frac{2}{d}} \leq \frac{1}{4}.$$

Set $R = 1$. Then, for sufficiently large T , we obtain

$$\|\Phi(v)\|_{X_{d,T,b}} < 1 = R$$

and

$$d(\Phi(u), \Phi(v)) \leq \frac{1}{2}d(u, v),$$

which shows $\Phi : X_{d,T,b,1} \rightarrow X_{d,T,b,1}$ is a contraction mapping. Then, we obtain a unique solution $v(t) \in X_{d,t,b,1}$. \square

APPENDIX A. LIPSCHITZ CONTINUITY OF g_F

In this appendix we show the following.

Lemma A.1. *Let $F(u)$ satisfy (1.1). Let $g(\theta)$ be a corresponding periodic function given by (1.6) and (1.7). Then, the following two statements are equivalent:*

- (1) *$g(\theta)$ is Lipschitz continuous.*
- (2) *There exists $C > 0$ such that*

$$(A.1) \quad |F(u) - F(v)| \leq C(|u|^{2/d} + |v|^{2/d})|u - v|$$

for all $u, v \in \mathbb{C}$.

Moreover, the constant C depends only on the Lipschitz constant of g .

Proof. By (1.7), it is easy to see that Lipschitz continuity of g is equivalent to existence of a constant C such that

$$(A.2) \quad |F(u) - F(v)| \leq C|u - v|$$

for all $u, v \in \mathbb{C}$ with $|u| = |v| = 1$. Hence, (2) \Rightarrow (1) is obvious.

We will show that (A.2) implies (A.1). We may suppose that $u \neq 0$ and $v \neq 0$. Otherwise (A.1) is immediate from (1.1). We have

$$|F(u) - F(v)| \leq \left| F(u) - F\left(\frac{|u|}{|v|}v\right) \right| + \left| F\left(\frac{|u|}{|v|}v\right) - F(v) \right|.$$

By (1.1) and (A.2), we have

$$\begin{aligned} \left| F(u) - F\left(\frac{|u|}{|v|}v\right) \right| &= |u|^{1+\frac{2}{d}} \left| F\left(\frac{u}{|u|}\right) - F\left(\frac{v}{|v|}\right) \right| \\ &\leq C|u|^{1+\frac{2}{d}} \left| \frac{u}{|u|} - \frac{v}{|v|} \right| \\ &\leq C|u|^{1+\frac{2}{d}} \frac{||v|u - v|v|| + ||v|v - |u|v|}{|u||v|} \\ &\leq C|u|^{\frac{2}{d}}|u - v|. \end{aligned}$$

Again by (1.1),

$$\left| F\left(\frac{|u|}{|v|}v\right) - F(v) \right| = \left| F\left(\frac{v}{|v|}\right) \right| \left| |u|^{1+\frac{2}{d}} - |v|^{1+\frac{2}{d}} \right|$$

$$\leq C(|u|^{\frac{2}{d}} + |v|^{\frac{2}{d}})|u - v|.$$

Thus, we obtain (A.1). \square

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