THE BAROTROPIC QUASI-GEOSTROPHIC EQUATION UNDER A FREE SURFACE*

QINGSHAN CHEN[†]

Abstract. The inviscid barotropic quasi-geostrophic equation with a free surface is considered. The free surface mandates a non-standard boundary condition. The global existence existence and uniqueness of a weak solution is established, thanks to the uniform in time bounds on the potential vorticity. The solution is also shown to satisfy the initial and boundary conditions in the classical sense.

Key words. Partial differential equations, inviscid model, geophysical fluid dynamics, global well-posedness

AMS subject classifications. 35Q35, 35Q86

1. Introduction. The inviscid barotropic quasi-geostrophic equation (QG) for large-scale geophysical flows takes the form of a scalar transport equation,

(1)
$$\frac{\partial}{\partial t}q + \boldsymbol{u} \cdot \nabla q = f, \qquad \mathcal{M}.$$

Here,

$$(2) q = \nabla \times \boldsymbol{u} + y - \psi$$

is the QG potential vorticity (QGPV), and the transport velocity u is given by

(3)
$$\boldsymbol{u} = \nabla^{\perp} \psi.$$

The streamfunction ψ physically represents small perturbations to the surface height. The equation is posed on a simply-connected two-dimensional domain \mathcal{M} . Within the expression (2) for the QGPV q, $\nabla \times \boldsymbol{u}$ is the relative vorticity of the velocity field, y the so-called beta term that arises thanks to the differential effect of the earth rotation along the meridional direction, and ψ the surface deformation.

For the simplicity of presentation, all the variables in (1)-(3) have been non-dimensionalized, and the dimensionless coefficients have been rounded to the constant 1, mandating that each term be of equal significance to the dynamics. In reality, of course, the situation is much more complex. We point out that, when the horizontal length scale is much smaller than the Rossby deformation radius, the fluctuations of the top surface are small and their impact on the vorticity dynamics is negligible, i.e. the classical "rigid lid" assumption holds. However, in non-homogeneous fluids, since the horizontal length scale of the flow are close in scale to the Rossby deformation radii of the interior layer interfaces, the deformations of the interior layer interface are greater, and so are their impact on the vorticity dynamics. For this reason, we want to study the well-posedness of the QG equation when the surface deformation is included. The current work can be considered a preparation for future efforts on more complex systems.

^{*}Submitted to the editors DATE.

Funding: This work was in part supported by the Simons Foundation contract no. #319070.

[†]Department of Mathematical Sciences, Clemson University, Clemson, SC 29631. (qsc@clemson.edu, http://discreteinfo.net).

Equation (1) implies that, in the absence of an external forcing, i.e. f = 0, the QGPV is conserved along the fluid paths. There is, of course, an analogue between the QG equation and the two-dimensional incompressible Euler equation, where the vorticity (called the relative vorticity in the above) is conserved along the fluid paths. The main difference between them is the appearance of the stream-function ψ in the QGPV, which reflects the fact that the QG actually describes a three-dimensional body of fluids, whose top surface is free to deform, while the two-dimensional Euler is strictly for the planar fluids. But the motions of the three-dimensional body of fluid are assumed to be uniform across the fluid depth, and therefore the velocity field u is two-dimensional. The third dimension only manifests itself through the varying fluid depth. A review of the derivation of the QG (see e.g. [41, 45]) reveals that the surface fluctuations affects the vorticity dynamics thanks to the mild compressibility of the full two-dimensional velocity field; the u in (1), which is incompressible, is the leading component in the asymptotic expansion of the full velocity field. Hence, the surface fluctuation ψ in (2) is a manifestation of the additional variability in the vertical dimension and the mild compressibility of the two-dimensional velocity field.

In a broader context, many geophysical models are derived based on the characteristics of large-scale geophysical flows, such as the small vertical to horizontal aspect ratio and the strong earth rotation. In most cases, the models demonstrate richer and more complex dynamics than the strictly two-dimensional models, but remain much simpler than the full three-dimensional fluid models, i.e. the three-dimensional Navier-Stokes equations (NSE) or the Euler equations. Mathematically speaking, the twodimensional fluid models, i.e. the two-dimensional Navier-Stokes or Euler equations, are well understood. The existence, uniqueness, and regularity of a global solution to these equations are known ([46, 26, 1]). But the same cannot be said about the three-dimensional NS or Euler equations; for a review of the limited results on these equations, see [44, 2]. Situated between the purely two-dimensional fluid models and the full three-dimensional models, large-scale geophysical flow models, such as the primitive equations (PEs), the shallow water equations, and the quasi-geostrophic equations, can offer valuable insights into the complex dynamics of fluid flows, and help bridge the knowledge gap between two-dimensional and three-dimensional fluid models.

Partly for the reason mentioned above, and partly for the practical interests in the evolution of large-scale geophysical flows, geophysical fluid models have been the subject of intense effort in the mathematical community for the past few decades. Lions, Temam, and Wang ([37, 36, 35, 34, 33]) offered the first systematic and rigorous treatment of the three-dimensional viscous primitive equations. Their results were followed and improved by many subsequent works; for a review of these progresses, see the review article [42]. In particular, Cao and Titi ([6]) and Kobelkov ([29, 28]) independently established the global existence and uniqueness of a strong solution to the three-dimensional viscous PEs under the rigid-lid assumption.

On the side of quasi-geostrophic equations, several authors have studied the three-dimensional QG equation under idealized settings, in the unbounded half space, or a rectangular box. An early work is by Dutton ([17, 18]), who considered the three-dimensional QG model in a rectangular box with periodic boundary conditions on the sides, and homogeneous Neumann boundary conditions on the top and bottom. The uniqueness of a classical solution, if it exists, and the global existence of a generalized solution were established. Bourgeois and Beale ([3]) studied the equation in a similar setting, and the existence of a global strong solution was proved. Desjardins and Greneier ([16]) also considered the equation in a similar setting, but included in their

model the Ekman pumping effect which effectively add diffusion to the flow. The existence of a global weak solution is given. Puel and Vasseur ([43]) considered the inviscid QG in the upper half space, with the non-penetration boundary condition at the bottom of the fluid. The global existence of a weak solution was proven. In these works, the issue of uniqueness of the solutions was left open. In a recent work, Novack and Vasseur ([40]) considered the three-dimensional QG in the same spatial setting as in [43], but with an added diffusion term in the boundary at z=0 due to the Ekman pumping effect. The existence and uniqueness of a global strong solution is proven.

Another related model is the surface QG equation (SQG). The SQG is in fact a generalization of the top surface boundary condition for the three-dimensional QG. The curl form of the SQG resembles that of the three-dimensional Euler equations. For this reason, the SQG has been intensely studied in the past twenty years or so ([11, 10, 23, 12, 13, 32, 5, 27, 20, 8, 15, 39, 31, 19, 9, 30, 7, 4]).

The current work studies the inviscid barotropic QG equation under a free surface and on a general bounded domain. Without the free surface, the barotropic QG is mathematically equivalent to the two-dimensional incompressible Euler equation, whose well-posedness has been established by various authors ([46, 26]). The introduction of the the free surface not only changes the relation between the potential vorticity q and the streamfunction ψ , but also mandates a new and slightly more complicated type of boundary condition for the stream function. For the twodimensional Euler equation, the homogeneous Dirichlet boundary conditions suffice for the streamfunction. But this is no longer true when the top surface is left free. The non-penetration boundary condition on the flow mandates that the streamfunction be constant along the boundary. Physically, the constant boundary value of the streamfunction should be left free to accommodate the free deformation of the top surface. Mathematically, that constant boundary value cannot be arbitrarily set without altering the shape of the solution, unlike in the case of the two-dimensional Euler equations. Thus, additional constraints have to be introduced to determine the value of the streamfunction on the boundary. In this work, we determine the constant by enforcing the mass conservation condition.

The constant but non-zero boundary condition gives rise to several technical difficulties that were absent in the case of the two-dimensional Euler equation. The main contribution of this work is to address these difficulties and establish the well-posedness of the barotropic QG equation with a free surface. The proof follows the approach that was originally laid down by Yudovich ([46]). However, for the construction of the flow map, the approach from Marchioro ([38]) is adopted. The simpler approach of Kato ([26]) does not apply because the solution of the current problem is not sufficiently smooth.

2. The initial and boundary conditions. It will become clear later in the analysis that the streamfunction ψ is a key quantity in the QG dynamics. In fact, the QG equation can be expressed entirely in this quantity,

(4)
$$\frac{\partial}{\partial t}(\Delta \psi + y - \psi) + \nabla^{\perp} \psi \cdot \nabla(\Delta \psi + y - \psi) = f, \quad \mathcal{M}.$$

Since the model is inviscid, it is natural to impose the no-flux boundary condition on the domain boundary $\partial \mathcal{M}$,

(5)
$$\mathbf{u} \cdot \mathbf{n} = -\frac{\partial \psi}{\partial \tau} = 0, \qquad \partial \mathcal{M},$$

where **n** and τ stand for the outer normal and positively oriented tangential vectors, respectively, on the boundary. The condition (5) is equivalent to the requirement that ψ be constant along the boundary, i.e. for some quantity l that depends on t only,

(6)
$$\psi(x,t) = l(t), \quad \forall x \in \partial \mathcal{M}.$$

The boundary value of ψ has been left free to accommodate the free movement of the top surface. In order to determine the value l for each t, we require that the fluctuation of the surface does not affect the overall volume of the fluid, that is,

(7)
$$\int_{\mathcal{M}} \psi dx = 0.$$

This is equivalent to the condition that mass is conserved.

The initial condition is specified on the streamfunction as well,

(8)
$$\psi(x,0) = \psi_0(x), \quad \forall x \in \mathcal{M}.$$

Thus, equations (1)–(3) and (6)–(8) (or, equivalently, (4) and (6)–(8)) constitute the complete initial and boundary value problem of the barotropic QG equation.

In the QG equation, the right-hand side forcing f is typically the curl of a vector field \mathbf{F} , representing, e.g., the wind in the physical world. Hence, we assume that f is derived from a given vector field \mathbf{F} via

$$(9) f = \nabla \times \mathbf{F}.$$

3. An non-standard elliptic boundary value problem. For regularity, we assume that the boundary of the domain, $\partial \mathcal{M}$, is at least C^2 smooth.

Once the QGPV q is known, the streamfunction ψ can be determined from an non-standard elliptic boundary value problem,

(10a)
$$\Delta \psi - \psi = q - y, \qquad \mathcal{M},$$

(10b)
$$\psi = l,$$
 $\partial \mathcal{M},$

(10c)
$$\int_{\mathcal{M}} \psi dx = 0.$$

We proceed by decomposition. This technique can be applied in more complex situations with holes inside the domain (see [26]). We let ψ_1 and ψ_2 be solutions of the following elliptic BVPs, respectively,

(11a)
$$\Delta \psi_1 - \psi_1 = q - y, \qquad \mathcal{M},$$

(11b)
$$\psi_1 = 0, \qquad \partial \mathcal{M},$$

and

(12a)
$$\Delta \psi_2 - \psi_2 = 0, \qquad \mathcal{M},$$

(12b)
$$\psi_2 = 1, \qquad \partial \mathcal{M}.$$

By the standard elliptic PDE theories, both BVPs (11) and (12) are well-posed under proper assumptions on the forcing on the right-hand side of (11a) and on the domain \mathcal{M} . The solution to the original BVP (10) can be expressed in terms of ψ_1 and ψ_2 ,

$$\psi = \psi_1 + l\psi_2.$$

The unknown constant l can be determined using the mass conservation constraint (10c)

$$\int_{\mathcal{M}} \psi dx = \int_{\mathcal{M}} \psi_1 dx + l \int_{\mathcal{M}} \psi_2 dx = 0,$$

which leads to

(14)
$$l = -\frac{\int_{\mathcal{M}} \psi_1 dx}{\int_{\mathcal{M}} \psi_2 dx}.$$

We point out that the expression (14) for l is valid because, as a consequence of the maximum principle, ψ_2 is positive in the interior of the domain, and the integral of ψ_2 in the denominator of (14) is strictly positive.

The elliptic PDE (10a) is called the Helmholtz equation. The associated differential operator, $\Delta - I$, has a fundamental solution that is all regular except for a logarithmic singularity ([14]), just like the Laplacian operator Δ . Thus, provided that the boundary $\partial \mathcal{M}$ is sufficiently smooth, the Green's function G(x, y) for the elliptic BVP (11) exists, and is smooth except for a logarithmic singularity. Specifically, G(x, y) has the following forms and estimates, for $\forall x, y \in \mathcal{M}$,

(15a)
$$G(x,y) = a(x,y) \ln|x-y| + b(x,y),$$

(15b)
$$\frac{\partial G}{\partial x_i}(x,y) = \frac{c(x,y)}{|x-y|} + d(x,y),$$

(15c)
$$|G(x,y)| \le C(\mathcal{M}) (1 + |\ln|x - y||),$$

(15d)
$$\left| \frac{\partial G}{\partial x_i}(x, y) \right| \le C(\mathcal{M})(1 + |x - y|^{-1}), \qquad i = 1, 2.$$

In the above, a(x,y), b(x,y), c(x,y), and d(x,y) are functions that are regular over the entire domain \mathcal{M} , and whose maximum values depend on \mathcal{M} only.

Using the Green's function G(x,y), the solution ψ_1 of (11) can be written as

(16)
$$\psi_1(x) = \int_{\mathcal{M}} G(x, y)(q(y) - y_2) dy.$$

The solution ψ_2 of (12) can be written as

(17)
$$\psi_2(x) = 1 + \int_{\mathcal{M}} G(x, y) dy.$$

Substituting (16) and (17) into (14), we obtain an expression for l, as a functional of the QGPV q,

(18)
$$l(q) = -\frac{\int_{\mathcal{M}} \int_{\mathcal{M}} G(x, y)(q(y) - y_2) dy dx}{|\mathcal{M}| + \int_{\mathcal{M}} \int_{\mathcal{M}} G(x, y) dy dx}.$$

The solution ψ to the non-standard BVP (10) can be expressed as

(19)
$$\psi(x) = \int_{\mathcal{M}} G(x, y)(q(y) - y_2)dy + l(1 + \int_{\mathcal{M}} G(x, y)dy).$$

As in the case of the two-dimensional Euler equation, the QGPV is simply being advected by the velocity field, and its maximum values are preserved in the absence of an external forcing. As noted above on (14), the denominator on the right-hand side of (18) is strictly positive. Thus, from (18), and making use of (15c), one derives an estimate on the magnitude of the constant value l of ψ on the boundary,

$$\begin{split} |l| &\leq C(\mathcal{M}, |q|_{\infty}) \left(1 + \int_{\mathcal{M}} \int_{\mathcal{M}} |\ln|x - y|| \, dy dx \right) \\ &\leq C(\mathcal{M}, |q|_{\infty}) \left(1 + \int_{\mathcal{M}} \left(\int_{\substack{|y - x| \geq 1 \\ y \in \mathcal{M}}} |\ln|x - y|| \, dy + \int_{\substack{|y - x| \leq 1 \\ y \in \mathcal{M}}} |\ln|x - y|| \, dy \right) \right) \\ &\leq C(\mathcal{M}, |q|_{\infty}) \left(1 + \int_{\mathcal{M}} \int_{\substack{|y - x| \leq 1 \\ y \in \mathcal{M}}} |\ln|x - y|| \, dy dx \right) \\ &\leq C(\mathcal{M}, |q|_{\infty}) \left(1 + \int_{\mathcal{M}} \int_{\substack{|y - x| \leq 1 \\ y \in \mathcal{M}}} |\ln|x - y|| \, dy dx \right) \\ &\leq C(\mathcal{M}, |q|_{\infty}) \left(1 + \int_{\mathcal{M}} \int_{0}^{1} 2\pi r |\ln r|| \, dr dx \right) \\ &\leq C(\mathcal{M}, |q|_{\infty}) \left(1 + 2\pi |\mathcal{M}| \int_{0}^{1} r |\ln r| \, dr \right) \\ &\leq C(\mathcal{M}, |q|_{\infty}) \left(1 + \frac{2\pi |\mathcal{M}|}{e} \right). \end{split}$$

To summarize, we have just shown that the value of ψ on the boundary is bounded by a constant that depends on the domain and the maximum norm of the QGPV q, that is,

$$(20) |l| \le C(\mathcal{M}, |q|_{\infty}).$$

Below, we shall formally state the regularity results for the elliptic boundary value problem (10). But, in order to do so, we need to first give the precise definitions of some relevant function spaces.

We denote by Q_T the spatial-temporal domain,

$$Q_T = \mathcal{M} \times (0, T).$$

We denote by $L^{\infty}(\mathcal{M})$, or $L^{\infty}(Q_T)$ when time is also involved, the space of functions that are essentially bounded. We denote by $C^{0,\gamma}(\mathcal{M})$, with $\gamma > 0$, the space of Hölder-continuous functions on \mathcal{M} , and similarly, $C^{0,\gamma}(Q_T)$ on Q_T . $C^{0,\gamma}(\mathcal{M})$ and $C^{0,\gamma}(Q_T)$ are both Banach spaces under the usual Hölder norms.

We denote by V the space of solutions to the elliptic boundary value problem (10) with $q \in L^{\infty}(\mathcal{M})$, i.e.,

$$V := \{ \psi \mid \psi \text{ solves (10) for some } q \in L^{\infty}(\mathcal{M}) \}.$$

The space V is equipped with the norm

$$\|\psi\|_V := \|\Delta\psi - \psi\|_{L^{\infty}(\mathcal{M})}.$$

By the continuity of the inverse elliptic operator $(\Delta - I)^{-1}$, V is a Banach space.

In the analysis, we will also encounter functions that are differentiable with continuous first derivatives. The space of these functions will be denoted as $C^1(\mathcal{M})$, equipped with the usual C^1 norm.

When time is involved, we use $L^{\infty}(0,T;V)$ to designate the space of functions that are essentially bounded with respect to the $\|\cdot\|_V$ norm, and $L^{\infty}(0,T;C^1(\mathcal{M}))$ for functions that are essentially bounded under the $\|\cdot\|_{C^1(\mathcal{M})}$ norm.

We can now formally state the regularity result for the elliptic boundary value problem (10).

LEMMA 3.1. Let the boundary $\partial \mathcal{M} \in C^2$, $q \in L^{\infty}(\mathcal{M})$. Then the solution ψ belongs to $W^{2,p}(\mathcal{M})$ for any p > 1, and the derivatives enjoy the following estimates,

$$||D^2\psi||_{L^p(\mathcal{M})} \le Cp||q-y||_{L^{\infty}(\mathcal{M})},$$

where D denotes the first-order differential operator, and the constant C depends on \mathcal{M} only, and not on p or the potential vorticity q. The first derivatives of the function ψ satisfy the Hölder condition with any $0 < \gamma < 1$,

(22)
$$||D\psi||_{C^{0,\gamma}(\mathcal{M})} \le \frac{C}{1-\gamma} ||q-y||_{L^{\infty}(\mathcal{M})},$$

and the quasi-Lipschitz condition,

$$(23) |D\psi(\xi) - D\psi(\eta)| \le C\chi(\delta) ||q - y||_{L^{\infty}(\mathcal{M})},$$

where $\delta = |\xi - \eta|$, and

$$\chi(\delta) = \begin{cases} (1 - \ln \delta)\delta & \text{if } \delta < 1, \\ 1 & \text{if } \delta \ge 1. \end{cases}$$

This result is similar to the one given in [46] for the Euler equation. What is new here is the presence of a free surface and its constant value on the boundary.

Proof of Lemma 3.1. The assertion (21) is part of the classical L^p regularity theory for the elliptic BVPs ([21]). We now verify that $\psi \in C^1(\mathcal{M})$ and the Hölder condition (22) and the quasi-Lipschitz condition (23) hold for ψ . We formally differentiate (19) with respect to x_i ,

(24)
$$\frac{\partial \psi}{\partial x_i}(x) = \int_{\mathcal{M}} \frac{\partial G}{\partial x_i}(x, y)(q(y) - y_2)dy + l \int_{\mathcal{M}} \frac{\partial G}{\partial x_i}(x, y)dy.$$

We call the right-hand side u(x), and we need to show that u(x) is well-defined. Using the fact that q is essentially bounded, we find that

$$|u(x)| \le \int_{\mathcal{M}} \left| \frac{\partial G}{\partial x_i}(x, y) \right| \cdot |q(y) - y_2| dy + l \int_{\mathcal{M}} \left| \frac{\partial G}{\partial x_i}(x, y) \right| dy$$

$$\le (|q|_{\infty} + |y_2|_{\infty} + |l|) \int_{\mathcal{M}} \left| \frac{\partial G}{\partial x_i}(x, y) \right| dy.$$

Using the estimate (15d) in the above, we proceed with the estimates,

$$|u(x)| \le (|q|_{\infty} + |y_{2}|_{\infty} + |l|) \int_{\mathcal{M}} \frac{1}{|y - x|} dy$$

$$\le (|q|_{\infty} + |y_{2}|_{\infty} + |l|) \left\{ \int_{\substack{|y - x| \ge 1 \ y \in \mathcal{M}}} \frac{1}{|y - x|} + \int_{\substack{|y - x| \le 1 \ |y - x| \le 1}} \frac{1}{|y - x|} dy \right\}$$

$$\le (|q|_{\infty} + |y_{2}|_{\infty} + |l|) \left\{ |\mathcal{M}| \cdot 1 + \int_{0}^{1} \int_{\substack{|y - x| = r \ r}} \frac{1}{r} ds dr \right\}$$

$$\le (|q|_{\infty} + |y_{2}|_{\infty} + |l|) \cdot (|\mathcal{M}| + 2\pi).$$

Combining this estimate with the estimate (20), we reach

$$(25) |u(x)| \le C(\mathcal{M}, |q|_{\infty}).$$

Hence, the right-hand side of (24) is well-defined, and the relation (24) holds.

Next, we show that the first derivative of ψ is quasi-Lipschitz continuous and satisfies the estimates (23). We let ξ and η be two arbitrary points in \mathcal{M} . Then, using the relation (24), we derive that

$$\left| \frac{\partial \psi}{\partial x_i}(\xi) - \frac{\partial \psi}{\partial x_i}(\eta) \right| \le \left(|q - y_2|_{\infty} + |l| \right) \int_{\mathcal{M}} \left| \frac{\partial G}{\partial x_i}(\xi, y) - \frac{\partial G}{\partial x_i}(\eta, y) \right| dy.$$

Substituting the form (15b) into (26), and using the estimate (20) on l, one derives that

$$(27) \qquad \left| \frac{\partial \psi}{\partial x_i}(\xi) - \frac{\partial \psi}{\partial x_i}(\eta) \right| \le C(\mathcal{M}, |q|_{\infty}) \left(|\xi - \eta| + \int_{\mathcal{M}} \left| \frac{1}{|y - \xi|} - \frac{1}{|y - \eta|} \right| dy \right).$$

Using the triangular inequality, one finds that, in a space of general dimension n,

(28)
$$\left| \frac{1}{|y - \xi|^{n-1}} - \frac{1}{|y - \eta|^{n-1}} \right| \le \frac{|\xi - \eta|}{|y - \xi^*(y)|^n}$$

where $\xi^*(y)$ is a point between ξ and η , and depends on y. We let R > 0 be such that $B(\xi, R)$, the ball centered at ξ with radius R, covers the entire domain \mathcal{M} . We then decompose the domain into two parts, one within the small ball $B(\xi, 2\delta)$, and the other within the annulus $2\delta \leq |y - \xi| \leq R$, where $\delta = |\xi - \|$. We estimate the integral of the left-hand side of (28) in these two sub-domains separately.

$$\begin{split} & \int_{\mathcal{M}} \left| \frac{1}{|y-\xi|^{n-1}} - \frac{1}{|y-\eta|^{n-1}} \right| dy \\ & \leq \int_{|y-\xi| \leq 2\delta} \left| \frac{1}{|y-\xi|^{n-1}} - \frac{1}{|y-\eta|^{n-1}} \right| dy + \int_{2\delta < |y-\xi| < R} \left| \frac{|\xi-\eta|}{|y-\xi|^{n-1}} - \frac{1}{|y-\eta|^{n-1}} \right| dy \\ & \leq \int_{|y-\xi| \leq 2\delta} \frac{1}{|y-\xi|^{n-1}} dy + \int_{|y-\eta| \leq 3\delta} \frac{1}{|y-\eta|^{n-1}} dy + \int_{2\delta < |y-\xi| < R} \left| \frac{|\xi-\eta|}{|y-\xi|^n} \frac{|y-\xi|^n}{|y-\xi^*|^n} dy \right| \\ & \leq \int_0^{2\delta} \int_{|y-\xi| = r} \frac{1}{r^{n-1}} ds dr + \int_0^{3\delta} \int_{|y-\eta| = r} \frac{1}{r^{n-1}} ds dr + \int_{2\delta}^R \int_{|y-\xi| = r} 2^n \frac{|\xi-\eta|}{r^n} ds dr \\ & \leq \int_0^{2\delta} \omega_n r^{-(n-1)} r^{n-1} dr + \int_0^{3\delta} \omega_n r^{-(n-1)} r^{n-1} dr + \int_{2\delta}^R |\xi-\eta| r^{-n} \omega_n r^{n-1} dr \\ & \leq (5+2^n (\ln R - \ln 2) - 2^n \ln \delta) \, \omega_n \delta. \end{split}$$

Using this estimate in (27), we obtain

(29)
$$\left| \frac{\partial \psi}{\partial x_i}(\xi) - \frac{\partial \psi}{\partial x_i}(\eta) \right| \le C(\mathcal{M}, |q|_{\infty})(1 - \ln \delta)\delta.$$

Thus, the claim of the quasi-Lipschitz continuity is proven.

The Hölder continuity is a consequence of the quasi-Lipschitz continuity. Indeed, for any $0 < \lambda < 1$,

$$\frac{\left|\frac{\partial \psi}{\partial x_i}(\xi) - \frac{\partial \psi}{\partial x_i}(\eta)\right|}{|\xi - \eta|^{\lambda}} \le C(\mathcal{M}, |q|_{\infty})(1 - \ln \delta)\delta^{1 - \lambda}.$$

The $0 < \delta < 1$, the expression $|\ln \delta| \cdot \delta^{1-\lambda}$ has a maximum value of $e^{-1}/(1-\lambda)$. Hence, we have that

$$\frac{\left|\frac{\partial \psi}{\partial x_i}(\xi) - \frac{\partial \psi}{\partial x_i}(\eta)\right|}{|\xi - \eta|^{\lambda}} \le C(\mathcal{M}, |q|_{\infty}) \frac{1}{1 - \lambda}.$$

The lemma is proven.

In the sequel, we will need the following regularity result, which can be easily derived from the classical L^p theory for elliptic equations with Dirichlet boundary conditions ([21]).

LEMMA 3.2. Let $g \in L^p(\mathcal{M})$ with p > 1, and let ψ be a solution of

(30a)
$$\Delta \psi - \psi = \frac{\partial g}{\partial x_i}, \qquad \mathcal{M},$$

(30b)
$$\psi = l, \qquad \partial \mathcal{M},$$

(30c)
$$\int_{\mathcal{M}} \psi dx = 0.$$

Then, ψ has one generalized derivative, and

(31)
$$\|\psi\|_{W^{1,p}(\mathcal{M})} \le Cp\|g\|_{L^p(\mathcal{M})}$$

4. Weak formulation and the uniqueness. We assume that ψ is a classical solution of (4) subjecting to the constraints (6)–(8). We let $\varphi \in C^{\infty}(Q_T)$ with $\varphi|_{\partial\mathcal{M}} = \varphi|_{t=T} = 0$. We multiply (4) with φ and integrate by parts to obtain

$$(32) - \int_{\mathcal{M}} (\Delta \psi_0 - \psi_0) \varphi(x, 0) dx - \int_0^T \int_{\mathcal{M}} (\Delta \psi - \psi) \frac{\partial \varphi}{\partial t} dx dt - \int_0^T \int_{\mathcal{M}} (\Delta \psi + y - \psi) \nabla^{\perp} \psi \cdot \nabla \varphi dx dt = \int_0^T \int_{\mathcal{M}} f \varphi dx dt.$$

Thus, every classical solution of the barotropic QG equation also solves the integral equation (32), but the converse is not true, for the QGPV $q = \Delta \psi + y - \psi$ may not be differentiable either in space x or in time t. Solutions of (32) are called weak solutions of the barotropic QG.

We establish the well-posedness of the barotropic QG (4), (6)–(8) by working with its weak formulation first, whose precise statement is given here.

Statement of the problem:

(33) Let $\psi_0 \in V$. Find $\psi \in L^{\infty}(0,T;V)$ such that (32) holds for every $\varphi \in C^{\infty}(Q_T)$ with $\varphi|_{\partial \mathcal{M}} = 0 = \varphi|_{t=T} = 0$.

We choose $\varphi(x,t) = g(t)\gamma(x)$ in (32) with $g \in C^{\infty}([0,T])$, g(T) = 0, and $\gamma \in C_c^{\infty}(\mathcal{M})$. Substituting this φ into (32), we have

$$(34) \quad -g(0) \int_{\mathcal{M}} (\Delta \psi_0 - \psi_0) \gamma dx - \int_0^T g'(t) \int_{\mathcal{M}} (\Delta \psi - \psi) \gamma(x) dx dt - \int_0^T g(t) \int_{\mathcal{M}} (\Delta \psi + y - \psi) \nabla^{\perp} \psi \cdot \nabla \gamma dx dt = \int_0^T g(t) \int_{\mathcal{M}} f \gamma dx dt.$$

If we take g(0) = 0 as well, then (34) becomes

(35)
$$-\int_{0}^{T} g'(t) \int_{\mathcal{M}} (\Delta \psi - \psi) \gamma(x) dx dt = \int_{0}^{T} g(t) \int_{\mathcal{M}} \left((\Delta \psi + y - \psi) \nabla^{\perp} \psi \cdot \nabla \gamma + f \gamma \right) dx dt.$$

This shows that

(36)
$$\frac{d}{dt} \int_{\mathcal{M}} (\Delta \psi - \psi) \gamma(x) dx dt = \int_{\mathcal{M}} \left((\Delta \psi + y - \psi) \nabla^{\perp} \psi \cdot \nabla \gamma + f \gamma \right) dx \quad \text{in } \mathcal{D}'(0, T).$$

Thanks to the fact that $C_c^{\infty}(\mathcal{M})$ is dense in $H_0^1(\mathcal{M})$, the above also holds for every $\varphi \in H_0^1(\mathcal{M})$. Thus, we conclude that $\Delta \psi - \psi$ is weakly continuous in time in the following sense,

$$\int_{\mathcal{M}} (\Delta \psi - \psi) \gamma dx \text{ is continuous in time for every } \gamma \in H_0^1(\mathcal{M}).$$

Integrating by parts in (34), we find

$$(37) \quad -g(0) \int_{\mathcal{M}} (\nabla \psi_0 \cdot \nabla \gamma + \psi_0 \gamma) dx + \int_0^T g'(t) \int_{\mathcal{M}} (\nabla \psi \cdot \nabla \gamma + \psi \gamma) dx dt = \int_0^T g(t) \int_{\mathcal{M}} \left((\Delta \psi + y - \psi) \nabla^{\perp} \psi \cdot \nabla \gamma + f \gamma \right) dx dt.$$

Again, taking g(0) = 0 yields

$$\int_{0}^{T} g'(t) \int_{\mathcal{M}} (\nabla \psi \cdot \nabla \gamma + \psi \gamma) dx dt = \int_{0}^{T} g(t) \int_{\mathcal{M}} \left((\Delta \psi + y - \psi) \nabla^{\perp} \psi \cdot \nabla \gamma + f \gamma \right) dx dt.$$

Since $C_c^{\infty}(\mathcal{M})$ is dense in the space $H_0^1(\mathcal{M})$ under the usual H^1 -norm, the above holds for every $\gamma \in H_0^1(\mathcal{M})$. Thus,

$$\frac{d}{dt} \int_{\mathcal{M}} (\nabla \psi \cdot \nabla \gamma + \psi \gamma) dx = -\int_{\mathcal{M}} \left((\Delta \psi + y - \psi) \nabla^{\perp} \psi \cdot \nabla \gamma + f \gamma \right) dx \quad \text{in } \mathcal{D}'(0, T).$$

This implies that ψ is weakly continuous in time for the H^1 -norm,

$$\int_{M} (\nabla \psi \cdot \nabla \gamma + \psi \gamma) dx \text{ is continuous in time for every } \gamma \in H_0^1(\mathcal{M}).$$

To investigate the initial value of ψ , we take $g \in \mathcal{C}^{\infty}([0,T])$ with $g(0) \neq 0$ and g(T) = 0. We multiply (36) by g(t) and integrate by parts in t to obtain

$$(40) \quad g(0) \int_{\mathcal{M}} (\Delta \psi(x,0) - \psi(x,0)) \gamma dx - \int_{0}^{T} g'(t) \int_{\mathcal{M}} (\Delta \psi - \psi) \gamma(x) dx dt - \int_{0}^{T} g(t) \int_{\mathcal{M}} (\Delta \psi + y - \psi) \nabla^{\perp} \psi \cdot \nabla \gamma dx dt = \int_{0}^{T} g(t) \int_{\mathcal{M}} f \gamma dx dt.$$

Comparing (40) with (34), we find that

(41)
$$\int_{\mathcal{M}} \left(\left(\Delta \psi(x,0) - \psi(x,0) \right) - \left(\Delta \psi_0 - \psi_0 \right) \right) \gamma dx = 0, \quad \forall \gamma \in C_c^{\infty}(\mathcal{M}).$$

Since $C_c^{\infty}(\mathcal{M})$ is dense in $L^2(\mathcal{M})$, the above holds for every $\gamma \in L^2(\mathcal{M})$. In particular, setting γ to the function in the parentheses, we reach

$$|\Delta \psi - \psi|_{t=0} = \Delta \psi_0 - \psi_0 \quad \text{in } L^2(\mathcal{M}).$$

Multiplying (39) by the same g(t) and integrating by parts in time, we obtain

$$(42) \quad -g(0) \int_{\mathcal{M}} (\nabla \psi(x,0) \cdot \nabla \gamma + \psi(x,0) \gamma) dx + \int_{0}^{T} g'(t) \int_{\mathcal{M}} (\nabla \psi \cdot \nabla \gamma + \psi \gamma) dx dt$$
$$\int_{0}^{T} g(t) \int_{\mathcal{M}} \left((\Delta \psi + y - \psi) \nabla^{\perp} \psi \cdot \nabla \gamma + f \gamma \right) dx dt.$$

Comparing this equation with (37), we easily see that

(43)
$$\int_{\mathcal{M}} ((\nabla \psi(x,0) - \nabla \psi_0) \cdot \nabla \gamma + (\psi(x,0) - \psi_0)\gamma) dx = 0, \quad \forall \gamma \in H_0^1(\mathcal{M}).$$

From the above, one can infer that the initial condition (8) is satisfied in the H^1 -norm. Indeed, we let

(44)
$$h(x) = \psi(x,0) - \psi_0(x).$$

Both ψ_0 and $\psi(\cdot,0)$ assume a constant value on the boundary, and so does h(x). If h(x) vanishes on the boundary, then we can simply set γ to h(x), and the conclusion follows. If h(x) does not vanish on the boundary, then let σ be its constant boundary value, and write

(45)
$$h(x) = h^{\#}(x) + \sigma.$$

The value σ is related to $h^{\#}$ via

(46)
$$\sigma = -\frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} h^{\#}(x) dx.$$

Then $h^{\#}$ vanishes on the boundary and belongs to $H_0^1(\mathcal{M})$. Substituting (45) and (46) into (43), and setting γ to $h^{\#}$, we obtain

(47)
$$\int_{\mathcal{M}} |\nabla h^{\#}(x,0)|^2 dx + \int_{\mathcal{M}} |h^{\#}(x,0)|^2 dx = \frac{1}{|\mathcal{M}|} \left(\int_{\mathcal{M}} h^{\#}(x,0) dx \right)^2.$$

It is an easy exercise to show that

(48)
$$\frac{1}{|\mathcal{M}|} \left(\int_{\mathcal{M}} h^{\#}(x,\tau) dx \right)^{2} \leq \int_{\mathcal{M}} |h^{\#}(x,\tau)|^{2} dx, \qquad \forall \tau.$$

In view of (48), the relation (47) is only possible if

$$|\nabla h^{\#}(\cdot,0)|_{L^{2}(\mathcal{M})} = |h^{\#}(\cdot,0)|_{L^{2}(\mathcal{M})} = 0.$$

Hence,

$$\psi|_{t=0} = \psi_0 \quad \text{in } H^1(\mathcal{M}).$$

We formally summarize these results in the following lemma.

LEMMA 4.1. The solution ψ to the weak formulation (32), if it exists, is weakly continuous in the following sense,

(49a)
$$\int_{\mathcal{M}} (\Delta \psi - \psi) \gamma dx \text{ is continuous in time for every } \gamma \in H_0^1(\mathcal{M}),$$

(49b)
$$\int_{M} (\nabla \psi \cdot \nabla \gamma + \psi \gamma) dx \text{ is continuous in time for every } \gamma \in H_0^1(\mathcal{M}).$$

The initial condition is satisfied in the sense that

(50a)
$$\Delta \psi - \psi|_{t=0} = \Delta \psi_0 - \psi_0 \qquad in \ L^2(\mathcal{M})$$

(50b)
$$\psi|_{t=0} = \psi_0 \qquad in \ H^1(\mathcal{M}).$$

By virtue of Lemma 3.1, any weak solutions of (32) automatically have second weak derivatives in space. In fact, it also has second temporal-spatial cross derivatives, according to the following lemma.

LEMMA 4.2. Let $\psi(x,t)$ be a generalized solution of (4), (6)–(8) in the sense of (32). Then there exists generalized derivatives $\partial^2 \psi/\partial x \partial t$ and, for any p > 1,

(51)
$$\sup_{0 < t < T} \| \frac{\partial^2 \psi}{\partial x \partial t} \|_{L^p(\mathcal{M})} \le Cp \sup_{0 < t < T} \left(\| F \|_{L^p(\mathcal{M})} + \| \psi \|_{L^{\infty}(0,T;V)} \cdot \| \nabla \psi \|_{L^p(\mathcal{M})} \right).$$

Proof. We can rewrite equation (4) as an elliptic equation,

(52)
$$(\Delta - I)\frac{\partial}{\partial t}\psi = \nabla \times F - \nabla \cdot \left(\nabla^{\perp}\psi(\Delta\psi + y - \psi)\right).$$

Then, by Lemma 3.2,

$$\left\| \frac{\partial \psi}{\partial t} \right\|_{W^{1,p}(\mathcal{M})} \le Cp \left(\|F\|_{L^p(\mathcal{M})} + \|\psi\|_{L^{\infty}(0,T:V)} \|\nabla \psi\|_{L^p(\mathcal{M})} \right).$$

Taking the supreme norm in time t on the right-hand side, and then on the left-hand side, we obtain

$$\sup_{0 < t < T} \left\| \frac{\partial \psi}{\partial t} \right\|_{W^{1,p}(\mathcal{M})} \le Cp \sup_{0 < t < T} \left(\|F\|_{L^p(\mathcal{M})} + \|\psi\|_{L^{\infty}(0,T:V)} \|\nabla \psi\|_{L^p(\mathcal{M})} \right).$$

Finally, we are in a position to address the uniqueness of the generalized solution of (32).

THEOREM 4.3. The generalized solution to the barotropic QG equation (4), (6)–(8) in the sense of (32), if exists, must be unique.

Proof. We let ψ^1 and ψ^2 be two solutions to the weak problem for the same initial data ψ_0 . Then, for any $t \in [0, T]$ and an arbitrary $\varphi \in C^{\infty}(Q_t)$ with $\varphi|_{\partial\mathcal{M}} = \varphi(\cdot, t) = 0$, ψ^1 and ψ^2 satisfy the following equations, respectively,

$$(53) \int_{\mathcal{M}} (\Delta \psi_0 - \psi_0) \varphi(x, 0) dx - \int_0^t \int_{\mathcal{M}} (\Delta \psi^1 - \psi^1) \frac{\partial \varphi}{\partial t} dx dt - \int_0^T \int_{\mathcal{M}} (\Delta \psi^1 + y - \psi^1) \nabla^{\perp} \psi^1 \cdot \nabla \varphi dx dt = \int_0^T \int_{\mathcal{M}} f \varphi dx dt,$$

$$(54) \int_{\mathcal{M}} (\Delta \psi_0 - \psi_0) \varphi(x, 0) dx - \int_0^t \int_{\mathcal{M}} (\Delta \psi^2 - \psi^2) \frac{\partial \varphi}{\partial t} dx dt - \int_0^T \int_{\mathcal{M}} (\Delta \psi^2 + y - \psi^2) \nabla^{\perp} \psi^2 \cdot \nabla \varphi dx dt = \int_0^T \int_{\mathcal{M}} f \varphi dx dt.$$

Subtracting these two equations, and denoting $h = \psi^1 - \psi^2$, we obtain

$$(55) - \int_0^t \int_{\mathcal{M}} (\Delta h - h) \frac{\partial \varphi}{\partial t} dx dt - \int_0^T \int_{\mathcal{M}} (\Delta h - h) \nabla^{\perp} \psi^1 \cdot \nabla \varphi dx dt + \int_0^T \int_{\mathcal{M}} (\Delta \psi^2 + y - \psi^2) \nabla^{\perp} h \cdot \nabla \varphi dx dt = 0.$$

An integration by parts in space in the first term leads to

(56)
$$\int_{0}^{t} \int_{\mathcal{M}} (\nabla h \cdot \nabla \partial_{t} \varphi + h \partial_{t} \varphi) dx dt - \int_{0}^{T} \int_{\mathcal{M}} (\Delta h - h) \nabla^{\perp} \psi^{1} \cdot \nabla \varphi dx dt + \int_{0}^{T} \int_{\mathcal{M}} (\Delta \psi^{2} + y - \psi^{2}) \nabla^{\perp} h \cdot \nabla \varphi dx dt = 0.$$

Both φ^1 and φ^2 assume space-independent values on the boundary $\partial \mathcal{M}$, and so does the difference h between them. Thus, after a shifting in the vertical direction, h will vanish on the boundary. We denote this shifted function by $h^{\#} \in L^{\infty}(0,t;H_0^1(\mathcal{M}))$. h and $h^{\#}$ are related via

(57)
$$h(x,\tau) = h^{\#}(x,\tau) + l(\tau), \qquad 0 \le \tau \le t$$

for some function $l(\tau)$. Both ψ^1 and ψ^2 have a zero average over \mathcal{M} , and so does their difference h. Integrating (57) over \mathcal{M} we establish a simple relation between l and $h^{\#}$,

(58)
$$l(\tau) = -\frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} h^{\#}(x,\tau) dx.$$

Since both ψ^1 and ψ^2 satisfy the same initial condition (8) in the sense of (50b), it is easy to see that

(59)
$$h(\cdot,0) = h^{\#}(\cdot,0) = 0.$$

Replacing h by $h^{\#} + l$ in the first and third integrals of (56) yields

(60)
$$\int_{0}^{t} \int_{\mathcal{M}} (\nabla h^{\#} \cdot \nabla \partial_{t} \varphi + h^{\#} \partial_{t} \varphi) dx d\tau + \int_{0}^{t} \int_{M} l \partial_{t} \varphi dx d\tau - \int_{0}^{T} \int_{\mathcal{M}} (\Delta h - h) \nabla^{\perp} \psi^{1} \cdot \nabla \varphi dx d\tau + \int_{0}^{T} \int_{\mathcal{M}} (\Delta \psi^{2} + y - \psi^{2}) \nabla^{\perp} h^{\#} \cdot \nabla \varphi dx d\tau = 0.$$

In view of the regularity results in the previous lemma, and the facts that $h^{\#}(\cdot,0) = h(\cdot,0) = 0$ and $\varphi(\cdot,t) = 0$, we integrate by parts in time in (60) and arrive at

(61)
$$-\int_0^t \int_{\mathcal{M}} (\partial_t \nabla h^\# \cdot \nabla \varphi + \partial_t h^\# \varphi) dx d\tau - \int_0^t \int_{\mathcal{M}} (\partial_t h - h) \nabla^\perp \psi^1 \cdot \nabla \varphi dx d\tau + \int_0^T \int_{\mathcal{M}} (\Delta \psi^2 + y - \psi^2) \nabla^\perp h^\# \cdot \nabla \varphi dx d\tau = 0.$$

We note that each of the integrals is linear and continuous with respect to φ in the norm of $L^2(0,T:H^1_0(\mathcal{M}))$. Thus, we can let φ tend to $h^\#$ in $L^2(0,T;H^1_0(\mathcal{M}))$, pass to the limit in (61), and notice the fact that $\nabla h^\# \cdot \nabla^\perp h^\# = 0$, we obtain

$$-\int_0^t \int_{\mathcal{M}} (\partial_t \nabla h^\# \cdot \nabla h^\# + \partial_t h^\# h^\#) dx d\tau - \int_0^t \int_{M} \partial_t l h^\# dx d\tau$$
$$-\int_0^T \int_{\mathcal{M}} (\Delta h - h) \nabla^\perp \psi^1 \cdot \nabla h^\# dx d\tau = 0.$$

(62)
$$-\int_0^t \frac{1}{2} \frac{d}{dt} \|h^{\#}\|_{H_0^1(\mathcal{M})}^2 dt d\tau - \int_0^t \frac{1}{2} \frac{d}{dt} \|h^{\#}\|_{L^2(\mathcal{M})}^2 d\tau$$
$$-\int_0^t \partial_t l \int_{\mathcal{M}} h^{\#} dx d\tau - \int_0^t \int_{\mathcal{M}} (\Delta h - h) \nabla^{\perp} \psi^1 \cdot \nabla h^{\#} dx d\tau = 0.$$

In the above, $\|\cdot\|_{H_0^1(\mathcal{M})} \equiv \|\nabla(\cdot)\|_{L^2(\mathcal{M})}$. Using the expression in (58) for $l(\tau)$, we derive that

(63)
$$-\frac{1}{2} \|h^{\#}(\cdot,t)\|_{H_{0}^{1}(\mathcal{M})}^{2} d\tau - \frac{1}{2} \|h^{\#}(\cdot,t)\|_{L^{2}(\mathcal{M})}^{2} d\tau + \frac{1}{|\mathcal{M}|} \int_{0}^{t} \frac{1}{2} \partial_{t} \left(\int_{\mathcal{M}} h^{\#} dx \right)^{2} d\tau$$
$$- \int_{0}^{t} \int_{\mathcal{M}} (\Delta h - h) \nabla^{\perp} \psi^{1} \cdot \nabla h^{\#} dx d\tau = 0.$$

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(64)
$$\|h^{\#}(\cdot,t)\|_{H_0^1(\mathcal{M})}^2 + \|h^{\#}(\cdot,t)\|_{L^2(\mathcal{M})}^2 - \frac{1}{|\mathcal{M}|} \left(\int_{\mathcal{M}} h^{\#} dx \right)^2 = -2 \int_0^t \int_{\mathcal{M}} (\Delta h - h) \nabla^{\perp} \psi^1 \cdot \nabla h^{\#} dx d\tau.$$

Using the estimate (48) for $\tau = t$ in the above, one derives that

(65)
$$||h^{\#}(\cdot,t)||_{H_0^1(\mathcal{M})}^2 \le -2 \int_0^t \int_{\mathcal{M}} (\Delta h - h) \nabla^{\perp} \psi^1 \cdot \nabla h^{\#} dx d\tau.$$

First, we note that, with the change of variable (57),

$$\Delta h - h = \Delta h^{\#} - h^{\#} - l(\tau).$$

Thanks to the non-penetration boundary conditions on $\partial \mathcal{M}$, we have

$$\int_{\mathcal{M}} h^{\#} \nabla^{\perp} \psi^{1} \cdot \nabla h^{\#} dx = 0,$$
$$\int_{\mathcal{M}} l(\tau) \nabla^{\perp} \psi^{1} \cdot \nabla h^{\#} dx = 0.$$

Therefore,

$$\int_{\mathcal{M}} (\Delta h - h) \nabla^{\perp} \psi^{1} \cdot \nabla h^{\#} dx = \int_{\mathcal{M}} \Delta h^{\#} \nabla^{\perp} \psi^{1} \cdot \nabla h^{\#} dx,$$

and (65) becomes

(66)
$$||h^{\#}(\cdot,t)||_{H_0^1(\mathcal{M})}^2 \le -2 \int_0^t \int_{\mathcal{M}} \Delta h^{\#} \nabla^{\perp} \psi^1 \cdot \nabla h^{\#} dx d\tau.$$

To further investigate the integral on the right-hand side, we adopt the notations

$$(u_1, u_2) = \nabla^{\perp} \psi^1 = (-\partial_2 \psi^1, \partial_1 \psi^1).$$

Then, using the Einstein convention of repeated indices for $1 \leq i, j \leq 2$ and the fact that $\nabla^{\perp}\psi^1 \cdot \nabla h^{\#}$ vanishes on the boundary $\partial \mathcal{M}$, we proceed by integration by parts and obtain

$$\begin{split} \int_{\mathcal{M}} \Delta h^{\#} \nabla^{\perp} \psi^{1} \cdot \nabla h^{\#} dx &= \int_{\mathcal{M}} \partial_{i}^{2} h^{\#} u_{j} \partial_{j} h^{\#} dx \\ &= - \int_{\mathcal{M}} \left(\partial_{i} h^{\#} \partial_{i} u_{j} \partial_{j} h^{\#} + \partial_{i} h^{\#} u_{j} \partial_{j} \partial_{i} h^{\#} \right) dx \\ &= - \int_{\mathcal{M}} \partial_{i} h^{\#} \partial_{i} u_{j} \partial_{j} h^{\#} dx. \end{split}$$

Reverting back to the standard index conventions and after rearrangements, we find that

(67)
$$\int_{\mathcal{M}} \Delta h^{\#} \nabla^{\perp} \psi^{1} \cdot \nabla h^{\#} dx = -\int_{\mathcal{M}} \left\{ \left[-(\partial_{1} h^{\#})^{2} + (\partial_{2} h^{\#})^{2} \right] \partial_{1} \partial_{2} \psi^{1} + \partial_{1} h^{\#} \partial_{2} h^{\#} (\partial_{1}^{2} \psi^{1} - \partial_{2}^{2} \psi^{1}) \right\} dx.$$

Hence, we have

$$\begin{split} &\|h^{\#}(\cdot,t)\|_{H^{1}_{0}(\mathcal{M})}^{2} \\ \leq & 2\int_{0}^{t} \int_{\mathcal{M}} \left\{ \left[-(\partial_{1}h^{\#})^{2} + (\partial_{2}h^{\#})^{2} \right] \partial_{1}\partial_{2}\psi^{1} + \partial_{1}h^{\#}\partial_{2}h^{\#}(\partial_{1}^{2}\psi^{1} - \partial_{2}^{2}\psi^{1}) \right\} dxd\tau \\ \leq & \int_{0}^{t} \int_{\mathcal{M}} \left\{ 2\left(|\partial_{1}h^{\#}|^{2} + |\partial_{2}h^{\#}|^{2} \right) \cdot |\partial_{1}\partial_{2}\psi^{1}| + (|\partial_{1}h^{\#}|^{2} + |\partial_{2}h^{\#}|^{2})(|\partial_{1}^{2}\psi^{1}| + |\partial_{2}^{2}\psi^{1}|) \right\} dxd\tau \\ \leq & \int_{0}^{t} \int_{\mathcal{M}} |\nabla h^{\#}|^{2} \left(|\partial_{1}^{2}\psi^{1}| + |\partial_{2}^{2}\psi^{1}| + 2|\partial_{1}\partial_{2}\psi^{1}| \right) dxd\tau. \end{split}$$

We note that, according to Lemma 3.1, both $\nabla \psi^1$ and $\nabla \psi^2$ are Hölder continuous in \mathcal{M} and essentially bounded in (0,T), and so is $\nabla h^{\#}$. We let $0 < \epsilon < 1$ be an arbitrary parameter. Then, we have

$$||h^{\#}(\cdot,t)||_{H_0^1(\mathcal{M})}^2 \le |\nabla h^{\#}|_{L^{\infty}(Q_T)}^{\epsilon} \int_0^t \int_{\mathcal{M}} |\nabla h^{\#}|^{2-\epsilon} \left(|\partial_1^2 \psi^1| + |\partial_2^2 \psi^1| + 2|\partial_1 \partial_2 \psi^1| \right) dx d\tau.$$

Applying Hölder's inequality to the spatial integral on the right-hand side, we obtain

$$(68) \quad \|h^{\#}(\cdot,t)\|_{H_0^1(\mathcal{M})}^2 \leq C|\nabla h^{\#}|_{L^{\infty}(Q_T)}^{\epsilon} \int_0^t \|h^{\#}(\cdot,\tau)\|_{H_0^1(\mathcal{M})}^{2-\epsilon} \cdot \|\psi^1(\cdot,\tau)\|_{W^{2,\frac{2}{\epsilon}}(\mathcal{M})} d\tau.$$

We note that both ψ^1 and ψ^2 belongs to $L^{\infty}(0,T;V)$, and so does h. By Lemma 3.1, $\psi^1(\cdot,\tau) \in W^{2,p}(\mathcal{M})$ with $p=2/\epsilon$, and $\nabla h^{\#}$, which equals ∇h , is Hölder continuous, for a.e. τ . We set

$$M_1 = \sup_{0 < t < T} \|\psi^1(\cdot, t)\|_V,$$

$$M_2 = \sup_{0 < t < T} \|h^1(\cdot, t)\|_V.$$

Then, it is inferred from (21) and (22) that

$$\sup_{0 < t < T} \|\psi^{1}(\cdot, t)\|_{W^{2, \frac{2}{\epsilon}}(\mathcal{M})} \le C \frac{2}{\epsilon} M_{1},$$

$$\sup_{0 < t < T} \|\nabla h^{\#}(\cdot, t)\|_{L^{\infty}\mathcal{M}} \le C M_{2},$$

where C designates generic constants that are independent of ψ^1 , ψ^2 and ϵ . Using these estimates in (68) leads to

(69)
$$||h^{\#}(\cdot,t)||_{H_0^1(\mathcal{M})}^2 \le \frac{C}{\epsilon} M_1 M_2^{\epsilon} \int_0^t ||h^{\#}(\cdot,\tau)||_{H_0^1(\mathcal{M})}^{2-\epsilon} d\tau.$$

We denote

$$\sigma(\cdot,t) \equiv \|h^{\#}(\cdot,t)\|_{H_0^1(\mathcal{M})}.$$

Then (69) can be written as

(70)
$$\sigma^{2}(t) \leq \frac{C}{\epsilon} M_{1} M_{2}^{\epsilon} \int_{0}^{t} \sigma^{2-\epsilon}(\tau) d\tau.$$

An estimate on σ can be obtained by the Gronwall inequality. Indeed, we let

$$F(t) = \frac{C}{\epsilon} M_1 M_2^{\epsilon} \int_0^t \sigma^{2-\epsilon}(\tau) d\tau.$$

Taking derivative of this function and using (70), we find

$$\frac{d}{dt}F(t) \le \frac{C}{\epsilon} M_1 M_2^{\epsilon} F^{1-\frac{\epsilon}{2}}(t).$$

Integration of this inequality yields

$$F(t) \le (CM_1 t)^{\frac{2}{\epsilon}} M_2^2.$$

Thus,

(71)
$$||h^{\#}(\cdot,t)||_{H_0^1(\mathcal{M})}^2 \le F(t) \le (CM_1t)^{\frac{2}{\epsilon}} M_2^2.$$

We take

$$t^* = \frac{1}{2CM_1}.$$

Then, for $0 \le t \le t^*$,

(72)
$$||h^{\#}(\cdot,t)||_{H_0^1(\mathcal{M})}^2 \le \left(\frac{1}{2}\right)^{\frac{2}{\epsilon}} M_2^2.$$

This estimate holds for arbitrary $\epsilon > 0$. Thus, $||h^{\#}(\cdot,t)||^2_{H^1_0(\mathcal{M})}$ must vanish for $0 \le t \le t^*$. The process can be repeated over subsequent time intervals of length t^* , and thus $||h^{\#}(\cdot,t)||^2_{H^1_0(\mathcal{M})} = 0$ for the whole time interval [0,T]. Combined with the relations (57) and (58), it implies that

$$h(\cdot, t) = 0$$
 for a.e. $0 < t < T$.

The generalized solution to the barotropic QG equation (4), (6)–(8) must be unique.

5. Existence of a solution to the weak problem. We establish the existence of a solution to the weak problem (32) through an iterative scheme. To get started, we set

(73)
$$q^0(x,t) = q_0(x),$$

where q_0 is the initial QGPV computed from the initial streamfunction ψ_0 . We assume that q^n , for a $n \geq 0$, is known, we compute q^{n+1} as follows. First, the streamfunction ψ^n corresponding to q^n is obtained from the non-standard elliptic BVP,

(74a)
$$\Delta \psi^n - \psi^n + \beta y = q^n, \qquad \mathcal{M},$$

(74b)
$$\psi^n = l^n, \qquad \partial \mathcal{M},$$

(74c)
$$\int_{\mathcal{M}} \psi^n dx = 0.$$

The corresponding velocity field u^n is obtained through

(75)
$$\boldsymbol{u}^n = \nabla^{\perp} \psi^n.$$

From this velocity field, a flow mapping (or the trajectory path) $\Phi_t^n(a)$ for each $a \in \mathcal{M}$ is constructed,

(76a)
$$\frac{d}{dt}\Phi_t^n(a) = \boldsymbol{u}^n(\Phi_t^n(a), t),$$

$$\Phi_0^n(a) = a.$$

Finally, the QGPV field is updated using the flow mapping,

(77)
$$q^{n+1}(\Phi_t^n(a), t) = q_0(a) + \int_0^t f(\Phi_s^n(a), s) ds, \quad \forall a \in \mathcal{M},$$

or, equivalently,

(78)
$$q^{n+1}(x,t) = q_0(\Phi_{-t}^n(x)) + \int_0^t f(\Phi_{s-t}^n(x), s) ds, \quad \forall x \in \mathcal{M}.$$

The iterative scheme (73)–(78) is straightforward except for the solution of the initial value problem (76). The elegant argument presented in [26] requires \boldsymbol{u} to be C^1 , which is not the case here. Given that $q^n(\cdot,t) \in L^{\infty}(\mathcal{M})$, u^n is quasi-Lipschitz according to Lemma 3.1. It is well known that (see [38]) the quasi-Lipschitz continuity is also *sufficient* to ensure the global existence and uniqueness of a solution to the IVP. Specifically, the following result is available.

LEMMA 5.1. Assume the velocity field \mathbf{u}^n is uniformly bounded in Q_T and quasi-Lipschitz continuous in \mathcal{M} , i.e. $\mathbf{u}^n(x,t) \leq C$ for $\forall (x,t) \in Q_T$, $|\mathbf{u}^n(x_1,t)-\mathbf{u}^n(x_2,t)| \leq C\chi(|x_1-x_2|)$, with C independent of x or t. Then the initial value problem (76) has a global unique solution.

The proof is recalled in Appendix A. We note that, since u^n is divergence free, the corresponding flow mapping preserves area, by virtue of the Liouville Theorem (see e.g. [22]).

We now show that the sequence $(q^n, \psi^n, \mathbf{u}^n)$ generated by the iterative scheme converges, and the limit solves the weak problem (32). We follow the procedure laid out in [38], and start with the convergence of the flow mapping $\Phi_t^n(\cdot)$.

Lemma 5.2. As $n \longrightarrow \infty$,

$$\Phi_t^n(a) \longrightarrow \Phi_t(a)$$
 strongly in $L^{\infty}(0,T;L^1(\mathcal{M}))$.

Proof. We first write the IVP (76) in the integral form, From (122),

(79)
$$\Phi_t^n(a) = a + \int_0^t \boldsymbol{u}^n(\Phi_s^n(a), s) ds.$$

We subtract $\Phi_t^n(a)$ and $\Phi_t^{n-1}(a)$,

(80)
$$\Phi_t^n(a) - \Phi_t^{n-1}(a) = \int_0^t (\boldsymbol{u}^n(\Phi_s^n(a), s) - \boldsymbol{u}^n(\Phi_s^n(a), s)) \, ds.$$

We define

(81)
$$K(x,y) = \nabla_x^{\perp} G(x,y),$$

where G(x,y) is the Green's function for $\Delta - I$ with Dirichlet boundary conditions on \mathcal{M} . Then, using the relation (19),

(82)
$$\mathbf{u}^{n}(\Phi_{s}^{n}(a),s) = \nabla^{\perp}\psi_{n}(\Phi_{s}^{n}(a),s) = \int_{\mathcal{M}} K(\Phi_{s}^{n}(a),y)(q^{n}(y,s) - y_{2})dy + l(q^{n}) \int_{\mathcal{M}} K(\Phi_{s}^{n}(a),y)dy.$$

Thus,

$$\begin{split} & \boldsymbol{u}^{n}(\Phi_{s}^{n}(a),s) - \boldsymbol{u}^{n-1}(\Phi_{s}^{n-1}(a),s) \\ & = \int_{\mathcal{M}} \left(K(\Phi_{s}^{n}(a),y) q^{n}(y,s) - K(\Phi_{s}^{n-1}(a),y) q^{n-1}(y,s) \right) dy - \\ & \int_{\mathcal{M}} \left(K(\Phi_{s}^{n}(a),y) - K(\Phi_{s}^{n-1}(a),y) \right) y_{2} dy + \\ & \int_{\mathcal{M}} \left(l(q^{n}) K(\Phi_{s}^{n}(a),y) - l(q^{n-1}) K(\Phi_{s}^{n-1}(a),y) \right) y_{2} dy \end{split}$$

Using (77), the difference between $K(\Phi_s^n(a), y)q^n(y, s)$ and $K(\Phi_s^{n-1}(a), y)q^{n-1}(y, s)$ can be split into three parts,

$$\int_{\mathcal{M}} \left(K(\Phi_{s}^{n}(a), y) q^{n}(y, s) - K(\Phi_{s}^{n-1}(a), y) q^{n-1}(y, s) \right) dy =$$

$$\int_{\mathcal{M}} \left(K(\Phi_{s}^{n}(a), y) - K(\Phi_{s}^{n-1}(a), y) \right) q^{n}(y, s) dy +$$

$$\int_{\mathcal{M}} K(\Phi_{s}^{n-1}(a), y) \left(q_{0}(\Phi_{-s}^{n-1}(y)) - q_{0}(\Phi_{-s}^{n-2}(y)) \right) dy +$$

$$\int_{\mathcal{M}} K(\Phi_{s}^{n-1}(a), y) \int_{0}^{s} \left(f(\Phi_{\tau-s}^{n-1}(y), \tau) - f(\Phi_{\tau-s}^{n-2}(y), \tau) \right) d\tau dy.$$

Thus, the difference in the velocity fields $u^n(\Phi^n_s(a), s)$ and $\boldsymbol{u}^{n-1}(\Phi^{n-1}_s(a), s)$ is split into four parts, labeled as I - IV below,

$$\begin{aligned} & u^{n}(\Phi_{s}^{n}(a),s) - u^{n-1}(\Phi_{s}^{n-1}(a),s) = \\ & \int_{\mathcal{M}} \left(K(\Phi_{s}^{n}(a),y) - K(\Phi_{s}^{n-1}(a),y) \right) \left(q^{n}(y,s) - y_{2} + l(q^{n}) \right) dy + \\ & \int_{\mathcal{M}} K(\Phi_{s}^{n-1}(a),y) \left(q_{0}(\Phi_{-s}^{n-1}(y)) - q_{0}(\Phi_{-s}^{n-2}(y)) \right) dy + \\ & \int_{\mathcal{M}} K(\Phi_{s}^{n-1}(a),y) \int_{0}^{s} \left(f(\Phi_{\tau-s}^{n-1}(y),\tau) - f(\Phi_{\tau-s}^{n-2}(y),\tau) \right) d\tau dy + \\ & \int_{\mathcal{M}} K(\Phi_{s}^{n-1}(a),y) \left(l(q^{n}) - l(q^{n-1}) \right) dy \\ & := I + II + III + IV. \end{aligned}$$

We now estimate each of these four parts. For the first term I, it is clear from the proof of Lemma (3.1) that $\int_{\mathcal{M}} K(x,y)dy$ is quasi-Lipschitz continuous. Also using the fact that q^n is essentially bounded, we proceed,

$$|I| \le |q^n - y_2 + l|_{\infty} \int_{\mathcal{M}} |K(\Phi_s^n(a), y) - K(\Phi_s^{n-1}(a), y)| dy$$

$$\le C|q^n - y_2 + l|_{\infty} \left(1 + \left|\ln|\Phi_s^n(a) - \Phi_s^{n-1}(a)|\right|\right) \cdot |\Phi_s^n(a) - \Phi_s^{n-1}(a)|.$$

We define

(84)
$$\delta_s^n(a) = \left| \Phi_s^n(a) - \Phi_s^{n-1}(a) \right|.$$

Then, for the term I we have

(85)
$$|I| \le C(\mathcal{M}, |q_0|_{\infty}, |f|_{\infty}) \chi \left(\delta_s^n(a)\right),$$

We apply change of variables in II, utilizing that fact that the mapping $y = \Phi_s^n(b)$ is area preserving,

$$II = \int_{\mathcal{M}} K\left(\Phi_{s}^{n-1}(a), y\right) q_{0}\left(\Phi_{-s}^{n-1}(y)\right) dy - \int_{\mathcal{M}} K\left(\Phi_{s}^{n-1}(a), y\right) q_{0}\left(\Phi_{-s}^{n-2}(y)\right) db$$

$$= \int_{\mathcal{M}} K\left(\Phi_{s}^{n-1}(a), \Phi_{s}^{n-1}(b)\right) q_{0}(b) db - \int_{\mathcal{M}} K\left(\Phi_{s}^{n-1}(a), \Phi_{s}^{n-2}(b)\right) q_{0}(b) db$$

$$= \int_{\mathcal{M}} \left(K\left(\Phi_{s}^{n-1}(a), \Phi_{s}^{n-1}(b)\right)\right) - K\left(\Phi_{s}^{n-1}(a), \Phi_{s}^{n-2}(b)\right) q_{0}(b) db.$$

Integrating II in a over \mathcal{M} , we find that

$$\int_{\mathcal{M}} |II| da \le |q_0|_{\infty} \int_{\mathcal{M}} \int_{\mathcal{M}} \left| K\left(x, \Phi_s^{n-1}(b)\right) - K\left(x, \Phi_s^{n-2}(b)\right) \right| dx db$$

Again, with the quasi-Lipschitz continuity of $\int_{\mathcal{M}} K(x,y)dy$, we derive that

(86)
$$\int_{\mathcal{M}} |II| da \le |q_0|_{\infty} \int_{\mathcal{M}} \chi(\delta_s^{n-1}(b)) db.$$

Also applying the change of variable and integration to III,

$$\begin{split} III &= \int_{0}^{s} \int_{\mathcal{M}} K\left(\Phi_{s}^{n-1}(a), y\right) f\left(\Phi_{\tau-s}^{n-1}(y), \tau\right) dy d\tau - \int_{0}^{s} \int_{\mathcal{M}} K\left(\Phi_{s}^{n-1}(a), y\right) f\left(\Phi_{\tau-s}^{n-2}(y), \tau\right) dy d\tau \\ &= \int_{0}^{s} \int_{\mathcal{M}} K\left(\Phi_{s}^{n-1}(a), \Phi_{s-\tau}^{n-1}(a)\right) f\left(b, \tau\right) db d\tau - \int_{0}^{s} \int_{\mathcal{M}} K\left(\Phi_{s}^{n-1}(a), \Phi_{s-\tau}^{n-2}(b)\right) f\left(b, \tau\right) db d\tau \\ &= \int_{0}^{s} \int_{\mathcal{M}} \left(K\left(\Phi_{s}^{n-1}(a), \Phi_{s-\tau}^{n-1}(b)\right)\right) - K\left(\Phi_{s}^{n-1}(a), \Phi_{s-\tau}^{n-2}(b)\right)\right) f\left(b, \tau\right) db d\tau. \end{split}$$

By integrating |III| over \mathcal{M} , and, again using the quasi-Lipschitz continuity of $\int_{\mathcal{M}} K(x,y)dy$, we reach

(87)
$$\int_{\mathcal{M}} |III| da \leq |f|_{\infty} \int_{\mathcal{M}} \int_{0}^{s} \chi(\delta_{s-\tau}^{n-1}(b)) d\tau db.$$

Using the formula (18) for l(q), we find that

(88)
$$l(q^n) - l(q^{n-1}) = -\frac{\int_{\mathcal{M}} \int_{\mathcal{M}} G(x, y) (q^n(y, s) - q^{n-1}(y, s)) dy dx}{|\mathcal{M}| + \int_{\mathcal{M}} \int_{\mathcal{M}} G(x, y) dy dx}.$$

Using the formula (78) for q^n and q^{n-1} in the above, and the mapping provided by Φ_t^n and Φ_t^{n-1} , we obtain that, after some changes of variables,

$$\begin{split} l(q^n) - l(q^{n-1}) &= -\frac{1}{|\mathcal{M}| + \int_{\mathcal{M}} \int_{\mathcal{M}} G(x, y) dy dx} \cdot \\ & \left(\int_{\mathcal{M}} \int_{\mathcal{M}} q_0(a) \left(G(x, \Phi_s^{n-1}(a)) - G(x, \Phi_s^{n-2}(a)) \right) dx \, da + \\ & \int_{\mathcal{M}} \int_{\mathcal{M}} \int_0^s \left(G(x, \Phi_{s-\tau}^{n-1}(a)) - G(x, \Phi_{s-\tau}^{n-2}(a)) \right) f(a, \tau) d\tau dx da \right). \end{split}$$

Using the assumptions that q_0 and f are essentially bounded, in \mathcal{M} and Q_T , respectively, we obtain

$$|l(q^{n}) - l(q^{n-1})| = \frac{|q_{0}|_{\infty} + |f|_{\infty}}{|\mathcal{M}| + \int_{\mathcal{M}} \int_{\mathcal{M}} G(x, y) dy dx} \cdot \left(\int_{\mathcal{M}} \int_{\mathcal{M}} |G(x, \Phi_{s}^{n-1}(a)) - G(x, \Phi_{s}^{n-2}(a))| dx da + \int_{\mathcal{M}} \int_{\mathcal{M}} \int_{0}^{s} |G(x, \Phi_{s-\tau}^{n-1}(a)) - G(x, \Phi_{s-\tau}^{n-2}(a))| d\tau dx da \right).$$

It is also clear from the proof of Lemma 3.1 that $\int_{\mathcal{M}} G(x,y) dx$ is C^1 . Therefore, we have that

$$|l(q^{n}) - l(q^{n-1})| \le C(\mathcal{M}, q_0, f) \left(\int_{\mathcal{M}} |\Phi_s^{n-1}(a) - \Phi_s^{n-2}(a)| da + \int_{\mathcal{M}} \int_0^s |\Phi_{s-\tau}^{n-1}(a) - \Phi_{s-\tau}^{n-2}(a)| d\tau da \right).$$

Using the notation introduced in (84), one can write the above estimate as

(89)
$$|l(q^n) - l(q^{n-1})| \le C(\mathcal{M}, q_0, f) \left(\int_{\mathcal{M}} \delta_s^{n-1}(a) da + \int_0^s \int_{\mathcal{M}} \delta_{s-\tau}^{n-1}(a) da d\tau \right).$$

Thus,

$$|IV| \le C(\mathcal{M}, |q_0|_{\infty}, |f|_{\infty}) \left(\int_{\mathcal{M}} \delta_s^{n-1}(a) da + \int_0^s \int_{\mathcal{M}} \delta_{s-\tau}^{n-1} da d\tau \right) \cdot \int_{\mathcal{M}} K(\Phi_s^{n-1}(a), y) dy$$

In the analysis leading to the estimate (25), it is clear that the last integral on the right-hand side is uniformly bounded, and the bound depends on the domain \mathcal{M} only. Therefore, we can simply write the above relation as

$$(90) \qquad |IV| \le C(\mathcal{M}, |q_0|_{\infty}, |f|_{\infty}) \left(\int_{\mathcal{M}} \delta_s^{n-1}(a) da + \int_0^s \int_{\mathcal{M}} \delta_{s-\tau}^{n-1}(a) da \, d\tau \right).$$

Using the relation (83) and the estimates (85), (86), (87), and (90) in (80), we obtain

$$(91) \int_{\mathcal{M}} \left| \Phi_t^n(a) - \Phi_t^{n-1}(a) \right| da \leq C(\mathcal{M}, |q_0|_{\infty}, |f|_{\infty}) \int_0^t \left(\int_{\mathcal{M}} \chi(\delta_s^n(a)) da + \int_{\mathcal{M}} \chi(\delta_s^{n-1}(a)) da + \int_0^s \int_{\mathcal{M}} \chi(\delta_{s-\tau}^{n-1}(a) da d\tau + \int_{\mathcal{M}} \delta_s^{n-1}(a) da + \int_0^s \int_{\mathcal{M}} \delta_{s-\tau}^{n-1}(a) da d\tau \right) ds.$$

We define

(92)
$$\rho_t^n = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} |\Phi_t^n(a) - \Phi_t^{n-1}(a)| da \equiv \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \delta_t^n(a) da.$$

We note that, thanks to the convexity of χ ,

(93)
$$\frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \chi\left(\delta_t^n(a)\right) da \le \chi\left(\frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \delta_t^n(a) da\right) \equiv \chi(\rho_s^n).$$

Then (91) can be written as (94)

$$\rho_t^n \le C(\mathcal{M}, |q_0|_{\infty}, |f|_{\infty}) \int_0^t \left(\chi(\rho_s^n) + \chi(\rho_s^{n-1}) + \rho_s^{n-1} + \int_s \chi(\rho_{s-\tau}^{n-1}) d\tau + \int_0^s \rho_{s-\tau}^{n-1} d\tau \right) ds.$$

For an arbitrary n > 0, we define

(95)
$$e^n(t) = \sup_{k > n} \rho_t^k.$$

Therefore, for a fixed t, $\rho_t^n \leq e^n(t)$, and $e^n(t)$ is monotonically decreasing in n. We also note that, since $\chi(\cdot)$ is a monotonically increasing function,

$$\chi(\rho_s^n) \le \chi(e^n(s)) \le \chi(e^{n-1}(s)).$$

Using these relations in (94), one finds that

(96)
$$e^{n}(t) \leq C(\mathcal{M}, |q_{0}|_{\infty}, |f|_{\infty}) \int_{0}^{t} \left(2\chi(e^{n-1}(s)) + e^{n-1}(s) + \int_{0}^{s} \chi(e^{n-1}(s-\tau))d\tau + \int_{0}^{s} e^{n-1}(s-\tau)d\tau\right) ds.$$

For an arbitrary scalar function $h(\cdot)$, we note that

$$\int_0^t \int_0^s h(s-\tau)d\tau ds = \int_0^t (t-s)h(s)ds.$$

Thus one can replace the double integrals in (96) with single integrals, and after rearrangements, one obtains

$$(97) e^{n}(t) \le C(\mathcal{M}, |q_0|_{\infty}, |f|_{\infty}) \int_0^t \left((t-s+2)\chi(e^{n-1}(s)) + (t-s+1)e^{n-1}(s) \right) ds.$$

Using the inequality (126) on the function χ , one obtains from (97) that

$$e^{n}(t) \leq C(\mathcal{M}, |q_{0}|_{\infty}, |f|_{\infty}) \int_{0}^{t} \left((t - s + 2) \left(-\ln \epsilon \cdot e^{n-1}(s) + \epsilon \right) + (t - s + 1) e^{n-1}(s) \right) ds$$

$$\leq C(\mathcal{M}, |q_{0}|_{\infty}, |f|_{\infty}) \int_{0}^{t} \left(3 \left(-\ln \epsilon \cdot e^{n-1}(s) + \epsilon \right) + 2e^{n-1}(s) \right) ds$$

$$\leq C(\mathcal{M}, |q_{0}|_{\infty}, |f|_{\infty}) \int_{0}^{t} \left((-3 \ln \epsilon + 2) \cdot e^{n-1}(s) + 3\epsilon \right) ds$$

Focusing on small t's, say $t \in [0, 1]$, one derive from the above that

(98)
$$e^{n}(t) \leq C(\mathcal{M}, |q_0|_{\infty}, |f|_{\infty}) \left(-\ln \epsilon + 1\right) \int_0^t e^{n-1}(s)ds + C\epsilon t.$$

This inequality is now very similar to the inequality (127). When n = 1, by the definition (95) and the relation (80), one derives that

$$e^{1}(t) = \sup_{k \geq 1} \rho^{k}(t) = \sup_{k \geq 1} \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \left| \Phi_{t}^{k}(a) - \Phi_{t}^{k-1}(a) \right| da$$

$$\leq \sup_{k \geq 1} \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \int_{0}^{t} \left| \boldsymbol{u}_{t}^{k}(\Phi_{s}^{k}(a), s) - \boldsymbol{u}^{k-1}(\Phi_{s}^{k-1}(a), s) \right| ds da.$$

From above, and the uniform boundedness of u^k , one deduces that

(99)
$$e^{1}(t) \leq C(\mathcal{M}, |q_0|_{\infty}, |f|_{\infty})t.$$

We now temporarily omit the dependence of $C(\mathcal{M}, |q_0|_{\infty}, |f|_{\infty})$, and simply write C, for the sake of conciseness. By induction, one can derive that

(100)
$$e^{n}(t) \leq C^{n} \left(-\ln \epsilon + 1\right)^{n-1} \frac{t^{n}}{n!} + \epsilon \sum_{k=1}^{n-1} C^{k} \left(-\ln \epsilon + 1\right)^{k-1} \frac{t^{k}}{k!}.$$

Majorizing the last summation on the right-hand side, one obtains that

(101)
$$e^{n}(t) \leq \frac{C \cdot t \left(C(-\ln \epsilon + 1)t\right)^{n-1}}{n!} + C \cdot t \cdot \epsilon^{1-Ct} \cdot e^{Ct}.$$

As ϵ can be arbitrary, we take $\epsilon = e^{-(n-1)}$, and find that

(102)
$$e^{n}(t) \le \frac{(C \cdot t)^{n} n^{n-1}}{n!} + C \cdot t \cdot e^{-n(1-Ct)} \cdot e.$$

Applying the Stirling formula (130) in the above, one obtains that

(103)
$$e^{n}(t) \le \frac{(C \cdot e \cdot t)^{n}}{\sqrt{2\pi}n^{\frac{3}{2}}} + C \cdot t \cdot e \cdot e^{-n(1-Ct)}.$$

We let

$$t^* = \min\left\{\frac{1}{2Ce}, 1, T\right\}.$$

Then, for all $0 \le t \le t^*$,

(104)
$$e^{n}(t) \leq \frac{1}{\sqrt{2\pi}n^{\frac{3}{2}}} \left(\frac{1}{2}\right)^{n} + C \cdot T \cdot e \cdot e^{-\frac{1}{2}n}.$$

Thus, $e^n(t)$ is a geometrically converging sequence independent of the time t, and so is $\rho^n(t)$. We therefore have that

(105)
$$\Phi_t^n(a) \longrightarrow \Phi_t(a)$$
 strongly in $L^{\infty}(0, t^*, L^1(\mathcal{M}))$.

We note that t^* can be taken independently of ϵ , and it depends on \mathcal{M} , $|q_0|_{\infty}$, and $|f|_{\infty}$, and therefore the above process can be repeated over intervals of length t^* , until the whole interval [0, T] is covered. The lemma is thus proven.

We now study the convergence of q^n . We define

(106)
$$q(x,t) := q_0(\Phi_{-t}(x)) + \int_0^t f(\Phi_{s-t}(x), s) ds, \quad \forall x \in \mathcal{M}.$$

It is clear that $q \in L^{\infty}(Q_T)$, provided that $f \in L^{\infty}(Q_T)$. We have the following result.

LEMMA 5.3. Assume that $f \in L^{\infty}(0,T;C(\mathcal{M}))$. Then, for any $g \in C(\mathcal{M})$,

(107)
$$\int_{\mathcal{M}} g(x)q^n(x,t)dx \longrightarrow \int_{\mathcal{M}} g(x)q(x,t)dx \quad \text{as } n \longrightarrow \infty.$$

The convergence is uniform in t.

Proof. We subtract (106) from (78),

$$q^{n+1}(x,t) - q(x,t) = q_0(\Phi_{-t}^n(x)) - q_0(\Phi_{-t}(x)) + \int_0^t \left(f(\Phi_{s-t}^n(x),s) - f(\Phi_{s-t}(x),s) \right) ds.$$

Multiplying the above equation with a $g \in C^1(\mathcal{M})$, and integrate over \mathcal{M} , we obtain

$$\int_{\mathcal{M}} g(x) \left(q^{n+1}(x,t) - q(x,t) \right) dx$$

$$= \int_{\mathcal{M}} g(x) \left(q_0(\Phi_{-t}^n(x)) - q_0(\Phi_{-t}(x)) \right) dx +$$

$$\int_0^t \int_{\mathcal{M}} g(x) \left(f(\Phi_{s-t}^n(x),s) - f(\Phi_{s-t}(x),s) \right) dx ds$$

$$= \int_{\mathcal{M}} q_0(a) \left(g(\Phi_t^n(a)) - g(\Phi_t(a)) dx +$$

$$\int_0^t \int_{\mathcal{M}} g(x) \left(f(\Phi_{s-t}^n(x),s) - f(\Phi_{s-t}(x),s) \right) dx ds.$$

For the moment, we assume that $g \in C^1(\mathcal{M})$ and $f \in L^{\infty}(0,T;C^1(\mathcal{M}))$. Then, we derive that

$$\left| \int_{\mathcal{M}} g(x) \left(q^{n+1}(x,t) - q(x,t) \right) dx \right|$$

$$\leq |\nabla g|_{\infty} \cdot |q_0|_{\infty} \int_{\mathcal{M}} |\Phi_t^n(a) - \Phi_t(a)| da +$$

$$|g|_{\infty} \cdot |\nabla f|_{\infty} \int_0^t \int_{\mathcal{M}} |\Phi_{s-t}^n(x) - \Phi_{s-t}(x)| dx ds.$$

By the strong convergence of Φ_t^n in $L^{\infty}(0,T;L^1(\mathcal{M}))$, the right-hand side above converges to zero as $n \to \infty$. Then, by a continuity argument, one can show that, for any $g \in C(\mathcal{M})$ and $f \in L^{\infty}(0,T;C(\mathcal{M}))$,

$$\int_{\mathcal{M}} g(x)(q^n(x,t) - q(x,t))dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

We now verify the convergence of the velocity field \boldsymbol{u}^n . Using the QGPV q, we define

(108)
$$u(x,t) = \int_{\mathcal{M}} K(x), y)(q(y,t) - y_2)dy + l(q) \int_{\mathcal{M}} K(x), y)dy.$$

Lemma 5.4. As $n \longrightarrow \infty$,

$$\mathbf{u}^n(x,t) \longrightarrow \mathbf{u}(x,t)$$
 strongly in $L^{\infty}(0,T;L^1(\mathcal{M}))$.

Proof. Subtracting (108) from (82), we have

(109)
$$\mathbf{u}^{n}(x,t) - \mathbf{u}(x,t) = \int_{\mathcal{M}} K(x), y) (q^{n}(y,t) - q(y,t)) dy + (l(q^{n}) - l(q)) \int_{\mathcal{M}} K(x,y) dy.$$

From (18), one obtains

$$l(q^n) - l(q) = -\frac{\int_{\mathcal{M}} \int_{\mathcal{M}} G(x, y) (q^n(y) - q(y)) dy dx}{|\mathcal{M}| + \int_{\mathcal{M}} \int_{\mathcal{M}} G(x, y) dy dx}.$$

We note that $\int_M G(x,y)dx$ is continuous in y. Thus, by (107), we confirm that

$$l(q^n) - l(q) \longrightarrow 0$$
 as $n \longrightarrow \infty$.

We also note that $\int_{\mathcal{M}} K(x,y) dy$ is uniformly bounded. Therefore, we have

(110)
$$(l(q^n) - l(q)) \int_{\mathcal{M}} K(x, y) dy \longrightarrow 0 \text{ uniformly in } x.$$

For the first term on the right-hand side of (109), we substitute specifications (78) and (106) for q^n and q, respectively, apply changes of variables, and we find that

$$\int_{\mathcal{M}} K(x), y) (q^{n}(y, t) - q(y, t)) dy = \int_{\mathcal{M}} \left(K(x, \Phi_{t}^{n-1}(a)) - K(x, \Phi_{t}(a)) \right) q_{0}(a) da + \int_{0}^{t} \int_{\mathcal{M}} \left(K(x, \Phi_{t-s}^{n-1}(b)) - K(x, \Phi_{t-s}(b)) \right) f(b, s) db ds.$$

Integrating the absolute value of the left-hand side on \mathcal{M} , and using the relation above, we derive that

$$\begin{split} &\int_{\mathcal{M}} \left| \int_{\mathcal{M}} K(x), y) (q^n(y, t) - q(y, t)) dy \right| dx \\ \leq &|q_0|_{\infty} \int_{\mathcal{M}} \int_{\mathcal{M}} \left| K(x, \Phi_t^{n-1}(a)) - K(x, \Phi_t(a)) \right| da dx + \\ &|f|_{\infty} \int_0^t \int_{\mathcal{M}} \int_{\mathcal{M}} \left| K(x, \Phi_{t-s}^{n-1}(b)) - K(x, \Phi_{t-s}(b)) \right| dx db ds \\ \leq &C(|q_0|_{\infty}, |f|_{\infty}, \mathcal{M}) \left(\int_{\mathcal{M}} \chi(\delta_t^{n-1}(a)) da + \int_0^t \int_{\mathcal{M}} \chi(\delta_{t-s}^{n-1}(b)) db ds \right). \end{split}$$

The function δ_s^n is defined in (84). Using the bound (126) on χ , we derive from the above that

$$\int_{\mathcal{M}} \left| \int_{\mathcal{M}} K(x), y) (q^{n}(y, t) - q(y, t)) dy \right| dx$$

$$\leq C(|q_{0}|_{\infty}, |f|_{\infty}, \mathcal{M}) \left(-\ln \epsilon \int_{\mathcal{M}} \delta_{t}^{n-1}(a) da - \ln \epsilon \int_{0}^{t} \int_{\mathcal{M}} \delta_{t-s}^{n-1}(b)) db \, ds + \epsilon |\mathcal{M}| + \epsilon \cdot t |\mathcal{M}| \right).$$

Thanks to the uniform (in t) convergence (105) of Φ_t^n , it is clear that, for any $\epsilon > 0$, there exists N such that, for any n > N,

(111)
$$\int_{\mathcal{M}} \left| \int_{\mathcal{M}} K(x), y \right| (q^n(y, t) - q(y, t)) dy \right| dx \le C(\mathcal{M}, |q_0|_{\infty}, |f|_{\infty}, T) \cdot \epsilon.$$

The constant C is independent of ϵ . Combining (110) and (111), we conclude that

(112)
$$u^n(x,t) \longrightarrow u(x,t)$$
 strongly in $L^{\infty}(0,T;L^1(\mathcal{M}))$.

It remains to show that the limit $\Phi_t(a)$ of $\Phi_t^n(a)$ is indeed the flow mapping for the limit velocity field $\boldsymbol{u}(x,t)$ of $\boldsymbol{u}^n(x,t)$, i.e. they satisfy the IVP (76) in a certain sense.

LEMMA 5.5. For a.e. $t \in (0,T)$, the flow mapping $\Phi_t(a)$ and the velocity field u satisfy the following relation,

(113)
$$\Phi_t(a) = a + \int_0^t \mathbf{u}(\Phi_s(a), s) ds \qquad \text{in } L^1(\mathcal{M}).$$

Proof. Formally, equation (113) is the limit of the integral equation (79). It has been shown that Φ_t^n converges to Φ_t in $L^1(\mathcal{M})$, uniformly for $t \in (0, T)$. We only need to show that the integral on the right-hand side of (79) converges to the integral on the right-hand side of (113) in $L^1(\mathcal{M})$ as well. To this end, we evaluate the L^1 -norm of the difference of these two integrals,

$$\int_{\mathcal{M}} \left| \int_{0}^{t} \left(\boldsymbol{u}^{n}(\Phi_{s}^{n}(a), s) - \boldsymbol{u}(\Phi_{s}(a), s) \right) ds \right| da \leq$$

$$\int_{\mathcal{M}} \int_{0}^{t} \left| \boldsymbol{u}^{n}(\Phi_{s}^{n}(a), s) - \boldsymbol{u}^{n}(\Phi_{s}(a), s) \right| ds da +$$

$$\int_{\mathcal{M}} \int_{0}^{t} \left| \boldsymbol{u}^{n}(\Phi_{s}(a), s) - \boldsymbol{u}(\Phi_{s}(a), s) \right| ds da.$$

By Lemma 3.1, each u^n is quasi-Lipschitz continuity, and the continuity parameter depends on \mathcal{M} , $|q_0|_{\infty}$ and $|f|_{\infty}$ only, and not on n. Therefore, we can write that

$$\int_{\mathcal{M}} \left| \int_{0}^{t} \left(\boldsymbol{u}^{n}(\Phi_{s}^{n}(a), s) - \boldsymbol{u}(\Phi_{s}(a), s) \right) ds \right| da$$

$$\leq C(\mathcal{M}, |q_{0}|_{\infty}) \int_{0}^{t} \int_{\mathcal{M}} \chi(|\Phi_{s}^{n}(a) - \Phi_{s}(a)| ds da + \int_{0}^{t} |\boldsymbol{u}^{n}(\cdot, s) - \boldsymbol{u}(\cdot, s)|_{L^{1}(\mathcal{M})} ds$$

$$\leq C(\mathcal{M}, |q_{0}|_{\infty}) |\mathcal{M}| \int_{0}^{t} \chi\left(\frac{|\Phi_{s}^{n} - \Phi_{s}|_{L^{1}(\mathcal{M})}}{|\mathcal{M}|}\right) ds + \int_{0}^{t} |\boldsymbol{u}^{n}(\cdot, s) - \boldsymbol{u}(\cdot, s)|_{L^{1}(\mathcal{M})} ds$$

The convexity of the scalar function $\chi(\cdot)$ has been used in deriving the last estimate. By the continuity of the function $\chi(\cdot)$, the L^1 convergence of Φ^n and \boldsymbol{u}^n , we conclude that the above expression goes to zero as n goes to infinity, uniformly in $t \in (0,T)$. The claim is thus proven.

It is straightforward to verify that, if everything is smooth, then the QGPV q solves the transport equation. Indeed,

$$\frac{\partial}{\partial t}q + \boldsymbol{u} \cdot \nabla q = \frac{\partial}{\partial t}q + \nabla q \cdot \frac{d}{dt}\Phi_t(a)$$
$$= \frac{d}{dt}q(\Phi_t(a), t)$$
$$= f(\Phi_t(a), t).$$

However, in general, Φ_t and u do not satisfy (76a) in the classical sense, and the so-defined QGPV q is not necessarily differentiable in time. We now show that q satisfies the transport equation in a weaker sense. Indeed, using the change of variable

 $x = \Phi_t(a)$, we verify that q(x,t) satisfies the weak formulation (32),

$$\begin{split} & \int_0^T \int_{\mathcal{M}} q(x,t) \left(\frac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla \varphi \right) dxdt \\ &= \int_0^T \int_{\mathcal{M}} \left(q_0(a) + \int_0^t f(\Phi_s(a),s) ds \right) \frac{d}{dt} \varphi(\Phi_t(a),t) dadt \\ &= - \int_{\mathcal{M}} q_0(a) \varphi(a,0) da - \int_0^T \int_{\mathcal{M}} f(\Phi_t(a),t) \varphi(\Phi_t(a),t) dadt \\ &= - \int_{\mathcal{M}} q_0(x) \varphi(x,0) dx - \int_0^T \int_{\mathcal{M}} f(x,t) \varphi(x,t) dxdt. \end{split}$$

The process goes through thanks to the fact that the mapping $x = \Phi_t(a)$ preserves area. Thus we have proven the following result.

THEOREM 5.6. Assume that $f \in L^{\infty}(0,T;C(\mathcal{M}))$. Then there exists a solution to the weak problem (33) in $L^{\infty}(0,T;V)$.

In the previous section, we have shown that the initial and boundary conditions are only satisfied in a weak sense. We will now show that the solution ψ actually enjoys better regularity, and satisfies the initial and boundary conditions in the classical sense.

THEOREM 5.7. The initial and boundary conditions (6)–(8) are satisfied in the classical sense, and $\Delta \psi$, $\partial^2 \psi / \partial x \partial t$ are strongly continuous with respect to t on [0, T] in $L^p(\mathcal{M})$ for any p > 1.

Proof. We let $\varphi \in \mathcal{C}_c^{\infty}(\mathcal{M})$. We multiply (4) by φ and integrate over $\mathcal{M} \times [\tau_1, \tau_2]$ for some $0 \le \tau_1 < \tau_2 \le T$,

$$(\Delta \psi - \psi, \varphi)|_{t=\tau_2} - (\Delta \psi - \psi, \varphi)|_{t=\tau_1} - \int_{\tau_1}^{\tau_2} \int_{\mathcal{M}} (\Delta \psi + y - \psi) \nabla^{\perp} \psi \cdot \nabla \varphi dx dt$$

$$= \int_{\tau_1}^{\tau_2} \int_{\mathcal{M}} f(x, t) \varphi(x) dx,$$

$$(q(\cdot, \tau_2), \varphi) - (q(\cdot, \tau_1), \varphi) = \int_{\tau_1}^{\tau_2} \int_{\mathcal{M}} q \nabla^{\perp} \psi \cdot \nabla \varphi + f \varphi dx dt.$$

We note that $\psi \in L^{\infty}(0,T;V)$, q is bounded in $L^{\infty}(Q_T)$, and $\nabla^{\perp}\psi$ is uniformly bounded in Q_T . Thus, as $\tau_2 \longrightarrow \tau_1$,

(114)
$$q(\cdot, \tau_2) \rightharpoonup q(\cdot, \tau_1)$$
 in any $L^p(\mathcal{M})$.

Writing (106) over the interval $[\tau_1, \tau_2]$, one can easily derive that, for $\forall p > 1$,

$$|q(\cdot, \tau_2)|_{L^p(\mathcal{M})} \le |q(\cdot, \tau_1)|_{L^p(\mathcal{M})} + \int_{\tau_1}^{\tau_2} |f(\cdot, t)|_{L^p(\mathcal{M})} dt.$$

From this estimate we conclude that

$$\overline{\lim}_{\tau_2 \to \tau_1} |q(\cdot, \tau_2)|_{L^p(\mathcal{M})} \le |q(\cdot, \tau_1)|_{L^p(\mathcal{M})}.$$

In view of this estimate and the weak convergence (114), the Radon-Riesz theorem applies, and we have

(115)
$$q(\cdot,t) \in C([0,T], L^p(\mathcal{M})), \quad \forall p > 1.$$

Concerning the continuity of $\partial^2/\partial x \partial t$, we rewrite (4) as

$$(\Delta - I)\frac{\partial}{\partial t}\psi = \nabla \times F - \nabla \cdot \left(\nabla^{\perp}\psi(\Delta\psi + y - \psi)\right).$$

Then,

$$\nabla \frac{\partial}{\partial t} \psi = \nabla (\Delta - I)^{-1} \nabla \times F - \nabla (\Delta - I)^{-1} \nabla \cdot \left(\nabla^{\perp} \psi (\Delta \psi + y - \psi) \right),$$

where $(\Delta - I)^{-1}$ is the solution operator of the elliptic boundary value problem (10). We note that $q = \Delta \psi + y - \psi$ is continuous in t in any $L^p(\mathcal{M})$ with p > 1, and $\nabla^{\perp} \psi$ is uniformly bounded in Q_T . Thus, thanks to the continuity of the differential operator $\nabla (\Delta - I)^{-1} \nabla (\cdot)$, and provided that F is continuous in t as well,

(116)
$$\nabla \frac{\partial \psi}{\partial t}(\cdot, t) \in C([0, T]; L^p(\mathcal{M})), \quad \forall p > 1.$$

By Lemma 3.1,

$$|\psi(\cdot,\tau_2)-\psi(\cdot,\tau_1)|_{W^{2,p}(\mathcal{M})} \le Cp|q(\cdot,\tau_2)-q(\cdot,\tau_1)|_{L^p(\mathcal{M})}.$$

Thus, as $\tau_2 \longrightarrow \tau_1$,

$$\psi(\cdot, \tau_2) \longrightarrow \psi(\cdot, \tau_1) \quad \text{in } W^{2,p}(\mathcal{M}),$$

Thus the initial condition (8) is satisfied in a stronger norm,

$$\psi(\cdot,0) = \psi_0(\cdot)$$
 in $W^{2,p}(\mathcal{M})$.

We also note that $\psi \in L^{\infty}(0,T;V)$ implies that

(117)
$$\frac{\partial \psi}{\partial x} \in L^{\infty}(0, T; W^{1,p}(\mathcal{M})).$$

From Lemma 4.2, we have

(118)
$$\frac{\partial^2 \psi}{\partial t \partial x} \in L^{\infty}(0, T; L^p(\mathcal{M})) \subset L^p(Q_T).$$

Combining (117) and (118), we derive that

(119)
$$\frac{\partial \psi}{\partial r} \in W^{1,p}(Q_T), \qquad \forall p > 1.$$

We take a p > 3. Then, by the Sobolev imbedding theorem,

(120)
$$\frac{\partial \psi}{\partial x} \in C^{0,\lambda}(Q_T) \quad \text{for some } 0 < \lambda < 1.$$

Thus, the streamfunction ψ is continuous in the spatial-temporal domain, and the initial and boundary conditions are satisfied in the classical sense.

Finally, (115) combined with (120) implies that

(121)
$$\Delta \psi \in C([0,T]; L^p(\mathcal{M})).$$

We point out that, thanks to (115), the QGPV q assumes its initial value q_0 in the L^p -norm, for any p > 1, which is an improvement over (50a).

6. Concluding remarks. So far, theoretical studies of geophysical models have largely focused on those with "rigid" lids on the top. Models with free or deformable top surfaces are much harder, and belong to the class of free boundary problems. Results on such problems are still scarce. The few published results concern the local existence and uniqueness for the viscous PEs with a free top surface ([25, 24]). This is not surprising. When a top surface is left free, it may break, as it does in reality. The current work deals with the inviscid barotropic QG equation. The free top surface enters the dynamics through its effect on the QGPV, thanks to Kelvin's Circulation Theorem. The current work confirms that, when the free surface is included in this way, the QG model remains globally well-posed.

The current work is part of a project to address the well-posedness of invicid QG equations. The other QG models that are being or will be considered include the multi-layer QG model and the three-dimensional QG model. Within the multi-layer QG model, besides the top surface, the interior layer interfaces are also free to deform. Physically, the deformation of the interior interfaces can be much more significant than those of the top surface, thanks to the reduced gravity in the interior of the fluid ([41]). A mathematical challenge posed by the multi-layer QG model is the fact that the linear differential operator in the QGPV specification is not negative definite, a departure from the barotropic case. The three-dimensional QG equations are posed on a truly three-dimensional domain, but its velocity field remains horizontal. Hence this system is much more complex than the barotropic or multi-layer QG equations, but are notably simpler than fully three-dimensional fluid models, including the Navier-Stokes/Euler equations, and the primitive equatons. These problems will be addressed in forthcoming papers.

Appendix A. Proof of Lemma 5.1. The proof is by the Picard iteration technique. For the moment, we drop the super index from (76), and write the initial boundary value problem in the integral form,

(122)
$$\Phi_t(a) = a + \int_0^t \boldsymbol{u}(\Phi_s(a), s) ds.$$

We let

(123)
$$\Phi_t^0(a) = a, \qquad t \ge 0.$$

Assuming that $\Phi_t^{k-1}(a)$ is known, we compute Φ_t^k by

(124)
$$\Phi_t^k(a) = a + \int_0^t \mathbf{u}(\Phi_s^{k-1}(a), s) ds.$$

When n = 1, using (123) and (124), and the uniform boundedness of \boldsymbol{u} , we obtain

(125)
$$|\Phi_t^1(a) - \Phi_t^0(a)| \le \int_0^t |\boldsymbol{u}(\Phi_s^{k-1}(a), s)| ds \le Ct.$$

We estimate the difference $\Phi_t^k(a) - \Phi_t^{k-1}(a)$ using (124) and the quasi-Lipschitz condition on \boldsymbol{u} ,

$$\begin{aligned} |\Phi_t^k(a) - \Phi_t^{k-1}(a)| &\leq \int_0^t |\boldsymbol{u}(\Phi_s^{k-1}(a), s) - \boldsymbol{u}(\Phi_s^{k-2}(a), s)| ds \\ &\leq C \int_0^t \chi\left(|\Phi_s^{k-1}(a) - \Phi_s^{k-2}(a)|\right) ds. \end{aligned}$$

Here, C is, by assumption, independent of a or t, and the scalar function χ is defined in Lemma 3.1. We note that, for an arbitrary $0 < \epsilon < 1$, this scalar function is bounded by

(126)
$$\chi(r) \le -\ln \epsilon \cdot r + \epsilon, \qquad \forall r \ge 0.$$

Hence, we have

$$(127) |\Phi_t^k(a) - \Phi_t^{k-1}(a)| \le -C \ln \epsilon \int_0^t |\Phi_s^{k-1}(a) - \Phi_s^{k-2}(a)| ds + C\epsilon t.$$

Using (125), we find that

$$|\Phi_t^2(a) - \Phi_t^1(a)| \le -\ln \epsilon \cdot C^2 \frac{t^2}{2} + C\epsilon t.$$

Then, by induction, we find that

$$(128) |\Phi_t^k(a) - \Phi_t^{k-1}(a)| \le C^k \left(-\ln \epsilon\right)^{k-1} \frac{t^k}{n!} + \epsilon \sum_{k=1}^{k-1} C^k \left(-\ln \epsilon\right)^{k-1} \frac{t^k}{k!}.$$

The summation on the right-hand side can be bounded by an exponential function,

$$\epsilon \sum_{k=1}^{k-1} C^k \left(-\ln \epsilon\right)^{k-1} \frac{t^k}{k!} \le C\epsilon t e^{-C\ln \epsilon t} = Ct \epsilon^{1-Ct}.$$

Thus,

(129)
$$|\Phi_t^k(a) - \Phi_t^{k-1}(a)| \le Ct \frac{(-C \ln \epsilon \cdot t)^{k-1}}{n!} + Ct \epsilon^{1-Ct}.$$

This estimate holds for arbitrary $0 < \epsilon < 1$. We take $\epsilon = e^{-k}$. We derive from (129) that

$$|\Phi_t^k(a) - \Phi_t^{k-1}(a)| \le C^k \cdot n^{k-1} \frac{t^k}{n!} + Cte^{-k(1-Ct)}.$$

By the Stirling formula,

(130)
$$\frac{n^{k-1}}{k!} \le \frac{e^k}{\sqrt{2\pi}k^{\frac{3}{2}}}.$$

Thus,

$$|\Phi^k_t(a) - \Phi^{k-1}_t(a)| \le \frac{(Cet)^k}{\sqrt{2\pi}k^{\frac{3}{2}}} + Cte^{-k(1-Ct)}.$$

We choose

$$t^* = \frac{1}{2Ce}.$$

Then,

$$Cet^* = \frac{1}{2}$$
 and $1 - Ct^* \ge \frac{1}{2}$,

and, for all $0 \le t \le t^*$,

$$|\Phi_t^k(a) - \Phi_t^{k-1}(a)| \le \frac{1}{\sqrt{2\pi}k^{\frac{3}{2}}} \left(\frac{1}{2}\right)^k + Ct^*e^{-\frac{1}{2}k}.$$

Thus, for $0 \le t \le t^*$, $|\Phi_t^k(a) - \Phi_t^{k-1}(a)|$ is a convergent geometric sequence, and therefore $\Phi_t^k(a)$ is Cauchy and converges to a limit function $\Phi_t(a)$, which solves the integral equation (122) and thus the IVP (76) on the interval $[0, t^*]$. The choice of t^* depends on the generic constant C only, and is independent of the initial position a. Hence, the same procedure can be applied to extend the solution $\Phi_t(a)$, for every $a \in \mathcal{M}$, to the whole time interval [0, T].

For uniqueness, we assume that $\Psi_t(a)$ is another solution satisfying

(131)
$$\Psi_t(a) = a + \int_0^t \mathbf{u}(\Psi_s(a), s) ds,$$

and $\Psi_t(a)$ differs from $\Phi_t(a)$ for every $t \in (0, \sigma)$ for some $\sigma > 0$. If this is not the case, we can always move the initial point to where $\Phi_t(a)$ and $\Psi_t(a)$ start to fork. Subtracting this equation from (122), and using the quasi-Lipschitz condition on \boldsymbol{u} , we find

$$|\Phi_t(a) - \Psi_t(a)| \le -C \ln \epsilon \int_0^t |\Phi_s(a) - \Psi_s(a)| ds + C\epsilon t, \quad \forall \, 0 < \epsilon < 1.$$

One can obtain an estimate on the difference using the Gronwall inequality,

$$|\Phi_t(a) - \Psi_t(a)| \le \frac{\epsilon}{-\ln \epsilon} \left(e^{-Ct \ln \epsilon} - 1 \right).$$

This inequality holds for arbitrary $0 < \epsilon < 1$, and for all $t \in [0, T]$. We take $\epsilon = e^{-k}$ for some integer n > 0. Then

$$|\Phi_t(a) - \Psi_t(a)| \le \frac{e^{-k}}{n} \left(e^{Ctn} - 1 \right) = \frac{1}{n} e^{(Ct-1)n} - \frac{e^{-k}}{n}.$$

We set $t^* = \frac{1}{2C}$. Then

$$Ct^* - 1 = -\frac{1}{2}$$

and for $0 \le t \le t^*$,

$$|\Phi_t(a) - \Psi_t(a)| \le \frac{1}{n} e^{-\frac{1}{2}k} - \frac{e^{-k}}{k}.$$

We note that this estimate holds for arbitrary k's. For this to be possible, $\Phi_t(a)$ and $\Psi_t(a)$ must agree on $[0, t^*]$, which contradicts the assumption on Ψ_t . Hence $\Phi_t(a)$ is a unique solution.

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